Tate Conjecture for Drinfeld Modules in Equal Characteristic

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Abstract

We prove that the ring of endomorphisms of the $\wp$-divisible group of a Drinfeld module of characteristic $\wp$ is canonically isomorphic to the ring of endomorphisms of the underlying Drinfeld module, completed in the $\wp$–adic topology. This completes the proof of the Tate Conjectures in the Drinfeld module setting.

1 Introduction

Fix a finite field ${}_q$. All constructions will take place within the category of ${}_q$–algebras. Hence in particular, any fields, rings, or schemes introduced in this paper will be over ${}_q$.

Suppose A is the ring of functions of a curve X, L a field, finitely–generated over ${}_q$, and $\iota : A \to L$ a morphism with $\wp := \ker \iota$ nonzero. For these data we can form the $\wp$–divisible group $\phi[\wp^\infty] := \varinjlim \phi[\wp^n]$, and the Tate Conjecture in this context asserts that the canonical map

$$\operatorname{End} \phi \otimes_A A_\wp \to \operatorname{End}(\phi[\wp^\infty])$$

is an isomorphism.

We can frame the equicharacteristic Tate Conjecture in the context of *τ–modules* on an ${}_q$–algebra B equipped with an endomorphisms τ. A τ-module on (B, τ) is a pair (M, σ) consisting of a free B–module M of finite rank together with an injective ${}_q$–linear *structure endomorphism* σ which is τ–semi-linear; morphisms of τ–modules are ordinary module homomorphisms that commute with the structure endomorphisms. Endowing the tensor product of the modules of two τ–modules with the obvious structure morphism permits us to form tensor products in the category of τ–modules over B, and if (M, σ) is a τ–module on B, then one can form the submodule M^σ of elements of M fixed by the structure morphism σ.

Now suppose K an algebraic function field in one variable over ${}_q$, $\wp$ a place of K, $K_\wp$ the completion of K at $\wp$, and L a field of finite type over ${}_q$. We set $K' := K \otimes_{{}_q} L$, and we denote by $K'_\wp$ the completion of K' with respect to the $\wp$–adic topology. We extend the Frobenius morphism τ on L to the endomorphism $\tau \otimes$ on K'; it extends naturally to an endomorphism (which by a slight abuse of notation we also call τ) of the completion $K'_\wp$. The Tate conjecture in this context is then the following assertion:

1.1 Conjecture (Tate Conjecture). *Suppose (M, σ) is a τ-module over K'. Then the natural map of $K_\wp$–vector spaces*

$$K_\wp \otimes_K M^\tau \to (K'_\wp \otimes_{K'} M)^\tau$$

is an isomorphism.

Fixing an isomorphism $M \cong K'^{\oplus r}$, the Tate Conjecture can be restated in the following fashion (after [14]):

1.2 Conjecture (Tate Conjecture, Reformulation). *Suppose A is a matrix in $\mathrm{Mat}_{r\times r}(K')$. Consider the linear Frobenius equation*

$$AX^{\tau} = X \tag{1.2.1}$$

where X is a variable taking values in ${K'_{\wp}}^{\oplus r}$, upon which τ acts component-wise. Let V (respectively, $V_{\wp}$) be the space of solutions of (1.2.1) in ${K'}^{\oplus r}$ (resp., in ${K'_{\wp}}^{\oplus r}$). Then the natural homomorphism of $K_{\wp}$-vector spaces $K_{\wp} \otimes_K V \to V_{\wp}$ is an isomorphism.

The injectivity of this map is easy, and one can reduce the problem without difficulty ([14]) to the case in which $K = \ _q(t)$, $\wp = (t)$, K' is the ring $L[t][S^{-1}]$, where S is the multiplicative system of all monic polynomials, and $K'_{\wp} = L((t))$.

Now the Tate Conjecture falls naturally in two cases. In the first, M comes from a t–motive of characteristic different from $\wp$. This is the *mixed characteristic case*. The first important result in this direction was proved by Anderson, who showed that any endomorphism of the t–divisible group over the generic fiber of a discrete valuation ring in mixed characteristic extends to the closed fiber. Taguchi then proved ([14]) a special case of the Conjecture. Shortly thereafter, Tamagawa verified the general mixed characteristic case (see [16] or [15] for a nice exposition).

In the second case, M comes from a t–motive of characteristic equal to $\wp$. This is the *equicharacteristic case* that we address in this paper. Our strategy will be as follows. We will use the main theorem of de Jong [10], and in particular the proof without the crystalline hypotheses given by Messing and Zink [12], translated into our context to prove the analogue of Anderson's theorem. We then will use the height–theoretic methods of Taguchi to prove the equicharacteristic case of the Conjecture for the endomorphisms of the motive of a Drinfeld module.

2 All sorts of τ–modules

We give a short review of the basic definitions of τ–modules here. For further details, see [6].

2.1 τ–modules over Frobenius rings

2.1 Definition. A **Frobenius ring** is a pair (R, τ) consisting of a commutative ring R and an endomorphism τ of R, which we call the **Frobenius** of R. A **morphism** of Frobenius rings is a ring homomorphism that commutes with the Frobenius.

A τ–**module** over a Frobenius ring (R, τ) is a pair (M, σ) consisting of an R–module M and a τ–semilinear endomorphism σ of M (i.e., σ is additive, and for any $a \in R$ and $m \in M$, $\sigma(am) = a^{\tau}\sigma(m)$). The map σ is called the **structure endomorphism** or (by an abuse of terminology) the **Frobenius** of the τ–module (M, σ). A **morphism** of τ–modules over (R, σ) is a homomorphism of R–modules that commutes with the structure endomorphisms.

2.2 Remark. Observe that we have chosen to work systematically with τ–modules instead of more general objects such as τ–sheaves. As a result, many of the theorems we quote here will be stated only for τ-modules though they frequently hold more generally.

2.2 τ–modules over curves

A somewhat more restricted notion is that of a τ–module over a curve.

2.3 Definition. Suppose R a commutative ${}_q$–algebra; set $X := \operatorname{Spec} R$. Suppose $C := \operatorname{Spec} A$ an absolutely irreducible smooth affine curve with constant field ${}_q$, and set $F := \operatorname{Frac} A$. For the product $C_X := C \times_{\operatorname{Spec} \ {}_q} X = \operatorname{Spec}(A \otimes_{{}_q} R)$, Denote by f the Frobenius endomorphism of R given by the assignment $s \mapsto s^q$. Then $A \otimes_{{}_q} R$ is a Frobenius ring with Frobenius $\tau := \ \otimes f$. Then a **τ–module of rank r over C_X** is a pair (M, σ) consisting of a projective module M over $A \otimes_{{}_q} R$ of rank r and an *injective* morphism $\sigma : \tau^* M \to M$.

2.4 Example. Suppose K a field extension of ${}_q$, and set $C := {}^1 = \operatorname{Spec} \ {}_q[t]$ and $X := \operatorname{Spec} K$. Then a τ–module over C_X is just a free finite module over $K[t]$ together with a τ–semilinear injective endomorphism, where τ acts as Frobenius on K and fixes t.

2.3 Formal and equicharacteristic τ–modules

2.5 Definition. Suppose R an ${}_q$–algebra, and let $R[[t]]$ be the usual t–adic completion of $R \otimes \ {}_q[[t]]$. Then a **formal τ–module over R** is a τ–module over $R[[t]]$, where τ acts as Frobenius on R and fixes t.

In this paper we will use a slightly more restrictive class of formal τ–modules that is more convenient for our purpose; in particular, the τ–modules we will consider will be defined to be *equicharacteristic τ–modules*.

2.6 Definition. A **frame** (A, τ) is a Frobenius ring (A, τ) such that A is an ${}_q[t]$–algebra without t–torsion, and τ satisfies

$$\tau(a) = a^q \mod tA$$

and fixes t. A **morphism** of frames is a morphism of Frobenius rings.

2.7 Definition. Now let R be a reduced ring of characteristic p. Set $A := R[[t]]$ and take for τ the endomorphism that acts as the q-th Frobenius on the coefficients of a power series in A and fixes t. Then (A, τ) is a frame, and an **equicharacteristic τ–module of rank r over (A, τ)** is a pair (M, σ) consisting of a finitely generated projective A–module M of rank r, together with a τ–linear endomorphism σ of M such that the linearized map

$$A \otimes_{(A,\tau)} \wedge^r M \to \wedge^r M$$

is of the form $t^s u$ for some integer s and an isomorphism $u : A \otimes_{(A,\tau)} \wedge^r M \to \wedge^r M$.

Observe that if M is free, then the last requirement says that, in some basis of M, $\det \sigma = t^s \epsilon$ for some $\epsilon \in A$. We will write $s = \operatorname{ord}_t \det F$.

2.4 Quasi–τ–modules

As mentioned earlier, the category of τ–modules possesses a tensor product; likewise, the category of equicharacteristic τ–modules possesses a tensor product, and the equicharacteristic τ–module (A, τ) (where R is a reduced ring of characteristic p, $A := R[[t]]$, and τ acts as the q-th Frobenius on R and fixes t) is a unit for $- \otimes -$. In order to guarantee the existence of internal Hom, it is necessary to work in a slightly bigger category, that of *quasi–τ–modules*:

2.8 Definition. A **quasi–τ–module over** A is a pair (M, σ) consisting of a projective A–module M and a τ–linear **structure homomorphism**

$$\sigma : M \to M[1/t],$$

such that for some $m \geq 0$, the pair $(M, t^m\sigma)$ is a τ–module. A **morphism** of quasi–τ–modules is of course a homomorphism of modules that commutes with the τ–linear structure homomorphisms.

2.9 Definition. A homomorphism $\alpha : (M_1, \sigma_1) \to (M_2, \sigma_2)$ of quasi–τ–modules of equal rank r is called an **isogeny** if $\wedge^r \alpha$ is of the form $t^s u$ for an integer $s \geq 0$ and an isomorphism $u : \wedge^r M_1 \to \wedge^r M_2$.

In this article we will concentrate on formal τ–modules, which are essentially Tate modules in mixed characteristic and the duals of t–divisible groups in equal characteristic. A quasi–τ–module should be thought of as the generic fiber a formal τ–module. So in particular, suppose L a perfect field with a valuation ν, and let $\overline{L}$ be the algebraic closure. Set $R := \overline{L}$, and let τ be the endomorphism of A such that $\tau(x) = x^q$ for any $x \in R$ and $\tau(t) = t$. The notion of a quasi–τ–module over $\overline{L}$ is thus the function field analogue of an *isocrystal* in the classical case.

If (M, σ) is a quasi–τ–module over $\overline{L}[[t]]$, then $M \otimes_{\overline{L}[[t]]} \overline{L}((t))$ is a finite–dimensional vector space over $\overline{L}((T))$, and τ is bijective. Note that we lose no information when working in the category of quasi–τ–modules over $k[[t]]$, where k is any field, by tensoring over $k[[t]]$ with $k((t))$. We record here a theorem from [11], Theorem 2.4.5, page 32, special cases of which we will need. Notice that in [11], quasi–τ–modules are called *Dieudonné modules.*

2.10 Theorem. *The category of quasi–τ–modules over $\overline{L}[[t]]$ is $\overline{L}((t))$–linear and semisimple. The simple objects (up to isomorphism) are the objects $(M_{r,s}, \sigma_{r,s})$, where r and s are integers with $r \geq 1$, $(r, s) = 1$, and*

$$M_{r,s} = S^{\oplus r}$$

and

$$\sigma_{r,s}(e_i) = \begin{cases} e_{i+1} & \text{if } i = 1, \ldots, r-1 \\ t^s & \text{if } i = r \end{cases}$$

where $\{e_i\}$ is the standard basis of $S^{\oplus r}$.

If the residue field L is not algebraically closed, then the category of quasi–τ–modules is L–linear, Artin, and noetherian. If L' is a field extension of L, we have an obvious base–change functor $L' \otimes_L -$. If (M, σ) is an indecomposable quasi–τ–module over a perfect field K, then there exists a unique triple of integers (r, s, t), with $r, s \geq 1$ and $(r, s) = 1$ such that $\overline{K} \otimes_K (M, \sigma)$ is isomorphic to $(M_{r,s}, \sigma_{r,s})^t$. We have the following definition:

2.11 Definition. The **slope** of (M, σ) is the rational number s/r.

2.5 Analytic τ–modules

We have discussed ordinary τ–modules, formal τ–modules, and what we have called equicharacteristic τ–modules. There are also *analytic* τ–modules, to which we now turn.

First we must recall some results and terminology from [6].

2.12 Definition. A τ–module over a discretely valued field K is said to have **good reduction** if there is a τ–module over R, the ring of integers of K, such that the generic fiber is the original τ–module. A τ–module over R is called **non–degenerate** if the fiber at the closed point contains a non–zero τ–module.

We can consider the formal spectrum of the ring $R[[t]]$ and associate to it a "generic fiber" $\mathcal{D}_K$ in the sense of [3], which is an affinoid subspace of $\operatorname{Spec} K[t]$. We will denote by $K\{\{t\}\}$ its ring of functions.Given a τ–module M over $K[t]$, we can base change to a τ–module $M^{an} := M \otimes_{K[t]} K\{\{t\}\}$. We call M^{an} the **analytic τ–module** associated to M.

Suppose M and M' two τ–modules on $C \times_{\mathrm{Spec}\ \ _q} \operatorname{Spec} K$. One can form the abelian group $\operatorname{Hom}^{\mathrm{an}}(M, N)$ of **analytic morphisms**, namely the abelian group of $K\{\{t\}\}$–linear homomorphisms $M^{\mathrm{an}} \to N^{\mathrm{an}}$ commuting with the τ–actions.

3 Results on formal τ–modules

Fix a perfect extension field k of $\ _q$. Our aim is now to prove the following analogue of a theorem of de Jong:

3.1 Theorem. *Suppose M and N be equicharacteristic τ–modules over $k[[t, x]]$, and suppose*

$$\phi : M \otimes_{k[[x,t]]} k((x))[[t]] \to N \otimes_{k[[x,t]]} k((x))[[t]]$$

a morphism of τ–modules. Then $\phi(M) \subset N$.

In order to proceed, we collect some of the key technical facts from the theory of formal τ–modules we will require in the sequel.

3.1 Formal τ–modules and t–divisible k–schemes

3.2 Definition. A **t–divisible k–scheme in $\ _q[[t]]$–modules** is an inductive system of finite k–schemes in $\ _q[[t]]$–modules

$$G = (G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} \dots) = \varinjlim G_n$$

such that, for each integer $n \geq 1$, the following axioms hold:

1. As a k–scheme in $\ _q$–vector spaces, G_n can be embedded into $\ ^N_{a,k}$ for some integer $N \geq 0$. Note that since there is a canonical map from $\ _q$ to k, then $\ _{a,k}$ can be considered as a k–scheme in $\ _q$ vector spaces.
2. There exists an integer $d \geq 0$—called the **rank** of G—such that as a k–scheme in $\ _q$–vector spaces, the dimension of G_n is nd.
3. the sequence of k–schemes in $\ _q[[t]]$–modules

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{t^n} G_{n+1}$$

is exact.

A **morphism** of t–divisible k–schemes in $\ _q[[t]]$–modules is a morphism of the corresponding inductive systems.

We will construct contravariant functors between this category and the category of τ–modules over k. More generally, let $\mathcal{M}(k)$ be the category whose objects are pairs (M, f) consisting of a finite–dimensional k–vector–space M and a Frobenius–linear endomorphism f of M, and whose morphisms are linear maps commuting with these endomorphisms. Let $\mathcal{S}(k)$ be the full subcategory of the category of finite k–schemes in $\ _q$–vector spaces spanned by those k–schemes in $\ _q$–vector spaces that can be embedded in $\ ^{\times N}_{a,k}$ for some $N \geq 0$.

Suppose $(M, f) \in \mathcal{M}(k)$. We define a functor $G_k(M, f)$ from the category of k–algebras to the category of $\ _q$–vector spaces with

$$G_k(M, f)(R) := \{g \in \operatorname{Hom}_k(M, R) \mid \forall m \in M \ g(f(m)) = g(m)^q\}$$

It is easy to see that $G_k(M,f)$ is in fact an object of $\mathcal{S}(k)$ (after taking a basis of M over k). This defines a contravariant functor G_k from $\mathcal{M}(k)$ to $\mathcal{S}(k)$.

If, conversely, G is an object of $\mathcal{S}(k)$, the group of homomorphisms of k–schemes in ${}_q$–vector spaces

$$M_k(G) = \mathrm{Hom}(G, \quad {}_{a,k})$$

is a finite dimensional k–vector space, and, moreover, the Frobenius endomorphism of ${}_{a,k} = \operatorname{Spec} k[t]$ over ${}_q$ induces a Frobenius–linear morphism $f_k(G) : M_k(G) \to M_k(G)$. This defines a contravariant functor M_k from $\mathcal{S}(k)$ to $\mathcal{M}(k)$.

We now state the following proposition from [11], Proposition 2.4.11, page 34. There the theorem is stated for $k \subset \overline{}_p$ but one sees from the proof, in exact analogy to the case of F–crystals, that one needs only that the field in question be perfect.

3.3 Proposition. *Let k be a perfect field containing ${}_q$. The contravariant functors G_k and M_k between $\mathcal{M}(k)$ and $\mathcal{S}(k)$ are exact and quasi–inverse to each other.*

We now extend the definition of the functors G_k and M_k by setting:

$$G_k(M,f) := \varinjlim G_k(M/t^nM, f \mod t^nM)$$

for a τ–module (M,f) over k and

$$(M_k(G), f_k(G)) = \varprojlim (M_k(G_n), f_k(G_n))$$

for any t–divisible k–scheme in ${}_q[[t]]$–modules $G = \varinjlim G_n$.

3.4 Corollary. *The functors G_k and M_k establish an anti–equivalence of categories between the category of formal τ–modules over k and the category of t–divisible k–schemes in ${}_q[[t]]$–vector spaces.*

This relates our main theorem to the Drinfeld module setting. More precisely, as an application of Theorem 3.1, it follows that the ring of endomorphisms of the t–divisible k–scheme associated to a Drinfeld module ϕ of positive characteristic over k is precisely the ring of endomorphisms of ϕ completed in the t–adic topology. This is the Tate Conjecture for Drinfeld modules of positive characteristic.

3.2 Base change for τ–modules

3.5 Proposition. *Let $\phi : (A, \tau_A) \to (B, \tau_B)$ be a homomorphism of frames. Assume that (t) is a prime ideal of A contained in the radical of A, and that $\phi(t)$ is not a unit of B.*

Let $\alpha : (M_1, \sigma_1) \to (M_2, \sigma_2)$ be a morphism of equicharacteristic τ–modules over A. If $\alpha \otimes {}_B$ is an isogeny (respectively, an isomorphism), then α is an isogeny (resp., an isomorphism).

Proof. The A–modules M_1 and M_2 have the same rank r. It suffices to show that $\wedge^r \alpha$ is an isomorphism. Consider the commutative diagram:

$$\begin{array}{ccc} \wedge^r M_1 & \xrightarrow{\wedge^r \alpha} & \wedge^r M_2 \\ \downarrow{\scriptstyle \wedge^r \det \sigma_1} & & \downarrow{\scriptstyle \wedge^r \det \sigma_2} \\ \wedge^r M_1 & \xrightarrow[\wedge^r \alpha]{} & \wedge^r M_2 \end{array}$$

Now we write $\det F_i = t^{a_i} u_i$, where $a_i \in \quad$, and u_i is a τ–linear isomorphism. If we tensor the diagram with B we obtain $a_1 = a_2$. Hence the diagram remains commutative if we replace $\det F_i$ by u_i. We divide $\wedge^r \alpha$ by the maximal possible power of t, and call the resulting homomorphism β. This power is bounded because it is bounded over B. If we divide the morphisms of the above commutative diagram by the maximal power of t and consider the result modulo t, we obtain the commutative diagram:

$$\begin{array}{ccc} L_1 & \xrightarrow{\overline{\beta}} & L_2 \\ {\scriptstyle \overline{u}_1}\downarrow & & \downarrow{\scriptstyle \overline{u}_2} \\ L_1 & \xrightarrow[\overline{\beta}]{} & L_2 \end{array}$$

We set $\overline{A} := A/tA$. The maps $\overline{u}_i$ are τ–linear isomorphisms of $\overline{A}$–modules. It suffices to show that $\overline{\beta}$ is an isomorphism. Take an open subset $\operatorname{Spec} \overline{A}_f \subset \operatorname{Spec} \overline{A}$ such that there are isomorphisms L_i is isomorphic to $\overline{A}_f$. Then for some $\rho \in \overline{A}_f$, we may write: $\overline{\beta}(x) = \rho x$ for any $x \in \overline{A}_f$. We note that $\rho \neq 0$; indeed, since $\overline{A}$ is an integral domain, $\rho = 0$ would imply that $\overline{\beta} = 0$, but then contrary to our assumption, β would be divisible by t. Hence we write $\overline{u}_i(x) = \epsilon_i x^q$, where $\epsilon_i \in \overline{A}_f$ are units for $i = 1, 2$. Because the last diagram is commutative we have $\epsilon_2 \rho^q = \rho \epsilon_1$. Since $\overline{A}$ is an integral domain, we may divide this equation by t. We obtain that ρ is a unit. □

In the analogue 3.1 of de Jong's theorem, we can replace equicharacteristic τ–modules by slightly more general objects. To see this we show:

3.6 Lemma. *Suppose M a finitely–generated torsion–free $k[[x,t]]$–module. We set:*

$$M_1 := \{m \in M_{(t)} \mid t^s m \in M, s \in \quad \}$$

Then M_1 is a free $k[[t,x]]$–module.

Proof. Since $k[[x,t]]$ is a UFD, we see that $M_1 = M$ if M is free. If M is not free, we find a free $k[[x,t]]$–module N, not of the same rank as M, such that $M \subset N \subset M_{(t)}$. The last $k[[x,t]]$–module is free, and since

$$N = N_1 := \{n \in N_{(t)} \mid t^s n \in N, s \in \quad \}$$

we find that $M_1 \subset N$.

By definition, N/M_1 has no t–torsion, and therefore $\operatorname{depth}_{k[[x,t]]} N/M_1 \geq 1$. However, since $k[[x,t]]$ is regular of dimension 2, this implies that the cohomological dimension of M_1 is zero. This proves the lemma. □

3.7 Remark. Let M be a torsion–free $k[[x,t]]$–module. Let σ be a τ–linear endomorphism of M such that $\sigma \otimes_{\ _q[t]} \ _q(t)$ is an isomorphism. Since τ operates on $k[[x,t]]_{(t)}$, the map σ extends to a τ–linear endomorphism (which by an abuse we also call σ) of M_1, which induces an isomorphism when tensored with $\ _q(t)$. Hence there is an $a \in k[[x,t]]$ and an integer $m \geq 0$ such that $a \det \tau = t^m$. Since $k[[x,t]]$ is a UFD, this implies that $\det \sigma = t^{m'} u$, where u is a unit, and thus (M_1, σ) is a τ–module. In this sense the pair (M, σ) is very nearly a τ–module. One sees easily that the analogue of de Jong's theorem holds if the pair (M, τ) is of this more general type, if it holds for τ–modules. We will not use this fact in the sequel.

3.3 Tate uniformization and semistability

We now recall Drinfeld's result on Tate uniformization. Consider R, a complete discrete valuation $_q$-algebra with finite residue field k and fraction field K. Suppose ϕ to be a Drinfeld module over K of rank r with bad reduction. There exists a finite totally ramified extension K' of K and the Drinfeld module ϕ' over K' obtained by base extension such that at the closed point, ϕ' is a Drinfeld module of rank s. Then the Tate uniformization theorem of Drinfeld states that there is an analytic morphism from a Drinfeld module ψ of rank s with good reduction over K to ϕ'. In [6], Gardeyn shows that there exists a way to think about this result in the τ–module setting. Here we state these results in a somewhat simpler setting, as that is all that is required.

3.8 Theorem. *In the notation introduced earlier, suppose M and N to be the τ–modules associated to ϕ and ψ, then the analytic morphism $e \in \mathrm{Hom}(\psi, \phi)$ of Drinfeld modules induces an analytic morphism*

$$e^* \in \mathrm{Hom}^{\mathrm{an}}(M, N)$$

Moreover there exists a τ–module P over $K[t]$, trivial over a finite extension of K, such that we have the following exact sequence:

$$0 \to \tilde{P} \to \tilde{M} \to \tilde{N} \to 0$$

of τ–modules over $K\{\{t\}\}$.

We now recall some more results of Gardeyn from [7]. In line with our approach, we phrase them in the language of modules rather than of sheaves, though they hold more generally.

3.9 Definition. Let K be a discretely valued field with ring of integers R and residue field k. We will say that any property holds **potentially** if it holds after a finite extension of the base field K.

1. If (M, σ) is a τ–module over $K[t]$, we say that a τ–module $(\mathcal{M}, \tilde{\sigma})$ over R is a **model** for (M, σ) if there is an isomorphism $(\mathcal{M} \otimes_R K, \tilde{\sigma} \otimes_R K) \to (M, \sigma)$. This model is called **good** if the induced map $\mathcal{M} \otimes_R k \to \mathcal{M} \otimes_R k$ is an injection.
2. If (M, σ) is a τ–module over $K\{\{t\}\}$, we say $(\mathcal{M}, \tilde{\sigma})$ is a **model** for (M, σ) if there is an isomorphism $(\mathcal{M} \otimes_{R[[t]]} K\{\{t\}\}, \tilde{\sigma} \otimes_{R[[t]]} K\{\{t\}\}) \to (M, \sigma)$. The model is called **good** if the induced map $M \otimes_R k \to M \otimes_R k$ is an injection.

An analytic τ–module M^{an} associated to a τ-module M over $K[t]$ is **semistable** if there exists a filtration

$$0 = N_0^{\mathrm{an}} \subset N_1^{\mathrm{an}} \subset \ldots \subset N_n^{\mathrm{an}} = M^{\mathrm{an}}$$

of M^{an} by analytic τ–modules N_i^{an} such that the modules M^{an}/N_i^{an} are torsion free and such that the subquotient τ–modules $M_i^{an} := N_i^{an}/N_{i-1}^{an}$ have good models over $R[[t]]$.

We have the following special case of a theorem from [8]:

3.10 Theorem. *Every analytic τ–module M^{an} coming from the motive M of a Drinfeld module is potentially semistable.*

3.11 Remark. The above result is actually true for any motive built from the motives of Drinfeld modules by the operations of direct sum and tensor product. In fact, the methods of this paper will easily extend to prove the Tate conjecture in this case.

4 Convergent subrings of power series rings

To prove our theorem 3.1 we need to establish results on some special subrings of Laurent series rings in two variables. We start from a more general viewpoint, and then specialize to the case we need. Thus we suppose A a ring equipped with a discrete valuation ν. We wish to investigate particular subrings of the power series ring $A[[t]]$.

4.1 Technical lemmas on convergent power series

4.1 Definition. A **convergent power series** with coefficients in A is an element f of $A[[t]]$, where $f = \sum a_i t^i$ and there exist constants $C_1 \in \quad$ and $C_2 \in \quad_{>0}$ with the property that for every $n \in \quad$, the inequality $\nu(a_n) \geq C_1 - C_2 n$ holds.

Another (simpler) characterization of a convergent power series is that the Newton polygon of f as a function of t be bounded in the plane by some line of negative slope. Clearly this property is preserved under multiplication and addition, and therefore convergent power series form a ring, which we denote $A[[t]]^c$. A particular example of such rings will be given in the following easy proposition. We remind the reader again that all rings introduced are actually $\quad_q$–algebras.

4.2 Proposition. *Let L be a field with $\nu : L \to \quad \cup \{\infty\}$ a nonarchimedean valuation. Then $L[[t]]^c$ is a discrete valuation ring with residue field L and uniformizer t. If L is perfect, then the Frobenius map $\tau : \sum a_i t^i \to \sum a_i^q t^i$ induces an automorphism of $L[[t]]^c$.*

Proof. If $f = \sum a_i t^i \in L[[t]]^c$, then we have $\nu(a_i) \geq -C_1 - C_2 i$ for some $C_1, C_2 \in \quad$. Applying τ to f we get $f^{(q)} := \sum {a_i}^q t^i$. Now $\nu({a_i}^q) = q\nu(a_i) \geq -qC_1 - qC_2 i$; so $f^{(q)}$ is convergent. □

4.3 Proposition. *Suppose $C = (c_{ij})$ be an $h \times h$ matrix with coefficients in $A[[t]]^c$. Let $y \in A[[t]]^{\oplus h}$ be a vector such that for some number $r \geq 0$, $Cy = \tau^r y$. Then we have $y \in (A[[t]]^c)^{\oplus h}$.*

Proof. Let us first define for some $f = \sum a_i t^i \in A[[t]]^c$:

$$\nu(f, n) := -\min\{\nu(a_i) \mid i = 0, \ldots, n\}$$

Note that this is an increasing function in n. We now write our equation in coordinates:

$$\sum_{k=1}^{h} c_{lk} y_k = \tau^r y_l, \quad l = 1, \ldots, h$$

We will show the following: Let $M \geq 0$ be a constant such that for all l, k:

$$\nu(c_{lk}, 0) \leq M, \quad \nu(c_{lk}, n) \leq Mn \quad \text{for}\, n \geq 1$$

Then for any $k = 1, \ldots, n$ we have:

$$\nu(y_k, 0) \leq M, \quad \nu(y_k, n) \leq Mn \quad \text{for}\, n \geq 1.$$

We show this by induction beginning with the case $n = 0$. Choose i such that $\nu(y_i, 0)$ is maximal among the $\nu(y_k, 0)$ for $k = 1, \ldots, h$. We obtain:

$$q^r \nu(y_i, 0) = \nu(\tau^r y_i, 0) \leq \nu(c_{ij} y_j, 0) = \nu(C_{ij}, 0) + \nu(y_j, 0) \leq \nu(c_{ij}, 0) + \nu(y_i, 0)$$

This implies

$$(q^r - 1)\nu(y_i, 0) \leq \nu(c_{ij}, 0) \leq M$$

Now we assume $n \geq 1$ and $\nu(y_k, u) \leq Mu$ for $1 \leq u < n$. Again we choose i such that $\nu(y_i, n)$ is maximal among $\nu(y_k, n)$, and j such that $\nu(c_{ij}y_j, n)$ is maximal among the $\nu(C_{ik}y_k, 0)$. Then we have:

$$q^r\nu(y_i, n) \leq \nu(c_{ij}y_j, n) \leq \max\{\nu(y_j, u) + \nu(c_{ij}, v) \mid u + v = n\}$$

Note that the leftmost term is $\nu(\tau^r y_i, n)$ and the rightmost term is $\leq Mu + Mv = Mn$. We are done. □

We set

$$\widetilde{A} = \{x \in A \mid \nu(a) \geq 0\}$$

Thus $\widetilde{A}[[t]]$ is the set of power series with coefficients in $\widetilde{A}$.

4.4 Corollary. *(of the proof). Assume that the coefficients of C are in $\widetilde{A}[[t]]$. Then $y \in \widetilde{A}[[t]]$.*

4.5 Proposition. *Let $\xi = \sum x_i t^i \in A[[t]]^c$ such that $x_0 \in A$ is a unit. Then ξ is a unit in $A[[t]]^c$.*

Proof. First let us state two easy inequalities. Let α, β be two elements of $A[[t]]$. Then we have that for all $n \in$:

(4.5.1) $$\nu(\alpha + \beta, n) \leq \max\{\nu(\alpha, n), \nu(\beta, n)\}$$

(4.5.2) $$\nu(\alpha\beta, n) \leq \max\{\nu(\alpha, l) + \nu(\beta, n - l) \mid \quad 0 \leq l \leq n\}$$

Moreover, if in either case there is a strict maximum on the right hand side, then these are equalities. Now let us take $\eta \in A[[t]]$ such that $\eta\xi = 1$. We must prove that η is actually in $A[[t]]^c$. Now, since we have that x_0 is a unit in A, we have that $\nu(\xi, 0) = \nu(\eta, 0) = 0$. Choose a constant C so that $\nu(\xi, n) \leq Cn$ and $\nu(\eta, 1) \leq C$. Then we will prove by induction that $\nu(\eta, n) \leq Cn$.

We have by our second inequality (4.5.1) that:

$$0 = \nu(1, n) \leq \max\{\nu(\eta, l) + \nu(\xi, n - l)\}$$

and that the above is an equality if there is a strict maximum. Thus we must have that:

$$\nu(\eta, n) + \nu(\xi, 0) =\leq \max\{0, \nu(\eta, l) + \nu(\xi, n - l) \mid \quad 0 \leq l \leq n - 1\}$$

But now, using our assumptions together with the induction hypothesis we have that:

$$\nu(\eta, n) \leq \max\{0, Cl + (n - l)C \mid \quad 0 \leq l \leq n - 1\}$$

Thus we have proved that $\nu(\eta, n) \leq nC$ and we are done. □

4.2 Convergent power series over Laurent series

We now specialize to the case that we need. Let $K := k((x))$, the field of algebraic Laurent series for some *perfect* field k of finite characteristic. We will study τ–modules over $K[[t]]^c$ in order to attack the Tate Conjecture. We have:

4.6 Corollary. *Let L be a perfect discretely valued field. Then $L[[t]]^c$ is a discrete valuation ring with residue field L and uniformizer t. Moreover τ induces an automorphism of $L[[t]]^c$.*

Suppose k is an algebraically closed field, and let K^{perf} be the perfection of $K := k((x))$. The set $\{x^\alpha \mid \alpha \in \quad [1/p], \quad 0 < \alpha < 1\}$ is a basis of K^{perf} as a K–vector space. We therefore have that an arbitrary element $\xi \in K^{\text{perf}}[[t]]$ can be written uniquely as a convergent sum:

$$\xi = \sum_{0<\alpha<1} x^\alpha f_\alpha$$

where the f_α are in $K[[t]]$. Convergence means that for any $m > 0$, almost all the f_α are in $t^m K[[t]]$. It is easy to check that if $\xi \in K^{\text{perf}}[[t]]^c$ then all the f_α are in $K[[t]]^c$, and conversely, if all the f_α have slopes bounded below by a uniform bound for α, then ξ is convergent.

4.7 Proposition. *Set $K := k((x))$ and $L := K^{\text{perf}}$. Then the natural map*

$$L[[t]]^c \otimes_{K[[t]]^c} K[[t]] \to L[[t]]$$

is injective.

Proof. Both $L[[t]]^c \otimes_{K[[t]]^c} K[[t]]$ and $L[[t]]$ inherit the t–adic topology, and the completion of the left hand side in the t–adic topology is equal to the right hand side. Therefore it suffices to show that the left hand side is separated in the t–adic topology.

Assume first that k is algebraically closed. We mentioned that, using the decomposition of ξ, there is an injection of $K[[t]]^c$–modules:

$$L[[t]]^c \to \prod_\alpha K[[t]]^c$$

Both source and target of the above map possess a natural t–adic topology. Thus it suffices to show that $(\prod_\alpha K[[t]]^c) \otimes_{K[[t]]^c} K[[t]]$ is t–adically separated. This in turn will follow from the injectivity of the map:

$$\left(\prod_\alpha K^c[[t]]\right) \otimes_{K[[t]]^c} K[[t]] \to \prod_\alpha K[[t]]$$

There is a standard proof for this: suppose M a finitely–generated $K[[t]]^c$–submodule of $K[[t]]$. Consider the commutative diagram

$$\begin{array}{ccc} (\prod_\alpha K[[t]]^c) \otimes_{K[[t]]^c} M & \xrightarrow{\iota} & \prod_\alpha M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ (\prod_\alpha K[[t]]^c) \otimes_{K[[t]]^c} K[[t]] & \longrightarrow & \prod_\alpha K[[t]] \end{array}$$

Since any finitely generated module can be expressed as the cokernel of a map ψ of finite free modules, and ψ commutes with $\otimes$, then one sees immediately that ι is an isomorphism. Clearly π_2 is injective. Now any element of $(\prod_\alpha K[[t]]^c) \otimes_{K[[t]]^c} K[[t]]$ is in the image of π_1 for some finitely generated module M. This proves our assertion, provided that k is algebraically closed.

Now suppose k is not algebraically closed, and $\overline{k}$ is the algebraic closure of k. Set $F := \overline{k}((x))$. We have an injection

$$F[[t]]^c \otimes_{k[[t]]^c} K[[t]] \to F[[t]]$$

Indeed, choose $u_i, i \in I$ to be a basis of $\overline{k}$ as a k vector space. Consider a series $f \in F[[t]]$:

$$f = \sum_{m \in} c_m x^m, \quad c_m \in \overline{k}[[t]]$$

We write $c_m = \sum_{i\in I} a_{m,i} u_i$ as a t–adic convergent sum. Then $c_m \in t^n \overline{k}[[t]]$ if and only if, for any $i \in I$, $a_{m,i} \in t^n k[[t]]$. We set $g_i := \sum_{m\in} a_{m,i} x^m$. Clearly $f \in F[[t]]^c$ only if for any $i \in I$, $g_i \in K[[t]]^c$. Thus we obtain an injection:

$$F[[t]]^c \to \prod_{i\in I} K[[t]]^c$$

This proves the result. □

4.8 Proposition. *Let M and N be $K[[t]]^c$–modules, so that N is torsion–free. Suppose $\phi : M \to N \otimes_{K[[t]]^c} K[[t]]$ is a homomorphism of $K[[t]]^c$–modules. Let $\tilde{\phi} := \phi \otimes_{K[[t]]^c} L[[t]]^c$, where $L = K^{\text{perf}}$. Note that, by Proposition 4.7, the target of this morphism injects to $N \otimes_{K[[t]]^c} L[[t]]$. The the natural injection:*

$$(\operatorname{Im}\phi \cap N) \otimes_{K[[t]]^c} L[[t]]^c \to \operatorname{Im}\tilde{\phi} \cap (N \otimes_{K[[t]]^c} L[[t]]^c)$$

is bijective.

Proof. All tensor products will be taken over $K[[t]]^c$. We may assume that $\phi : M \to N \otimes K[[t]]$ is injective. Then by the previous proposition we have that the map

$$\tilde{\phi} : M \otimes L[[t]]^c \to N \otimes K[[t]] \otimes L[[t]]^c \to N \otimes L[[t]]$$

is also injective. It is enough to show that the obvious inclusion:

$$\phi^{-1}(N) \otimes L[[t]]^c \subset \tilde{\phi}^{-1}(N \otimes L[[t]]^c)$$

is actually an equality.

Consider the following injection induced by the map ϕ:

$$M/\phi^{-1}(N) \to (N \otimes K[[t]])/N$$

If we tensor this map with $L[[t]]^c$, we obtain two injections, where the second comes from the previous proposition:

$$M \otimes L[[t]]^c/\phi^{-1}(N) \otimes L[[t]]^c \to (N \otimes K[[t]]) \otimes L[[t]]^c/N \otimes L[[t]]^c$$
$$(N \otimes K[[t]]) \otimes L[[t]]^c/N \otimes L[[t]]^c \to N \otimes L[[t]]/N \otimes L[[t]]^c$$

Composing these two injections gives an injection induced by $\tilde{\phi}$ (since we have tensored a map induced by ϕ with $L[[t]]^c$) from $M \otimes L[[t]]^c/\phi^{-1}(N) \otimes L[[t]]^c$ to $N \otimes L[[t]]/N \otimes L[[t]]^c$. Thus we have that:

$$\tilde{\phi}^{-1}(N \otimes L[[t]]^c) \subset \phi^{-1}(N) \otimes L[[t]]^c$$

since otherwise the composed map would have a non-trivial kernel. But this is what we needed to show. □

5 The slope filtration over $L[[t]]^c$

In this section, L will be a perfect field with valuation ν. Then $L[[t]]$ and $L[[t]]^c$ will be discrete valuation rings whose maximal ideals are each generated by t. We will denote the Frobenius automorphism $\sum a_i t^i \to \sum {a_i}^q t^i$ by σ.

Let (M, τ) be a τ–module over $L[[t]]$. By [11] Appendix B, Theorem 2.4.5, we have that up to isogeny, (M, τ) is a direct sum of isoclinic τ–modules. This implies in particular that M has a filtration by τ–invariant submodules:

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m,$$

such that M_i/M_{i-1} is a τ–module isoclinic of slope λ_i. Moreover we may arrange $\lambda_1 > \ldots > \lambda_m$, and then the filtration is unique. We will show that such a filtration exists for every τ–module over $L[[t]]^c$. Let (M, τ) and (N, τ) be quasi–τ–modules over $L[[t]]$. Let $\lambda_1, \ldots, \lambda_h$ be the slopes of M with multiplicities, where $h = \operatorname{rank}_{L[[t]]} M$, and let $\mu_1, \ldots, \mu_l$ be the slopes of N with multiplicities, where $l = \operatorname{rank}_{L[[t]]} N$. One sees easily that:

- The slopes of $M \otimes N$ with multiplicities are
$$\lambda_i + \mu_j, \quad i = 1, \ldots, h, \quad j = 1, \ldots, l;$$
- the slopes of $\operatorname{Hom}_{L[[t]]}(M, N)$ with multiplicities are
$$\mu_j - \lambda_i, \quad i = 1, \ldots, h, \quad j = 1, \ldots, l;$$
- the slopes of $\wedge^k M$ with multiplicities are
$$\lambda_{i_1} + \ldots + \lambda_{i_k}$$

where the indices run through all tuples, such that $1 \leq i_1 < \ldots < i_k \leq h$. Before we turn to τ–modules over $L[[t]]^c$ we give some slightly more general results.

5.1 Technical lemmas

We now recall some technical results from [12] that we will need later on.

5.1 Lemma. *Let $R \to R'$ be an injective ring homomorphism. Let M be a finitely generated projective R–module, and let $N' \subset M \otimes_R R'$ be a direct summand of constant rank r.*

Then there is a direct summand $N \subset M$ such that $N' = N \otimes_R R'$ if and only if there is a direct summand $L \subset \wedge^r M$ such that $L \otimes_R R' = \wedge^r N'$.

Proof. Let X be the Grassmannian of rank r submodules of M and Y be $(\wedge^r M^\vee)$, where $M^\vee := \operatorname{Hom}_R(M, R)$. The operation $\wedge^r$ induces the Plücker morphism $X \to Y$ which is a closed immersion. Corresponding to N' and L we obtain a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array}$$

The lemma says that there is a lift $\operatorname{Spec} R \to X$. As $X \to Y$ is a monomorphism, this arrow is unique if it exists. Therefore the question of existence is local, and hence we may assume that $\operatorname{Spec} R \to Y$ factors through an affine open $\operatorname{Spec} A \subset Y$. Let us denote by $\operatorname{Spec} B \subset X$ the preimage of $\operatorname{Spec} A$. Then we obtain a commutative diagram of rings:

(5.1.1)
$$\begin{array}{ccc} B & \xleftarrow{\pi} & A \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ R' & \longleftarrow & R \end{array}$$

We want to show that there is a $w : B \to R$ making the diagram commutative. Since $R \to R'$ is injective, we have $u(\operatorname{Ker} \pi) = 0$. But this suffices for the existence of w since π is surjective. This concludes the proof of the lemma. □

Again following [12], Corollary 18, we show:

5.2 Corollary. *Let S be a local ring, and G a set of endomorphisms of S. We denote by $R = S^G$ the ring of invariants. Let M be a finitely generated free R–module. Assume that we are given a direct summand N' of the S–module $M \otimes_R S$, such that $gN' \subset N'$ for each $g \in G$. Then there is a unique direct summand N of the R–module M, such that $N' = N \otimes_R S$*

Proof. We note that R is also a local ring. The uniqueness follows as in lemma 5.1. The lemma also shows that we may assume that N' is of rank 1. Let $e_1, \ldots, e_r$ be a basis of M and let n be a generator of N'. Then we write:

$$n = a_1 e_1 + \ldots + a_r e_r \tag{5.2.1}$$

where $a_i \in S$. Since S is local we may assume without loss of generality that $a_1 = 1$. By assumption there is for each $g \in G$ an element $\lambda(g) \in S$, such that

$$g(n) = \lambda(g)n$$

Inserting for n the right hand side of (5.2.1) and comparing coefficients gives $\lambda(g) = 1$ and $g(a_i) = a_i$. But this implies $a_i \in R$. Then $Rn \subset M$ is the desired direct summand. □

5.2 Existence of the slope filtration

Let (M, τ) be a quasi τ–module over $L[[t]]^c$, where we recall that L is assumed to be a perfect field.

5.3 Proposition. *Let*

$$\lambda_1 > \lambda_2 > \ldots > \lambda_m$$

be the slopes of the quasi τ–module (M, τ); then M has a unique filtration by quasi τ–submodules:

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m$$

such that M_i/M_{i-1} is a non–zero isoclinic quasi τ–module of slope λ_i for $i = 1, \ldots, m$.

We need first the following lemma:

5.4 Lemma. *Let (M, τ) be a quasi–τ–module over $L[[t]]^c$. Let $\lambda = r/s$ where r and s are integers.*

1. *Assume that all slopes of (M, τ) are greater or equal to λ; then there is a quasi–τ–module (N, τ) which is isogenous to (M, τ), and such that $\tau^s N \subset t^r N$.*
2. *Assume that all slopes of (M, τ) are less than or equal to λ; then there is a quasi–τ–module (N, τ) which is isogenous to (M, τ) and such that $\tau^s N \supset t^r N$.*
3. *Assume that (M, τ) is isoclinic of slope λ. Then there is a quasi–τ–module (N, τ) which is isogenous to (M, τ) and such that $\tau^s N = t^r N$.*

Proof. The corresponding statement is true for $M' = L[[t]] \otimes_{L[[t]]^c} M$. In the first case we find a submodule of finite index $N' \subset M'$ such that $\tau^s N' \subset t^r N'$. We set $N = N' \cap M \otimes \ {}_q(t)$. Then we find $\tau^s N \subset t^r N' \cap M \otimes \ {}_q(t) = t^r N$. The other cases are similar. □

5.5 Remark. We note that $\tau^s N \subset t^r N$, if and only if $t^r N^\vee \subset \tau^s N^\vee$ holds for the dual module. This is because $\tau^s N^\vee$ is the dual module to $\tau^s N$ with respect to the pairing:

$$(N \otimes_{\ q[t]} \ _q(t)) \times (N^\vee \otimes_{\ q[t]} \ _q(t)) \to L[[t]]^c \otimes_{\ q[t]} \ _q(t)$$

Proof of Proposition 5.3. To prove the proposition we may change M within its isogeny class. Let $\lambda_1 = \lambda = r/s$ be the highest slope. Then we find $M \subset N \subset M \otimes_{\ q[t]} \ _q(t)$ such that $t^r N \subset \tau^s N$. Let $N' = L[[t]] \otimes_{L[[t]]^c} N$. Let $N'_1 \subset N'$ be the isoclinic part of slope λ. We have to show that there is a direct summand $N_1 \subset N$, such that $N'{}_1 = L[[t]] \otimes_{L[[t]]^c} N_1$. Applying lemma 5.1 we may assume that N'_1 has rank 1. We set $\Phi = t^r \tau^{-s} : N \to N$. Let n be a generator of N'_1. Then we find:

$$\Phi n = un$$

for some unit $u \in L[[t]]$. Let us first assume that L is algebraically closed. Then we may write $u = a^{1-(q)}$. Then we find that $\Phi(an) = an$. Then proposition 4.3 shows that $an \in N$. This proves the proposition in the case that L is algebraically closed. In the general case it shows that there is a direct summand $\overline{N}_1 \subset \overline{L}[[t]]^c \otimes_{L[[t]]^c} N$, where $\overline{L}$ is the algebraic closure of L, such that $\overline{L}[[t]] \otimes_{L[[t]]} N'_1 = \overline{L}[[t]] \otimes_{\overline{L}[[t]]^c} \overline{N}_1$. Since the left hand side of this equality is invariant under the $\mathrm{Gal}(\overline{L}/L)$, so is the direct summand $\overline{N}_1 \subset \overline{L}[[t]]^c \otimes_{L[[t]]^c} N$. It is here that we use the fact that L is perfect. Hence this direct summand descends to a submodule $\overline{N}_1 \subset N$, by corollary 5.2. □

5.3 Descent for convergent τ–modules

5.6 Proposition. *Let M, N be τ–modules over $L[[t]]^c$. Assume that M is isoclinic of slope λ, and that all the slopes of N are less than or equal to λ. Let*

$$\alpha : M \otimes_{L[[t]]^c} L[[t]] \to N \otimes_{L[[t]]^c} L[[t]]$$

be a homomorphism of τ–modules. Then we have $\alpha(M) \subset N$.

Proof. We set $U = (L[[t]]^c, \sigma)$, the unit τ–module. Tensoring α with the dual quasi τ–module $M^\vee$, we obtain a morphism $U \otimes_{L[[t]]^c} L[[t]] \to N \otimes_{L[[t]]^c} M^\vee \otimes_{L[[t]]^c} L[[t]]$ of quasi τ–modules. Twisting this morphism by $U(\ell)$—the ℓ-th "Tate twist", replacing τ by $t^m\tau$—we obtain a morphism of τ–modules. Hence we assume without loss of generality that $M = U(\ell)$. Since the slopes of N are less than or equal to ℓ, we find an isogeny $N \to N'$ such that $t^\ell \tau^{-1} N' \subset N'$. We set $n = \alpha(1) \in N' \otimes_{L[[t]]^c} L[[t]]$. Then we have $t^\ell \tau^{-1} n = n$. By proposition 4.3 we find that $n \in N'$. This shows that $\alpha(M \otimes_{\ q[t]} \ _q(t)) \subset N \otimes_{\ q[t]} \ _q(t)$. But this suffices since:

$$(N \otimes_{\ q[t]} \ _q(t)) \cap (L[[t]] \otimes_{L[[t]]^c} N) = N$$

Indeed, this follows because $x \in L[[t]]$ is in $L[[t]]^c$ if and only if $tx \in L^{[}[t]]$ □

We recall that in our notation $K = k((x))$, the field of Laurent series over a perfect field k.

5.7 Proposition. *Let M and N be τ–modules over $K[[t]]^c$. Assume that N is isoclinic of slope λ. Suppose we are given a morphism of τ–modules*

$$\phi : M \otimes_{K[[t]]^c} K[[t]] \to N \otimes_{K[[t]]^c} K[[t]]$$

such that $\phi \otimes_{\ q[t]} \ _q(t)$ surjective. We set $E = \phi^{-1}(N) \cap M$ and consider the map $\psi : E \to N$ induced by ϕ. Then the map $\psi \otimes_{\ q[t]} \ _q(t)$ is surjective too. Assume moreover that the map $M \to N \otimes_{K[[t]]^c} K[[t]]$ induced by ϕ is injective. Then all slopes of M are less than or equal to λ.

Proof. Let $\phi_0 : M \to N \otimes_{K[[t]]^c} K[[t]]$ be the restriction of ϕ. Clearly we may assume that this map is injective. If we tensor ϕ_0 by $\otimes_{K[[t]]^c} L[[t]]^c$, where L is the perfection of K, we obtain by 4.7 an injection:

$$\phi_1 : M \otimes_{K[[t]]^c} L[[t]]^c \to N \otimes_{K[[t]]^c} L[[t]]^c \to N \otimes_{K[[t]]^c} L[[t]]$$

Let μ be the highest slope of $M \otimes_{K[[t]]^c} L[[t]]^c$ and consider the first step of the slope filtration $M(\mu) \subset M \otimes_{K[[t]]^c} L[[t]]^c$. Then $M(\mu)$ is a τ–module which is isoclinic of slope μ. Consider the injection

(5.7.1) $$M(\mu) \to N \otimes_{K[[t]]^c} L[[t]]^c$$

induced by ϕ_1. If $\mu \neq \lambda$, the map $M(\mu) \otimes_{L[[t]]^c} L[[t]] \to N \otimes_{K[[t]]^c} L[[t]]$ would be zero. This contradicts the injectivity of (5.7.1). Hence all slopes of M are less than or equal to λ.

It follows from proposition 5.6 that ϕ_1 maps $M(\lambda)$ to $N \otimes_{K[[t]]^c} L[[t]]^c$. Looking for the slope decomposition over $L[[t]]$, we find that the map $M(\lambda) \otimes_{\ q[t]} \ {}_q(t) \to N \otimes_{K[[t]]^c} L[[t]]^c \otimes_{\ q[t]} \ {}_q(t)$ is surjective. This shows that $\mathrm{Im}(M \otimes_{K[[t]]^c} L[[t]]^c) \cap (N \otimes_{K[[t]]^c} L[[t]]^c)$ has the same rank as N. It follows by corollary 4.8 that $\mathrm{Im}(M) \cap N$ has the same rank as N. □

6 Extension of endomorphisms

Our theorem depends on two further results, the first of which seems to require the theory of rigid analytic functions for its proof. After having stated them, we show how they imply the theorem, and then give the proofs in the following two sections.

6.1 Theorem. *Let M be a τ–module over $k[[t]][[x]]$. Let $N^c \subset M \otimes_{k[[x,t]]} k((x))[[t]]^c$ be an τ–invariant direct summand. Then there is a unique τ–submodule $N \subset M$, such that $N^c = N \otimes_{k[[x,t]]} k((x))[[t]]^c$.*

6.2 Corollary. *Given (M_1, τ_{M_1}) and (M_2, τ_{M_2}) τ–modules over $k[[x,t]]$ and a morphism*

$$\phi^c : (M_1 \otimes_{k[[t,x]]} K[[t]]^c, \tau_{M_1}) \to (M_2 \otimes_{k[[x,t]]} K[[t]]^c, \tau_{M_2})$$

there is a morphism $\phi : (M_1, \tau_{M_1}) \to (M_2, \tau_{M_2})$ such that $\phi^c = \phi \otimes K[[t]]^c$.

Proof. Apply the theorem to the graph of ϕ^c. □

6.3 Proposition. *Let $M \to N$ be a homomorphism of quasi–τ–modules over $k[[x,t]]$. Assume $M \otimes_{k[[x,t]]} K[[t]]$ is isoclinic of slope λ and that all slopes of $N \otimes_{k[[x,t]]} K[[t]]$ are less than or equal to λ. If the map*

$$M \otimes_{\ q[t]} \ {}_q(t) \to N \otimes_{\ q[t]} \ {}_q(t)$$

is injective, then it admits a τ–equivariant retraction.

Proof of Theorem 3.1. : We note first that it is enough to prove the "rational version" of the statement, namely that $\phi(M \otimes_{\ q[t]} \ {}_q(t)) \subset N \otimes_{\ q[t]} \ {}_q(t)$. To prove this claim we need to check that

(6.3.1) $$(N \otimes_{\ q[t]} \ {}_q(t)) \cap (N \otimes_{k[[x,t]]} k((x))[[t]]) = N$$

We can assume for this that $N = k[[x,t]]$. Since $k[[x,t]]$ is a UFD we see that $k[[x,t]]_{(t)}/k[[x,t]]$ has no t-torsion. Since $k((x))[[t]]/k[[x,t]]_{(t)}$ has no t Let $\hat{N}$ be the quasi τ–module dual to N. Denote by U the unit τ–module. Then, in the category of quasi τ–modules over $k[[x,t]]$ we have that (6.3.1) holds.

$$\mathrm{Hom}(M,N) \cong \mathrm{Hom}(M \otimes \hat{N}, U)$$

Thus we may assume that $N = U(\ell)$, which is a rank one τ–module of slope ℓ.
Consider the map $\hat{\phi} : M \to N \otimes_{k[[x,t]]} K[[t]]$. Dividing by the kernel, we may assume that the map is injective. To show this consider M', the factor of M by this kernel. If M' is not free we consider the module M'_1 given by Lemma 3.6. For a suitable integer a we obtain an injective map $t^a\hat{\phi} : M'_1 \to N \otimes K[[t]]$.

Assume then that $\hat{\phi}$ is injective, and denote by M^c the module $M \otimes_{k[[x,t]]} K[[t]]^c$. The map

$$M^c \to N \otimes_{k[[x,t]]} K[[t]]$$

is also injective. Indeed by theorem (6.1), the kernel would be defined over $k[[x,t]]$. Thus it is zero since $\hat{\phi}$ is injective. Applying proposition (5.7) to the above morphism, we obtain that all slopes of M^c are less than or equal to λ. Moreover $E^c = M^c \cap N^c$ is a τ–submodule of rank 1 and slope ℓ of M^c.

By theorem (6.1) there is an $E \subset M$, such that $E^c = E \otimes_{k[[x,t]]} k((x))[[t]]^c$. Applying the corollary to theorem (6.1) to the map $E^c \to N^c$ we obtain that $\hat{\phi}(E) \subset N$. Since the slopes of M^c are less than or equal to ℓ, the τ–submodule $E \subset M$ has by Proposition 6.3 a complement E' up to isogeny—i.e., the map $E \oplus E' \to M$ has cokernel supported on the closed point. Now we conclude $\hat{\phi}(E') \subset N$ by induction on the rank of M. □

7 Proof of Theorem 6.1

In his original paper [10] de Jong proved his theorem for F–crystals rather than for Frobenius modules as in [12]. Thus he made use of the differential operator θ, given in the data of a crystal, in the proof of this result. The author suspected, given the analogy to the Drinfeld module case, that this was not needed, but it was only in discovering the paper of Zink and Messing that it became clear how to make this result apply.

To prove this theorem we must first recall some facts about non–archimedean rigid analysis. Let k be a perfect field, and let Ω be the algebraic closure of the field $k((t))$. We set $|a| = p^{-ord_t a}$ for $a \in \Omega$.

7.1 Definition. Let I be a nonempty interval of nonnegative real numbers. We denote by $\mathcal{L}(I)$ the set of all formal Laurent series

$$\sum_{n\in} a_n x^n, \quad a_n \in k((t))$$

such that for all $\rho \in I \setminus 0$

$$\lim_{|n|\to\infty} |a_n|\rho^n = 0$$

and if $0 \in I$ then $a_n = 0$ for $n < 0$. An element of $\mathcal{L}(I)$ is called a Laurent series convergent in I.

We have that $\mathcal{L}(I)$ has a ring structure, whereas the set $\mathcal{L}$ (no convergence condition) does not. Let $f \in \mathcal{L}(I)$ for arbitrary I. An I–root of f will be an element $z \in \Omega$ such that $|z| \in I$, and $\sum_n a_n z^n = 0$. The following (see [12]) is a consequence of the theory of Newton polygons.

7.2 Theorem. *Assume that a non–zero $f \in \mathcal{L}(I)$ has only finitely many I–roots. Then there is a unique monic polynomial $P \in k((t))[x]$ whose roots in Ω have absolute values in I, and a unit u in the ring $\mathcal{L}(I)$ such that $fu = P$. In particular $f \in \mathcal{L}(I)^*$ iff f has no I–roots.*
If the interval I is compact, then $f \in \mathcal{L}(I)$ has only finitely many I–roots.

7.3 Notation. We will have need of the following rings of Laurent series:

$$\mathcal{D}_\eta = \mathcal{L}([\eta, 1)), \text{for } 0 \leq \eta < 1$$

We set $\mathcal{D} = \cup_\eta \mathcal{D}_\eta$. We have $k((x))[[t]]^c \subset \mathcal{D}$. One can show that $k((x))[[t]]^c[t^{-1}]$ consists of all the Laurent series $f \in \mathcal{D}$ such that $|a_n|$ is bounded, though we do not need this here. For any power series α with coefficients in t, $\quad \alpha = \sum \alpha_i t^i$ we denote by $\alpha^{(q^d)}$ the power series$\sum \alpha_i^{q^d} t^i$ for any integer d.

7.4 Lemma. *Assume that a non–zero $f \in \mathcal{L}$ satisfies an equation $\tau^u f = cf$ for some number $u \geq 1$, and for some $c \in K((t))$. Then we have $f \in K((t))$ and $c \in K[[t]]^*$.*

Proof. The assertion follows by comparing the coefficients of the equation:

$$\sum_{n\in} a_n^{(q^u)} x^{nq^u} = \sum_{n\in} ca_n x^n$$

□

The following proposition is an analogue of a result discovered by Dwork.

7.5 Proposition. *Let k be algebraically closed. Let (M, τ_M) be a τ–module over $k[[x,t]]$. The $\mathcal{D}_0$–module $M \otimes_{k[[x,t]]} \mathcal{D}_0$ possesses a basis $\{d_1, \ldots, d_r\}$ which satisfies $\tau_M^n d_i = t^{a_i} d_i$ for suitable integers $n > 0, a_1, \ldots, a_r \geq 0$.*

Proof. M/xM with the operator τ is a τ–module over k. Hence by the Dieudonné theory for Drinfeld modules [11], page 35, there exist elements $e_1, \ldots e_r \in M/xM$ which form a basis of $M/xM \otimes_{k[[t]]} k((t))$ and such that the following equations

$$\tau_M^n e_i = t^{a_i} e_i, \quad i = 1, \ldots, r \tag{7.5.1}$$

hold for suitable integers $n > 0$ and $a_1, \ldots, a_r \geq 0$. Choose arbitrary lifts a'_i of these elements. We write

$$\tau_M^n e'_i = t^{a_i} e'_i + x z_i$$

for suitable $z_i \in M$. Inductively we find for each number N:

$$\tau^{Nn} e_i = t^{Nn} e_i \sum_{j=1}^{N} t^{(N-j)a_i} x^{q^{n(j-1)}} \tau^{n(j-1)}(z_i)$$

Consider for fixed i the sequence:

$$f_{i,n} = t^{-Na_i} \tau^{Nn} e_i$$

By the last equation we obtain the congruence:

$$t^{-Na_i} \tau^{Nn} e_i = t^{-(N-1)a_i} \tau^{(N-1)n} e_i \mod t^{-Na_i} x^{q^{n(N-1)}} M$$

We choose an isomorphism $M \cong k[[x,t]]^r$. Clearly the sequence $f_{i,n}$ converges in the x–adic topology to an element $\tilde{e}_i$ in $M \otimes_{k[[x,t]]} k((t))[[x]]$.

Both sides of the above congruence differ by a vector with components $t^{-Na_i}x^{q^{n(N-1)}}w_i$ where $w_i \in k[[x,t]]$. Since $|t^{-Na_i}x^{q^{n(N-1)}}|_\rho = Na_i\rho^{q^{n(N-1)}}$ converges to zero for any $\rho < 1$ the components of the sequence $f_{i,N}$ converge in the Banach algebra $\mathcal{L}(\rho)$. Since this is true for any $\rho < 1$ we find $\tilde{e}_i \in M \otimes_{k[[x,t]]} \mathcal{D}_0$. From the equation $t^{-a_i}\tau^n f_{i,n} = f_{i,n+1}$ we obtain:

$$\tau_M^n \tilde{e}_i = t^{a_i}\tilde{e}_i$$

It remains to be shown that the elements $\tilde{e_i}$ for $i = 1, \ldots, r$ form a basis of $M \otimes \mathcal{D}_0$, or equivalently that $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r$ is a basis of $\bigwedge^r M \otimes_{k[[x,t]]} \mathcal{D}_0$. By the lemma below, there exists a generator $y \in \bigwedge^r M$ such that $\tau_M(y) = t^\ell y$ for some number ℓ. We write:

$$\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r = y \otimes f, f \in \mathcal{D}_0$$

Applying τ^u to this equation gives the equality:

$$t^i(\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r) = t^j y \otimes f^{(q^u)}$$

for some numbers i and j. We deduce that $t^{i-j}f^{(q^u)} = f$. By lemma 7.4 we find $i = j$ and $f \in k((t))$ It remains to verify that $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r \neq 0$. But by our construction we have that $\tilde{e_i} = e_i \mod x$ and hence $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ gives a basis for the k–vector space

$$M \otimes_{k[[x,t]]} \mathcal{D}_0 / x(M \otimes_{k[[x,t]]} \mathcal{D}_0)$$

Therefore $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r$ is not zero. □

7.6 Lemma. *Let k be algebraically closed. Let (M, τ) be a τ–module over $k[[x,t]]$ of rank 1. Then there is an element $m \in M$, such that $\tau m = t^l m$ for some number l.*

Proof. We fix an isomorphism $M \cong k[[x,t]]$. We set $\tau 1 = \lambda$. By assumption, $\lambda = t^l \eta$ for some unit $\eta \in k[[x,t]]$. By the next lemma we find a unit $u \in k[[x,t]]$ such that $u^{(q)}\eta/u = 1$. This implies $\tau u = u^{(q)}t^l\eta = t^l u$. □

7.7 Lemma. *Let $\eta \in k[[x,t]]$ be a unit. Then there exists a unit $\xi \in k[[x,t]]$ which solves the equation:*

$$\xi^{(q)}/\xi = \eta$$

Proof. We write the equation modulo t:

$$\xi^{q-1} - \eta = 0 \qquad \mod t$$

This has a solution in the algebraically closed field k which lifts by Hensel's lemma to a solution in $k[[x]]$. Hence we obtain a solution modulo t. Let us assume by induction that we have a solution ξ modulo t^n:

$$\xi^{(q)} = \xi\eta \qquad \mod t^n$$

It is enough to show that ξ lifts to a solution modulo t^{n+1}. We set $\xi^{(q)} - \xi\eta = t^n\mu$ for $\mu \in k[[x,t]]$. We have to find $\rho \in k[[x,t]]$, such that

$$(\xi + t^n\rho)^{(q)} - (\xi + t^n\rho)\eta = 0 \mod t^{n+1}$$

This amounts to finding a solution of the following congruence:

$$\rho^q - \rho\eta + \mu = 0 \mod t$$

This congruence is an algebraic equation over $k[[x]]$ which can be solved by Hensel's lemma as above. □

7.8 Lemma. *Let k be algebraically closed. Let (M, τ_M) be a τ–module over $k[[x,t]]$. Let $N' \subset M \otimes_{k[[x,t]]} \mathcal{D}$ be a τ–invariant direct summand, which is free of rank 1. Then there is a free direct summand $N \subset M \otimes_{k[[x,t]]} \mathcal{D}_0$ of rank 1, such that $N' = N \otimes_{\mathcal{D}_0} \mathcal{D}$.*

Proof. By the above proposition there exists a basis $\{d_1, \ldots, d_r\}$ of $M \otimes_{k[[x,t]]} \mathcal{D}_0$ such that for suitable numbers u and a_i:

$$\tau^u d_i = t^{a_i} d_i$$

Let n be a generator of N'. We can write in $M \otimes_{k[[x,t]]} \mathcal{D} = M \otimes_{k[[x,t]]} \mathcal{D}_0 \otimes_{\mathcal{D}_0} \mathcal{D}$:

$$n = \sum_{i=1}^{r} d_i \otimes h_i, h_i \in \mathcal{D} \tag{7.8.1}$$

Since N' is invariant under τ we have

$$\tau^u n = \sum_{i=1}^{r} d_i \otimes t^{a_i} {h_i}^{(q^u)} = hn$$

for some $h \in \mathcal{D}$. comparing coefficients we find that $t^{a_i} {h_i}^{(q^u)} = h h_i$. then we see that:

$$t^{a_i} {h_i}^{(q^u)} h_j = t^{a_j} {h_j}^{(q^u)} h_i \tag{7.8.2}$$

Assume we have chosen j such that h_j is not identically zero. we choose η such that all Laurent series h_i and ${h_i}^{(q^u)}$ are in $\mathcal{D}_\eta$. Then we choose a transcendental ρ such that $\eta < \rho < 1$. We set $s = q^u$. One sees that $x \to x^{(q^u)}$ induces a homomorphism of fields $\mathcal{L}(\rho) \to \mathcal{L}(\rho^{1/s})$. Therefore we obtain in $\mathcal{L}(\rho^{1/s})$ the equation

$${h_j}^{(q^u)} (h_i h_j^{-1})^{(q^u)} = {h_i}^{(q^u)}$$

Dividing equation (7.8.2)in the field $\mathcal{L}(\rho^{1/s})$ by ${h_j}^{(q^u)} h_j$ we obtain:

$$t^{a_j} h_i {h_j}^{-1} = t^{a_i} {h_i}^{(q^u)} {h_j}^{-(q^u)} = t^{a_i} (h_i h_j^{-1})^{(q^u)}$$

. By lemma 7.4 we obtain $h_i {h_j}^{-1} = \lambda_i \in k((t))$, where $\lambda_i = 0$ if $a_i \neq a_j$. We write $f = h_j$ and rewrite (7.8.1) as:

$$n = \sum_{i=1}^{r} \lambda_i d_i \otimes f$$

In the basis $d_1, \ldots, d_r$ we may replace the element d_j by $\sum_{i=1}^{r} \lambda_i d_i$. then we can write $n = d_j \otimes f$. Since n generates a $\mathcal{D}$ submodule of $\oplus_{i=1}^{r} \mathcal{D} d_i$ that is in fact a direct summand, it follows that f is a unit in $\mathcal{D}$. Hence $N = \mathcal{D}_0 d_j$ is the direct summand that we wanted. □

We have a Cartesian diagram of rings

$$\begin{array}{ccc} k[[x,t]] & \longrightarrow & K[[t]]^c \\ \downarrow & & \downarrow \\ \mathcal{D}_0 & \longrightarrow & \mathcal{D} \end{array}$$

which, when localized at the prime (t) of $k[[x,t]]$ gives rise to a Cartesian diagram:

$$\begin{array}{ccc} B & \longrightarrow & K[[t]]^c \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{D} \end{array}$$

where here $B = k[[x,t]]_{(t)}$ and $\mathcal{M} = \mathcal{D}_0 \otimes_{k[[x,t]]} B$.

7.9 Lemma. *Let M be a finitely generated free B–module. Let $N^c \subset M \otimes_B K[[t]]^c$ be a rank 1 direct summand, and let $N' \subset M \otimes_B \mathcal{M}$ be a direct summand, which is free of rank 1 as an $\mathcal{M}$–module. We assume that $N^c \otimes_{K[[t]]^c} \mathcal{D} = N' \otimes_{\mathcal{M}} \mathcal{D}$. Then there exists a unique direct summand $N \subset M$ which induces N^c and N'.*

Proof. Let $f \in \mathcal{M}$ be an element which becomes a unit in $\mathcal{D}$. We claim that f is a unit in $\mathcal{M}$. Assume $f \in \mathcal{D}_0$. By assumption f is a unit in $\mathcal{L}([\rho, 1))$ for some ρ. By theorem 7.2 we may write in $\mathcal{L}([0, \rho])$:

$$fu = P$$

In $\mathcal{L}(\rho)$ we obtain $u = P/f$. The right hand side of this equation converges in $[\rho, 1)$. We conclude that u converges in $[\rho, 1)$ too, and finally $u \in \mathcal{D}_0$. Therefore we find in $\mathcal{M}$ the equation $f(u/P) = 1$. Let n^c respectively n' be generators of N^c respectively N'. Let $e_1, \ldots, e_r$ be a basis for the B–module M. Then we write

$$n^c = e_1 \otimes g_1 + \ldots + e_r \otimes g_r \tag{7.9.1}$$

$$N' = e_1 \otimes f_1 + \ldots + e_r \otimes f_r \tag{7.9.2}$$

where the $g_i \in K[[t]]^c$, and $f_i \in \mathcal{M}$. Since $K[[t]]^c$ is a local ring we may assume without loss of generality that $g_1 = 1$. By assumption there is a unit $h \in \mathcal{D}$ such that $n' = hn$. We conclude that $f_1 = h$, and by the assertion shown above that f_1 is a unit in $\mathcal{M}$, on dividing by f_1 we may assume $f_1 = h = 1$. Then we see from our Cartesian diagram that $f_i = g_i \in B$. □

Proof. of theorem (6.1): Suppose given (M, τ_M) as before, and $N^c \subset M \otimes_{k[[x,t]]} k((x))[[t]]^c$ a τ_M–invariant direct summand. We must show that there is a unique τ–submodule N of M so that $N^c = N \otimes_{k[[x,t]]} K[[t]]^c$. If the N in the statement of the theorem exists, then $N_1 = N^c \cap M$ is by lemma 3.6 another τ–submodule, which fulfills the theorem and contains N. Since $N \to N_1$ becomes an isomorphism over $K[[t]]^c$, by proposition 3.5, it is already an isomorphism. Thus N is unique.

Let $B = k[[x,t]]_{(t)}$ be the localization in the prime (t). We set $M' = M \otimes B$. It is equivalent to show that there is a direct summand N' of M' as B–module such that $N' \otimes_B K[[t]]^c = N^c$. Then we have that $N' = M' \cap N^c$, and thus N' is τ–invariant. By lemma 5.1 we may thus assume N^c has rank 1. Moreover we can assume that k is algebraically closed. Indeed, let $\overline{k}$ be the algebraic closure of k. We denote by $\overline{B}$ etc... the objects corresponding to $\overline{k}$. Then the Galois group $G = \mathrm{Gal}(\overline{k}/k)$ acts on $\overline{B}$ and the invariants are B. Here again we use the fact that k is assumed perfect. If $\overline{N}' \subset M' \otimes_B \overline{B}$ exists, it is stable by G since it is uniquely determined. Thus we may apply corollary 5.2 . Hence assume that k is algebraically closed and that N^c has rank 1. By lemma (7.8) we find a direct summand $N' \subset M \otimes_{k[[x,t]]} \mathcal{D}_0$ of rank 1, such that $N^c \otimes_{K[[t]]^c} \mathcal{D} = N' \otimes_{\mathcal{D}_0} \mathcal{D}$. Thus by lemma (7.9) we are done. □

8 Existence of Section

We now prove proposition 6.3 in its dual version:

8.1 Proposition. *Let k be algebraically closed. Let $M \to N$ be a morphism of quasi–τ–modules over $k[[x]]$, such that N is isoclinic of slope λ, and all slopes of M are greater or equal to λ. If the map $M \otimes_{k[[t]]} k((t)) \to N \otimes_{k[[t]]} k((t))$ is surjective, then it admits a τ–equivariant section.*

Proof. We set $\lambda = r/s$ where r and s are integers, $s > 0$. We set $U = t^{-r}\tau^s$. We may assume that $UM \subset M, UN \subset N$ and that the cokernel of the map $M \to N$ is annihilated by a power of the maximal ideal of $k[[x,t]]$. Indeed, we set $B = k[[x,t]]_{(t)}, M' = M \otimes B$, and $N' = N \otimes B$. Let $M'_0 \subset M' \otimes k((t))$ be a finitely generated free B–submodule which is mapped surjectively to $N'_0 = N'$ and such that $M'_0 \otimes k((t)) = M' \otimes k((t))$. we set $M'_1 = \sum_{i=1}^{\infty} U^i M'_0$ and $N'_1 = \sum_{i=1}^{\infty} U^i N'_0$. Then M'_1, N'_1 are both U–invariant and finitely generated over B. The map $M'_1 \to N'_1$ is surjective. Finally we consider $M_1 = M'_1 \cap (M \otimes k((t)))$ and $N_1 = N'_1 \cap (N \otimes k((t)))$. These $k[[x,t]]$–modules are free by lemma 3.6. Then we have $UM_1 \subset M_1$ and $UN_1 \subset N_1$ because N_1 is isoclinic of slope λ. Finally the map $M_1 \to N_1$ becomes surjective if we tensor it with B. Hence the cokernel is annihilated by a power of the maximal ideal of $k[[x,t]]$. We define X and Y by the exact sequence:

$$0 \to Y \to M \to N \to X \to 0$$

We deduce an exact sequence:

$$0 \to (Y/xY) \otimes k((t)) \to (M/xM) \otimes k((t)) \to (N/xN) \otimes k((t)) \to 0$$

Then Y/xY is a U–invariant submodule of M/xM and $\overline{Y} = (Y/xY) \otimes k((t)) \cap (M/xM)$ is U–invariant too. Let us consider the cokernel $\overline{N}$:

$$0 \to \overline{Y} \to M/xM \to \overline{N} \to 0$$

The operator U acts on $\overline{N}$. On the other hand, $\overline{N}$ is isoclinic of slope λ, since $\overline{N} \otimes k((t)) = N/xN \otimes k((t))$ is. This shows that U induces an isomorphism $\overline{N}^{(q^s)} \to \overline{N}$. By the Dieudonné classification ([11], Theorem 2.4.5) we have a U–equivariant section $\overline{N} \otimes k((t)) \to M/xM$. Its image $\overline{E}$ lifts by the Lemma 8.2 to a U–invariant direct summand E of M. Then the U–equivariant map:

$$Y \oplus E \to M$$

is an isogeny modulo x. Therefore it is an isogeny by proposition 3.5 and the remark after it. We find that $E \to N$ is an isogeny, and hence $E \otimes k((t)) \to N \otimes k((t))$ is an isomorphism. This shows that $M \otimes k((t)) \to N \otimes k((t))$ has a U–equivariant section β. But then

$$1/s \sum_{i=0}^{s-1} \tau^{-i} \beta \tau^i$$

is the desired τ–equivariant section. □

Finally we must prove that $\overline{E}$ lifts. We fix an integer $s > 0$ and we consider the endomorphism $z \to z^{(q^s)}$ of $k[[x,t]]$

8.2 Lemma. *Let P be a projective finitely generated $k[[x,t]]$–module and $U : P \to P$ a (q^s)–linear endomorphism. We set $P_0 = P/xP$. Then U induces a (q^s)–linear endomorphism of the $k[[t]]$–module P_0.*

Let E_0 be a direct summand of P_0, such that U induces a (q^s)–linear isomorphism

$$\varphi_0 : E_0 \to E_0$$

Then there exists a direct summand $E \subset P$ which is uniquely determined by the following properties:

1. *$U(E) \subseteq E$.*
2. *E lifts E_0*
3. *$U : E \to E$ is a (q^s)–linear isomorphism.*

We prove this in a more general situation: Let $f \in \Gamma^c$. There exist constants $C_1, C_2 \in$ such that $\nu(f,n) \leq C_1 + nC_2$ for all nonnegative integers n. Let A be a commutative ring and $\mathfrak{a} \subset A$ an ideal consisting of nilpotent elements. We set $A_0 = A/\mathfrak{a}$ and more generally we denote for an A–module M the A_0–module $M/\mathfrak{a}M$ by M_0. Let $\sigma : M \to M$ be a ring homomorphism, such that $\sigma(\mathfrak{a}) \subset \mathfrak{a}$ and such that there exists a natural number r with $\sigma^r(\mathfrak{a}) = 0$. We denote by $\sigma_0 : A_0 \to A_0$ the induced morphism. We will be interested in the particular case where $A = k[[x,t]]/(x^m)$ for an arbitrary natural number m, the endomorphism σ induced by σ on $k[[x,t]]$ and the ideal $\mathfrak{a} = xA$. Thus it is enough to prove the following proposition

8.3 Proposition. *Let P be a finitely generated projective A–module and $\varphi : P \to P$ a σ–linear endomorphism. Let $P_0 = P \otimes_A A/\mathfrak{a}$. Then φ induces a σ_0–linear endomorphism $\varphi_0 : P_0 \to P_0$ of the A_0–module P_0. Let E_0 be a direct summand of P_0, such that φ_0 induces a σ_0–linear isomorphism :*

$$\varphi_0 : E_0 \to E_0$$

Then there exists a direct summand $E \subset P$ which is uniquely determined by the following properties:

- *$\varphi(E) \subset E$.*
- *E lifts E_0*
- *$\varphi : E \to E$ is a σ–linear isomorphism.*
- *Let C be an A–module, which is equipped with a σ–linear isomorphism $\psi : C \to C$. Let $\alpha : (C, \psi) \to (P, \varphi)$ be an A–module homomorphism such that $\alpha \circ \psi = \varphi \circ \alpha$. Let us assume that $\alpha_0(C_0) \subset E_0$. Then we have $\alpha(C) \subset E$.*

Proof. This is just [12], Proposition 36. □

9 Reduction to transcendence degree 1

We have so far only proved the analogue of de Jong's theorem for the case of $K = k[[x]]$ where k is a perfect field. We now show that it holds more generally for any discrete valuation. The proof is somewhat interesting for the following reason. In the analogous reduction of [10], de Jong makes critical use of the differential operator associated to an F-crystal to get around the fact that in the construction $R \to W(R)$, where R is a ring of characteristic p and $W(R)$ is the ring of Witt vectors in R, there is no canonical lift of the Frobenius of R. However the non-canonicality of the lift is exactly made up for by the differential operator. This leads one to suspect that a classical statement about F-crystals that does not *explicitly* refer to the differential structure, will have an analogous statement for formal τ-modules. Moreover, the same proof should carry over, with the modification that any step classically that uses the differential operator should be made up for by the use of the canonical Frobenius lift in the function field case.

Now let R be a discrete equal characteristic discrete valuation ring with K its fraction field. By standard commutative algebra, we can find a discrete valuation ring R' which is complete with algebraically closed residue field k' and so that $e(R'/R) = 1$, together with a local injective homomorphism $R \to R'$. We have an obvious map $R[[t]] \to R'[[t]]$ together with the obvious lift of the absolute q-Frobenius on R, unlike the situation in [10]. Now if M_1, M_2 are formal τ–modules over R then we claim that

$$\mathrm{Hom}_R(M_1, M_2) = \mathrm{Hom}_K(M_1 \otimes_R K, M_2 \otimes_R K) \cap \mathrm{Hom}_{R'}(M_1 \otimes_R R', M_2 \otimes_R R')$$

But this follows from looking at the cofficients of these maps in terms of a basis for M_1 and M_2 over $R[[t]]$. Thus our proof of 3.1 for a complete discrete valuation over an algebraically closed field (actually we only used perfect) suffices to prove the result for any discrete valuation ring.

Now we can reduce the proof of the equicharacteristic Tate Conjecture for Drinfeld modules to the case of a Drinfeld module over a field of transcendence degree one over a finite field. The case of a finite field is was proved by Drinfeld (see for instance: [11], Theorem 2.5.6).

Let us therefore take a Drinfeld module ϕ over a field L of transcendence degree greater than 1 over its (finite) field of constants. We know that ϕ has a model ϕ_U defined over an irreducible variety U with residue field L at the generic point. We can find an irreducible divisor $Z \subset U$ so that the specialization map $\mathrm{End}(\phi_L) \to \mathrm{End}(\phi_{\kappa(Z)})$ is an isomorphism, where $\kappa(Z)$ is the residue field at the generic point of Z. For instance we could use the ℓ-adic representations for some prime ℓ away from the characteristic, where the Tate Conjecture is known from [14] and then use Bertini. On the other hand our extension of 3.1 to all discrete valuation rings gives us a specialization map

$$\mathrm{End}(\phi_L) \to \mathrm{End}(\phi_{\kappa(Z)})$$

Thus if we prove the theorem for $\kappa(Z)$ then the theorem for L follows. By induction we can assume that the transcendence degree of L is one.

10 End of the proof

We need one last proposition before we are ready to prove the promised version of the Tate Conjecture.

Let ϕ be a Drinfeld module for a ring A over the fraction field K of a discrete valuation ring R finite residue field, and let $\wp$ be a prime of A. Suppose moreover that ϕ has semistable reduction over R given by $\phi = \psi/\Lambda$ where ψ is a Drinfeld module over K with good reduction over R and Λ is a lattice in ψ (see [4] for further details). Consider the $\wp$–divisible group $\phi_K[\wp^\infty]$ of ϕ over K. The semistable structure of ϕ induces a semistable filtration

$$\phi[\wp^\infty]^c \subset \phi[\wp^\infty]^{st} \subset \phi[\wp^\infty] \tag{10.0.1}$$

where $\phi[\wp^\infty]$ is the sub–$\wp$–divisible group of $\phi[\wp^\infty]$ induced from the $\wp$–divisible group of ψ. We now claim:

10.1 Proposition. *Every endomorphism of $\phi_K[\wp^\infty]$ respects the filtration* (10.0.1)

Proof. The fact that any endomorphism of the $\wp$–divisible group takes the connected component $\phi[\wp^\infty]$ to itself follows immediately from the fact that there are no non–trivial morphisms from a purely inseparable scheme to an étale one. Thus in order to prove the proposition we need to find a characterization *over* K of the second step of the filtration. Since everything in question is now étale, we may view it as a question about the Galois representation ρ of $G = \mathrm{Gal}(\overline{K}/K)$ into the étale $\wp$–torsion of ϕ.

$$\rho : G \to \mathrm{End}(\lim_{\leftarrow} \phi[\wp^n]^{et})$$

Now there is a subrepresentation of ρ corresponding to the étale $\wp$ torsion of ψ. We know that after possibly passing to a finite extension of the base field K, the induced quotient representation is trivial, from the Tate uniformization theory. Since ψ has a good model over R, its $\wp$–torsion is étale, and the action of G is induced from the action of Frobenius on the $\wp$–torsion of the fibre of ψ over the residue field k of R. We assumed k to be finite, and in this situation it is well known that the $\wp$-adic valuations of Frobenius are all equal and greater than 0 (see for instance [9], Theorem 4.12.8, page 105). Thus we have characterized the semistable filtration purely in terms of the action of $\mathrm{Gal}(\overline{K}/K)$, and thus it must be preserved by endomorphisms over K. □

10.2 Theorem. *Let L be a field of finite type over ${}_q$, and let M be the τ–module over ${}_q[t] \otimes L$ corresponding to a Drinfeld module ϕ over L. Then the natural map of ${}_q[[t]]$–modules:*

$${}_q[[t]] \otimes_{{}_q[t]} \mathrm{End}(M, \tau) \to \mathrm{End}(L[[t]] \otimes_{L[t]} M, \tau)$$

is an isomorphism.

Proof. (Equicharacteristic Tate Conjecture)
We will proceed by the methods of [14]. We will first show the rational version of 10.2; namely that the map of ${}_q((t))$ vector spaces:

$${}_q((t)) \otimes_{{}_q[t]} \mathrm{End}(M, \tau) \to \mathrm{End}(M \otimes_{L[t]} L((t)), \tau)$$

is an isomorphism. The integral statement then follows easily. For convenience we will denote by V the ${}_q(t)$–vector space $\mathrm{End}(M, \tau) \otimes_{L[t]} L(t)$ and by V_t the ${}_q((t))$–vector space $\mathrm{End}(M \otimes_{L[t]} L((t)), \tau)$.

We know that there are only finitely many places of L where ϕ has bad reduction and that this reduction becomes good or semistable after a finite extension of L. Thus we can pass to a finite extension K of L such that ϕ has semistable reduction at a finite number of places of K, and good reduction everywhere else. Now let v be a place of K where ϕ has semistable reduction, and set $M_v = M \otimes_{K[t]} K_v[t]$. By the results of Gardeyn, we know that $\tilde{M}_v := M_v \otimes_{K[t]} K\{\{t\}\}$ decomposes into an exact sequence:

$$0 \to \tilde{M}_v^{triv} \to \tilde{M}_v \to \tilde{M}_v^{good} \to 0 \tag{10.2.1}$$

where the map $M_v \to M_v^{good}$ is the map induced by the exponential map from the good reduction of ψ to ϕ, where ψ is the good model of ϕ. This decomposition of analytic motives corresponds to a decomposition of the t–divisible group associated to ϕ. Moreover by Proposition 10.1 any endomorphism of the t–divisible group respects this decomposition. By the equivalence of Corollary 3.4 this means that any endomorphism of the analytic motive respects the decomposition. Thus if we have that τ_M is given by

$$\tau_{\mathbf{M}} = \begin{pmatrix} \tau^{triv} & \tau^{ext} \\ 0 & \tau^{good} \end{pmatrix}$$

Then any endomorphism of the t–divisible group corresponds to a matrix A with coefficients in $K[[t]]$ such that

$$(10.2.2) \qquad \begin{pmatrix} \tau^{triv} & \tau^{ext} \\ 0 & \tau^{good} \end{pmatrix} \begin{pmatrix} A_{11}^{(q)} & A_{21}^{(q)} \\ A_{12}^{(q)} & A_{22}^{(q)} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} \tau^{triv} & \tau^{ext} \\ 0 & \tau^{good} \end{pmatrix}$$

The fact that a morphism of the t–divisible groups respects the Tate data, gives us that $A_{12} = 0$. We now conclude the following:

1. A_{11} has bounded coefficients. That is $A_{11} = (a_{ij})$ where the $a_{ij} \in K[[t]], a_{ij} = \sum b_{ijk}t^k$ and $v(b_{ijk}) \geq c$ for some fixed c.
2. A_{22} has integral coefficients.
3. A_{21} has bounded coefficients.

Proof. 1. We will introduce the following notation for convenience. Given a discretely valued field K with a valuation ν, we will say that an element $\alpha = \sum \alpha_i t^i \in K[[t]]$ lies in $K[[t]]^{bdd}$ if α has bounded coefficients. Now by a result of Gardeyn we know that M_n^{triv} becomes trivial after base change to a finite extension of K, and a change of basis of bounded valuation. That is, there exists a matrix $L = (l_{ij})$ with coefficients in $K'[[t]]^{bdd}$, such that

$$L\tau^{triv}L^{-(q)} = \mathrm{Id}$$

In fact we can choose L to be a matrix of constant functions, though we do not need this fact. Now let N be a square matrix with coefficients in $\ _q[[t]]$ corresponding to an endomorphism of M_v^{triv}, written in this new basis. Then in our old basis, this morphism is given by: $L^{-1}NL = A_{11}$. The fact that the coefficients have bounded valuation follows immediately from the corresponding fact for L.

2. This follows immediately from the analogue of de Jong's theorem, since the determinant of the τ–module corresponding to the good reduction of ϕ is equal to tu where u is a unit in K.
3. Let L be the change of basis matrix for M_V^{triv} such that $L\tau^{triv}L^{-(q)} = \mathrm{Id}$. We change basis for M_v by the matrix:

$$\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}$$

Then by (10.2.2), we have the following identity:

$$A_{21}^{(q)} + L\tau^{ext}A_{22}^{(q)} = A_{11}L\tau^{ext} + AA_{21}\tau^{good}$$

The result will follow from the following more general lemma.

□

10.3 Lemma. *Let A_i, B_i, X be matrices with coefficients in $K[[t]]$ for K, ρ a valued field and suppose that the coefficients of A_i and B_i actually lie in $M_{n\times n}(K[[t]]^{bdd})$. Suppose that X satisfies a "τ–equation" of the form:*

$$X^{(q^n)} + A_nX^{(q^{n-1})}B_n + \ldots + A_1XB_1 + A_0$$

Then $X \in M_{n\times n}(K[[t]]^{bdd})$

Proof. We proceed by induction on k, where we assume that the hypothesis is true for the coefficients of t^k in X. More precisely, let N be $\min\{\rho(\text{coefficients of } A_i, B_i)\}$. We claim that the valuation all coefficients of X are bounded below by $2N$. Suppose this is false for the constant coefficients in X. Denote by x^0_{ij} the constant coefficient of X having minimum valuation, and let $v = \rho(x^0_{ij})$ be this valuation. We have that

$$q^n v = \min\{\alpha_n, \ldots, \alpha_0\} \tag{10.3.1}$$

Where here α_i is the valuation of the constant coefficient of the ijth entry of the matrix $A_i X^{(q^{i-1})} B_i$. Clearly $\alpha_i \geq q^{i-1}v + 2N$. Thus we have that:

$$q^n v \geq 2N + q^{n-1}v \tag{10.3.2}$$

We conclude that v is bounded below by $2N/q^{n-1}(q-1)$ and a fortiori by $2N$.

Now the general case follows easily. For the induction step we suppose that $\rho(x^l_{ij}) \geq 2N \quad \forall l < k$ where here x^l_{ij} is the coefficient of t^l in the ij position of the matrix X. Then if now v is taken to be the minimal valuation of the coefficient of t^k in X, we have the same equation as (10.3.1) for the new v. The result follows immediately. □

Now for the next step. The decomposition of (10.2.1) takes place over $K\{\{t\}\}$, the ring of analytic functions with coefficients in K. These are power series in $K[[t]]$ that converge at any point of K. In particular they lie $K[[t]]^{bdd}$. Thus, Gardeyn shows that if (M, τ_M) is the t–motive associated to ϕ, then there exists a matrix $P \in \mathrm{GL}(K\{\{t\}\})$, such that:

$$P\tau_M P^{-(q)} = \begin{pmatrix} \tau^{triv} & \tau^{ext} \\ 0 & \tau^{good} \end{pmatrix}$$

Thus if A is the matrix in (10.2.2), then $P^{-1}AP$ is the matrix representing an endomorphism of (M, τ_M). Since we have proved that A and P have coefficients in $K[[t]]^{bdd}$, so does $B := P^{-1}AP$. Thus we have that B satisfies

$$\tau_M B^{(q)} = B\tau_M \tag{10.3.3}$$

. Writing τ_M as $\sum_{i=0}^{N} C_i t^i$ and $B = \sum_{i=0}^{\infty}$ where $C_i, B_i \in M_{n\times n}$ and setting equal the coefficients of t^k we get

$$\sum_{i=0}^{N} C_i B^q_{k-i} = \sum_{i=0}^{N} B_{k-i} C_i \tag{10.3.4}$$

Now suppose that this equation has a solution $\overline{B}$ in $\mathrm{M}_{n\times n}(L((t)))$. We will show that it has a solution B in $L(t)$ sufficiently close in the t–adic topology to $\overline{B}$, so that, if $(\overline{B}_i)$ is a basis for V_t, then B_i is a basis for V.

For every valuation ν on L, let us denote by $\nu(B)$ the minimum of the valuations of the entries of $B_i, \forall i \geq 0$. We have shown that $\nu(B)$ is bounded below for all valuations ν, and bounded by 0 for all but finitely many places ν. Thus by the bounded height theorem, there appear only finitely many matrices in the sequence $B_0, B_1, \ldots$ We call a solution $B = \sum B_i t^i$ to (10.3.4) **periodic** if, apart from a finite number of initial terms we have for some integer I the relation $B_i = B_{i+I}$. However, since there appear only finitely many equations in (10.3.4), then one can choose a periodic solution $B = \sum B_i t^i$ in $M_{n\times n}(L((t)))$ close enough to the original $\overline{B}$. Periodicity implies that B is rational with denominator in $\quad_q[t]$, and hence $B \in M_{n\times n}(L(t))$. □

10.4 Remark. The above methods should extend without too much difficulty to prove the Tate conjecture for τ-modules of equal characteristic which are analytically semistable. These will include for instance t-motives formed out of natural operations such as tensor and direct sum from the t-motives of Drinfeld modules. Unfortunately very little is known about the analytic structure of general τ-modules at the characteristic. In particular, the methods of Gardeyn do not apply there. The condition of analytic semistability is maybe stronger than one needs however. In particular, rather than requiring that the functions converge on the entire plane, we need only that they have bounded denominator – i.e., that they converge on the open unit circle.

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