REAL-ROOTED POLYNOMIALS IN COMBINATORICS

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ABSTRACT

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Combinatorics is full of examples of generating polynomials that have only real roots. At the same time, only a few classical methods are known to prove that a polynomial, or a family of polynomials, has this curious property. In this dissertation we advocate the use of a multivariate approach that relies on the theory of stable polynomials, which has recently received much attention.

We first present a proof of the Monotone Column Permanent conjecture of Haglund, Ono and Wagner, which asserts that certain polynomials obtained as permanents of a special class of matrices have only real roots. This proof is joint work with Brändén, Haglund and Wagner.

A special case of this result reduces to the well-known fact that all roots of the Eulerian polynomials are real. Furthermore, it gives rise to a multivariate stable refinement of these polynomials. The (univariate) Eulerian polynomials have been generalized in various directions and some of their variants have also been showed to have only real roots. With the use of the multivariate methodology we are able to strengthen these real-rootedness results to their stable counterparts. We give meaningful combinatorial descriptions for such

refinements over Stirling permutations (joint work with Haglund) and for some reflection groups (joint work with Williams). Our approach provides a unifying framework with often simpler proofs.

Finally, a partial result on a further refinement of Eulerian polynomials is also included in order to emphasize that the multivariate approach can also be successfully employed to tackle problems other than real-rootedness or stability.

Contents

> "Disparate problems in combinatorics ... do have at least one common feature: their solution can be reduced to the problem of finding the roots of some polynomial or analytic function."
>
> Gian-Carlo Rota

Introduction

This dissertation is concerned with generating polynomials in combinatorics whose roots are all real. This is of particular interest due to the deep connections between the roots and the coefficients of a polynomial. The body of work presented here was largely motivated by the Monotone Column Permanent (MCP) conjecture of Haglund, Ono and Wagner which asserts that certain polynomials obtained from a special class of matrices have only real roots. I was introduced to this conjecture by my advisor, J. Haglund while I was a master's student at the University of Pennsylvania; see [Vis07].

Chapter 1 of this dissertation contains the proof of MCP conjecture which is based on two papers [HV09, BHVW11]. The first one is joint work with J. Haglund, and contains a multivariate generalization of the conjecture and its proof for $n \leqslant 4$. The multivariate conjecture asserts that a certain multivariate refinement of the polynomials in question is *stable*, in the sense that it does not vanish whenever all variables lie in the open upper half-plane. The second paper—joint work with P. Brändén, J. Haglund and D. G. Wagner—establishes the multivariate conjecture proposed in the first paper for all n. The techniques from the theory of stable polynomials that were successfully used in the proof of the (multivariate) MCP conjecture are then applied to a related conjecture of Haglund on hafnians. This last, partial, result which appeared in [Vis11], concludes the first chapter.

Eulerian polynomials are ubiquitous in mathematics; they play an important role in Combinatorics, Analysis, Number Theory, and Algebraic Geometry. In fact, they naturally arise in connection to the above mentioned problem as permanents of monotone column $\{0, 1\}$ matrices of "staircase shape". The multivariate version of the MCP theorem refines this correspondence and leads to a multivariate refinement of the Eulerian polynomials that is stable—generalizing a classical result that the (univariate) Eulerian polynomials have only real roots. On the other hand, from the enumerative combinatorics perspective, Eulerian polynomials can be viewed as generating functions of the descent statistic over the set of permutations. Using this interpretation, their multivariate refinements can be viewed as generating functions of some refined descent statistics. The descent statistic can also be extended to a wider set of objects (other than permutations). This gives rise to the so-called generalized Eulerian polynomials.

Chapter 2 of the dissertation deals with generalizations of the Eulerian polynomials and their stable multivariate refinements. This chapter is based on two papers [HV12, VW12]. The first one is joint work with J. Haglund, and gives stable Eulerian polynomials for restricted multiset permutations, such as the Stirling permutations. The second one is joint work with N. Williams, and extends the stability results of Eulerian polynomials to groups other than the symmetric group, namely, the hyperoctahedral group, the generalized symmetric group and some affine Weyl groups. The key contribution of this chapter is that the multivariate ideology provides a unifying framework to prove real-rootedness results using stability.

The final chapter is concerned with a further refinement of the Eulerian polynomials, the so-called two-sided Eulerian polynomials. These are generating functions of the joint distribution of descents and inverse descents (or can be thought of as the joint left- and right-descent generating polynomials in the more general setting of Coxeter groups). A conjecture of Gessel is discussed here, that asserts that the two-sided Eulerian polynomials have a certain expansion in a given basis with nonnegative integer coefficients. This would generalize the univariate result of γ-positivity—a property closely related to real-rootedness. We obtain some preliminary results here, by deriving a recurrence for the coefficients in the expansion.

“...the interesting problems tend to be open precisely because the established techniques cannot easily be applied.”

Timothy Gowers

Chapter 1

The MCP conjecture

In this chapter we give a proof of the MCP conjecture. In Section 1.1, we formulate the conjecture and briefly mention its history. In Section 1.2, we collect results from the theory of stable polynomials that will be needed for the proof. In Section 1.3, we give the proof of the MCP conjecture—in fact, we prove a stronger multivariate version of it. Finally, we apply some of the techniques used in the proof to a related conjecture of Haglund, and obtain some partial results in Section 1.4.

The results in this chapter are taken from the following three papers [HV09, BHVW11, Vis11]. The main result, presented in Section 1.3, was obtained in collaboration with P. Brändén, J. Haglund and D. G. Wagner. Sections 1.2 and 1.3 are taken verbatim with minor modifications from Sections 2 and 3 of [BHVW11]. Section 1.4 is taken in parts from [Vis11].

1.1 The origin of the conjecture

Let us begin with stating the conjecture. Before we can do that we need to review some definitions.

Definition 1.1.1. The *permanent* of an n-by-n matrix $M = (m_{ij})$ is the “signless determinant”, that is,

$$\mathrm{per}(M) = \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{n} m_{i,\pi_i} ,$$

where $\mathfrak{S}_n$ denotes the symmetric group on n elements.

Definition 1.1.2. An n-by-n matrix $A = (a_{ij})$ is a *monotone column matrix* if its entries are weakly increasing down columns, i.e., $a_{ij} \leqslant a_{i+1,j}$ for all $1 \leqslant i \leqslant n-1$ and all $1 \leqslant j \leqslant n$.

In [HOW99], Haglund, Ono, and Wagner conjectured the following.

Conjecture 1.1.3. *Let J_n denote the matrix of all ones of size n-by-n. If A is an n-by-n monotone column matrix, then $\mathrm{per}(zJ_n + A)$, as a polynomial in z, has only real roots.*

We will refer to this conjecture as the Monotone Column Permanent (MCP) conjecture. Important special cases of monotone column matrices are the ones with $\{0, 1\}$ entries, which we call *Ferrers matrices*. These matrices appear frequently in rook theory and in algebraic combinatorics. Haglund, Ono, and Wagner arrived at the MCP conjecture after having settled it for Ferrers matrices. In fact, they proved that the so-called hit polynomial of any Ferrers board has only real roots, which is equivalent to the above conjecture for $\{0, 1\}$ monotone column matrices. Relaxing integrality—much like in the case of the celebrated Heilmann–Lieb theorem—seemed a natural extension. We refer the reader to [HOW99, Hag00, Vis07] for the definition of Ferrers board, hit polynomial, and their connection to the Heilmann–Lieb theorem and further details on the history of the problem.

1.2 Theory of stable polynomials

In this section, we introduce elements from the theory of stable polynomials recently developed by Borcea and Brändén. These results lie at the heart of the proof of the MCP conjecture and will be used in Chapter 2 as well.

The basic idea of using a multivariate generalization to prove real-rootedness results is not new. For example, it was applied in one of the beautiful (and probably lesser known) proofs of the Heilmann–Lieb theorem [HL72, Theorem 4.6, Lemma 4.7]. The recent developments in stability-preserving operators, which we are about to discuss, made this approach even more powerful and applicable.

First, we set up some multivariate notation for convenience. For a positive integer n, let $[n]$ denote the set $\{1, \ldots, n\}$ and let $\mathbf{x}$ be the shorthand for the n-tuple $x_1, \ldots, x_n$. Similarly, $\mathbf{x} + \mathbf{y}$ denotes the n-tuple $x_1 + y_1, \ldots, x_n + y_n$. For $\mathcal{U}$ a set (or multiset) with entries from $[n]$, we let $\mathbf{x}^{\mathcal{U}} = \prod_{i \in \mathcal{U}} x_i$; for example, $(\mathbf{x} + \mathbf{y})^{[n]} = \prod_{i=1}^{n} (x_i + y_i)$. The cardinality of $\mathcal{U}$ is written $|\mathcal{U}|$. We will denote the concatenation of $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x}, \mathbf{y}$ or sometimes by $\mathbf{x}; \mathbf{y}$ in case we want to emphasize the different roles played by $\mathbf{x}$ and $\mathbf{y}$. We use $\mathcal{H} = \{w \in \mathbb{C} \mid \Im(w) > 0\}$ to denote the open upper half of the complex plane, and $\overline{\mathcal{H}}$ denote its closure in $\mathbb{C}$. For $f \in \mathbb{C}[\mathbf{z}]$ and $1 \leqslant j \leqslant n$, let $\deg_{z_j}(f)$ denote the degree of z_j in f.

Definition 1.2.1. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is *stable* if $f(\mathbf{z})$ does not vanish, whenever the imaginary part of each z_i for $1 \leqslant i \leqslant n$ is positive.

Definition 1.2.2. A linear transformation $T : \mathbb{R}[\mathbf{z}] \to \mathbb{R}[\mathbf{z}]$ *preserves stability* if $T[f(\mathbf{z})]$ is stable whenever $f(\mathbf{z})$ is stable.

The results presented here are taken from [BB10, BB09, Brä07]; see also Sections 2 and 3 of [Wag11]. Our presentation is an abridged version of Section 2 of [BHVW11], we repeat the following results and proofs for the sake of completeness with the only difference that most of the theorems are stated for real polynomials only. That suffices for the results in presented in this dissertation.

Lemma 1.2.3 (see Lemma 2.4 of [Wag11]). *These operations preserve stability of polynomials in* $\mathbb{R}[\mathbf{z}]$*.*

(a) Permutation: *for any permutation* $\sigma \in \mathfrak{S}_n$, $f \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$.

(b) Scaling: *for* $c \in \mathbb{C}$ *and* $\mathbf{a} \in \mathbb{R}^n$ *with* $\mathbf{a} > 0$, $f \mapsto cf(a_1 z_1, \ldots, a_n z_n)$.

(c) Diagonalization: *for* $1 \leqslant i < j \leqslant n$, $f \mapsto f(\mathbf{z})|_{z_i = z_j}$.

(d) Specialization: *for* $a \in \overline{\mathcal{H}}$, $f \mapsto f(a, z_2, \ldots, z_n)$.

(e) Inversion: *if* $\deg_{z_1}(f) = d$, $f \mapsto z_1^d f(-z_1^{-1}, z_2, \ldots, z_n)$.

(f) Translation: $f \mapsto f(z_1 + t, z_2, \ldots, z_n) \in \mathbb{R}[\mathbf{z}, t]$.

(g) Differentiation: $f \mapsto \partial f(\mathbf{z}) / \partial z_1$.

Proof. Only part (f) is not made explicit in [BB10, BB09, Wag11]. But if both $z_1 \in \mathcal{H}$ and $t \in \mathcal{H}$ then clearly $z_1 + t \in \mathcal{H}$, from which the result follows. □

Of course, parts (d), (e), (f), (g) apply to any index j as well, by permutation. Part (g) is the only difficult one—it is essentially the Gauss-Lucas Theorem.

Lemma 1.2.4. *Let* $f(\mathbf{z}, t) \in \mathbb{C}[\mathbf{z}, t]$ *be stable, and let*

$$f(\mathbf{z}, t) = \sum_{k=0}^{d} f_k(\mathbf{z}) t^k$$

with $f_d(\mathbf{z}) \not\equiv 0$*. Then* $f_k(\mathbf{z})$ *is stable for all* $0 \leqslant k \leqslant d = \deg_t(f)$*.*

Proof. Clearly $f_k(\mathbf{z})$ is a constant multiple of $\partial^k f(\mathbf{z}, t) / \partial t^k|_{t=0}$, for $0 \leqslant k \leqslant d$, which is stable by Lemma 1.2.3 parts (d) and (g). □

Definition 1.2.5. Polynomials $g(\mathbf{z}), h(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$ are in *proper position*, denoted by $g \ll h$, if the polynomial $h(\mathbf{z}) + \imath g(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is stable.

This is the multivariate analogue of interlacing roots for univariate polynomials with only real roots.

Proposition 1.2.6 (Lemma 2.8 of [BB09] and Theorem 1.6 of [BB10]). *Let* $g, h \in \mathbb{R}[\mathbf{z}]$*.*

(a) *Then* $h \ll g$ *if and only if* $g + th \in \mathbb{R}[\mathbf{z}, t]$ *is stable.*

(b) *Then* $ag + bh$ *is stable for all* $a, b \in \mathbb{R}$ *if and only if either* $h \ll g$ *or* $g \ll h$.

It then follows from parts (d) and (g) of Lemma 1.2.3 that if $h \ll g$ then both h and g are stable (or identically zero).

Proposition 1.2.7 (Lemma 2.6 of [BB10])**.** *Suppose that* $g \in \mathbb{R}[\mathbf{z}]$ *is stable. Then the sets*

$$\{h \in \mathbb{R}[\mathbf{z}] : g \ll h\} \quad \textit{and} \quad \{h \in \mathbb{R}[\mathbf{z}] : h \ll g\}$$

are convex cones containing g.

Proposition 1.2.8. *Let* V *be a real vector space,* $\phi : V^n \to \mathbb{R}$ *a multilinear form, and* $e_1, \ldots, e_n, v_2, \ldots, v_n$ *fixed vectors in* V. *Suppose that the polynomial*

$$\phi(e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$$

in $\mathbb{R}[\mathbf{z}]$ *is not identically zero. Then the set of all* $v_1 \in V$ *for which the polynomial*

$$\phi(v_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$$

is stable is either empty or a convex cone (with apex 0*) containing* e_1 *and* $-e_1$.

Proof. Let C be the set of all $v_1 \in V$ for which the polynomial $\phi(v_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$ is stable. For $v \in V$ let $F_v = \phi(v, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$. Since

$$\phi(v_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n) = F_{v_1} + z_1 F_{e_1},$$

we have $C = \{v \in V : F_{e_1} \ll F_v\}$. Moreover since $F_{\lambda v + \mu w} = \lambda F_v + \mu F_w$ it follows from Proposition 1.2.7 that C is a convex cone provided that C is non-empty. If C is nonempty then $F_v + z_1 F_{e_1}$ is stable for some $v \in V$. But then F_{e_1} is stable, and so is

$$(\pm 1 + z_1) F_{e_1} = \phi(\pm e_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$$

which proves that $\pm e_1 \in C$. □

(Of course, by permuting the indices Proposition 1.2.8 applies to any index j as well.)

Definition 1.2.9. A polynomial $f \in \mathbb{R}[\mathbf{z}]$ is *multiaffine* if it has degree at most one in each variable.

Let $\mathbb{R}[\mathbf{z}]^{\mathrm{ma}}$ denote the vector subspace of multiaffine polynomials: that is, polynomials of degree at most one in each indeterminate.

Proposition 1.2.10 (Theorem 5.6 of [Brä07]). *Let $f \in \mathbb{R}[\boldsymbol{z}]^{\mathrm{ma}}$ be multiaffine. Then the following are equivalent:*

(a) f *is stable.*

(b) *For all $1 \leqslant i < j \leqslant n$ and all $\boldsymbol{a} \in \mathbb{R}^n$,*

$$\frac{\partial f}{\partial z_i}(\boldsymbol{a})\frac{\partial f}{\partial z_j}(\boldsymbol{a}) - f(\boldsymbol{a})\frac{\partial^2 f}{\partial z_i \partial z_j}(\boldsymbol{a}) \geqslant 0.$$

Definition 1.2.11. For a linear operator $T : \mathbb{R}[\boldsymbol{z}]^{\mathrm{ma}} \to \mathbb{R}[\boldsymbol{z}]$ define its *algebraic symbol* as

$$G_T(\boldsymbol{z}, \boldsymbol{w}) = T\left[(\boldsymbol{z} + \boldsymbol{w})^{[n]}\right] \in \mathbb{R}[\boldsymbol{z}, \boldsymbol{w}].$$

The importance of this definition lies in the following characterization of stability-preserving linear operators.

Proposition 1.2.12 (Theorem 2.2 of [BB09]). *Let $T : \mathbb{R}[\boldsymbol{z}]^{\mathrm{ma}} \to \mathbb{R}[\boldsymbol{z}]$ be a linear transformation. Then T preserves stability if and only if either*

(a) $T(f) = \eta(f) \cdot p$ *for some linear functional* $\eta : \mathbb{R}[\boldsymbol{z}]^{\mathrm{ma}} \to \mathbb{R}$ *and stable* $p \in \mathbb{R}[\boldsymbol{z}]$, *or*

(b) *the polynomial* $G_T(\boldsymbol{z}, \boldsymbol{w})$ *is stable, or*

(c) *the polynomial* $G_T(\boldsymbol{z}, -\boldsymbol{w})$ *is stable.*

1.3 Proof of the MCP conjecture

The following multivariate generalization of the MCP conjecture, which appeared in [HV09], turned out to be a key idea in the resolution of the conjecture.

Conjecture 1.3.1. *Let J_n denote the n-by-n matrix of all ones, and let*

$$Z_n = \mathrm{diag}(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{pmatrix}$$

be the n-by-n diagonal matrix of indeterminates $\boldsymbol{z} = z_1, \ldots, z_n$. If A is an n-by-n monotone column matrix then $\mathrm{per}(J_n Z_n + A)$ is a stable polynomial.

At first glance, this might seem as an unnecessary generalization of the problem (multiple variables, stability vs real-rootedness). But we also gain something. It turns out that it is sufficient to verify the above multivariate MCP conjecture (MMCPC) for Ferrers matrices only. By introducing multiple variable we are able to reduce the problem to checking stability for a finite number of cases for each fixed n.

1.3.1 Reduction to Ferrers matrices

Lemma 1.3.2. *If* $\mathrm{per}(J_n Z_n + A)$ *is stable for all Ferrers matrices* A*, then the MMCPC (Conjecture 1.3.1) is true.*

Proof. If $\mathrm{per}(z_j + a_{ij})$ is stable for all Ferrers matrices, then by permuting the columns of such a matrix, the same is true for all monotone column $\{0,1\}$-matrices. Now let $A = (a_{ij})$ be an arbitrary n-by-n monotone column matrix. We will show that $\mathrm{per}(z_j + a_{ij})$ is stable by n applications of Proposition 1.2.8.

Let V be the vector space of column vectors of length n. The multilinear form ϕ we consider is the permanent of an n-by-n matrix obtained by concatenating n vectors in V. Let each of $e_1, \ldots, e_n$ be the all-ones vector in V.

Initially, let $v_1, v_2, \ldots, v_n$ be arbitrary monotone $\{0,1\}$-vectors. Then $\phi(v_1 + z_1 e_1, \ldots, v_n + z_n e_n) = \mathrm{per}(J_n Z_n + H)$ for some monotone column $\{0,1\}$-matrix H. One can specialize any number of v_j to the zero vector, and any number of z_j to 1, and the result is not identically zero. By hypothesis, all these polynomials are stable.

Now we proceed by induction. Assume that if $v_1, \ldots, v_{j-1}$ are the first $j-1$ columns of A, and if $v_j, \ldots, v_n$ are arbitrary monotone $\{0,1\}$-columns, then $\phi(v_1 + z_1 e_1, \ldots, v_n + z_n e_n)$ is stable. (The base case, $j = 1$, is the previous paragraph.) Putting $v_j = 0$ and $z_j = 1$, the resulting polynomial is not identically zero. By Proposition 1.2.8 (applied to index j), the set of vectors v_j such that $\phi(v_1 + z_1 e_1, \ldots, v_n + z_n e_n)$ is stable is a convex cone containing $\pm e_j$. Moreover, it contains all monotone $\{0,1\}$-columns, by hypothesis. Now, any monotone column of real numbers can be written as a nonnegative linear combination of $-e_1$ and monotone $\{0,1\}$-columns, and hence is in this cone. Thus, we may take $v_1, \ldots, v_{j-1}, v_j$ to be the first j columns of A, $v_{j+1}, \ldots, v_n$ to be arbitrary monotone $\{0,1\}$-columns, and the resulting polynomial is stable. This completes the induction step.

After the n-th step we find that $\mathrm{per}(J_n Z_n + A)$ is stable. $\square$

1.3.2 A more symmetrical problem

Let $A = (a_{ij})$ be an n-by-n Ferrers matrix, and let $\mathbf{z} = \{z_1, \ldots, z_n\}$. For each $1 \leqslant j \leqslant n$, let $y_j = (z_j + 1)/z_j$, and let $Y_n = \mathrm{diag}(y_1, \ldots, y_n)$. The matrix obtained from $J_n Z_n + A$ by factoring z_j out of column j for all $1 \leqslant j \leqslant n$ is $AY_n + J_n - A = (a_{ij} y_j + 1 - a_{ij})$. It follows that

$$\mathrm{per}(z_j + a_{ij}) = z_1 z_2 \cdots z_n \cdot \mathrm{per}(a_{ij} y_j + 1 - a_{ij}). \tag{1.3.1}$$

Lemma 1.3.3. *For a Ferrers matrix* $A = (a_{ij})$*,* $\mathrm{per}(z_j + a_{ij})$ *is stable if and only if* $\mathrm{per}(a_{ij} y_j + 1 - a_{ij})$ *is stable.*

Proof. The polynomials are not identically zero. Notice that $\Im(z_j) > 0$ if and only if $\Im(y_j) = \Im(1 + z_j^{-1}) < 0$. If $\mathrm{per}(z_j + a_{ij})$ is stable, then $\mathrm{per}(a_{ij}y_j + 1 - a_{ij}) \neq 0$ whenever $\Im(y_j) < 0$ for all $1 \leqslant j \leqslant n$. Since this polynomial has real coefficients, it follows that it is stable. The converse is similar. $\square$

The set of n-by-n Ferrers matrices has the following duality $A \mapsto A^\vee = J_n - A^\top$: transpose and exchange zeros and ones. That is, $A^\vee = (a_{ij}^\vee)$ in which $a_{ij}^\vee = 1 - a_{ji}$ for all $1 \leqslant i, j \leqslant n$. Note that $(A^\vee)^\vee = A$. However, the form of the expression $\mathrm{per}(a_{ij}y_j + 1 - a_{ij})$ is not preserved by this symmetry. To remedy this defect, introduce new indeterminates $\mathbf{x} = \{x_1, \ldots, x_n\}$ and consider the matrix $B(A) = (b_{ij})$ with entries $b_{ij} = a_{ij}y_j + (1 - a_{ij})x_i$ for all $1 \leqslant i, j \leqslant n$. For example, if

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then} \quad B(A) = \begin{bmatrix} x_1 & y_2 & y_3 & y_4 & y_5 \\ x_2 & x_2 & y_3 & y_4 & y_5 \\ x_3 & x_3 & y_3 & y_4 & y_5 \\ x_4 & x_4 & x_4 & y_4 & y_5 \\ x_5 & x_5 & x_5 & x_5 & x_5 \end{bmatrix}.$$

For emphasis, we may write $B(A; \mathbf{x}; \mathbf{y})$ to indicate that the row variables are $\mathbf{x}$ and the column variables are $\mathbf{y}$. The matrices $B(A)$ and $B(A^\vee)$ have the same general form, and in fact

$$\mathrm{per}(B(A^\vee; \mathbf{x}; \mathbf{y})) = \mathrm{per}(B(A; \mathbf{y}; \mathbf{x})). \tag{1.3.2}$$

Clearly $\mathrm{per}(B(A))$ specializes to $\mathrm{per}(a_{ij}y_j + 1 - a_{ij})$ by setting $x_i = 1$ for all $1 \leqslant i \leqslant n$. We will show that $\mathrm{per}(B(A))$ is stable, for any Ferrers matrix A. By Lemmas 1.2.3(d), 1.3.3, and 1.3.2, this will imply the MMCPC.

1.3.3 A differential recurrence relation

Next, we derive a differential recurrence relation for polynomials of the form $\mathrm{per}(B(A))$, for A an n-by-n Ferrers matrix. There are two cases: either $a_{nn} = 0$ or $a_{nn} = 1$. Replacing A by $A^\vee$ and using (1.3.2), if necessary, we can assume that $a_{nn} = 0$.

Lemma 1.3.4. *Let $A = (a_{ij})$ be an n-by-n Ferrers matrix with $a_{nn} = 0$, let $k \geqslant 1$ be the number of 0's in the last column of A, and let A° be the matrix obtained from A by deleting the last column and the last row of A. Then*

$$\mathrm{per}(B(A)) = kx_n \,\mathrm{per}(B(A^\circ)) + x_n y_n \,\boldsymbol{\partial}\mathrm{per}(B(A^\circ)),$$

$$
\begin{bmatrix}
\cdot & \cdot & \square & \cdot & \cdot & \cdot \\
\square & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \square \\
\cdot & \cdot & \cdot & \cdot & \square & \cdot \\
\cdot & \square & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \square & \cdot & \cdot
\end{bmatrix}
\mapsto
\begin{bmatrix}
\cdot & \cdot & \square & \cdot & \cdot \\
\square & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \square & \cdot \\
\cdot & \cdot & \cdot & \cdot & \square \\
\cdot & \square & \cdot & \cdot & \cdot
\end{bmatrix}
$$

Figure 1.1: $\sigma = 3\ 1\ 6\ 5\ 2\ 4$ maps to $\pi(\sigma) = 3\ 1\ 4\ 5\ 2$.

in which

$$\boldsymbol{\partial} = \sum_{i=1}^{n-k} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j}.$$

Proof. In the permutation expansion of $\mathrm{per}(B(A))$ there are two types of terms: those that do not contain y_n and those that do. Let T_σ be the term of $\mathrm{per}(B(A))$ indexed by $\sigma \in \mathfrak{S}_n$. For each $n-k+1 \leqslant i \leqslant n$, let $\mathcal{C}_i$ be the set of those terms T_σ such that $\sigma(i) = n$; for a term in $\mathcal{C}_i$ the variable chosen in the last column is x_i. Let $\mathcal{D}$ be the set of all other terms; for a term in $\mathcal{D}$ the variable chosen in the last column is y_n.

For every permutation $\sigma \in \mathfrak{S}_n$, let (i_σ, j_σ) be such that $\sigma(i_\sigma) = n$ and $\sigma(n) = j_\sigma$, and define $\pi(\sigma) \in \mathfrak{S}_{n-1}$ by putting $\pi(i) = \sigma(i)$ if $i \neq i_\sigma$, and $\pi(i_\sigma) = j_\sigma$ (if $i_\sigma \neq n$). Let $T_{\pi(\sigma)}$ be the corresponding term of $\mathrm{per}(B(A^\circ))$. See Figure 1 for an example. Informally, $\pi(\sigma)$ is obtained from σ, in word notation, by replacing the largest element with the last element, unless the largest element is last, in which case it is deleted.

For each $n-k+1 \leqslant i \leqslant n$, consider all permutations σ indexing terms in $\mathcal{C}_i$. The mapping $T_\sigma \mapsto T_{\pi(\sigma)}$ is a bijection from the terms in $\mathcal{C}_i$ to all the terms in $\mathrm{per}(B(A^\circ))$. Also, for each $\sigma \in \mathcal{C}_i$, $T_\sigma = x_n T_{\pi(\sigma)}$. Thus, for each $n-k+1 \leqslant i \leqslant n$, the sum of all terms in $\mathcal{C}_i$ is $x_n \mathrm{per}(B(A^\circ))$.

Next, consider all permutations σ indexing terms in $\mathcal{D}$. The mapping $T_\sigma \mapsto T_{\pi(\sigma)}$ is $(n-k)$-to-one from $\mathcal{D}$ to the set of all terms in $\mathrm{per}(B(A^\circ))$, since one needs both $\pi(\sigma)$ and i_σ to recover σ. Let ν_σ be the variable in position (i_σ, j_σ) of $B(A^\circ)$. Then $\nu_\sigma T_\sigma = x_n y_n T_{\pi(\sigma)}$. It follows that for any variable w in the set $\{x_1, \ldots, x_{n-k}, y_1, \ldots, y_{n-1}\}$, the sum over all terms in $\mathcal{D}$ such that $\nu_\sigma = w$ is

$$x_n y_n \frac{\partial}{\partial w} \mathrm{per}(B(A^\circ)).$$

Since ν_σ is any element of the set $\{x_1, \ldots, x_{n-k}, y_1, \ldots, y_{n-1}\}$, it follows that the sum of all terms in $\mathcal{D}$ is $x_n y_n \boldsymbol{\partial} \mathrm{per}(B(A^\circ))$.

The preceding paragraphs imply the stated formula. $\square$

1.3.4 Proof of the multivariate MCP conjecture

Theorem 1.3.5. *For any* n*-by-*n *Ferrers matrix* A, $\mathrm{per}(B(A))$ *is stable.*

Proof. As above, by replacing A by $A^\vee$ if necessary, we may assume that $a_{1n} = 0$. We proceed by induction on n, the base case $n = 1$ being trivial. For the induction step, let A° be as in Lemma 1.3.4. By induction, we may assume that $\mathrm{per}(B(A^\circ))$ is stable; clearly this polynomial is multiaffine. Thus, by Lemma 1.3.4, it suffices to prove that the linear transformation $T = k + y_n \partial$ maps stable multiaffine polynomials to stable polynomials if $k \geqslant 1$. This operator has the form $T = k + z_n \sum_{j=1}^{n-1} \partial/\partial z_j$ (renaming the variables suitably). By Proposition 1.2.12 it suffices to check that the polynomial

$$\begin{aligned} G_T(\boldsymbol{z}, \boldsymbol{w}) &= T\left[(\boldsymbol{z} + \boldsymbol{w})^{[n]}\right] \\ &= \left(k + z_m \sum_{j=1}^{m-1} \frac{1}{z_j + w_j}\right)(\boldsymbol{z} + \boldsymbol{w})^{[n]} \end{aligned}$$

is stable. If z_j and w_j have positive imaginary parts for all $1 \leqslant j \leqslant n$ then

$$\xi = \frac{k}{z_m} + \sum_{j=1}^{m-1} \frac{1}{z_j + w_j}$$

has negative imaginary part (since $k \geqslant 0$). Thus $z_m \xi \neq 0$. Also, $z_j + w_j$ has positive imaginary part, so that $z_j + w_j \neq 0$ for each $1 \leqslant j \leqslant n$. It follows that $G_T(\boldsymbol{z}, \boldsymbol{w}) \neq 0$, so that G_T is stable, completing the induction step and the proof. □

Proof of the MMCPC. Let A be any n-by-n Ferrers matrix. By Theorem 1.3.5, $\mathrm{per}(B(A))$ is stable. Specializing $x_i = 1$ for all $1 \leqslant i \leqslant n$, Lemma 1.2.3(d) implies that $\mathrm{per}(a_{ij} y_j + 1 - a_{ij})$ is stable. Now Lemma 1.3.3 implies that $\mathrm{per}(z_j + a_{ij})$ is stable. Finally, Lemma 1.3.2 implies that the MMCPC is true. □

Remark 1.3.6. There are several interesting consequences of the multivariate MCP theorem. These include multivariate stable Eulerian polynomials, a new proof of Grace's apolarity theorem and some permanental inequalities. We will discuss the Eulerian polynomials in depth in Chapter 2, for the other two results we refer the reader to Section 4 of [BHVW11].

1.4 On a related conjecture of Haglund

In [Hag00], Haglund introduced another conjecture closely related to the MCP conjecture. The key motive was that the permanent is essentially counting perfect matchings in a complete bipartite graph. Equivalently, one could consider perfect matchings in the complete graph on even number of vertices instead.

In order to state this conjecture, first we will define this matrix function—the analog of the permanent, and then define a condition on matrices that will replace the monotone column condition.

Definition 1.4.1. Let $C = (c_{ij})$ be a $2n \times 2n$ a symmetric matrix, the *hafnian* of C is defined as

$$\mathrm{haf}(C) = \frac{1}{n!2^n} \sum_{\sigma \in \mathfrak{S}_{2n}} \prod_{k=1}^{n} c_{\sigma(2k-1),\sigma(2k)}, \tag{1.4.1}$$

where $\mathfrak{S}_{2n}$ denotes the symmetric group on $2n$ elements.

Remark 1.4.2. Originally, Caianiello defined the hafnian for (upper) triangular arrays as the "signless Pfaffian" [Cai59, Equation (11)]. There are slight variations in the literature on how to extend the original definition to matrices; the one above is in agreement with the notation in [Hag00].

The analogue of the monotone column property is best explained using triangular arrays (which are essentially matrices with the entries below the diagonal discarded).

Definition 1.4.3. Let $1 \leqslant m \leqslant n$. The m*th hook* of a triangular array $A = (a_{ij})_{1 \leqslant i<j \leqslant n}$ is the set of cells given by

$$\mathrm{hook}_m = \{(i,m) \mid i = 1,\dots,m-1\} \cup \{(m,j) \mid j = m+1,\dots,n\}. \tag{1.4.2}$$

Furthermore, the direction *along* the mth hook is the one in which the quantity $i+j$ is increasing where $(i,j) \in \mathrm{hook}_m$.

Definition 1.4.4. A *monotone hook triangular array* has real entries decreasing along at least $n-1$ of its hooks, or possibly along all n of them. Analogously, a *monotone hook matrix* is a real symmetric matrix whose entries above the diagonal form a monotone hook triangular array.

We are in position to state the following conjecture of Haglund.

Conjecture 1.4.5 (Conjecture 2.3 in [Hag00])**.** *Let* A *be a* $2n \times 2n$ *monotone hook matrix, and* J_{2n} *the* $2n \times 2n$ *matrix of all ones. Then the polynomial* $\mathrm{haf}(zJ_{2n} + A)$ *has only real roots.*

We will refer to this conjecture as the Monotone Hook Hafnian (MHH) conjecture. Haglund proved that the MHH conjecture holds for adjacency matrices of a class of graphs called threshold graphs (see Definition 1.4.6), and as a corollary for all monotone hook $\{0,1\}$ matrices [Hag00, Theorem 2.2]. In addition, the MHH conjecture was verified for all $2n \times 2n$ monotone hook matrices for $n \leqslant 2$ [Hag00, Theorem 4.4].

Threshold graphs have been widely studied and are known to have several equivalent definitions (see Theorem 1.2.4 in [MP95] for a couple). For our purposes, the following definition will come in handy.

Definition 1.4.6. A graph G on n vertices is a *threshold graph* if it that can be constructed starting from a one-vertex graph by adding vertices one at a time in the following way. At step i, the vertex being added is either isolated (has degree 0) or dominating (has degree $i-1$ at the time when added).

Let A_G denote the adjacency matrix of a threshold graph G. From Definition 1.4.1 it is clear, that $\mathrm{haf}(zJ_{2n} + A_G)$ is invariant under the permutation of the vertices of G. Hence, we can assume that the vertices of G are labeled in the order of the above construction. This means that in every column i, for $2 \leqslant i \leqslant 2n$, the entries above the diagonal element (i,i) are either equal to z (if v_i was added as an isolated vertex) or $z+1$ (if v_i was added as dominating vertex). This suggests the multivariate generalization that we show next. The idea essentially is to add a new variable for each vertex (see construction of the matrix B in Theorem 1.4.7 below).

The following theorem is a (multivariate) generalization of the MHH conjecture for the special case of adjacency matrices of threshold graphs.

Theorem 1.4.7. *Let $z_1, \ldots, z_{2n}$ denote commuting indeterminates and let $B = (b_{ij})$ denote the $2n \times 2n$ symmetric matrix with entries $b_{ij} = z_{\max(i,j)}$. Then $\mathrm{haf}(B)$ is a stable polynomial in the variables $z_2, \ldots, z_{2n}$ (z_1 only appears on the diagonal).*

Proof. The proof is a direct application of the idea of the proof of Theorem 1.3.5. to hafnians. We use induction. Clearly,

$$\mathrm{haf}\begin{pmatrix} z_1 & z_2 \\ z_2 & z_2 \end{pmatrix} = z_2$$

is stable, which settles the base case. Let B° denote the $(2n-2)$-by-$(2n-2)$ matrix obtained from B by deleting the last two rows and two last columns. Next we show that $\mathrm{haf}(B)$ is stable if $\mathrm{haf}(B^\circ)$ was stable. This follows from the differential recursion:

$$\mathrm{haf}(B) = z_{2n}\mathrm{haf}(B^\circ) + 2z_{2n-1}z_{2n}\partial\mathrm{haf}(B^\circ)$$

in which

$$\partial = \sum_{i=2}^{2n-2} \frac{\partial}{\partial z_i}.$$

This recursion can be seen from the expansion of the hafnian along the last column. The differential operator in the right-hand side preserves stability. It can be seen, analogously to the proof of Theorem 1.3.5, that the algebraic symbol of the linear operator $T = z_{2n-1}z_{2n}\left(\frac{1}{z_{2n-1}} + 2\sum \frac{\partial}{\partial z_i}\right)$ is a stable polynomial. $\square$

The proposition gives a multivariate version of the MHH conjecture for certain matrices. Unfortunately, it is not clear how to transition from here to the general MHH conjecture. Nevertheless, Theorem 1.4.7 does imply the following theorem of Haglund, the special case of the MHH conjecture for threshold graphs.

Theorem 1.4.8 (Theorem 2.2 of [Hag00])**.** *Let* A_G *denote the adjacency matrix of a (non-weighted) threshold graph* G *on* $2n$ *vertices. Then* $\mathrm{haf}(zJ_{2n} + A_G)$ *is stable.*

Proof. Let G be a threshold graph on $2n$ vertices. Assume that the vertices of G are ordered as in the definition of the vertex-by-vertex construction above. It is easy to see that matrix B in Theorem 1.4.7 specializes to $zJ_{2n} + A_G$ if for all i, $2 \leqslant i \leqslant 2n$, we set

$$z_i = \begin{cases} z, & \text{if } v_i \text{ was added as an isolated vertex} \\ z+1, & \text{if } v_i \text{ was added as a dominating vertex} \end{cases} \tag{1.4.3}$$

These last operations of translating, specializing and diagonalizing the variables preserves stability by Lemma 1.2.3 parts (c), (d) and (f). $\square$

Remark 1.4.9. The same proof goes through if we allow edges to have any weights (as opposed to only zero or one). The only restriction that applies is that when we add a dominating vertex all edges incident to it must have the same weight.

"Lisez Euler, lisez Euler, c'est notre maître à tous."

Pierre-Simon Laplace

Chapter 2

Stable Eulerian polynomials

Eulerian polynomials play an important role in enumerative combinatorics. In the context of the MCP conjecture (now theorem), these polynomials naturally arise as permanents of the so-called "staircase" shaped Ferrers matrices. In this chapter, we will examine the combinatorial interpretation of these polynomials in terms of permutation statistics.

We begin with some review of terminology. In Section 2.1, we introduce descents and excedances in permutations. These statistics are often called Eulerian, since their distribution over $\mathfrak{S}_n$ agrees with the coefficients of the Eulerian polynomials $\mathfrak{S}_n(x)$. We then define refinements of these statistics which will give rise to multivariate Eulerian polynomials. As a consequence of the MMCP theorem these multivariate polynomials are stable which generalizes a classical result that the Eulerian polynomials are real-rooted. This observation serves as a starting point for the results presented in this chapter.

The notion of descent can be extended to various generalizations of permutations, such as multi-permutations, or Coxeter groups. Eulerian polynomials arising from such generalizations are often known—in some cases only conjectured—to be *real-rooted*. Our contribution is a unifying framework that allows us to strengthen these results to multivariate stability. This is achieved by understanding the differential recurrences and taking advantage of the fact that they turn out to be given by stability-preserving linear operators.

In Section 2.2, we provide stable Eulerian polynomials for Stirling permutations, and their generalizations. In Section 2.3, we prove stability for Eulerian polynomials for signed and colored permutations and Eulerian-like polynomials for some affine Weyl groups.

This chapter is based on the following two papers [HV12, VW12]. These results were obtained in collaboration with J. Haglund and N. Williams, respectively. Our presentation closely follows these papers, some parts are taken verbatim, some with minor modifications.

2.1 Eulerian polynomials as generating functions

The polynomials

$$
\begin{aligned}
``\alpha &= x \\
\beta &= x + x^2 \\
\gamma &= x + 4x^2 + x^3 \\
\delta &= x + 11x^2 + 11x^3 + x^4 \\
\varepsilon &= x + 26x^2 + 66x^3 + 26x^4 + x^5 \\
\zeta &= x + 57x^2 + 302x^3 + 302x^4 + 57x^5 + x^6 \text{ etc.''}
\end{aligned}
$$

appeared in Euler's work on a method of summation of series [Eul36].

Since then these polynomials, known as the Eulerian polynomials and their coefficients, the so-called Eulerian numbers have been widely studied in enumerative combinatorics, especially within the combinatorics of permutations. They serve as generating functions of several permutation statistics such as descents, excedances, or runs of permutations. They are intimately connected to Stirling numbers, and also the binomial coefficients, via the famous Worpitzky-identity [Wor83]. See the notes of Foata and Schützenberger [FS70, Foa10], Carlitz ([Car59, Car73]) for a survey and the history of these polynomials.

2.1.1 Statistics on permutations

Let n be a positive integer, and recall that $\mathfrak{S}_n$ denotes the set of all permutations of the set $[n] = \{1, \ldots, n\}$.

Definition 2.1.1. For a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$, let

$$
\begin{aligned}
ASC(\pi) &= \{ i \mid \pi_{i-1} < \pi_i \}, \\
DES(\pi) &= \{ i \mid \pi_i > \pi_{i+1} \},
\end{aligned}
$$

denote the *ascent set* and the *descent set* of π, respectively.

For convenience, we will also define a slight variant of the above statistics.

Definition 2.1.2. For a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$, let

$$
\begin{aligned}
ASC^0(\pi) &= \{ i \mid \pi_{i-1} < \pi_i \} \cup \{1\}, \\
DES^0(\pi) &= \{ i \mid \pi_i > \pi_{i+1} \} \cup \{n\},
\end{aligned}
$$

denote the *extended ascent set* and the *extended descent set* of π, respectively.

Remark 2.1.3. One way to think about the latter extended sets is to consider statistics on the permutation π with a zero prepended to the beginning and a zero appended to the end, i.e., $\pi_0 = \pi_{n+1} = 0$.

Definition 2.1.4. For a permutation π in $\mathfrak{S}_n$ let

$$\begin{aligned} \mathrm{des}(\pi) &= |\mathrm{DES}(\pi)| \\ \mathrm{asc}(\pi) &= |\mathrm{ASC}(\pi)| \\ \mathrm{edes}(\pi) &= |\mathrm{DES}^0(\pi)| \\ \mathrm{easc}(\pi) &= |\mathrm{ASC}^0(\pi)| \end{aligned}$$

denote the cardinality of these sets, the number of descents, ascents and extended descents and extended ascents in π, respectively.

Remark 2.1.5. Obviously, $\mathrm{edes}(\pi) = \mathrm{des}(\pi) + 1$ and $\mathrm{easc}(\pi) = \mathrm{asc}(\pi) + 1$, but sometimes it will be more convenient to use the notation for the extended descents (and ascents).

We also mention two other well-known permutation statistics.

Definition 2.1.6. Let

$$\mathcal{EXC}(\pi) = \{i \mid \pi_i > i\}$$

denote the *set of excedances*, and let $\mathrm{exc}(\pi) = |\mathcal{EXC}(\pi)|$.

Definition 2.1.7. Let

$$\mathcal{WEXC}(\pi) = \{i \mid \pi_i \geqslant i\}$$

denote the *set of weak excedances*, and let $\mathrm{wexc}(\pi) = |\mathcal{WEXC}(\pi)|$.

It is well-known that excedances are equidistributed with descents, and weak excedances are equidistributed with the extended descents.

2.1.2 Eulerian numbers

Eulerian numbers (see sequence A008292 in the OEIS [OEI12]) denoted by $\left\langle {n \atop k} \right\rangle$, or sometimes by $A(n, k)$, are amongst the most studied sequences of numbers in enumerative combinatorics. They count, for example, the number of permutations of $\{1, \dots, n\}$ with k extended descents (or $k - 1$ descents).

Definition 2.1.8. For $1 \leqslant k \leqslant n$, let

$$\left\langle {n \atop k} \right\rangle = |\{\pi \in \mathfrak{S}_n \mid \mathrm{edes}(\pi) = k\}| \,. \tag{2.1.1}$$

From this definition it is then clear that the Eulerian numbers satisfy the following recursion. Observe what happens to the number of descents when $n + 1$ is inserted in a permutation $\pi \in \mathfrak{S}_n$.

Proposition 2.1.9. *For* $1 \leqslant k < n+1$,

$$\left\langle {n+1 \atop k} \right\rangle = k \left\langle {n \atop k} \right\rangle + (n+2-k) \left\langle {n \atop k-1} \right\rangle, \tag{2.1.2}$$

with initial condition $\left\langle {1 \atop 1} \right\rangle = 1$ *and boundary conditions* $\left\langle {n \atop k} \right\rangle = 0$ *for* $k \leqslant 0$ *or* $n < k$.

In this chapter, we will investigate the ordinary generating function of Eulerian numbers along with several generalizations of them.

Definition 2.1.10. The generating polynomials of the Eulerian numbers, also known as the *classical Eulerian polynomials* are defined as:

$$\mathfrak{S}_n(x) = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle x^k = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{edes}(\pi)} . \tag{2.1.3}$$

Remark 2.1.11. We note here that there is no standard notation for Eulerian polynomials unfortunately. In the works of Euler, Carlitz, Comtet the above notation is more prevalent, however, recently the descent generating polynomial (which is off by a factor of x) is also commonly used.

The recursion for the Eulerian numbers (2.1.2) translates into a recursion for the Eulerian polynomials.

Proposition 2.1.12. *Let* $A_0(x) = 1$. *For* $n \geqslant 0$,

$$\mathfrak{S}_{n+1}(x) = (n+1)x\mathfrak{S}_n(x) + x(1-x)\frac{\partial}{\partial x}\mathfrak{S}_n(x) . \tag{2.1.4}$$

Remark 2.1.13. See an intuitive proof of this statement in REF (CH3).

This recursion serves as the basis of several proofs of the following folklore result in combinatorics, already noted by Frobenius [Fro10, p. 829].

Theorem 2.1.14. *The classical Eulerian polynomial* $\mathfrak{S}_n(x)$ *has only real roots. Furthermore, the roots are all distinct and nonpositive.*

2.1.3 Polynomials with only real roots

There are several properties of a sequence implied by the fact that its generating polynomial has only real roots. Real-rootedness of a polynomial $\sum_{i=0}^{n} b_i x^i$ with nonnegative coefficients is often used to show that the sequence $\{b_i\}_{i=0\dots n}$ is log-concave ($b_i^2 \geqslant b_{i+1}b_{i-1}$) and unimodal ($b_0 \leqslant \cdots \leqslant b_k \geqslant \cdots \geqslant b_n$). In fact, even more is true. For example, the sequence $\{b_i\}_{i=0\dots n}$ can have at most two modes, and the coefficients satisfy several nice properties such as Newton's inequalities [New07], Darroch's theorem [Dar64], etc. Furthermore, the normalized coefficients $b_i/(\sum_i b_i)$ – viewed as a probability distribution – converge to a normal distribution as n goes to infinity, under the additional constraint that the variance tends to infinity [Ben73, Har67]. See also [Pit97] for a review and further references.

2.2 Eulerian polynomials for Stirling permutations

Gessel and Stanley defined and studied the following restricted subset of multiset permutations, called *Stirling permutations*, in [GS78]. Their motivation was to find a combinatorial interpretation of the coefficients of a polynomial related to the Stirling numbers. Eventually, Stirling permutations turned out to be interesting enough combinatorial objects to be studied in their on right.

Definition 2.2.1. Consider the multiset $\mathcal{M}_n = \{1, 1, 2, 2, \ldots, n, n\}$, where each number from 1 to n appears twice. *Stirling permutations* of order n, denoted by $\mathcal{Q}_n$, are permutations of $\mathcal{M}_n$ in which for all $1 \leqslant i \leqslant n$, all entries between the two occurrences of i are larger than i.

For instance, $\mathcal{Q}_1 = \{11\}$, $\mathcal{Q}_2 = \{1122, 1221, 2211\}$, and from a recursive construction rule—observe that n and n have to be adjacent in $\mathcal{Q}_n$—it is not difficult to see that $|\mathcal{Q}_n| = 1 \cdot 3 \cdot \cdots \cdot (2n-1) = (2n-1)!!$.

2.2.1 Statistics on Stirling permutations

Gessel and Stanley also studied the descent statistic over $\mathcal{Q}_n$ (the number of descents gave the interpretations of the coefficients they were looking for). The notions of ascents and descents can be easily extended to Stirling permutations. Bóna in [Bón09] introduced an additional statistic called *plateau* and studied the distribution of the following three statistics over Stirling permutations.

Definition 2.2.2. For $\sigma = \sigma_1\sigma_2 \ldots \sigma_{2n} \in \mathcal{Q}_n$, let

$$\begin{aligned}
\mathcal{ASC}^0(\sigma) &= \{\, i \mid \sigma_{i-1} < \sigma_i \} \cup \{1\}, \\
\mathcal{DES}^0(\sigma) &= \{\, i \mid \sigma_i > \sigma_{i+1} \} \cup \{2n\}, \\
\mathcal{PLAT}(\sigma) &= \{\, i \mid \sigma_i = \sigma_{i+1} \}
\end{aligned}$$

denote the set of extended ascents, extended descents and plateaux of a Stirling permutation σ, respectively.

As for permutations, we can think of these statistics as "padding" the Stirling permutation with zeros, i.e., defining $\sigma_0 = \sigma_{2n+1} = 0$.

Definition 2.2.3. For $\sigma \in \mathcal{Q}_n$, let

$$\begin{aligned} \operatorname{easc}(\sigma) &= |\mathcal{ASC}^0(\sigma)|\,, \\ \operatorname{edes}(\sigma) &= |\mathcal{DES}^0(\sigma)|\,, \\ \operatorname{plat}(\sigma) &= |\mathcal{PLAT}(\sigma)| \end{aligned}$$

denote the number of extended ascents, extended descents, and plateaux in σ.

Remark 2.2.4. $\operatorname{edes}(\sigma) + \operatorname{easc}(\sigma) + \operatorname{plat}(\sigma) = 2n + 1$, for any $\sigma \in \mathcal{Q}_n$ (since the number of gaps, counting the padding zeros, is exactly $2n + 1$).

We conclude by yet another reason why the extended statistics is a convenient shorthand.

Proposition 2.2.5 (Proposition 1 in [Bón09]). *The three statistics are equidistributed over* $\mathcal{Q}_n$.

2.2.2 Second-order Eulerian numbers

We adopt the following double angle-bracket notation suggested in [GKP94, p. 256].

Definition 2.2.6. For $1 \leqslant k \leqslant n$, let

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = |\{\sigma \in \mathcal{Q}_n \mid \operatorname{edes}(\sigma) = k\}|\,. \tag{2.2.1}$$

Following [GKP94], we refer to these numbers as the "second-order Eulerian numbers"[1] since they satisfy a recursion very similar to (2.1.2).

Proposition 2.2.7 (Equation (14) in [Car65]). *For* $1 \leqslant k < n + 1$, *we have*

$$\left\langle\!\!\left\langle {n+1 \atop k} \right\rangle\!\!\right\rangle = k \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle + (2n + 2 - k) \left\langle\!\!\left\langle {n \atop k-1} \right\rangle\!\!\right\rangle\,, \tag{2.2.2}$$

with initial condition $\left\langle\!\left\langle {1 \atop 1} \right\rangle\!\right\rangle = 1$ *and boundary conditions* $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = 0$ *for* $k \leqslant 0$ *or* $n < k$.

Remark 2.2.8. Note that our indexing is in agreement with sequence A008517 in [OEI12] and with the definition of the statistics adopted from [Bón09], however it differs from the one in [GKP94].

The second-order Eulerian numbers have not been as widely studied as the Eulerian numbers. Nevertheless, they are known to have several interesting combinatorial interpretations. Apart from counting Stirling permutations $\mathcal{Q}_n$ with k descents [GS78], k ascents, k plateau [Bón09], these numbers also count

[1] Not to be confused with $2^n - 2n$ (see sequence A005803 in [OEI12]).

the number of Riordan trapezoidal words of length n with k distinct letters [Rio76, p. 9], the number of rooted plane trees on $n+1$ nodes with k leaves [Jan08], and matchings of the complete graph on $2n$ vertices with $(n-k)$ left-nestings (Claim 6.1 of [Lev10]).

Definition 2.2.9. The *second-order Eulerian polynomial*, denoted by $\mathfrak{S}_n^{(2)}(x)$, is the generating function of the second-order Eulerian numbers, formally,

$$\mathfrak{S}_n^{(2)}(x) = \sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle x^k = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{edes}(\sigma)} .$$

The following theorem is an analog of Theorem 2.1.14 for the second-order Eulerian numbers.

Theorem 2.2.10 (Theorem 1 of [Bón09]). *The second-order Eulerian polynomial $\mathfrak{S}_n^{(2)}(x)$ has only real (simple, nonnegative) roots.*

This result can be seen as the consequence of that the following recursion satisfied by these generating polynomials which is strikingly similar to (2.1.4).

Proposition 2.2.11 (Equation (13) in [Car65]). *Let $\mathfrak{S}_1^{(2)}(x) = 1$. For $n \geqslant 0$,*

$$\mathfrak{S}_{n+1}^{(2)}(x) = (2n+1)x\mathfrak{S}_n^{(2)}(x) + x(1-x)\frac{\partial}{\partial x}\mathfrak{S}_n^{(2)}(x) . \tag{2.2.3}$$

2.2.3 A stable refinement of the classical Eulerian polynomial

Brändén and Stembridge suggested finding a stable multivariate generalization of the Eulerian polynomials. In [HV09], the multivariate polynomial

$$\mathfrak{S}_n(\mathbf{x}) = \sum_{\pi \in \mathfrak{S}_n} \left(\prod_{\pi_i \geqslant i} x_{\pi_i} \right), \tag{2.2.4}$$

was shown to be stable for $n \leqslant 5$, and was conjectured to be stable for all n.

Next we show that the multivariate refinement in (2.2.4) refines the weak excedance set and the extended descent set statistics simultaneously. See Proposition 2.2.13 below for a formal statement. This correspondence is key as the extended descents allow for an easy recurrence. So, for convenience, we will be working with extended descents instead of weak excedances.

Let us continue with the introduction of some notation for the refinement of the extended descent sets.

Definition 2.2.12. For a permutation π in $\mathfrak{S}_n$, define its *descent top* set to be

$$\mathcal{DT}(\pi) = \{\pi_i : 0 \leqslant i \leqslant n+1, \pi_i > \pi_{i+1}\},$$

and similarly, let its *ascent top* set be

$$\mathcal{AT}(\pi) = \{\pi_{i+1} : 0 \leqslant i \leqslant n+1, \pi_i < \pi_{i+1}\},$$

where $\pi_0 = \pi_{n+1} = 0$.

Proposition 2.2.13 (Proposition 3.1 in [HV12])**.**

$$\mathfrak{S}_n(\mathbf{x}) = \sum_{\pi \in \mathfrak{S}_n} \mathbf{x}^{\mathcal{DT}(\pi)},$$

Proof. Follows from a bijection of Riordan [Rio58, page], see also Theorem 3.15 of [Bre94]. □

Following the idea of P. Brändén for the proof of stability of type A Eulerian polynomials (given in detail in Section 2.3.2) consider the following homogenization of $\mathfrak{S}_n(\mathbf{x})$:

$$\mathfrak{S}_n(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in \mathfrak{S}_n} \mathbf{x}^{\mathcal{DT}(\pi)} \mathbf{y}^{\mathcal{AT}(\pi)}. \tag{2.2.5}$$

Theorem 2.2.14 (Theorem 3.2 of [HV12])**.** $\mathfrak{S}_n(\mathbf{x}, \mathbf{y})$ *is a stable polynomial.*

Proof. $\mathfrak{S}_1(x_1, y_1) = x_1 y_1$ which is stable. Note that the following recursion

$$\mathfrak{S}_{n+1}(\mathbf{x}, \mathbf{y}) = x_{n+1} y_{n+1} \boldsymbol{\partial} \mathfrak{S}_n(\mathbf{x}, \mathbf{y}), \tag{2.2.6}$$

holds for $n \geqslant 1$, where $\boldsymbol{\partial}$ again denotes the sum of all partials. Note that $\boldsymbol{\partial}$ is a stability preserving operator by Proposition 1.2.12, since its algebraic symbol

$$G_{\boldsymbol{\partial}} = \left(\sum_{i=1}^{n} \frac{1}{x_i + u_i} + \sum_{i=1}^{n} \frac{1}{y_i + v_i} \right) (\mathbf{x} + \mathbf{u})^{[n]} (\mathbf{y} + \mathbf{v})^{[n]}$$

is a stable polynomial.

□

We will often deal with functions that have $\mathbf{x}$ and $\mathbf{y}$ as variables, and so unless specified otherwise the special symbol $\boldsymbol{\partial}$ will always denote $\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right)$, the sum of partial derivatives with respect to all x and y variables.

Remark 2.2.15. Using the ascent tops and descent tops for subscripts in the multivariate refinement is crucial. If we were to use the position of descents and ascents as subscripts, the resulting polynomials would fail to be stable.

From Theorem 2.2.14 we get the following corollary, which is also a special case of Theorem 3.4 of [BHVW11] for Ferrers boards of staircase shape.

Corollary 2.2.16. $\mathfrak{S}_n(\mathbf{x})$ *is a stable polynomial.*

In [BHVW11] the stability results for staircase shape boards were further extended to obtain a stable multivariate generalization of the multiset Eulerian polynomial previously studied by Simion [Sim84]. We continue along a similar direction as well, and extend Theorem 2.2.14 to a restricted subset of multiset permutations introduced in this section, the Stirling permutations.

2.2.4 Multivariate second-order Eulerian polynomials

Janson in [Jan08] suggested studying the trivariate polynomial that simultaneously counts all three statistics of the Stirling permutations (see also [Dum80]):

$$\mathfrak{S}_n^{(2)}(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\operatorname{des}(\sigma)}\, y^{\operatorname{asc}(\sigma)}\, z^{\operatorname{plat}(\sigma)}. \tag{2.2.7}$$

We go a step further, and introduce a refinement of this polynomial obtained by indexing each ascent, descent and plateau by the value where they appear, i.e., ascent top, descent top, plateau:

$$\mathfrak{S}_n^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\sigma \in \mathcal{Q}_n} \mathbf{x}^{\mathcal{DT}(\sigma)} \mathbf{y}^{\mathcal{AT}(\sigma)} \mathbf{z}^{\mathcal{PT}(\sigma)}, \tag{2.2.8}$$

where $\mathcal{AT}(\sigma)$ and $\mathcal{DT}(\sigma)$ are extended to Stirling permutations σ in the obvious way and the plateaux top set is defined as $\mathcal{PT}(\sigma) = \{\sigma_i : \sigma_i = \sigma_{i+1}\}$. For example,

$$\begin{aligned}
\mathfrak{S}_1^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= x_1 y_1 z_1\,, \\
\mathfrak{S}_2^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= x_2 y_1 y_2 z_1 z_2 + x_1 x_2 y_1 y_2 z_2 + x_1 x_2 y_2 z_1 z_2\,.
\end{aligned}$$

These polynomials are multiaffine, since any value $\nu \in \{1, \ldots, n\}$ can only appear at most once as an ascent top (similarly, at most once as a descent top or a plateau, respectively). This is immediate from the restriction in the definition of a Stirling permutation. Furthermore, each gap $(j, j+1)$ for $0 \leqslant j \leqslant 2n$ in a Stirling permutation $\sigma \in \mathcal{Q}_n$ is either a descent, or an ascent or a plateau. This implies that $\mathfrak{S}_n^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is also homogeneous, and of degree $2n + 1$.

Theorem 2.2.17. *The polynomial $\mathfrak{S}_n^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defined in* (2.2.8) *is stable.*

Proof. Note that

$$\mathfrak{S}_{n+1}^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_{n+1} y_{n+1} z_{n+1} \boldsymbol{\partial} \mathfrak{S}_n^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \tag{2.2.9}$$

where $\boldsymbol{\partial} = \sum_{i=1}^n \partial/\partial x_i + \sum_{i=1}^n \partial/\partial y_i + \sum_{i=1}^n \partial/\partial z_i$. The recursion follows from the fact that each Stirling permutation in $\mathcal{Q}_{n+1}$ is obtained by inserting two consecutive $(n+1)$'s into one of the $2n+1$ gaps of some $\sigma \in \mathcal{Q}_n$. This insertion introduces a new ascent, a new plateau, a new descent, and removes the statistic—either ascent, plateau, or descent—that existed in the gap before. From here, the proof is analogous to that of Theorem 2.2.14. □

There are some interesting corollaries of this theorem. First, note that, diagonalizing variables preserves stability (see part c in Lemma 1.2.3). Hence, by setting $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = y$, and $z_1 = \cdots = z_n = z$, we immediately have the following.

Corollary 2.2.18. *The trivariate generating polynomial $\mathfrak{S}_n^{(2)}(x, y, z)$ defined in* (2.2.7) *is stable.*

Specializing variables also preserves stability (part d of Lemma 1.2.3). Thus, by setting $y = z = 1$, we get back Theorem 2.2.10.

If we specialize variables first, without diagonalizing, namely set $y_1 = \cdots = y_n = z_1 = \cdots = z_n = 1$ we get a different corollary.

Corollary 2.2.19. *The multivariate descent polynomial for Stirling permutations*

$$\mathfrak{S}_n^{(2)}(\mathbf{x}) = \sum_{\sigma \in \mathcal{Q}_n} \mathbf{x}^{\mathcal{DT}(\sigma)} \tag{2.2.10}$$

is stable.

Corollary 2.2.20 (Theorem 2.1 of [Jan08])**.** *The trivariate polynomial $\mathfrak{S}_n^{(2)}(x, y, z)$ defined in* (2.2.7) *is symmetric in the variables x, y, z.*

Proof. Follows from the symmetry of the recursion (2.2.9) and the fact that $\mathfrak{S}_1^{(2)}(x, y, z) = xyz$. □

2.2.5 Generalized Stirling permutations and Pólya urns

One can model the differential recursion in (2.2.9) as follows (see the Urn I model in [Jan08]). Step 1: start with $r = 3$ balls in an urn. Each ball has a different color: red, green, blue. At each step i, for $i = 2 \dots n$, we remove one ball (chosen uniformly at random) from the urn and put in three new balls, one of each color. The distribution of the balls of each color corresponds to the distribution of the ascents (red), descents (green), and plateau (blue) in a Stirling permutation. Our multivariate refinement can be thought of as simply labeling each ball by a number i that represents the step i when we placed the ball in the urn. Clearly, this method can be further generalized to r colors, as was done by Janson, Kuba and Panholzer in [JKP11], which led them to consider statistics over generalizations of Stirling permutations.

We begin with one such generalization suggested by Gessel and Stanley [GS78], called r-Stirling permutations, which have also been studied by Park in [Par94c, Par94a, Par94b] under the name r-multipermutations.

Definition 2.2.21. Let r and n be a positive integers. The set of r-*Stirling permutations* of order n, denoted by $\mathcal{Q}_n(r)$, is the set of multiset permutations of $\{1^r, \ldots, n^r\}$ with the property that all elements between two occurrences of i are at least i. In other words, every element that appears between "consecutive" occurrences of i is larger than i, or in pattern avoidance terminology, $\mathcal{Q}_n$ consists of multiset permutations of $\{1^r, \ldots, n^r\}$ that are 212-avoiding.

Janson, Kuba and Panholzer in [JKP11] considered various statistics over r-Stirling permutations. We define the ascent top sets and descent top sets identically as in the two previous sections (with the convention of padding with zeros, $\sigma_0 = \sigma_{rn+1} = 0$). In addition, we will adopt their definition of the j-plateau, and define the analogous j-plateau top set.

Definition 2.2.22. For an r-Stirling permutation σ, a j-*plateau top set* of σ, denoted by $\mathcal{PT}_j(\sigma)$, is the set of values σ_i such that $\sigma_i = \sigma_{i+1}$ where $\sigma_1 \ldots \sigma_{i-1}$ contains $j-1$ instances of σ_i.

In other words, a j-plateau counts the number of times the jth occurrence of an element is followed immediately by the $(j+1)$st occurrence of it. We note that there are j-plateaux for $j = 1 \ldots r-1$ in $\mathcal{Q}_n(r)$.

Now we can define a multivariate polynomial that is analogous to the previously studied $\mathfrak{S}_n^{(1)}(\boldsymbol{x}, \boldsymbol{y})$ and $\mathfrak{S}_n^{(2)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. For $r \geqslant 1$, let

$$\mathfrak{S}_n^{(r)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{r-1}) = \sum_{\sigma \in \mathcal{Q}_n(r)} \boldsymbol{x}^{\mathcal{DT}(\sigma)} \boldsymbol{y}^{\mathcal{AT}(\sigma)} \prod_{j=1}^{r-1} \boldsymbol{z}_j^{\mathcal{PT}_j(\sigma)} \tag{2.2.11}$$

where $\boldsymbol{z}_j = z_{j,1}, \ldots, z_{j,n}$ for all $j = 1, \ldots, r-1$.

Theorem 2.2.23. $\mathfrak{S}_n^{(r)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{r-1})$ *is a stable polynomial.*

Proof. The proof is identical to that of $\mathfrak{S}_n$ and $\mathfrak{S}_n^{(2)}$ (see Theorems 2.2.14 and 2.2.17) and therefore omitted. □

As a corollary of this theorem, we obtain that the diagonalized polynomial, $\mathfrak{S}_n^{(r)}(x, y, z_1, \ldots, z_{r-1})$ is symmetric in the variables $x, y, z_1, \ldots, z_{r-1}$ which implies the results of Theorem 9 in [JKP11]. Analogously, we could define the rth order Eulerian numbers as the number of r-Stirling permutations with exactly k descents (or equivalently, k ascents or k j-plateau for some fixed j). We suggest the notation $\left\langle {n \atop k} \right\rangle_r$, r being the shorthand for the r angle parentheses. The results for the special cases of $r = 1$ and $r = 2$ give the results for permutations and Stirling permutations, respectively.

Janson, Kuba and Panholzer in [JKP11] also studied statistics over generalized Stirling permutations. These permutations were previously investigated by Brenti in [Bre89, Bre98].

Definition 2.2.24. Let $k_1, \ldots, k_n$ be nonnegative integers. The set of *generalized Stirling permutations* of rank n, denoted by $\mathcal{Q}_n^*$, is the set of all permutations of the multiset $\{1^{k_1}, \ldots, n^{k_n}\}$ with the same restriction as before: for each i, for $1 \leqslant i \leqslant n$, the elements occurring between two occurrences of i are at least i.

We can further generalize the multivariate Eulerian polynomials by simply extending the above defined statistics to generalized Stirling permutations. This corresponds to an urn model with balls colored with $\kappa = \max_{i=1}^{n} k_i + 1$ many colors: $c_1, c_2, \ldots, c_\kappa$. We start with $k_1 + 1$ balls in the urn colored with $c_1, \ldots, c_{k_1+1}$ (each ball has a different color). In each round i, for $2 \leqslant i \leqslant n$, we remove one ball and put in $k_i + 1$ balls, one from each of the first $k_i + 1$ colors, $c_1, \ldots, c_{k_i+1}$. We can then define a multivariate polynomial counting all statistics simultaneously,

$$\mathfrak{S}_n^{(*)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{\kappa-2}) = \sum_{\sigma \in \mathcal{Q}_n^*} \boldsymbol{x}^{\mathcal{DT}(\sigma)} \boldsymbol{y}^{\mathcal{AT}(\sigma)} \prod_{j=1}^{\kappa-2} \boldsymbol{z}_j^{\mathcal{PT}_j(\sigma)} \tag{2.2.12}$$

Theorem 2.2.25. $\mathfrak{S}_n^{(*)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{\kappa-2})$ *is stable.*

Proof.

$$\mathfrak{S}_{n+1}^{(*)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{\lambda-1}) = x_{n+1} y_{n+1} \left(\prod_{\ell=1}^{k_{n+1}-1} z_{\ell, n+1} \right) \partial \mathfrak{S}_n^{(*)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{\kappa-2}),$$

where $\lambda = \max(\kappa - 1, k_{n+1})$, and ∂, as before, denotes the sum of all first-order partials (with respect to all variables in $\mathfrak{S}_n^{(*)}$). $\square$

Theorem 2.2.23 is a special case of Theorem 2.2.25 with $k_i = r$, for all $1 \leqslant i \leqslant n$. Note that the diagonalized version of the polynomial defined in (2.2.12) need not be a symmetric function in all variables $x, y, z_1, z_2, \ldots z_{\kappa-2}$. Nevertheless, if we specialize all variables except x, i.e., by letting $y = z_1 = \cdots = z_{\kappa-2} = 1$ we get the following result of Brenti.

Theorem 2.2.26 (Theorem 6.6.3 in [Bre89])**.** *The descent generating polynomial over generalized Stirling permutations*

$$\mathfrak{S}_n^{(*)}(x) = \sum_{\sigma \in \mathcal{Q}_n^*} x^{\mathrm{edes}(\sigma)}$$

has only real roots.

Another interesting generalization could be obtained using the urn model. Consider a scenario when instead of removing one ball, $s \geqslant 2$ balls are removed in each round. This way, we could define (r, s)-Eulerian numbers, polynomials, and investigate whether they are stable or not.

Several related questions concerning q-analogs, Legendre–Stirling numbers, Durfee squares are posed in [HV12].

2.3 Stable W-Eulerian polynomials

In this section, we discuss how the stability results of the Eulerian polynomials can be further extended in another direction. Reiner [Rei93, Rei95] and Brenti [Bre94] noted that the notion of descent can be defined for groups other than the symmetric groups. In this section we focus mainly on Coxeter groups.

We begin by revising some basic Coxeter group terminology, and show how descents can be defined in such groups. That is essentially all we need to define the so-called W-Eulerian polynomials, which we will denote as $P(W; x)$ following Brenti (mainly to avoid confusion between the type A Eulerian polynomials and the classical ones).

Brenti showed that the type B Eulerian polynomials have only real roots and conjectured that for any Coxeter group W, the W-Eulerian polynomial is real-rooted. We follow our multivariate approach. In particular, we define refinements of the descent statistics that will give rise to stable multivariate W-Eulerian polynomials for some W. For type A, we present Brändén's proof [Brä10]. We then show how this can be extended to type B (hyperoctahedral group or signed permutations), generalizing the univariate result of Brenti. With a little more work we give a stable refinement for the generalized symmetric group (colored permutations). Finally, we show stability results for affine Eulerian polynomials, introduced by Dilks, Petersen, and Stembridge in [DPS09]. Our multivariate stability results generalize the already known univariate real-rootedness results, but we are not able to settle the cases where real-rootedness is only conjectured (see Conjectures 2.3.3 and 2.4.3, and Remark 2.4.9).

2.3.1 Eulerian polynomials for Coxeter groups

We follow the notation in [BB05]. Let S be a set of Coxeter generators, m be a Coxeter matrix, and

$$W = \langle S : (ss')^{m(s,s')} = e, \text{ for } s, s' \in S,\ m(s, s') < \infty \rangle$$

be the corresponding Coxeter group. Given such a Coxeter system (W, S) and $\sigma \in W$, we denote by $\ell_W(\sigma)$ the length of σ in W with respect to S.

Definition 2.3.1. For W a finite Coxeter group, the *descent* set of $\sigma \in W$ is

$$\mathcal{D}_W(\sigma) = \{s \in S : \ell_W(\sigma s) < \ell_W(\sigma)\}.$$

Definition 2.3.2. For W a finite Coxeter group, the *W-Eulerian polynomial* is the descent generating polynomial

$$P(W; x) = \sum_{\sigma \in W} x^{|\mathcal{D}_W(\sigma)|}.$$

The study of these polynomials was originated by Brenti, who also made the following conjecture.

Conjecture 2.3.3 (Conjecture 5.2 of [Bre94]). *For every finite Coxeter group W, the polynomial* $P(W; x)$ *has only real roots.*

2.3.2 Stable Eulerian polynomials for type A

Let A_n denote the Coxeter group of type A of rank n. We can regard A_n as $\mathfrak{S}_{n+1}$, the group of all permutations on $[n+1]$ with generators $S = \{s_1, \dots, s_n\}$, where s_i is the transposition $(i, i+1)$ for $1 \leqslant i \leqslant n$.

Proposition 2.3.4 (Proposition 1.5.3 in [BB05]). *Given* $\sigma = \sigma_1 \dots \sigma_{n+1} \in A_n$,

$$\mathcal{D}_A(\sigma) = \{s_i \in S : \sigma_i > \sigma_{i+1}\},$$

Definition 2.3.5. The *Eulerian polynomials of type* A are defined as

$$P(A_n; x) = \sum_{\sigma \in A_n} x^{|\mathcal{D}_A(\sigma)|}.$$

Using this notation we have that

$$P(A_n; x) = \frac{\mathfrak{S}_{n-1}(x)}{x},$$

where $\mathfrak{S}_n(x)$ denotes the classical Eulerian polynomial defined in Definition 2.1.10. Hence, by Theorem 2.1.14, we immediately have the following.

Theorem 2.3.6. $P(A_n; x)$ *has only real roots.*

To give a multivariate refinement of this polynomial we make use of following refined type A statistics.

Definition 2.3.7. Given $\sigma \in A_n$, define the type A *descent top* set to be

$$\mathcal{DT}_A(\sigma) = \{\max(\sigma_i, \sigma_{i+1}) : 1 \leqslant i \leqslant n, \sigma_i > \sigma_{i+1}\},$$

and similarly, let the type A *ascent top* set be

$$\mathcal{AT}_A(\sigma) = \{\max(\sigma_i, \sigma_{i+1}) : 1 \leqslant i \leqslant n, \sigma_i < \sigma_{i+1}\}.$$

Remark 2.3.8. This definition is slightly different from the descent top set defined in Definition 2.2.12 for extended descents. For example, when $\sigma = 31452 \in A_4$, $\mathcal{DT}_A(\sigma) = \{3, 5\}$ and $\mathcal{AT}_A(\sigma) = \{4, 5\}$.

Note the seemingly superfluous notation $\max(\sigma_i, \sigma_{i+1})$ simply reduces to σ_i and σ_{i+1} in the case of type A descent top and ascent top sets, respectively. Its significance will become apparent when we introduce the type B descent top and ascent top sets (see Definition 2.3.15).

Theorem 2.3.9 (Brändén [Brä10])**.**

$$P(A_n; \mathbf{x}, \mathbf{y}) = \sum_{\sigma \in A_n} \mathbf{x}^{\mathcal{DT}_A(\sigma)} \mathbf{y}^{\mathcal{AT}_A(\sigma)} \tag{2.3.1}$$

is stable.

Proof. We proceed by induction. Note that $P(A_0; x_1, y_1) = 1$ is stable. By observing the effect of inserting $n+1$ into a permutation $\sigma \in A_{n-1}$ on the type A ascent top and descent top sets, we obtain the following recursion. For $n > 0$, we have

$$P(A_n; \mathbf{x}, \mathbf{y}) = (x_{n+1} + y_{n+1}) P(A_{n-1}; \mathbf{x}, \mathbf{y}) + x_{n+1} y_{n+1} \boldsymbol{\partial} P(A_{n-1}; \mathbf{x}, \mathbf{y}). \tag{2.3.2}$$

We remind the reader here that $\boldsymbol{\partial} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right)$. It is easy to check using Proposition 1.2.12 that the linear operator $T = (x_{n+1} + y_{n+1}) + x_{n+1} y_{n+1} \boldsymbol{\partial}$ is stability-preserving, because its algebraic symbol

$$G_T = x_{n+1} y_{n+1} \underbrace{\left[\frac{1}{y_{n+1}} + \frac{1}{x_{n+1}} + \sum_{i=1}^{n} \left(\frac{1}{x_i + u_i} + \frac{1}{y_i + v_i} \right) \right]}_{\text{In } \mathcal{H}_- \text{ when } \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathcal{H}^+} (\mathbf{x} + \mathbf{u})^{[n]} (\mathbf{y} + \mathbf{v})^{[n]}$$

is a stable polynomial. □

Specializing the y_i variables to 1, it follows that

Corollary 2.3.10.

$$P(A_n; \mathbf{x}) = \sum_{\sigma \in A_n} \mathbf{x}^{\mathcal{DT}_A(\sigma)}$$

is stable.

Diagonalizing $\mathbf{x}$, we obtain

Corollary 2.3.11.

$$P(A_n; x) = \sum_{\sigma \in A_n} x^{|\mathcal{DT}_A(\sigma)|} = \sum_{\sigma \in A_n} x^{|\mathcal{D}_A(\sigma)|}$$

is stable.

2.3.3 Stable Eulerian polynomials of type B

Let B_n denote the Coxeter group of type B of rank n. We regard B_n as the group of all signed permutations on $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$ with generators $S = \{s_0, s_1, \dots, s_{n-1}\}$, where s_0 is the transposition $(-1, 1)$ and $s_i = (i, i+1)$ for $1 \leqslant i \leqslant n-1$. Given $\sigma = (\sigma_1, \dots, \sigma_n) \in B_n$, we let

$$N(\sigma) = |\{i \in [n] : \sigma_i < 0\}|$$

denote the number of negative entries in the signed permutation σ.

Type B descents have a simple combinatorial description that we will exploit.

Proposition 2.3.12 (Corollary 3.2 of [Bre94], also Proposition 8.1.2 of [BB05])**.** *Given* $\sigma \in B_n$*,*

$$\mathcal{D}_B(\sigma) = \{s_i \in S : \sigma_i > \sigma_{i+1}\},$$

where $\sigma_0 \stackrel{\text{def}}{=} 0$*.*

Analogously to type A, the type B Eulerian polynomials have only real roots.

Theorem 2.3.13 (Brenti [Bre94])**.**

$$P(B_n; x) = \sum_{\sigma \in B_n} x^{|\mathcal{D}_B(\sigma)|} \tag{2.3.3}$$

has only real roots.

In [Bre94], Brenti introduced a "q-analog"[2] of the univariate Eulerian polynomials and showed the following.

Theorem 2.3.14 (Corollary 3.7 of [Bre94])**.** *For* $q \geqslant 0$*,*

$$\mathcal{B}_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{|\mathcal{D}_B(\sigma)|}. \tag{2.3.4}$$

has only real roots.

[2]We adopt Brenti's notation and use the variable q here. At the same time we would like to point out that q-analog is usually used when the statistic we keep track of by q is Mahonian, i.e., equidistributed with the number of inversions, or the major index.

These $\mathcal{B}_n(x;q)$ polynomials specialize to Eulerian polynomials $P(A_{n-1};x)$ and $P(B_n;x)$—when $q = 0$ and $q = 1$, respectively—so that Theorem 2.3.14 simultaneously generalizes Theorem 2.3.6 and 2.3.13.

We will proceed in the same way that Theorem 2.3.9 extends Theorem 2.3.6. Recall that the stability of the multivariate refinement of the type A Eulerian polynomials in (2.3.1) came from the choice of the statistic. Choosing the larger index from each ascent and descent allowed for the simple stability-preserving recursion.

Next we extend this idea to signed permutations in such a way that the definitions remain consistent with the definitions for ordinary permutations.

Definition 2.3.15. Given $\sigma \in B_n$, define the type B *descent top* set to be

$$\mathcal{DT}_B(\sigma) = \{\max(|\sigma_i|, |\sigma_{i+1}|) : 0 \leqslant i \leqslant n-1,\ \sigma_i > \sigma_{i+1}\}.$$

Analogously, we define the type B *ascent top* set to be

$$\mathcal{AT}_B(\sigma) = \{\max(|\sigma_i|, |\sigma_{i+1}|) : 0 \leqslant i \leqslant n-1,\ \sigma_i < \sigma_{i+1}\}.$$

For example, for $\sigma = (3, 1, -4, -5, 2) \in B_5$, we have $\mathcal{DT}_B(\sigma) = \{3, 4, 5\}$ and $\mathcal{AT}_B(\sigma) = \{3, 5\}$.

Now we are in position to give a multivariate strengthening of Theorem 2.3.14.

Theorem 2.3.16 (Theorem 3.9 of [VW12]). *For* $q \geqslant 0$,

$$\mathcal{B}_n(\mathbf{x}, \mathbf{y}; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} \mathbf{x}^{\mathcal{DT}_B(\sigma)} \mathbf{y}^{\mathcal{AT}_B(\sigma)} \tag{2.3.5}$$

is stable.

Proof. As in the proof of Theorem 2.3.9, we proceed by induction. $B_1(x_1, y_1; q) = qx_1 + y_1$ is stable when $q \geqslant 0$, which settles the base case. By observing the effect on the ascent top and descent top sets of type B of inserting $n+1$ or $-(n+1)$ into a signed permutation $\sigma \in B_n$, we obtain the following recursion. For $n > 0$, we have

$$\mathcal{B}_{n+1}(\mathbf{x}, \mathbf{y}; q) = (qx_{n+1} + y_{n+1})\mathcal{B}_n(\mathbf{x}, \mathbf{y}; q) + (1+q)x_{n+1}y_{n+1}\boldsymbol{\partial}\mathcal{B}_n(\mathbf{x}, \mathbf{y}; q). \tag{2.3.6}$$

To complete the proof, we note that for a fixed $q \geqslant 0$, the linear operator acting on the right hand side, $T = (qx_n + y_n) + (1+q)x_n y_n \boldsymbol{\partial}$ preserves stability by Proposition 1.2.12, since

$$G_T = \left[\frac{q}{y_{n+1}} + \frac{1}{x_{n+1}} + \sum_{i=1}^{n} \left(\frac{1+q}{x_i + u_i} + \frac{1+q}{y_i + v_i}\right)\right] (\mathbf{x} + \mathbf{u})^{[n]} (\mathbf{y} + \mathbf{v})^{[n]}$$

is stable whenever $q \geqslant 0$. $\square$

By looking at (2.3.6) it is clear that we can have a strengthening of Theorem 2.3.16 for $\mathcal{B}_n(\mathbf{x}, \mathbf{y}; p, q)$ where the additional variable p counts the positive values in $\sigma_1 \ldots \sigma_n \in B_n$.

Remark 2.3.17. It is possible to define excedance for hyperoctahedral groups as well (see [Bre94]). The multivariate refinement used in (2.3.5) simultaneously refines the type B descent and the type B excedance statistic. See Theorem 3.15 of [Bre94] for a bijective proof using an extension of Foata's "fundamental transformation".

Theorem 2.3.16 has some noteworthy consequences. The value of $\mathcal{B}_n(\mathbf{x}, \mathbf{y}; q)$ at $q = -1$ is immediate from the recursion.

Corollary 2.3.18.

$$\mathcal{B}_n(\mathbf{x}, \mathbf{y}; -1) = (\mathbf{y} - \mathbf{x})^{[n]}$$

If we set $q = 1$, we obtain the analogue of Theorem 2.3.9 for type B.

Corollary 2.3.19.

$$\mathcal{B}_n(\mathbf{x}, \mathbf{y}; 1) = \sum_{\sigma \in B_n} \mathbf{x}^{\mathcal{DT}_B(\sigma)} \mathbf{y}^{\mathcal{AT}_B(\sigma)}$$

is stable.

Remark 2.3.20. We would like to point out that when we plug in $q = 0$ into (2.3.5), we get a *homogenized* polynomial that is not equal to the polynomial $P(A_{n-1}; \mathbf{x}, \mathbf{y})$ from (2.3.1), since their recursions differ. Rather, $\mathcal{B}_n(\mathbf{x}, \mathbf{y}; 0)$ is the permanent of the so-called staircase Ferrers matrix—a special case of the matrices considered in Chapter 1. Let $M = (m_{ij})$ be the following $n \times n$ matrix. For $i, j \in [n]$, let $m_{ij} = x_i$, whenever $i < j$ and $m_{ij} = y_j$, otherwise. When we expand the permanent by the last column, we obtain the recurrence in (2.3.6) with $q = 0$ (cf. the proof of Lemma 1.3.4).

Specializing all the y variables in $\mathcal{B}_n(\mathbf{x}, \mathbf{y}; q)$ to 1 preserves stability.

Corollary 2.3.21. *For* $q \geqslant 0$,

$$\mathcal{B}_n(\mathbf{x}; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} \mathbf{x}^{\mathcal{DT}_B(\sigma)}$$

is stable.

Remark 2.3.22. Finally, observe that (the non-homogeneous) $\mathcal{B}_n(\mathbf{x}; q)$ polynomial does reduce to $P(A_{n-1}; \mathbf{x})$ and $P(B_n; \mathbf{x})$—the multivariate Eulerian polynomial of type A and type B—when $q = 0$ and $q = 1$, respectively.

Diagonalizing $\mathbf{x}$ in $\mathcal{B}_n(\mathbf{x}; q)$ yields the polynomial $\mathcal{B}_n(x; q)$ defined in (2.3.4). We therefore recover Theorem 2.3.14 as a corollary.

Corollary 2.3.23. *For* $q \geqslant 0$, $B_n(x; q)$ *is stable.*

2.3.4 Stable Eulerian polynomials for colored permutations

Theorem 2.3.16 can be extended in two directions independently: from signed permutations to colored permutations, and from a single q variable to several.

Let $\mathbb{Z}_r$ denote the cyclic group of order r with generator ζ. We will take ζ to be an rth primitive root of unity. The wreath product $G_n^r = \mathbb{Z}_r \wr A_{n-1}$ is the semidirect product $(\mathbb{Z}_r)^{\times n} \rtimes A_{n-1}$. Its elements can be thought of as $\sigma = (\zeta^{e_1}\tau_1, \ldots, \zeta^{e_n}\tau_n)$, where $e_i \in \{0, 1, \ldots, r-1\}$ and $\tau \in A_{n-1}$. G_n^r is sometimes called the generalized symmetric group. Its elements are also known as r-colored permutations, which reduce to signed permutations and ordinary permutations when $r = 2$ and $r = 1$, respectively. In other words, $B_n = G_n^2$ and $A_{n-1} = G_n^1$.

Definition 2.3.24. Given $\sigma = (\zeta^{e_1}\tau_1, \ldots, \zeta^{e_n}\tau_n) \in G_n^r$, let $\mathcal{N}(\sigma)$ be the multiset in which each $i \in [n]$ appears e_i times. Note that for $\sigma \in B_n$, $|\mathcal{N}(\sigma)| = N(\sigma)$, the number of negative entries in $\sigma = (\sigma_1, \ldots, \sigma_n)$.

We adopt the following total order on the elements of $(\mathbb{Z}_r \times [n]) \cup \{0\}$ (see [Ass10], for example):

$$\zeta^{r-1}n < \cdots < \zeta n < \zeta^{r-1}(n-1) < \cdots < \zeta(n-1) < \cdots < \zeta^{r-1}1 < \cdots < \zeta 1 < 0$$

$$0 < 1 < \cdots < n.$$

Using this ordering, the definitions of descent, descent top set, and ascent top set all extend verbatim from B_n to G_n^r. We shall therefore, by slightly abusing the notation, use the same symbols to denote them. For example, when $\sigma = (3, \zeta^2 1, \zeta^2 4, \zeta^4 5, \zeta 2) \in G_n^5$, then $\mathcal{DT}_B(\sigma) = \{3, 5\}$, and $\mathcal{AT}_B(\sigma) = \{3, 4, 5\}$.

Brändén generalized Brenti's $\mathcal{B}_n(x; q)$ polynomial, defined in (2.3.4), to multiple q variables, and proved the following.

Theorem 2.3.25 (Corollary 6.5 in [Brä06]). *Let* $\mathbf{q} = (q_1, ..., q_n)$. *If* $q_i \geqslant 0$, *for* $1 \leqslant i \leqslant n$, *then*

$$\mathcal{B}_n(x; \mathbf{q}) = \sum_{\sigma \in B_n} \mathbf{q}^{\mathcal{N}(\sigma)} x^{|\mathcal{D}_B(\sigma)|}. \tag{2.3.7}$$

has only simple real roots.

Next, we extend this result simultaneously to G_n^r and to multiple x variables.

Theorem 2.3.26. *If* $q_i \geqslant 0$, *for all* $1 \leqslant i \leqslant n$, *then the multivariate* $\mathbf{q}$*-Eulerian polynomial for the generalized symmetric group,* G_n^r, *defined as*

$$G_n^r(\mathbf{x}, \mathbf{y}; \mathbf{q}) = \sum_{\sigma \in G_n^r} \mathbf{q}^{\mathcal{N}(\sigma)} \mathbf{x}^{\mathcal{DT}_B(\sigma)} \mathbf{y}^{\mathcal{AT}_B(\sigma)} \tag{2.3.8}$$

is stable.

Proof. $G_n^r(x_1, y_1; q_1) = (q_1 + \cdots + q_1^{r-1})x_1 + y_1$ is clearly stable when $q_1 \geqslant 0$. The theorem follows immediately from the following recursion. For $n > 1$,

$$G_n^r(\mathbf{x}, \mathbf{y}; \mathbf{q}) = \left[(q_n + \cdots + q_n^{r-1})x_n + y_n + (1 + \cdots + q_n^{r-1})x_n y_n \boldsymbol{\partial}\right] G_{n-1}^r(\mathbf{x}, \mathbf{y}; \mathbf{q}).$$

□

As a consequence, we obtain a generalization of Corollary 2.3.18 to G_n^r.

Corollary 2.3.27. *Let $r \geqslant 2$. For an rth root of unity, $\zeta \neq 1$, we have*

$$G_n^r(\mathbf{x}, \mathbf{y}; \underbrace{\zeta, \ldots, \zeta}_{n}) = (\mathbf{y} - \mathbf{x})^{[n]}.$$

Letting $r = 2$ also generalizes Theorem 2.3.16 to multiple q variables.

Corollary 2.3.28. *If $q_i \geqslant 0$, for all $1 \leqslant i \leqslant n$, then $B_n(\mathbf{x}, \mathbf{y}; \mathbf{q})$ is stable.*

Diagonalizing $\mathbf{q}$ gives us a result for G_n^r with a single q variable.

Corollary 2.3.29. *If $q \geqslant 0$, then $G_n^r(\mathbf{x}, \mathbf{y}; q)$ is stable.*

2.4 Stable $\widetilde{W}$-Eulerian polynomials

Dilks, Petersen and Stembridge studied Eulerian-like polynomials associated to affine Weyl groups. They defined the so-called "affine" $\widetilde{W}$-Eulerian polynomials as the "affine descent"-generating polynomials over the corresponding finite Weyl group. In [DPS09], it was shown that the (univariate) $\widetilde{W}$-Eulerian polynomials have only real roots for types A and C, and also for the exceptional types. We strengthen these results for types A and C by giving multivariate stable refinements of these polynomials as well.

Definition 2.4.1. For W a finite Weyl group, the *affine descent* set of $\sigma \in W$ is

$$\widetilde{\mathcal{D}}_W(\sigma) = \mathcal{D}_W(\sigma) \cup \{s_0 : \ell_W(\sigma s_0) > \ell_W(\sigma)\},$$

where s_0 is the reflection corresponding to the lowest root in the underlying crystallographic root system. See [DPS09] for further details and the motivation behind this definition.

Definition 2.4.2. For W a finite Weyl group, the *$\widetilde{W}$-Eulerian polynomial* is the affine descent generating polynomial

$$\widetilde{P}(W, x) = \sum_{\sigma \in W} x^{|\widetilde{\mathcal{D}}_W(\sigma)|}.$$

Note that the summation is over the corresponding finite Weyl group W.

Dilks, Petersen and Stembridge proposed a companion to Conjecture 2.3.3.

Conjecture 2.4.3 (Conjecture 4.1 of [DPS09])**.** *For every finite Weyl group* W*, the affine descent generating polynomial* $\widetilde{P}(W, x)$ *has only real roots.*

2.4.1 Stable affine Eulerian polynomials of type A

Let A_n denote the Coxeter group of type A of rank n. The affine descents of type A contain the (ordinary) descents of type A and an extra "affine" descent at 0 if and only if, $\sigma_{n+1} > \sigma_1$, where $\sigma = (\sigma_1, \dots, \sigma_{n+1}) \in A_n$. Formally,

$$\widetilde{\mathcal{D}}_A(\sigma) = \mathcal{D}_A(\sigma) \cup \{s_0 : \sigma_{n+1} > \sigma_1\}.$$

See Section 5.1 in [DPS09] for further details.

The definitions of descent top and ascent top sets for type A can be extended in the obvious way. For $\sigma \in A_n$,

$$\begin{aligned}\widetilde{\mathcal{DT}}_A(\sigma) &= \mathcal{DT}_A(\sigma) \cup \{\sigma_{n+1} : \sigma_{n+1} > \sigma_1\},\\ \widetilde{\mathcal{AT}}_A(\sigma) &= \mathcal{AT}_A(\sigma) \cup \{\sigma_1 : \sigma_{n+1} < \sigma_1\}\end{aligned}$$

and we obtain the following result.

Theorem 2.4.4.
$$\widetilde{P}(A_n; \mathbf{x}, \mathbf{y}) = \sum_{\sigma \in A_n} \mathbf{x}^{\widetilde{\mathcal{DT}}_A(\sigma)} \mathbf{y}^{\widetilde{\mathcal{AT}}_A(\sigma)}$$
is stable.

Proof. This statement is immediate once we establish the following lemma. Stability then follows, since $P(A_n; \mathbf{x}, \mathbf{y})$ is stable and the operator on the right-hand side is clearly stability-preserving. □

Lemma 2.4.5. *For* $n > 0$*, we have*

$$\widetilde{P}(A_n; \mathbf{x}, \mathbf{y}) = (n+1) x_{n+1} y_{n+1} P(A_{n-1}; \mathbf{x}, \mathbf{y}).$$

Proof. Consider a permutation $\sigma \in A_{n-1}$. We will modify it to obtain a permutation in $\widetilde{A}_n$. Append $(n+1)$ to the end of σ and pick a cyclic rotation of the newly obtained permutation. The new permutation will have the same affine ascent top and affine descent top sets as the ascent top and descent top sets σ had and in addition it will have $(n+1)$ both as an ascent top and as a descent top. To conclude the proof, note that there are exactly $n+1$ cyclic rotations. This is essentially a refinement of the proof of Proposition 1.1 of [Pet05]. □

By diagonalizing $\mathbf{x}$ and specializing $\mathbf{y}$ to 1, Lemma 2.4.5 reduces to an identity discovered by Fulman (Corollary 1 in [Ful00]) and we recover a result of Dilks, Petersen, and Stembridge:

Corollary 2.4.6 (see Section 4 of [DPS09])**.**

$$\widetilde{P}(A_n; x) = \sum_{\sigma \in A_n} x^{|\widetilde{\mathcal{D}}_A(\sigma)|}$$

has only real roots.

One can construct a recurrence from Lemma 2.4.5 for the affine Eulerian polynomials of type A as well, but this recurrence will not preserve stability.

2.4.2 Stable affine Eulerian polynomials of type C

Let C_n denote the Coxeter group of type C of rank n. Affine descents of type C consist of the ordinary descent set of type C, which coincides with the descent set of type B (see Proposition 2.3.12 for type B descents) and an extra "affine" descent at 0 when $\sigma_n > 0$. Formally,

$$\widetilde{\mathcal{D}}_C(\sigma) = \mathcal{D}_C(\sigma) \cup \{s_0 : \sigma_n > 0\}.$$

See Section 5.2 of [DPS09] for further details.

As in type A, the definition of the type C affine ascent and descent top sets can be adapted from those of type C (equivalently, type B). For $\sigma \in C_n$, let

$$\widetilde{\mathcal{DT}}_C(\sigma) = \mathcal{DT}_B(\sigma) \cup \{\sigma_n : \sigma_n > 0\},$$
$$\widetilde{\mathcal{AT}}_C(\sigma) = \mathcal{AT}_B(\sigma) \cup \{\sigma_n : \sigma_n < 0\}.$$

Theorem 2.4.7.

$$\widetilde{P}(C_n; \mathbf{x}, \mathbf{y}) = \sum_{\sigma \in C_n} \mathbf{x}^{\widetilde{\mathcal{DT}}_C(\sigma)} \mathbf{y}^{\widetilde{\mathcal{AT}}_C(\sigma)}$$

is stable.

Proof. $\widetilde{P}(C_1; x_1, y_1) = 2x_1y_1$ and the following recurrence holds for $n > 1$:

$$\widetilde{P}(C_n; \mathbf{x}, \mathbf{y}) = 2x_ny_n\boldsymbol{\partial}\widetilde{P}(C_{n-1}; \mathbf{x}, \mathbf{y}). \qquad \square$$

We note that a very similar recurrence also appeared in Theorem 2.2.14 (without the factor of two and with a different initial value).

There is also a direct connection between the polynomials $\widetilde{P}(C; \mathbf{x}, \mathbf{y})$ and $P(A; \mathbf{x}, \mathbf{y})$.

Proposition 2.4.8.

$$\widetilde{P}(C_n; x_1, \ldots, x_n, y_1, \ldots, y_n) = 2^n x_n y_n P(A_{n-1}; x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}).$$

Proof. Follows by a similar argument as Lemma 2.4.5. □

Once again, this gives a refinement of the univariate identity by Fulman [Ful01]. We mention that multivariate refinements (different from the above) and respective refinements for the identities for $\widetilde{P}(A; \mathbf{x}, \mathbf{y})$ and $\widetilde{P}(C; \mathbf{x}, \mathbf{y})$ have appeared in [DPS09].

Remark 2.4.9. At the time of writing, Conjecture 2.3.3 is still open for type D and Conjecture 2.4.3 is open for types B and D. It would be nice to find a unifying (possibly multivariate) proof that could handle all these cases as well.

Good problems and mushrooms of certain kinds
have something in common; they grow in clusters.

Pólya György

Chapter 3

On two-sided Eulerian polynomials

In this chapter we study the joint distribution of descents and inverse descents over the set of permutations of n letters. Gessel conjectured that the two-variable generating function of this distribution can be written in a given basis with positive integer coefficients. By examining the Eulerian operators that give the recurrence for these generating functions we are able to devise a recurrence for these coefficients but are unable to settle the conjecture.

We also propose a multivariate approach that might lead to a refinement of the conjecture, and a combinatorial model in terms of statistics on inversion sequences. Finally, we discuss further generalizations and a type B analog of a recurrence of Carlitz, Roselle, and Scoville.

3.1 A conjecture of Gessel

As in the previous chapters, we use $\mathfrak{S}_n$ to denote the set of permutations on n letters. Recall that the number of *descents* in a permutation $\pi = \pi_1 \dots \pi_n$ is defined as $\mathrm{des}(\pi) = |\{i : \pi_i > \pi_{i+1}\}|$. Our object of study is the two-variable generating function of descents and *inverse descents*:

$$A_n(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})+1} t^{\mathrm{des}(\pi)+1}. \tag{3.1.1}$$

The specialization of this polynomial to a single variable reduces to the *classical Eulerian polynomial*:

$$\mathfrak{S}_n(t) = A_n(1,t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle t^k .$$

Even though the Eulerian polynomials have been widely studied, it is only recently that pairs of Eulerian statistics such as $(\mathrm{des}, \mathrm{ides})$ have drawn attention.

Throughout this chapter we will use the shorthand $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1})$. Our main motivation to study this joint distribution is the following conjecture of Gessel which appeared in a recent article [Brä08]; see also a nice exposition in [Pet11].

Conjecture 3.1.1 (Gessel). *For all* $n \geqslant 1$,

$$A_n(s,t) = \sum_{i,j} \gamma_{n,i,j} (st)^i (s+t)^j (1+st)^{n+1-j-2i},$$

where $\gamma_{n,i,j}$ *are nonnegative integers for all* $i, j \in \mathbb{N}$.

If true, this decomposition would refine the following classical result, the γ-*nonnegativity* for the Eulerian polynomials $A_n(t)$.

Theorem 3.1.2 (Théorème 5.6 of [FS70]).

$$A_n(t) = \sum_{i=1}^{\lceil n/2 \rceil} \gamma_{n,i} t^i (1+t)^{n+1-2i},$$

where $\gamma_{n,i}$ *are nonnegative integers for all* $i \in \mathbb{N}$.

In [FS70, Chapitre V], Foata and Schützenberger give a purely algebraic proof by considering the *homogenized* Eulerian polynomial,

$$A_n(t;y) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} y^{\mathrm{asc}(\pi)+1},$$

where $\mathrm{asc}(\pi)$ denotes the number of ascents ($\pi_i < \pi_{i+1}$) in the permutation π. Note that this polynomial is the same as the one in (2.2.5), but different from and therefore should not be confused with $A_n(s,t)$ defined in (3.1.1). To avoid confusion we use a semicolon and different variables. We include their proof next, as we will be applying the same idea to the joint generating polynomial of descents and inverse descents in Section 3.3.

Proof of Theorem 3.1.2. The homogenized Eulerian polynomials satisfy the recurrence

$$A_n(t;y) = ty\left(\frac{\partial}{\partial t}A_{n-1}(t;y) + \frac{\partial}{\partial y}A_{n-1}(t;y)\right), \text{ for } n \geqslant 2, \tag{3.1.2}$$

which follows from observing the effect of inserting the letter n into a permutation of $\{1,\ldots,n-1\}$ or equivalently, into a circular permutation of length n with the first entry fixed.

It is clear from symmetry observations that $A_n(t;y)$ can be written in the basis

$$\{(ty)^i(t+y)^{n+1-2i}\}_{i=1\ldots\lceil\frac{n}{2}\rceil}$$

with some coefficients $\gamma_{n,i}$. To show that $\gamma_{n,i}$ are in fact nonnegative integers consider the action of the operator $T = ty\,(\partial/\partial t + \partial/\partial y)$ on a basis element. We get that

$$T[(ty)^i(t+y)^{n+1-2i}] = i(ty)^i(t+y)^{n+2-2i} + 2(n+1-2i)(ty)^{i+1}(t+y)^{n-2i},$$

which in turn implies the following recurrence on the coefficients:

$$\gamma_{n+1,i} = i\gamma_{n,i} + 2(n+3-2i)\gamma_{n,i-1}. \tag{3.1.3}$$

The statement of Theorem 3.1.2 now follows, since the initial values are nonnegative integers (in particular, $\gamma_{1,1} = 1$ and $\gamma_{1,i} = 0$ for $i \neq 1$), and the constraint $i \leqslant \lceil\frac{n}{2}\rceil$ assures that both positivity and integrality are preserved by the recurrence.

□

The study of these so-called *Eulerian operators* goes back to Carlitz as was pointed out to the author by I. Gessel. See, for example, [Car73] for a slightly different variant of the operator T (which corresponds to the type A descents discussed in Section 2.3).

We must also mention the so-called "valley-hopping" proof of Theorem 3.1.2 by Shapiro, Woan, and Getu. In [SWG83, Proposition 4], they give a beautiful construction that proves that the coefficients $\gamma_{n,i}$ are not only nonnegative integers but that they are, in fact, cardinalities of certain equivalence classes of permutations.

3.2 Symmetries and a homogeneous recurrence

The $A_n(s,t)$ polynomials were first studied by Carlitz, Roselle and Scoville in [CRS66]. They proved a recurrence for the coefficients of $A_n(s,t)$ (see equation (7.8) in their article—note there is an obvious typo in the last row of the equation, cf. equation (7.7) in the same article). That recurrence for the coefficients is equivalent to the following one for the generating functions.

Theorem 3.2.1 (Corollary 3 in [Pet11]). *For $n \geqslant 2$,*

$$\begin{aligned} nA_n(s,t) = & \left(n^2 st + (n-1)(1-s)(1-t)\right) A_{n-1}(s,t) \\ & + nst(1-s)\frac{\partial}{\partial s}A_{n-1}(s,t) + nst(1-t)\frac{\partial}{\partial s}A_{n-1}(s,t) \\ & + st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}A_{n-1}(s,t), \end{aligned}$$

with initial value $A_1(s,t) = st$.

At first glance, this recurrence might not seem very useful at all. However, if we introduce additional variables—to count ascents (asc) and inverse ascents (iasc)—we obtain a more transparent recurrence. So, let us define

$$\begin{aligned} A_n(s,t;x,y) &= \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{ides}(\pi)+1} t^{\mathrm{des}(\pi)+1} x^{\mathrm{iasc}(\pi)+1} y^{\mathrm{asc}(\pi)+1} \\ &= \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{ides}(\pi)+1} t^{\mathrm{des}(\pi)+1} x^{n-\mathrm{ides}(\pi)} y^{n-\mathrm{des}(\pi)} \\ &= (xy)^{n+1} A_n(s/x, t/y)\,. \end{aligned}$$

Proposition 3.2.2. *$A_n(s,t;x,y)$ is homogeneous of degree $2n+2$ and is invariant under the action of the Klein 4-group $V = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$, where the action of $\sigma \in V$ on $A_n(p,q;s,t)$ is permutation of the variables accordingly (e.g., $\sigma = (13)(24)$ swaps x with s and y with t, simultaneously).*

Proof. The homogeneity is immediate from the second line of the equation above. The invariance is a consequence of the symmetry properties of $A_n(s,t)$, such as $A_n(s,t) = A_n(t,s)$ (see, e.g., [Pet11, Observation 1]). Note that, due to the introduction of the new variables, for $n \geqslant 4$, the polynomial $A_n(s,t;x,y)$ is *not* symmetric. □

Now we are in position to give our homogeneous recurrence.

Theorem 3.2.3. *For $n \geqslant 2$,*

$$\begin{aligned} nA_n(s,t;x,y) = & (n-1)(s-x)(t-y)A_{n-1}(s,t;x,y) \\ & + stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)A_{n-1}(s,t;x,y) \end{aligned} \tag{3.2.1}$$

with initial value $A_1(s,t;x,y) = stxy$.

Proof. Recall the recurrence satisfied by the classical Eulerian polynomials:

$$A_n(t) = ntA_{n-1}(t) + t(1-t)\frac{\partial}{\partial t}A_{n-1}(t)\,. \tag{3.2.2}$$

A combinatorial interpretation for this recurrence is the following. Start with a permutation π in $\mathfrak{S}_{n-1}$. Insert the letter n in one the n possible positions and for a moment pretend that all of these choices have increased the number of descents by one. This corresponds to the operation of multiplying $A_{n-1}(t)$ by nt. Of course, not all choices increase the descent count; if we have inserted n between two consecutive numbers where the first one was bigger, i.e., they already formed a descent in π, then the total number of descents does not change. Hence, the second term in the left-hand side can be regarded as the correction factor, which is composed of two parts. The $t(\partial/\partial t)$ part "counts" the number of descents, it multiplies the monomial t^d corresponding to a permutation with d descents by d. Another way to think of it as an indicator which assumes the value one for each descent and zero for each non-descent. Finally, for each descent, the $(1-t)$ factor decreases the number of descents by one, replaces t^d by t^{d-1} to compensate for the initial overcount.

Comparing recurrences (3.1.2) and (3.2.2) it is clear that the homogenized counterpart of the operator $t\,(n+(1-t)(\partial/\partial t))$ is given by $yt(\partial/\partial y + \partial/\partial t)$. The former can be factored out the same way as it is done in the homogeneous case. In some sense, the homogenized operator seems more natural to work with.

□

Remark 3.2.4. The operator $T = (n+(1-t)(\partial/\partial t))$ is closely related to the so-called *polar derivative with respect to a point* α, denoted by D_α, and defined as

$$D_\alpha[p_n](t) = np_n(t) + (\alpha - t)p_n'(t)\,,$$

where $p_n(t) \in \mathbb{R}[t]$ is a polynomial of degree n. Since T is acting on polynomials of degree $n-1$, we have that $T[p_n] = (1+D_\alpha)[p_n]$ with respect to the point $\alpha = 1$.

Remark 3.2.5. Invariance of $A_n(s,t;x,y)$ under the Klein-group action also follows easily from recurrence (3.2.1) directly. Clearly, $A_1(s,t;x,y) = stxy$ is symmetric and the operator acting on $A_n(s,t;x,y)$ denoted by

$$T_n = n(s-x)(t-y) + stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right).$$

itself is invariant under the same action.

Remark 3.2.6. It is not too hard to see that Theorem 3.2.3 is, in fact, equivalent to Theorem 3.2.1. At the same time, the symmetric nature of the homogeneous operator is more suggestive to combinatorial interpretation. It would be nice to find such an interpretation (perhaps in terms of non-attacking rook placements on a rectangular board).

Finally, Theorem 3.2.3 allows us to give a (homogenized) restatement of Gessel's conjecture:

Conjecture 3.2.7.

$$A_n(s,t;x,y) = \sum_{i,j} \gamma_{n,i,j}(stxy)^i(st+xy)^j(tx+sy)^{n+1-2i-j},$$

where $\gamma_{n,i,j} \in \mathbb{N}$ *for all* $i, j \in \mathbb{N}$.

For example, we have (cf. page 18 of [Pet11]):

$$\begin{aligned}
A_1(s,t;x,y) &= stxy \\
A_2(s,t;x,y) &= stxy(st+xy) \\
A_3(s,t;x,y) &= stxy(st+xy)^2 + 2(stxy)^2 \\
A_4(s,t;x,y) &= stxy(st+xy)^3 + 7(stxy)^2(st+xy) + (stxy)^2(tx+sy) \\
A_5(s,t;x,y) &= stxy(st+xy)^4 + 16(stxy)^2(st+xy)^2 + 6(stxy)^2(st+xy)(tx+sy) \\
&\quad + 16(stxy)^3.
\end{aligned}$$

3.3 A recurrence for the coefficients $\gamma_{n,i,j}$

Following the ideas of [FS70, Chapitre V] that were used to devise a recurrence for $\gamma_{n,i}$, we apply the operator T_n to the basis elements to obtain a recurrence for the coefficients $\gamma_{n,i,j}$. As a result, we obtain the following recurrence.

Theorem 3.3.1. *Let* $n \geqslant 1$. *For all* $i \geqslant 1$ *and* $j \geqslant 0$, *we have*

$$\begin{aligned}
(n+1)\gamma_{n+1,i,j} = \quad & (n + i(n+2-i-j))\gamma_{n,i,j-1} + (i(i+j)-n)\gamma_{n,i,j} \\
& + (n+4-2i-j)(n+3-2i-j)\gamma_{n,i-1,j-1} + \\
& + (n+2i+j)(n+3-2i-j)\gamma_{n,i-1,j} \\
& + (j+1)(2n+2-j)\gamma_{n,i-1,j+1} + (j+1)(j+2)\gamma_{n,i-1,j+2}\,,
\end{aligned} \tag{3.3.1}$$

with $\gamma_{1,1,0} = 1$, $\gamma_{1,i,j} = 0$ *(unless* $i = 1$ *and* $j = 0$*) and* $\gamma_{n,i,j} = 0$ *if* $i < 1$ *or* $j < 0$.

Proof. Denote the basis elements by $B^{(n)}_{i,j} = (stxy)^i(st+xy)^j(tx+sy)^{n+1-2i-j}$ for convenience, and let $T_n = M_n + D_1 + D_2$, where $M_n = n(s-x)(t-y)$, $D_1 = stxy\left(\frac{\partial^2}{\partial s \partial t} + \frac{\partial^2}{\partial x \partial y}\right)$ and $D_2 = stxy\left(\frac{\partial^2}{\partial s \partial y} + \frac{\partial^2}{\partial t \partial x}\right)$.

A quick calculation shows that

$$M_n[B^{(n)}_{i,j}] = n\left(B^{(n+1)}_{i,j+1} - B^{(n+1)}_{i,j}\right). \tag{3.3.2}$$

To calculate the action of the differential operators on the basis elements, we use the product rule, which for second-order partial derivatives is given by the following formula:

$$
\begin{aligned}
\partial_{zw}(fgh) = \quad & \partial_{zw}(f)gh + \partial_z(f)\partial_w(g)h + \partial_z(f)g\partial_w(h) \\
& + \partial_w(f)\partial_z(g)h + f\partial_{zw}(g)h + f\partial_z(g)\partial_w(h) \\
& + \partial_w(f)g\partial_z(h)gh + f\partial_w(g)\partial_z(h) + fg\partial_{zw}(h)\,,
\end{aligned}
$$

where f, g, h are functions, $\partial_z = \partial/\partial z$ and $\partial_w = \partial/\partial w$ denote the partial differential operators with respect to z and w, and $\partial_{zw} = \partial_z\partial_w$ is the second-order differential operator.

After some calculations, this gives the following:

$$
\begin{aligned}
D_1[B^{(n)}_{i,j}] &= i(n+1-i-j)B^{(n+1)}_{i,j+1} + j(2n+3-j)B^{(n+1)}_{i+1,j-1} \\
&\quad + (n+1-2i-j)(n-2i-j)B^{(n+1)}_{i+1,j+1}\,. \qquad (3.3.3)\\
D_2[B^{(n)}_{i,j}] &= i(i+j)B^{(n+1)}_{i,j} + j(j-1)B^{(n+1)}_{i+1,j-2} \\
&\quad + (n+1-2i-j)(n+2+2i+j)B^{(n+1)}_{i+1,j}\,. \qquad (3.3.4)
\end{aligned}
$$

Summing (3.3.2), (3.3.3) and (3.3.4) we arrive at the following expression.

$$
\begin{aligned}
T_n[B^{(n)}_{i,j}] = & (n + i(n+1-i-j))B^{(n+1)}_{i,j+1} \\
& + (i(i+j)-n)B^{(n+1)}_{i,j} \\
& + (n+1-2i-j)(n-2i-j)B^{(n+1)}_{i+1,j+1} \\
& + (n+2+2i+j)(n+1-2i-j)B^{(n+1)}_{i+1,j} \\
& + j(2n+3-j)B^{(n+1)}_{i+1,j-1} \\
& + j(j-1)B^{(n+1)}_{i+1,j-2}.
\end{aligned}
$$

Finally, collecting together all terms $T_n[B^{(n)}_{k,\ell}]$ which contribute to $B^{(n+1)}_{i,j}$ we obtain (3.3.1).

□

Remark 3.3.2. If we sum up both sides of (3.3.1) for all possible j then we get (3.1.3) back.

It would be of interest to study the generating function

$$G(u, v, w) = \sum_{i,j} \gamma_{n,i,j} u^n v^i w^j$$

with coefficients satisfying the above recurrence. Gessel's conjecture is equivalent to saying that its coefficients are nonnegative integers. Unfortunately, these properties are not immediate from the recurrence (3.3.1).

3.4 Generalizations of the conjecture

[Ges12] noted that the following equality of [CRS66]

$$\sum_{i,j=0}^{\infty} \binom{ij+n-1}{n} s^i t^j = \frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}$$

can be generalized as follows.

Let $\tau \in \mathfrak{S}_n$ with $\operatorname{des}(\tau) = k-1$. Define $A_n^{(k)}(t)$ by

$$\sum_{i,j=0}^{\infty} \binom{ij+n-k}{n} s^i t^j = \frac{A_n^{(k)}(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}.$$

Then the coefficient of $s^i t^j$ in $A_n^{(k)}$ is the number of pairs of permutations (π, σ) such that $\pi\sigma = \tau$, $\operatorname{des}(\pi) = i$ and $\operatorname{des}(\sigma) = j$. [Ges12] also pointed out that these polynomials arise implicitly in [MP70]; compare (11.10) there with the above equation.

This suggests that Conjecture 3.1.1 holds in a more general form (this version of the conjecture appeared as Conjecture 10.2 in [Brä08]).

Conjecture 3.4.1 (Gessel). *Let $\tau \in \mathfrak{S}_n$. Then*

$$\sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi)+1} t^{\operatorname{des}(\pi^{-1}\tau)} = \sum_{i,j} \gamma_{n,i,j}^{\tau} (st)^i (s+t)^j (1+st)^{n+1-j-2i},$$

where $\gamma_{n,i,j}^{\tau}$ are nonnegative integers for all $i, j \in \mathbb{N}$. Furthermore, the coefficients $\gamma_{n,i,j}^{\tau}$ do not depend on the actual permutation τ, only on the number of descents in τ.

In the special case when $\tau = n(n-1)\ldots 21$ (and hence $\operatorname{des}(\tau) = n-1$) the roles of descents and ascents interchange.

Theorem 3.4.2. *For $n \geqslant 2$,*

$$\begin{aligned} nA_n^{(n)}(s,t;x,y) =& (n-1)(x-s)(t-y)A_{n-1}^{(n-1)}(s,t;x,y) \\ &+ stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) A_{n-1}^{(n-1)}(s,t;x,y) \end{aligned} \tag{3.4.1}$$

with initial value $A_1^{(1)}(s,t;x,y) = stxy$.

In particular, we have the following identity.

Corollary 3.4.3.

$$A_n^{(n)}(s,t;x,y) = A_n(t,s;x,y).$$

[Ges12] also noted that there is an analogous definition for the hyperoctahedral group $\mathfrak{B}_n$.

$$\sum_{i,j=0}^{\infty} \binom{2ij+i+j+n-k}{n} s^i t^j = \frac{B_n^{(k)}(s,t)}{(1-s)^{n+1}(1-t)^{n+1}},$$

where

$$B_n^{(k)}(s,t) = \sum_{\sigma \in \mathfrak{B}_n} s^{\mathrm{des}(\sigma)+1} t^{\mathrm{des}(\sigma^{-1}\tau)+1},$$

with $\tau \in \mathfrak{B}_n$ such that $\mathrm{des}(\tau) = k-1$.

Therefore, mimicking the proof of Theorem 3.2.1 given by [Pet11, Corollary 3], we get an analog of Theorem 3.2.3 for the type B homogenous two-sided Eulerian polynomials.

Theorem 3.4.4. *For* $n \geqslant 2$,

$$\begin{aligned} nB_n(s,t;x,y) =&(n-1)(s-x)(t-y)B_{n-1}(s,t;x,y) \\ &+ sx\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right) B_{n-1}(s,t;x,y) \\ &+ ty\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) B_{n-1}(s,t;x,y) \\ &+ 2stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) B_{n-1}(s,t;x,y) \end{aligned} \tag{3.4.2}$$

with initial value $B_1(s,t;x,y) = stxy(st+xy)$.

3.5 The multivariate approach

There are several ways one can refine the Eulerian polynomials. For example, instead of just keeping track of number of descents in the exponent one can use a different variable for each descent indicating its position. Another way to obtain a multivariate refinement of the Eulerian polynomials is to keep track of the *values* at the "descent tops" instead: $\{\pi_i : \pi_i > \pi_{i+1}\}$. Consider the following multi-affine homogeneous Eulerian polynomial:

$$A_n(t_1,\ldots,t_n;y_1,\ldots,y_n) = \sum_{\pi \in \mathfrak{S}_n} \left(t_{\pi_n} \prod_{\pi_k > \pi_{k+1}} t_{\pi_k} \right) \left(y_{\pi_1} \prod_{\pi_{k-1} < \pi_k} y_{\pi_k} \right), \tag{3.5.1}$$

where a variable y_m appears in the monomial associated to a permutation π, if m is a so-called *ascent top*, that is, it is either the first entry: $\pi_1 = m$, or $\pi_k = m$ preceded by a smaller entry in the one line notation of π: $\pi_k > \pi_{k-1}$.

Similarly, t_m appears in the monomial if m is a *descent top* in the corresponding permutation (with the analogous definition).

The polynomial defined in (3.5.1) was studied in [HV12] and was shown to enjoy several useful properties. Indeed, (3.5.1) is a refinement of the homogenized Eulerian polynomials, since (with slight abuse of notation):

$$A_n(\underbrace{t, \ldots, t}_{n}; \underbrace{y, \ldots, y}_{n}) = A_n(t; y)\,.$$

Furthermore, these polynomials satisfy a recurrence relationship similar to (3.1.2). Again, one could argue that this multivariate recurrence is a more natural one to work with.

Proposition 3.5.1 (Equation (14) of [HV12])**.** *For $n \geqslant 2$,*

$$A_n(t_1, \ldots, t_n; y_1, \ldots, y_n) = t_n y_n \boldsymbol{\partial} A_{n-1}(t_1, \ldots, t_{n-1}; y_1, \ldots, y_{n-1})\,,$$

where $\boldsymbol{\partial} = \sum_{k=1}^{n-1} \left(\frac{\partial}{\partial t_k} + \frac{\partial}{\partial y_k}\right)$.

This recurrence suggests a multivariate strengthening of Theorem 3.1.2.

Theorem 3.5.2. *For $n \geqslant 1$, we have*

$$A_n(t_1, \ldots, t_n; y_1, \ldots, y_n) = \sum_{\substack{U, V \subset \{1, \ldots, n\} \\ U \cap V = \emptyset}} \gamma_{n,U,V} \left(\prod_{i \in U} t_i y_i\right) \left(\prod_{j \in V} (t_j + y_j)\right)\,,$$

where $\gamma_{n,U,V} \in \mathbb{N}$ for all $U, V \subset \{1, \ldots, n\}$ with $U \cap V = \emptyset$.

Proof. We use induction. Note that $A_1(t_1; y_1) = t_1 y_1$ is clearly of this form. It is easy to see that the operator $t_n y_n \sum_{k=1}^{n-1} \left(\frac{\partial}{\partial t_k} + \frac{\partial}{\partial y_k}\right)$ preserves this property. □

A natural question is whether it is possible to extend this idea to obtain a multivariate refinement for $A_n(s, t; x, y)$ in a similar way?

Remark 3.5.3. Theorem 3.5.2 can also be seen as the corollary of the "valley-hopping" argument of [SWG83]. The elements i of U correspond to the so-called *peaks* $\{\pi_i : \pi_{i-1} < \pi_i > \pi_{i+1}\}$, the values which are preceded and followed by a smaller entry in the one-line notation of the permutation. Similarly, the elements j of the complement set of $U \cup V$ are the *valleys* $\{\pi_j : \pi_{j-1} > \pi_j < \pi_{j+1}\}$. Finally, the elements of V are the ones which can hop from one side to the other, changing each double descent $\{\pi_i : \pi_{i-1} > \pi_i > \pi_{i+1}\}$ to a double ascent $\{\pi_i : \pi_{i-1} < \pi_i < \pi_{i+1}\}$, and vice versa.

The peak, valley, double descent, double ascent statistic appeared in the work of [Vie84] and more recently in [VW12] where the multivariate recurrences were extended to type B and other groups (generalizing the results of [DPS09] on γ-nonnegativity for the W-Eulerian polynomials).

The linear differential operators in the recursion in Proposition 3.5.1 were studied in the context of multivariate stability, a generalization of real-rootedness. We say that a multivariate polynomial is *stable* if it does not vanish whenever all the variables have positive imaginary parts. These operators were shown to be stability-preserving and this property was used to prove that the homogenized Eulerian polynomials $A_n(t;y)$ are stable. One might ask whether the polynomial $A_n(s,t;x,y)$ is also stable, in the sense that it does not vanish when the imaginary parts of the four variables are all positive. The answer to this question is negative, because stability of a multivariate generating polynomial implies negative correlation of the statistics counted by the different variables. However, this would contradict the following observation, which is an easy consequence of Theorem 3.2.3.

Remark 3.5.4. The descents and inverse descents are positively correlated.

3.6 Connection to inversion sequences

We conclude by proposing a combinatorial model for the joint distribution of descents and inverse descents.

Definition 3.6.1. Let I_n denote the set of *inversion sequences* by

$$I_n = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leqslant e_i \leqslant i-1\}.$$

Recently, [SV12] defined an *ascent* statistic for inversion sequences (in fact, they defined ascents for generalized inversion sequences, where $i-1$ in the above formula can be replaced by some positive integer s_i) as

$$\mathrm{asc}_I(e) = \left| \left\{ i \in \{1, \ldots, n\} : \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \right\} \right| .$$

We will use the subscript I to emphasize that this is a statistic for inversion sequences which is different from the ascent statistic for permutations.

We define two additional statistics for inversion sequences.

Definition 3.6.2. For $e = (e_1, \ldots, e_n) \in I_n$, let $\mathrm{row}(e) = |\{e_i : 1 \leqslant i \leqslant n\}|$ and $\mathrm{diag}(e) = |\{e_i - i : 1 \leqslant i \leqslant n\}|$ denote the number of distinct components of $e = (e_1, \ldots, e_n)$ and $(e_1 - 1, \ldots, e_n - n)$, respectively.

The names of these statistics stem from the graphical representations of the inversion sequences. An inversion sequence can be though of as a rook placement on staircase board (a board with n columns with heights $1, 2, \ldots, n$, respectively) where each column can have exactly one rook. Two examples are depicted in Figure 3.1.

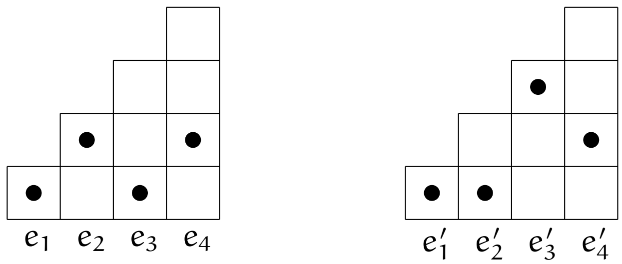

Figure 3.1: The left figure represents inversion sequence $e = (0, 1, 0, 1)$ with $\mathrm{row}(e) = 2$, $\mathrm{diag}(e) = 2$, $\mathrm{asc}_I(e) = 2$ and the right figure represents $e' = (0, 0, 2, 1)$ with $\mathrm{row}(e') = 3$, $\mathrm{diag}(e') = 3$, $\mathrm{asc}_I(e') = 1$,

Somewhat surprisingly all three statistics are *Eulerian*.

Proposition 3.6.3.

$$A_n(x) = \sum_{e \in I_n} x^{\mathrm{asc}_I(e)+1} \tag{3.6.1}$$

$$= \sum_{e \in I_n} x^{\mathrm{row}(e)} \tag{3.6.2}$$

$$= \sum_{e \in I_n} x^{\mathrm{diag}(e)} . \tag{3.6.3}$$

Proof. (3.6.1) follows from recent work of [SV12]. They give a bijection between inversion sequences and permutations that maps the $\mathrm{asc}_I(e)$ statistic to the descent statistic over permutations.

It is not too hard to see that the row statistic is Eulerian. One way to see that, is to examine what happens to the statistic when we add a component e_{n+1} to our inversion sequence e (a new column in the picture) and realize that it is the same recurrence that the Eulerian numbers satisfy:

$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n+1-k) \left\langle {n-1 \atop k-1} \right\rangle .$$

We can prove (3.6.3) in a similar way, using the above recurrence or just considering an involution on the staircase diagrams (or inversion sequences) which exchanges diagonals with rows of the same length. □

Even more interestingly, the joint distribution $(\mathrm{asc}_I, \mathrm{row})$ seems to agree with the joint distribution $(\mathrm{des}, \mathrm{ides})$ of descents and inverse descents which makes inversion sequences and their combinatorial representations, the staircase diagrams even more interesting to study.

Conjecture 3.6.4.

$$A_n(t;y) = \sum_{e \in I_n} s^{\mathrm{row}(e)} t^{\mathrm{asc}_I(e)+1} .$$

This observation clearly deserves a bijective proof. This might shed light on a combinatorial proof of recurrence (3.2.1).

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