ON BERGLUND-HÜBSCH-KRAWITZ MIRROR SYMMETRY

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ABSTRACT

ON BERGLUND-HÜBSCH-KRAWITZ MIRROR SYMMETRY

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We provide various suites of results for the Calabi-Yau orbifolds that have Berglund-Hübsch-Krawitz (BHK) mirrors. These Calabi-Yau orbifolds are certain finite symplectic quotients of hypersurfaces in weighted-projective space. First, we will describe their birational geometry using Shioda maps and prove that so-called alternate mirrors are birational. Next, we will compute algebraic invariants of the orbifolds and their crepant resolutions in the case where they are orbifold K3 surfaces, both over the complex numbers and fields of positive characteristic. Finally, we provide a conjectural framework that unifies the toric mirror construction of Batyrev and Borisov with the BHK construction in the context of Kontsevich's Homological Mirror Symmetry Conjecture.

Contents

1 Introduction **1**

2 Background **9**

2.1 What is Mirror Symmetry? 9

2.2 Mirror Constructions 11

2.2.1 Greene-Plesser Mirrors 12

2.2.2 Batyrev-Borisov Duality 14

2.3 Berglund-Hübsch-Krawitz Mirror Symmetry 15

2.3.1 Delsarte Hypersurfaces in Weighted Projective Space 15

2.3.2 Group of Diagonal Automorphisms 17

2.3.3 The BHK Mirror 20

2.3.4 Classical Mirror Symmetry for BHK Mirrors 22

2.3.5 Comparison between Mirror Constructions 24

3 Birational Geometry of BHK Mirrors **27**

3.1 Shioda Maps 27

3.1.1 Reinterpretation of the Dual Group 29

3.1.2 Birational Geometry of BHK Mirrors 30

3.2 Multiple Mirrors 32

3.3 Example: BHK Mirror 3-folds, Shioda Maps, Chen-Ruan Hodge Numbers 36

3.3.1 BHK Mirrors 37

3.3.2 Shioda Maps 38

3.3.3 Chen-Ruan Hodge Numbers 40

4 On the Picard Ranks of Surfaces of BHK-Type 45

4.1 Introduction 45

4.2 Surfaces of BHK-Type 49

4.2.1 Weighted Delsarte Surfaces 49

4.2.2 Symplectic Group Actions 51

4.2.3 The Berglund-Hübsch-Krawitz Mirror 54

4.3 Picard Ranks of Surfaces of BHK-Type 54

4.3.1 Hodge Theory on Fermat Surfaces 55

4.3.2 Picard Ranks of K3 Surfaces of BHK Type 56

4.4 An Example 60

5 Toric reformulations of BHK Mirrors 68

5.1 Introduction 68

5.2 Dual Gorenstein Cones and the Unification Framework 69
    5.2.1 Notation and the Geometry of Dual Convex Cones 69
    5.2.2 Gorenstein Cones and their Splittings 73
    5.2.3 Cone Closures and Vector Bundles 75
5.3 Batyrev Duality 81
    5.3.1 Total spaces of line bundles 81
    5.3.2 The case of normal fans to reflexive polytopes 84
5.4 Batyrev-Borisov Duality 86
    5.4.1 Toric Vector Bundles and Gorenstein Cones 86
    5.4.2 Nef Partitions and Batyrev-Borisov Duality 90
5.5 Berglund-Hübsch Duality 92
5.6 Examples of higher index Gorenstein cones with respect to Berglund-Hübsch Symmetry 96
    5.6.1 Example One 96
    5.6.2 Example Two 99
    5.6.3 Example Three 101

Chapter 1

Introduction

The mirror symmetry conjecture predicts that for a Calabi-Yau variety, M, there exists another Calabi-Yau variety, W, so that various geometric and physical data is exchanged between M and W. A classical relationship found between so-called mirror pairs is that on the level of cohomology

$$H^{p,q}(M,\mathbb{C}) \cong H^{N-p,q}(W,\mathbb{C}),$$

provided that both Calabi-Yau varieties M and W are N-dimensional. Suppose F_A is a polynomial,

$$F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}, \tag{1.0.1}$$

where $a_{ij} \in \mathbb{N}$, so that there exist positive integers q_j and d so that $\sum_j a_{ij} q_j = d$ for all i (i.e., F_A is quasihomogeneous). The polynomial F_A cuts out a hypersurface $X_A := Z(F_A) \subset W\mathbb{P}^n(q_0, \dots, q_n)$ of dimension $N = n - 1$. Further assume that this hypersurface is a quasi-smooth Calabi-Yau variety (the Calabi-Yau condition is equivalent to $\sum_i q_i = d$ and see Subsection 2.3.1 for details about the quasismooth condition). Greene and Plesser proposed a mirror to X_A when the polynomial F_A was Fermat [26]. Their proposed mirror for the hypersurface X_A was a quotient of X_A by all its phase symmetries of X_A leaving the cohomology $H^{n,0}(X_A)$ invariant. The problem was that their proposal does not work well for the case when X_A was not a Fermat hypersurface. Berglund and Hübsch proposed that the mirror of the hypersurface X_A should relate to a hypersurface X_{A^T} cut out by

$$F_{A^T} = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ji}}. \tag{1.0.2}$$

The hypersurface X_{A^T} sits inside a different weighted-projective 4-space, denoted $W\mathbb{P}^n(r_0, \ldots, r_n)$. Berglund and Hübsch proposed that the mirror of X_A should be a quotient of this new hypersurface X_{A^T} by a suitable subgroup P of the phase symmetries. In several examples, they showed that X_A and X_{A^T}/P satisfy the classical mirror symmetry relation in that

$$h^{p,q}(X_A, \mathbb{C}) = h^{n-1-p,q}(X_{A^T}/P, \mathbb{C}).$$

This proposal fell out of favor when Batyrev and Borisov developed the powerful toric approach (see [3], [4], and [5]). In the 2000s, Krawitz revived Berglund and Hübsch's proposal by giving a rigorous mathematical description of their mirror and proving a mirror symmetry theorem on the level of Frobenius algebra structures [33].

Krawitz also generalized the Berglund-Hübsch mirror proposal by introducing the notion of a dual group: We start with a polynomial F_A. Consider the group

$SL(F_A)$ of phase symmetries of F_A leaving $H^{n,0}(X_A)$ invariant. Define the subgroup J_{F_A} of $SL(F_A)$ to be the group consisting of the phase symmetries induced by the $\mathbb{C}^*$ action on weighted-projective space (so that all elements of J_{F_A} act trivially on the weighted-projective space). Take the group G to be some subgroup of $SL(F_A)$ containing J_{F_A}, i.e., $J_{F_A} \subseteq G \subseteq SL(F_A)$. We obtain a Calabi-Yau orbifold $Z_{A,G} := X_A/\tilde{G}$ where $\tilde{G} := G/J_{F_A}$. Consider the analogous groups $SL(F_{A^T})$ and $J_{F_{A^T}}$ for the polynomial F_{A^T}. Krawitz defined the dual group G^T relative to G so that $J_{F_{A^T}} \subseteq G^T \subseteq SL(F_{A^T})$. For precise definitions of these groups, we direct the reader to Section 2.3. Take the quotient $\tilde{G}^T := G^T/J_{F_{A^T}}$. The Berglund-Hübsch-Krawitz mirror to the orbifold $Z_{A,G}$ is the orbifold $Z_{A^T,G^T} := X_{A^T}/\tilde{G}^T$. Chiodo and Ruan proved the classical mirror symmetry statement for the mirror pair $Z_{A,G}$ and Z_{A^T,G^T} is satisfied on the level of Chen-Ruan cohomology [16]:

$$H^{p,q}_{\mathrm{CR}}(Z_{A,G}, \mathbb{C}) \cong H^{n-1-p,q}_{\mathrm{CR}}(Z_{A^T,G^T}, \mathbb{C}). \tag{1.0.3}$$

This dissertation is a suite of projects discussing Berglund-Hübsch-Krawitz mirrors.

Chapter 2 provides the background about the field, starting with a short exposition on the field of mirror symmetry and mirror constructions, then moving to describing Berglund-Hübsch-Krawitz (BHK) Mirror Symmetry in detail. The chapter finishes with a discussion comparing BHK mirror mirrors with Batyrev-Borisov mirrors.

Chapter 3 discusses the birational geometry of BHK Mirrors. A feature of BHK mirror symmetry is that it proposes possibly distinct mirrors of isolated points of the family in the Calabi-Yau moduli space–not mirrors of families like the work of Batyrev and Borisov. These BHK mirrors of the isolated points may not live in the same family. Suppose one starts with two quasihomogeneous potentials F_A and $F_{A'}$

$$F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}; \qquad F_{A'} = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a'_{ij}}. \tag{1.0.4}$$

Assume that there exist positive integers q_i, q'_i so that

$$\begin{aligned} X_A &= Z(F_A) \subseteq W\mathbb{P}^n(q_0, \ldots, q_n); \\ X_{A'} &= Z(F_{A'}) \subseteq W\mathbb{P}^n(q'_0, \ldots, q'_n) \end{aligned} \tag{1.0.5}$$

and that X_A and $X_{A'}$ are Calabi-Yau. Take G and G' to be subgroups of the group of phase symmetries that leave the respective cohomologies $H^{n,0}(X_A, \mathbb{C})$ and $H^{n,0}(X_{A'}, \mathbb{C})$ to be invariant. We obtain two Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G'}$. One can find examples $Z_{A,G}$ and $Z_{A',G'}$ in the same family where their BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ will be quotients of hypersurfaces in different weighted-projective spaces. See Section 3.3 for an explicit example.

Since the mirrors proposed by BHK and Batyrev-Borisov mirror symmetry are different, we ask the question of how we can relate them. Iritani suggested to look at the birational geometry of the mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$. In this paper, we prove the following theorem.

Theorem 1.0.1. *Let $Z_{A,G}$ and $Z_{A',G'}$ be Calabi-Yau orbifolds as above. If the groups G and G' are equal, then the BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ of these orbifolds are birational.*

In dimension 3, one then has that the BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ are derived equivalent, due to a theorem of Bridgeland. A key technical idea for proving Theorem 1.0.1 is using Shioda maps. Originally, Shioda used these maps to compute Picard numbers of Delsarte surfaces in [38]. These maps entered the multiple mirror literature in [10] where they were generalized and then used to investigate Picard-Fuchs equations of different pencils of quintics in $\mathbb{P}^4$. The Shioda maps were then further generalized to look at GKZ hypergeometric systems for certain families of Calabi-Yau varieties in weighted-projective space in [9]. This chapter provides a concrete description of how Shioda maps relate to BHK mirror symmetry than the previous two papers, and explains the groups used in the theorems of [10] and [9] in the context of BHK mirrors. The chapter ends with an explicit example that shows explicitly a full example where the Calabi-Yau orbifolds Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ are birational and the Chen-Ruan cohomology decomposes in different ways.

Chapter 4 focuses on studying BHK mirror surfaces. In this section, we specialize to surfaces while dropping the condition that we work over the complex numbers to working over an algebraically closed field. There has been recent work of Artebani, Boissière and Sarti that tries to unify the BHK mirror story with Dolgachev-Voisin mirror symmetry in the case where the hypersurface X_A is a double cover of $\mathbb{P}^2$ [1]. Their work has been extended by Comparin, Lyons, Priddis, and Suggs to prime covers of $\mathbb{P}^2$ [18]. In this corpus of work, the authors focus on proving that the Picard groups of the K3 surfaces $Z_{A,G}$ and Z_{A^T,G^T} have polarizations by so-called mirror lattices. In particular, these lattices embed into the subgroup of the Picard groups of the BHK mirrors that are invariant under the non-symplectic automorphism induced on the K3 surface due to the fact of it being a prime cover of $\mathbb{P}^2$. The fact that it is a prime cover of $\mathbb{P}^2$ requires that the polynomial be of the form

$$F_A := x_0^{a_{00}} + \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}, \tag{1.0.6}$$

where a_{00} is a prime number.

In this paper, we drop this hypothesis and investigate the Picard ranks. The key tools that we use are Shioda maps and information about the middle cohomology of Fermat varieties. We use a Shioda map to relate each surface of BHK-type birationally to a quotient of a higher degree Fermat hypersurface in projective space by a finite group H. We then describe the H-invariant part of the transcendental lattice of the Fermat hypersurface, which gives us the rank of the transcendental lattice of the surface of BHK type, i.e., the Lefschetz number. Recall that for an algebraic surface X, the Lefschetz number $\lambda(X)$ is defined to be

$$\lambda(X) := b_2(X) - \rho(X). \tag{1.0.7}$$

Take BHK mirrors surfaces $Z_{A,G}$ and Z_{A^T,G^T} as above over a field of characteristic p. Take d to be a positive integer so that the matrix dA^{-1} has only integer entries. Let $\mathfrak{I}_d(p)$ be the subset of symmetries on a degree d Fermat hypersurface X_d that correspond to elements in the transcendental lattice of X_d tensored with

$\mathbb{Q}$, $T^n(X_d)$ (See Section 4.3.1 for an explicit description of $\mathfrak{I}_d(p)$ that is very computable). We remark that the subset $\mathfrak{I}_d(p)$ depends on the characteristic p of the field k. We then describe the rank of the Picard group. In particular, we prove the following theorem:

Theorem 1.0.2. *The Lefschetz numbers of the BHK mirrors* $Z_{A,G}$ *and* Z_{A^T,G^T} *are:*

$$\begin{aligned} \lambda(Z_{A,G}) &= \#(\mathfrak{I}_d(p) \cap G^T) \text{ and} \\ \lambda(Z_{A^T,G^T}) &= \#(\mathfrak{I}_d(p) \cap G). \end{aligned} \tag{1.0.8}$$

The surprise is that the dual group G^T associated to the BHK Mirror Z_{A^T,G^T} actually plays a role in the computation of the Lefschetz number of the original K3 orbifold $Z_{A,G}$. We can see explicitly a nice correspondence between the mirrors in this fashion. This theorem has the following corollary:

Corollary 1.0.3. *The Picard ranks of the BHK mirrors* $Z_{A,G}$ *and* Z_{A^T,G^T} *are:*

$$\begin{aligned} \rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d(p) \cap G^T) \text{ and} \\ \rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_d(p) \cap G). \end{aligned} \tag{1.0.9}$$

An added quick corollary is a lower bound on the Picard number of a BHK mirror is by the order of dual group G^T. Also, a great benefit to this is that the Picard number of each BHK mirror surface is now computable, once one chooses over which field one works. We end the chapter with an example where we compute the Picard ranks of a BHK mirror pair over various fields and see that there are examples of primes p where either one, both or neither of the surfaces $Z_{A,G}$ and Z_{A^T,G^T} are supersingular (i.e., have Picard rank 22).

Chapter 5 focuses on a toric reinterpretation of BHK mirror symmetry. We focus on looking at how reinterpreting BHK mirror symmetry into a toric language immediately lends to generalizations of it. We explore various examples that are immediately in the framework from doing an intuitive toric translation of the mirror formulation. Two works-in-progress, one with C. Siegel and one with C. F. Doran and D. Favero will solidify this framework over the next few years.

Chapter 2

Background

2.1 What is Mirror Symmetry?

The field of mirror symmetry has been a focal point in the last twenty years of interaction between geometry and physics. Mirror symmetry first started as a duality amongst two different (2,2) superconformal field theories (SCFT) and then provided a conjectural framework for two Calabi-Yau varieties M and W to interchange various geometric and physical data. Consider a Calabi-Yau threefold M. When one looks at a local neighborhood of the Calabi-Yau moduli space near M, one views the moduli space as a product of the moduli space of complex structure $\mathcal{M}_{\text{c-x}}(X)$ and the moduli space of Kähler structure $\mathcal{M}_K(X)$. The original mirror symmetry intuition comes from the fact that there is a choice in the gauge group for the (2,2) SCFT, which manifests in a prediction that there exists a Calabi-Yau threefold W

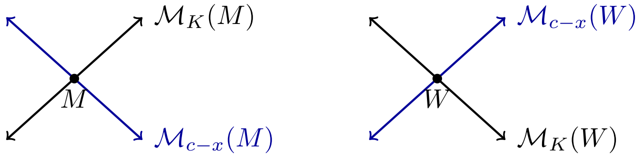

Figure 2.1: (2,2) Mirror Symmetry for M and its mirror W

so that $\mathcal{M}_{\text{c-x}}(Y)$ corresponds to $\mathcal{M}_K(X)$ and $\mathcal{M}_K(Y)$ corresponds to $\mathcal{M}_{\text{c-x}}(X)$:

On the level of cohomology, for two Calabi-Yau threefolds M and W, this picture means that their Hodge diamonds 'flip', i.e., there is an isomorphism among cohomology groups

$$H^p(M, \Omega^q_M) \cong H^{3-p}(W, \Omega^q_W),$$

where Ω_M and Ω_W are the cotangent bundles of M and W, respectively. Generalizing this framework, mirror symmetry predicts that, given an n-dimensional Calabi-Yau varieties M, there exists another W so that their Hodge diamonds 'flip' and one has:

$$H^p(M, \Omega^q_M) \cong H^{n-p}(W, \Omega^q_W).$$

Such a relationship as above is itself surprising and exciting when it was first introduced in 1989 by Greene and Plesser in [26]. In the early 1990s, the celebrated work of Candelas, de la Ossa, Green, and Parkes [13] went further to observe that the variation of Hodge structure of a family of Calabi-Yau varieties, the quintic mirror family, provided valuable data about the enumerative geometry of the quintic threefold. Since then, many deep connections between mirror families have been observed.

In his 1994 ICM plenary lecture, Kontsevich introduced the notion of Homological Mirror Symmetry, a conjectural formulation that mirror symmetry categorically exchanges the symplectic data of M with the complex geometry of W. In [32], Kontevich proposed the following equivalence. Take (M, ω) to be a $2n$-dimensional symplectic manifold with vanishing first Chern class $c_1(M) = 0$ and W to be a dual n-dimensional complex algebraic manifold.

Conjecture 2.1.1 (Kontevich's Homological Mirror Conjecture as stated in [32])*.* The derived category constructed from the Fukaya category $F(V)$ (or a suitably enlarged one) is equivalent to the derived category of coherent sheaves on a complex algebraic variety W.

Over the past twenty years, there has been very promising work on proving this conjecture and many refinements of this conjecture have been stated. For example, Sheridan recently proved Homological Mirror Symmetry for Calabi-Yau hypersurfaces in projective space [39].

2.2 Mirror Constructions

One may notice right now that we have so far only discussed what type of mirror statements one may expect, given a mirror pair M and W. The question still remains on how to get the mirror manifold W, given a Calabi-Yau manifold M. There has been many proposals over the years, some of which have had more development in the literature.

2.2.1 Greene-Plesser Mirrors

Greene and Plesser introduced the idea that one can find a mirror to certain Calabi-Yau hypersurfaces in weighted-projective space by quotienting by certain discrete symmetries of certain type and then taking the crepant resolution of the singularities. In their work [26], they look at the example of the Fermat hypersurface $X_5 = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}^4$. There are many choices of a group G by which one can quotient X_5 that will create a Calabi-Yau orbifold X_5/G, i.e., when one quotients the hypersurface X_5 by the group G and then take the minimal resolution $\widetilde{X_5/G} \dashrightarrow (X_5/G)$ so that $\widetilde{X_5/G}$ is still a Calabi-Yau manifold.

Let us comment on the possible groups G. Let H be the group of automorphisms found by multiplying by the roots of unity on each coordinate while keeping the nonvanishing holomorphic 3-form invariant, i.e.,

$$H = \left\{ (e^{2\pi i a_1/5}, \ldots, e^{2\pi i a_5/5}) \middle| a_i \in \mathbb{Z}, \sum_{j=1}^{5} a_i \in 5\mathbb{Z} \right\}.$$

This does not act nontrivially on the hypersurface. The subgroup J that acts trivially is generated by the element $(e^{2\pi i/5}, \ldots, e^{2\pi i/5})$. Quotient H by J to make a group $\bar{H} = H/J$. For notation, we will denote the element $(e^{2\pi i a_1/5}, \ldots, e^{2\pi i a_5/5})$ by $[a_1, a_2, a_3, a_4, a_5]$. One can look at all the possible quotients of the hypersurface X_5 by subgroups of $\bar{H}$. Greene and Plesser compute the cohomology values $h^{1,1}$ and $h^{2,1}$ of the minimal resolutions of the quotients $X_5/\bar{G}$, where $\bar{G} \subset \bar{H}$ (See Table 2.1). As one can see, there's a correspondence between taking different quotients of discrete symmetries. For every choice of $\bar{G}$ there is another choice of $\bar{G}'$ so that the

Generators of $\bar{G}$	$h^{2,1}$	$h^{1,1}$
0	101	1
$[0,0,0,1,4]$	49	5
$[0,1,2,3,4]$	21	1
$[0,1,1,4,4]$, $[0,1,2,3,4]$	21	17
$[0,1,1,4,4]$	17	21
$[0,1,3,1,0]$, $[0,1,1,0,3]$	1	21
$[0,1,4,0,0]$, $[0,3,0,1,1]$	5	49
$[0,1,2,3,4]$, $[0,1,1,4,4]$, $[0,0,0,1,4]$	1	101

Table 2.1: **Quotients of X_5**

Hodge numbers are exchanged, providing evidence of a mirror pair.

This phenomenon was made rigorous for polynomials of the form

$$F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}$$

where $A = (a_{ij})_{i,j=0}^{n}$ is an $(n+1)$ by $(n+1)$ matrix satisfying certain properties. Quotients of the zero locus of F_A in weighted projective space are related to quotients of the zero locus of F_{A^T}. This mirror duality was proposed in 1992 by Berglund and Hübsch in [8]. Unfortunately, Berglund and Hübsch's mirror duality was overlooked and consequently underinvestigated at the time due to the powerful toric mirror constructions proposed by Batyrev and Borisov. In the past five years, it has been studied in more detail after the proposal of Berglund and Hübsch was made rigorous by Krawitz in [33]. For an explicit description of Berglund-Hübsch-Krawitz mirror symmetry, we direct the reader to Section 2.3.

2.2.2 Batyrev-Borisov Duality

Next, let us explain the toric approach of Batyrev and Borisov. Let M and N be dual lattices. One starts with a Gorenstein Fano toric variety $\mathbb{P}_\Delta$ associated to a lattice polytope $\Delta \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$, i.e., all the vertices of the polytope Δ are in the lattice M. The polytope associated to a Gorenstein Fano toric variety is reflexive. Recall the definition of reflexive is that the dual polytope

$$\Delta^* := \{n \in N | \langle m, n \rangle \geq -1 \text{ for all } m \in \Delta\}$$

is a lattice polytope in N. Note that this requires that the polytopes Δ and Δ^* to both have exactly one lattice point in their interiors. Moreover, note that if the polytope Δ is reflexive, then the dual polytope Δ^* is reflexive as well. This implies that the toric variety $\mathbb{P}_{\Delta^*}$ associated to the polytope Δ^* is One can construct families $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta^*)$ of hypersurfaces in $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^*}$ that satisfy certain regularity conditions that imply that the singularities of the hypersurface are induced by the ambient toric varieties $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^*}$. In [3], Batyrev proposes the combinatorial involution $\Delta \longrightarrow \Delta'$ agrees with the mirror involution on conformal field theories associated to the Calabi-Yau varieties $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta^*)$. His evidence of this proposal for Calabi-Yau threefolds that were maximal projective crepant partial desingularizations of hypersurfaces in $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta^*)$ included showing that the

Hodge numbers $h^{2,1}$ and $h^{1,1}$ were exchanged amongst the two hypersurfaces. This proposal was generalized by Batyrev and Borisov to explain the case of a regular Calabi-Yau complete intersections in Gorenstein Fano toric varieties [4] and [5].

We now give an exposition about an alternate mirror construction to that of Batyrev and Borisov, the Berglund-Hübsch-Krawitz mirror construction, for certain hypersurfaces in quotients of weighted projective space.

2.3 Berglund-Hübsch-Krawitz Mirror Symmetry

2.3.1 Delsarte Hypersurfaces in Weighted Projective Space

We start with a matrix A with nonnegative integer entries $(a_{ij})_{i,j=0}^{n}$. Define a polynomial

$$F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}$$

and impose the following conditions:

1. the matrix A is invertible;

2. the polynomial F_A is quasihomogeneous, i.e., there exist positive integers q_j, d so that

   $$\sum_{j=0}^{n} a_{ij} q_j = d,$$

   for all i; and

3. the polynomial F_A is a non-degenerate potential away from the origin, i.e., we are assuming that, when viewing F_A as a polynomial in $\mathbb{C}^{n+1}$, $Z(F_A)$ has exactly one singular point (at the origin).

Remark 2.3.1. These conditions are restrictive. By Theorem 1 of [34], there is a classification of such polynomials. That is, F_A can be written as a sum of invertible potentials, each of which must be of one of the three so-called *atomic types*:

$$\begin{aligned} W_{\text{Fermat}} &:= x^a, \\ W_{\text{loop}} &:= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1, \text{ and} \\ W_{\text{chain}} &:= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}. \end{aligned} \tag{2.3.1}$$

Using Condition (1), we define the matrix $B = dA^{-1}$, where d is a positive integer so that all the entries of B are integers (note that d is not necessarily the smallest such d). Take $e := (1, \ldots, 1)^T \in \mathbb{R}^n$ and

$$q := Be, \text{ i.e., } q_i = \sum_j b_{ij}.$$

Then the polynomial F_A defines a zero locus $X_A = Z(F_A) \subseteq W\mathbb{P}^n(q_0, \ldots, q_n)$. Indeed, with these weights, the polynomial F_A is quasihomogeneous: each monomial in F_A has degree $\sum_{j=0}^n a_{ij}q_j = d$, as $Aq = ABe = de$. Condition (2) above is used to ensure that each integer q_i is positive.

Assume further that $\sum_i q_i = d$ is the degree of the polynomial, which implies that the hypersurface X_A is a Calabi-Yau variety. Define $\text{Sing}(V)$ to be the singular locus of any variety V, we say the hypersurface X_A is quasi-smooth if $\text{Sing}(X_A) \subseteq$

$\mathrm{Sing}(W\mathbb{P}^n(q_0,\ldots,q_n)) \cap X_A$. Condition (3) above implies that our hypersurface X_A is quasi-smooth. We remark that Condition (1) is used once again when we introduce the BHK mirror in Section 2.2: it ensures that the matrix A^T is a matrix of exponents of a polynomial with $n+1$ monomials and $n+1$ variables that also satisfies Conditions (1), (2), and (3).

2.3.2 Group of Diagonal Automorphisms

Let us discuss the groups of symmetries of the Calabi-Yau variety X_{F_A}. Firstly, consider the scaling automorphisms of the set $\mathbb{C}^{n+1} \setminus \{0\}$ when $n \geq 2$. There is a subgroup, $(\mathbb{C}^*)^{n+1}$, of the automorphisms of $\mathbb{C}^{n+1} \setminus \{0\}$. Explicitly, an element $(\lambda_0,\ldots,\lambda_n) \in (\mathbb{C}^*)^{n+1}$ acts on any element $\mathbf{x} = (x_0,\ldots,x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ by:

$$(\lambda_0,\ldots,\lambda_n) \times (x_0,\ldots,x_n) \longmapsto (\lambda_0 x_0,\ldots,\lambda_n x_n).$$

We view the weighted projective n-space $W\mathbb{P}^n(q_0,\ldots,q_n)$ as a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by a subgroup $\mathbb{C}^* \subset (\mathbb{C}^*)^{n+1}$ consisting of the elements that can be written $(\lambda^{q_0/d},\ldots,\lambda^{q_n/d})$ for some $\lambda \in \mathbb{C}^*$.

Moreover, there is a subgroup of $(\mathbb{C}^*)^{n+1}$, denoted $\mathrm{Aut}(F_A)$, which can be defined as

$$\begin{aligned}\mathrm{Aut}(F_A) := \big\{(\lambda_0,\ldots,\lambda_n) \in (\mathbb{C}^*)^{n+1} \quad | \quad & F_A(\lambda_0 x_0,\ldots,\lambda_n x_n) = F_A(x_0,\ldots,x_n) \\ & \text{for all } (x_0,\ldots,x_n)\big\}.\end{aligned} \tag{2.3.2}$$

This group is sometimes referred to as the group of diagonal automorphisms or the group of scaling symmetries. Note that for $(\lambda_0, \ldots, \lambda_n)$ to be an element of $\mathrm{Aut}(F_A)$, each monomial $\prod_{j=0}^{n} x_j^{a_{ij}}$ in the polynomial F_A must be invariant under the action of $(\lambda_0, \ldots, \lambda_n)$.

Using the classification of Kreuzer and Skarke (see Remark 2.3.1), we can see that for any polynomial of one of the atomic types that each λ_i must have modulus 1. If the polynomial F_A is of Fermat-type, then $\lambda^a x^a = x^a$ hence $\lambda^a = 1$. If F_A is of loop-type, then $\lambda_i^{a_i}\lambda_{i+1} = 1$ for all $i < a_m$, hence $\lambda_{i+1} = \lambda_i^{-a_i}$. Moreover, $\lambda_m^{a_m}\lambda_1 = 1$ hence $\lambda_1 = \lambda_m^{-a_m} = \lambda_{m-1}^{a_m a_{m-1}} = \cdots = \lambda_1^{(-1)^m a_1 \cdots a_m}$. If $|\lambda| \neq 1$ then $(-1)^m a_1 \ldots a_m = 1$. This would require m to be even and a_i to be 1 for all i. However, then the degree of the polynomial, d, must be $q_1 + q_2$; however $d = \sum_{i=0}^{n} q_i$, $n \geq 2$, and $q_i > 0$, hence a contradiction is reached. Lastly, if F_A is of chain-type, $\lambda_m^{a_m} x_m^{a_m} = x_m^{a_m}$, hence $|\lambda_m|^{a_m} = 1$. This implies that $|\lambda_{m-1}^{a_{m-1}}\lambda_m| = |\lambda_{m-1}^{a_{m-1}}| = 1$, and so on, hence $|\lambda_i| = 1$. Any polynomial that is a combination of such types has an analogous argument.

Since each λ_i can be written as $e^{i\theta_i}$, for some $\theta_i \in \mathbb{R}$, we can then see that $(\lambda_0, \ldots, \lambda_n) \in \mathrm{Aut}(F_A)$ if and only if we have that $\prod_{j=0}^{n} e^{i a_{ij}\theta_j} = 1$ for all i. The map $(\lambda_0, \ldots, \lambda_n) \mapsto (\frac{1}{2\pi i}\log(\lambda_0), \ldots, \frac{1}{2\pi i}\log(\lambda_n))$ induces an isomorphism

$$\mathrm{Aut}(F_A) \cong \left\{ (z_0, \ldots, z_n) \in (\mathbb{R}/\mathbb{Z})^n \;\middle|\; A \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{Z}^{n+1} \right\}. \tag{2.3.3}$$

We then observe that we can describe $\mathrm{Aut}(F_A)$ as being generated by the elements

$$\rho_i = (e^{2\pi i b_{0i}/d}, \ldots, e^{2\pi b_{ni}/d}) \in (\mathbb{C}^*)^{n+1}.$$

Moreover, there is a characterization by Artebani, Boissière, and Sarti of the group $\mathrm{Aut}(F_A)$ (Proposition 2 of [1]):

Proposition 2.3.2. $\mathrm{Aut}(F_A)$ *is a finite abelian group of order* $|\det A|$. *If we think of* F_A *as a sum of atomic types,* $F_{A_1}(x_0, \ldots, x_{i_1}) + \ldots + F_{A_k}(x_{i_{k-1}+1}, \ldots, x_n)$, *then we may characterize the elements of* $\mathrm{Aut}(F_A)$ *as being the product of the* k *groups* $\mathrm{Aut}(F_{A_i})$. *The groups* $\mathrm{Aut}(F_{A_i})$ *are determined based on the atomic types:*

1. *For a summand of Fermat type* $W_{\mathit{Fermat}} = x^a$, *the group* $\mathrm{Aut}(W_{\mathit{Fermat}})$ *is isomorphic to* $\mathbb{Z}/a\mathbb{Z}$ *and generated by* $\varphi = e^{2\pi i/a} \in \mathbb{C}^*$.

2. *For a summand of loop type* $W_{\mathit{loop}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1$, *the group* $\mathrm{Aut}(W_{\mathit{loop}})$ *is isomorphic to* $\mathbb{Z}/\Gamma\mathbb{Z}$ *where* $\Gamma = a_1 \cdots a_m + (-1)^{m+1}$ *and generated by* $(\varphi_1, \ldots, \varphi_m) \in (\mathbb{C}^*)^m$, *where*

$$\varphi_1 := e^{2\pi i(-1)^m/\Gamma}, \text{ and } \varphi_i := e^{2\pi i(-1)^{m+1-i}a_1\cdots a_{i-1}/\Gamma}, i \geq 2.$$

3. *For a summand of chain type,* $W_{\mathit{chain}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}$, *the group* $\mathrm{Aut}(W_{\mathit{chain}})$ *is isomorphic to* $\mathbb{Z}/(a_1 \cdots a_m)\mathbb{Z}$, *and generated by* $(\varphi_1, \ldots, \varphi_m) \in (\mathbb{C}^*)^m$, *where*

$$\varphi_i = e^{2\pi i(-1)^{m+i}/a_i\cdots a_m}.$$

Note that the subgroups $\mathbb{C}^*$ and $\mathrm{Aut}(F_A)$ have nontrivial intersection. Let $J_{F_A} := \mathrm{Aut}(F_A) \cap \mathbb{C}^*$. The group J_{F_A} is generated by $(e^{2\pi i q_0/d}, \ldots, e^{2\pi i q_n/d})$, which is clearly in $\mathrm{Aut}(F_A)$ because $\sum_{j=0}^n a_{ij} q_j = d$ and the alternate description provided by the isomorphism above in Equation 2.3.3 (moreover, $(e^{2\pi i q_0/d}, \ldots, e^{2\pi i q_n/d}) = \prod_{i=0}^n \rho_i \in \mathrm{Aut}(F_A)$).

We now introduce the group

$$SL(F_A) := \left\{ (\lambda_0, \ldots, \lambda_n) \in \mathrm{Aut}(F_A) \middle| \prod_{j=0}^n \lambda_j = 1 \right\}.$$

The group J_{F_A} is a subgroup of $SL(F_A)$ as a generator of J_{F_A} is the element $(e^{2\pi i q_j/d})_j$ and $\prod_j e^{2\pi i q_j/d} = e^{\frac{2\pi i}{d} \sum_j q_j} = 1$. Fix a group G so that $J_{F_A} \subseteq G \subseteq SL(F_A)$ and put $\tilde{G} := G/J_{F_A}$. To help summarize, we have the following diagram of groups:

$$\begin{tikzcd}
J_{F_A} \arrow[r, hook] \arrow[d, hook] & J_{F_A} \arrow[r, hook] \arrow[d, hook] & J_{F_A} \arrow[r, hook] \arrow[d, hook] & \mathbb{C}^* \arrow[d, hook] \\
G \arrow[r, hook] \arrow[d] & SL(F_A) \arrow[r, hook] & \mathrm{Aut}(F_A) \arrow[r, hook] & (\mathbb{C}^*)^{n+1} \arrow[d] \\
\tilde{G} := G/J_{F_A} & & & (\mathbb{C}^*)^{n+1}/\mathbb{C}^*
\end{tikzcd}$$

Consider the Calabi-Yau orbifold, $Z_{A,G} := X_{F_A}/\tilde{G} \subset W\mathbb{P}^n(q_0, \ldots, q_n)/\tilde{G}$. We now will describe the Berglund-Hübsch-Krawitz mirror to it.

2.3.3 The BHK Mirror

In this section, we construct the BHK mirror to the Calabi-Yau orbifold $Z_{A,G}$ defined above. Take the polynomial

$$F_{A^T} = \sum_{i=0}^n \prod_{j=0}^n X_j^{a_{ji}}. \tag{2.3.4}$$

It is quasihomogeneous because there exist positive integers $r_i := \sum_j b_{ji}$ so that

$$F_{A^T}(\lambda^{r_0}X_0, \dots, \lambda^{r_n}X_n) = \lambda^d F_{A^T}(X_0, \dots, X_n). \tag{2.3.5}$$

Note that the polynomial F_{A^T} cuts out a well-defined Calabi-Yau hypersurface $X_{A^T} \subseteq W\mathbb{P}^n(r_0, \dots, r_n)$. Define the diagonal automorphism group, $\mathrm{Aut}(F_{A^T})$, analogously to $\mathrm{Aut}(F_A)$. By the analogous isomorphism to that in Equation 2.3.3, the group $\mathrm{Aut}(F_{A^T})$ is generated by $\rho_i^T := \mathrm{diag}(e^{2\pi i b_{ij}/d})_{j=0}^n \in (\mathbb{C}^*)^{n+1}$. Define the dual group G^T relative to G to be

$$G^T := \left\{ \prod_{i=0}^n (\rho_i^T)^{s_i} \middle| s_i \in \mathbb{Z}, \text{ where } \prod_{i=0}^n x_i^{s_i} \text{ is } G\text{-invariant} \right\} \subseteq \mathrm{Aut}(F_{A^T}). \tag{2.3.6}$$

Lemma 2.3.3. *If the group G is a subgroup of $SL(F_A)$, then the dual group G^T contains the group $J_{F_{A^T}}$.*

Proof. It is sufficient to show that the element $\prod_{j=0}^n \rho_j^T$ is in the dual group G^T. This is equivalent to $\prod_{j=0}^n x_j$ to be G-invariant. Any element $(\lambda_0, \dots, \lambda_n)$ of G acts on the monomial $\prod_{j=0}^n x_j$ by $\prod_{j=0}^n \lambda_j = 1$ (as $G \subseteq SL(F_A)$). ☐

Lemma 2.3.4. *If the group G contains J_{F_A}, then the dual group G^T is contained in $SL(F_{A^T})$.*

The proof of this lemma is analogous to the lemma above. As the dual group G^T sits between $J_{F_{A^T}}$ and $SL(F_{A^T})$, define the group $\tilde{G}^T := G^T/J_{F_{A^T}}$. We have a well-defined Calabi-Yau orbifold $Z_{A^T,G^T} := X_{A^T}/\tilde{G}^T \subset W\mathbb{P}^n(r_0, \dots, r_n)/\tilde{G}^T$. The Calabi-Yau orbifold Z_{A^T,G^T} is the BHK mirror to $Z_{A,G}$.

2.3.4 Classical Mirror Symmetry for BHK Mirrors

In this section, we summarize some results of Chiodo and Ruan for BHK mirrors. This section is based on Section 3.2 of [16]. We recommend the exposition there. Recall that we can view the weighted projective n-space $W\mathbb{P}^n(q_0, \ldots, q_n)$ as a stack

$$\left[\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*\right] \tag{2.3.7}$$

where a group element λ of the torus $\mathbb{C}^*$ acts by

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n). \tag{2.3.8}$$

The quotient stack $W\mathbb{P}^n(q_0, \ldots, q_n)/\tilde{G}$ is equivalent to the stack

$$\left[\mathbb{C}^{n+1} \setminus \{0\}/G\mathbb{C}^*\right] \tag{2.3.9}$$

so we can view the Calabi-Yau orbifold $Z_{A,G}$ as the (smooth) Deligne-Mumford stack

$$[Z_{A,G}] := \left[\left\{x \in \mathbb{C}^{n+1} \setminus \{0\} | F_A(x) = 0\right\} /G\mathbb{C}^*\right] \subseteq \left[\mathbb{C}^{n+1} \setminus \{0\}/G\mathbb{C}^*\right]. \tag{2.3.10}$$

We now review the Chen-Ruan orbifold cohomology for such a stack. Intuitively speaking, it consists of a direct sum over all elements of $G\mathbb{C}^*$ of G-invariant cohomology of the fixed loci of each element.

If γ is an element of $G\mathbb{C}^*$, take the fixed loci

$$\begin{aligned} \mathbb{C}^{n+1}_\gamma &: = \{\mathbf{x} \in \mathbb{C}^{n+1} \setminus \{0\} | \gamma \cdot \mathbf{x} = \mathbf{x}\}; \text{ and} \\ X_A^\gamma &: = \{F_A|_{\mathbb{C}^{n+1}_\gamma} = 0\} \subset \mathbb{C}^{n+1}_\gamma. \end{aligned} \tag{2.3.11}$$

Fix a point $\mathbf{x} \in X_A^\gamma$. The action of γ on the tangent space $T_{\mathbf{x}}(\{F_A = 0\})$ can be written as a diagonal matrix (when written with respect to a certain basis), $\Lambda_\gamma = \operatorname{diag}(e^{2\pi i a_1^\gamma}, \ldots, e^{2\pi i a_n^\gamma})$, for some real numbers $a_i^\gamma \in [0, 1)$. We then define the *age shift* of γ,

$$a(\gamma) := \frac{1}{2\pi i} \log(\det \Lambda_\gamma) = \sum_{j=1}^{n} a_j^\gamma. \tag{2.3.12}$$

We now may define the bigraded Chen-Ruan orbifold cohomology as a direct sum of twisted sector ordinary cohomology groups:

$$H^{p,q}_{\mathrm{CR}}([Z_{A,G}], \mathbb{C}) = \bigoplus_{\gamma \in G\mathbb{C}^*} H^{p-a(\gamma), q-a(\gamma)}(X_A^\gamma / G\mathbb{C}^*, \mathbb{C}). \tag{2.3.13}$$

The degree d Chen-Ruan orbifold cohomology is defined to be the direct sum

$$H^{d}_{\mathrm{CR}}([Z_{A,G}], \mathbb{C}) = \bigoplus_{p+q=d} H^{p,q}_{\mathrm{CR}}([Z_{A,G}], \mathbb{C}). \tag{2.3.14}$$

Continue to assume that the group G contains J_{F_A} and is a subgroup of $SL(F_A)$ and the hypersurface X_A is Calabi-Yau. Chiodo and Ruan prove:

Theorem 2.3.5 (Theorem 2 of [16]). *Given the Calabi-Yau orbifold $Z_{A,G}$ and its BHK mirror Z_{A^T,G^T} as above, one has the standard relationship between the Hodge diamonds of mirror pairs on the level of the Chen-Ruan cohomology of the orbifolds:*

$$H^{p,q}_{CR}([Z_{A,G}], \mathbb{C}) \cong H^{n-1-p,q}_{CR}([Z_{A^T,G^T}], \mathbb{C}).$$

This is a classical mirror symmetry theorem for such orbifolds. We remark that in the case of orbifolds the dimension of the bigraded Chen-Ruan orbifold cohomology vector spaces and stringy Hodge numbers agree. Moreover, we have:

Corollary 2.3.6 (Corollary 4 of [16]). *Suppose both Calabi-Yau orbifolds* $Z_{A,G}$ *and* Z_{A^T,G^T} *admit smooth crepant resolutions* M *and* W *respectively, then we have the equality*

$$h^{p,q}(M,\mathbb{C}) = h^{n-1-p,q}(W,\mathbb{C}),$$

where $h^{p,q}$ *is the ordinary* (p,q) *Hodge number.*

2.3.5 Comparison between Mirror Constructions

In this subsection we will comment on the different mirrors that BHK and Batyrev-Borisov mirror symmetry propose for hypersurfaces in quotients of weighted projective space. In Sections 5.4 and 5.5 of [3], Batyrev pays close attention to the case where the reflexive polytopes Δ and Δ^* are simplices, which corresponds to the toric varieties $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^*}$ being quotients of weighted projective spaces. Let Δ be a polytope associated to a weighted projective space with weights q_i. Batyrev proves that the family of Calabi-Yau hypersurfaces $\mathcal{F}(\Delta)$ consists of deformations of Fermat-type hypersurfaces. He also proves that the toric variety Δ^* will be a quotient of the weighted projective space with weights q_i. This is in contrast to the BHK construction where the mirrors can be hypersurfaces in quotients of different weighted projective spaces. See for example in Table 2.2 that is taken from [23] that computes the BHK mirrors to quintics in projective 4-space.

As one can see, not all BHK mirrors to quintics in $\mathbb{P}^4$ are hypersurfaces in quotients of projective 4-space, providing a contrast between BHK mirror duality and

Batyrev-Borisov mirror duality. When one has multiple mirrors for a Calabi-Yau manifold, one can ask a variety of questions. In his 1994 paper outlining homological mirror symmetry, Kontsevich explains that "the B-models on birationally equivalent Calabi-Yau manifolds W and W' are believed to be isomorphic.... we expect that the derived categories of coherent sheaves on W and on W' are equivalent.

Kontsevich's viewpoint provides the following question:

Question 2.3.7. If F_A and $F_{A'}$ are polynomials as above and G and G' are groups so that $J_{F_A} \subset G \subset SL(F_A)$ and $J_{F_{A'}} \subset G' \subset SL(F_{A'})$ so that the Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G'}$ are in the same toric variety, do we have a birational or derived equivalence between Z_{A^T,G^T} and $Z_{(A')^T,G'^T}$?

This question was first posed in this fashion by Iritani. In the next chapter, we investigate the question in the context of birational geometry and answer the question affirmatively. Indeed, in the manner that we solve it, we get a more general theorem than what Iritani asked. A very concrete corollary to what we prove in Chapter 3 is that all the orbifolds in the BHK Mirror column in Table 2.2 are birational.

	Quintic $Z_{A,G} \subseteq \mathbb{P}^4$	BHK Mirror Z_{A^T,G^T}
1	$\{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^5 + y_1^5 + y_2^5 + y_3^5 + y_4^5 = 0\} \subset \mathbb{P}^4}{(\mathbb{Z}_5)^3 : [4,1,0,0,0], [4,0,1,0,0], [4,0,0,1,0]}$
2	$\{x_0^4 x_1 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^4 + y_0 y_1^5 + y_2^5 + y_3^5 + y_4^5 = 0\} \subset W\mathbb{P}^4(5,3,4,4,4)}{(\mathbb{Z}_5)^2 : [0,0,4,1,0], [0,0,4,0,1]}$
3	$\{x_0^4 x_1 + x_1^4 x_2 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^4 + y_0 y_1^4 + y_1 y_2^5 + y_3^5 + y_4^5 = 0\} \subset W\mathbb{P}^4(20,15,13,16,16)}{\mathbb{Z}_5 : [0,0,0,4,1]}$
4	$\{x_0^4 x_1 + x_1^4 x_2 + x_2^4 x_3 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\{y_0^4 + y_0 y_1^4 + y_1 y_2^4 + y_2 y_3^5 + y_4^5 = 0\} \subset W\mathbb{P}^4(80,60,65,51,64)$
5	$\{x_0^4 x_1 + x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_4 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\{y_0^4 + y_0 y_1^4 + y_1 y_2^4 + y_2 y_3^4 + y_3 y_4^5 = 0\} \subset W\mathbb{P}^4(64,48,52,51,41)$
6	$\{x_0^4 x_1 + x_1^5 + x_2^4 x_3 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\{y_0^4 + y_0 y_1^5 + y_2^4 + y_2 y_3^5 + y_4^5 = 0\} \subset W\mathbb{P}^4(5,3,5,3,4)$
7	$\{x_0^4 x_1 + x_1^4 x_2 + x_2^5 + x_3^4 x_4 + x_4^5 = 0\} \subset \mathbb{P}^4$	$\{y_0^4 + y_0 y_1^4 + y_1 y_2^5 + y_3^4 + y_3 y_5^5 = 0\} \subset W\mathbb{P}^4(20,15,13,20,12)$
8	$\{x_0^5 + x_1^5 + x_2^5 + x_3^4 x_4 + x_4^4 x_3 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^5 + y_1^5 + y_2^5 + y_3^4 y_4 + y_3 y_4^4 = 0\} \subset \mathbb{P}^4}{(\mathbb{Z}_5)^2 \times \mathbb{Z}_3 : [1,4,0,0,0], [0,4,1,0,0], [0,0,0,2,1]}$
9	$\{x_0^5 + x_1^4 x_2 + x_2^5 + x_3^4 x_4 + x_4^4 x_3 = 0\} \subset \mathbb{P}^4$	$\{y_0^5 + y_1^4 + y_1 y_2^5 + y_3^4 y_4 + y_3 y_4^4 = 0\} \subset W\mathbb{P}^4(20,15,13,16,16)$
10	$\{x_0^4 x_1 + x_1^4 x_2 + x_2^5 + x_3^4 x_4 + x_4^4 x_3 = 0\} \subset \mathbb{P}^4$	$\frac{y_0^4 + y_0 y_1^4 + y_1 y_2^5 + y_3^4 y_4 + y_3 y_4^4 = 0\} \subset W\mathbb{P}^4(20,15,13,16,16)}{\mathbb{Z}_{15} : [0,0,3,11,1]}$
11	$\{x_0^5 + x_1^4 x_2 + x_2^4 x_1 + x_3^4 x_4 + x_4^4 x_3 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^5 + y_1^4 y_2 + y_1 y_2^4 + y_3^4 y_4 + y3 y_4^4 = 0\} \subset \mathbb{P}^4}{\mathbb{Z}_3 : [0,2,1,2,1]}$
12	$\{x_0^5 + x_1^5 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_2 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^5 + y_1^5 + y_2^4 y_4 + y_2 y_3^4 + y_3 y_4^4 = 0\} \subset \mathbb{P}^4}{\mathbb{Z}_5 \times \mathbb{Z}_{13} : [1,4,0,0,0], [0,0,3,9,1]}$
13	$\{x_0^4 x_1 + x_1^5 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_2 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^4 + y_0 y_1^5 + y_2^4 y_4 + y_2 y_3^4 + y_3 y_4^4 = 0\} \subset W\mathbb{P}^4(5,3,4,4,4)}{\mathbb{Z}_{65} : [0,52,16,61,1]}$
14	$\{x_0^4 x_1 + x_1^4 x_0 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_2 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^4 y_1 + y_0 y_1^4 + y_2^4 y_4 + y_2 y_3^4 + y_3 y_4^4 = 0\} \subset \mathbb{P}^4}{\mathbb{Z}_3 \times \mathbb{Z}_{13} : [1,2,0,0,0], [0,0,3,9,1]}$
15	$\{x_0^5 + x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_1 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^5 + y_1^4 y_4 + y_1 y_2^4 + y_2 y_3^4 + y_4^4 y_3 = 0\}}{\mathbb{Z}_{51} : [0,38,16,47,1]}$
16	$\{x_0^4 x_1 + x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_0 = 0\} \subset \mathbb{P}^4$	$\frac{\{y_0^4 y_4 + y_0 y_1^4 + y_1 y_2^4 + y_2 y_3^4 + y_3 y_4^4 = 0\} \subset \mathbb{P}^4}{\mathbb{Z}_{41} : [10,18,16,37,1]}$

Table 2.2: **BHK mirrors of quintics in $\mathbb{P}^4$**

Chapter 3

Birational Geometry of BHK Mirrors

3.1 Shioda Maps

We now introduce the Shioda map and relate it to BHK mirrors. Recall the hypersurfaces X_A and X_{A^T} as above. Define the matrix B to be dA^{-1} where d is a positive integer so that B has only integer entries. The Shioda maps are the rational maps

$$\begin{aligned} &\phi_B : \mathbb{P}^n \dashrightarrow W\mathbb{P}^n(q_0, \ldots, q_n), \text{ and} \\ &\phi_{B^T} : \mathbb{P}^n \dashrightarrow W\mathbb{P}^n(r_0, \ldots, r_n), \end{aligned} \tag{3.1.1}$$

where

$$\begin{aligned} (y_0 : \ldots : y_n) &\overset{\phi_B}{\mapsto} (x_0 : \ldots : x_n), \quad x_j = \prod_{k=0}^{n} y_k^{b_{jk}}, \text{ and} \\ (y_0 : \ldots : y_n) &\overset{\phi_{B^T}}{\mapsto} (z_0 : \ldots : z_n), \quad z_j = \prod_{k=0}^{n} y_k^{b_{kj}}. \end{aligned} \tag{3.1.2}$$

Consider the polynomial

$$F_{dI} := \sum_{i=0}^{n} y_i^d \tag{3.1.3}$$

and the Fermat hypersurface cut out by it, $X_{dI} := Z(F_{dI}) \subset \mathbb{P}^n$. Note that the Shioda maps above restrict to rational maps $X_{dI} \overset{\phi_B}{\dashrightarrow} X_A$ and $X_{dI} \overset{\phi_B}{\dashrightarrow} X_{A^T}$, respectively, allowing us to obtain the diagram:

$$\begin{tikzcd} & X_{dI} \arrow[dl, dashed, "\phi_B"'] \arrow[dr, dashed, "\phi_{B^T}"] & \\ X_A & & X_{A^T} \end{tikzcd} \tag{3.1.4}$$

We now reinterpret the groups G and G^T in the context of the Shioda map. Any element of $\mathrm{Aut}(F_{dI})$ is of the form $g = (e^{2\pi i h_j/d})_j$, for some integers h_j. When we push forward the action of g via ϕ_B, we obtain the diagonal automorphism

$$(\phi_B)_*(g) := (e^{\frac{2\pi i}{d} \sum_{j=0}^{n} b_{ij} h_j})_i \in \mathrm{Aut}(F_A). \tag{3.1.5}$$

The element $(\phi_B)_*(g)$ is a generic element of $\mathrm{Aut}(F_A)$, namely $\prod_{j=0}^{n} \rho_j^{h_j}$. We turn our attention to describing the dual group G^T to G. If we push the element $g^T := (e^{2\pi i s_i/d})_i \in \mathrm{Aut}(F_{dI})$ down via the map ϕ_{B^T}, then we get the action

$$(\phi_{B^T})_*(g^T) = (e^{\frac{2\pi i}{d} \sum_{i=0}^{n} s_i b_{ij}})_j = \prod_{i=1}^{n} (\rho_i^T)^{s_i}. \tag{3.1.6}$$

In other words, we have (surjective) group homomorphisms

$$\begin{aligned} (\phi_B)_* &: \mathrm{Aut}(F_{dI}) \to \mathrm{Aut}(F_A); \text{ and} \\ (\phi_{B^T})_* &: \mathrm{Aut}(F_{dI}) \to \mathrm{Aut}(F_{A^T}). \end{aligned} \tag{3.1.7}$$

This gives us a new interpretation of the choice of groups G and G^T: both are pushforwards of subgroups of $\mathrm{Aut}(F_{dI})$ via the Shioda maps ϕ_B and ϕ_{B^T}, respectively.

3.1.1 Reinterpretation of the Dual Group

We now reformulate the relationship between the groups G and G^T via a bilinear pairing. Consider the map

$$\langle , \rangle_B : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}$$

where $\langle \mathbf{s}, \mathbf{h} \rangle_B := \mathbf{s}^T B \mathbf{h}$. Choose a subgroup $G \subset \mathrm{Aut}(F_A)$, so that $J_{F_A} \subseteq G$. Then set $H := ((\phi_B)_*)^{-1}(G)$. Note that the map $(h_j)_j \mapsto (e^{2\pi i h_j/d})_j$ induces a natural, surjective group homomorphism

$$\mathbb{Z}^{n+1} \xrightarrow{pr} \mathrm{Aut}(F_{dI}). \tag{3.1.8}$$

Take $\tilde{H}$ to be the inverse image $\tilde{H} := pr^{-1}(H)$ of H under this map. We can then define the subgroup $\tilde{H}^{\perp_B} \subseteq \mathbb{Z}^{n+1}$ to be

$$\tilde{H}^{\perp_B} := \left\{ \mathbf{s} \in \mathbb{Z}^{n+1} \middle| \langle \mathbf{s}, \mathbf{h} \rangle_B \in d\mathbb{Z} \text{ for all } \mathbf{h} \in \tilde{H} \right\}. \tag{3.1.9}$$

Define $H^{\perp_B}$ to be the image of $\tilde{H}^{\perp_B}$ under pr, $pr(\tilde{H}^{\perp_B})$.

We remark that it is clear that the group $J_{F_{dI}}$ is contained by H as

$$(\phi_B)_*(e^{2\pi i/d}, \ldots, e^{2\pi i/d}) = \prod_j \rho_j \tag{3.1.10}$$

is a generator of J_{F_A}.

We have assumed that the group G is a subgroup of $SL(F_A)$. This requires that, for all group elements $\mathbf{h} = (h_k)_k \in \tilde{H}$, the product $\prod_{j=0}^n e^{\frac{2\pi i}{d}\sum_k b_{jk}h_k}$ equals 1. This implies that the sum $\sum_{j,k=0}^n b_{jk}h_k$ is an integer divisible by d; therefore, $(1, \ldots, 1) \in \tilde{H}^{\perp_B}$. So, its image $pr(1, \ldots, 1)$ must be in $H^{\perp_B}$. The element $pr(1, \ldots, 1) = (e^{2\pi i/d}, \ldots, e^{2\pi i/d})$ is a generator of the group $J_{F_{dI}}$, hence $H^{\perp_B}$ contains $J_{F_{dI}}$.

Moreover, if one unravels all the definitions, one can see that $(\phi_{B^T})_*(H^{\perp_B}) = G^T$. In order for a monomial $\prod_{i=0}^n x_i^{s_i}$ to be G-invariant, we will need, for any $\prod_{i=1}^n \rho_i^{h_i} = (e^{\frac{2\pi i}{d}\sum_{i=0}^n b_{ij}h_j})_i \in G$, that $\prod_{i=0}^n x_i^{s_i} = \prod_{i=0}^n (e^{\frac{2\pi i}{d}\sum_{i=0}^n b_{ij}h_j} x_i)^{s_i}$. This is equivalent to $\sum_{i,j} s_i b_{ij} h_j$ being a multiple of d.

3.1.2 Birational Geometry of BHK Mirrors

We now give a Theorem of Bini, written in our notation (Theorem 3.1 of [9]):

Theorem 3.1.1. *Let all the notation be as above. Then the hypersurfaces X_A and X_{A^T} are birational to the quotients of the Fermat variety $X_{dI}/(((\phi_B)_*)^{-1}(J_{F_A})/J_{F_{dI}})$ and $X_{dI}/(((\phi_{B^T})_*)^{-1}(J_{F_{A^T}})/J_{F_{dI}})$, respectively.*

We now give a few comments about the proof of the above theorem. It is proven via composing ϕ_B with the map

$$\begin{aligned} q_A : W\mathbb{P}^n(q_0, \ldots, q_n) &\dashrightarrow \mathbb{P}^{n+1}; \\ (x_0 : \ldots : x_n) &\longmapsto \left(\prod_j x_j : \prod_j x_j^{a_{1j}} : \ldots : \prod_j x_j^{a_{nj}} \right). \end{aligned} \tag{3.1.11}$$

Note that the composition $q_A \circ \phi_B : X_{dI} \dashrightarrow \mathbb{P}^{n+1}$ gives the map

$$(y_0 : \ldots : y_n) \longmapsto \left(\prod_j y_j^{q_j'} : y_1^d : \ldots : y_n^d \right). \tag{3.1.12}$$

Letting $m = \gcd(d, q_1', \ldots, q_n')$, we describe the closure of the image as

$$\overline{M_A} := Z\left(\sum_{i=1}^n u_i, u_0^{d/m} = \prod_{i=1}^n u_i^{q_i'/m} \right) \subset \mathbb{P}^{n+1}. \tag{3.1.13}$$

Bini then proves that the map $q_A \circ \phi_B$ is birational to a quotient map, which in our notation implies the birational equivalence

$$\overline{M_A} \simeq X_{dI}/(\phi_B^{-1}(SL(F_A))/J_{F_{dI}}). \tag{3.1.14}$$

Bini then refers the reader to the proof of Theorem 2.6 in [10] to see why the other two maps are birational to quotient maps as well. Note that Bini requires d to be the smallest positive integer so that dA^{-1} is an integral matrix, but the requirement that d is the smallest such integer is unnecessary. One can just use the first part of Theorem 2.6 of [10] to eliminate this hypothesis.

An upshot of this reinterpretation of the theorem is that the mirror statement of BHK duality is a relation of two orbifolds birational to different orbifold quotients of the same Fermat hypersurface in projective space. Namely,

$X_A/\bar{G}$ is birational to $X_{dI}/(((\phi_B)_*)^{-1}(J_{F_A}) + H/J_{F_{dI}})$ while $X_{A^T}/\bar{G}^T$ is birational to $X_{dI}/(((\phi_{B^T})_*)^{-1}(J_{F_{A^T}}) + H^{\perp_B}/J_{F_{dI}})$. As $J_{F_A} \subseteq G$ and $J_{F_{A^T}} \subseteq G^T$,

$$\begin{gathered} ((\phi_B)_*)^{-1}(J_{F_A}) \subseteq ((\phi_B)_*)^{-1}(G) = H; \text{ and} \\ ((\phi_{B^T})_*)^{-1}(J_{F_{A^T}}) \subseteq ((\phi_{B^T})_*)^{-1}(G^T) \subseteq H^{\perp_B} \end{gathered} \tag{3.1.15}$$

which gives us the following corollary:

Corollary 3.1.2. *The Calabi-Yau orbifold $Z_{A,G}$ is birational to $X_{dI}/(H/J_{F_{dI}})$ and its BHK mirror Z_{A^T,G^T} is birational to $X_{dI}/(H^{\perp_B}/J_{F_{dI}})$.*

3.2 Multiple Mirrors

As stated in the introduction, one can take two polynomials

$$F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}; \qquad F_{A'} = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a'_{ij}}. \tag{3.2.1}$$

that cut out two hypersurfaces in weighted-projective n-spaces,

$$\begin{gathered} X_A \subseteq W\mathbb{P}^n(q_0, \ldots, q_n) \text{ and} \\ X_{A^T} \subseteq W\mathbb{P}^n(q'_0, \ldots, q'_0), \end{gathered} \tag{3.2.2}$$

respectively. Take two groups G and G' so that $J_{F_A} \subseteq G \subseteq SL(F_A)$ and $J_{F_{A^T}} \subseteq G^T \subseteq SL(F_{A^T})$. We then obtain two Calabi-Yau orbifolds $Z_{A,G} := X_A/\bar{G}$ and $Z_{A',G'} := X_{A'}/\bar{G}'$.

Even if these two orbifolds are in the same family of Calabi-Yau varieties, they may have BHK mirrors that are not in the same quotient of weighted-projective

n-space (see Section 5 for an explicit example or Tables 5.1-3 of [23]). Take the polynomials

$$F_{A^T} = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ji}}; \qquad F_{(A')^T} = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a'_{ji}}. \tag{3.2.3}$$

They are quasihomogeneous polynomials but not necessarily with the same weights. So they cut out hypersurfaces X_{A^T} and $X_{(A')^T}$. Take the dual groups G^T and $(G')^T$ to G and G', respectively. We quotient each hypersurface by their respective groups, $\bar{G}^T := G^T/J_{F_{A^T}}$ and $(\bar{G}')^T := (G')^T/J_{F_{(A')^T}}$. We then have the BHK mirror dualities:

$$\begin{gathered} Z_{A,G} \overset{\text{BHK mirrors}}{\longleftrightarrow} Z_{A^T,G^T} \\ Z_{A',G'} \overset{\text{BHK mirrors}}{\longleftrightarrow} Z_{(A')^T,(G')^T} \end{gathered} \tag{3.2.4}$$

In this section, we will investigate and compare the birational geometry of the BHK mirrors of the Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G'}$ by using the Shioda maps. Take positive integers d and d' so that $B := dA^{-1}$ and $B' := d'(A')^{-1}$ are matrices with integer entries. Then we can form a "tree" diagram of Shioda maps:

$$\begin{tikzcd} & & & X_{dd'I} \arrow[dll, dashed, "\phi_{d'I}"'] \arrow[drr, dashed, "\phi_{dI}"] & & & \\ & X_{dI} \arrow[dl, dashed, "\phi_B"'] \arrow[dr, dashed, "\phi_{B^T}"] & & & & X_{d'I} \arrow[dl, dashed, "\phi_{B'}"'] \arrow[dr, dashed, "\phi_{(B')^T}"] & \\ X_A & & X_{A^T} & & X_{A'} & & X_{(A')^T} \end{tikzcd} \tag{3.2.5}$$

One can then calculate that $\phi_M \circ \phi_{cI} = \phi_{cM}$ for any integer valued matrix M

and positive integer c. This means we can simplify our tree to just the diagram:

$$\begin{tikzcd}
& & X_{dd'I} \arrow[dll, dashed, "\phi_{d'B}"'] \arrow[dl, dashed, "\phi_{d'B^T}"] \arrow[dr, dashed, "\phi_{dB'}"'] \arrow[drr, dashed, "\phi_{d(B')^T}"] & & \\
X_A & X_{A^T} & & X_{A'} & X_{(A')^T}
\end{tikzcd} \tag{3.2.6}$$

The Calabi-Yau orbifolds are just finite quotients of the hypersurfaces X_A, X_{A^T}, $X_{A'}$ and $X_{(A')^T}$, so we can view them in the context of the diagram:

$$\begin{tikzcd}
& & X_{dd'I} \arrow[dll, dashed, "\phi_{d'B}"'] \arrow[dl, dashed, "\phi_{d'B^T}"] \arrow[dr, dashed, "\phi_{dB'}"'] \arrow[drr, dashed, "\phi_{d(B')^T}"] & & \\
X_A \arrow[d] & X_{A^T} \arrow[d] & & X_{A'} \arrow[d] & X_{(A')^T} \arrow[d] \\
Z_{A,G} & Z_{A^T,G^T} & & Z_{A',G'} & Z_{(A')^T,(G')^T}
\end{tikzcd} \tag{3.2.7}$$

Letting H and H' be the groups $H := (\phi_{d'B})_*^{-1}(G)$ and $H' := (\phi_{dB'})_*^{-1}(G')$, we know that:

Proposition 3.2.1. *The following birational equivalences hold:*

$$\begin{gathered}
Z_{A,G} \simeq X_{dd'I}/(H/J_{F_{dd'I}}); \\
Z_{A^T,G^T} \simeq X_{dd'I}/(H^{\perp_{d'B}}/J_{F_{dd'I}}); \\
Z_{A',G'} \simeq X_{dd'I}/(H'/J_{F_{dd'I}});\ \textit{and} \\
Z_{(A')^T,(G')^T} \simeq X_{dd'I}/((H')^{\perp_{dB'}}/J_{F_{dd'I}}).
\end{gathered} \tag{3.2.8}$$

Proof. Follows directly from Corollary 3.1.2. □

Recall that we are asking for the conditions in which Z_{A^T,G^T} and $Z_{(A'),(G')^T}$ are birational. This question can be answered if we can show that the groups $H^{\perp_{d'B}}$

and $(H')^{\perp_{dB'}}$ are equal. We now prove that such an equality holds, if we assume that the groups G and G' are equal.

Lemma 3.2.2. *If the groups G and G' are equal, then $H^{\perp_{d'B}}$ and $(H')^{\perp_{dB'}}$ are equal.*

Proof. Set $\tilde{H} := pr^{-1}(H)$ and $\tilde{H}' := pr^{-1}(H')$ (Recall these groups from Section 3.2). Note that we have an equality of groups $(\phi_{d'B})_* \circ pr(\tilde{H}) = G = G' = (\phi_{dB'})_* \circ pr(\tilde{H}')$. This implies that, for any element $\mathbf{h} \in \tilde{H}$, there exists an element $\mathbf{h}' \in \tilde{H}'$ so that $d'B\mathbf{h} = dB'\mathbf{h}'$.

Suppose that $\mathbf{s} \in (\tilde{H}')^{\perp_{dB'}}$. We claim that $\mathbf{s}$ is in $\tilde{H}^{\perp_{d'B}}$, i.e., for every $\mathbf{h} \in \tilde{H}$, that $\langle \mathbf{s}, \mathbf{h} \rangle_{d'B} \in d\mathbb{Z}$. Indeed, this is true. Given any $\mathbf{h} \in \tilde{H}$, there exists some $\mathbf{h}'$ as above where $d'B\mathbf{h} = dB'\mathbf{h}'$, hence $\langle \mathbf{s}, \mathbf{h} \rangle_{d'B} = \langle \mathbf{s}, \mathbf{h}' \rangle_{dB'} \in d\mathbb{Z}$, as $\mathbf{s} \in (\tilde{H}')^{\perp_{dB'}}$. This proves that $(\tilde{H})^{\perp_{d'B}} \subseteq (\tilde{H}')^{\perp_{dB'}}$. By symmetry, we now have the equality of the groups, $\tilde{H}^{\perp_{d'B}} = (\tilde{H}')^{\perp_{dB'}}$.

This implies that the images of the groups $\tilde{H}^{\perp_{d'B}}$ and $(\tilde{H}')^{\perp_{dB'}}$ under the homomorphism pr are equal, hence $H^{\perp_{d'B}}$ and $(H')^{\perp_{dB'}}$ are equal. □

We then have the proof of following theorem:

Theorem 3.2.3. *Let $Z_{A,G}$ and $Z_{A',G'}$ be Calabi-Yau orbifolds as above. If the groups G and G' are equal, then the BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ of these orbifolds are birational.*

Proof. Follows directly from Proposition 4.1 and Lemma 4.2. □

This answers Question 2.3.7 affirmatively in the birational case.

3.3 Example: BHK Mirror 3-folds, Shioda Maps, Chen-Ruan Hodge Numbers

In this section, we give an example of two Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G'}$ that are in the same family, but have two different BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ that are not in the same family. As mentioned before, this is a feature of BHK mirror duality that differentiates it from the mirror construction of Batyrev and Borisov. We will show that the BHK mirrors are birational to each other and that their Chen-Ruan Hodge numbers match.

Consider the polynomials

$$\begin{aligned} F_A &:= y_1^8 + y_2^8 + y_3^4 + y_4^3 + y_5^6; \text{ and} \\ F_{A'} &:= y_1^8 + y_2^8 + y_3^4 + y_4^3 + y_4 y_5^4. \end{aligned} \tag{3.3.1}$$

The polynomials F_A and $F_{A'}$ cut out hypersurfaces $X_A = Z(F_A)$ and $X_{A'} = Z(F_{A'})$, two well-defined hypersurfaces in the (Gorenstein) weighted projective 4-space $W\mathbb{P}^4(3,3,6,8,4)$. Note that they are in the same family.

We now address the groups involved in the BHK mirror construction. Set ζ to be a primitive 24th root of unity. The groups J_{F_A} and $J_{F_{A'}}$ are equal and are generated by the element $(\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4) \in (\mathbb{C}^*)^5$. We take G and G' to be the same group, namely we define it to be

$$G = G' := \langle (\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4), (\zeta^{18}, 1, \zeta^6, 1, 1), (1, 1, \zeta^{12}, 1, \zeta^{12}) \rangle. \tag{3.3.2}$$

Note that each of the generators of the group G are also in $SL(F_A)$ and $SL(F_{A'})$, hence the group G satisfies the conditions required for BHK duality. We quotient both the hypersurfaces by X_A and $X_{A'}$ by the group G/J_{F_A} to obtain the Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G}$ which are in the same family of hypersurfaces in $W\mathbb{P}^4(3,3,6,8,4)/(G/J_{F_A})$.

3.3.1 BHK Mirrors

Next, we describe the BHK mirrors to $Z_{A,G}$ and $Z_{A',G'}$. The polynomials associated to the matrices A and A^T are

$$\begin{aligned} F_{A^T} = F_A :&= y_1^8 + y_2^8 + y_3^4 + y_4^3 + y_5^6; \text{ and} \\ F_{A'^T} :&= z_1^8 + z_2^8 + z_3^4 + z_4^3 z_5 + z_5^4. \end{aligned} \tag{3.3.3}$$

While the hypersurface $X_{A^T} = Z(F_{A^T})$ is in $W\mathbb{P}^4(3,3,6,8,4)$, the hypersurface $X_{(A')^T} = Z(F_{A'^T})$ is in a different (Gorenstein) weighted projective 4-space, namely $W\mathbb{P}^4(1,1,2,2,2)$. We can compute the following groups:

$$\begin{aligned} J_{F_{A^T}} &= \langle (\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4) \rangle; \\ J_{F_{(A')^T}} &= \langle (\zeta^3, \zeta^3, \zeta^6, \zeta^6, \zeta^6) \rangle; \\ G^T &= \langle (\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4) \rangle; \text{ and} \\ (G')^T &= \langle (\zeta^3, \zeta^3, \zeta^6, \zeta^6, \zeta^6), (1, 1, 1, \zeta^{12}, \zeta^{12}) \rangle. \end{aligned} \tag{3.3.4}$$

Note that the groups G^T and $J_{F_{A^T}}$ are equal, so the BHK mirror Z_{A^T,G^T} is the hypersurface X_{A^T}. On the other hand the quotient group $(G')^T/J_{F_{(A')^T}}$ is isomorphic to $\mathbb{Z}_2$, hence the BHK mirror $Z_{(A')^T,(G')^T}$ is the Calabi-Yau orbifold $X_{(A')^T}/\mathbb{Z}_2$. Note that the Calabi-Yau orbifold Z_{A^T,G^T} is a hypersurface in $W\mathbb{P}^4(3,3,6,8,4)$, while $Z_{(A')^T,(G')^T}$ is in $W\mathbb{P}(1,1,2,2,2)/\mathbb{Z}_2$. The two BHK mirrors are not hypersurfaces of the same quotient of weighted-projective spaces, hence not sitting inside the same family of Calabi-Yau orbifolds.

3.3.2 Shioda Maps

Even though the two BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ do not sit in the same family of hypersurfaces of the same quotient of weighted-projective space, we can show that they are birational. Take the matrices $B := 24A^{-1}$ and $B' := 24(A')^{-1}$. Let X_{24I} be the hypersurface $Z(x_1^{24} + x_2^{24} + x_3^{24} + x_4^{24} + x_5^{24}) \subset \mathbb{P}^4$. We then have the Shioda maps

$$\begin{tikzcd}
 & & X_{24I} \arrow[dll, "\phi_B"'] \arrow[dl, "\phi_{B^T}"] \arrow[dr, dashed, "\phi_{B'}"'] \arrow[drr, dashed, "\phi_{(B')^T}"] & & \\
X_A & X_{A^T} & & X_{A'} & X_{(A')^T}
\end{tikzcd} \tag{3.3.5}$$

The maps then can be described explicitly:

$$\begin{aligned}
\phi_B(x_1 : \ldots : x_5) &= (x_1^3 : x_2^3 : x_3^6 : x_4^8 : x_5^4) \in X_A \\
\phi_{B^T}(x_1 : \ldots : x_5) &= (x_1^3 : x_2^3 : x_3^6 : x_4^8 : x_5^4) \in X_{A^T} \\
\phi_{B'}(x_1 : \ldots : x_5) &= (x_1^3 : x_2^3 : x_3^6 : x_4^8 : x_4^{-2}x_5^6) \in X_{A'} \\
\phi_{(B')^T}(x_1 : \ldots : x_5) &= (x_1^3 : x_2^3 : x_3^6 : x_4^8x_5^{-2} : x_5^6) \in X_{(A')^T}
\end{aligned} \tag{3.3.6}$$

The four Shioda maps are rational maps that are birational to quotient maps. Take the following four subgroups to $\mathrm{Aut}(F_{24I})$:

$$\begin{aligned}
H &:= \big\langle(\zeta,\zeta,\zeta,\zeta,\zeta),(\zeta^8,1,1,1,1),(\zeta^2,1,\zeta^{-1},1,1),(1,1,\zeta^2,1,\zeta^3), \\
&\qquad (1,1,1,\zeta,\zeta^4)\big\rangle; \\
H' &:= \big\langle(\zeta,\zeta,\zeta,\zeta,\zeta),(\zeta^8,1,1,1,1),(\zeta^2,1,\zeta^{-1},1,1),(1,1,\zeta^2,1,\zeta^2), \\
&\qquad (1,1,1,\zeta^3,\zeta)\big\rangle; \\
H^{\perp_B} = (H')^{\perp_{B'}} &:= \big\langle(\zeta,\zeta,\zeta,\zeta,\zeta),(\zeta^8,1,1,1,1),(1,1,\zeta^4,1,1),(\zeta^2,\zeta^2,\zeta^2,1,1), \\
&\qquad (1,1,1,\zeta,\zeta^4)\big\rangle; \\
J_{F_{24I}} &= \big\langle(\zeta,\zeta,\zeta,\zeta,\zeta)\big\rangle.
\end{aligned} \tag{3.3.7}$$

By Proposition 3.2.1, we have the following birational equivalences

$$\begin{aligned}
Z_{A,G} &\simeq X_{24I}\Big/\big\langle(\zeta^8,1,1,1,1),(\zeta^2,1,\zeta^{-1},1,1),(1,1,\zeta^2,1,\zeta^3),(1,1,1,\zeta,\zeta^4)\big\rangle;\\
Z_{A',G'} &\simeq X_{24I}\Big/\big\langle(\zeta^8,1,1,1,1),(\zeta^2,1,\zeta^{-1},1,1),(1,1,\zeta^2,1,\zeta^2),(1,1,1,\zeta^3,\zeta)\big\rangle;\\
Z_{A^T,G^T} &\simeq X_{24I}\Big/\big\langle(\zeta^8,1,1,1,1),(1,1,\zeta^4,1,1),(\zeta^2,\zeta^2,\zeta^2,1,1),(1,1,1,\zeta,\zeta^4)\big\rangle;\\
Z_{(A')^T,(G')^T} &\simeq X_{24I}\Big/\big\langle(\zeta^8,1,1,1,1),(1,1,\zeta^4,1,1),(\zeta^2,\zeta^2,\zeta^2,1,1),(1,1,1,\zeta,\zeta^4)\big\rangle.
\end{aligned} \tag{3.3.8}$$

So we can see that the BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ are birational.

3.3.3 Chen-Ruan Hodge Numbers

As the Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G}$ are quasismooth varieties in the same toric variety, namely $W\mathbb{P}^4(3,3,6,8,4)/\langle(-i,1,i,1,1),(1,1,-1,1,-1)\rangle$, they have the same Chen-Ruan Hodge numbers. By the theorem of Chiodo and Ruan, this means that their BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ must have the same Chen-Ruan Hodge numbers. We now check this explicitly.

Consider the hypersurface $X_{A^T} \subseteq W\mathbb{P}^4(3,3,6,8,4)$. The dual group G^T is equal to the group $J_{F_{A^T}}$. The only elements of the group $G^T\mathbb{C}^*$ that will have nontrivial fixed loci are in $J_{F_{A^T}}$ as the weighted projective space is Gorenstein. The group $J_{F_{A^T}}$ has exactly six elements which have fixed loci that have nonempty intersections with the hypersurface (see Table 3.1).

We can just then compute the Hodge numbers by using the Griffiths-Dolgachev-Steenbrink formulas (see [22]). This computation gives us that X_{A^T} has a Hodge

Element of $J_{F_{A^T}}$	Fixed Locus
$(1,1,1,1,1)$	X_{A^T}
$(\zeta^{18},\zeta^{18},\zeta^{12},1,1)$	$Z(y_1,y_2,y_3)\cap X_{A^T}$
$(1,1,1,\zeta^{16},\zeta^{8})$	$Z(y_4,y_5)\cap X_{A^T}$
$(\zeta^{12},\zeta^{12},1,1,1)$	$Z(y_1,y_2)\cap X_{A^T}$
$(1,1,1,\zeta^{8},\zeta^{16})$	$Z(y_4,y_5)\cap X_{A^T}$
$(\zeta^{6},\zeta^{6},\zeta^{12},1,1)$	$Z(y_1,y_2,y_3)\cap X_{A^T}$

Table 3.1: **Elements of $J_{F_{A^T}}$ with fixed loci**

diamond of:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 36 & & 36 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The remaining fixed loci are simpler: $Z(y_1,y_2,y_3)\cap X_{A^T}$ consists of three points, $Z(y_4,y_5)\cap X_{A^T}$ is a curve of genus nine, and $Z(y_1,y_2)\cap X_{A^T}$ is a curve of genus one. After considering the age shift, one obtains the Chen-Ruan Hodge diamond of the Calabi-Yau orbifold $Z_{A,G}$:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 7 & & 0 & \\
1 & & 55 & & 55 & & 1 \\
 & 0 & & 7 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Next, we check that these are the same Chen-Ruan Hodge numbers as the Calabi-Yau orbifold $Z_{(A')^T,(G')^T}$. Recall that $X_{A'^T} \subset W\mathbb{P}^4(1,1,2,2,2)$, so we will have a different $\mathbb{C}^*$ action. The group $(G')^T$ is the group $J_{F_{(A')^T}} \cdot \langle(1,1,1,-1,-1)\rangle$. As the weighted-projective space is Gorenstein, we can only look at $(G')^T$ to find the nontrivial fixed loci of elements. The group $(G')^T$ only has five elements that will have nonempty intersections between the hypersurface and the fixed loci of the elements (see Table 3.2).

One then computes the cohomology of each fixed locus and finds the piece invariant under the action of the group $\mathbb{Z}_2$ generated by $(1,1,1,-1,-1)$. The $(\mathbb{Z}_2)$-

Element of $(G')^T$	Fixed Locus
$(1,1,1,1,1)$	$X_{(A')^T}$
$(\zeta^{12},\zeta^{12},1,1,1)$	$Z(z_1,z_2)\cap X_{(A')^T}$
$(1,1,1,\zeta^{12},\zeta^{12})$	$Z(z_4,z_5)\cap X_{(A')^T}$
$(\zeta^{6},\zeta^{6},\zeta^{12},1,1)$	$Z(z_1,z_2,z_3)\cap X_{(A')^T}$
$(\zeta^{18},\zeta^{18},\zeta^{12},1,1)$	$Z(z_1,z_2,z_3)\cap X_{(A')^T}$

Table 3.2: **Elements of $(G')^T$ with fixed loci**

invariant part of the cohomology of $X_{(A')^T}$ gives the Hodge diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 45 & & 45 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$Z(z_1,z_2)\cap X_{(A')^T}$ is a curve with a $\mathbb{Z}_2$ invariant $h^{0,1}=1$, $Z(z_4,z_5)\cap X_{(A')^T}$ is a $\mathbb{Z}_2$-invariant curve of genus nine, and $Z(z_1,z_2,z_3)\cap X_{(A')^T}$ is a set of four $\mathbb{Z}_2$-invariant points. After considering the age shift, one obtains the Chen-Ruan Hodge diamond of $Z_{(A')^T,(G')^T}$:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 7 & & 0 & \\
1 & & 55 & & 55 & & 1 \\
 & 0 & & 7 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Note that this Chen-Ruan Hodge diamond matches that of the Calabi-Yau orbifold $Z_{A,G}$. To summarize, what we have given here is two Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G'}$ that live in a family of hypersurfaces in a finite quotient of a weighted-projective space. Their BHK mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ do not sit in a single family, unlike the mirrors proposed by Batyrev and Borisov. However, the two BHK mirrors have the same Chen-Ruan Hodge number and are birationally equivalent to one another, as both are birational to the same finite quotient of a Fermat hypersurface of $\mathbb{P}^4$.

Chapter 4

On the Picard Ranks of Surfaces of BHK-Type

4.1 Introduction

A classical problem is to compute the Picard rank of a given algebraic surface. Much work has been done in recent years in order to understand a generic hypersurface in a toric Fano 3-fold [12], but one may ask about the highly symmetric hypersurfaces in a weighted projective space. In this note, we will give an explicit description in order to compute the Picard rank of certain symmetric K3 surfaces that are hypersurfaces in weighted projective spaces. This is a generalization of the case of Delsarte surfaces answered by Shioda in [38].

In order to solve this problem, we introduce a mirror symmetry viewpoint. A generalization to the Delsarte surfaces in the K3 case are K3 surfaces of BHK-type, that is, K3 surfaces that have a Berglund-Hübsch-Krawitz mirror. Let k be an algebraically closed field. Take F_A is a polynomial that is a sum of four monomials with four variables over k

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}} \tag{4.1.1}$$

where A is an invertible matrix with entries $(a_{ij})_{i,j=0}^{n}$. Assume that F_A cuts out a quasismooth K3 hypersurface in a weighted projective space $X_A := Z(F_A) \subseteq W\mathbb{P}^n(q_0, \ldots, q_n)$ for some positive integers q_i. This implies that the weighted degree d of the polynomial F_A is the sum $\sum_i q_i$.

The group $(k^*)^{n+1}$ acts on the space $W\mathbb{P}^n(q_0, \ldots, q_n)$ by coordinate-wise multiplication with a subgroup k^* that acts trivially. Define the group $SL(F_A)$ to be the elements of $(k^*)^{n+1}$ that preserve the polynomial F_A and the holomorphic 3-form. Choose a subgroup G of $SL(F_A)$ such that it contains $k^* \cap SL(F_A)$. Setting $\bar{G}$ to be $G/(SL(F_A) \cap k^*)$, the orbifold quotient $Z_{A,G} := X_A/\bar{G}$ is a K3 orbifold.

Berglund-Hübsch-Krawitz mirror symmetry proposes that the mirror to the orbifold $Z_{A,G}$ is related to the polynomial associated to the transposed matrix A^T, $F_{A^T} := \sum_{i=0}^{n} \prod_{j=0}^{n} z_j^{a_{ji}}$. The polynomial F_{A^T} cuts out a Calabi-Yau hypersurface $X_{A^T} := Z(F_{A^T}) \subseteq W\mathbb{P}^n(r_0, \ldots, r_n)$ for some positive integers r_i. The dual group introduced by Krawitz in [33] is a group G^T which satisfies the analogous conditions for X_{A^T} that the group G does for X_A. Define the quotient group $\bar{G}^T := G^T/(SL(F_{A^T}) \cap \mathbb{C}^*)$. The *BHK mirror* Z_{A^T,G^T} to the Calabi-Yau orbifold

$Z_{A,G}$ to be the quotient $X_{A^T}/\bar{G}^T$.

There has been a flurry of activity on BHK mirrors. Chiodo and Ruan in [16] prove that the mirror symmetry theorem for these $(n-1)$-dimensional Calabi-Yau orbifolds on the level of Chen-Ruan Hodge cohomology:

$$H^{p,q}_{CR}(Z_{A,G},\mathbb{C}) = H^{(n-1)-p,q}_{CR}(Z_{A^T,G^T},\mathbb{C}).$$

This is evidence that the orbifolds $Z_{A,G}$ and Z_{A^T,G^T} form a mirror pair in dimensions 3 and greater; however it does not tell us anything in the case of surfaces. There has been recent work of Artebani, Boissière and Sarti that tries to unify the BHK mirror story with Dolgachev-Voisin mirror symmetry in the case where the hypersurface X_A is a double cover of $\mathbb{P}^2$. Their work has been extended by Comparin, Lyons, Priddis, and Suggs to prime covers of $\mathbb{P}^2$. In this corpus of work, the authors focus on proving that the Picard groups of the K3 surfaces $Z_{A,G}$ and Z_{A^T,G^T} have polarizations by so-called mirror lattices. In particular, these lattices embed into the subgroup of the Picard groups of the BHK mirrors that are invariant under the non-symplectic automorphism induced on the K3 surface due to the fact of it being a prime cover of $\mathbb{P}^2$. The fact that it is a prime cover of $\mathbb{P}^2$ requires that the polynomial look like

$$F_A := x_0^{a_{00}} + \sum_{i=1}^{n}\prod_{j=1}^{n} x_j^{a_{ij}}. \tag{4.1.2}$$

In this paper, we drop this hypothesis and the hypothesis of working over the complex numbers and investigate the Picard numbers. The key tools that we use are relating each surface of BHK-type birationally to a quotient of a higher degree Fermat hypersurface in projective space by a finite group H. We then describe the H-invariant part of the transcendental lattice of the Fermat hypersurface, which gives us the rank of the transcendental lattice of the surface of BHK type and the Lefschetz number.

Take BHK mirrors surfaces $Z_{A,G}$ and Z_{A^T,G^T} as above. Take d to be a positive integer so that the matrix dA^{-1} has only integer entries. Let $\mathfrak{I}_d$ be the subset of symmetries on a degree d Fermat hypersurface X_d that correspond to elements in the transcendental lattice of X_d tensored with $\mathbb{Q}$, $T^n(X_d)$ (See Section 3.1 for an explicit description of $\mathfrak{I}_d$ that is very computable). We remark that the subset $\mathfrak{I}_d$ depends on the characteristic of the field k. We then describe the rank of the Picard group. In particular, we prove the following theorem:

Theorem 4.1.1. *The Lefschetz numbers of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned} \lambda(Z_{A,G}) &= \#(\mathfrak{I}_d \cap G^T) \text{ and} \\ \lambda(Z_{A^T,G^T}) &= \#(\mathfrak{I}_d \cap G). \end{aligned} \tag{4.1.3}$$

The surprise is that the dual group G^T associated to the BHK Mirror Z_{A^T,G^T} actually plays a role in the computation of the Lefschetz number of the original K3 orbifold $Z_{A,G}$. We can see explicitly a nice correspondence between the mirrors in this fashion. This theorem has the following corollary:

Corollary 4.1.2. *The Picard ranks of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned}\rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d \cap G^T) \text{ and} \\ \rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_d \cap G).\end{aligned} \tag{4.1.4}$$

An added quick corollary is a lower bound on the Picard number of a BHK mirror is by the order of dual group G^T. Also, a great benefit to this is that the Picard number of each BHK mirror surface is now computable, once one chooses over which field one works.

4.2 Surfaces of BHK-Type

In this section, we will introduce the K3 surfaces which have a BHK Mirror, so called surfaces of BHK type. We will start by introducing weighted Delsarte surfaces that are K3 surfaces and then move to quotienting by certain symplectic quotients.

4.2.1 Weighted Delsarte Surfaces

Let k be an algebraically closed field. Take F_A to be a sum of four monomials in four variables

$$F_A := \sum_{i=0}^{3} \prod_{j=0}^{3} x_j^{a_{ij}}.$$

We require that the polynomial F_A satisfy three conditions:

1. the matrix $A = (a_{ij})_{i,j=0}^{3}$ is invertible;

2. there exists positive integers q_i and m so that, for all i, m equals the sum $\sum_{j=0}^{3} a_{ij}q_j$; and

3. when the polynomial is viewed as a map $F_A : k^{n+1} \to k$, its only critical point is the origin.

These conditions are invertibility, quasihomogeneity and quasismoothness, respectively. The zero locus of the polynomial F_A cuts out a well-defined degree m hypersurface $X_A := \{F_A = 0\}$ in the weighted-projective space $W\mathbb{P}^3(q_0, \ldots, q_3)$.

Remark 4.2.1. When $q_i = 1$ for all i, the hypersurface $X_A \subseteq \mathbb{P}^3$ is called a Delsarte surface. For this reason, the hypersurface X_A is called a weighted Delsarte surface.

When in characteristic zero, condition (3) above implies that the singular locus of the hypersurface X_A is exactly the singular locus of the weighted projective space intersected with the hypersurface itself, i.e.,

$$\mathrm{Sing}(X_A) = X_A \cap \mathrm{Sing}(W\mathbb{P}^3(q_0, q_1, q_2, q_3).$$

Remark 4.2.2 ([21]). Recall that there is an explicit description of the singular locus of a weighted projective space. A point $x = (x_0, x_1, x_2, x_3) \in W\mathbb{P}^3(q_0, q_1, q_2, q_3)$ is in the singular locus of the weighted projective space if and only if the quantity $\gcd(q_i : i \in I(x))$ is greater than one, where the set $I(x)$ is $\{i : x_i \neq 0\}$.

When over an arbitrary algebraically closed field, we will add an additional condition to our hypersurface:

Definition 4.2.3 ([20])**.** We say X_A is in general position if

$$\operatorname{codim}_{X_A}(X_A \cap \operatorname{Sing}(W\mathbb{P}^3(q_0, q_1, q_2, q_3))) = 2. \tag{4.2.1}$$

Lemma 4.2.4 ([20])**.** *Let X_A be a quasismooth hypersurface in general position in $W\mathbb{P}^3(q_0, q_1, q_2, q_3)$, then*

$$Sing(X_A) = X_A \cap Sing(W\mathbb{P}^3(q_0, q_1, q_2, q_3)).$$

From here on, we will assume that X_A is in general position, if over a field of positive characteristic. Given a weighted Delsarte surface X_A, we can now calculate the canonical class of its (minimal) resolution $\tilde{X}_A \dashrightarrow X_A$ to be

$$\omega_{\tilde{X}_A} \cong \mathcal{O}_{\tilde{X}_A}(m - q_0 - q_1 - q_2 - q_3).$$

So, X_A is a (possibly singular) K3 surface if $\sum_{i=0}^{3} q_i = m$, or, equivalently the sum $\sum_{i=0}^{3} a^{ij}$ equals 1.

4.2.2 Symplectic Group Actions

In this section, we introduce the symplectic group actions on a weighted Delsarte surface X_A that are outlined in the Berglund-Hübsch-Krawitz mirror construction. We first will start by defining what we mean by symplectic group actions over fields that are not the complex numbers.

Definition 4.2.5 ([27])**.** Let X be a normal surface over k. Let G be a finite group of k-automorphisms of X. Denote by Y the quotient surface X/G and by $\pi : X \longrightarrow Y$ the quotient map.

1. A surface X is said to be an *orbifold K3 surface* if the canonical sheaf ω_X is isomorphic to the structure sheaf $\mathcal{O}_X$, the first cohomology class $H^1(X, \mathcal{O}_X)$ of the structure sheaf vanishes, and the canonical sheaf of the minimal resolution $\sigma : \tilde{X} \to X$ is just the pullback of the canonical sheaf of X along σ, i.e., $\omega_{\tilde{X}} \cong \sigma^*(\omega_X)$.

2. We say that the quotient map $\pi : X \to Y$ *contains no wild codimension one ramification* if the characteristic of k does not divide the order of the inertia group of the map π at every prime divisor of X.

3. The group action of G on X is called *symplectic* if every element of G fixes the nowhere vanishing 2-form on X, i.e., $g^*\omega_X = \omega_X$ for all $g \in G$.

Lemma 4.2.6. *Assume $\pi : X \to Y$ as above to contain no wild codimension one ramification. Then the canonical sheaf of Y ω_Y is isomorphic to $(\pi_*\omega_X)^G$. If additionally, the surface X is a K3, then $\omega_Y \cong \mathcal{O}_Y$.*

We now would like to give a few facts about the above objects:

Remark 4.2.7. If the characteristic of the field k does not divide the order of the group G, then the map π contains no wild codimension one ramification.

Remark 4.2.8. When working over a field of positive characteristic, there exists examples of a K3 surface X and finite group G such that G is a symplectic group acting on X and the quotient X/G is not an orbifold K3 surface. (See Example 2.8 of [27]).

Consider the group of automorphisms of the polynomial F_A, denoted $\mathrm{Aut}(F_A)$:

$$\mathrm{Aut}(F_A) := \left\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in (k^*)^4 \middle| F_A(\lambda_0 x_0, \lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) = F_A(x_0, x_1, x_2, x_3)\right\}$$

We can describe the elements of $\mathrm{Aut}(F_A)$ easily as being generated by the elements of the torus $(\mathbb{C}^*)^4$ generated by

$$\rho_j = (e^{2\pi i a^{0j}}, e^{2\pi i a^{1j}}, e^{2\pi i a^{2j}}, e^{2\pi i a^{3j}}).$$

This group does not act symplectically on the hypersurface X_A but it has a subgroup $SL(F_A)$ that does, which can be described as:

$$SL(F_A) := \left\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathrm{Aut}(F_A) \middle| \prod_{i=0}^{3} \lambda_i = 1\right\}.$$

The group $SL(F_A)$ contains a subgroup, the exponential grading operator group J_{F_A}, that acts trivially on the hypersurface X_A. We can describe this group as:

$$J_{F_A} := \left\{(\lambda^{q_0}, \lambda^{q_1}, \lambda^{q_2}, \lambda^{q_3}) \in SL(F_A) | \lambda \in k^*\right\}.$$

Take a group G so that $J_{F_A} \subseteq G \subseteq SL(F_A)$. Then the quotient $\bar{G} := G/J_{F_A}$ is a subgroup of the (nontrivial) automorphisms of X_A that leave the nonvanishing 2-form invariant. We then take the orbifold

$$Z_{A,G} = X_A/\bar{G}.$$

Any orbifold K3 surface that can be written as $Z_{A,G}$ for an appropriate choice of A and G are defined to be K3 surfaces of BHK type. In the next section, we will describe what these orbifolds look like birationally and then compute their Picard numbers, which will relate to their BHK mirrors.

4.2.3 The Berglund-Hübsch-Krawitz Mirror

In this section, we construct the BHK mirror to the Calabi-Yau orbifold $Z_{A,G}$ defined above. Take the polynomial

$$F_{A^T} = \sum_{i=0}^{3} \prod_{j=0}^{3} X_j^{a_{ji}}. \tag{4.2.2}$$

It is quasihomogeneous because there exist positive integers $r_i := \sum_j b_{ji}$ so that

$$F_{A^T}(\lambda^{r_0} X_0, \ldots, \lambda^{r_3} X_3) = \lambda^m F_{A^T}(X_0, \ldots, X_3), \tag{4.2.3}$$

for all $\lambda \in k^*$. Note that the polynomial F_{A^T} cuts out a well-defined Calabi-Yau hypersurface $X_{A^T} \subseteq W\mathbb{P}^n(r_0, \ldots, r_3)$. Define the diagonal automorphism group, $\mathrm{Aut}(F_{A^T})$, analogously to $\mathrm{Aut}(F_A)$. The group $\mathrm{Aut}(F_{A^T})$ is generated by $\rho_i^T := \mathrm{diag}(e^{2\pi i b_{ij}/d})_{j=0}^{3} \in (k^*)^4$. Define the dual group G^T relative to G to be

$$G^T := \left\{ \prod_{i=0}^{n} (\rho_i^T)^{s_i} \middle| s_i \in \mathbb{Z}, \text{ where } \prod_{i=0}^{n} x_i^{s_i} \text{ is } G\text{-invariant} \right\} \subseteq \mathrm{Aut}(F_{A^T}). \tag{4.2.4}$$

Note that the dual group G^T sits between $J_{F_{A^T}}$ and $SL(F_{A^T})$ (for details, see [30]). Define the group $\tilde{G}^T := G^T / J_{F_{A^T}}$. We have a well-defined K3 orbifold $Z_{A^T,G^T} := X_{A^T}/\tilde{G}^T \subset W\mathbb{P}^n(r_0, \ldots, r_3)/\tilde{G}^T$. The K3 orbifold Z_{A^T,G^T} is the *BHK mirror* to $Z_{A,G}$.

4.3 Picard Ranks of Surfaces of BHK-Type

In this section, we compute the Picard numbers of the K3 surfaces of BHK type $Z_{A,G}$ described above. We will do this by showing the surfaces are birational to certain quotients of Fermat varieties, and then relating the transcendental part of the middle cohomology of the Fermat variety to the transcendental lattice of the K3 surface. We then obtain a surprising result where the dual group G^T related to the BHK mirror Z_{A^T,G^T} is directly related to the Picard number of the surface $Z_{A,G}$.

4.3.1 Hodge Theory on Fermat Surfaces

In this subsection, we review Shioda's treatment of Hodge theory on Fermat Surfaces as a minor digression (see [38] and [40] for more details). Rest assured, this very computational description of Fermat surfaces will be used in the next section in a very concrete manner. Let X_d be the degree d Fermat surface in $\mathbb{P}^3$. Define the following groups

$$M_d = \big\{(a_0, a_1, a_2, a_3) \in (\mathbb{Z}/d\mathbb{Z})^4 | a_0 + a_1 + a_2 + a_3 \equiv 0 (\text{mod } d)\big\}$$

and

$$\mathfrak{A}_d = \{(a_0, a_1, a_2, a_3) \in M_d | a_i \not\equiv 0 \text{ (mod } d\text{), all } i\}\,.$$

If the characteristic of the field k is $p > 0$, then consider the subset of $\mathfrak{A}_d$, $\mathfrak{B}_d(p)$, that is used in the study of Fermat surfaces:

$$\mathfrak{B}_d := \left\{(b_0, b_1, b_2, b_3) \in \mathfrak{A}_d | \sum_{i=0}^{3} \sum_{j=0}^{f-1} \langle \frac{ta_ip^j}{d} \rangle = 2f \text{ for all } t \text{ such that } (t, d) = 1\right\},$$

where f is the order of p in $(\mathbb{Z}/d\mathbb{Z})^\times$. When the field k is of characteristic zero, then we define the set $\mathfrak{B}_d$ as:

$$\mathfrak{B}_d := \left\{ (b_0, b_1, b_2, b_3) \in \mathfrak{A}_d | \sum_{i=0}^{3} \langle \frac{ta_i}{d} \rangle = 2 \text{ for all } t \text{ such that } (t, d) = 1 \right\}.$$

Also, we define the subset $\mathfrak{I}_d$ as the complement of $\mathfrak{B}_d$ in $\mathfrak{A}_d$, i.e., $\mathfrak{I}_d = \mathfrak{A}_d - \mathfrak{B}_d$.

We can describe the cohomology of the hypersurface X_d by using the symmetries of the variety [40]:

$$H^2(X_d, \mathbb{Q}) = \bigoplus_{\alpha \in \mathfrak{A}_d \cup \{0\}} V(\alpha), \quad \dim V(\alpha) = 1.$$

We can decompose this cohomology to be the Neron-Severi group tensored with $\mathbb{Q}$, denoted $NS(X_d)$ and the transcendental cycles tensored with $\mathbb{Q}$, denoted $T^n(X_d)$. We can describe these groups as

$$NS(X_d) = \bigoplus_{\alpha \in \mathfrak{B}_d \cup \{0\}} V(\alpha)$$

and

$$T^n(X_d) = \bigoplus_{\alpha \in \mathfrak{I}_d} V(\alpha).$$

4.3.2 Picard Ranks of K3 Surfaces of BHK Type

In this section, we use the Shioda map to understand the birational geometry of the K3 surfaces of BHK type. We will compute their Lefschetz numbers which are invariant under the birational transformation. This in part will tell us the rank of the transcendental lattice when tensored with $\mathbb{Q}$ and consequently the Picard rank of any K3 surface of the form $Z_{A,G}$ as above.

Consider a K3 hypersurface X_A defined by the polynomial F_A as above, that sits in a weighted-projective 3-space $W\mathbb{P}^3(q_0, q_1, q_2, q_3)$. Set a positive integer d so that the matrix $B := dA^{-1}$ has only integer values. We define the *Shioda map* ϕ_B to be the rational map

$$\phi_B : \mathbb{P}^n \dashrightarrow W\mathbb{P}^3(q_0, q_1, q_2, q_3); \tag{4.3.1}$$

where

$$(y_0; y_1 : y_2 : y_3) \xmapsto{\phi_B} (x_0 : x_1 : x_2 : x_3); \qquad x_j = \prod_{k=0}^{3} y_k^{b_{jk}}. \tag{4.3.2}$$

Note that this map is regular if and only A is diagonal. Denote the degree d Fermat hypersurface $X_d := Z(x_0^d + x_1^d + x_2^d + x_3^d) \subset \mathbb{P}^n$. We also denote the defining polynomial of the Fermat hypersurface to be F_d. When we restrict the map ϕ_B to this Fermat hypersurface, we get the map

$$\phi_B : X_d \dashrightarrow X_A.$$

Moreover, we can define a map where this is composed with the quotient by the group $\bar{G}$ as above, to obtain a map

$$\phi_{B,G} : X_d \dashrightarrow Z_{A,G}$$

which is just composition of the map ϕ_B with the quotient map $X_A \to Z_{A,G}$.

A natural question is to now investigate the action of an element of the diagonal automorphism group $\mathrm{Aut}(F_d)$ with respect to the Shioda map. By a linear algebra computation, one can see that we have the following commutative diagram

$$\begin{tikzcd} X_d \arrow[r, "\mu_j"] \arrow[d, dashed, "\phi_B"'] & X_d \arrow[d, dashed, "\phi_B"'] \\ X_A \arrow[r, "\rho_j"] & X_A \end{tikzcd} \tag{4.3.3}$$

where μ_j is the element of $\mathrm{Aut}(F_A)$ that is associated to the map that maps $y_j \longmapsto e^{2\pi i/d} y_j$ and $y_k \longmapsto y_k$ for all $k \neq j$. Note that the elements μ_j generate the group $\mathrm{Aut}(F_d)$. One obtains the (surjective) group homomorphism

$$(\phi_B)_* : \mathrm{Aut}(F_d) \longrightarrow \mathrm{Aut}(F_A);\ \text{where } \mu_j \mapsto \rho_j.$$

Define the quotient groups $G_d := \mathrm{Aut}(F_d)/J_{F_d}$ and $G_A := \mathrm{Aut}(F_A)/J_{F_A}$ where each element of these groups act nontrivially on X_d and X_A, respectively. We have the induced map

$$\overline{(\phi_B)_*} : G_d \longrightarrow G_A.$$

We have the following proposition

Proposition 4.3.1 ([9], [30])**.** *The maps ϕ_B and $\overline{\phi_B}$ are birational to quotient maps. In particular, the map ϕ_B is birational to the quotient map*

$$X_d \longrightarrow X_d/(\ker \overline{(\phi_B)_*}),$$

and the map $\overline{\phi_B}$ is birational to the quotient map

$$X_d \longrightarrow X_d/(\overline{(\phi_B)_*}^{-1}(G/J_{F_A})).$$

This result helps us understand the transcendental lattice of the K3 surface tensored with $\mathbb{Q}$. Recall the following specialization of a proposition of Shioda:

Proposition 4.3.2 (Proposition 5 of [38]). *For any nonsingular, complete variety X of dimension r over $\mathbb{C}$ and for any n, $T^n(X)$ is a birational invariant. Further, if Γ is a finite group of automorphisms of X such that the quotient $Y = X/\Gamma$ exists, then for any resolution Y' of Y, one has:*

$$T^n(Y') \cong T^n(Y) \cong T^n(X)^{\Gamma}.$$

As we know that $Z_{A,G}$ is birational to $X_d/\overline{(\phi_B)_*}^{-1}(G/J_{F_A})$, we can apply the above proposition in the context of

$$\begin{aligned} X &:= X_d \\ \Gamma &:= \overline{(\phi_B)_*}^{-1}(G/J_{F_A}); \text{ and} \\ Y &:= Z_{A,G}. \end{aligned} \tag{4.3.4}$$

Recall that $T^n(X_d)$ can be decomposed as the direct sum

$$T^n(X_d) = \bigoplus_{\alpha \in \mathfrak{I}_d} V(\alpha),$$

hence, we denote the elements of $\mathrm{Aut}(X_d)$ that are invariant under Γ as the subgroup $L(\Gamma)$. We now will describe $L(\Gamma)$.

$$\langle , \rangle_B : \mathbb{Z}_d^4 \times \mathbb{Z}_d^4 \to \mathbb{Z}_d^4$$

so that $\langle \mathbf{s}, \mathbf{h} \rangle_B := \mathbf{s}^T B \mathbf{h}$. For any group $H \subseteq \mathbb{Z}_d^4$, we can define the group

$$H^{\perp_B} := \{\mathbf{s} \in \mathbb{Z}_d^4 | \langle \mathbf{s}, \mathbf{h} \rangle_B \equiv 0 \text{ for all } \mathbf{h} \in H\}.$$

Here, we set H to be the group $((\phi_B)_*)^{-1}(G)$. Then

$$L(\Gamma) = H^{\perp_B} B = G^T.$$

So now, we have that

$$T^n(Z_{A,G}) = \left(\bigoplus_{\alpha\in\mathfrak{I}_d} V(\alpha)\right)^{\Gamma} = \bigoplus_{\alpha\in\mathfrak{I}_d\cap G^T} V(\alpha).$$

Now, we remark that by [30], the BHK mirror Z_{A^T,G^T} is birational to a quotient of X_d by $H_B^{\perp}/J_{F_d}$, hence

$$T^n(Z_{A^T,G^T}) = \bigoplus_{\alpha\in\mathfrak{I}_d\cap G} V(\alpha).$$

So there is a mirror duality on the level of Lefschetz numbers for the BHK mirrors:

Theorem 4.3.3. *The Lefschetz numbers of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned}\lambda(Z_{A,G}) &= \#(\mathfrak{I}_d\cap G^T) \text{ and}\\ \lambda(Z_{A^T,G^T}) &= \#(\mathfrak{I}_d\cap G).\end{aligned} \tag{4.3.5}$$

As the Lefschetz numbers and Picard ranks sum to 22 for any K3 surface, we then have the following Corollary:

Corollary 4.3.4. *The Picard ranks of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned}\rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d\cap G^T) \text{ and}\\ \rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_d\cap G).\end{aligned} \tag{4.3.6}$$

4.4 An Example

In this section we will give an explicit example of the computation of the Picard ranks of a K3 surface of BHK type and its BHK mirror. We will do this by following the proof above: describing them explicitly as birational to quotients of a Fermat hypersurface in projective 3-space $\mathbb{P}^3$ and then looking at the invariant part of the transcendental lattice of the Fermat hypersurface.

Consider the polynomial F_A defined to be

$$F_A := x_0^2x_1 + x_1^2x_2 + x_2^6x_3 + x_3^7.$$

This polynomial cuts out a well-defined hypersurface $X_A := \{F_A = 0\}$ in the weighted projective space $W\mathbb{P}^3(2,3,1,1)$. Note that we can check that the only critical point that it has when viewed as a map $F_A : \mathbb{C}^4 \to \mathbb{C}$ is at the origin. Note that the matrix A associated to the polynomial F_A is

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix},$$

which is invertible.

We will now comment on the symmetry groups associated to the polynomial F_A. The group of automorphisms $\mathrm{Aut}(F_A)$ can be described by being generated by one element,

$$\mathrm{Aut}(F_A) = \langle(\zeta, \zeta^{-2}, \zeta^4, \zeta^{-24})\rangle,$$

where ζ is a primitive root of unity of order 168. This group does not act symplectically on the hypersurface X_A. The group that acts symplectically on X_A is the subgroup $SL(F_A)$ that is generated by one element:

$$SL(F_A) = \langle(\zeta^8, \zeta^{-16}, \zeta^{32}, \zeta^{-24})\rangle.$$

Note that this group has elements that act trivially on the hypersurface. We note that the subgroup J_{F_A}, the so-called exponential grading operator, can be described as the subgroup

$$J_{F_A} = \langle(\zeta^{48}, \zeta^{72}, \zeta^{24}, \zeta^{24})\rangle.$$

We now have a choice of choosing a group G so that it sits between the groups J_{F_A} and $SL(F_A)$, i.e.,

$$J_{F_A} \subseteq G \subseteq SL(F_A).$$

For the purposes of this example, we will choose the group G to be equal to J_{F_A}. We then have the K3 surface $Z_{A,G} = X_A/(G/J_{F_A}) = X_A$. We now compute the BHK mirror to $Z_{A,G}$. We start by looking at the transposed polynomial

$$F_{A^T} = x_0^2 + x_0x_1^2 + x_1x_2^6 + x_2x_3^7.$$

This polynomial cuts out a well-defined hypersurface $X_{A^T} := \{F_{A^T} = 0\}$ in the weighted projective space $W\mathbb{P}^3(4, 2, 1, 1)$. We can then compute the symmetry groups on the polynomial F_{A^T}:

$$\begin{aligned}
\mathrm{Aut}(F_{A^T}) &= \langle(\zeta^{84}, \zeta^{-42}, \zeta^{7}, \zeta^{-1})\rangle; \\
SL(F_{A^T}) &= \langle(\zeta^{84}, \zeta^{42}, \zeta^{49}, \zeta^{161})\rangle; \text{ and} \qquad (4.4.1) \\
J_{F_{A^T}} &= \langle(\zeta^{84}, \zeta^{42}, \zeta^{21}, \zeta^{21})\rangle.
\end{aligned}$$

Note that when one computes the dual group G^T to G, one finds that G^T is exactly $SL(F_{A^T})$. Take the group $\bar{G}^T$ to be the quotient $G^T/J_{F_{A^T}}$. Then the BHK mirror to the K3 surface $Z_{A,G}$ is the K3 surface $Z_{A^T,G^T} := X_{A^T}/\bar{G}^T$.

Now let us rearticulate this picture in a birational setting using Shioda maps. First, let us note that we choose $d = 168$ a positive integer so that the matrix $B := dA^{-1}$ has only integer entries. Then we see that

$$B = \begin{pmatrix} 84 & -42 & 7 & -1 \\ 0 & 84 & -14 & 2 \\ 0 & 0 & 28 & -4 \\ 0 & 0 & 0 & 24 \end{pmatrix}$$

We can now define the Shioda maps associated to the matrices B and B^T to be:

$$\begin{aligned} \phi_B &: \mathbb{P}^n \dashrightarrow W\mathbb{P}^3(2,3,1,1) \\ \phi_B^T &: \mathbb{P}^n \dashrightarrow W\mathbb{P}^3(4,2,1,1) \end{aligned} \tag{4.4.2}$$

defined by

$$\begin{aligned} (y_0 : y_1 : y_2 : y_3) &\xmapsto{\phi_B} (y_0^{84}y_1^{-42}y_2^{7}y_3^{-1} : y_1^{84}y_2^{-14}y_3^{2} : y_2^{28}y_3^{-4} : y_3^{24}) \\ (y_0 : y_1 : y_2 : y_3) &\xmapsto{\phi_{B^T}} (y_0^{84} : y_0^{-42}y_1^{84} : y_0^{7}y_1^{-14}y_2^{28} : y_0^{-1}y_1^{2}y_2^{-4}y_3^{24}). \end{aligned} \tag{4.4.3}$$

Take the degree $d = 168$ Fermat hypersurface X_{168} in $\mathbb{P}^3$, i.e., $X_{168} = \{F_{168} := y_0^{168} + y_1^{168} + y_2^{168} + y_3^{168} = 0\} \subset \mathbb{P}^3$. Note that if we restrict the maps ϕ_B and ϕ_{B^T} to just the hypersurface X_{168} we get the maps $X_{168} \overset{\phi_B}{\dashrightarrow} X_A$ and $X_{168} \overset{\phi_{B^T}}{\dashrightarrow} X_{A^T}$. Further, let us construct the maps $\phi_{B,G}$ and ϕ_{B^T,G^T} by composing the maps ϕ_B and ϕ_{B^T} by the quotient maps that quotient X_A and X_{A^T} by the groups (G/J_{F_A}) and $(G^T/J_{F_{A^T}})$, respectively. We then have the following diagram of rational maps:

$$\begin{tikzcd} & X_{168} \arrow[dl, dashed, "\phi_{B,G}"'] \arrow[dr, dashed, "\phi_{B^T,G^T}"] & \\ Z_{A,G} & & Z_{A^T,G^T} \end{tikzcd}$$

One can compute the following groups:

$$\begin{aligned} H &:= \langle(\zeta,\zeta,\zeta,\zeta),(\zeta^2,1,1,1),(\zeta,\zeta^2,1,1),(1,\zeta,\zeta^6,1)\rangle; \\ H^{\perp_B} &= \langle(\zeta,\zeta,\zeta,\zeta),(\zeta^7,1,1,1),(\zeta,1,\zeta^{-2},1),(1,\zeta,\zeta^{-3},1)\rangle; \text{ and} \\ J_{F_{168}} &= \langle(\zeta,\zeta,\zeta,\zeta)\rangle. \end{aligned} \tag{4.4.4}$$

The maps $\phi_{B,G}$ and ϕ_{B^T,G^T} are birational to quotient maps yielding the following birational equivalences:

$$\begin{aligned} Z_{A,G} &\simeq X_{168}/(H/J_{F_{168}}); \\ Z_{A^T,G^T} &\simeq X_{168}/(H^{\perp_B}/J_{F_{168}}). \end{aligned} \tag{4.4.5}$$

So, we recall that we know a lot about the Picard and transcendental lattices of Fermat hypersurfaces. Note that $\mathrm{Aut}(F_{168})$ is isomorphic to $(\mathbb{Z}/168\mathbb{Z})^4$. Recall that we have the sets of elements in the group $\mathrm{Aut}(F_{168})$:

$$\begin{aligned} M_{168} &= \left\{(a_0,a_1,a_2,a_3) \in (\mathbb{Z}/168\mathbb{Z})^4 | a_0+a_1+a_2+a_3 \equiv 0 (\text{mod } 168)\right\}; \\ \mathfrak{A}_{168} &= \{(a_0,a_1,a_2,a_3) \in M_{168} | a_i \not\equiv 0 \text{ (mod } d\text{), all } i\}; \\ \mathfrak{B}_{168}(p) &= \left\{(b_0,b_1,b_2,b_3) \in \mathfrak{A}_{168} | \sum_{i=0}^{3}\sum_{j=0}^{f-1} \langle \frac{ta_i p^j}{d} \rangle = 2f \text{ for all } t \text{ so that } (t,d)=1\right\}; \\ \mathfrak{I}_{168}(p) &= \mathfrak{A}_{168} - \mathfrak{B}_{168}(p); \end{aligned} \tag{4.4.6}$$

where f is the order of $\mathrm{char} k = p$ in $(\mathbb{Z}/168\mathbb{Z})^\times$ if p is positive. When the field k is of characteristic zero, then we define the set $\mathfrak{B}_d(0)$ as:

$$\mathfrak{B}_d(0) := \left\{(b_0,b_1,b_2,b_3) \in \mathfrak{A}_{168} | \sum_{i=0}^{3} \langle \frac{ta_i}{d} \rangle = 2 \text{ for all } t \text{ such that } (t,d)=1\right\}.$$

In order to assume we have no wild codimension one ramification and we have that the orbifolds $Z_{A,G}$ and Z_{A^T,G^T} are K3 orbifolds, we assume that we are working over a field of characteristic zero or p where p is not 2, 3, or 7 (so that it does not divide 168, or the order of any group by which we quotient).

Recall that we have a description of the transcendental lattice of X_{168} tensored with $\mathbb{Q}$:

$$T^2(X_{168}) = \bigoplus_{\alpha \in \mathfrak{I}_{168}(p)} V(\alpha).$$

So, recalling Proposition 4.3.2 and the birational equivalences in Equation 4.4.5, we have:

$$\begin{aligned} T^2(Z_{A,G}) &= \left(\bigoplus_{\alpha \in \mathfrak{I}_{168}(p)} V(\alpha) \right)^{H/J_{F_{168}}} \\ T^2(Z_{A^T,G^T}) &= \left(\bigoplus_{\alpha \in \mathfrak{I}_{168}(p)} V(\alpha) \right)^{H^{\perp_B}/J_{F_{168}}} . \end{aligned} \tag{4.4.7}$$

One can see that the elements of $\mathfrak{I}_{168}(p)$ that are invariant under the action of any element of $H/J_{F_{168}}$ are exactly those also in $H^{\perp_B}B = G^T$, by the definition of $H^{\perp_B}$. One can do the analogous thing and notice that the elements of $\mathfrak{I}_{168}(p)$ that are invariant under the action of any element of $H^{\perp_B}/J_{F_{168}}$ are those also in $(H^{\perp_B})^{\perp_{B^T}}B^T = HB^T = G$. Consequently, one has that

$$\begin{aligned} T^2(Z_{A,G}) &= \bigoplus_{\alpha \in \mathfrak{I}_{168}(p) \cap G^T} V(\alpha) \\ T^2(Z_{A^T,G^T}) &= \bigoplus_{\alpha \in \mathfrak{I}_{168}(p) \cap G} V(\alpha). \end{aligned} \tag{4.4.8}$$

This means that the Lefschetz numbers $\lambda(Z_{A,G})$ and $\lambda(Z_{A^T,G^T})$ are exactly the number of elements in the sets $\mathfrak{I}_{168}(p) \cap G^T$ and $\mathfrak{I}_{168}(p) \cap G$, respectively. As both orbifolds are K3s, we then have that the Picard numbers of each are:

$$\begin{aligned} \rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_{168}(p) \cap G^T); \\ \rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_{168}(p) \cap G). \end{aligned} \tag{4.4.9}$$

We now will compute this for a few examples of p, which is just to take every element in G^T or G and then check computationally if they are in $\mathfrak{B}_{168}(p)$ or not. We now construct a table to illustrate some potential values of Picard ranks over various fields. Note first an observation that was first observed by Tate [43] that if $p \equiv 1 (\text{mod } 168)$, then $\mathfrak{I}_{168}(0) = \mathfrak{I}_{168}(p)$. Otherwise, one must actually compute $\mathfrak{I}_{168}(p)$ explicitly. We now provide a table of (small) primes p that do not divide 168 and the corresponding elements that are in the sets $G \cap \mathfrak{I}_{168}(p)$ and $G^T \cap \mathfrak{I}_{168}(p)$. The primes clustered into four different groups (see Table 4.1).

It is interesting to note that there exists certain values of p where either one, neither or both of the K3 surfaces are supersingular (Picard rank is 22). Also, the order of p in $(\mathbb{Z}/168\mathbb{Z})^*$ does not indicate the cluster of values of p that a specific value of p belongs to.

p	elts in $G \cap \mathfrak{I}_{168}(p)$	elts in $G^T \cap \mathfrak{I}_{168}(p)$	$\rho(Z_{A,G})$	$\rho(Z_{A^T,G^T})$
0, 11, 29, 37, 43, 53, 67, 107, 109, 113, 137, 149, 163	(48, 72, 24, 24) (96, 144, 48, 48) (144, 48, 72, 72) (24, 120, 96, 96) (72, 24, 120, 120) (120, 96, 144, 144)	(84, 126, 147, 147) (84, 42, 105, 105) (84, 126, 63, 63) (84, 42, 21, 21)	18	16
23, 71, 79 127, 151	(48, 72, 24, 24) (96, 144, 48, 48) (144, 48, 72, 72) (24, 120, 96, 96) (72, 24, 120, 120) (120, 96, 144, 144)	none	22	16
5, 13, 17, 19, 41, 59, 61, 83 89, 97, 101, 131, 139, 157	none	(84, 126, 147, 147) (84, 42, 105, 105) (84, 126, 63, 63) (84, 42, 21, 21)	18	22
31, 47, 103, 167	none	none	22	22

Table 4.1: **Picard Ranks of $Z_{A,G}$ and Z_{A^T,G^T} over a field of char p**

Chapter 5

Toric reformulations of BHK Mirrors

5.1 Introduction

In this chapter, we present a toric framework using Gorenstein cones that attempts to unify the two mirror construction frameworks of Batyrev-Borisov and BHK. There has been some work in the past on this approach ([17], [41], and [11]). We provide an alternate approach that is done by using the approach given by [11] and fitting it into an explicit context of toric vector bundles. We take dual Gorenstein cones σ and $\sigma^\vee$ of index r. In this work, we find fans Σ and $\Sigma^\vee$ which have corresponding toric varieties X_Σ and $X_{\Sigma^\vee}$ that are vector bundles over two other toric varieties. By taking the dual vector bundles to X_Σ and $X_{\Sigma^\vee}$, V and Λ and generic zero sections of them, we find two varieties M and W that are conjectural mirrors to one another.

The new change to the framework in order to include both frameworks involves dropping the hypothesis on both σ and $\sigma^{\vee}$ that they are completely split Gorenstein cones. This allows both Borisov's interpretation of BHK mirror duality and the Batyrev-Borisov mirror constructions to sit in the same framework. The consequence with dropping this hypothesis is that there will be vector bundles V as above so that the zero loci of generic global sections in the base space will be Fano varieties, not necessarily Calabi-Yau varieties.

For this reason, we have to pass from classical mirror symmetry to homological mirror symmetry.

5.2 Dual Gorenstein Cones and the Unification Framework

5.2.1 Notation and the Geometry of Dual Convex Cones

In this section, we will introduce some notation that we will use throughout the paper and review some convex geometry. We will more or less follow the appendix in [35]. The statements and proofs presented here are also in [25] and the statements are in [19].

Let M and N be free abelian groups of rank n with a nondegenerate inner pairing $\langle , \rangle : M \times N \to \mathbb{Z}$. Take the real vector spaces generated by each, denoted $M_{\mathbb{R}} := M \otimes \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes \mathbb{R}$. For any element $v \in N_{\mathbb{R}}$ and an integer r, we define the hyperplane, r-shifted hyperplane, and half-space relative to v

$$\begin{aligned} H_v &:= \{w \in M_{\mathbb{R}} : \langle w, v\rangle = 0\}; \\ H_v(r) &:= \{w \in M_{\mathbb{R}} : \langle w, v\rangle = r\}; \text{ and} \\ H_v^+ &:= \{w \in M_{\mathbb{R}} : \langle w, v\rangle = 0\}, \end{aligned} \tag{5.2.1}$$

respectively. We say that the orthogonal complement of a set $S \subseteq N_{\mathbb{R}}$ is $S^{\perp} := \bigcap_{s \in S} H_s$. Recall the following definitions

Definition 5.2.1. Let $S \subset N_{\mathbb{R}}$ be a finite set. A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$C = \operatorname{Cone}(S) = \left\{ \sum_{v \in S} \lambda_v v | \lambda_v \geq 0 \right\} \subseteq N_{\mathbb{R}}.$$

A *strictly convex polyhedral cone* is a convex polyhedral cone so that $C \cap (-C) = 0$.

In this paper, all cones will be polyhedral so we drop the word polyhedral throughout. Given a convex cone C in $N_{\mathbb{R}}$, we define the dual cone $C^{\vee}$ in $M_{\mathbb{R}}$ to be

$$C^{\vee} := \{w \in W : \langle w, v\rangle \geq 0 \text{ for all } v \in C\} = \bigcap_{v \in C} H_v^+.$$

Note that the duality is inclusion reversing, i.e., if C and C' are two convex cones and $C \subset C'$ then $(C')^{\vee} \subset C^{\vee}$. Let the relative interior $\operatorname{relint}(C)$ of a convex cone be the interior of C regarded as a subset of $\mathbb{R}C$. We define the relative boundary ∂C of C to be $\partial C := C \setminus \operatorname{relint}(C)$.

Definition 5.2.2. A subset F of a convex cone C is called a *face* of C, denoted $F \prec C$, if there exists a $u \in C^\vee \subseteq M_\mathbb{R}$ so that

$$F = C \cap H_u.$$

Note that the cone C is a face of itself as $C = C \cap H_0$. The set of all proper faces of the cone C are contained in the relative boundary ∂C. One has the decomposition of the cone C as the disjoint union of relative interiors of the faces

$$C = \coprod_{F \prec C} \mathrm{relint}(F).$$

Lemma 5.2.3. *The following are equivalent for a convex cone $C \subseteq N_\mathbb{R}$ and $v \in C$:*

1. $v \in$ *relint*(C)*;*

2. $\langle w, v \rangle > 0$ *for any element w in the dual cone that is not in the intersection of all hyperplanes relative to all $u \in C$, i.e. $w \in C^\vee \setminus \bigcap_{u \in C} H_u$; and*

3. *the intersection of the dual cone $C^\vee$ and the hyperplane relative to v H_v is the intersection of all hyperplanes relative to all $u \in C$:*

$$C^\vee \cap H_v = \bigcap_{u \in C} H_u.$$

Proof. We first prove (i) and (ii) are equivalent. Suppose (ii) does not hold. Then there is a $w \in C^\vee \setminus \bigcap_{u \in C} H_u$ so that $\langle w, v \rangle = 0$. Then v is contained in the proper face $F = C \cap H_w$, hence in the relative boundary and not in the relative interior of C. Suppose (i) does not hold. Then v is in the proper face $F := C \cap H_w$ of the cone

C for some $w \in C^\vee$ so that $H_w \subsetneq C$. We know that $w \notin \bigcap_{u \in C} H_u$ because if it were then the cone C would be contained in H_u. This means that $w \in C^\vee \setminus \bigcap_{u\in C} H_u$ and $\langle w, v\rangle = 0$. This proves that (i) and (ii) are equivalent.

We now prove that (ii) and (iii) are equivalent. Suppose (ii) holds. It is clear that $\bigcap_{u\in C} H_u \subset C^\vee \cap H_v$. For any element $w \in C^\vee \setminus \bigcap_{u\in C} H_u$, the inner pairing $\langle w, v\rangle$ is positive, hence we know the set $C^\vee \cap H_v \setminus \bigcap_{u\in C} H_u$ is empty, hence $C^\vee \cap H_v \subseteq \setminus \bigcap_{u\in C} H_u$. This proves that (ii) implies (iii). Now suppose (iii) holds. If $w \in C^\vee \setminus \bigcap_{u\in C} H_u$, then $w \notin H_v$ which proves the inequality $\langle w, v\rangle > 0$. This proves that (ii) and (iii) are equivalent. $\square$

Let $\mathcal{F}(C)$ denote the set of faces of a convex cone C.

Proposition 5.2.4 (Proposition A.16 of [35]). *The set $\mathcal{F}(C)$ is a finite partially ordered set with respect to the face relation $\prec$. The maximal element of the set $\mathcal{F}(C)$ is the entire cone C and the smallest element is the intersection $C \cap (-C)$. If one takes a face $F \in \mathcal{F}(C)$ and a face F' of F, then $F' \in \mathcal{F}(C)$. Moreover, if one takes two fans $F_1, F_2 \in \mathcal{F}(C)$ then the intersection $F_1 \cap F_2$ is a face of both faces F_1 and F_2.*

Proposition 5.2.5 (Proposition A.17 in [35]). *Let F be a face of a convex polyhedral cone C. Define the dual face F^* relative to the cone C to be the face of the dual cone $F^* := C^\vee \cap F^\perp$. There exists a bijection $\mathcal{F}(C) \to \mathcal{F}(C^\vee)$ between the faces of the cones C and $C^\vee$ defined by the map $F \longmapsto F^*$. This bijection is inclusion reversing when one views the sets as posets with respect to the relation $\prec$.*

5.2.2 Gorenstein Cones and their Splittings

In this subsection, we will explain what the supports of the fans we are interested in look like. Let M_0 and N_0 be free abelian groups of rank n with nondegenerate inner pairing $\langle , \rangle : M_0 \times N_0 \to \mathbb{Z}$ and vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ as above. Take σ to be a strictly convex cone in $N_{\mathbb{R}}$ of maximal dimension. The dual cone $\sigma^{\vee}$ is then a strictly convex cone in $M_{\mathbb{R}}$ of maximal dimension. We assume that $\sigma^{\vee}$ is polyhedral in the free abelian group M_0. Take overlattices $M \supset M_0$ and $N \subset N_0$ so that M and N are dual. Since the cone σ is strictly convex, there must exist some element $\deg \in M$ so that σ is contained in the halfspace $H^{+}_{\deg}$ and the intersection of the hyperplane with the cone σ is trivial, $H_{\deg} \cap \sigma = \{0\}$. We now make our first assumption on the cone σ.

Assumption 5.2.6. There exists some element $\deg \in \sigma^{\vee}$ as above and positive integer r so that there's a unique element in the intersection of the relative interior of the cone σ, the r-shifted hyperplane relative to deg and the lattice N, i.e.,

$$\#(\mathrm{relint}(\sigma) \cap H_{\deg}(r) \cap N) = 1.$$

We call this unique element in the intersection $\deg^{\vee} \in N$. We can see that the dual cone $\sigma^{\vee}$ is contained in the halfspace relative to the element $\deg^{\vee}$, $H^{+}_{\deg^{\vee}}$. Also, the intersection of the hyperplane $H_{\deg^{\vee}}$ and the dual cone $\sigma^{\vee}$ is trivial. We note that if one looks at the element deg, it is an element of the analogous intersection to that in Assumption 5.2.6, i.e., $\deg \in \mathrm{relint}(\sigma^{\vee}) \cap H_{\deg^{\vee}}(r) \cap M$. We now make the second assumption on the cone σ.

Assumption 5.2.7. The element $\deg^\vee \in \sigma$ is the unique element in the intersection of the relative interior of the dual cone $\sigma^\vee$, the r-shifted hyperplane relative to $\deg^\vee$ and the lattice M, i.e.,

$$\#(\mathrm{relint}(\sigma^\vee) \cap H_{\deg^\vee}(r) \cap N) = 1.$$

We can naturally put these assumptions in the definitions previously introduced in the literature [6].

Definition 5.2.8. We say σ is a *Gorenstein cone* if there exists a set of generators S_σ of the cone σ and an element $\deg \in M$ that is contained in the 1-shifted hyperplane $H_{\deg}(1)$.

Definition 5.2.9. The cone σ is a *reflexive Gorenstein cone* if both cones σ and $\sigma^\vee$ are Gorenstein cones. That is, the cone σ is Gorenstein and there exists a set of generators $S_{\sigma^\vee}$ of the cone $\sigma^\vee$ in M and a lattice element $\deg^\vee \in N$ so that $S_{\sigma^\vee}$ is contained in the 1-shifted hyperplane $H_{\deg^\vee}(1)$.

Definition 5.2.10. If σ is a reflexively Gorenstein cone, then we say the *index* of the cone σ is the positive integer $r = \langle \deg, \deg^\vee \rangle$ as above.

Assumptions 1 and 2 hold for a cone σ if and only if σ is a reflexive Gorenstein cone of index r. In this subsection we will discuss the combinatorial notion of a splitting of reflexive Gorenstein cone.

Definition 5.2.11. Take a reflexive Gorenstein cone σ and a finite set of lattice elements $E = \{e_1, \dots, e_k\}$ in $\sigma \cap N$. We say E is a *proper splitting* if the following hold

1. the pairing of the element deg and E is positive, i.e., $\langle \deg, e_i \rangle > 0$ for all i;
2. the sum of all elements in E is $\deg^\vee$, i.e., $\sum_{i=1}^{k} e_i = \deg^\vee$;
3. there exists a $\mathbb{Z}$-basis $\mathcal{B}$ for the lattice N such that the set E is contained in $\mathcal{B}$; and
4. the set $M_{e_i} := \{m \in \sigma^\vee : \langle m, e_i \geq 0\}$ is not zero-dimensional for all i.

Note that the cardinality of a proper splitting E of σ must be less than or equal to the index r of σ (this is a consequence of conditions 1 and 2 of the definition). We call the cardinality of the proper splitting E its *length*.

Definition 5.2.12. We say that a reflexive Gorenstein cone σ *completely splits* if there exists a proper splitting E of σ where the proper splitting is of length r.

Remark 5.2.13. There exists a reflexive Gorenstein cone σ where σ completely splits while its dual cone $\sigma^\vee$ does not. See Section 5.6.3 for an example.

5.2.3 Cone Closures and Vector Bundles

We will now discuss fans whose supports are Gorenstein cones as described above.

Definition 5.2.14. We say a fan Σ is a cone closure of a cone σ if Σ is the set $\mathcal{F}(\sigma)$ of all faces of the cone.

Take σ to be a maximal dimension, strictly convex, Gorenstein cone σ in the vector space $N_{\mathbb{R}}$ with lattice points in N. Let Σ be the cone closure of σ and $\Sigma^{\vee}$ be the cone closure of $\sigma^{\vee}$. We now will take refinements of these fans that will correspond to affine bundles. Recall the following definition.

Definition 5.2.15 (Page 515 of [19])**.** Take a fan Σ in $N_{\mathbb{R}}$ and a primitive lattice element $v \in |\Sigma| \cap N$. The generalized star subdivision of the fan Σ at v is the set $\Sigma(v)$ of the following cones:

1. the cones $\tau \in \Sigma$ that do not contain the lattice element v and
2. the cones $\tau(v) = \operatorname{Cone}(\tau, v)$ where $v \notin \tau \in \Sigma$ and $\{v\} \cup \tau \subseteq \tau'$ for some cone $\tau' \in \Sigma$.

Lemma 5.2.16 (Lemma 11.1.3 of [19])**.** *The fan $\Sigma(v)$ is a refinement of Σ.*

If the set E is a proper splitting of σ, then every element is primitive in N (due to condition (3) of the definition of proper splitting) and one can iterate the procedure of generalized star subdividing at each element of E, obtaining new fans $\Sigma(e_1)$, then $(\Sigma(e_1))\,(e_2)$ and so on. Denote by Σ_E the fan obtained after star subdividing by all elements of the set E, which we call the star subdivision of the fan Σ by the proper splitting E.

Given the cones σ and $\sigma^{\vee}$ as above and proper splittings E and F respectively, we obtain the fans Σ_E and $\Sigma_F^{\vee}$ by taking the cone closures and star subdividing. Now, take the projections $\pi_E : N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}/(E)$ and $\pi_F : M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/(F)$. Construct two new fans by just taking the collection of cones in $N_{\mathbb{R}}/(E)$ and $M_{\mathbb{R}}/(F)$ that are images of the cones in the fans Σ_E and $\Sigma_F^{\vee}$, respectively. Call these new fans $\underline{\Sigma_E}$ and $\underline{\Sigma_F^{\vee}}$. Note that, by construction, the fans Σ_E and $\underline{\Sigma_E}$ (respectively $\Sigma_F^{\vee}$ and $\underline{\Sigma_F^{\vee}}$) are compatible under the projection π_E (π_F) hence they induce toric morphisms

$$\begin{tikzcd}
X_{\Sigma_E} \arrow[d, "\pi_E"'] & X_{\Sigma_F} \arrow[d, "\pi_F"'] \\
X_{\underline{\Sigma_E}} & X_{\underline{\Sigma_F^{\vee}}}
\end{tikzcd}$$

Take the cone closures Θ_E and Θ_F that are associated to the cones $\text{Cone}(E)$ and $\text{Cone}(F)$. Note that $\Theta_E \subset \Sigma_E$ and $\Theta_F \subset \Sigma_F^{\vee}$. Recall the following definition:

Definition 5.2.17 (Definition 3.3.18 of [19])**.** Let N and N' be lattices and Σ and Σ' be fans in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ respectively. Suppose there is a surjective $\mathbb{Z}$-linear mapping $\phi : N \to N'$ that is compatible with the fans Σ and Σ'. Take N_0 to be the lattice that is the kernel of the map ϕ and the subfan

$$\Sigma_0 := \{\tau \in \Sigma : \tau \subseteq (N_0)_{\mathbb{R}}\}$$

of Σ. We say that the fan Σ is split by Σ' and Σ_0 if there exists a subfan $\hat{\Sigma} \subseteq \Sigma$ so that

1. $\phi_{\mathbb{R}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ maps each cone $\hat{\tau} \subseteq \hat{\Sigma}$ bijectively to a cone $\tau' \in \Sigma'$ such that $\phi(\hat{\tau} \cap N) = \tau' \cap N'$.
2. Given cones $\hat{\tau} \in \hat{\Sigma}$ and $\tau_0 \in \Sigma_0$, the sum $\hat{\tau} + \tau_0$ lies in Σ, and every cone of Σ arises this way.

In our context, we can see that Σ_E is split by $\underline{\Sigma_E}$ and Θ_E, by construction. The subfan $\hat{\Sigma}_E$ is the set of all cones in the fan Σ_E that is disjoint from the set E. By Theorem 3.3.19, this implies that X_{Σ_E} is a rank k affine bundle over $X_{\underline{\Sigma_E}}$ where k is the length of the proper splitting E. Analogously, the toric variety $X_{\Sigma_F^\vee}$ is a rank ℓ affine bundle over $X_{\underline{\Sigma_F^\vee}}$ where ℓ is the length of the proper splitting F. We now add the last assumption:

Assumption 5.2.18. The affine bundles X_{Σ_E} and $X_{\Sigma_F^\vee}$ are vector bundles over $X_{\underline{\Sigma_E}}$ and $X_{\underline{\Sigma_F^\vee}}$, respectively.

According to [19], Oda notes in [35] that if a toric vector bundle V is a toric variety, then the bundle is a direct sum of line bundles. This is proven by using the classification of toric vector bundles found in [31]. This implies that the vector bundles X_{Σ_E} and $X_{\Sigma_F^\vee}$ both split as a direct sum of line bundles over $X_{\underline{\Sigma_E}}$ and $X_{\underline{\Sigma_F^\vee}}$, respectively.

Proposition 5.2.19. *Each line bundle that is a direct summand in the vector bundle X_{Σ_E} corresponds to a divisor D so that $-D$ is nef.*

Proof. The support functions associated to each divisor will be convex, yielding it being associated to an anti-nef divisor. □

Due to this proposition, we look at the dual vector bundles V and Λ to X_{Σ_E} and $X_{\Sigma_F^\vee}$, respectively. Take generic global sections $f \in \Gamma(X_{\underline{\Sigma_E}}, V)$ and $g \in \Gamma(X_{\underline{\Sigma_F^\vee}}, \Lambda)$ so that in each cone of the fan, they cut out either a a quasismooth or empty zero loci in the part of the toric variety corresponding to that cone. These zero loci are complete intersections of k and ℓ polynomials each. Denote the complete intersections $\mathcal{M}_E := Z(f)$ and $\mathcal{W}_F := Z(g)$. Note that the variety $\mathcal{M}_E$ has dimension $n - 2k$ and $\mathcal{W}_F$ has dimension $n - 2\ell$. We now propose the following question:

Question 5.2.20. Are $\mathcal{M}_E$ and $\mathcal{W}_F$ mirrors in the sense of Kontsevich's Homological Mirror Conjecture (Conjecture 2.1.1)?

We now will break down some historical results about some cases of this construction. The first result in this theme was that of Batyrev:

Theorem 5.2.21 ([3])**.** *If the cone σ is a Gorenstein cone of index 1, then proper splittings exist and are unique. Moreover, the Calabi-Yau manifolds $\mathcal{M}_E$ and $\mathcal{W}_F$ are mirrors on the level of stringy cohomology, i.e.,*

$$h_{st}^{p,q}(\mathcal{M}_E, \mathbb{C}) = h_{st}^{(n-2)-p,q}(\mathcal{W}_F, \mathbb{C})$$

Batyrev and Borisov then generalized this result to Calabi-Yau complete intersections:

Theorem 5.2.22 ([5])**.** *Suppose the cone σ is a dual Gorenstein cone of index r and one has complete splittings E and F for the cones σ and $\sigma^\vee$ respectively. Then the Calabi-Yau manifolds $\mathcal{M}_E$ and $\mathcal{W}_F$ are mirrors on the level of stringy cohomology, i.e.,*

$$h_{st}^{p,q}(\mathcal{M}_E, \mathbb{C}) = h_{st}^{(n-2r)-p,q}(\mathcal{W}_F, \mathbb{C})$$

Since in our setup we do not necessarily have the Calabi-Yau condition, we state our claims on the level of homological mirror symmetry. We propose a conjecture for what happens when we loosen the hypotheses to those stated above.

Conjecture 5.2.23. Let σ and $\sigma^\vee$ be dual reflexive Gorenstein cones of index r, imposing the assumptions taken in Subsection 5.2.2. Take E to be a complete splitting of σ and F to be proper splitting of $\sigma^\vee$ of maximal length. We then obtain varieties $\mathcal{M}_E$ and $\mathcal{W}_F$ as above. We claim that they are mirrors in the sense that the dimension of the Hochschild homology of the (derived) Fukaya category $\mathrm{Fuk}(\mathcal{M}_E)$ of $\mathcal{M}_E$ is equal to the dimension of the Hochschild homology of the largest Calabi-Yau category $\mathcal{CY}_{\mathcal{W}_F}$ that is a subcategory of the bounded derived category $D^b(\mathrm{coh}\ \mathcal{W}_F)$ of coherent sheaves on $\mathcal{W}_F$, i.e.,

$$\dim HH_\bullet(\mathrm{Fuk}(\mathcal{M}_E)) = \dim HH_\bullet(\mathcal{CY}_{\mathcal{W}_F}).$$

Note that since $\mathcal{M}_E$ is a Calabi-Yau variety, we know that

$$\dim HH_k(\mathrm{Fuk}(\mathcal{M}_E)) = \sum_{p+q=k} h_{st}^{p,q}(\mathcal{M}_E, \mathbb{C}).$$

In the context of our conjecture, we then can focus on cohomology and the B-side of mirror symmetry and sidestep looking at the Fukaya category. While the hypotheses of the conjecture are weaker, the claim is also weaker in that we do not have a bigrading. We are currently in a sense looking at Betti numbers of the Hodge diamond.

We remark that due to the recent work, there is a bigrading of these cohomology theories that give Hodge-like numbers to these categories due to Katzarkov, Kontsevich and Pantev [29]. In the next few sections we will be showing how both Batyrev-Borisov and Berglund-Hübsch-Krawitz mirror dualities fit into this toric construction.

5.3 Batyrev Duality

We first review the construction of total spaces of line bundles over toric varieties. Then we will place the story of section 2 into the context of Batyrev duality. This section will serve as a warm-up to the combinatorial arguments used for the full generality of vector bundles that one will find in the next section. For an expert reader on the combinatorial aspects of toric vector bundles, this section can be skipped with an eye towards the more general discussion in Section 5.4.

5.3.1 Total spaces of line bundles

Let M, N be dual lattices of rank n. Let Σ be a fan in N. Set $r := \#(\Sigma(1))$ and enumerate these r 1-rays so that we denote them by $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$. Set u_i to be the minimal generator of each ρ_i in N, respectively. We will then denote an extension of the lattice N and M by a lattice of rank 1 by $\bar{N} := N \oplus \mathbb{Z}$ and $\bar{M} := M \oplus \mathbb{Z}$. There is a natural nondegenerate bilinear pairing $\langle , \rangle : \bar{M} \times \bar{N} \to \mathbb{Z}$ induced by the original inner pairing where $\langle (m,t), (n,s) \rangle = \langle m, n \rangle + ts$.

Let $u_0 := (0, 1) \in \bar{N}$ and ν be the ray in $\bar{N}_{\mathbb{R}}$ generated by u_0. Take a Cartier Divisor of X_Σ, D. Decompose D into a sum of Weil divisors $D = \sum_i a_i D_{\rho_i}$ where

D_{ρ_i} is the divisor associated to the 1-ray ρ_i. Now set $\bar{u}_i^D := (u_i, -a_i) \in \bar{N}$. If $\sigma \in \Sigma$, then we can construct $\bar{\sigma}_D := \text{Cone}((u_\rho, -a_\rho)|\rho \in \sigma(1))$ and $\tilde{\sigma}_D :=$ $\text{Cone}((0,1),(u_\rho, -a_\rho)|\rho \in \sigma(1))$. Take $\Sigma \times D$ to be the collection of cones $\bar{\sigma}_D$ and $\tilde{\sigma}_D$ for all $\sigma \in \Sigma$. Take $\pi : N \times \mathbb{Z} \to N$, which is easily seen to be compatible with the fans $\Sigma \times D$ and Σ.

Proposition 5.3.1 (Prop 7.3.1 in [19])**.** *$\pi : X_{\Sigma \times D} \to X_\Sigma$ is a rank 1 vector bundle whose sheaf of sections is $\mathcal{O}_{X_\Sigma}(D)$.*

In other words, $X_{\Sigma \times D} = \text{tot}(\mathcal{O}_{X_\Sigma}(D))$. We construct the dual fan to $\Sigma \times D$. Take any cone $\tau \in \Sigma \times D$. Then define the dual relative to $\Sigma \times D$ to be the cone

$$\tau^*_{\Sigma \times D} := \tau^\perp \cap |\Sigma \times D|^\vee.$$

We then define the dual collection of cones, $(\Sigma \times D)^\vee$, to be the collection of all such $\tau^*_{\Sigma \times D}$ where $\tau \in \Sigma \times D$.

The next step is that we want $(\Sigma \times D)^\vee$ to be a fan. We first prove the following proposition.

Proposition 5.3.2. *If Σ has convex support of full dimension and $-D$ is a basepoint-free divisor, then $\Sigma \times D$ is a fan whose support is convex and of full dimension.*

Proof. Let $S \subset N_\mathbb{R}$ be a convex set. Recall that a function $\varphi : S \to \mathbb{R}$ is convex if $\varphi(tu + (1-t)v) \geq t\varphi(u) + (1-t)\varphi(v)$ for all $u, v \in S$ and $t \in [0,1]$. By Theorem 6.1.7 of [19], a support function of a Cartier divisor is convex on the support of a fan whose support is convex and of full dimension if and only if the Cartier divisor is basepoint free. Say $-D$ is basepoint free, then for all $u, v \in |\Sigma|$, $t \in [0,1]$, $\varphi_{-D}(tu+(1-t)v) \geq t\varphi_{-D}(u)+(1-t)\varphi_{-D}(v)$. Note that $\varphi_{-D}(u) = -\varphi_D(u)$. This implies that the support function for D is 'anti-convex' in the sense that, for all $u, v \in |\Sigma|$, $t \in [0,1]$,

$$\varphi_D(tu+(1-t)v) \leq t\varphi_D(u)+(1-t)\varphi_D(v).$$

We now describe the elements of $|\Sigma \times D|$. One can see that

$$|\Sigma \times D| = \{(u, \varphi_D(u)+a) \in N_{\mathbb{R}} \oplus \mathbb{R} : u \in |\Sigma| \text{ and } a \in \mathbb{R}_{\geq 0}\}.$$

which means that in order to prove that $|\Sigma \times D|$ is convex, we need that, for all $(u,r),(v,s) \in |\Sigma| \times \mathbb{R}_{\geq 0}$, that $(tu+(1-t)v, t\varphi_D(u)+(1-t)\varphi_D(v)+r+s) \in |\Sigma \times D|$. Well, as Σ has convex support, we know that $tu+(1-t)v \in |\Sigma|$. Now we need to show that $t\varphi_D(u)+(1-t)\varphi_D(v)+r+s \geq \varphi_D(tu+(1-t)v)$, but that is just implied by the 'anti-convexity' of the support function, φ_D. This implies that the support is convex.

The fact that the support of $\Sigma \times D$ is of full dimension is clear as the support of Σ is of full dimension so there is a cone of full dimension, n, hence when we add the ray $(0,1)$ it adds a dimension as it will be linearly independent (this is equivalent to the Cartier condition on the divisor). $\square$

This proves that $|\Sigma \times D|$ is convex, hence it has a dual cone closure $(\Sigma \times D)^\vee$. If $a_i < 0$ for all i, then one can take $\deg := (0,1)$ and $\deg^\vee := (0,1)$.

5.3.2 The case of normal fans to reflexive polytopes

Take Δ to be a reflexive polytope in $M_{\mathbb{R}}$, i.e., Δ contains the origin as its (unique) interior point and if its dual polytope

$$\Delta^* = \{u \in N_{\mathbb{R}} : \langle m, u \rangle \geq -1 \text{ for all } m \in \Delta\} \subset N_{\mathbb{R}}$$

is also integral. As it is clear that $0 \in \Delta^*$ is an interior point, one can then see $(\Delta^*)^* = \Delta$ and that the reflexive polytopes come in pairs. One can note that each facet, F, of Δ is given by the equation $\langle m, u_F \rangle = -1$ for some $u_F \in N$. Now, let us consider the method of constructing toric varieties via polytopes outlined in [2]. Take the (Gorenstein) cone, C, over the set $\Delta \times \{1\} \subset \bar{M}_{\mathbb{R}}$ and set Θ to be the cone closure fan of C. The dual cone to C is the cone over $\Delta^* \times \{1\}$, i.e., $C^\vee = \text{Cone}(\Delta^* \times \{1\})$. The dual fan $\Theta^\vee$ is the cone closure of $C^\vee$ by Proposition 2.2. The dual fan has support that is $\Delta^* \times \{1\}$. Take Σ_N to be the normal fan to Δ and K to be the canonical divisor on $\Sigma_{N(\Delta)}$. One can check that this dual fan $\Theta^\vee$ is actually just the cone closure of the support of the fan $\Sigma_{N(\Delta)} \times K$. If one (generalized) star subdivides the dual fan by the ray $(0, 1)$, then the fan is exactly $\Sigma_{N(\Delta)} \times K$.

The mirror pair is found when one 'exchanges' M and N, and then the mirror toric variety is the fan given by the generalized star subdivision along $(0, 1)$ of Θ.

So, the mirror duality $\Delta \longrightarrow \Delta^*$ can be thought of as

$$\Sigma_{N(\Delta)} \times K \overset{\text{Batyrev}}{\longleftrightarrow} \Sigma_{N(\Delta^*)} \times K.$$

Or, equivalently in the context of the proposed dual fans, one can see that $\ell = \ell^\vee = 1$ and $\mu = \mathrm{Cone}(0,1)$ and $\nu = \mathrm{Cone}(0,1)$ hence

$$\text{Star subdivision along } (0,1) \text{ of } \Theta^\vee \longleftrightarrow \text{Star subdivision along } (0,1) \text{ of } \Theta,$$

where we chose Θ to be $\mathrm{Cone}(\Delta \times \{1\})$. Note that Batyrev's proposed mirror duality then finds an $\mathcal{A}$-triangulation (resp., say $\mathcal{B}$-triangulation) of the polytopes Δ (resp., Δ^*), which subdivides the polytopes into simplices. This is equivalent to us using Carathéodory's Theorem to subdivide each cone in Θ and $\Theta^\vee$ into a finite union of simplicial cones.

So the toric varieties end up corresponding to the duality above

$$\Theta_{\mu,\mathcal{A}} \longleftrightarrow \Theta_{\nu,\mathcal{B}}.$$

Remark 5.3.3. When one takes D to be a Cartier divisor where $-D$ is nef, and then takes the dual cones $|\Sigma \times D|$ and $|\Sigma \times D|^\vee$, then one has the cones of which subdivisions are taken to be analyzed in [28] in their approach for mirror symmetry of general type with regards to homological mirror symmetry. The refinement of the fans is more technical than underscored here in this case. Also, this type of duality requires stripping the requirement that the cone $\sigma := |\Sigma \times D|$ satisfies Assumption 2 in Section 2.

5.4 Batyrev-Borisov Duality

In the previous section, we gave a way to find the total space of a line bundle associated to a Cartier divisor, D. This extends easily to decomposable toric vector bundles (see pp. 337-8 of [19]). We will review the construction in the first subsection and then in the second subsection we will fit the Batyrev-Borisov duality into the construction outlined in section 2.

5.4.1 Toric Vector Bundles and Gorenstein Cones

Let N, M be dual lattices of rank n and Σ a fan of convex support of full dimension in $N_{\mathbb{R}}$. Take k Cartier divisors $D_i = \sum_{\rho\in\Sigma(1)} a_{i\rho} D_\rho$, $i = 1, \ldots, k$. This gives the locally free sheaf

$$\mathcal{E} = \mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_k)$$

or rank k. We now will construct the corresponding fan $\Sigma_{\mathcal{E}}$ so that $X_{\Sigma_{\mathcal{E}}}$ is the total space of the vector bundle $\mathcal{E}$ in $N_{\mathbb{R}} \times \mathbb{R}^k$. Take $e_1, \ldots, e_k$ to be the standard basis for $\mathbb{R}^k$. Given $\sigma \in \Sigma$, we construct the cone

$$\begin{aligned} \tilde{\sigma} &:= \{u + \lambda_1 e_1 + \ldots + \lambda_r e_r : u \in \sigma, \lambda_i \geq \varphi_{D_i}(u) \text{ for } i = 1, \ldots, r\} \\ &= \text{Cone}(u_\rho - a_{1\rho} e_1 - \cdots - a_{r\rho} e_r : \rho \in \sigma(1)) + \text{Cone}(e_1, \ldots e_r). \end{aligned} \tag{5.4.1}$$

Take the set consisting of the cones $\tilde{\sigma}$ for $\sigma \in \Sigma$ and their faces, call it $\Sigma_{\mathcal{E}}$. “One can show without difficulty that $[\Sigma_{\mathcal{E}}]$ is a fan in $N_{\mathbb{R}} \times \mathbb{R}^k$ such that the toric variety is the vector bundle over X_Σ whose sheaf of sections is ” $\mathcal{E}$ (page 337 of [19]). We give the following definition.

Definition 5.4.1. Take a variety V with a map $\pi : V \to X_\Sigma$. We say π is a toric vector bundle if the torus of X_Σ acts on V so that the action is linear on the fibers and π is equivariant.

According to [19], Oda notes in [36] that if a toric vector bundle, V, is a toric variety in its own right, then the bundle is a direct sum of line bundles as above, which one can prove from the classification of toric vector bundles found in [31].

We now want to know for which $D_1, \ldots, D_k$ is $|\Sigma_\mathcal{E}|$ convex.

Proposition 5.4.2. *Suppose Σ has convex support of full dimension and $-D_i$ is nef for all i, then $\Sigma_\mathcal{E}$ has convex support of full dimension.*

Proof. We first describe the elements in $|\Sigma_\mathcal{E}|$. One can see that

$$|\Sigma_\mathcal{E}| = \{(u, \varphi_{D_1}(u) + a_1, \ldots, \varphi_{D_k}(u) + a_k) \in N_\mathbb{R} \oplus \mathbb{R}^k : u \in |\Sigma| \text{ and } a_i \in \mathbb{R}_{\geq 0}\}$$

which means that, in order to prove that $|\Sigma_\mathcal{E}|$ is convex, we need that, for all $(u, c_1, \ldots, c_k), (v, d_1, \ldots, d_k) \in |\Sigma| \times \mathbb{R}^k_{\geq 0}$, that $(tu+(1-t)v, t\varphi_{D_1}(u)+(1-t)\varphi_{D_1}(v)+ c_1 + d_1, \ldots, t\varphi_{D_k}(u) + (1-t)\varphi_{D_k}(v) + c_k + d_k) \in |\Sigma_\mathcal{E}|$. As Σ has convex support, we know that $tu + (1-t)v \in |\Sigma|$. This implies the proof is now reduced to showing that $t\varphi_{D_i}(u) + (1-t)\varphi_{D_i}(v) + c_i + d_i \geq \varphi_{D_i}(tu + (1-t)v)$, but that is just implied by the 'anti-convexity' of the support function φ_{D_i} (see proof of 3.2). This implies that the support is convex. The fact that the support is of full dimension is implied by the fact that Σ is of full dimension and each ray e_i adds a dimension, meaning an addition of k dimensions. $\square$

This proves that $|\Sigma_{\mathcal{E}}|$ is convex, hence it has a dual cone closure $(\Sigma_{\mathcal{E}})^\vee$ in $N \oplus (\mathbb{R}^k)^*$.

We now need to prove some lemmas about polytopes in order to understand the dual cone.

Lemma 5.4.3. *Let Σ be complete in $N_{\mathbb{R}}$. Take two nef divisors, $D_1 = \sum_\rho a_\rho D_\rho$ and $D_2 = \sum_\rho b_\rho D_\rho$. Then $P_{D_1} + P_{D_2} = P_{D_1+D_2}$.*

Proof. Note that $D_1 + D_2$ is nef. Note that by definition

$$\begin{aligned} P_{D_1} &= \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle = -a_\rho \text{ for all } \rho \in \Sigma(1)\}; \\ P_{D_2} &= \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle = -b_\rho \text{ for all } \rho \in \Sigma(1)\}; \text{ and} \\ P_{D_1+D_2} &= \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle = -a_\rho - b_\rho \text{ for all } \rho \in \Sigma(1)\} \end{aligned} \tag{5.4.2}$$

It is clear from the definition that $P_{D_1} + P_{D_2} \subseteq P_{D_1+D_2}$. We now focus on proving the other containment. Note that we can construct the Cartier data for each of the above divisors. For every $\sigma \in \Sigma_{\max}$, there exists $m_\sigma, m'_\sigma \in M$ so that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ and $\langle m'_\sigma, u_\rho \rangle = -b_\rho$ for all $\rho \in \sigma(1)$. The sets of all $\{m_\sigma\}_\sigma$ and $\{m'_\sigma\}_\sigma$ give the Cartier data for D_1 and D_2 respectively. Note the m_σ and m'_σ are unique as $\sigma^\perp = \{0\}$ for all $\sigma \in \Sigma_{\max}$. Then the Cartier data for $D_1 + D_2$ is precisely $\{m_\sigma + m'_\sigma\}_\sigma$. Now, by Theorem 6.1.7 of [19], $P_{D_1} = \mathrm{Conv}(m_\sigma | \sigma \in \Sigma(n))$, $P_{D_2} = \mathrm{Conv}(m'_\sigma | \sigma \in \Sigma(n))$, and $P_{D_1+D_2} = \mathrm{Conv}(m_\sigma + m'_\sigma | \sigma \in \Sigma(n))$. This implies that for any $m \in P_{D_1+D_2}$, there are $t_\sigma \in \mathbb{R}_{\geq 0}$ such that $\sum_{\sigma \in \Sigma_{\max}} t_\sigma = 1$ and $m = \sum_{\sigma \in \Sigma_{\max}} t_\sigma (m_\sigma + m'_\sigma) = \sum_{\sigma \in \Sigma_{\max}} t_\sigma m_\sigma + \sum_{\sigma \in \Sigma_{\max}} t_\sigma m'_\sigma \in P_{D_1} + P_{D_2}$. □

Definition 5.4.4. Let $P_1, \dots, P_k \subseteq M_{\mathbb{R}}$ be lattice polytopes. The Cayley polytope $P_1 * \dots * P_k$ is defined as

$$\operatorname{conv}(P_1 \times \{e_1\}, \dots, P_k \times \{e_k\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}^k.$$

Proposition 5.4.5. *Let Σ be a complete fan in $N_{\mathbb{R}}$. Let $-D_i$ be nef. Take the Cayley polytope $P_{-D_1} * \cdots * P_{-D_k} := Conv(P_{-D_i} + e_i)_{i=1}^k \subseteq M_{\mathbb{R}} \oplus (\mathbb{R}^k)^*$. Then $|(\Sigma_{\mathcal{E}})^\vee| = Cone(P_{-D_1} * \cdots * P_{-D_k})$.*

Proof. We first prove that $\operatorname{Cone}(P_{-D_1} * \cdots * P_{-D_k}) \subseteq |(\Sigma_{\mathcal{E}})^\vee|$. Take an element of the Cayley polytope, namely, $(\sum_{i=1}^k t_i u_i, t_1, \dots, t_k) \in P_{-D_1} * \cdots * P_{-D_k}$, where $u_i \in P_{-D_i}$ and $\sum_i t_i = 1$ where $t_i \in \mathbb{R}_{\geq 0}$ for all i. For any $(v, \varphi_{D_1}(u) + a_1, \dots, \varphi_{D_k}(u) + a_k) \in |\Sigma_{\mathcal{E}}|$ Then we can see that

$$\begin{aligned} &\langle (\sum_{i=1}^k t_i u_i, t_1, \dots, t_k), (v, \varphi_{D_1}(v) + a_1, \dots, \varphi_{D_k}(v) + a_k) \rangle \\ &\qquad\qquad = \sum_{i=1}^k t_i \langle u_i, v \rangle + \sum_{i=1} t_i (\varphi_{D_i}(v) + a_i) \qquad (5.4.3) \\ &\qquad\qquad \geq \sum_{i=1}^k t_i a_i \geq 0. \end{aligned}$$

We just need to prove that $|(\Sigma_{\mathcal{E}})^\vee| \subseteq \operatorname{Cone}(P_{-D_1} * \cdots * P_{-D_k}))$.

Suppose $(m, t_1, \dots, t_k) \in |(\Sigma_{\mathcal{E}})^\vee| \cap (M_{\mathbb{Q}} \oplus \mathbb{Q}^k)$. Then, for all $v \in |\Sigma|$,

$$\langle (m, t_1, \dots, t_k), (v, \varphi_{-D_1}(v), \dots, \varphi_{-D_k}(v)) = \langle m, v \rangle - \sum_{i=1}^k t_i \varphi_{-D_i}(v) \geq 0. \quad (5.4.4)$$

Then, take T to be the smallest integer so that $Tt_i \in \mathbb{Z}$ for all i. Then we just want to prove that $(Tm, Tt_1, \dots, Tt_k) \in \operatorname{Cone}(P_{-D_1} * \cdots * P_{-D_k}))$. Note that

$Tm \in P_{-Tt_1D_1-\dots-Tt_kD_k}$. As $-D_i$ is nef, $-Tt_iD_i$ is nef, hence by the previous lemma $Tm \in Tt_1P_{-D_1} + \cdots + Tt_kP_{-D_k}$. By Proposition 4.3, this implies that $m \in t_1P_{-D_1} + \cdots + t_kP_{-D_k}$, which proves the desired containment. □

5.4.2 Nef Partitions and Batyrev-Borisov Duality

Nef-partitions were a concept that Borisov introduced in order to understand how to look at Calabi-Yau complete intersections in toric fano varieties.

Definition 5.4.6. Let X be a Gorenstein toric Fano variety. A *nef partition* is a partition of the torus-invariant prime divisors of X into effective, nef, Cartier divisors $D_1, \dots, D_r$. In other words,

$$-K_X = D_1 + \ldots + D_r.$$

The associated generic anticanonical complete intersection in the crepant resolution of X is a (possibly singular) Calabi-Yau.

Now to find the polytope equivalent, we first look at the fan. Let D_ρ denote the toric divisor associated to $\rho \in \Sigma_{N(\Delta)}(1)$, where $\Sigma_{N(\Delta)}$ is the normal fan of Δ. We then partition $\Sigma_{N(\Delta)}(1)$ so that $\Sigma_{N(\Delta)}(1) = I_1 \cup \ldots \cup I_k$ into k disjoint subsets, we get the divisors $E_j := \sum_{\rho \in I_j} D_\rho$.

Definition 5.4.7. We say the decomposition $\Sigma_{N(\Delta)}(1) = I_1 \cup \ldots \cup I_k$ is a nef-partition if, for each j, E_j is a nef Cartier divisor (equivalently, basepoint free).

Note that we can now associate the nef-partition to Minkowski sums of polytopes. Suppose D is a divisor of the form $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$. Take P_{E_i} to be the lattice polytope associated to the divisor E_i. Then:

$$\Delta = P_{-K_X} = P_{E_1} + \ldots + P_{E_k} \text{ reflexive.}$$

By abuse of notation, $-K_X = \sum E_i$ and $\Delta = \sum P_i$ are both also called nef-partitions.

We can construct a dual polytope to P_i, Q_i, as follows:

$$Q_i := \{y \in N_{\mathbb{R}} : \langle P_j, y \rangle \geq -\delta_{ij} \text{ for all } j = 1, \ldots, k\}$$

for all i.

Define $\nabla := Q_1 + \ldots + Q_k$. Note that ∇ is reflexive, $\nabla^* = \text{conv}(P_1, \ldots, P_k)$ is reflexive. Analogously, $\Delta^* = \text{conv}(Q_1, \ldots, Q_k)$. Note that $\langle P_i, Q_j \rangle \geq -\delta_{ij}$.

We say a nef-partition is proper if $\dim P_i > 0$ for all i. Note that any nef-partition can be reduced to a proper nef partition. We now assume that the nef-partition is proper.

Fix a nef partition E_j of length k of nef divisors and now look at the vector bundle

$$\mathcal{E} = \mathcal{O}(-E_1) \oplus \cdots \oplus \mathcal{O}(-E_k). \tag{5.4.5}$$

Then take $\Theta = (\Sigma_{N(\Delta)})_{\mathcal{E}}$, which is a fan in $(N \oplus \mathbb{Z}^k) \otimes \mathbb{R}$. Then, as Σ is complete and E_j for all j is nef, then $|\Theta|$ is convex, hence we have a dual fan $\Theta^\vee$. Note that $\Theta^\vee$ is just the cone over Cayley polytope $P_{E_1} * \cdots * P_{E_k}$, call it σ_P. By taking the dual cone to $\Theta^\vee$, one can then see that $|(\Theta^\vee)^\vee|$ is the cone over the Cayley polytope $Q_1 * \cdots * Q_k$, say σ_Q [4]. This makes the duality

$$\sigma_P \overset{\text{Batyrev-Borisov}}{\longleftrightarrow} \sigma_Q$$

the same as

$$((\Sigma_{N(\Delta)})_{\mathcal{E}})^\vee \longleftrightarrow (\Sigma_{N(\Delta)})_{\mathcal{E}}$$

in our context.

Moreover, one can take $\deg = e_1 + \ldots + e_k$, $\deg^\vee = e_1^* + \ldots e_k^*$. We now star subdivide $(((\Sigma_{N(\Delta)})_{\mathcal{E}})^\vee)^\vee$ and $((\Sigma_{N(\Delta)})_{\mathcal{E}})^\vee$ by the rays e_i and e_i^* respectively. Note that when one does this, the resulting fans are of toric vector bundles $\mathcal{E}$ and a nef partition vector bundle of the canonical divisor of the toric variety has polytope ∇.

One can now take a $\mathcal{A}$-triangulation of the polytopes P_i and a $\mathcal{B}$-triangulation of the polytopes Q_i in order to give a maximally partial crepant (MPCP) desingularization (as explained in Section 2, or see [2]). This makes the mirror duality

$$((\Sigma_{N(\Delta)})_{\mathcal{E}})_{\mathcal{B}}^\vee \longleftrightarrow ((\Sigma_{N(\Delta)})_{\mathcal{E}})_{\mathcal{A}}$$

as proposed by Borisov.

5.5 Berglund-Hübsch Duality

A combinatorial formulation of Berglund-Hübsch is discussed by Borisov in [11]. The novel idea for Berglund-Hübsch duality is that one constructs the lattices M

and N as overlattices of free abelian groups in order to encapsulate the data of the superpotentials. Take M_0 and N_0 to be free abelian groups with generators u_i and v_i, $1 \leq i \leq n+1$, respectively.

The trick to Borisov's toric reinterpretation of Berglund-Hübsch is that we will need to choose overlattices $M \supseteq M_0$ and $N \supseteq N_0$ so that they are dual to one another, which is equivalent to choosing the groups G and $G^\vee$ that one wants to quotient the variety by, but we need to be sure that they are groups of diagonal automorphisms and contain the exponential grading operator [11]. We define the elements $\deg \in N_0^\vee$ and $\deg^\vee \in M_0^\vee$ so that, for all i,

$$\langle u_i, \deg^\vee \rangle = 1; \qquad \langle \deg, v_i \rangle = 1.$$

We have two chains of overlattices $M_0 \subset M \subset N_0^\vee$ where $\deg \in N$ and $N_0 \subset N \subset M_0^\vee$ where $\deg^\vee \in M$. The fact that one has $\deg \in M$ and $\deg^\vee \in N$ is equivalent to the group G be in the group of diagonal automorphisms of W and $G^\vee$ containing the exponential grading operator, two conditions for the group choices for Berglund-Hübsch.

We want to describe a few things about this setup, namely about the elements deg and $\deg^\vee$. Firstly, note that we can consider the $\mathbb{Q}$-vector space over M_0 and N_0 and have dual vector spaces. Note that we can then describe $\deg = \sum_{j=1}^{n+1} v_j^* = \sum_{i,j=1}^{n+1} (A^{-1})_{ji} u_i$ and $\deg^\vee = \sum_{i=1}^{n+1} u_i^* = \sum_{i,j=1}^{n+1} (A^{-1})_{ji} v_i$. This leads us to know that

$$\langle \deg, \deg^\vee \rangle = \sum_{i,j=1}^{n+1} (A^{-1})_{ij}. \tag{5.5.1}$$

Note that the Calabi-Yau condition is equivalent to $\langle \deg, \deg^\vee \rangle \in \mathbb{Z}_{>0}$ (Corollary 2.3.5 of [11]), this number is known as the index of the pair. In "good cases," one side or both sides of the duality may be related to a Calabi-Yau complete intersection of $\langle \deg, \deg^\vee \rangle$ hypersurfaces (Remark 2.4.4 of [11]). Also, Borisov states that the best case scenario corresponds to the nef-partition case. In Section 6, we give an example of a cone constructed in this fashion in which it is not a "good case" in the spirit of Borisov's remark. We set $k := \langle \deg, \deg^\vee \rangle$.

Proposition 5.5.1. *Consider the cones $C_M = Cone(\{u_i\}) \subset M_\mathbb{R}$ and $C_N = Cone(\{v_i\}) \subset N_\mathbb{R}$. Consider the hyperplanes $H_{\deg}(k) = \{m \in M : \langle m, \deg^\vee \rangle = k\}$ and $H_{\deg^\vee}(k) = \{n \in N : \langle \deg, n \rangle = k\}$. Then $H_{\deg}(k) \cap relint(C_N^\vee) \cap M = \{\deg\}$ and $H_{\deg^\vee}(k) \cap relint(C_M^\vee) \cap N = \{\deg^\vee\}$. Moreover, if $l < k$ then $H_{\deg}(l) \cap relint(C_N^\vee) \cap M = \varnothing$ and $H_{\deg^\vee}(l) \cap relint(C_M^\vee) \cap N = \varnothing$.*

Proof. Note that, as $\deg^\vee \in \operatorname{relint}(C_N)$, we know that we can rewrite

$$\deg^\vee = c_1 v_1 + \ldots + c_{n+1} v_{n+1} \text{ for some } c_j \in \mathbb{R}_{>0}.$$

So then we may noted that $\langle \deg, \deg^\vee \rangle = k$ and $\langle \deg, v_j \rangle = 1$ for all j. This requires that

$$k = \langle \deg, \deg^\vee \rangle = \sum_{j=1}^{n+1} c_j \langle \deg, v_j \rangle = \sum_{j=1}^{n+1} c_i.$$

Now suppose $m \in \operatorname{relint}(C_N^\vee) \cap M \setminus \{\deg\}$. Then $\langle m, v_j \rangle > 0$ since otherwise it would be on a proper face or not in the dual cone. Note that this requires $\langle m, v_j \rangle \geq 1$ and for there to exist some j so that $\langle m, v_j \rangle \geq 2$. This is because deg is the only element in M so whose inner product with all the v_j equates to 1. So then

$$\langle m, \deg^\vee \rangle = \sum_{j=1}^{n+1} c_i \langle m, v_j \rangle > \sum_{j=1}^{n+1} c_i \langle \deg, v_j \rangle = \langle \deg, \deg^\vee \rangle = k$$

which proves that $(\operatorname{relint}(C_N^\vee) \cap M \backslash \{\deg\}) \cap H_{\deg^\vee}(j) = \varnothing$ for all $j \leq k$, which proves that $\operatorname{relint}(C_N^\vee) \cap M \cap H_{\deg^\vee}(k) = \{\deg\}$ and $\operatorname{relint}(C_N^\vee) \cap M \cap H_{\deg^\vee}(l) = \{\deg\}$ for all $l < k$. The other claim is also done by symmetry. $\square$

Corollary 5.5.2. *$H_{\deg}(k) \cap relint(C_M) \cap M = \{\deg\}$ and $H_{\deg^\vee}(k) \cap relint(C_N) \cap N = \{\deg^\vee\}$.*

Proof. Notice that $\langle C_M, C_N \rangle \geq 0$, hence $C_M \subset C_N^\vee$ and $C_N \subset C_M^\vee$. Then $\operatorname{relint}(C_M) \subset \operatorname{relint}(C_N^\vee)$ and $\operatorname{relint}(C_N) \subset \operatorname{relint}(C_M^\vee)$. The rest of the proof just relies on noting that is clear that deg and $\deg^\vee$ both belong to their respective intersections. $\square$

We now take a (simplicial) cone C so that $C_N \subseteq C \subseteq C_M^\vee$ whose generators are in N. Then the dual cone to C will be $C^\vee$ and we know that $C_M \subseteq C^\vee \subseteq C_N^\vee$. Take the complete closure of C and star subdivide by the ray generated by deg. This gives a fan, $\Sigma_{C,\deg}$. Doing an analogous construction, one obtains $\Sigma_{C^\vee,\deg^\vee}$. One can then look at projections $\pi_{\deg} : N \to H_{\deg}$ and $\pi_{\deg^\vee} : M \to H_{\deg^\vee}$.

Corollary 5.5.3. *Take $\underline{\Sigma}_{C,\deg}$ and $\underline{\Sigma}_{C^\vee,\deg^\vee}$ to be the collection of images of cones of $\Sigma_{C,\deg}$ and $\Sigma_{C^\vee,\deg^\vee}$ under the maps $\pi_{\deg}$ and $\pi_{\deg^\vee}$, respectively. Then $\pi_{\deg} : \Sigma_{C,\deg} \to \underline{\Sigma}_{C,\deg}$ and $\pi_{\deg^\vee} : \Sigma_{C^\vee,\deg^\vee} \to \underline{\Sigma}_{C^\vee,\deg^\vee}$ are line bundles.*

Proof. This is a Corollary to Proposition 2.4. $\square$

Moreover, they are simplicial fans, i.e., fans with at most orbifold singularities. We can then see that this fits into the proposed duality where μ and ν are $\deg^\vee$ and $\deg$ respectively, and we already have a simplicial resolution, so we do not need to take an $\mathcal{A}$-triangulation of the cones in the fans.

This means that Berglund-Hübsch can be related to a duality of the form

$$\Theta^C_\nu \longleftrightarrow (\Theta^C)^\vee_\mu$$

by just choosing Θ^C to be the cone closure of a simplicial cone, C, so that $C_N \subseteq C \subseteq C^\vee_M$. The cones C and $C^\vee$ are dual reflexive Gorenstein cones of index $\langle \deg, \deg^\vee \rangle$.

One can then ask what the matrix pairing associated to the 1-dimensional faces of the cones C and $C^\vee$ are. By Proposition 2.3, one can see that there is an enumeration of the 1-faces of C (resp. $C^\vee$) to be the set $\{u'_0, \ldots, u'_n\}$ (resp. $\{v'_0, \ldots, v'_n\}$) so that the matrix $M = (m_{ij})$, $m_{ij} := \langle u'_i, v'_i \rangle$ is diagonal, i.e., the matrix is associated to a Fermat-like superpotential.

5.6 Examples of higher index Gorenstein cones with respect to Berglund-Hübsch Symmetry

5.6.1 Example One

Consider the Berglund-Hübsch setting where M_0 and N_0 are free abelian groups, generated by u_i and v_i, respectively, where $i = 1, \ldots, 8$. We define a non-degenerate inner pairing $\langle , \rangle : M_0 \times N_0 \longrightarrow \mathbb{Z}$ via the pairing $\langle u_i, v_i \rangle = a_{ij}$ governed by the matrix

$$A = (a_{ij}) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 \end{pmatrix}.$$

Note that $\deg = \sum_{i=1}^{8} \frac{1}{a_{ii}} u_i$, $\deg^\vee = \sum_{i=1}^{8} \frac{1}{a_{ii}} v_i$, and $\langle \deg, \deg^\vee \rangle = 2$, hence satisfying the Calabi-Yau condition, as defined in [11]. By Proposition 2.3.4 and Corollary 2.3.5 of [11], one can take overlattices of finite index $M \supseteq M_0$ and $N \supset N_0$ such that $\deg \in M$ and $\deg^\vee \in N$, which is equivalent to choosing a group, G, that lies in the $\mathrm{SL}_d \cap \mathrm{Aut}(W)$ and contains the exponential grading operator.

Now take the cones $C_M = \mathrm{Cone}(\{u_i\}) \subset M_\mathbb{R}$ and $C_N = \mathrm{Cone}(\{v_i\}) \subset N_\mathbb{R}$, as specified in Section 5. They are Gorenstein cones of index 2. As A is Fermat-like, we see that $C_M = C_N^\vee$ and $C_N = C_M^\vee$, so there is no choice of intermediate cone, $C_M \subseteq C \subseteq C_N^\vee$.

We now prove that these cones are not completely split. Recall that if a reflexive Gorenstein cone σ of index r is completely split, then it is the Cayley cone associated to r lattice polytopes. We now recall a fact from [7]:

Proposition 5.6.1 ([7], Corollary 2.5)**.** *Let $\sigma \subseteq M_{\mathbb{R}}$ be a reflexive Gorenstein cone of index r with support Δ. Then the following are equivalent:*

1. *σ is completely split;*
2. *there exist lattice points $e_1^*, \ldots, e_r^* \in \Delta^* \cap N$ such that*

$$e_1^* + \ldots + e_r^* = n_\sigma.$$

Note that in Proposition 5.2, we proved that $\deg^\vee = n_\sigma$. This would require that there exists $e_1^*, e_2^* \in C_N \cap N$ such that $e_1^* + e_2^* = \deg^\vee$. However, this would require $\langle \deg, e_1^* \rangle, \langle \deg, e_2^* \rangle \in \mathbb{Z}$ and $e_1^*, e_2^* \in C_N \cap H_{\deg}(1)$. Then $e_1^* = \sum_i s_i v_i$, $e_2^* = \sum_i t_i v_i$, for some $s_i, t_i \in \mathbb{R}_{\geq 0}$, and since both $e_1^*, e_2^* \in C_N$ and sum to $\deg^\vee$, one must have that $t_i, s_i \leq 1/a_{ii}$. Also, $\langle u_i, e_j^* \rangle, \langle \deg, e_j^* \rangle \in \mathbb{Z}$ for all i, j. This requires $t_i a_{ii}, s_i a_{ii} \in \mathbb{Z}$, but as $t_i < 1/a_{ii}$ this implies that $t_i a_{ii} \leq 1$ for all i, hence $t_i = 1/a_{ii}$ or 0 and, analogously, $s_i = 1/a_{ii}$ or 0. However, as $\langle \deg, e_1^* \rangle = \sum_i t_i = 1$, this requires a partition of the entries of the sum $\sum_i \frac{1}{a_{ii}}$, such that there is a subset $I \subset \{1, \ldots 8\}$ such that $\sum_{i \in I} \frac{1}{a_{ii}} = 1$, but no such subset exists. This implies that the cone C_N is not completely split.

One can construct a suite of examples of such Fermat-like Berglund-Hübsch superpotential functions that yield non-completely split Gorenstein cones of index $k > 1$ via constructing sequences of positive integers whose reciprocals sum to a positive integer not equal to one that do not partition into sums of 1. This can sometimes be a difficult question in additive number theory; however, there is some work in factorization theory that yields such examples.

As we have proven in Section 5, we know that C_M and C_N are Gorenstein cones of index 2; however, the issue as to whether the deg or $\deg^\vee$ element lies in the semigroup generated by $\{u_i\}$ and $\{v_i\}$ leads to showing that one can not encompass this example as a nef-partition (see Remark 2.4.4 of [11]).

When one considers this example in the context of [16], it can be interpreted as a hypersurface in a Gorenstein weighted projective space, namely

$$Z(x_1^2 + x_2^3 + x_3^3 + x_4^5 + x_5^5 + x_6^5 + x_7^5 + x_8^{30}) \subseteq W\mathbb{P}^7(15, 10, 10, 6, 6, 6, 6, 1),$$

but it is not Calabi-Yau as $\sum q_i = 60$ while the degree of the polynomial is 30. It is however a Fano Calabi-Yau, meaning its Hodge diamond contains a subdiamond that corresponds to the Hodge diamond that a Calabi-Yau manifold could have.

5.6.2 Example Two

We also can give an example of a completely split Gorenstein cone of index $k > 1$ whose deg element does not lie in the semigroup generated by $\{u_i\}$ but is a nef-partition.

First let us note that the cartesian product $\sigma_1 \times \sigma_2 \subset M_{1,\mathbb{R}} \oplus M_{2,\mathbb{R}}$ of two reflexive Gorenstein cones is again a reflexive Gorenstein cone. Take the dimension of σ_i to be d_i and its index r_i then $\sigma_1 \times \sigma_2$ has dimension $d_1 + d_2$ and index $r_1 + r_2$ with dual cone $\sigma_1^\vee \times \sigma_2^\vee \subset N_{1,\mathbb{R}} \oplus N_{2,\mathbb{R}}$ [42].

Take $M = \mathbb{Z}^9$ and $N = \mathrm{Hom}(\mathbb{Z}^9, \mathbb{Z}) \cong \mathbb{Z}^9$. Take the standard $\mathbb{Z}$-basis of elements

e_i for M and f_i for N. We will take the product of the cone over the polytope corresponding to $\mathcal{O}_{\mathbb{P}^2}(3)$, i.e., $\mathrm{Conv}((2,-1,1),(-1,2,1),(-1,-1,1))$. Explicitly, we take the u_i and v_i to be the following: $u_1 = 2e_1 - e_2 + e_3$, $u_2 = -e_1 + 2e_2 + e_3$, $u_3 = -e_1 - e_2 + e_3$, $u_4 = 2e_4 - e_5 + e_6$, $u_4 = -e_4 + 2e_5 + e_6$, $u_6 = -e_4 - e_5 + e_6$, $u_7 = 8e_7 - e_8 + e_9$, $u_8 = -e_7 + 8e_8 + e_9$, and $u_9 = -e_7 - e_8 + e_9$. Then take $v_1 = f_1 + f_3$, $v_2 = f_2 + f_3$, $v_3 = -f_1 - f_2 + f_3$, $v_4 = f_4 + f_6$, $v_5 = f_5 + f_6$, $v_6 = -f_4 - f_5 + f_6$, $v_7 = f_7 + f_9$, $v_8 = f_8 + f_9$, and $v_9 = -f_7 - f_8 + f_9$.

By taking the sublattices M_0 and N_0 generated by the u_i and v_i one gets the pairing matrix of Berglund-Hübsch type

$$A = (\langle u_i, v_j\rangle)_{ij} = A = (a_{ij}) = 3I_9 \in M_{9,9}(\mathbb{Z})$$

The degree operator is $\deg = \sum_i \frac{1}{3}u_i = e_3 + e_6 + e_9$ and $\deg^\vee = \sum_i \frac{1}{3}v_i = f_3 + f_6 + f_9$, hence $\langle \deg, \deg^\vee\rangle = 3$. It is clear that if one takes C_N is a split Gorenstein cone as it is a Cayley sum where if one takes $\Delta_1 = \mathrm{Conv}(2e_1 - e_2, -e_1 + 2e_2, -e_1 - e_2)$, $\Delta_2 = \mathrm{Conv}(2e_4 - e_5, -e_4 + 2e_5, -e_4 - e_5)$ and $\Delta_3 = \mathrm{Conv}(2e_7 - e_8, -e_7 + 2e_8, -e_7 - e_8)$ and $C_N = \mathrm{Cone}(\mathrm{Conv}(\Delta_1 + e_3, \Delta_2 + e_6, \Delta_3 + e_9))$. One can explicitly create an equivalent decomposition for the v_i via the analogous decomposition. Although we have a completely split Gorenstein cone of index 3, we do not have deg as an element of the semigroup generated by the $\{u_i\}$.

We have now shown that whether or not deg lies in the semigroup generated by the $\{u_i\}$ does not determine if the cone is completely split and hence yield a nef-partition. We also shown that whether or not deg lies in the lattice elements intersection with the cone generated by the $\{u_i\}$. This provides evidence that shows this notion is much more delicate than Remark 2.4.4 of [11] alludes.

5.6.3 Example Three

We now give a second pair of cones with the same Berglund-Hübsch pairing matrix as Example Two above.

Take $M = \mathbb{Z}^9$ and $N = \mathrm{Hom}(\mathbb{Z}^9, \mathbb{Z}) \cong \mathbb{Z}^9$. Take the standard $\mathbb{Z}$-basis of elements e_i for M and f_i for N. We will take $u_i = 3e_i + e_9$ for $i = 1, \ldots, 8$ and $u_9 = e_9$. Then set $v_i = f_i$ for $i = 1, \ldots, 8$ and $v_9 = -f_1 - \ldots - f_8 + 3f_9$.

By taking the sublattices M_0 and N_0 generated by the u_i and v_i one gets the pairing matrix of Berglund-Hübsch type

$$A = (\langle u_i, v_j \rangle)_{ij} = A = (a_{ij}) = 3I_9 \in M_{9,9}(\mathbb{Z}).$$

As in Example Two, the degree operator is $\deg = \sum_i \frac{1}{3} u_i$ and $\deg^\vee = \sum_i \frac{1}{3} v_i$, hence $\langle \deg, \deg^\vee \rangle = 3$. Note that $\langle \deg, v_i \rangle = 1$ and $\langle u_i, \deg^\vee \rangle = 1$ for all i hence this situation satisfies the Berglund-Hübsch framework as described in [11]. Put $\sigma^\vee = C_M = \mathrm{Cone}(u_i)_i \subset M$ and $\sigma = C_N = \mathrm{Cone}(v_i)_i \subset N$. These are dual cones.

We will show that C_M is completely split; however, C_N is not. Recall the following proposition from [7].

Proposition 5.6.2 (Proposition 2.3 of [7]). *Let $\sigma \subseteq M_{\mathbb{R}}$ be a Gorenstein cone with support $\sigma_{(1)} \subseteq H_{\deg}(1)$. Then the following are equivalent:*

1. *σ is a Cayley cone associated to r lattice polytopes*
2. *$\sigma_{(1)}$ is a Cayley polytope of length r;*
3. *There are nonzero $e_1^*, \ldots, e_r^* \in \sigma^\vee \cap N$ such that*

$$e_1^* + \ldots + e_r^* = n_\sigma.$$

Moreover, the lattice vectors $e_1^*, \ldots, e_r^*$ form part of a basis of N and the Cayley structure of $\sigma_{(1)}$ is uniquely determined by the r polytopes

$$\Delta_i := \{x \in \sigma_{(1)} : \langle x, e_j^* \rangle = 0 \textit{ for } j \neq i\} \textit{ for all } i = 1, \ldots, r.$$

These polytope have the property that $\langle \Delta_i, e_i^* \rangle = 1$ hence $\sigma_{(1)} = Conv(\Delta_i)_i$.

Note that in the context of our cones, $m_{\sigma^\vee} = \deg^\vee = f_9$ and $n_\sigma = \deg = e_1 + \ldots + e_8 + 3e_9$. We first prove σ is a Cayley cone associated to at most 1 lattice polytope. Assume that there exists some $e_i^* \in \sigma^\vee \cap N$, $1 \leq i \leq 3$ such that $\sum_i e_i^* = m_{\sigma^\vee} = f_9$. Then $e_i^* = \sum_{j=1}^{9} c_{ij} v_j$ for some $c_{ij} \geq 0$. Then, as $\langle u_j, e_i^* \rangle \in \mathbb{Z}$ for all i, j, we have that $3c_{ij} \in \mathbb{Z}$ for all i, j. Moreover, $\sum_j c_{9j} = \frac{1}{3}$, hence without loss of generality, assume $c_{91} = \frac{1}{3}$ and $c_{92} = c_{93} = 0$. This implies that, for all $i \leq 8$, $\sum_j c_{ij} = \frac{1}{3}$, yet $a_{ij} - a_{9j} \in \mathbb{Z}$ (as $e_j^* \in N$), hence, as $a_{ij} < 1$, we know that $a_{ij} = a_{9j}$, proving that $e_1^* = f_9$ and $e_i^* = 0$ for all $i > 1$. This proves that $\sigma^\vee$ must not be a Cayley cone associate to more than one lattice polytope.

On the other hand, we can see that $\sigma^\vee$ is a Cayley cone associated to 3 lattice polytopes. Namely, take $f_1^* = \frac{1}{3}(u_1+u_2+u_3) = e_1+e_2+e_3+e_9$, $f_2^* = \frac{1}{3}(u_4+u_5+u_6) =$

$e_4+e_5+e_6+e_9$, and $f_3^* = \frac{1}{3}(u_7+u_8+u_9) = e_7+e_8+e_9$. Note $f_1^*+f_2^*+f_3^* = m_{\sigma^\vee} = \deg$. Now, define

$$\nabla_i := \{y \in \sigma^\vee_{(1)} : \langle f_j^*, y\rangle = 0 \text{ for } j \neq i\} \text{ for all } i = 1, \ldots, 3.$$

Here, we claim

$$\begin{aligned} \nabla_1 &= \mathrm{Conv}(v_1, v_2, v_3); \\ \nabla_2 &= \mathrm{Conv}(v_4, v_5, v_6); \text{ and} \\ \nabla_3 &= \mathrm{Conv}(v_7, v_8, v_9). \end{aligned} \tag{5.6.1}$$

This can be proven by direct inspection.

We see that there is a fan Θ whose support is σ. Take the star subdivision with respect to the 1-ray, ν, corresponding to the cone over $\deg^\vee$, then project down with respect to $\deg^\vee$ we can see that the map $\pi_{\nu,\mathbb{R}} : X_{\Theta_\nu} \longrightarrow X_{\underline{\Theta_\nu}}$ is the line bundle associated to $\mathcal{O}_{\mathbb{P}^8}(-3)$. Take the dual line bundle and a global section of it. The zero locus of that global section is exactly a cubic hypersurface in $\mathbb{P}^8$, call it $\mathcal{M}$.

We also claim that there is a fan $\Theta^\vee$ whose support is $\sigma^\vee$ and which is a rank three split vector bundle composed of line bundles whose dual line bundles direct sum to give a nef-partition of an orbifold quotient of the product of three copies of $\mathbb{P}^2$. Take a regular complete intersection of global sections of these three dual line bundles.

So, take $\sigma^\vee$ and star subdivide by the cones generated by the f_i^*. Now project down $\pi : M \longrightarrow M/(f_i^*)$. $M/(f_i^*)$ is generated by the $\mathbb{Z}$-basis $g_j = \pi(e_j) = e_j + (f_i^*)$ for $j \in \{1, 2, 4, 5, 7, 8\}$. Set $\bar{u}_i = \pi(u_i)$. Then we compute that $\bar{u}_i = 3g_i - g_7 - g_8$ for all $i \in \{1, 2, 4, 5\}$, $\bar{u}_3 = -\bar{u}_1 - \bar{u}_2$, $\bar{u}_6 = -\bar{u}_4 - \bar{u}_5$, $\bar{u}_7 = 2g_7 - g_8$, $\bar{u}_8 = -g_7 + g_8$ and $\bar{u}_9 = -g_7 - g_8$. These lattice elements each correspond to a minimal generator of the nine 1-rays ρ_i of our fan $\underline{\Theta}_{f_1^*, f_2^*, f_3^*}$. This fan is complete. The primitive collections are thus $\{\rho_1, \rho_2, \rho_3\}$, $\{\rho_4, \rho_5, \rho_6\}$ and $\{\rho_7, \rho_8, \rho_9\}$.

We construct the variety $X_{\underline{\Theta}_{f_1^*, f_2^*, f_3^*}}$ via the quotient construction, i.e., $X_{\underline{\Theta}_{f_1^*, f_2^*, f_3^*}} = \left((\mathbb{C}^*)^{\underline{\Theta}_{f_1^*, f_2^*, f_3^*}(1)} \setminus Z(\underline{\Theta}_{f_1^*, f_2^*, f_3^*})\right)/G$, where

$$G = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\underline{\Theta}_{f_1^*, f_2^*, f_3^*}(1)} | \prod_\rho t_\rho^{\langle u_\rho, n\rangle} = 1 \text{ for all } n \in N \right\}.$$

Take $t_i := t_{\rho_i}$ if $(t_1, \ldots, t_9) \in G$ then the following must be satisfied:

$$\begin{aligned} t_1^3 t_3^{-3} &= 1 \\ t_2^3 t_3^{-3} &= 1 \\ t_4^3 t_6^{-3} &= 1 \\ t_4^3 t_6^{-3} &= 1 \\ t_1^{-1} t_2^{-1} t_3^2 t_4^{-1} t_5^{-1} t_6^2 t_7^2 t_8^{-1} t_9^{-1} &= 1 \\ t_1^{-1} t_2^{-1} t_3^2 t_4^{-1} t_5^{-1} t_6^2 t_7^{-1} t_8^2 t_9^{-1} &= 1. \end{aligned} \tag{5.6.2}$$

Via inspection, we can see that $G \cong (\mathbb{C}^*)^3 \times (\mathbb{Z}/3\mathbb{Z})^8$ and the action $G \times (C^*)^9 \setminus Z(\underline{\Theta}_{f_1^*, f_2^*, f_3^*})$ by:

$$\begin{aligned} &(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \ldots, \zeta_8) \times (x_1, \ldots, x_9) \mapsto \\ &\quad (\lambda_1\zeta_1 x_1, \lambda_1\zeta_2 x_2, \lambda_1\zeta_3 x_3, \lambda_2\zeta_4 x_4, \lambda_2\zeta_5 x_5, \lambda_2\zeta_6 x_6, \lambda_3\zeta_7 x_7, \lambda_3\zeta_8 x_8, \lambda_3(\zeta_1^2 \cdots \zeta_8^2) x_9). \end{aligned} \tag{5.6.3}$$

Note that this is a orbifold quotient of $(\mathbb{P}^2)^3$ by the group $(\mathbb{Z}/3\mathbb{Z})^8$. The manifold $\mathcal{W}$ is a complete intersection generated by global sections of the sheaves

$\mathcal{O}_{X_{\Theta_{f_1^*,f_2^*,f_3^*}}}(D_{\rho_1} + D_{\rho_2} + D_{\rho_3}), \mathcal{O}_{X_{\Theta_{f_1^*,f_2^*,f_3^*}}}(D_{\rho_4} + D_{\rho_5} + D_{\rho_6})$, and $\mathcal{O}_{X_{\Theta_{f_1^*,f_2^*,f_3^*}}}(D_{\rho_7} + D_{\rho_8} + D_{\rho_9})$. This corresponds to cubic hypersurfaces inside each copy of $\mathbb{P}^2$, then their complete intersection and the $(\mathbb{Z}/3\mathbb{Z})^8$ quotient of it. Call this orbifold $\mathcal{W}$. Note that if one just factored through by just the $\mathbb{Z}/3\mathbb{Z}$ action where all the ζ_i were equal, then one would obtain the $\mathcal{Z}$-manifold as introduced in [15] and further investigated in [14] and [24]. Note that the actions where $\zeta_{3i-2} = \zeta_{3i-1} = \zeta_{3i}$ can be thought of as absorbed by the homogeneity of the λ_i. What we have is an action of a group $\tilde{G} := (\mathbb{Z}/3\mathbb{Z})^5$ on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Let $\zeta^3 = 1$ be a primitive root of unity, and let $(a_1, \ldots, a_9)$ represent the action

$$\begin{aligned}(x_1 :&x_2 : x_3), (x_4 : x_5 : x_6), (x_7 : x_8 : x_9) \mapsto \\ &(\zeta^{a_1}x_1 : \zeta^{a_2}x_2 : \zeta^{a_2}x_2), (\zeta^{a_4}x_4 : \zeta^{a_5}x_5 : \zeta^{a_6}x_6), (\zeta^{a_7}x_7 : \zeta^{a_8}x_8 : \zeta^{a_9}x_9).\end{aligned} \tag{5.6.4}$$

Then the action of the group $\tilde{G}$ is generated by the following elements:

$$\begin{aligned} z &= (1, 0, 0, 1, 0, 0, 1, 0, 0); \\ g &= (2, 0, 0, 1, 0, 0, 0, 0, 0); \\ f_1 &= (1, 2, 0, 0, 0, 0, 0, 0, 0); \\ f_2 &= (0, 0, 0, 1, 2, 0, 0, 0, 0);\ \text{and} \\ f_3 &= (0, 0, 0, 0, 0, 0, 1, 2, 0). \end{aligned} \tag{5.6.5}$$

It is worth remarking that if we add the trivial actions $(1, 1, 1, 0, 0, 0, 0, 0, 0)$, $(0, 0, 0, 1, 1, 1, 0, 0, 0)$, and $(0, 0, 0, 0, 0, 0, 1, 1, 1)$ then we have the subgroup generated by the condition that $\sum_i a_i = 0$.

We now will compute the Chen-Ruan cohomology of our orbifold. Here, we are looking at the complete intersection $Z(x_1^3 + x_2^3 + x_3, x_4^3 + x_5^3 + x_6^3, x_7^3 + x_8^3 + x_9^3) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. This is a product of three Fermat cubic curves, so we will write it as $F_1 \times F_2 \times F_3 \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. We now will look at the fixed loci of the elements $\tilde{G}$. It is important to note that the action of $(1, 2, 0)$ on one Fermat cubic curve is free, so that greatly reduces the number of fixed loci one will have. Namely, there are three types of fixed loci coming from three types of elements in the group $\tilde{G}$:

Case One: the trivial element,

Case Two: elements of the form $\alpha z + b_1 f_1 + b_2 f_2 + b_3 f_3$ where $\alpha \neq 0$ mod 3, and

Case Three: elements of the form $\alpha z + \beta g + b_1 f_1 + b_2 f_2 + b_3 f_3$ where $\beta \neq 0$ and one of the following are true:

(a) $\alpha \equiv \beta$ and $b_1 \equiv 0$,

(b) $\alpha + \beta \equiv 0$ and $b_2 \equiv 0$, or

(c) $\alpha \equiv 0$ and $b_3 \equiv 0$.

If a group element does not fit into any of these cases, then one can check that it will have no fixed loci (this is because it will have one copy of $\mathbb{P}^2$ being acted upon by an action (1,2,0) which will act on one of the cubic curves freely, hence on the entire manifold). We now define the following three subsets of $\mathbb{P}^2$ of three elements apiece:

$$\begin{aligned} S_1 &= \{(0:1:-\zeta^i):\zeta^3=1\}; \\ S_2 &= \{(1:0:-\zeta^i):\zeta^3=1\};\ \text{and} \\ S_3 &= \{(1:-\zeta^i:0):\zeta^3=1\}. \end{aligned} \tag{5.6.6}$$

Then if we look at these cases one by one, the fixed locus of Case One is $F_1 \times F_2 \times F_3$. In the second case, the fixed locus of an element $\alpha z + b_1 f_1 + b_2 f_2 + b_3 f_3$ is $S_{\alpha b_1 - 1} \times S_{\alpha b_2 - 1} \times S_{\alpha b_3 - 1}$ when one views the subindices of the sets S_i modulo 3. In the third case, we have three subcases. In Case 3.a, the fixed locus for an element $\alpha z + \alpha g + b_2 f_2 + b_3 f_3$ is exactly $F_1 \times S_{-\alpha b_2 + 1} \times S_{-\alpha b_3 + 1}$. In Case 3.b, the fixed locus for an element $\alpha z - \alpha g + b_1 f_1 + b_3 f_3$ is $S_{-\alpha b_1 + 1} \times F_2 \times S_{\alpha \beta + 1}$. Finally, in Case 3.c, the fixed locus for an element $\alpha g + b_1 f_1 + b_2 f_2$ is the set $S_{\alpha b_1 + 1} \times S_{-\alpha b_2 + 1} \times F_3$.

We now need to compute the $\tilde{G}$-invariant cohomology pieces of the fixed loci for each group element, i.e., $H^{p,q}(\mathcal{W}_{\tilde{g}})^{\tilde{G}}$ for all $\tilde{g} \in \tilde{G}$.

For the trivial element, we start with the cohomology generated by all the differential forms $dz_I \wedge d\bar{z}_I$ generating the diamond represented by

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 3 & & 3 & & \\
 & 3 & & 9 & & 3 & \\
1 & & 9 & & 9 & & 1 \\
 & 3 & & 9 & & 3 & \\
 & & 3 & & 3 & & \\
 & & & 1 & & &
\end{array}$$

This is because we can view the Fermat cubic F_i as the elliptic curve viewed as $\mathbb{C}$ with the equivalence relations $z_i \simeq z_i + 1 \simeq z_i + e^{2\pi i/3}$. Now, when we impose that actions of $\tilde{G}$ keep the differential forms $\tilde{G}$-invariant, then the only forms that are invariant are generated by 1, $dz_i \wedge d\bar{z}_i$, $dz_1 \wedge dz_2 \wedge dz_3$ and $d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$, hence the $\tilde{G}$-invariant Hodge diamond is represented by

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Now for case two, let us note that for any diagonal automorphism $(a, b, c) \in (\mathbb{Z}/3\mathbb{Z})^2$ which acts via $(x_1 : x_2 : x_3) \mapsto (\zeta^a x_1 : \zeta^b x_2 : \zeta^c x_3)$, the set S_i is invariant for all

i, though it may permute the elements inside (for example $(1, 2, 0)$ will permute the elements of each S_i). A fixed locus of an element that satisfies the hypotheses of Case 2 will be of the form $S_i \times S_j \times S_k$ which is a set of 27 points. Its Hodge diamond is hence just

$$27,$$

generated by the 0-forms y_i, one for each element of the set. However, we only want the $\tilde{G}$-invariant forms and we can see that via using the elements generated by the f_i that $\tilde{G}$ acts transitively on our set, hence the $\tilde{G}$-invariant 0-forms are generated by $\sum_i y_i$, hence our $\tilde{G}$ -invariant cohomology Hodge diamond is

$$1.$$

Via looking at the age shift, we can see that an element $\tilde{g}$ and its inverse $2\tilde{g}$ will add a factor of

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

to the Chen-Ruan orbifold Hodge diamond of $\mathcal{W}$. As there are precisely 54 such elements that satisfy Case Two, we get a total contribution to the Chen-Ruan orbifold cohomology of

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 27 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 27 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

Now, for Case 3, we will just do case 3.a and the other cases will work in the same manner. The fixed locus $F_1 \times S_i \times S_j$ is 9 disjoint copies of elliptic curves, hence the Hodge diamond is

$$\begin{array}{ccc}
 & 9 & \\
9 & & 9 \\
 & 9 &
\end{array}$$

We now discuss which elements of this are $\tilde{G}$-invariant. Denote the 0-forms of the nine elliptic curves by w_i and their (1,0)-forms (resp. (0,1)-forms) dw_i ($d\bar{w}_i$). We note that any 1-form is not $\tilde{G}$-invariant for the same reason as they weren't for the $\mathcal{Z}$-manifold. The only $\tilde{G}$-invariant pieces for the 0-forms (resp. 2-forms) are generated by $\sum_i w_i$ ($\sum_i dw_i \wedge d\bar{w}_i$). This means that the $\tilde{G}$-invariant Hodge diamond is just

$$\begin{array}{ccc}
 & 1 & \\
0 & & 0 \\
 & 1 &
\end{array}$$

There are exactly 54 group elements in the entirety of Case 3, so it will give a contribution to the Chen-Ruan orbifold cohomology of :

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

We now sum together and get the Hodge Diamond of $H^{p,q}_{\mathrm{CR}}(\mathcal{W}, \mathbb{C})$:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 84 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 84 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Recall the middle Hodge piece of the cubic sevenfold, $\mathcal{M}$, is

$$0 \quad 0 \quad 1 \quad 84 \quad 84 \quad 1 \quad 0 \quad 0$$

So the orbifolds $\mathcal{M}$ and $\mathcal{W}$ are mirror orbifolds in the sense of a flip of the middle 3-by-3 (primitive) Hodge diamond, just as generalized mirrors were interpreted in [14] (on the cohomological level).

But note that what we want to look at is the Hochschild homology of the category DGrB($\mathcal{M}$). Recall the following theorem:

Theorem 5.6.3 (Abbreviated Version of Theorem 3.11 of [37])**.** *Let X be the affine space $\mathbb{A}^N$ and let W be a homogeneous polynomial of degree d. Let $Y \subset \mathbb{P}^{N-1}$ be the hypersurface of degree d that is given by the equation $\{W = 0\}$. Then, there is a following relation between the triangulated category of graded B-branes and the derived category of coherent sheaves $D^b(coh(Y))$:*

1. *If $d < N$, i.e., if Y is Fano, then there is a semiorthogonal decomposition:*

$$D^b(coh(Y)) = \langle \mathcal{O}_Y(d - N + 1), \ldots, \mathcal{O}_Y, DGrB(W)\rangle.$$

2. *If $d = N$, i.e., if Y is a Calabi-Yau, then there is an equivalence*

$$DGrB(W) \xrightarrow{\sim} D^b(coh(Y)).$$

This requires if we want to compute the Hochschild Homology associated to our cubic sevenfold, we will need to "quotient out" by the chern characters in $\bigoplus_p H^{p,p}(\mathcal{W})$ associated to our sheaves $\mathcal{O}_{\mathcal{W}}, \ldots, \mathcal{O}_{\mathcal{W}}(-5)$.

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