THE CLASSIFICATION OF AUTOMORPHISM GROUPS OF RATIONAL ELLIPTIC SURFACES WITH SECTION

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A DISSERTATION
in
MATHEMATICS

Presented to the Faculties of the University of Pennsylvania
in
Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy
2011

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ABSTRACT

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In this dissertation, we give a classification of (regular) automorphism groups of relatively minimal rational elliptic surfaces with section over the field $\mathbb{C}$ which have non-constant J-maps. The automorphism group $Aut(B)$ of such a surface B is the semi-direct product of its Mordell-Weil group $MW(B)$ and the subgroup $Aut_\sigma(B)$ of the automorphisms preserving the zero section σ of the rational elliptic surface B. The configuration of singular fibers on the surface determines the Mordell-Weil group as has been shown by Oguiso and Shioda, and $Aut_\sigma(B)$ also depends on the singular fibers. The classification of automorphism groups in this dissertation gives the group $Aut_\sigma(B)$ in terms of the configuration of singular fibers on the surface. In general, $Aut_\sigma(B)$ is a finite group of order less than or equal to 24 which is a $\mathbb{Z}/2\mathbb{Z}$ extension of either $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, D_n (the Dihedral group of order $2n$) or A_4 (the Alternating group of order 12). The configuration of singular fibers does not determine the group $Aut_\sigma(B)$ uniquely; however we list explicitly all the possible groups $Aut_\sigma(B)$ and the configurations of singular fibers for which each group occurs.

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1 Introduction

The goal of this study is to give a description of the automorphisms, and the structure of the automorphism groups, of relatively minimal rational elliptic surfaces with section over the field $\mathbb{C}$. For such a surface B, $Aut(B)$ denotes the group of regular isomorphisms on B, or equivalently the group of biholomorphic maps on the complex surface B. Note that by $Aut(B)$ we do not mean the birational automorphism group of B.

The motivation for this study comes from the work of Bouchard and Donagi [2], *On a Class of Non-simply Connected Calabi-Yau 3folds.* In that paper, they obtain non-simply connected Calabi-Yau threefolds as the quotients of Schoen threefolds (which are the fibered products of two rational elliptic surfaces) by free actions of finite abelian groups. This is done by studying pairs of automorphisms of rational elliptic surfaces that induce a free action on the fibered product of the elliptic surfaces. The classification of such pairs of automorphisms is given in their paper, and the general question of classifying all automorphisms of rational elliptic surfaces is asked. Achieving this more general classification can extend the results to non-simply connected Calabi-Yau threefolds with finite non-abelian fundamental groups. Non-simply connected Calabi-Yau threefolds have a significance in string theory, particularly in the study of finding low energy limits of the theory.

Given a relatively minimal rational elliptic surface $\beta : B \to \mathbb{P}^1$ with section, the generic fiber is an elliptic curve, and there are singular fibers which are of one of the Kodaira types I_n ($n > 0$), I_n^* ($n \geq 0$), II, III, IV, II^*, III^* or IV^*(described in the next section). These singular fibers play an important role in determining the automorphism group $Aut(B)$ of the surface B. To study how $Aut(B)$ is affected by the singular fibers of B, we use the work of Persson [10] and Miranda [6] where they produce the list of all possible configurations of singular fibers on relatively minimal rational elliptic surfaces with section. The Mordell-Weil group $MW(B)$ of an elliptic surface B with section is the group of sections of the elliptic surface, and it can be identified with a subgroup of $Aut(B)$ by defining the action of a section as follows: If ζ is a section of B, then let $t_\zeta \in Aut(B)$ be the automorphism of B which acts on every smooth fiber F as the translation by $\zeta \cap F$ determined by the group law on the elliptic curve F. Oguiso and Shioda [9] have shown that $MW(B)$ is determined by the configuration of singular fibers on B for a relatively minimal rational elliptic surface with section, and they have calculated $MW(B)$ for each configuration of singular fibers. $MW(B)$ is a finitely generated abelian group of rank at most 8, and the order of its torsion subgroup is at most 9.

In this work, it is proved that $Aut(B)$ is the semi-direct product $MW(B) \rtimes Aut_\sigma(B)$ of the Mordell-Weil group of B and the subgroup $Aut_\sigma(B)$ of the automorphisms of B which preserve the zero section σ of B (Theorem 3.0.1). Together with the classification of $MW(B)$ by Oguiso and Shioda [9], the following theorem gives the classification of $Aut(B)$ for surfaces B which have non-constant J-maps.

Theorem 6.0.1: *Let B be a relatively minimal rational elliptic surface with section over the field $\mathbb{C}$. If the J-map of B is not constant, then Table 11 lists all the groups $Aut_\sigma(B)$ and all the possible configurations of singular fibers of B corresponding to each group.*

In general, $|Aut_\sigma(B)| = |Aut(B) : MW(B)| \leq 24$ if the J-map of the surface B is not constant. For the constant J-map case, this order is unbounded.

The outline of this work is as follows. After giving some basic information about elliptic surfaces in Section 2, we define the subgroup $Aut_\sigma(B)$ of automorphisms preserving the zero section σ of the elliptic surface $\beta : B \mapsto \mathbb{P}^1$ in Section 3, and give the homomorphisms $\phi : Aut(B) \mapsto Aut(\mathbb{P}^1)$ which gives the induced automorphisms group $Aut_B(\mathbb{P}^1)$ as its image; and $\psi : Aut(B) \mapsto Aut_\sigma(B)$ which maps every automorphism to an automorphism preserving the zero section by composing with a translation by a section. We prove in Theorem 3.0.1 that $Aut(B) = MW(B) \rtimes Aut_\sigma(B)$.

In Section 4, we first study the orders of induced automorphisms for an elliptic surface with a given configuration of singular fibers. Table 2 and 3 show the possible values of orders of induced automorphisms in terms of configurations of singular fibers (here, all the values shown in the tables may not occur). Also, we prove in Proposition 4.2.3 that if the J-map is not constant, $Aut_B(\mathbb{P}^1)$ can be one of the groups $\mathbb{Z}/n\mathbb{Z}$ ($1 \leq n \leq 12$, $n \neq 11$), $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_3, D_4, D_6 or A_4; and give the list of the configurations of singular fibers for which these groups may arise (again this is not an existence result).

In Section 5, we show in Lemma 5.0.4 that if the J-map is not constant, $Aut_\sigma(B)$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of $Aut_B(\mathbb{P}^1)$. Then, if the order of an induced automorphism is n, this automorphism is induced by either an order n automorphism of the surface B, or an order $2n$ automorphism. In the subsections Construction 1 and Construction 2, we examine these two situations and give the tables (Table 6 and 9) of the configurations of singular fibers for which such automorphisms actually exist for specific orders of the automorphisms. In the the last subsection, we show the existence of the non-cyclic $Aut_B(\mathbb{P}^1)$ groups following a generalization of the thechniques used in the subsection Construction 2.

In Section 6, we collect together all the results obtained in the previous sections and give the full classification in Table 11.

2 Preliminaries

In this section, we give some basic information about elliptic surfaces relevant to the following sections. The reader can consult the book [8] for more details.

An elliptic surface is a smooth algebraic surface B together with a map $\beta : B \to C$ to a smooth algebraic curve C such that the generic fiber is a smooth genus 1 curve. If there is a section of the map β, then every smooth genus 1 fiber has a marked point, hence it is an elliptic curve. An elliptic surface is *relatively minimal* if it has no smooth rational (-1) curves in the fibers. By blowing down such (-1) curves, every elliptic surface can be transformed to a birational relatively minimal elliptic surface. A rational elliptic surface is an elliptic surface birational to $\mathbb{P}^2$. A rational elliptic surface with section is necessarily fibered over the base curve $C = \mathbb{P}^1$. A relatively minimal rational elliptic surface with section is the blow-up of $\mathbb{P}^2$ at the 9 base points of a pencil of generically smooth cubics.

Singular Fibers of Elliptic Surfaces : While the generic fiber is a smooth genus 1 curve, elliptic surfaces usually have singular fibers. Kodaira [5] has shown that fibers of a relatively minimal elliptic surface are of one of the following types:

Name	Description
I_0	Smooth genus 1 curve
I_1	Nodal rational curve
I_2	2 copies of $\mathbb{P}^1$ meeting at 2 distinct points transversally
I_3	3 copies of $\mathbb{P}^1$ meeting at 3 distinct points with dual graph $\bar{A}_2$
I_n , $(n \geq 4)$	n copies of $\mathbb{P}^1$ meeting in a cycle, i.e. meeting with dual graph $\bar{A}_{n-1}$
${}_mI_n$, $(n \geq 0)$	Multiple fiber, I_n with multiplicity m
I_n^* , $n \geq 0$	$n+5$ copies of $\mathbb{P}^1$ meeting with dual graph $\bar{D}_{n+4}$
II	Cuspidal rational curve
III	2 copies of $\mathbb{P}^1$ meeting at a single point to order 2
IV	3 copies of $\mathbb{P}^1$ all meeting at a single point
II^*	9 copies of $\mathbb{P}^1$ meeting with dual graph $\bar{E}_8$
III^*	8 copies of $\mathbb{P}^1$ meeting with dual graph $\bar{E}_7$
IV^*	7 copies of $\mathbb{P}^1$ meeting with dual graph $\bar{E}_6$

The graphs referred to in the above descriptions are the extended Dynkin diagrams given below. Each graph describes a singular fiber where the singular fiber has a $\mathbb{P}^1$ component with self intersection (-2) corresponding to each vertex of the graph, and two components have intersection number k if there are k edges between the corresponding vertices. A singular fiber is a divisor of the elliptic surface and the multiplicities of each component are denoted next to the graphs below.

Note that if an elliptic surface has a section, then the intersection number of the section with each fiber is 1, hence elliptic surfaces with section do not have singular fibers of type ${}_mI_n$ for $m > 1$.

Name	Graph	Multiplicities
$\bar{A}_{n-1}$	A cycle of n vertices	1 for each vertex
$\bar{D}_n$	$^{\circ}_{\circ}\!>\circ-\circ-\cdots-\circ<^{\circ}_{\circ}$ $n+1$ vertices	$^{1}_{1}\!>2-2-\cdots-2<^{1}_{1}$
$\bar{E}_6$	$^{\circ-\circ}_{\circ-\circ}\!>\circ-\circ-\circ$	$^{1-2}_{1-2}\!>3-2-1$
$\bar{E}_7$	$\circ-\circ-\circ-\underset{\overset{\vert}{\circ}}{\circ}-\circ-\circ-\circ$	$1-2-3-\underset{\overset{\vert}{2}}{4}-3-2-1$
$\bar{E}_8$	$\circ-\circ-\underset{\overset{\vert}{\circ}}{\circ}-\circ-\circ-\circ-\circ-\circ$	$2-4-\underset{\overset{\vert}{3}}{6}-5-4-3-2-1$

Configurations of Singular Fibers : Since the Euler characteristic of a genus 1 curve is zero, an Euler characteristic argument shows that for any elliptic surface B

$$\sum_{S \text{ singular fiber}} \chi(S) = \chi(B)$$

But this equation does not suffice to determine the configurations of singular fibers of elliptic surfaces. Persson [10] and Miranda [6] have shown that if B is a relatively minimal rational elliptic surface with section, then the configuration of singular fibers of B is one of those shown in Table 1 below.

Notation and Ordering : The notation $IVII^2I_2I_1^2$ indicates that there is one singular fiber of type IV, one of type I_2, two of type II and two of type I_1. The exponents in the notation are the numbers of each type of singular fiber in the configuration. When writing configurations and ordering them in the lists and tables below, a decreasing lexicographic order is followed according to the following:

$$S^{n+1} > S^n \text{ for any fiber } S\text{, and} \quad T > S^n \quad \text{if} \quad T > S$$

and

$$I_{n+1}^* > I_n^* > IV^* > III^* > II^* > IV > III > II > I_{n+1} > I_n.$$

The J-map : Given a relatively minimal rational elliptic surface $\beta : B \to \mathbb{P}^1$ with section; and fixing a section σ of B, the *Weierstrass fibration* B' of B is obtained by collapsing all the components of singular fibers which do not intersect the section σ. B' may be a singular surface, its fibers are either elliptic curves, cuspidal rational curves or nodal rational curves. There are two sections

D and E of the line bundles $\mathcal{O}_{\mathbb{P}^1}(4H)$ and $\mathcal{O}_{\mathbb{P}^1}(6H)$, respectively (where H is the hyperplane), such that B' is a divisor on the $\mathbb{P}^2$ bundle $P(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2H) \oplus \mathcal{O}_{\mathbb{P}^1}(-3H))$ given by the *Weierstrass equation*

$$Y^2 = X^3 + DX + E.$$

Then

$$J = \frac{4D^3}{4D^3 + 27E^2}.$$

defines a meromorphic map on $\mathbb{P}^1$. If the fiber over $z \in \mathbb{P}^1$ is a smooth elliptic curve, then $J(z)$ is the j-invariant of that elliptic curve. Note that there is a singular fiber over z if the section $4D^3 + 27E^2$ vanishes at z. Since this is a section of a line bundle with degree 12, there are at most 12 singular fibers. The type of the singular fiber of B can be determined by the orders of vanishing of the sections D, E and $\Delta = 4D^3 + 27E^2$, which is known as Tat's algorithm (p.41 in [8]).

Double Cover of F_2 : If the Weierstrass fibration B' is given by $Y^2 = X^3 + DX + E$ in the $\mathbb{P}^2$ bundle over $\mathbb{P}^1$ as above, then $(X, Y) \mapsto X$ is an involution on B', which maps it to the rational ruled surface $P(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2H)) = F_2$ as a double cover branched over the minimal section of the F_2 and a trisection T given by the equation $X^3 + DX + E = 0$ in F_2.

Mordell-Weil Group : The set of sections of an elliptic surface B with section is a group called the Mordell-Weil group of B, and denoted by $MW(B)$. If ζ_1 and ζ_2 are two sections, using the group law on each smooth fiber, which is an elliptic curve, one can define $\zeta_1 + \zeta_2$ over the open set of the base curve corresponding to smooth fibers. Then by taking the closure in B, this can be extended uniquely to a section of B. For a relatively minimal rational elliptic surface B with section, Oguiso and Shioda [9] have shown that $MW(B)$ (even its lattice structure) is determined by the configuration of singular fibers on the surface, and they have listed the lattice structure of $MW(B)$ corresponding to each possible configuration. Their results show that $MW(B)$ is a finitely generated abelian group of rank at most 8 and torsion group of size at most 9.

$deg(J)$	Configuration of Singular Fibers
0	$I_0^* I_0^*$, $I_0^* IV II$, $I_0^* III^2$, $I_0^* II^3$, $IV^* IV$, $IV^* II^2$, $III^* III$, $II^* II$, IV^3 , $IV^2 II^2$, $IV II^4$, III^4 , II^6
1	$I_1^* III\ II$, $I_0^* III\ II\ I_1$, $IV^* III\ I_1$, $III^* II\ I_1$
2	$I_2^* II^2$, $I_1^* IV I_1$, $I_1^* II^2 I_1$, $I_0^* IV I_1^2$, $I_0^* II^2 I_2$, $I_0^* II^2 I_1^2$, $IV^* II\ I_2$, $IV^* II\ I_1^2$, $II^* I_1^2$, $IV III^2 I_2$, $IV III^2 I_1^2$, $III^2 II^2 I_2$, $III^2 II^2 I_1^2$
3	$I_2^* III\ I_1$, $I_1^* III\ I_2$, $I_1^* III\ I_1^2$, $I_0^* III\ I_2 I_1$, $I_0^* III\ I_1^3$, $III^* I_2 I_1$, $III^* I_1^3$, $IV III\ II\ I_3$, $IV III\ II\ I_2 I_1$, $IV III\ II\ I_1^3$, $III^3 I_3$, $III^3 I_2 I_1$, $III^3 I_1^3$, $III\ II^3 I_3$, $III\ II^3 I_2 I_1$, $III\ II^3 I_1^3$

$deg(J)$	Configuration of Singular Fibers
4	$I_3^*II\ I_1$, $I_2^*II\ I_1^2$, $I_1^*II\ I_3$, $I_1^*II\ I_2I_1$, $I_1^*II\ I_1^3$, $I_0^*II\ I_3I_1$, $I_0^*II\ I_2I_1^2$, $I_0^*II\ I_1^4$, $IV^*I_3I_1$, $IV^*I_2I_1^2$, $IV^*I_1^4$, $IV^2I_2^2$, $IV^2I_2I_1^2$, $IV^2I_1^4$, $IVII^2I_4$, $IVII^2I_3I_1$, $IVII^2I_2^2$, $IVII^2I_2I_1^2$, $IVII^2I_1^4$, $III^2II\ I_4$, $III^2II\ I_3I_1$, $III^2II\ I_2^2$, $III^2II\ I_2I_1^2$, $III^2II\ I_1^4$, II^4I_4 , $II^4I_3I_1$, $II^4I_2^2$, $II^4I_2I_1^2$, $II^4I_1^4$
5	$IVIII\ I_4I_1$, $IVIII\ I_3I_2$, $IVIII\ I_3I_1^2$, $IVIII\ I_2^2I_1$, $IVIII\ I_2I_1^3$, $IVIII\ I_1^5$, $III\ II^2I_5$, $III\ II^2I_4I_1$, $III\ II^2I_3I_2$, $III\ II^2I_3I_1^2$, $III\ II^2I_2^2I_1$, $III\ II^2I_2I_1^3$, $III\ II^2I_1^5$
6	$I_4^*I_1^2$, $I_3^*I_1^3$, $I_2^*I_2^2$, $I_2^*I_2I_1^2$, $I_2^*I_1^4$, $I_1^*I_4I_1$, $I_1^*I_3I_1^2$, $I_1^*I_2^2I_1$, $I_1^*I_2I_1^3$, $I_1^*I_1^5$, $I_0^*I_4I_1^2$, $I_0^*I_3I_1^3$, $I_0^*I_2^3$, $I_0^*I_2^2I_1^2$, $I_0^*I_2I_1^4$, $I_0^*I_1^6$, $IVII\ I_5I_1$, $IVII\ I_4I_2$, $IVII\ I_4I_1^2$, $IVII\ I_3I_2I_1$, $IVII\ I_3I_1^3$, $IVII\ I_2^3$, $IVII\ I_2^2I_1^2$, $IVII\ I_2I_1^4$, $IVII\ I_1^6$, $III^2I_5I_1$, $III^2I_4I_2$, $III^2I_4I_1^2$, $III^2I_3^2$, $III^2I_3I_2I_1$, $III^2I_3I_1^3$, $III^2I_2^3$, $III^2I_2^2I_1^2$, $III^2I_2I_1^4$, $III^2I_1^6$, II^3I_6 , $II^3I_5I_1$, $II^3I_4I_2$, $II^3I_4I_1^2$, $II^3I_3^2$, $II^3I_3I_2I_1$, $II^3I_3I_1^3$, $II^3I_2^3$, $II^3I_2^2I_1^2$, $II^3I_2I_1^4$, $II^3I_1^6$
7	$III\ II\ I_6I_1$, $III\ II\ I_5I_2$, $III\ II\ I_5I_1^2$, $III\ II\ I_4I_3$, $III\ II\ I_4I_2I_1$, $III\ II\ I_4I_1^3$, $III\ II\ I_3^2I_1$, $III\ II\ I_3I_2^2$, $III\ II\ I_3I_2I_1^2$, $III\ II\ I_3I_1^4$, $III\ II\ I_2^3I_1$, $III\ II\ I_2^2I_1^3$, $III\ II\ I_2I_1^5$, $III\ II\ I_1^7$
8	$IVI_6I_1^2$, $IVI_5I_2I_1$, $IVI_5I_1^3$, $IVI_4I_2I_1^2$, $IVI_4I_1^4$, $IVI_3^2I_2$, $IVI_3^2I_1^2$, $IVI_3I_2^2I_1$, $IVI_3I_2I_1^3$, $IVI_3I_1^5$, $IVI_2^3I_1^2$, $IVI_2^2I_1^4$, $IVI_2I_1^6$, IVI_1^8 , $II^2I_7I_1$, $II^2I_6I_2$, $II^2I_6I_1^2$, $II^2I_5I_2I_1$, $II^2I_5I_1^3$, $II^2I_4^2$, $II^2I_4I_3I_1$, $II^2I_4I_2^2$, $II^2I_4I_2I_1^2$, $II^2I_4I_1^4$, $II^2I_3^2I_2$, $II^2I_3^2I_1^2$, $II^2I_3I_2^2I_1$, $II^2I_3I_2I_1^3$, $II^2I_3I_1^5$, $II^2I_2^4$, $II^2I_2^3I_1^2$, $II^2I_2^2I_1^4$, $II^2I_2I_1^6$, $II^2I_1^8$
9	$III\ I_7I_1^2$, $III\ I_6I_2I_1$, $III\ I_6I_1^3$, $III\ I_5I_3I_1$, $III\ I_5I_2I_1^2$, $III\ I_5I_1^4$, $III\ I_4I_3I_2$, $III\ I_4I_3I_1^2$, $III\ I_4I_2^2I_1$, $III\ I_4I_2I_1^3$, $III\ I_4I_1^5$, $III\ I_3^2I_2I_1$, $III\ I_3^2I_1^3$, $III\ I_3I_2^3$, $III\ I_3I_2^2I_1^2$, $III\ I_3I_2I_1^4$, $III\ I_3I_1^6$, $III\ I_2^4I_1$, $III\ I_2^3I_1^3$, $III\ I_2^2I_1^5$, $III\ I_2I_1^7$, $III\ I_1^9$
10	$II\ I_8I_1^2$, $II\ I_7I_2I_1$, $II\ I_7I_1^3$, $II\ I_6I_2I_1^2$, $II\ I_6I_1^4$, $II\ I_5I_4I_1$, $II\ I_5I_3I_2$, $II\ I_5I_3I_1^2$, $II\ I_5I_2^2I_1$, $II\ I_5I_2I_1^3$, $II\ I_5I_1^5$, $II\ I_4^2I_1^2$, $II\ I_4I_3I_2I_1$, $II\ I_4I_3I_1^3$, $II\ I_4I_2^2I_1^2$, $II\ I_4I_2I_1^4$, $II\ I_4I_1^6$, $II\ I_3^2I_2I_1$, $II\ I_3^2I_1^3$, $II\ I_3I_2^3I_1$, $II\ I_3I_2^2I_1^3$, $II\ I_3I_2I_1^5$, $II\ I_3I_1^7$, $II\ I_2^4I_1^2$, $II\ I_2^3I_1^4$, $II\ I_2^2I_1^6$, $II\ I_2I_1^8$, $II\ I_1^{10}$
12	$I_9I_1^3$, $I_8I_2I_1^2$, $I_8I_1^4$, $I_7I_2I_1^3$, $I_7I_1^5$, $I_6I_3I_2I_1$, $I_6I_3I_1^3$, $I_6I_2^2I_1^2$, $I_6I_2I_1^4$, $I_6I_1^6$, $I_5^2I_1^2$, $I_5I_4I_1^3$, $I_5I_3I_2I_1^2$, $I_5I_3I_1^4$, $I_5I_2^2I_1^3$, $I_5I_2I_1^5$, $I_5I_1^7$, $I_4^2I_2^2$, $I_4^2I_2I_1^2$, $I_4^2I_1^4$, $I_4I_3I_2^2I_1$, $I_4I_3I_2I_1^3$, $I_4I_3I_1^5$, $I_4I_2^4$, $I_4I_2^3I_1^2$, $I_4I_2^2I_1^4$, $I_4I_2I_1^6$, $I_4I_1^8$, I_3^4 , $I_3^3I_2I_1$, $I_3^3I_1^3$, $I_3^2I_2^2I_1^2$, $I_3^2I_2I_1^4$, $I_3^2I_1^6$, $I_3I_2^4I_1$, $I_3I_2^3I_1^3$, $I_3I_2^2I_1^5$, $I_3I_2I_1^7$, $I_3I_1^9$, I_2^6 , $I_2^5I_1^2$, $I_2^4I_1^4$, $I_2^3I_1^6$, $I_2^2I_1^8$, $I_2I_1^{10}$, I_1^{12}

Table 1: Persson's list of configurations of singular fibers for relatively minimal rational elliptic surfaces with section [6].

3 Automorphisms of Rational Elliptic Surfaces

In this section, we examine the structure of the group $Aut(B)$ of the automorphisms of the relatively minimal rational elliptic surface B with section. The groups $Aut_B(\mathbb{P}^1)$ of induced automorphisms on the base curve $\mathbb{P}^1$ and the group $Aut_\sigma(B)$ of the automorphisms preserving the zero section of B are defined together with the group homomorphisms ϕ and ψ from $Aut(B)$ onto each of these groups, respectively. Theorem 3.0.1 shows that $Aut(B) = MW(B) \rtimes Aut_\sigma(B)$.

Let $\beta : B \to \mathbb{P}^1$ be a relatively minimal rational elliptic surface with section and $\tau : B \to B$ be an automorphism of B (a biholomorphic map). The canonical divisor class K_B of B is preserved by every automorphism. The canonical class formula for elliptic surfaces gives $K_B = -F$ where F is the class of the fiber of the map β for *rational* elliptic surfaces B. Furthermore, $|F|$ is a pencil, hence τ maps every fiber of β to a fiber. This shows that every automorphism τ of B induces an automorphism τ_B on $\mathbb{P}^1$ making the following diagram commute:

$$\begin{array}{ccc} B & \xrightarrow{\tau} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbb{P}^1 & \xrightarrow[\tau_{\mathbb{P}^1}]{} & \mathbb{P}^1 \end{array}$$

This gives a group homomorphism:

$$\phi \, : \, Aut(B) \to Aut(\mathbb{P}^1) \, , \; \tau \mapsto \tau_{\mathbb{P}^1}. \tag{1}$$

We denote the group of induced automorphisms on $\mathbb{P}^1$ by

$$Aut_B(\mathbb{P}^1) = \phi(Aut(B)). \tag{2}$$

The sections of the map $\beta : B \to \mathbb{P}^1$ form an abelian group called the Mordell-Weil group of B, and denoted by $MW(B)$. If σ_1 and σ_2 are two sections $\sigma_1 + \sigma_2$ in $MW(B)$ is obtained by performing the group law in every smooth fiber of β, which is an elliptic curve. The intersection of $\sigma_1 + \sigma_2$ with singular fibers is determined by taking the closure of what is obtained in smooth fibers. Each element ζ of $MW(B)$ can be identified with an automorphism t_ζ of B, namely *the translation by the section* ζ, which acts on every smooth fiber as the translation in the elliptic curve by the intersection of the section with the elliptic curve. With this identification the Mordell-Weil group of B embeds in the automorphism group of B as the group of translations by sections. We will see $MW(B)$ as a subgroup of $Aut(B)$.

$$MW(B) \hookrightarrow Aut(B) \, , \, \zeta \mapsto t_\zeta. \tag{3}$$

t_ζ induces the identity $\mathbb{I}_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

$$\phi(t_\zeta) = \mathbb{I}_{\mathbb{P}^1}. \tag{4}$$

For every rational elliptic surface B with section, there is an automorphism $-\mathbb{I} \in Aut(B)$ which acts on every smooth fiber as the inversion of the group law. This automorphism also induces the identity on $\mathbb{P}^1$.

$$\phi(-\mathbb{I}) = \mathbb{I}_{\mathbb{P}^1}. \tag{5}$$

$-\mathbb{I}$ is an involution.

$$(-\mathbb{I})^2 = (-\mathbb{I}) \circ (-\mathbb{I}) = \mathbb{I} \tag{6}$$

where $\mathbb{I}$ denotes the identity map on B.

We define a subgroup of $Aut(B)$ by:

$$Aut_\sigma(B) = \{\tau \in Aut(B) | \tau(\sigma) = \sigma\} \tag{7}$$

where σ is the zero section of B, the zero of $MW(B)$. $Aut_\sigma(B)$ is the group of automorphisms preserving the zero section of B. We can define a map

$$\begin{aligned} \psi \,:\, Aut(B) &\to Aut_\sigma(B) \\ \tau &\mapsto \alpha = t_{-\tau(\sigma)} \circ \tau \end{aligned}. \tag{8}$$

Composing τ with the translation by the inverse of the section $\tau(\sigma)$ gives an automorphism which maps the zero section σ to itself as a set. $\alpha = \psi(\tau)$ is called the *linearization* of τ.

$\alpha = \psi(\tau)$ preserves the zero section σ, thus when restricted to a smooth fiber it maps the zero of this elliptic curve to the zero of another elliptic curve, hence is a group isomorphism between the elliptic curves. Then for any section ζ we have;

$$\alpha \circ t_\zeta = t_{\alpha(\zeta)} \circ \alpha. \tag{9}$$

Using this we can show that ψ is a group homomorphism since

$$\begin{aligned} \psi(\tau_1) \circ \psi(\tau_2) &= \psi(\tau_1) \circ t_{-\tau_2(\sigma)} \circ \tau_2 = t_{-\psi(\tau_1)(\tau_2(\sigma))} \circ \psi(\tau_1) \circ \tau_2 \\ &= t_{-(-\tau_1(\sigma)+\tau_1(\tau_2(\sigma)))} \circ t_{-\tau_1(\sigma)} \circ \tau_1 \circ \tau_2 = t_{-\tau_1(\tau_2(\sigma))} \circ \tau_1 \circ \tau_2 \\ &= \psi(\tau_1 \circ \tau_2). \end{aligned} \tag{10}$$

The kernel of this group homomorphism ψ consists of the automorphisms τ such that $\psi(\tau) = t_{-\tau(\sigma)} \circ \tau = \mathbb{I}$, thus $\tau = t_{\tau(\sigma)}$. Then the kernel is the group of translations by sections

$$Ker(\psi) = MW(B). \tag{11}$$

This gives the following short exact sequence

$$1 \to MW(B) \hookrightarrow Aut(B) \xrightarrow{\psi} Aut_\sigma(B) \to 1. \tag{12}$$

Theorem 3.0.1. *The automorphism group of a relatively minimal rational elliptic surface B with section is isomorphic to the semi-direct product of the Mordell-Weil group of B and the subgroup of automorphisms preserving the zero section.*

$$Aut(B) = MW(B) \rtimes Aut_\sigma(B).$$

The action of $Aut_\sigma(B)$ on MW(B) is given by:

$$\alpha \cdot t_\zeta = t_{\alpha(\zeta)} \textit{ for all } \alpha \in Aut_\sigma(B), \zeta \in MW(B)$$

so that the group operation in the semi-direct product is:

$$\begin{aligned}(t_{\zeta_1} \circ \alpha_1)(t_{\zeta_2} \circ \alpha_2) &= ((t_{\zeta_1} \circ (\alpha_1 \cdot t_{\zeta_2})) \circ (\alpha_1 \circ \alpha_2)) \\ &= (t_{\zeta_1 + \alpha_1(\zeta_2)} \circ (\alpha_1 \circ \alpha_2)).\end{aligned}$$

Proof. Both $MW(B)$ and $Aut_\sigma(B)$ are subgroups of $Aut(B)$, hence the short exact sequence above gives the semi-direct product statement. For the action we have

$$\alpha \cdot t_\zeta = \alpha \circ t_\zeta \circ \alpha^{-1} = t_{\alpha(\zeta)} \circ \alpha \circ \alpha^{-1} = t_{\alpha(\zeta)}.$$

□

The Mordell-Weil groups of relatively minimal rational elliptic surfaces with section have been completely classified in terms of the configurations of singular fibers on the surfaces by Oguiso and Shioda [9]. Our aim in this dissertation is to give the other component of the semi-direct product structure of $Aut(B)$, namely the subgroup $Aut_\sigma(B)$.

Unless stated otherwise, elliptic surfaces are assumed relatively minimal and with section in the following sections.

4 Induced Automorphisms on $\mathbb{P}^1$

In this section, we first study the orders of automorphisms induced on the base curve $\mathbb{P}^1$ by the automorphisms of the rational elliptic surface B. The points on $\mathbb{P}^1$ corresponding to the singular fibers of B are special points which bring useful combinatorial criteria on possible orders of induced automorphisms. Using these criteria, we present the possible orders of induced automorphisms (without proving existence) for each configuration of singular fibers in Table 2 and 3. In the subsection 4.2, we examine the group $Aut_B(\mathbb{P}^1)$. We show in Proposition 4.2.3 that if the J-map is not constant, $Aut_B(\mathbb{P}^1)$ is one of the groups $\mathbb{Z}/n\mathbb{Z}$ ($n \leq 12$, $n \neq 11$), $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_3, D_4, D_6 or A_4, and we give the lists of the configurations of singular fibers corresponding to the non-cyclic ones.

4.1 Orders of induced automorphisms

Lemma 4.1.1. *For every automorphism $\tau \in Aut(B)$, the following diagram commutes.*

$$\begin{array}{ccc} B & \xrightarrow{\tau} & B \\ \downarrow \beta & & \downarrow \beta \\ \mathbb{P}^1 & \xrightarrow{\tau_{\mathbb{P}^1}} & \mathbb{P}^1 \\ \Big\downarrow J & J \swarrow & \\ \mathbb{P}^1 & & \end{array}$$

Proof. As discussed before, τ maps the fibers of β to fibers. τ has an inverse. When restricted to a fiber, τ becomes an invertible map between two fibers. Then the singular fibers map to singular fibers of the same type (II maps to a II fiber, I_n maps to an I_n etc.) since each type has a different homeomorphism class. On smooth fibers τ is biholomorphic, hence the J-values of the elliptic curves permuted by τ are the same. There is only one J-value that a singular fiber a given type can take except for the singular fiber type I_0^* which can take any J-value. But B can have at most two I_0^* fibers, in which case the J-map is constant. In any case, the J-values of the fibers which are mapped to each other by τ are the same. □

Proposition 4.1.2. *If $\tau_{\mathbb{P}^1} = \phi(\tau)$ is the automorphism induced on $\mathbb{P}^1$ by $\tau \in Aut(B)$, $n = ord(\tau_{\mathbb{P}^1})$ and $d = deg(J) > 0$, then n is finite and $n|d$.*

Proof. $\tau_{\mathbb{P}^1}$ permutes the points in $J^{-1}(x)$ for all x by the commutativity of the above diagram. Pick three distinct points x. Some finite power of $\tau_{\mathbb{P}^1}$ will fix every point in $J^{-1}(x)$ for these three values of x, hence will fix at least three points in $\mathbb{P}^1$, then it is the identity. Therefore, $n = ord(\tau_{\mathbb{P}^1})$ is finite. Then $\tau_{\mathbb{P}^1}$ fixes two points of $\mathbb{P}^1$ and all other orbits have n points. For generic x, $J^{-1}(x)$ consists of d distinct points and the permutation of these d points by $\tau_{\mathbb{P}^1}$ is by n-cycles. Thus, $n|d$. □

If we know the configuration of singular fibers on B, then using the above lemma and proposition, we can list the possible values of $n = ord(\tau_{\mathbb{P}^1})$. Below, we give a table of the configurations of singular fibers and the possible values of n which is the order of an automorphism induced on $\mathbb{P}^1$ by an automorphism of an elliptic surface B with this configuration of singular fibers. The criteria we use for determining these possible values of n are as follows:

• The singular fibers of the same type are permuted by τ, so the points in $\mathbb{P}^1$ corresponding to those singular fibers are permuted by $\tau_{\mathbb{P}^1}$.

• If i_k and i_k^* are the numbers of the I_k and I_k^* fibers on B, then $deg(J) = d = \Sigma k(i_k + i_k^*)$ and $n|d$.

• If $n > 1$ then there are 2 fixed points of $\tau_{\mathbb{P}^1}$. Then after a change of coordinates, $\tau_{\mathbb{P}^1}$ is the rotation of order n on $\mathbb{P}^1$, so all the orbits except for the fixed points must have n points.

• The multiplicity of the J-map is the same on x and $\tau_{\mathbb{P}^1}(x)$ for all x.

• If there are three distinct singular fiber types each appearing only once in the configuration of singular fibers of the surface B, then $n = 1$ since an automorphism of $\mathbb{P}^1$ fixing three points must be the identity. For such B, we have $Aut_B(\mathbb{P}^1) = 0$.

Note that all possible configurations of singular fibers have been classified by Persson [10] and Miranda [6]. In Table 2 below, we only include the configurations for which $n > 1$ is possible subject to the above criteria. This table does not give the *existing* values of n, but just the candidate values of n. Those are the numbers we get by applying only the criteria described above.

Table 3 lists all the configurations of singular fibers for which n must be 1 subject to the above criteria. For elliptic surfaces B with such configurations $Aut_B(\mathbb{P}^1) = 0$.

d	n	Configuration of Singular Fibers
0	-	II^*II , III^*III , IV^*IV , $I_0^*I_0^*$
	2	$I_0^*III\,III$, $IV^*II\,II$, $IV\,IV\,II\,II$
	2 or 3	$I_0^*II\,II\,II$, $IV\,IV\,IV$
	2,3 or 4	$IV\,II\,II\,II\,II$, $III\,III\,III\,III$
	2,3,4,5 or 6	$II\,II\,II\,II\,II\,II$
2	2	$I_2^*II^2$, $I_0^*IV\,I_1^2$, $I_0^*II^2I_2$, $I_0^*II^2I_1^2$, $II^*I_1^2$, $IV\,III^2I_2$, $IV\,III^2I_1^2$, $III^2II^2I_2$, $III^2II^2I_1^2$
3	3	$I_0^*III\,I_1^3$, $III^*I_1^3$, III^3I_3, $III\,II^3I_1^3$, $III\,II^3I_3$, $III^3I_1^3$
4	2	$I_2^*II\,I_1^2$, $IV^*I_2I_1^2$, $IV^2I_2^2$, $IV^2I_2I_1^2$, $IV^2I_1^4$, $IV\,II^2I_4$, $IV\,II^2I_2^2$, $IV\,II^2I_2I_1^2$, $IV\,II^2I_1^4$, $III^2II\,I_2^2$, $III^2II\,I_1^4$, $II^4I_3I_1$, $II^4I_2^2$, $II^4I_2I_1^2$
	2 or 4	$I_0^*II\,I_1^4$, $IV^*I_1^4$, II^4I_4 , $II^4I_1^4$
5	5	$IV\,III\,I_1^5$
6	2	$I_4^*I_1^2$, $I_2^*I_2^2$, $I_2^*I_1^4$, $I_0^*I_4I_1^2$, $I_0^*I_2^2I_1^2$, $I_0^*I_2I_1^4$, $III^2I_5I_1$, $III^2I_4I_2$, $III^2I_4I_1^2$, $III^2I_3^2$, $III^2I_3I_1^3$, $III^2I_2^3$, $III^2I_2^2I_1^2$, $III^2I_2I_1^4$, $III^2I_1^6$, $II^3I_3^2$, $II^3I_2^2I_1^2$
	3	$I_3^*I_1^3$, $I_0^*I_3I_1^3$, $IV\,II\,I_2^3$, $IV\,II\,I_1^6$, II^3I_6, $II^3I_3I_1^3$, $II^3I_2^3$
	2 or 3	$I_0^*I_2^3$, $II^3I_1^6$
	2,3 or 6	$I_0^*I_1^6$
7	7	$III\,II\,I_1^7$
8	2	$IV\,I_6I_1^2$, $IV\,I_3^2I_2$, $IV\,I_3^2I_1^2$, $IV\,I_2^3I_1^2$, $IV\,I_2^2I_1^4$, $IV\,I_2I_1^6$, $II^2I_7I_1$,$II^2I_6I_2$, $II^2I_6I_1^2$, $II^2I_5I_1^3$, $II^2I_4^2$, $II^2I_4I_2^2$, $II^2I_4I_2I_1^2$, $II^2I_4I_1^4$, $II^2I_3^2I_2$, $II^2I_3^2I_1^2$, $II^2I_3I_2^2I_1$, $II^2I_3I_1^5$, $II^2I_2^3I_1^2$, $II^2I_2^2I_1^4$, $II^2I_2I_1^6$
	2 or 4	$IV\,I_4I_1^4$, $II^2I_2^4$, $II^2I_1^8$
	2,4 or 8	$IV\,I_1^8$
9	3	$III\,I_6I_1^3$, $III\,I_3I_2^3$, $III\,I_3I_1^6$, $III\,I_2^3I_1^3$
	3 or 9	$III\,I_1^9$
10	2	$II\,I_4^2I_1^2$, $II\,I_3^2I_2^2$, $II\,I_3^2I_1^4$, $II\,I_2^4I_1^2$, $II\,I_2^2I_1^6$
	5	$II\,I_5I_1^5$
	2,5 or 10	$II\,I_1^{10}$
12	2	$I_8I_2I_1^2$, $I_7I_1^5$, $I_6I_2^2I_1^2$, $I_6I_2I_1^4$, $I_5^2I_1^2$, $I_5I_3I_1^4$, $I_5I_2^2I_1^3$, $I_5I_1^7$, $I_4^2I_2^2$, $I_4^2I_2I_1^2$, $I_4^2I_1^4$, $I_4I_2^3I_1^2$, $I_4I_2^2I_1^4$, $I_4I_2I_1^6$, $I_3^2I_2^2I_1^2$, $I_3^2I_2I_1^4$, $I_3^2I_1^6$, $I_3I_2^4I_1$, $I_3I_2^2I_1^5$, $I_2^5I_1^2$, $I_2^2I_1^8$, $I_2I_1^{10}$
	3	$I_3I_2^3I_1^3$
	2 or 3	$I_9I_1^3$, $I_3^3I_1^3$, $I_3I_1^9$, $I_2^3I_1^6$
	2 or 4	$I_8I_1^4$, $I_4I_2^4$, $I_4I_1^8$, $I_2^4I_1^4$
	2,3 or 4	I_3^4
	2,3 or 6	$I_6I_1^6$, I_2^6
	2,3,4,6 or 12	I_1^{12}

Table 2: The possible orders n of induced automorphisms on $\mathbb{P}^1$, $d = deg(J)$.

d	Configuration of Singular Fibers
1	$I_1^* III\, II$, $I_0^* III\, II\, I_1$, $IV^* III\, I_1$, $III^* II\, I_1$
2	$I_1^* IV I_1$, $I_1^* II^2 I_1$, $IV^* II\, I_2$, $IV^* II\, I_1^2$
3	$I_2^* III\, I_1$, $I_1^* III\, I_2$, $I_1^* III\, I_1^2$, $I_0^* III\, I_2 I_1$, $III^* I_2 I_1$, $IV III\, II\, I_3$, $IV III\, II\, I_2 I_1$, $IV III\, II\, I_1^3$, $III^3 I_2 I_1$, $III\, II^2 I_2 I_1$
4	$I_3^* II\, I_1$, $I_1^* II\, I_3$, $I_1^* II\, I_2 I_1$, $I_1^* II\, I_1^3$, $I_0^* II\, I_2 I_1^2$, $IV^* I_3 I_1$, $IV II^2 I_3 I_1$, $III^2 II\, I_4$, $III^2 II\, I_3 I_1$, $III^2 II\, I_2 I_1^2$
5	$IV III\, I_4 I_1$, $IV III\, I_3 I_2$, $IV III\, I_3 I_1^2$, $IV III\, I_2^2 I_1$, $IV III\, I_2 I_1^3$, $III\, II^2 I_5$, $III\, II^2 I_4 I_1$, $III\, II^2 I_3 I_2$, $III\, II^2 I_3 I_1^2$, $III\, II^2 I_2^2 I_1$, $III\, II^2 I_2 I_1^3$, $III\, II^2 I_1^5$
6	$I_1^* I_4 I_1$, $I_1^* I_3 I_1^2$, $I_1^* I_2 I_1^3$, $I_1^* I_1^5$, $IV II\, I_5 I_1$, $IV II\, I_4 I_2$, $IV II\, I_4 I_1^2$, $IV II\, I_3 I_2 I_1$, $IV II\, I_3 I_1^3$, $IV II\, I_2^2 I_1^2$, $IV II\, I_2 I_1^4$, $III^2 I_3 I_2 I_1$, $II^3 I_5 I_1$, $II^3 I_4 I_2$, $II^3 I_4 I_1^2$, $II^3 I_3 I_2 I_1$, $II^3 I_2 I_1^4$
7	$III\, II\, I_6 I_1$, $III\, II\, I_5 I_2$, $III\, II\, I_5 I_1^2$, $III\, II\, I_4 I_3$, $III\, II\, I_4 I_2 I_1$, $III\, II\, I_4 I_1^3$, $III\, II\, I_3^2 I_1$, $III\, II\, I_3 I_2^2$, $III\, II\, I_3 I_2 I_1^2$, $III\, II\, I_3 I_1^4$, $III\, II\, I_2^3 I_1$, $III\, II\, I_2^2 I_1^3$, $III\, II\, I_2 I_1^5$
8	$IV I_5 I_2 I_1$, $IV I_5 I_1^3$, $IV I_4 I_2 I_1^2$, $IV I_3 I_2^2 I_1$, $IV I_3 I_2 I_1^3$, $IV I_3 I_1^5$, $II^2 I_5 I_2 I_1$, $II^2 I_4 I_3 I_1$, $II^2 I_4 I_2 I_1^2$, $II^2 I_3 I_2 I_1^3$
9	$III\, I_7 I_1^2$, $III\, I_6 I_2 I_1$, $III\, I_5 I_3 I_1$, $III\, I_5 I_2 I_1^2$, $III\, I_5 I_1^4$, $III\, I_4 I_3 I_2$, $III\, I_4 I_3 I_1^2$, $III\, I_4 I_2^2 I_1$, $III\, I_4 I_2 I_1^3$, $III\, I_4 I_1^5$, $III\, I_3^2 I_2 I_1$, $III\, I_3^2 I_1^3$, $III\, I_3 I_2^2 I_1^2$, $III\, I_3 I_2 I_1^4$, $III\, I_2^4 I_1$, $III\, I_2^2 I_1^5$, $III\, I_2 I_1^7$
10	$II\, I_8 I_1^2$, $II\, I_7 I_2 I_1$, $II\, I_7 I_1^3$, $II\, I_6 I_2 I_1^2$, $II\, I_6 I_1^4$, $II\, I_5 I_4 I_1$, $II\, I_5 I_3 I_2$, $II\, I_5 I_3 I_1^2$, $II\, I_5 I_2^2 I_1$, $II\, I_5 I_2 I_1^3$, $II\, I_4 I_3 I_2 I_1$, $II\, I_4 I_3 I_1^3$, $II\, I_4 I_2^2 I_1^2$, $II\, I_4 I_2 I_1^4$, $II\, I_4 I_1^6$, $II\, I_3^2 I_2 I_1^2$, $II\, I_3 I_2^3 I_1$, $II\, I_3 I_2^2 I_1^3$, $II\, I_3 I_2 I_1^5$, $II\, I_3 I_1^7$, $II\, I_2^3 I_1^4$, $II\, I_2 I_1^8$
12	$I_7 I_2 I_1^3$, $I_6 I_3 I_2 I_1$, $I_6 I_3 I_1^3$, $I_5 I_4 I_1^3$, $I_5 I_3 I_2 I_1^2$, $I_5 I_2 I_1^5$, $I_4 I_3 I_2^2 I_1$, $I_4 I_3 I_2 I_1^3$, $I_4 I_3 I_1^5$, $I_3^3 I_2 I_1$, $I_3 I_2 I_1^7$

Table 3: Configurations for which $n = 1$ subject to Lemma 4.1.1 and Proposition 4.1.2.

Here, we want to give some examples on how we determine the possible orders of induced automorphisms on $\mathbb{P}^1$ as listed in Table 2 and 3. In the following examples, we make use of Table 4 which shows the multiplicity of the J-map on the points of the base curve $\mathbb{P}^1$ corresponding to each fiber type.

Fiber type over $x \in \mathbb{P}^1$	$J(x)$	Multiplicity of the J-map at x
I_n or I_n^* $(n \geq 1)$	∞	n
III or III*	1	$1 \mod 2$
II or IV*	0	$1 \mod 3$
IV or II*	0	$2 \mod 3$
I_0 or I_0^*	0	$0 \mod 3$
	1	$0 \mod 2$
	$\neq 0, 1, \infty$	no restriction

Table 4: The multiplicity and the value of the J-map on the points corresponding to each fiber type.

Using this table, we can determine the possible ramifications of the J-map at the points of $\mathbb{P}^1$ corresponding to each fiber for a given configuration of singular fibers. The total ramification of the J-map is $2 \cdot deg(J) - 2$ by the Hurwitz's Formula. The multiplicities of the J-map at the points with the same J-value add up to $deg(J)$.

4.1.1 Examples

1) $IVI_3I_2I_1^3$**:** The singular fibers IV, I_3 and I_2 appear once in the configuration. Then the points on $\mathbb{P}^1$ corresponding to these three fibers must be fixed by any induced automorphism, hence every induced automorphism is the identity. Thus, $n = 1$.

2) $IV^*II\ I_1^2$ **:** The degree of the J-map is 2. There must be I_0 fibers over $J = 1$ (I_0 fibers with J-value 1) since there are no III , III^* or I_0^* fibers in the configuration, and these are the only fibers which can have J-value 1. Multiplicities of the J-map at the points corresponding to the I_0 fibers over $J = 1$ are congruent to $0 \mod 2$ from Table 4. These multiplicities should add up to the degree of the J-map, which is 2. Then, there is only one I_0 with J-value 1, and the multiplicity of the J-map is 2 at the point corresponding to this I_0. An induced automorphism must fix that point since there is only one fiber with J-value 1, and induced automorphisms permute the points corresponding to the fibers with the same J-value. Induced automorphisms also fix the points corresponding to the IV^* and II fibers since these fibers appear only once in the configuration. Hence, there are already three fixed points. Thus, $n = 1$.

3) $I_4I_3I_1^5$ **:** The points corresponding to I_4 and I_3 are fixed. The points corresponding to the five I_1 fibers are then permuted by n-cycles. The degree of the J-map is $d = 12$. So $n|d = 12$ and $n|5$. Then $n = 1$.

4) $III^2II\ I_4$ The degree of the J-map is $d = 4$. The total ramification of the J-map is 6. Then, there are I_0 fibers over $J = 1$ and $J = 0$ with the multiplicity of the J-map 2 and 3, respectively. The multiplicity of the J-map at (the points corresponding to) III and II must be 1. Together with the (points corresponding to) I_4 and II, (the points corresponding to) these two I_0 with J-values 0 and 1 must be fixed. There are already four fixed points. $n = 1$.

5)$III^2 II\ I_2^2$ **:** $d = 4$, the total ramification is 6. II is fixed (i.e. the point corresponding to the fiber II is fixed by every induced automorphism. From this point on, we will write so for short). Over $J = 1$, we either have two III fibers with the multiplicities of J 3 and 1; or two III fibers and an I_0 with the multiplicities of J 1,1,2. In the former case, both of the III are fixed since the multiplicities of J are different, so they cannot be permuted. Then we get three fixed points and $n = 1$. In the latter case, I_0 is fixed. Over $J = 0$ we may have a II with the multiplicity of J 4; or we may have a II and an I_0 with the multiplicities of J 1 and 3. In the latter case, together with the I_0 over $J = 1$, the II and I_0 over $J = 0$ are fixed and $n = 1$. In the former case, the total ramification is already 6 with the ramifications at the I_0 over $J = 1$, II and the two I_2. So there is no more ramification. There are four I_0 fibers over any J-value except 0,1 and ∞. The I_0 over $J = 1$ and II are fixed and all the other fibers can be permuted by 2-cycles. Hence, $n = 2$ is possible.

6) $I_0^* II\ I_1^4$ **:** $d = 4$, the total ramification of J is 6. n can be 1,2 or 4. If we have I_0^* over $J = 1$ with the multiplicity of J 4, and II with the multiplicity of J 4, then there is no other ramification except for the I_0^* and II. There are four I_0 fibers over each $J \neq 1, 0, \infty$. Fixing I_0^* and II, and permuting the other fibers by 4-cycles is possible. $n = 4$ is possible. Then, if τ is an induced automorphism of order $n = 4$, τ^2 has order 2. $n = 2$ is also possible. (Generally, if $n = k$ is possible and $m|k$, then $n = m$ is also possible.). $n = 2$ or 4.

7) $I_0^* I_2^3$ **:** $d = 6$, the total ramification is 10. n may be 2, 3 or 6. 6 is not possible since there are three I_2 fibers which must be permuted. We may have three I_0 fibers over $J = 1$ with the multiplicity of J 2 at each, and one I_0^* and an I_0 over $J = 0$ with the multiplicity of J 3 at each. These numbers complete the total ramification to 10 with the ramifications of the I_2 fibers. Fixing the I_0 and I_0^* over $J = 0$ and permuting the other fibers by 3-cycles is possible. $n = 3$ is possible. Or, we may have two I_0 over $J = 0$ and one I_0^* and two I_0 over $J = 1$. In this case, I_0^* and I_2 can be fixed and the other fibers can be permuted by 2-cycles. So, $n = 2$ or 3.

8) $II^3 I_4 I_1^2$ **:** $d = 6$, the total ramification is 10. I_4 is fixed and there are two I_1 fibers. If one of the I_1 fibers is fixed, the other is also fixed since they are permuted. Then, only $n = 2$ may be possible. Over $J = 0$, we may have three II fibers and one I_0 with the multiplicities of J 1,1,1,3; or we may have only three II fibers with the multiplicities of J 1,1,4. In the former case I_0 is fixed and to have $n = 2$, one of the II fibers must be fixed, but then we have three fixed points and $n = 1$. In the latter case, the II fiber with the multiplicity of J 4 is fixed. But here, over $J = 1$ we can have three I_0; or two I_0 with the multiplicities of J 2,4; or one I_0. In all of those cases, we must have another fixed point in order to get $n = 2$. We again have 3 fixed points and so $n = 1$.

9) $II^3 I_3 I_1^3$ **:** $d = 6$, the total ramification is 10. I_3 is fixed. Since there are three II and three I_1 fibers, $n = 6$ and $n = 2$ are not possible. We may have three II and one I_0 over $J = 0$ with the multiplicities of J 1,1,1,3; three I_0 over $J = 1$ and another $J \neq 1, 0, \infty$ with the multiplicities of J 2,2,2. Fixing the I_0 over $J = 0$ together with I_3 and permuting the others by 3-cycles is possible. $n = 3$ is possible.

10) $II^3 I_1^6$ **:** $d = 6$, the total ramification is 10. Since there are three II fibers, $n = 6$ is not possible. We can have $n = 3$ if there are three II and one I_0 over $J = 0$ with the multiplicities of J 1,1,1,3; one I_0 over $J = 1$ with the multiplicity of J 6; and three I_0 over a $J \neq 0, 1, \infty$ with the multiplicities of J 2,2,2. Here, the I_0 fibers over $J = 0, 1$ are fixed and the other fibers are permuted in 3-cycles. To have $n = 2$ possible, we may have three II fibers over $J = 0$ with the multiplicities of J 1,1,4; one I_0 over $J = 1$ with the multiplicity of J 6; and four I_0 fibers over a $J \neq 0, 1, \infty$ with the multiplicities of J 1,1,2,2. Here, the I_0 over $J = 1$ and the II with the multiplicity of J 4 is fixed and the others are permuted by 2 cycles. $n = 2$ or 3 is possible.

11) $I_2^3 I_1^6$ **:** $d = 12$, the total ramification is 22. Since there are three I_2 fibers, $n = 4, 6, 12$ are not possible. We can have $n = 3$ if there is one I_0 over $J = 1$ with the multiplicity of J 12, and four I_0 over $J = 0$ with the multiplicity of J 3 for each. Here, the I_0 over $J = 1$ and one of the four I_0 fibers over $J = 0$ are fixed while the others are permuted by 3-cycles. We can have $n = 2$ if there are three I_0 fibers over $J = 1$ with the multiplicities of J 4,4,4 , and two I_0 over $J = 0$ with the multiplicities of J 6,6. Here, one of the I_0 over $J = 1$ and one of the I_2 are fixed while the others are permuted by 2-cycles. $n = 2$ or 3 is possible.

12) I_3^4 **:** $d = 12$, the total ramification is 12. There are six I_0 over $J = 1$ with the multiplicity of J 2 for each, and four I_0 over $J = 0$ with the multiplicity of J 3 for each. Since there are four I_3 fibers, $n = 6, 12$ are not possible. We can have $n = 4$ if two of the I_0 over $J = 1$ are fixed and the other fibers are permuted by 4-cycles. We can have $n = 3$ if one of the I_0 over $J = 0$ and one of I_3 are fixed while the other fibers are permuted by 3-cycles. $n =$ 2,3 and 4 are possible.

Using similar arguments to those in the above examples, the reader can easily check that possible values of n (order of an induced automorphism) for each configuration of singular fibers are as listed in Table 2 and Table 3.

4.2 $Aut_B(\mathbb{P}^1)$: the group of induced automorphisms

After discussing orders of induced automorphisms, we now turn to discuss the group $Aut_B(\mathbb{P}^1)$. The elements in the orbits of $Aut_B(\mathbb{P}^1)$ have the same J-value by Lemma 4.1.1. For a relatively minimal rational elliptic surface with section, the degree of the J-map can be at most 12. Then the maximum orbit size of $Aut_B(\mathbb{P}^1)$ is 12.

Lemma 4.2.1. *If $deg(J) > 0$ and if there is a point of $\mathbb{P}^1$ which is fixed by every element in $Aut_B(\mathbb{P}^1)$ (in particular, if there is only one singular fiber of a particular type in the configuration of singular fibers of B), then there is another point which is also fixed by every element of $Aut_B(\mathbb{P}^1)$, and $Aut_B(\mathbb{P}^1)$ is a cyclic group.*

Proof. Assume x is fixed by every element, and y and z ($y \neq z$) are fixed by two distinct non-identity elements γ_1 and γ_2. Note that a non-identity automorphism of $\mathbb{P}^1$ has exactly two fixed points. We may choose coordinates on $\mathbb{P}^1$ so that $x = \infty$ and $y = 0$. By taking powers if necessary, we may assume that the orders of γ_i are the primes p_i. γ_1 is the rotation around 0 of order p_1, and γ_2 is the rotation around $z \neq 0$ of order p_2. We are only concerned with $p_i < 12$, hence with 2, 3, 5 or 7 (we do not have 11 here since the degree of the J-map is never 11 for a rational elliptic surface). It is easy to see that in every case, the orbit of z will have infinitely many elements, contradicting the maximum orbit size being 12. Then $y = z$, i.e. the second fixed point of all the non-identity elements of $Aut_B(\mathbb{P}^1)$ is y. The orders of elements of $Aut_B(\mathbb{P}^1)$ are restricted by 12 (the orbit size is less than or equal to 12). Pick one element with highest order in that group, it will generate the whole group, hence $Aut_B(\mathbb{P}^1)$ is cyclic. ☐

Lemma 4.2.2. *The subgroups of $Aut(\mathbb{P}^1)$ with orbit size less than or equal to 12 are $\mathbb{Z}/n\mathbb{Z}$ $n \leq 12$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_3, D_4, D_5, D_6 and A_4, where D_k is the dihedral group of $2k$ elements and A_4 is the alternating group.*

Proof. If G is such a subgroup of $Aut(\mathbb{P}^1)$, then if $\gamma \in G$ is a non-identity automorphism fixing x and y in $\mathbb{P}^1$, all the automorphisms in G fixing x must also fix y, otherwise the orbit of y will be infinite as in the proof of the above lemma. Then, the stabilizer of x in G is finite since automorphisms of order at most 12 which fix x and y can form only cyclic subgroups of order at most 12. Since the orbit of x also has at most 12 elements, G is finite. Then, there are only finitely many points which can be fixed by a non-identity element of G since each such automorphism has exactly 2 fixed points. Then, a generic point of $\mathbb{P}^1$ has a stabilizer consisting of just the identity. Then, $|G|$ is at most $deg(J)$, hence $|G| \leq 12$. If we take the Moebius transformation $z \mapsto \mu_n z$, where μ_n is a primitive n-th root of unity, this automorphism of $\mathbb{P}^1$ generates a subgroup of $Aut(\mathbb{P}^1)$ isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and the subgroup of automorphisms generated by the two Moebius transformations $z \mapsto 1/z$ and $z \mapsto \mu_n z$ ($n = 2, 3, 4, 5, 6$), is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $n = 2$ and D_n for $n = 3, 4, 5, 6$. Considering

$\mathbb{P}^1$ as the sphere, the orientation preserving transformations of a regular tetrahedron inside the sphere induces an automorphism subgroup of $\mathbb{P}^1$ isomorphic to A_4. This proves the existence of the listed groups. Note that this last group A_4 is generated by a rotation of order 3, fixing a vertex of the tetrahedron and rotating the opposite face, and two rotations of order 2, each interchanging two pairs of vertices of the tetrahedron and mapping the faces accordingly.

It is a well-known fact that the only finite subgroups of the automorphism group of $\mathbb{P}^1$ are cyclic groups, dihedral groups or the groups of rotations of a regular tetrahedron, octahedron or dodecahedron (p.184 in [1]). The only groups of order less than or equal to 12 in that list are the groups listed in the lemma. This completes the proof. □

Proposition 4.2.3. *The only configurations of singular fibers which may allow a non-cyclic $Aut_B(\mathbb{P}^1)$ and the corresponding non-cyclic groups $Aut_B(\mathbb{P}^1)$ are as listed below:*

∗ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$: $IV^2I_2^2$, $IV^2I_1^4$, $II^4I_2^2$, $II^4I_1^4$, $II^2I_4^2$, $II^2I_3^2I_1^2$, $II^2I_2^4$, $II^2I_2^2I_1^4$, $II^2I_1^8$, $I_5^2I_1^2$, $I_4^2I_2^2$, $I_4^2I_1^4$, I_3^4, $I_3^2I_1^6$, I_2^6, $I_2^4I_1^4$, $I_2^2I_1^8$, I_1^{12}.

∗ D_3 : $I_3^3I_1^3$, $I_2^3I_1^6$, I_2^6, I_1^{12}.

∗ D_4 : $II^2I_2^4$, $II^2I_1^8$.

∗ D_6 : I_2^6, I_1^{12}.

∗ A_4 : I_3^4, I_1^{12}.

If $Aut_B(\mathbb{P}^1) = \mathbb{Z}/n\mathbb{Z}$, then $n|deg(J)$. In particular, $n \leq 12$ and $n \neq 11$.

Proof. The last statement about the order of a cyclic $Aut_B(\mathbb{P}^1)$ group follows directly from Lemma 4.1.1 and the fact that $deg(J) \leq 12$ and $deg(J) \neq 11$ for rational elliptic surfaces. By Lemma 4.2.1, we should consider only those configurations of singular fibers for which singular fiber types appear at least twice. The proof of Lemma 4.2.2 gives the generators of the possible groups which can arise as $Aut_B(\mathbb{P}^1)$ and the relative positions of their fixed points. Using this, we can look at the orbits of each possible group on $\mathbb{P}^1$. From now on, we will denote a Moebius transformation by its formula alone.

For each group, we determine which configurations of singular fibers may allow a faithful action of this group on the base curve $\mathbb{P}^1$ subject to the criteria that the points of $\mathbb{P}^1$ corresponding to the singular fibers of the same type are permuted, the order of an induced automorphism divides the degree of the J-map, and the value and the multiplicity of the J-map is the same for the points in the same orbit. We compare the sizes of the orbits allowed with respect to these criteria to the actual orbit sizes of the group actions on $\mathbb{P}^1$ under question, and eliminate the configurations for which the orbit sizes are not compatible.

The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ consists of the transformations z, $-z$, $1/z$ and $-1/z$, where $\{0, \infty\}$, $\{1, -1\}$ and $\{i, -i\}$ are the only 2-element orbits. All the other orbits have 4 elements. Then, if $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $4|deg(J)$ since the generic orbit has 4 elements, $J^{-1}(x)$ consists of $deg(J)$ elements for a generic x, and every element in the same orbit must have the same J-value.

If $deg(J) = 4$, the only configurations where every singular fiber type appears at least twice are $IV^2 I_2^2$, $IV^2 I_1^4$, $II^4 I_2^2$ and $II^4 I_1^4$. For $IV^2 I_2^2$, there are two I_0 over $J = 1$, and over any $J \neq 0, 1, \infty$ there are four I_0 fibers. The two I_0 over $J = 1$, the two IV and the two I_2 can form the three orbits of size 2; and the four I_0 over any $J \neq 0, 1, \infty$ can form the orbits of size 4. Thus, $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ can occur for $IV^2 I_2^2$. Similarly, we can give the ramifications of the J-map for the other configurations which are appropriate to have $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as follows:
• $IV^2 I_1^4$: two I_0 fibers over $J = 1$ and some $J = j_0 \neq 0, 1, \infty$, which form three orbits of size 2 together with the two IV fibers. The other orbits all have four elements, which are the four fibers over any $J \neq 0, 1, j_0$ explicitly.
• $II^4 I_2^2$: Two I_0 over $J = 1$ and some $J = j_0 \neq 0, 1, \infty$.
• $II^4 I_1^4$: Two I_0 over $J = 1$, and two other values of $J \neq 0, 1, \infty$

If $deg(J) = 8$, the configurations and the ramifications of the J-map allowing three orbits of size 2 while all the other orbits have size 4 are as follows:
• $II^2 I_4^2$: Four I_0 over $J = 1$ and two I_0 over $J = 0$, eight I_0 over any $J \neq 0, 1, \infty$.
• $II^2 I_3^2 I_1^2$: Four I_0 over $J = 1$.
• $II^2 I_2^4$: Two I_0 over $J = 1$ with the multiplicity of J 4 at each, and two I_0 over $J = 0$.
• $II^2 I_2^2 I_1^4$: Two I_0 over $J = 1$ with the multiplicity of J 4 at each.
• $II^2 I_1^8$: Four I_0 over $J = 1$, two I_0 over $J = 0$, and two I_0 over a $J \neq 0, 1, \infty$ with the multiplicity of J 4 at each.

For $deg(J) = 8$, there is one more configuration, $II^2 I_2^3 I_1^2$, where each singular fiber type appears at least twice. But for this configuration we cannot have three orbits of size 2 while the other orbits have size 4 since there are three I_2 fibers which have to be permuted by the action. These cannot form an orbit of size 4, and if two of them are in one orbit, then the other has to form a singleton orbit, which is not allowed. Then, $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not possible for $II^2 I_2^3 I_1^2$.

If $deg(J) = 12$, the configurations and the ramifications of the J-map allowing three orbits of size 2 while all the other orbits have size 4 are as follows:
• $I_5^2 I_1^2$ and $I_4^2 I_2^2$: Six I_0 over $J = 1$, four I_0 over $J = 0$, twelve I_0 over any $J \neq 0, 1, \infty$. Besides the two I_5 and the two I_1 for the first configuration, the third orbit of size 2 is given by two of the six I_0 over $J = 1$. Similarly for the

second configuration.
• $I_4^2 I_1^4$: Six I_0 over $J = 1$, and two I_0 over $J = 0$ with the multiplicity of J 6 at both. Together with the two I_4 and the two I_0 over $J = 0$, the third orbit of size 2 consists of the two of the six I_0 over $J = 1$.
• $I_3^2 I_1^6$: Two I_0 over $J = 1$ with the multiplicity of J 6 at each, four I_0 over $J = 0$. Two of the six I_1, the two I_3 and the two I_0 over $J = 1$ are the three orbits of size 2.
• $I_2^2 I_1^8$: Two I_0 over $J = 1$ and $J = 0$ with the multiplicity of J 6 at each.
• $I_2^4 I_1^4$: Four I_0 over $J = 1$ with multiplicities of J 2,2,4,4; and two I_0 over $J = 0$ with the multiplicity of J 6 at each. These six I_0 form three orbits of size 2.
• I_3^4 : Six I_0 over $J = 1$ and four I_0 over $J = 0$. The six I_0 over $J = 1$ give the three orbits of size 2.
• I_2^6 : Six I_0 over $J = 1$; and two I_0 over $J = 0$ with the multiplicity of J 6 at each. Two of the six I_2, two of the six I_0 over $J = 1$ and the two I_0 over $J = 0$ give the three orbits of size 2.
• I_1^{12} : Six I_0 over $J = 1$, four I_0 over $J = 0$, and four I_0 over a $J \neq 0, 1, \infty$ with multiplicities of J 1,1,5,5. These last four I_0 and two of the six I_0 over $J = 1$ give the three orbits of size 2.

For the configurations $I_2^5 I_1^2$, $I_3^3 I_1^3$ and $I_2^3 I_1^6$, the singular fibers which appear an odd number of times in the configuration do not allow orbits of size 2 or 4. For the configuration $I_3^2 I_2^2 I_1^2$, we already have three orbits of size 2 coming from the singular fibers, but over $J = 1$, we cannot have four or eight I_0 to get orbits of size 4. Eight I_0 is impossible since multiplicity of J is congruent to $0 \mod 2$ for such I_0. If there are four I_0, the multiplicities of J cannot be the same for all four. Thus, they cannot be in the same orbit. For these four configurations, $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not possible.

D_3 is generated by $1/z$ and $\mu_3 z$ where all elements of the group are $\mu_3^j z$, μ_3^k/z, $0 \leq j, k \leq 2$. The orbits $\{0, \infty\}$, $\{1, \mu_3, \mu_3^2\}$ and $\{-1, -\mu_3, -\mu_3^2\}$ are the only ones which have fewer than six elements. All the other orbits have size 6. If $Aut_B(\mathbb{P}^1) = D_3$, then $6 | deg(J)$, and there are two orbits of size 3, one orbit of size 2 and all the other orbits have size 6.

For the case $deg(J) = 6$, the only configuration which can admit both order 2 and order 3 induced automorphisms, and for which all the singular fiber types in the configuration appear at least twice, is $II^3 I_1^6$ (evident from Table 2). For this configuration to admit an order 3 induced automorphism, over $J = 0$, there must be one I_0 fiber with the multiplicity of J 3. Otherwise, the three II cannot be permuted by an order 3 automorphism since the multiplicities of J are different. Then, this I_0 will also be fixed by any induced automorphism, hence $Aut_B(\mathbb{P}^1)$ is cyclic by Lemma 4.2.1. D_3 is not possible.

For the $deg(J) = 12$ case, the configurations which can admit order 2 and 3 induced automorphisms, and for which every singular fiber type appears at least twice, are as follows. For each, we determine if having two orbits of size 3 and one orbit of size 2 is possible under the action of two induced automorphisms of orders 2 and 3 , while the generic orbit has size 6.

• $I_3^3 I_1^3$: D_3 is possible. We may have six I_0 over $J=1$, and two I_0 over $J=0$ with the multiplicity of J 6 for each. The order 3 generator of D_3 should fix the two I_0 over $J=0$, and the order 2 generator of D_3 should fix one of the three I_3 and one of the three I_1. This way, the two I_0 over $J=0$ form an orbit of size 2, and the I_3 and the I_1 fibers form two orbits of size 3. All the other orbits will have size 6.

• $I_2^3 I_1^6$: D_3 is possible. We may have three I_0 over $J=1$ with the multiplicity of J 4 at each, and two I_0 over $J=0$ with the multiplicity of J 6 at each. If the order 3 generator of D_3 fixes the two I_0 over $J=0$, and the order 2 generator of D_3 fixes one of the three I_0 over $J=1$ and one of the three I_2, then we have the orbit sizes as desired. The two I_0 over $J=0$; the three I_0 over $J=1$ and the three I_2 form orbits of sizes 2,3,3. All the other orbits have size 6.

• I_3^4 : D_3 is not possible. The four I_3 should split into orbits. We cannot have a partition of 4 using 2,3,3.

• I_2^6 : D_3 is possible. We can have two I_0 over $J=0$ with the multiplicity of J 6 at each, and six I_0 over $J=1$. The order 3 generator of D_3 should fix the two I_0 over $J=0$. This will divide the six I_2 into two 3-cycles. If we pick one I_2 from each 3-cycle and have the order 2 generator of D_3 fix these two I_2, then the two 3-cycles described will form two orbits of size 3. The two I_0 over $J=0$ is also an orbit, of size 2. All the other orbits will have size 6.

• I_1^{12} : D_3 is possible. We may have two I_0 over $J=0$ with the multiplicity of J 6 at each, three I_0 over $J=1$ with the multiplicity of J 4 at each, and nine I_0 over a $J \neq 0,1,\infty$ where multiplicity of J is 1 at six of them, and is 2 at the remaining three. The order 3 generator of D_3 should fix the two I_0 over $J=0$, and the order 2 generator should fix one of the three I_0 over $J=1$ and one of the three I_0 over the specified $J \neq 0,1,\infty$ with the multiplicity of J 2. These last three I_0; the three I_0 over $J=1$ and the two I_0 over $J=0$ form orbits of sizes 3,3,2. All the other orbits have size 6.

D_4 is generated by $1/z$ and iz. The group consists of $\pm z$, $\pm iz$, $\pm 1/z$ and $\pm i/z$. The orbits $\{0,\infty\}$, $\{\pm 1, \pm i\}$ and $\{\pm\mu_8, \pm\mu_8^3\}$ are the only ones which have size less than 8. All the other orbits have size 8. Thus, if $Aut_B(\mathbb{P}^1) = D_4$ then $8|deg(J)$ and the action has one orbit of size 2 and two orbits of size 4 while all the other orbits have size 8. The only configurations with $deg(J) = 8$ which can admit order 4 induced automorphisms, and for which every singular fiber type appears at least twice, are $II^2 I_2^4$ and $II^2 I_1^8$ (evident from Table 2). $Aut_B(\mathbb{P}^1) = D_4$ is possible for both:

• $II^2 I_2^4$: We can have four I_0 over $J=1$. If the order 4 generator of D_4 fixes the two II, and the order 2 generator fixes two of the four I_2, then we have the four I_0 over $J=1$; the four I_2 and the two II as orbits of sizes 4,4,2. All the other orbits have size 8.

• $II^2 I_1^8$: Having four I_0 over $J=1$ and another $J \neq 0,1,\infty$ with the multiplicity of J 2 at each, allows to have orbits of sizes 4,4,2 as in the above case.

To have $Aut_B(\mathbb{P}^1) = D_5$, we must have $10|deg(J)$, hence $deg(J) = 10$. The only such configuration which can have both an order 2 and an order 5 induced automorphism is $II\ I_1^{10}$ (From Table 2). But, II is fixed by every induced automorphism since it appears once in the configuration. Then by Lemma 4.2.1, $Aut_B(\mathbb{P}^1)$ is cyclic, it cannot be D_5.

Similarly as above, if $Aut_B(\mathbb{P}^1) = D_6$, there are two orbits of size 6 and one orbit of size 2 while all the other orbits have size 12. Then $12|deg(J)$. The configurations with $deg(J) = 12$ which can have order 6 induced automorphisms, and for which every singular fiber in the configuration appears at least twice, are I_2^6 and I_1^{12}. $Aut_B(\mathbb{P}^1) = D_6$ is possible for both:

• I_2^6 : We may have six I_0 over $J = 1$, and two I_0 over $J = 0$ with the multiplicity of J 6 at both. Then, if the order 6 generator of D_6 fixes the two I_0 over $J = 0$, and the order 2 generator fixes two of the six I_2, then we have orbits of sizes 2,6,6 which are the fibers over $J = 0$, $J = 1$ and $J = \infty$, respectively. All the other orbits have size 12.

• I_1^{12} : We may have six I_0 over $J = 1$ and another $J \neq 0, 1, \infty$ with the multiplicity of J 2 at each; and two I_0 over $J = 0$ with the multiplicity of J 6 at each, then as in the above case, we get orbits of sizes 6,6,2. All the other orbits have size 12.

For the action of A_4 on $\mathbb{P}^1$, if we think about the orientation preserving transformations of the regular tetrahedron inside the sphere, order 3 rotations fix a vertex of the tetrahedron and the opposite point on the sphere to that vertex. A vertex can be mapped to any vertex by an element of A_4. The four vertices form an orbit and the opposite points of the vertices also form an orbit of size 4. The fixed points of the three order 2 elements of A_4 also form an orbit of size 6. All the other orbits have size 12. If we look at $deg(J) = 12$ configurations which can induce order 2 and 3 automorphisms on $\mathbb{P}^1$, the only configurations which allow two orbits of size 4, one orbit of size 6 and generic orbits of size 12 are I_3^4 and I_1^{12}.

• I_3^4 : There are six I_0 over $J = 1$ and four I_0 over $J = 8$.

• I_1^{12} : If there are four I_0 over $J = 0$ and another $J \neq 0, 1, \infty$ with the multiplicity of J 3 at each; and six I_0 over $J = 1$, these can form the orbits of sizes 4,4,6.

A_4 is not possible for the configurations $I_3^3 I_1^3$ and $I_2^3 I_1^6$ because of the fibers which appear three times. A_4 is not possible for I_2^6, either. Otherwise, the six I_2 will be an orbit, but over $J = 1$, we cannot have six I_0 since they will then form another orbit of size 6. Then, to have an order 3 induced automorphism, there should be four I_0 with multiplicities of J 2,2,2,6. But then, the I_0 with the multiplicity of J 6 must be fixed by every induced automorphism. Thus, A_4 cannot occur for I_2^6, either. □

5 $Aut_\sigma(B)$: Automorphisms Preserving the Zero Section

Our aim in this dissertation is to give $Aut_\sigma(B)$ for each configuration of singular fibers. In this section, we show that if the J-map is not constant, then $Aut_\sigma(B)$ is an extension of the group $Aut_B(\mathbb{P}^1)$ by $\mathbb{Z}/2\mathbb{Z}$ (Lemma 5.0.4). Then, if the order of the induced automorphism $\phi(\alpha)$ of $\alpha \in Aut_\sigma(B)$ is n, the order of α is either n or $2n$. These two cases are studied in the subsections "Construction 1" and "Construction 2", and the existence of all such automorphisms is proved. The configurations of singular fibers which have such automorphisms with specified orders are listed in Tables 5 and 8. In the subsection 5.3, we prove the existence of non-cyclic $Aut_B(\mathbb{P}^1)$ groups (which were predicted in Proposition 4.2.3) and show what the group $Aut_\sigma(B)$ is for each such non-cyclic group and corresponding configurations of singular fibers.

Lemma 5.0.4. *If the J-map of the relatively minimal rational elliptic surface B with section is not constant ($deg(J) > 0$), then*

$$Ker(\phi|_{Aut_\sigma(B)}) = \mathbb{Z}/2\mathbb{Z} = \langle -\mathbb{I} \rangle.$$

Proof. An automorphism of B which induces the identity on $\mathbb{P}^1$, and which also preserves the zero section, must act on each smooth fiber as the identity or the inversion since it fixes the zero of each elliptic curve. The complex multiplications of orders $3, 4$ or 6 are not possible since the J-map is not constant, hence there are only finitely many elliptic curves among the fibers which admit such complex multiplications. If the automorphism acts as the inversion on a fiber, then it acts as the inversion on every smooth fiber. Thus, it is the map $-\mathbb{I}$ or it is the identity on B. $\square$

We get the short exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Aut_\sigma(B) \xrightarrow{\phi} Aut_B(\mathbb{P}^1) \to 1. \tag{13}$$

$Aut_\sigma(B)$ is an extension of the group $Aut_B(\mathbb{P}^1)$ by $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.0.5. *If $Aut_B(\mathbb{P}^1) = \mathbb{Z}/n\mathbb{Z}$, then $Aut_\sigma(B) = \mathbb{Z}/2n\mathbb{Z}$, or $Aut_\sigma(B) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.*

Proof. If α induces an order n automorphism on $\mathbb{P}^1$, then α^k induces the identity on $\mathbb{P}^1$ iff $n|k$. α^n is either $\mathbb{I}$ or $-\mathbb{I}$. In the latter case, we have $ord(\alpha) = 2n$. In the former case, α and $-\mathbb{I}$ generate $Aut_\sigma(B)$. They have orders n and 2, respectively, and they commute (restricted to smooth fibers, α is an elliptic curve isomorphism, and $-\mathbb{I}$ is the inversion of the group law of the elliptic curve). This gives $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. $\square$

In the above discussions, we did not give any existence results. The arguments above restrict the possible groups to a small list. Below, we give the existence results by constructing automorphisms preserving the zero section.

5.1 Construction 1

By Lemma 5.0.4, if the J-map is not constant, then the order of an automorphism $\alpha \in Aut_\sigma(B)$ is equal to either the order of the induced automorphism $\phi(\alpha) \in Aut_B(\mathbb{P}^1)$, or twice the order of $\phi(\alpha)$. In this subsection, we study the former case and give the construction of all such automorphisms α using a process called pull-back described below. Such an automorphism exists if B can be obtained by pulling back another rational elliptic surface by the degree n map $z \mapsto z^n$ on $\mathbb{P}^1$, and reducing to the minimal model (Theorem 5.1.1). Table 6 shows all configurations of singular fibers for which such an α exists together with the configurations of singular fibers of the surfaces from which B is obtained by a pull-back. Table 7 lists the configurations for which such an α does not exist although Table 2 predicts a non-trivial induced automorphism for these configurations. Some examples are given to illustrate how the results in the tables are obtained.

Assume that B is a relatively minimal rational elliptic surface with section and α is an automorphism preserving the zero section, i.e. $\alpha \in Aut_\sigma(B)$. Assume further that the order of α and the order of the induced automorphism $\tau_{\mathbb{P}^1} = \phi(\alpha)$ are equal.

$$ord(\alpha) = ord(\tau_{\mathbb{P}^1}) = n. \tag{14}$$

Consider the action of the cyclic group $\langle\alpha\rangle$ generated by α on the surface B, and let the quotient of this action be the surface $\tilde{B}$ (which may have singularities). If $\pi : B \to \tilde{B}$ is the projection of this quotient, and

$$g_n : \mathbb{P}^1 \to \mathbb{P}^1 \ , \ z \mapsto z^n, \tag{15}$$

then the following diagram commutes if we choose coordinates on $\mathbb{P}^1$ such that the fixed points of $\tau_{\mathbb{P}^1}$ are 0 and ∞:

$$\begin{array}{ccc} B & \xrightarrow{\pi} & \tilde{B} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\tilde{\beta}} \\ \mathbb{P}^1 & \xrightarrow[g_n]{} & \mathbb{P}^1 \end{array}$$

Note here that since α maps the fibers of β to fibers, β induces the map $\tilde{\beta} : \tilde{B} \to \mathbb{P}^1$ above. The base curve is again $\mathbb{P}^1$ since the induced map on $\mathbb{P}^1$ by α is the map $z \mapsto \mu_n z$ where μ_n is the primitive n-th root of 1, and the quotient of $\mathbb{P}^1$ under the action of this map is again $\mathbb{P}^1$ with the projection map of the quotient g_n.

The zero section σ of B maps to a section $\tilde{\sigma}$ of $\tilde{\beta} : \tilde{B} \to \mathbb{P}^1$ by π. The smooth fibers of β are mapped to smooth fibers (elliptic curves) with the same J-value by α. Taking the quotient by the action of α identifies n elliptic curves on B to one elliptic curve on $\tilde{B}$, hence the generic fiber of $\tilde{\beta} : \tilde{B} \to \mathbb{P}^1$ is an elliptic curve. However, $\tilde{B}$ may not be smooth; there may be singularities over 0 and ∞. Let $\tilde{\tilde{B}}$ be the Kodaira Model of $\tilde{B}$, which is obtained by first resolving the singularities of $\tilde{B}$, and then blowing down all the (-1)-curves in the fibers. Then, $\tilde{\tilde{B}}$ is a relatively minimal rational elliptic surface with section. It is rational since it is birational to a quotient of a rational surface.

Starting with the pair (B, α), we obtained a relatively minimal rational elliptic surface $\tilde{\tilde{B}}$. This process can be reversed to obtain (B, α) back as follows. Let $\hat{B}$ be the fibered product $\mathbb{P}^1 \times_{\mathbb{P}^1} \tilde{\tilde{B}}$ of $g_n : \mathbb{P}^1 \to \mathbb{P}^1$ with $\tilde{\tilde{\beta}} : \tilde{\tilde{B}} \to \mathbb{P}^1$. Then, $\hat{B}$ is birational to B and by the uniqueness of Kodaira Models, we can obtain B from $\hat{B}$ by resolving its singularities and then blowing down the (-1)-curves in the fibers. To obtain α, first note that the map f_n given by $z \mapsto \mu_n z$ is a deck transformation for the map g_n ($g_n \circ f_n = g_n$), and together with the identity map on $\tilde{\tilde{B}}$, f_n induces an automorphism on the fibered product $\hat{B}$, and this automorphism in turn gives an automorphism on the Kodaira Model B. Since f_n has order n, this automorphism also has order n, and it preserves the zero section of B. If we start with (B, α) and obtain $\tilde{\tilde{B}}$, and apply the last process, then we recover (B, α) [2].
The above arguments prove the following:

Theorem 5.1.1. *A relatively minimal rational elliptic surface B with section has an automorphism $\alpha \in Aut_\sigma(B)$ with ord(α) = ord($\tau_{\mathbb{P}^1}$) = n , where $\tau_{\mathbb{P}^1}$ is the automorphism induced on $\mathbb{P}^1$ by α, if and only if B is the Kodaira Model of the fibered product $\mathbb{P}^1 \times_{\mathbb{P}^1} B'$ of a relatively minimal rational elliptic surface B' with section and the map $g_n : \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^n$ (0 and ∞ are the fixed points of the induced automorphism $\tau_{\mathbb{P}^1}$).*
In such a case, B is said to be obtained from the pull-back of B' by g_n.

Now, we have a characterization of the rational elliptic surfaces which have automorphisms $\alpha \in Aut_\sigma(B)$ inducing automorphisms of the same order on $\mathbb{P}^1$. Since we are particularly interested in the configurations of singular fibers, we should know the relationship between the singular fibers of B and B'. If $p \in \mathbb{P}^1$ is not 0 or ∞, then g_n is not ramified at p, so the fiber of B over p and the fiber of B' over $g_n(p)$ are the same. If p is 0 or ∞, then g_n is ramified of order n, and in this case, Table 5 gives the fibers of B over p in terms of n and the fibers of B' over $g_n(p)$. For detailed information how this table is computed, see [7] (Table 7.1, p.555).

Fiber of B' over $g_n(p)$	Fiber of B over p ($p = 0, \infty$)
I_0	I_0
I_M	I_{nM}
I_M^*	I_{nM} if n even $\quad I_{nM}^*$ if n odd
II	I_0 if $n = 0 \mod 6$ $\quad II$ if $n = 1 \mod 6$ IV if $n = 2 \mod 6$ $\quad I_0^*$ if $n = 3 \mod 6$ IV^* if $n = 4 \mod 6$ $\quad II^*$ if $n = 5 \mod 6$
III	I_0 if $n = 0 \mod 4$ $\quad III$ if $n = 1 \mod 4$ I_0^* if $n = 2 \mod 4$ $\quad III^*$ if $n = 3 \mod 4$
IV	I_0 if $n = 0 \mod 3$ $\quad IV$ if $n = 1 \mod 3$ IV^* if $n = 2 \mod 3$
IV^*	I_0 if $n = 0 \mod 3$ $\quad IV^*$ if $n = 1 \mod 3$ IV if $n = 2 \mod 3$
III^*	I_0 if $n = 0 \mod 4$ $\quad III^*$ if $n = 1 \mod 4$ I_0^* if $n = 2 \mod 4$ $\quad III$ if $n = 3 \mod 4$
II^*	I_0 if $n = 0 \mod 6$ $\quad II^*$ if $n = 1 \mod 6$ IV^* if $n = 2 \mod 6$ $\quad I_0^*$ if $n = 3 \mod 6$ IV if $n = 4 \mod 6$ $\quad II$ if $n = 5 \mod 6$

Table 5: Transformation of the fibers over the ramified points in the pull-back process.

If we pull-back a relatively minimal rational elliptic surface B' by the map g_n and get the relatively minimal elliptic surface B, B is not necessarily rational. There is a simple criterion for determining the rationality of B:

$$\sum_{S \text{ singular fiber}} \chi(S) = 12 = \chi(B). \tag{16}$$

If B is rational, the sum of the Euler characteristics of the singular fibers of B should be 12, which is the Euler characteristic of B. This comes from the fact that B is an elliptic curve fibration and $\chi(E) = 0$ for any elliptic curve E. This criterion suffices for the rationality of B since any relatively minimal elliptic surface with section which is fibered over $\mathbb{P}^1$ is rational if its Euler characteristic is 12.

Table 6 gives the configurations of singular fibers of all relatively minimal rational elliptic surfaces B and B', where B is obtained from the pull-back of B' by some g_n. By Theorem 5.1.1, we can say that the B column of Table 6 lists all possible configurations of singular fibers of rational elliptic surfaces B which admit an automorphism $\alpha \in Aut_\sigma(B)$ such that $ord(\alpha) = ord(\tau_{\mathbb{P}^1}) = n$. In Table 2, we have listed the possible values of $n \geq 2$ for every configuration of singular fibers. To obtain Table 6, it suffices to check the configurations in Table 2 for the corresponding values of n. From Table 5, we know how singular fibers transform under the pull-back process by the map g_n. To check for a specific configuration of singular fibers and the corresponding value of n from Table 2, we need to find another configuration of singular fibers which transforms to the desired configuration of singular fibers after the pull-back by g_n. Note that if J_B and $J_{B'}$ are the J-maps of the two surfaces, where B is obtained from the pull-back of B' by g_n, then:

$$deg(J_B) = n \cdot deg(J_{B'}). \tag{17}$$

Table 7 shows the configurations for which $n = ord(\tau_{\mathbb{P}^1}) > 1$ was predicted in Table 2, but which cannot be obtained after a pull-back by a g_n from any other rational elliptic surface.

5.1.1 Examples

1) For the cases $n = deg(J_B)$, we get $deg(J_{B'}) = 1$. So B is obtained from a pull-back of one of $I_0^* III\, II\, I_1$, $IV^* III\, I_1$, $III^* II\, I_1$ or $I_1^* III\, II$ by the map $g_{deg(J_B)}$. The fibers of B' over 0 or ∞ transform as shown in Table 5, and we get $n = deg(J_B)$ copies of other fibers. Take $n = 7$; the only candidate configuration which can have $n = 7$ is $III\, II\, I_1^7$ from Table 2. If we have the III^* and II fibers of the configuration $III^* II\, I_1$ over 0 and ∞, then these transform to III and II (from Table 5, n=7), and the I_1 gives 7 copies of I_1 after the pull-back by g_7. Hence, $III\, II\, I_1^7$ is obtained from a pull-back of $III^* II\, I_1$ by g_7. None of the other $deg(J) = 1$ configurations can give $III\, II\, I_1^7$ after a pull-back by g_7, as can be checked easily.

2) If $n = 2$, none of the fibers over 0 or ∞ transforms to a II fiber. In Table 2, all $deg(J) = 10$ configurations for which $n = 2$ is predicted have a II fiber which appears once. Thus, none of such configurations can arise from a pull-back by g_2.

3) If $n = 4$ and $deg(J_B) = 8$, then $deg(J_{B'}) = 2$. The candidates for configurations of singular fibers of B are $IV I_4 I_1^4$, $IV I_1^8$, $II^2 I_2^4$ and $II^2 I_1^8$ from Table 2 (These are the only configurations for which $n = 4$ is possible according to the criteria discussed before Table 2). None of the fibers over 0 or ∞ is transformed to a II fiber by g_4, and four copies of each fiber over a point different from 0 and ∞ are obtained after a pull-back by g_4. Therefore, the last 2 configurations including two II fibers cannot arise from a pull-back by g_4. Since the fibers IV and II^* are the only fibers which can give a IV fiber after a pull-back by g_4, we should look at $deg(J) = 2$ configurations including only one IV or II^* fiber in order to obtain $IV I_4 I_1^4$ or $IV I_1^6$ after a pull-back by g_4. $IV I_4 I_1^4$ is obtained from a pull back of $I_1^* IV I_1$ by g_4 if fibers over 0 and ∞ are I_1^* and IV; or from $II^* I_1^2$ if fibers over 0 and ∞ are II^* and I_1. $IV I_1^8$ is obtained from $I_0^* IV I_1^2$ if fibers over 0 and ∞ are I_0^* and IV; or from $II^* I_1^2$ if fibers over 0 and ∞ are II^* and any smooth fiber I_0.

n	d	Singular Fibers of B	Singular Fibers of B'
12	12	I_1^{12}	$III^*II\ I_1$ or $IV^*III\ I_1$
9	9	$III\ I_1^9$	$IV^*III\ I_1$
8	8	IVI_1^8	$IV^*III\ I_1$ or $III^*II\ I_1$
7	7	$III\ II\ I_1^7$	$III^*II\ I_1$
6	12	$I_6I_1^6$	$II^*I_1^2$ or $I_1^*IVI_1$
		I_2^6	$IV^*II\ I_2$
		I_1^{12}	$I_0^*IVI_1^2$, $II^*I_1^2$ or $IV^*II\ I_1^2$
	6	$I_0^*I_1^6$	$IV^*III\ I_1$ or $III^*II\ I_1$
5	10	$II\ I_5I_1^5$, $II\ I_1^{10}$	$II^*I_1^2$
	5	$IVIII\ I_1^5$	$IV^*III\ I_1$
4	12	$I_8I_1^4$	$III^*I_2I_1$ or $I_2^*III\ I_1$
		$I_4I_2^4$	$III^*I_2I_1$ or $I_1^*III\ I_2$
		$I_4I_1^8$	$III^*I_1^3$ or $I_1^*III\ I_1^2$
		$I_2^4I_1^4$	$I_0^*III\ I_2I_1$ or $III^*I_2I_1$
		I_1^{12}	$I_0^*III\ I_1^3$ or $III^*I_1^3$
	8	$IVI_4I_1^4$	$I_1^*IVI_1$ or $II^*I_1^2$
		IVI_1^8	$I_0^*IVI_1^2$ or $II^*I_1^2$
	4	$IV^*I_1^4$	$IV^*III\ I_1$ or $III^*II\ I_1$
		II^4I_4	$I_1^*III\ II$ or $III^*II\ I_1$
		$II^4I_1^4$	$I_0^*III\ II\ I_1$ or $III^*II\ I_1$
3	12	$I_9I_1^3$, I_3^4 , $I_3^3I_1^3$	$IV^*I_3I_1$
		$I_6I_1^6$, $I_3I_2^3I_1^3$	$IV^*I_2I_1^2$
		$I_3I_1^9$	$IV^*I_1^4$
		I_2^6	$IV^2I_2^2$
		$I_2^3I_1^6$	$IV^*I_2I_1^2$ or $IV^2I_2I_1^2$
		I_1^{12}	$IV^2I_1^4$ or $IV^*I_1^4$
	9	$III\ I_6I_1^3$, $III\ I_3I_2^3$, $III\ I_2^3I_1^3$	$III^*I_2I_1$
		$III\ I_3I_1^6$, $III\ I_1^9$	$III^*I_1^3$
	6	$I_3^*I_1^3$	$I_1^*IVI_1$
		$I_0^*I_3I_1^3$	$II^*I_1^2$
		$I_0^*I_2^3$, II^3I_6 , $II^3I_2^3$	$IV^*II\ I_2$
		$I_0^*I_1^6$	$I_0^*IVI_1^2$, $II^*I_1^2$ or $IV^*II\ I_1^2$
		$II^3I_3I_1^3$, $II^3I_1^6$	$IV^*II\ I_1^2$
	3	$III^*I_1^3$, III^3I_3 , $III^3I_1^3$	$IV^*III\ I_1$
		$I_0^*III\ I_1^3$, $III\ II^3I_3$, $III\ II^3I_1^3$	$III^*II\ I_1$
2	12	$I_8I_2I_1^2$	$I_4^*I_1^2$ or $I_1^*I_4I_1$
		$I_8I_1^4$	$I_0^*I_4I_1^2$ or $I_4^*I_1^2$
		$I_6I_2I_1^4$	$I_3^*I_1^3$ or $I_1^*I_3I_1^2$
		$I_6I_1^6$	$I_3^*I_1^3$ or $I_0^*I_3I_1^3$
		$I_4^2I_2^2$	$I_1^*I_4I_1$ or $I_2^*I_2^2$
		$I_4^2I_2I_1^2$	$I_0^*I_4I_1^2$ or $I_1^*I_4I_1$
		$I_4^2I_1^4$	$I_0^*I_4I_1^2$ or $I_2^*I_2I_1^2$
		$I_4I_2^4$	$I_0^*I_2^3$ or $I_2^*I_2^2$
		$I_4I_2^3I_1^2$	$I_2^*I_2I_1^2$ or $I_1^*I_2^2I_1$

n	d	Singular Fibers of B	Singular Fibers of B'
2	12	$I_4I_2^2I_1^4$	$I_0^*I_2^2I_1^2$ or $I_2^*I_2I_1^2$
		$I_4I_2I_1^6$	$I_2^*I_1^4$ or $I_1^*I_2I_1^3$
		$I_4I_1^8$	$I_0^*I_2I_1^4$ or $I_2^*I_1^4$
		$I_3^2I_2^2I_1^2$	$I_1^*I_3I_1^2$
		$I_3^2I_2I_1^4$	$I_1^*I_3I_1^2$ or $I_0^*I_3I_1^3$
		$I_3^2I_1^6$	$I_0^*I_3I_1^3$
		I_2^6	$I_0^*I_2^3$ or $I_1^*I_2^2I_1$
		$I_2^5I_1^2$	$I_0^*I_2^2I_1^2$ or $I_1^*I_2^2I_1$
		$I_2^4I_1^4$	$I_0^*I_2^2I_1^2$ or $I_1^*I_2I_1^3$
		$I_2^3I_1^6$	$I_0^*I_2I_1^4$ or $I_1^*I_2I_1^3$
		$I_2^2I_1^8$	$I_0^*I_2I_1^4$ or $I_1^*I_1^5$
		$I_2I_1^{10}$	$I_0^*I_1^6$ or $I_1^*I_1^5$
		I_1^{12}	$I_0^*I_1^6$
	8	$IV\,I_6I_1^2$	$IV^*I_3I_1$ or $I_3^*II\ I_1$
		$IV\,I_4I_1^4$	$IV^*I_2I_1^2$ or $I_2^*II\ I_1^2$
		$IV\,I_3^2I_2$	$IV^*I_3I_1$ or $I_1^*II\ I_3$
		$IV\,I_3^2I_1^2$	$I_0^*II\ I_3I_1$ or $IV^*I_3I_1$
		$IV\,I_2^3I_1^2$	$IV^*I_2I_1^2$ or $I_1^*II\ I_2I_1$
		$IV\,I_2^2I_1^4$	$IV^*I_2I_1^2$ or $I_0^*II\ I_2I_1^2$
		$IV\,I_2I_1^6$	$IV^*I_1^4$ or $I_1^*II\ I_1^3$
		$IV\,I_1^8$	$I_0^*II\ I_1^4$ or $IV^*I_1^4$
		$II^2I_6I_2$	$I_3^*II\ I_1$ or $I_1^*II\ I_3$
		$II^2I_6I_1^2$	$I_3^*II\ I_1$ or $I_0^*II\ I_3I_1$
		$II^2I_4I_2I_1^2$	$I_2^*II\ I_1^2$ or $I_1^*II\ I_2I_1$
		$II^2I_4I_1^4$	$I_2^*II\ I_1^2$ or $I_0^*II\ I_2I_1^2$
		$II^2I_3^2I_2$	$I_0^*II\ I_3I_1$ or $I_1^*II\ I_3$
		$II^2I_3^2I_1^2$	$I_0^*II\ I_3I_1$
		$II^2I_2^4$	$I_1^*II\ I_2I_1$
		$II^2I_2^3I_1^2$	$I_0^*II\ I_2I_1^2$ or $I_1^*II\ I_2I_1$
		$II^2I_2^2I_1^4$	$I_0^*II\ I_2I_1^2$ or $I_1^*II\ I_1^3$
		$II^2I_2I_1^6$	$I_0^*II\ I_1^4$ or $I_1^*II\ I_1^3$
		$II^2I_1^8$	$I_0^*II\ I_1^4$
	6	$I_0^*I_4I_1^2$	$III^*I_2I_1$ or $I_2^*III\ I_1$
		$I_0^*I_2^3$	$III^*I_2I_1$ or $I_1^*III\ I_2$
		$I_0^*I_2^2I_1^2$	$I_0^*III\ I_2I_1$ or $III^*I_2I_1$
		$I_0^*I_2I_1^4$	$III^*I_1^3$ or $I_1^*III\ I_1^2$
		$I_0^*I_1^6$	$I_0^*III\ I_1^3$ or $III^*I_1^3$
		$III^2I_4I_2$	$I_2^*III\ I_1$ or $I_1^*III\ I_2$
		$III^2I_4I_1^2$	$I_0^*III\ I_2I_1$ or $I_2^*III\ I_1$
		$III^2I_2^3$	$I_0^*III\ I_2I_1$ or $I_1^*III\ I_2$
		$III^2I_2^2I_1^2$	$I_0^*III\ I_2I_1$ or $I_1^*III\ I_1^2$
		$III^2I_2I_1^4$	$I_0^*III\ I_1^3$ or $I_1^*III\ I_1^2$
		$III^2I_1^6$	$I_0^*III\ I_1^3$

n	d	Singular Fibers of B	Singular Fibers of B'
2	4	$IV^*I_2I_1^2$	$II^*I_1^2$ or $I_1^*IVI_1$
		$IV^*I_1^4$	$I_0^*IVI_1^2$ or $II^*I_1^2$
		$IV^2I_2^2$	$I_1^*IVI_1$ or $IV^*II\,I_2$
		$IV^2I_2I_1^2$	$I_0^*IVI_1^2$ or $I_1^*IVI_1$
		$IV^2I_1^4$	$I_0^*IVI_1^2$ or $IV^*II\,I_1^2$
		$IVII^2I_4$	$IV^*II\,I_2$ or $I_2^*II^2$
		$IVII^2I_2^2$	$I_0^*II^2I_2$ or $IV^*II\,I_2$
		$IVII^2I_2I_1^2$	$IV^*II\,I_1^2$ or $I_1^*II^2I_1$
		$IVII^2I_1^4$	$I_0^*II^2I_1^2$ or $IV^*II\,I_1^2$
		II^4I_4	$I_0^*II^2I_2$ or $I_2^*II^2$
		$II^4I_2^2$	$I_0^*II^2I_2$ or $I_1^*II^2I_1$
		$II^4I_2I_1^2$	$I_0^*II^2I_1^2$ or $I_1^*II^2I_1$
		$II^4I_1^4$	$I_0^*II^2I_1^2$
	2	$I_0^*IVI_1^2$	$IV^*III\,I_1$ or $III^*II\,I_1$
		$I_0^*II^2I_2$	$III^*II\,I_1$ or $I_1^*III\,II$
		$I_0^*II^2I_1^2$	$III^*II\,I_1$ or $I_0^*III\,II\,I_1$
		$IVIII^2I_2$	$IV^*III\,I_1$ or $I_1^*III\,II$
		$IVIII^2I_1^2$	$I_0^*III\,II\,I_1$ or $IV^*III\,I_1$
		$III^2II^2I_2$	$I_1^*III\,II$ or $I_0^*III\,II\,I_1$
		$III^2II^2I_1^2$	$I_0^*III\,II\,I_1$

Table 6: Configurations of singular fibers of relatively minimal rational elliptic surfaces B which can be obtained by a pull-back of another relatively minimal rational elliptic surface B'.

n	$d = deg(J)$	Configuration of singular fibers
10	10	$II\,I_1^{10}$
4	12	I_3^4
	8	$II^2I_2^4$, $II^2I_1^8$
	4	$I_0^*II\,I_1^4$
3	6	$IVII\,I_2^3$, $IVII\,I_1^6$
2	12	$I_9I_1^3$, $I_7I_1^5$, $I_6I_2^2I_1^2$, $I_5^2I_1^2$, $I_5I_3I_1^4$, $I_5I_2^2I_1^3$, $I_5I_1^7$ I_3^4 , $I_3^3I_1^3$, $I_3I_2^4I_1$, $I_3I_2^2I_1^5$, $I_3I_1^9$
	10	$II\,I_4^2I_1^2$, $II\,I_3^2I_2^2$, $II\,I_3^2I_1^4$, $II\,I_2^4I_1^2$, $II\,I_2^2I_1^6$, $II\,I_1^{10}$
	8	$II^2I_7I_1$, $II^2I_5I_1^3$, $II^2I_4^2$, $II^2I_4I_2^2$, $II^2I_3I_2^2I_1$, $II^2I_3I_1^5$
	6	$I_4^*I_1^2$, $I_2^*I_2^2$, $I_2^*I_1^4$, $III^2I_5I_1$, $III^2I_3^2$, $III^2I_3I_1^3$ $II^3I_3^2$, $II^3I_2^2I_1^2$, $II^3I_1^6$
	4	$I_2^*II\,I_1^2$, $I_0^*II\,I_1^4$, $III^2II\,I_2^2$, $III^2II\,I_1^4$, $II^4I_3I_1$
	2	$I_2^*II^2$, $II^*I_1^2$

Table 7: Configurations of singular fibers of relatively minimal rational elliptic surfaces which cannot be obtained from a pull-back.

4) $I_4^2 I_2^2$ can be obtained from a pull-back by g_2 of either $I_1^* I_4 I_1$ if I_1^* and I_1 are over 0 and ∞; or $I_2^* I_2^2$ if I_2^* and I_2 are over 0 and ∞.

5) Since none of the fibers over 0 or ∞ can transform to an I_k with k odd or an I_m^* by a pull-back by g_2, none of the configurations including an I_k with k odd or odd number of I_m^* fibers can arise from a pull-back by g_2.

Remark: If n is odd, then $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = ord(\phi(\alpha)) = n$ exists if and only if $\alpha' \in Aut_\sigma(B)$ with $ord(\alpha') = 2 \cdot ord(\phi(\alpha')) = 2n$ exists. This comes from $Aut_\sigma(B)$ being a $\mathbb{Z}/2\mathbb{Z}$ extension of $Aut_B(\mathbb{P}^1)$. Here, we have $\alpha' = \alpha \circ (-\mathbb{I})$. Hence, basically we are done with the case of odd n after the results obtained in this section. For the case of even n, we need a different construction to show the existence of $\alpha \in Aut_\sigma(B)$ whose order is $2n$, twice the order of its induced automorphism.

5.2 Construction 2

In this section, we are concerned with showing the existence of $\alpha \in Aut_\sigma(B)$ with the property that $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ for even n. We will list the configurations of singular fibers for which such automorphisms exist. Before describing the general construction of such automorphisms, we will first discuss some conditions under which such automorphisms do not exist. This will decrease the number of cases to be considered for the general construction. The construction is performed by lifting the automorphisms of the rational ruled surface F_2 to a double cover of F_2 branched over the minimal section and a trisection which is preserved under that automorphism. The results are shown in Table 9.

Assume that $\beta : B \mapsto \mathbb{P}^1$ is a relatively minimal rational elliptic surface with section, $\alpha \in Aut_\sigma(B)$, $ord(\alpha) = 2n$, $ord(\phi(\alpha)) = n$, n is even, and $\phi(\alpha)$, which is the induced map on $\mathbb{P}^1$, is given by $z \mapsto \mu_n z$. Then

$$\alpha^n = -\mathbb{I}, \tag{18}$$

where $-\mathbb{I}$ is the automorphism of B which acts on every smooth fiber by the inversion of the group law on that fiber which is an elliptic curve.

Assume that C is a fiber over 0 or ∞ of $\mathbb{P}^1$. Then $\alpha(C) = C$, and furthermore, since α preserves the zero section σ, the point $C \cap \sigma$ is fixed by α. To understand how α acts on C better, consider the following.

If we denote by $B^\sharp$ the surface B minus the singular points of the fibers of B and the components of the fibers with multiplicity greater than 1, and also denote by $B^\sharp_z$ the fiber of $B^\sharp$ over the point $z \in \mathbb{P}^1$, then there is an abelian group structure on $B^\sharp_z$ as follows (p.73-74 in [8]): Consider the germ of the fiber $B^\sharp_z$ and let $\mathcal{S}$ be the set of all local sections in that germ. $\mathcal{S}$ is an abelian group with the addition of local sections fiber by fiber on the smooth fibers and taking the closure on the singular fibers. If $\mathcal{S}_0$ denotes the set of the local sections in the germ passing through $B^\sharp_z \cap \sigma$, which is clearly a subgroup, then the quotient $\mathcal{S}/\mathcal{S}_0$ can be identified with $B^\sharp_z$ (since a section must intersect B_z with intersection number 1, and through every point of $B^\sharp_z$ there is a local section), and this identification gives the group structure on $B^\sharp_z$. If we denote by $B^\sharp_{z0}$ the component of $B^\sharp_z$ intersecting the zero section σ, then with the group structure defined, $B^\sharp_{z0}$ is the component of the identity in $B^\sharp_z$, and $B^\sharp_z/B^\sharp_{z0}$ is a finite abelian group whose order is the number of multiplicity 1 components of the fiber B_z of B over z. The table below gives that group in terms of the fiber type of B_z(VII.3.4 in [8]):

B_z	$B^\sharp_{z0}$	$B^\sharp_z/B^\sharp_{z0}$
I_0	Elliptic curve	0
I_M	$\mathbb{C}^*$	$\mathbb{Z}/M\mathbb{Z}$
I^*_M	$\mathbb{C}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if M is even $\mathbb{Z}/4\mathbb{Z}$ if M is odd
II , II^*	$\mathbb{C}$	0
III III^*	$\mathbb{C}$	$\mathbb{Z}/2\mathbb{Z}$
IV , IV^*	$\mathbb{C}$	$\mathbb{Z}/3\mathbb{Z}$

Table 8: The groups $B^\sharp_{z0}$ and $B^\sharp_z/B^\sharp_{z0}$.

If C is a fiber over 0 or ∞, then any $\alpha \in Aut_\sigma(B)$ induces an automorphism on the group structure of $C^\sharp = B^\sharp_z$ ($z = 0$ or ∞) and the finite group $B^\sharp_z/B^\sharp_{z0}$. This is because α maps the zero of every elliptic curve fiber to the zero of another elliptic curve, hence restricted to smooth fibers α is an elliptic curve isomorphism, thus it gives an isomorphism on the group of both local and global sections of the elliptic surface. Since the group structure is defined using the local sections, α induces the mentioned automorphism on the specified groups.

If in particular we take $\alpha = -\mathbb{I}$, then α acts on the local sections as the inversion map, so the induced group automorphism on $B^\sharp_z$ or $B^\sharp_z/B^\sharp_{z0}$ is also the inversion of these groups.

For $C = IV$ or IV^*, α induces an automorphism of the group $B^\sharp_z/B^\sharp_{z0}$, which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Then, if n is even, α^n induces the identity on $\mathbb{Z}/3\mathbb{Z}$ since $Aut(\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Then $\alpha^n \neq -\mathbb{I}$.

For $C = I_M$ $M > 0$, α induces an automorphism of the group $B^\sharp_{z0}$ which is isomorphic to $\mathbb{C}^*$. Then α induces either the identity or the inversion on $\mathbb{C}^*$. If n is even, then α^n induces the identity, hence $\alpha^n \neq -\mathbb{I}$. These arguments prove the following lemma.

Lemma 5.2.1. *If there is a IV , IV^* or I_M fiber ($M > 0$) over 0 or ∞ of $\mathbb{P}^1$, then for even n, there is no $\alpha \in Aut_\sigma(B)$ such that $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ and the fixed points of the induced map $\phi(\alpha)$ are 0 and ∞.*

For $C = I_0$, α induces an automorphism on the elliptic curve C. The automorphism group of an elliptic curve is $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ depending on the j-invariant of the curve being 0,1 or anything else , respectively. When an elliptic curve is considered as the quotient of $\mathbb{C}$ by a rank 2 lattice, any automorphism comes from the complex multiplication maps $z \mapsto mz$ ($m = -1$, μ_3, μ_4 or μ_6, where μ_k is any primitive k-th root of 1). If n is even, then α^n cannot be the inversion map, which corresponds to the unique order 2 element in each of those groups, unless $n = 2 \mod 4$ and the j-invariant is 1. Hence, we get the following lemma.

Lemma 5.2.2. *If there is an I_0 fiber over 0 or ∞ of $\mathbb{P}^1$, then for even n, there is no $\alpha \in Aut_\sigma(B)$ such that $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ and the fixed points of the induced map $\phi(\alpha)$ are 0 and ∞; unless $n = 2 \mod 4$ and the j-invariant of I_0 is 1.*

Using the two lemmas above, we can consult Table 2 giving the candidate values of n, the order of an induced automorphism on $\mathbb{P}^1$, and eliminate the configurations for which the existence of an $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ is impossible. Some examples are as follows:

1) I_1^{12} **,** $n = 12$ **:** If $n = 12$ exists for I_1^{12}, then there are two I_0 fibers, one over $J = 0$ and one over $J = 1$. Then these I_0 fibers should be fibered over the points 0 and ∞ of $\mathbb{P}^1$ fixed by the induced automorphism (see Section 4). Since one of the I_0 have j-invariant 0 (or since $12 \neq 2 \mod 4$ even though the other I_0 has j-invariant 1), there is no $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 24$. Thus, this configuration is eliminated.

2) I_1^{12} **or** I_2^{6} **,**$n = 6$ **:** If $n = 6$ exists for these cases, then at least one I_0 over $J = 0$ is fibered over 0 or ∞ of $\mathbb{P}^1$ (see Section 4). Since the j-invariant is 0 and n is even, these cases are eliminated by Lemma 5.2.2.

3) I_1^{12} **,** I_3^{4} **,** $I_2^4 I_1^4$ **or** $II^4 I_1^4$ **,** $n = 4$ **:** If $n = 4$ exists, then at least one I_0 fiber over $J = 1$ should be fibered over 0 or ∞ of $\mathbb{P}^1$ (see Section 4). Since $4 \neq 2 \mod 4$, we can eliminate these cases by Lemma 5.2.2..

4) $III^2 II^2 I_1^2$ **,** $n = 2$**:** If an induced automorphism has order 2, then there should be two I_0 fibers, one over $J = j_1$ and the other over $J = j_2$ where $J_i \neq 0, 1$ with the multiplicity of the J-map 2 at each I_0. Then these I_0 fibers are fibered over the points 0 and ∞ fixed by that induced automorphism. Since the j-invariants are not 1, Lemma 5.2.2 eliminates this case from the discussion of this section.

5) If in a configuration of singular fibers there is an odd number of IV , IV^* or I_M $(M > 0)$ fibers, then for even n , if order of an induced automorphism is n, one of the fibers which appear an odd number of times in the configuration must be fibered over a point of $\mathbb{P}^1$ which is fixed by this induced automorphism (see Section 4). Then Lemma 5.2.1 eliminates this configuration of singular fibers from our discussion in this section.

Going through the information in Table 2, and using Lemma 5.2.1 and Lemma 5.2.2, we can eliminate a number of configurations. The configurations which are left to be examined after this elimination are listed in Table 9. Note that the existence of α for each configuration in Table 9 is proved in the rest of this section.

n	$deg(J)$	Configuration of Singular Fibers
10	10	$II\,I_1^{10}$
6	6	$I_0^*I_1^6$
4	8	$II^2I_2^4$, $II^2I_1^8$
	4	$I_0^*II\,I_1^4$
2	12	$I_5^2I_1^2$, $I_4^2I_2^2$, $I_4^2I_1^4$, I_3^4 , $I_3^2I_2^2I_1^2$, $I_3^2I_1^6$, I_2^6 , $I_2^4I_1^4$, $I_2^2I_1^8$, I_1^{12}
	10	$II\,I_4^2I_1^2$, $II\,I_3^2I_2^2$, $II\,I_3^2I_1^4$, $II\,I_2^4I_1^2$, $II\,I_2^2I_1^6$, $II\,I_1^{10}$
	8	$II^2I_4^2$, $II^2I_3^2I_1^2$, $II^2I_2^4$, $II^2I_2^2I_1^4$, $II^2I_1^8$
	6	$I_4^*I_1^2$, $I_2^*I_2^2$, $I_2^*I_1^4$, $I_0^*I_2^2I_1^2$, $I_0^*I_1^6$, $III^2I_3^2$, $III^2I_2^2I_1^2$, $III^2I_1^6$, $II^3I_3^2$, $II^3I_2^2I_1^2$, $II^3I_1^6$
	4	$I_2^*II\,I_1^2$, $I_0^*II\,I_1^4$, $IV^2I_2^2$, $IV^2I_1^4$, $III^2II\,I_2^2$, $III^2II\,I_1^4$, $II^4I_2^2$, $II^4I_1^4$
	2	$I_2^*II^2$, $I_0^*II^2I_1^2$, $II^*I_1^2$

Table 9: Configurations for which $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2{\cdot}ord(\phi(\alpha)) = 2n$ exists.

Remark: If $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ exists for $II\,I_1^{10}$ when $n = 10$, then $ord(\alpha^4) = ord(\phi(\alpha^4)) = 5$, so the elliptic surface is obtained from a pull-back of another surface by the map g_5 (see Section 5.1). Then α induces an automorphism $\bar{\alpha}$ of that second surface with $ord(\bar{\alpha}) = 4$ and $ord(\phi(\bar{\alpha})) = 2$. Table 6 shows that $II\,I_1^{10}$ pulls back only from $II^*I_1^2$. Conversely, if $\bar{\alpha}$ with the given orders exists for an elliptic surface with the configuration $II^*I_1^2$, then after pulling back by the map g_5, $\bar{\alpha}$ induces an automorphism α on the pull-back surface which satisfies $ord(\alpha) = 20$ and $ord(\phi(\alpha)) = 10$. Thus, such an α exists for $II\,I_1^{10}$ if and only if an $\bar{\alpha}$ with the specified orders exists for $II^*I_1^2$. Similarly, for $n = 6$, α exists for $I_0^*I_1^6$ if and only if $\bar{\alpha}$ with $ord(\bar{\alpha}) = 4$ and $ord(\phi(\bar{\alpha})) = 2$ exists for $II^*I_1^2$ (Since the former is a pull-back of the latter by g_3, and if α exists α^4 and $\phi(\alpha^4$ both have order 3).

5.2.1 General construction

We now construct the automorphisms $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2{\cdot}ord(\phi(\alpha)) = 2n$ by taking the double cover of the rational ruled surface F_2 branched over the minimal section and a trisection T, where T is preserved under an order n automorphism of F_2 which also induces an order n automorphism on $\mathbb{P}^1$. The automorphism of F_2 induces an automorphism on the double cover, which is the Weierstrass fibration of a rational elliptic surface.

If B is a relatively minimal rational elliptic surface (with section), then its Weierstrass fibration B' (which is obtained by collapsing all the components of the singular fibers except for those intersecting the zero section) is a double cover of the rational ruled surface F_2 branched over the minimal section E_∞ (the unique divisor with self intersection -2 in F_2) and a trisection T (a divisor which has intersection number 3 with the fiber F of the ruled surface F_2) (p.179 in [6]). The involution of this cover is the map induced on B' by the automorphism $-\mathbb{I}$ on B acting as the inversion of the group law in every smooth fiber. Hence, F_2 is the quotient of B' by the action of this automorphism (p.30 in [8]). The trisection T and the minimal section E_∞ are disjoint. T intersects the fiber F of F_2 at three distinct points exactly when the fiber G of B' over F is a smooth elliptic curve. The Kodaira type of the singular fibers of B are determined by the singularities of T and the intersection points of T and F as the following proposition shows (Proposition IV.2.2 in [8]). Note that T can have only simple singularities of types A_n , D_n or E_n (p.39 in [8]).

Proposition 5.2.3. *Let B be a relatively minimal rational elliptic surface with section, $\pi : B' \to F_2$ be the double cover of its Weierstrass fibration, and T the trisection of the branch locus of that cover. If a fiber G of B projects to the fiber F of F_2, then*
a) G has type I_0 if $T \cap F$ has 3 distinct points.
b) If $T \cap F = p + 2q$ where p and q are distinct points, then
b1) G has type I_1 if T is smooth at q.
b2) G has type I_n if T has an A_{n-1} singularity at q.
c) If $T \cap F = 3p$, then
c1) G has type iI If T is smooth at p.
c2) G has type III if p is a double point of T with an A_1 singularity.
c3) G has type IV if p is a double point of T with an A_2 singularity.
c4) G has type I_n^* if p is a triple point of T with a D_{n+4} singularity.
c5) G has type II^* , III^* or IV^* if p is a triple point of T with an E_8 , E_7 or E_6 singularity, respectively.

The table below describes the types of simple curve singularities of curves on smooth surfaces. Here C is the curve, p is the singular point, E is the exceptional divisor and C' is the proper transform of C after the blow-up at p

Type	Equation	Description
A_0	$x = 0$	smooth
A_1	$x^2 = y^2$	ordinary node
A_2	$x^2 = y^3$	ordinary cusp
A_n	$x^2 = y^{n+1}$	higher order cusp or tacnode ($n \geq 3$)
D_4	$yx^2 = y^3$	ordinary triple point
$D_{n\geq 5}$	$yx^2 = y^{n-1}$	triple point of C where C' meets E at two points; one smooth, one singular of type A_{n-5}
E_6	$x^3 = y^4$	triple point of C with one tangent, C' is smooth and meets E at one point to order 3
E_7	$x^3 = xy^3$	triple point of C with one tangent, C' has an ordinary node with E one of the tangents
E_8	$x^3 = y^5$	triple point of C with one tangent, C' has an ordinary cusp with E tangent

Table 10: Simple curve singularities for curves on surfaces.

If $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ exists, then α induces an automorphism $\bar{\alpha}$ of F_2 with $ord(\bar{\alpha}) = n$ since $\alpha^n = -\mathbb{I}$ and F_2 is the quotient of the Weierstrass fibration B' by the action of the automorphism induced by $-\mathbb{I}$. Note that $\bar{\alpha}$ acts on F_2 by mapping the fibers to fibers, and the induced map on $\mathbb{P}^1$ by $\bar{\alpha}$ is the same as the induced map of α. $\bar{\alpha}$ preserves the branch locus of the double cover $\pi : B' \to F_2$. Since the minimal section E_∞ is preserved by every automorphism of F_2, $\bar{\alpha}$ preserves the trisection T.

Conversely, if any order n automorphism $\bar{\alpha}$ of F_2 inducing an order n automorphism on $\mathbb{P}^1$ preserves a trisection T, then $\bar{\alpha}$ lifts to an automorphism of the double cover branched over T and E_∞ which is a Weierstrass fibration. The automorphism $\bar{\alpha}$ of F_2 lifts since the double covers of F_2 branched over a given branch locus are unique. One can define this lifting by pulling-back the unique cover branched over T and E_∞ by the map $\bar{\alpha}$. Since $\bar{\alpha}$ preserves T and E_∞, the pull-back cover will be branched over the same locus, and by the uniqueness of the covers, the map induced by the pull-back gives a lifting of $\bar{\alpha}$ to an automorphism of the double cover. This lifted automorphism then gives an automorphism α of the corresponding relatively minimal rational elliptic surface B. Here, the induced automorphisms of α and $\bar{\alpha}$ are the same, hence have order n. Whether $ord(\alpha) = 2n$ or n should be checked.

We are then searching for pairs (Γ, T) satisfying $\Gamma \in Aut(F_2)$, $ord(\Gamma) = n$ and $\Gamma(T) = T$ (Γ preserves T as a set), where T is a trisection such that the elliptic surface B whose Weierstrass fibration is a double cover of F_2 branched over this T and E_∞ has a configuration of singular fibers which is listed in Table 9. If Θ is another automorphism of F_2 ,then $\Theta(T)$ gives rise to the same configuration of singular fibers by the above proposition since the type of singularities of T and the intersection multiplicities are preserved by Θ, and a fiber of F_2 maps to another fiber by Θ. Then $(\Theta\circ\Gamma\circ\Theta^{-1}, \Theta(T))$ is also a suitable pair. Hence, it suffices to work on the conjugacy classes of the automorphisms of F_2. We have the following lemma which is proved in the appendix.

Lemma 5.2.4. *There are only two conjugacy classes of order 2 automorphisms of F_2 whose representatives induce an order 2 automorphism on $\mathbb{P}^1$. If F_2 is mapped to $\mathbb{P}^3$ after collapsing the minimal section E_∞ to the singular point of the quadric cone $Y^2 = XZ$ in $\mathbb{P}^3$, these two conjugacy classes are represented by the automorphisms of the cone induced by the following automorphisms of $\mathbb{P}^3$:*

$$\Gamma_1 : \mathbb{P}^3 \to \mathbb{P}^3\ ,\ [X, Y, Z, W] \mapsto [X, -Y, Z, W] \tag{19}$$

$$\Gamma_2 : \mathbb{P}^3 \to \mathbb{P}^3\ ,\ [X, Y, Z, W] \mapsto [X, -Y, Z, -W]. \tag{20}$$

Remark: If $\alpha \in Aut(B)$, $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 2n$ where n is even, and C is the fiber of B over 0 or ∞ (fixed points of $\phi(\alpha)$), then α^n acts on $C^\sharp = B^\sharp_z$ $(z = 0, \infty)$ as the inversion of the group structure on $C^\sharp$. Since n is even α cannot act on $C^\sharp$ as the inversion or the identity. Hence, the induced automorphism on F_2 does not act as the identity on the fibers over 0 or ∞. In particular, for $n = 2$, α cannot induce the automorphism Γ_1 in the above lemma. For $n = 4$, $ord(\alpha^2) = 2 \cdot ord(\phi(\alpha^2)) = 4$, hence α^2 induces an automorphism of F_2 which is conjugate to Γ_2. The following lemma is also proved in the appendix.

Lemma 5.2.5. *There are only two conjugacy classes of order 4 automorphisms of F_2 such that the square of any representative is conjugate to Γ_2. If F_2 is mapped to $\mathbb{P}^3$ after collapsing the minimal section E_∞ to the singular point of the quadric cone $Y^2 = XZ$ in $\mathbb{P}^3$, these two conjugacy classes are represented by the automorphisms of the cone induced by the following automorphisms of $\mathbb{P}^3$:*

$$\Delta_1 : \mathbb{P}^3 \to \mathbb{P}^3\ ,\ [X, Y, Z, W] \mapsto [X, iY, -Z, iW] \tag{21}$$

$$\Delta_2 : \mathbb{P}^3 \to \mathbb{P}^3\ ,\ [X, Y, Z, W] \mapsto [X, iY, -Z, -iW]. \tag{22}$$

In the rest of this section, we will consider the three automorphisms Γ_2, Δ_1 and Δ_2. Our goal is to show the existence of the trisections T of F_2 which are preserved by these automorphisms, and which give rise to a particular configuration of singular fibers listed in Table 9 when the double cover of F_2 branched over T and E_∞ is considered. If F is the divisor class of a fiber of F_2 and E_∞ is the minimal section (the unique divisor with self intersection (-2) in F_2), then $T = 3F + 6E_\infty$ in the Picard group of F_2 since $T \cdot F = 3$ and $T \cdot E_\infty = 0$. $h^0(F_2, 3F + 6E_\infty) = 16$ (Theorem 1.8 in [4]). When we consider F_2 as the quadric cone Q given by $Y^2 = XZ$ (after collapsing E_∞ to the vertex of the cone), the trisections are given by cubics nonvanishing on Q, which has dimension $h^0(\mathbb{P}^3, 3H) - h^0(\mathbb{P}^3, 3H - 2H) = 20 - 4 = 16$. The automorphisms Γ_2, Δ_1 and Δ_2 of $\mathbb{P}^3$ act as linear transformations on the vector space of these cubics, and the trisections of F_2 preserved by the automorphism are the ones corresponding to the eigenvectors of that linear transformation.

Lemma 5.2.6. *A basis for the cubics P(X,Y,Z,W) that do not vanish on the quadric cone Q given by* $Y^2 = XZ$ *is*
$\{X^3, Z^3, W^3, X^2Z, X^2W, Z^2X, Z^2W, W^2X, W^2Z, XZW, X^2Y, Z^2Y, W^2Y, XZY, XWY, ZWY\}$
and when the linear action of an automorphism θ *of* $\mathbb{P}^3$ *on that vector space by* $\theta(P(X,Y,Z,W)) = P(\theta^{-1}(X), \theta^{-1}(Y), \theta^{-1}(Z), \theta^{-1}(W))$ *is considered (so that the hypersurface* $P = 0$ *maps to* $\theta(P) = 0$*), the eigenspaces containing the cubics not passing through the vertex [0,0,0,1] of the cone Q are as follows:*

θ	*Eigenspace*
Γ_2	$\langle X^2Y, XYZ, YZ^2, X^2W, Z^2W, XZW, YW^2, W^3\rangle$
Δ_1	$\langle XYZ, XZW, YW^2, W^3\rangle$
Δ_2	$\langle X^2Y, YZ^2, XZW, W^3\rangle$

5.2.2 Trisections for Γ_2:

We now show the existence of the trisections giving rise to the configurations in Table 9 for $n = 2$.

Since we are interested in the trisections not passing through the vertex $[0,0,0,1]$ of the cone (which corresponds to the minimal section E_∞ of F_2), the equation of the trisection is of the form:

$$Y^2 = XZ$$
$$W^3 + bYW^2 + cX^2W + dXZW + eZ^2W + fX^2Y + gXYZ + hYZ^2 = 0. \tag{23}$$

If we denote the cubic above by $P(X,Y,Z,W)$, and the trisection given by the above equations by T, then if $\theta : [X,Y,Z,W] \mapsto [X,Y,Z,W + bY/3]$, θ gives an automorphism of F_2 (seen as the cone $Y^2 = XZ$ in $\mathbb{P}^3$) that induces the identity on $\mathbb{P}^1$, and $\theta(T)$ is given by $P(X,Y,Z,W - bY/3)$. Replacing Y^2 by XZ, this last cubic is of the same form as $P(X,Y,Z,W)$ and we have $b = 0$. Then $\theta(T)$ is also preserved by Γ_2. Note that the singular fibers of the elliptic surfaces corresponding to T and $\theta(T)$ are the same and they are fibered over the same points of $\mathbb{P}^1$. Thus, we may assume that $b = 0$, hence T is given by

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + cX^2W + dXZW + eZ^2W + fX^2Y + gXYZ + hYZ^2 = 0. \end{aligned} \tag{24}$$

Denote the lines on the cone Q, $Y^2 = XZ$, by

$$L_t = \{[1, t, t^2, w] | w \in \mathbb{C}\} \cup \{[0,0,0,1]\} \ , \ t \in \mathbb{C}, \tag{25}$$

$$L_\infty = \{[0,0,1,w] | w \in \mathbb{C}\} \cup \{[0,0,0,1]\}. \tag{26}$$

Note that L_t is the fiber of F_2 over $t \in \mathbb{P}^1$ (including $t = \infty$).
We can use the affine chart on $Q - L_\infty$ given by

$$(t, w) \mapsto [1, t, t^2, w].$$

On this chart, the trisection T is given by the equation

$$T \ : \ w^3 + (c + dt^2 + et^4)w + (ft + gt^3 + ht^5) = 0 \tag{27}$$

and the line L_{t_0} is given by

$$L_{t_0} \ : \ t = t_0. \tag{28}$$

Recall that the rational elliptic surface corresponding to T has a singular fiber over $t \in \mathbb{P}^1$ if $L_t \cap T$ does not have 3 distinct points (Proposition 5.2.3). Viewing the above equation as a cubic in the variable w, there are repeated roots iff the discriminant vanishes:

$$\text{There is a singular fiber over } t \text{ iff } \Delta(t) = 4(c+dt^2+et^4)^3+27(ft+gt^3+ht^5)^2 = 0. \tag{29}$$

From Proposition 5.2.3, the singular fiber over t is of type I_n iff $L_t \cap T$ consists of two distinct points, and of type I_n^*, II, III, IV, IV^*, III^* or II^* iff $L_t \cap T$ consists of a single point.

$$L_t \cap T \text{is a single point iff } c + dt^2 + et^4 = 0 \text{ and } ft + gt^3 + ht^5 = 0. \tag{30}$$

Then, there is an I_n $(n > 0)$ over t_0 iff

$$\Delta(t_0) = 0 \quad \text{and} \quad t_0 \quad \text{is not a common root of} \quad c+dt^2+et^4 \quad \text{and} \quad ft+gt^3+ht^5. \tag{31}$$

Specifically, for the line L_0 we have

$$L_0 \cap T \quad \text{is a single point iff} \quad c = 0\,, \text{ and three distinct points otherwise.} \tag{32}$$

$$T \quad \text{is singular at} \quad L_0 \cap T \quad \text{iff} \quad c = f = 0. \tag{33}$$

For the line L_∞ which is not in that affine chart we have:

$$L_\infty \cap T \quad \text{is a single point iff} \quad e = 0 \quad \text{and three distinct points otherwise.} \tag{34}$$

$$T \quad \text{is singular at} L_\infty \cap T \quad \text{iff} \quad e = h = 0. \tag{35}$$

The order of vanishing of $\Delta(t)$ at t_0 gives important information about the type of the singular fiber over t_0 (p.30 and 41 in [8]). The order of vanishing of $\Delta(t)$ for each type of singular fiber is as follows:

$$I_n : n \quad I_n^* : n{+}6 \quad II : 2 \quad III : 3 \quad IV : 4 \quad IV^* : 8 \quad III^* : 9 \quad II^* : 10. \tag{36}$$

Remark: For a relatively minimal elliptic surface B with section given by the

Weierstrass data (L, A, B) where L is a line bundle over the base curve C of the elliptic surface B, and A and B are two sections of L^4 and L^6, respectively, such that B is given by the Weierstrass equation $Y^2Z = X^3 + AXZ^2 + BZ^3$ in the $\mathbb{P}^2$ bundle $P(C, \mathcal{O}_C \oplus L^{-2} \oplus L^{-3})$ over C, the types of the singular fibers of B are determined by the orders of vanishing of the line bundles A, B, and the discriminant $\Delta = 4A^3 + 27B^2$ at the points of C corresponding to the singular fibers (basically the points over which Δ vanishes). This is known as the *Tate's Algorithm* (p.40 in [8]). In the above discussion, for the rational elliptic surface whose Weierstrass fibration is the double cover of F_2 branched over the minimal section and the trisection T given by the equation (27), the Weierstrass data is as follows: L is the line bundle $\mathcal{O}_{\mathbb{P}^1}(H)$, $A = cs^4 + ds^2t^2 + et^4$, and $B = fs^5t + gs^3t^3 + hst^5$.

Using the above criteria, we now show the existence of the trisections giving rise to each configuration of singular fibers in Table 9.

• $II^*I_1^2$**:** If $e \neq 0$ in (24), there is an I_0 fiber over ∞ by (34). If there is a II^* fiber over $t = 0$, then by Proposition 5.2.3, (27) has a singular point at $(0, 0)$, hence $c = f = 0$ by (33); and the type of the singularity is E_8. Thus, there is a unique tangent to (27) at $(0, 0)$, hence $d = g = 0$. Then the discriminant is $\Delta(t) = t^{10}(4e^3t^2 + 27h^2)$ from (29). If $e = -3$ and $h = 2$ then the order of vanishing of $\Delta(t)$ at 0 is 10, and it is 1 at ± 1. By (29) and (36), this gives a II^* over 0 and two I_1 fibers over 1 and -1. Therefore, if $c = d = f = g = 0$, $e = -3$ and $h = 2$; the trisection given by (24) corresponds to the configuration $II^*I_1^2$, and $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 4$ exists for $II^*I_1^2$.

• $I_4^*I_1^2$ **:** If α exists, there is an I_0 over $t = \infty$ and an I_4^* over $t = 0$. Then, by (34) and (33), $e \neq 0$ and $c = f = 0$. The type of singularity of T corresponding to the singular fiber I_4^* is D_8, hence (27) must have two tangent lines at $(0, 0)$; thus, $4d^3 + 27g^2 = 0$ and $g \neq 0$. Then $\Delta(t) = t^8[(12d^2e + 54gh) + (12de^2 + 27h^2)t^2 + 4e^3t^4]$ from (29). If $d = -3$, $e = 1$, $g = 2$ and $h = -1$, then $\Delta(t)$ vanishes to order 10 at zero and order 1 at $\pm 3/2$. By (36), there is an I_4^* fiber over $t = 0$ and two I_0 fibers over $t = \pm 3/2$. Thus, α exists for $I_4^*I_1^2$.

• $I_2^*I_1^4$ **,** $I_2^*I_2^2$ **,** $I_2^*II^2$ **:** To have an I_0 over $t = \infty$ and an I_2^* over $t = 0$, $e \neq 0$ by (34) and $c = f = 0$ by (33). I_2^* corresponds to a D_6 singularity, so (27) has two tangent lines at $(0, 0)$, thus, $4d^3 + 27g^2 = 0$ and $g \neq 0$. Let $e = 1$, $d = -3$ and $g = 2$, then from (29), $\Delta(t) = t^8[108(1+h) + (27h^2 - 36)t^2 + 4t^4]$. If $h \neq -1$, the order of vanishing of $\Delta(t)$ at $t = 0$ is 8, which gives an I_2^* over 0 by (36). If $((27h^2 - 36)^2 - 1728(h + 1) \neq 0$, then $\Delta(t)$ has four other distinct roots with the order of vanishing 1 at each, which corresponds to four I_1 fibers. Thus, α exists for $I_2^*I_1^4$. If $(27h^2 - 36)^2 - 1728(h + 1) = 0$, then $h = 2$ or $h = -2/3$, and $\Delta(t)$ has two more distinct roots with multiplicity 2. By (36), the fibers corresponding to these two roots are either I_2 or II. If $h = -2/3$, these two roots are $\pm i\sqrt{3}$ and these are common roots of $c + dt^2 + et^4$ and $ft + gt^3 + ht^5$, hence by (31), we get two II fibers. For $h = 2$, $c + dt^2 + et^4$ and $ft + gt^3 + ht^5$

do not have a common non-zero root, hence we get two I_2 fibers by (31). Thus, α exists for $I_2^* I_2^2$ and $I_2^* II^2$.

• $I_0^* I_1^6$, $I_0^* I_2^2 I_1^2$, $I_0^* II^2 I_1^2$: As in the above cases, since we have an I_0 over $t = \infty$ and I_0^* over $t = 0$, we must have $e \neq 0$ and $c = f = 0$. Since I_0^* corresponds to a D_4 singularity, (27) has three distinct tangent lines at $(0, 0)$, hence $4d^3 + 27g^2 \neq 0$. Then $\Delta(t) = t^6[4(d + et^2)^3 + 27(g + ht^2)^2]$. The root $t = 0$ corresponds to I_0^* since it is a root of $\Delta(t)$ with multiplicity 6. To determine the other singular fibers, we should examine the multiplicities of the other roots of $\Delta(t)$. Note that $\Delta(t)$ is a polynomial in t^2. To simplify the notation, let $S = t^2$, $D = \sqrt[3]{4}d$, $E = \sqrt[3]{4}e$, $G = 3\sqrt{3}g$ and $H = 3\sqrt{3}h$. Then $\Delta = S^3[(ES + D)^3 + (HS + G)^2]$ where $E \neq 0$ and $D^3 + G^2 \neq 0$. If $E = G = H = 1$ and $D = 0$, we get $S^3(S^3 + S^2 + 2S + 1)$. $S^3 + S^2 + 2S + 1$ has three distinct roots with multiplicity 1. Then $\Delta(t)$ has six distinct non-zero roots with multiplicity 1 which gives six I_0 fibers. Then α exists for $I_0^* I_1^6$. If $E = 1$, $D = H = -3$, then $\Delta = S^3[S^3 + (27 - 6G)S + G^2 - 27]$. There are non-zero multiple roots iff $4(27-6G)^3 + 27(G^2-27)^2 = 0$ and $G^2 - 27 \neq 0$. Then $G = 5$ or $G = 9$. If $G = 9$, $(ES + D)$ and $HS + G$ have a common root, and if $G = 5$, there is no common root. Then, by (31), $G = 5$ corresponds to I_2 and $G = 9$ corresponds to II fibers (since there is one multiplicity 2 and one multiplicity 1 root of $(ES+D)^3 + (HS+G)^2$). Hence, α exists for $I_0^* I_2^2 I_1^2$ and $I_0^* II^2 I_1^2$.

• $I_0^* II\ 1^4$: There is a II fiber over $t = \infty$ iff $e = 0$ and $h \neq 0$ by Proposition 5.2.3, (34) and (35). If there is an I_0^* over $t = 0$, then $c = f = 0$ by (33), and since I_0^* corresponds to a D_4 singularity, (27) has three tangent lines at $(0, 0)$, hence $4d^3 + 27g^2 \neq 0$. If $d = h = 1$ and $g = 0$, then $\Delta(t) = t^6(27t^4 + 4)$. $\Delta(t)$ vanishes to order 6 at $t = 0$ and to order 1 at four other distinct points. By (36), these roots of $\Delta(t)$ correspond to one I_0^* and four I_1 fibers. Then α exists for $I_0^* II\ I_1^4$.

• $I_2^* II\ I_1^2$: There is a II fiber over $t = \infty$ iff $e = 0$ and $h \neq 0$ as above. If there is an I_2^* over $t = 0$, then $c = f = 0$, and since I_2^* corresponds to a D_6 singularity, (27) has two tangent lines at $(0, 0)$, hence $4d^3 + 27g^2 = 0$ and $g \neq 0$. If $d = -3$, $g = 2$ and $h = 1$, then $\Delta(t) = t^8(108 + 27t^2)$. The roots have multiplicities 8, 1 and 1, hence there are one I_2^* and two I_1 fibers corresponding to those roots. Then α exists for $I_2^* II\ I_1^2$.

• $II^3 I_3^2$, $II^3 I_2^2 I_1^2$, $II^3 I_1^6$: There is an I_0 over $t = \infty$ iff $e \neq 0$ by (32), and there is a II over $t = 0$ iff $c = 0$ and $f \neq 0$. There are II fibers over $t = \pm 1$ iff $L_t \cap T$ is a smooth point of T and the intersection multiplicity is 3 at that point for $t = \pm 1$. By (30) $L_{\pm 1}$ intersects T at a single point (with intersection multiplicity 3) iff $d + e = 0$ (since c=0) and $f + g + h = 0$. If these hold, then (27) is smooth at $(t, w) = (\pm 1, 0)$ iff $f \neq h$. If this also holds, $\Delta(t) = 4(-et^2 + et^4)^3 + 27(ft - (f+h)t^3 + ht^5)^2 = t^2(t^2-1)^2[4e^3(t^6 - t^4) + 27(ht^2 - f)^2]$. Except for $t = 0, \pm 1$, L_t cannot intersect T at a single point by (30), hence all the other roots of $\Delta(t)$ correspond to I_n fibers where n is the multiplicity of $\Delta(t)$

at that root. Writing $S = t^2$ and $e = 3E$, we are examining the multiplicities of the roots of $\Delta^\sharp(S) = 4E^3S^3 + (h^2 - 4E^3)S^2 - 2fhS + f^2$. If $E = 1$, $f = -2$ and $h = 2$, $\Delta^\sharp(S)$ has three distinct roots, hence there are six distinct roots of $\Delta(t)$ corresponding to I_1 fibers. Then α exists for $II^3I_1^6$. It can be checked that for $E = 9$, $f = 64$ and $h = 54$, $\Delta^\sharp(S)$ has $8/9$ as a double root and $-16/9$ as a simple root. If $E = -3$, $f = 16$ and $h = 18$, then $4/3$ is a triple root. Then α exists for $II^3I_2^2I_1^2$ and $II^3I_3^2$.

• $III^2II\ I_2^2$ **,** $III^2II\ I_1^4$ **:** Similar to the above case, $e \neq 0$ to have an I_0 over $t = \infty$; $c = 0$ and $f \neq 0$ to have a II over $t = 0$; $d + e = 0$ and $f + g + h = 0$ in order for $L_{\pm 1} \cap T$ to be a single point. Differently from the above case, this time $f = h$ for T to be singular at $L_{\pm 1} \cap T$. With these conditions, $\Delta(t) = t^2(t^2 - 1)^3(4e^3t^4 + 27h^2t^2 - 27h^2)$. $t = \pm 1$ gives III by (36) and $t = 0$ gives II. The other roots of $\Delta(t)$ give I_n where n is the multiplicity of the root. Writing $S = t^2$, $E = 4e^3$ and $H = 27h^2$ then we are concerned with the multiplicities of the roots of $\Delta^*(S) = ES^2 + HS - H$. For $E = H = 1$, there are two distinct roots and for $E = 1$ and $H = -2$, there is a double root. Hence, α exists for both $III^2II\ I_1^4$ and $III^2II\ I_2^2$.

• $II^2I_2^4$ **,** $II^2I_2^2I_1^4$ **,** $II^2I_1^8$ **:** To have II fibers over $t = 0$ and $t = \infty$, we have $c = e = 0$, $f \neq 0$ and $h \neq 0$ by (32), (33), (34) and (35). If $d \neq 0$, $L_t \cap T$ is not a single point unless $t = 0$ or $t = \infty$ by (30). Then the non-zero roots of $\Delta(t)$ correspond to I_n fibers where n is the multiplicity of the root by (31) and (36). $\Delta(t) = t^2[4d^3t^4 + 27(f + gt^2 + ht^4)^2]$. Writing $S = t^2$, $d = -3D$, $f = 2F$, $g = 2G$ and $h = 2H$, we get $\Delta(S) = 108S[(F + GS + HS^2)^2 - D^3S^2]$. We are concerned with the multiplicities of the roots of $\Delta^*(S) = (F + GS + HS^2)^2 - D^3S^2 = (F + (G + D^{3/2})S + HS^2)(F + (G - D^{3/2})S + HS^2)$. Since $D \neq 0$, $\Delta^*(S)$ cannot have a root with multiplicity 4 or 3. If $F = H = 1$, $G = 0$ and $D^{3/2} = 2$, then $\Delta^*(S)$ has two roots with multiplicities 2, hence $\Delta(t)$ has four non-zero roots with multiplicity 2, giving four I_2 fibers. Then α exists for $II^2I_2^4$. If $D = F = G = H = 1$, then $\Delta^*(S)$ has one root with multiplicity 2 and two roots with multiplicity 1, hence α exists for $II^2I_2^2I_1^4$. If $F = G = H = 1$ and $D^{3/2} = 2$, then $\Delta^*(S)$ has four distinct roots with multiplicity 1, hence α exists for $II^2I_1^8$.

• $II^2I_4^2$ **,** $II^2I_3^2I_1^2$ **:** To have I_0 fibers over $t = 0$ and $t = \infty$, we have $c \neq 0$ and $e \neq 0$ by (32) and (34). In order to have II fibers over $t = \pm 1$, $L_{\pm 1} \cap T$ is a single smooth point of T, hence by (30), $c + d + e = f + g + h = 0$ and the intersection point is a smooth point iff $f \neq h$ ((27) is smooth at $(t, w) = (\pm 1, 0)$). With these conditions, we get $\Delta(t) = (t^2 - 1)^2[4(et^2 - c)^3(t^2 - 1) + 27t^2(ht^2 - f)^2]$.Writing $S = t^2$, $e = 3E$, $c = 3C$, $f = 2F$ and $h = 2H$, we get $\Delta(t) = \Delta^*(S) = 108(s - 1)^2[(ES - C)^3(S - 1) + S(HS - F)^2]$. Note that a root t_0 of $\Delta(t)$ gives an I_n fiber iff $c + dt^2 + et^4$ and $ft + gt^3 + ht^5$ do not vanish simultaneously at t_0 by (31). Since $t_0 = \pm 1$ are already common roots, if another common root exists then $(f, g, h) = \lambda(c, d, e)$. If this is not the case, there is an I_n fiber over the root t_0 of $\Delta(t)$ where n is the multiplicity of the root. Then we are concerned with the multiplicities of the roots of $\Delta^\sharp(S) = (ES-C)^3(S-1)+S(HS-F)^2$. It can be checked that $\Delta^\sharp(S) = E^3(S-a)^4$ holds with $a \neq 1$, $a \neq 0$, hence $F \neq H$ and $C \neq 0$ (a particular solution can be obtained with $E = 2$). Then α exists for $II^2 I_4^2$. It can also be checked that $\Delta^\sharp(S) = E^3(S-a)^3(S-b)$ where $a \neq b$ holds for some $C \neq 0$, $E \neq 0$, $F \neq H$, $a \neq 0, 1$ and $b \neq 0, 1, a$. A particular solution exists with $E = 1$ and $a = 2$. Then since there is a root with multiplicity 3 and a root with multiplicity 1, α exists for $II^2 I_3^2 I_1^2$.

• $II^4 I_2^2$ **,** $II^4 I_1^4$ **:** To have I_0 fibers over $t = 0, \infty$ and II fibers over $t = \pm 1$, we must have $c \neq 0$, $e \neq 0$, $f \neq h$ and $c+d+e = f+g+h = 0$ as in the above case. As also explained above, if $(f, g, h) = \lambda(c, d, e)$, then there is a $t_0 \neq \pm 1$ such that $L_{\pm t_0} \cap T$ is a single point. Then there is a II fiber over $t = \pm t_0$ if T is smooth at that point. With these conditions and the notation in the above case, we get $\Delta(t) = \Delta^*(S) = 108(S-1)^2(ES-C)^2[(ES-C)(S-1)+\lambda^2 S]$. $t = \pm 1$ and $t = \pm\sqrt{C/E}$ give II fibers and the other roots of $\Delta(t)$ give I_n fibers where n is the multiplicity of the root. If $E = 1$, $C = 4$ and $\lambda = 3$, then $(ES-C)(S-1)+\lambda^2 S$ has a double root, while it has two distinct roots if $\lambda = E = 1$ and $C = 4$. Then α exists for $II^4 I_2^2$ and $II^4 I_1^4$.

• $III^2 I_3^2$ **,** $III^2 I_2^2 I_1^2$ **,** $III^2 I_1^6$ **:** To have I_0 fibers over $t = 0, \infty$, we have $c \neq 0$ and $e \neq 0$. To have III fibers over $t = \pm 1$, $L_{\pm 1} \cap T$ is a single point and T is singular at that point. Then as explained before, $c+d+e = f+g+h = 0$ and for the singularity, we have $f = h$. Then $\Delta(t) = (t^2-1)^3[4(et^2-c)^3+27h^2t^2(t^2-1)]$. Writing $S = t^2$, $e = 3E$, $c = 3C$ and $h = 2H$, we have $\Delta(t) = \Delta^\sharp(S) = 108(S-1)^3[(ES-C)^3+H^2S(S-1)]$. If $e \neq c$, then by (36), $t = \pm 1$ gives III fibers since the multiplicity of the root $t = \pm 1$ of $\Delta(t)$ is 3. The other roots give I_n fibers if $h \neq 0$ by (31) where n is the multiplicity of the root by (36). We are then concerned with the multiplicities of the roots of $\Delta^*(S) = (ES-C)^3 + H^2S(S-1)$. If $E = 1$, $C = -\mu_3$ and $H^2 = -3\sqrt{3}i$ where μ_3 is a primitive third root of unity, then $\Delta^*(S) = (S-\mu_3 C)^3$, hence it has a root with multiplicity 3 and α exists for $III^2 I_3^2$. If $E = 1$ and $H^2 = 3C$, then $\Delta^*(S) = S^3 + (3C^2-3C)S - C^3$, and there is a multiple root iff $4(3C^2-3C)^3 + 27C^6 = 0$. There is a root $C \neq 0$ and $C \neq E = 1$, then for this value of C, there is a multiple root and its multiplicity is 2 since a multiplicity 3 root cannot occur in this case unless $C = 0$. Then α exists for $III^2 I_2^2 I_1^2$. Also, if $4(3C^2-3C)^3 + 27C^6 \neq 0$ and $C \neq E = 1$, then there are three roots with multiplicity 1. Then α exists for $III^2 I_1^6$.

• $IV^2 I_2^2$ **,** $IV^2 I_1^4$ **:** As in the case of $III^2 I_1^6$ above, we have $e \neq 0$, $c \neq 0$, $f = h \neq 0$, $c+d+e = f+g+h = 0$; and to have IV fibers instead of III fibers we should have $e = c$ in which case the discriminant (using the same notation) becomes $\Delta(t) = \Delta^\sharp(S) = 108(S-1)^4[E^3(S-1)^2 + H^2 S]$. $t = \pm 1$ gives IV fibers. The roots of $E^3(S-1)^2 + H^2 S$ give I_n fibers. If $E = 1$ and $H = 2$, then $S = -1$ is a double root; and if $E = H = 1$, there are two distinct roots. Then α exists for both $IV^2 I_2^2$ and $IV^2 I_1^4$.

• $II\ I_4^2 I_1^2$, $II\ I_3^2 I_2^2$, $II\ I_3^2 I_1^4$, $II\ I_2^4 I_1^2$, $II\ I_2^2 I_1^6$, $II\ I_1^{10}$ **:** There is a II fiber over $t = 0$ iff $c = 0$ and $f \neq 0$. There is an I_0 over $t = \infty$ iff $e \neq 0$. The other singular fibers are of type I_n iff $dt^2 + et^4$ and $ft + gt^3 + ht^5$ do not have a non-zero common solution by (31). $\Delta(t) = t^2[4t^4(et^2 + d)^3 + 27(f + gt^2 + ht^4)^2]$. Since $e \neq 0$, we can write $e = 3E$, $d = 3DE$, $f = 2E^{3/2}F$, $g = 2E^{3/2}G$, $h = 2E^{3/2}H$ and $S = t^2$ which gives $\Delta(t) = \Delta^\sharp(S) = 108E^3 S[S^2(S + D)^3 + (F + GS + H)^2]$. If $S = -D$ is not a root of $F + GS + HS^2$, then all the singular fibers except for the II over $t = 0$ are of type I_n where n is the multiplicity of the root of $\Delta(t)$. Then we are reduced to examine the multiplicities of the roots of $\Delta^*(S) = S^2(S + D)^3 + (F + GS + HS^2)^2$ under the constraints $F \neq 0$ and $\Delta^*(-D) \neq 0$.
If $D = G = H = 0$ and $F = 1$, there are five distinct roots with multiplicity 1, α exists for $II\ I_1^{10}$.
If $D = H = 0$, $F = 48$ and $G = 20$, then $\Delta^*(S) = (S+4)^2(S^3 - 8S^2 + 48S + 144)$ where $S = -4$ is a double root and the other three roots are distinct. Then α exists for $II\ I_2^2 I_1^6$.
If $D = -19/3$, $F = 6\sqrt{3}$, $G = -26\sqrt{3}/9$ and $H = 4\sqrt{3}$, then
$\Delta^*(S) = (S - 1)^2(S + 2)^2(S + 27)$, hence α exists for $II\ I_2^4 I_1^2$.
If $D = -5/4$, $F = 1/4$, $G = -5/8$ and $H = 5/2$; then
$\Delta^*(S) = (S + 1)^3(S - 1/4)^2$, hence α exists for $II\ I_3^2 I_2^2$.
If $D = -5/6$, $F = i\dfrac{3\sqrt{6}}{8}$, $G = -i\dfrac{35\sqrt{6}}{36}$ and $H = i\dfrac{5\sqrt{6}}{8}$; then
$\Delta^*(S) = (x - 1)^4(x - \dfrac{27}{32})$, hence α exists for $II\ I_4^2 I_1^2$.
To explain how these specific values for D, F, G, H and the corresponding roots are obtained, here we give an outline of the calculations for the case $II\ I_3^2 I_1^4$. For this case, we need $\Delta^*(S)$ to have a root with multiplicity 3 and two roots with multiplicity 1. So we want to have

$$\Delta^*(S) = (S - a)^3(S - b)(S - c)$$

for distinct and non-zero a, b, c. We also require that $\Delta^*(-D) \neq 0$, hence a, b and c are not equal to $-D$. Expanding the above equation and equating the coefficients gives five equations in seven variables. Note that if a solution with distinct a, b and c exists, then we have $b \neq -D$ and $c \neq -D$ automatically satisfied since If $-D$ is a root of $\Delta^*(S) = S^2(S + D)^3 + (F + GS + HS^2)^2$, then it is a root with multiplicity 2 or 3. If we can solve the five equations substituting $b = 1$ and $c = -1$, then if a solution exists and if $\Delta^*(-D) = 0$, then $a = -D$. Thus, since $-D$ is a triple root, we have $\Delta^*(S) = (S + D)^3[S^2 + H^2(S + D)]$, then $b = 1$ and $c = -1$ are roots of $S^2 + H^2(S + D)$ which is impossible. With this contradiction, if $b = 1$ and $c = -1$, then $a \neq -D$ provided a is not equal to b or c. With $b = 1$ and $c = -1$, the five equations we have are:
$F^2 = a^3 \qquad 2FG = -3a^2$
$D^3 + G^2 + 2FH = 3a - a^3$
$2GH + 3D^2 = 3a^2 - 1 \qquad H^2 + 3D = -3a$
Assuming $a \neq 0$, we get $G = \dfrac{-3}{2a}F$, and using this and fourth equation,

$H = -(3a^3 - a - 3aD^2)/(3F)$. Substituting these in the third and fifth equations we obtain $12D^3 + 24aD^2 - 12a^3 - a = 0$ and $9D^4 - 6(3a^2-1)D^2 + 27aD + (3a^2-1)^2 + 27a^2 = 0$. Then the whole system has a solution if these last two equations have a solution. Viewing these as polynomials in D with coefficients in $\mathbb{C}[a]$, there is a solution iff the resultant vanishes. The resultant has a non-zero solution which is not ± 1. Hence, a solution with the desired properties exists. Then α exists for $II\ I_3^2 I_1^4$.

• **Configurations which have I_n fibers only :** There are I_0 fibers over $t = 0, \infty$ iff $c \neq 0$ and $e \neq 0$ by (34) and (32). All the singular fibers are of I_n type iff $c + dt^2 + et^4$ and $ft + gt^3 + ht^5$ do not have a common root in t. Then $\Delta(t) = 4(c + dt^2 + et^4)^3 + 27t^2(f + gt^2 + ht^4)^2$. If $S = t^2$, $c = 3C$, $d = 3D$, $e = 3E$, $f = 2F$, $g = 2G$ and $h = 2H$, we get $\Delta(t) = \Delta^\sharp(S) = 108[(ES^2 + DS + C)^3 + S(HS^2 + GS + F)^2]$ where $C \neq 0$ and $E \neq 0$. If $ES^2 + DS + C$ and $HS^2 + GS + F$ do not have a common root, then all the singular fibers are of I_n type. If $\Delta^\sharp(S)$ has a root at $S = S_0$ with multiplicity n_0, then there are I_{n_0} fibers over $t = \pm\sqrt{S_0}$. Then for each configuration in our list, we must show the existence of a polynomial $\Delta^\sharp(S)$ with the corresponding multiplicities of roots. Since $E \neq 0$, we can write $C = EC^\sharp$, $D = ED^\sharp$, $(F, G, H) = E^{3/2}(F^\sharp, G^\sharp, H^\sharp)$ which gives $\Delta^\sharp(S) = 108E^3[(S^2 + D^\sharp S + C^\sharp)^2 + S(H^\sharp S^2 + G^\sharp S + F^\sharp)^2]$. Since $C^\sharp \neq 0$, if S_0 is a root of $\Delta^\sharp(S)$, then $S_0/\sqrt{C^\sharp}$ is a root of $\Delta^\sharp(S\sqrt{C^\sharp}) = 108E^3[(C^\sharp S^2 + D^\sharp\sqrt{C^\sharp}S + C^\sharp)^3 + S\sqrt{C^\sharp}(H^\sharp C^\sharp S^2 + G^\sharp\sqrt{C^\sharp}S + F^\sharp)^2]$ whose roots have the same multiplicities as $\Delta^\sharp(S)$. Letting $D^\sharp = D^*\sqrt{C^\sharp}$, $H^\sharp = H^*(C^\sharp)^{1/4}$, $G^\sharp = G^*(C^\sharp)^{3/4}$ and $F^\sharp = F^*(C^\sharp)^{5/4}$ we get $\Delta^\sharp(S\sqrt{C^\sharp}) = 108E^3(C^\sharp)^3[(S^2 + D^*S + 1)^3 + S(H^*S^2 + G^*S + F^*)^2]$. Then we can reduce to examining the multiplicities of the roots of

$$\Delta^*(S) = (S^2 + DS + 1)^3 + S(HS^2 + GS + F)^2$$

under the constraint that $S^2 + DS + 1$ and $HS^2 + GS + F$ do not have a common root in S.

Below, the existence of such polynomials with appropriate root multiplicities corresponding to each configuration are shown. Then α exists for all of them.

I_1^{12} **:** If $D = G = H = 0$ and $F^2 = -24$, then $\Delta^*(S) = (S^2+1)^3 - 24S$ has six distinct roots, each with multiplicity 1.

$I_2^2 I_1^8$ **:** If $D = F = H = 0$ and $G^2 = -8$, $\Delta^*(S) = (S^2+1)^3 - 8S^3 = (S-1)^2(S^4 + 2S^3 + 6S^2 + 2S + 1)$ which has one root with multiplicity 2 and four roots with multiplicity 1.

$I_2^4 I_1^4$ **:**

$$\begin{aligned}(\frac{S^2 - (2 + 4\sqrt[3]{4})S - 3}{4})^3 + S(\sqrt{3}\sqrt[3]{2}\frac{S^2 - (2 + 2\sqrt[3]{4})S - 3}{4})^2 \\ = \frac{S^2 - 6S - 3}{4}(\frac{S^2 + 3}{4})^2\end{aligned}$$

I_2^6 **:**

$$((\frac{S-1}{\sqrt[3]{2}})^2)^3 + S(3S^2+10S+3)^2 = (\frac{S^3+15S^2+15S+1}{2})^2$$

has three roots with multiplicity 2.
I_3^4 **:** Since

$$(-\frac{S^2}{12} - \frac{S}{2} + \frac{1}{4})^3 + S(\frac{\sqrt{3}S^2}{12} + \frac{\sqrt{3}}{4})^2 = (-\frac{S^2}{12} + \frac{S}{2} + \frac{1}{4})^3,$$

α exists for I_3^4.
$I_3^2 I_1^6$ **:** If $D = 0$, $F = i87\sqrt{10}/80$, $G = -i77\sqrt{10}/80$ and $H = i267\sqrt{10}/320$, then

$$\Delta^*(S) = (S-2)^3(S^3 - \frac{9849}{10240}S^2 + \frac{6609}{5120}S - \frac{1}{8})$$

which has one root with multiplicity 3 and three roots with multiplicity 1.
$I_3^2 I_2^2 I_1^2$**,** $I_4^2 I_2^2$**,** $I_4^2 I_1^4$**,** $I_5^2 I_1^2$**:** For these configurations, if one equates $\Delta^*(S)$ to a general polynomial of degree 6 in S with the appropriate root multiplicities, then it can be shown that the system of equations given by the coefficients of the polynomials can be solved similarly as illustrated in the previous case. α exists for all of these configurations.

5.2.3 Trisections for Δ_2

We are concerned with the trisections not passing through the vertex $[0,0,0,1]$ of the cone $Y^2 = XZ$, we restrict our attention to the trisections T given in the general form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aXZW + bX^2Y + cYZ^2 = 0 \end{aligned} \tag{37}$$

which are preserved under the action of the order 4 automorphism Δ_2 on F_2 (Lemma 5.2.6).
L_0 and L_∞ intersect T at a single point for all a, b and c. T is smooth at $L_0 \cap T$ iff $b \neq 0$. T is smooth at $L_\infty \cap T$ iff $c \neq 0$. Using the local chart $[X,Y,Z,W] = [1,t,t^2,w]$ on $Q - L_\infty$, where Q is the cone $Y^2 = XZ$, we have the equation of $T - L_\infty$ given by

$$w^3 + at^2w + bt + ct^5 = 0. \tag{38}$$

The singular fibers except for the one over $t = \infty$ occur over the roots of

$$\Delta(t) = 4a^3t^6 + 27(bt+ct^5)^2 = t^2[4a^3t^4 + 27(b+ct^4)^2] \tag{39}$$

since there is a singular fiber over $t = t_0$ iff $L_{t_0} \cap T$ has less than three distinct points by Proposition 5.2.3, and this happens iff $t = t_0$ is a multiple root of (38), hence the discriminant (39) vanishes at t_0. By Proposition 5.2.3 again, there is a II fiber over $t = 0$ iff $b \neq 0$ and there is a II fiber over $t = \infty$ iff $c \neq 0$. If $b = 0$, since $t = 0$ is a root of $\Delta(t)$ with multiplicity 6, by (36) there is an I_0^* fiber over $t = 0$. Using these criteria, we give the equations of the trisections giving rise to the configurations with $n = 4$ in Table 9:

$II^2 I_1^8$, $II^2 I_2^4$: There are II fibers over $t = 0, \infty$ iff $b \neq 0$ and $c \neq 0$ as shown above. Writing $a = 3A$, $b = 2B$, $c = 2C$ and $S = t^2$, we get $\Delta(t) = \Delta^*(S) = 108S[A^3S^2 + (B + CS^2)^2]$ from (39). Since S^2 and $B + CS^2$ do not have a common root when $B \neq 0$; there are I_n fibers over $t = \pm\sqrt{S_0}$ for $S_0 \neq 0$ iff $\Delta^*(S_0) = 0$ with the multiplicity of the root S_0 equal to n. If $A^3 = -2$ and $B = C = 1$, then $\Delta^*(S) = 108S(S^4 + 1)$ which has five distinct roots with multiplicity 1 (including $S = 0$ corresponding to a II fiber over $t = 0$). Then α exists for $II^2 I_1^8$. If $A^3 = -4$ and $B = C = 1$, then $\Delta^*(S) = 108S(S^2 - 1)^2$ and this trisection gives rise to the configuration $II^2 I_2^4$.

$I_0^* II\, I_1^4$: There is a II fiber over $t = \infty$ iff $c \neq 0$, and there is an I_0^* over $t = 0$ iff $b = 0$. If $a = c = 1$ and $b = 0$, then $\Delta(t) = t^6[4 + 27t^4]$. 0 is a root with multiplicity 6, and there are four more distinct roots with multiplicity 1. Then there are four I_1 fibers over these roots. α exists for $I_0^* II\, I_1^4$.

5.3 The non-cyclic $Aut_B(\mathbb{P}^1)$ case

In Proposition 4.2.3, we have given the the configurations of singular fibers which can have non-cyclic $Aut_B(\mathbb{P}^1)$ groups. In this subsection, we prove the existence of such non-cyclic $Aut_B(\mathbb{P}^1)$ and give the corresponding $Aut_\sigma(B)$ groups. The technique used here is basically the generalization of the techniques of the previous subsection. We show the existence of the trisections of F_2 which are invariant under the action of a non-cyclic group, and we characterize the configurations of singular fibers corresponding to each such trisection.

5.3.1 $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ case

If we consider the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on the quadric cone Q (given by $Y^2 = XZ$ in $\mathbb{P}^3$ and viewed as F_2 after blowing down the minimal section to the vertex of the cone) generated by the automorphisms

$$\begin{aligned} \Gamma_2 &: [X,Y,Z,W] \mapsto [X,-Y,Z,-W] \\ \Omega_1 &: [X,Y,Z,W] \mapsto [Z,Y,X,W] \end{aligned} \tag{40}$$

on $\mathbb{P}^3$, then the trisections (not passing through the vertex of the cone) preserved by this action are of the form

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + bYW^2 + cX^2W + dXZW + cZ^2W + fX^2Y + gXYZ + fZ^2Y = 0. \end{aligned} \tag{41}$$

As in the previous subsection, we may assume $b = 0$. Note that using the same notation from the previous subsection, $t = 0, \infty$, $t = \pm 1$ and $t = \pm i$ are the fixed points of the automorphisms induced on the base curve $\mathbb{P}^1$ by Γ_2, Ω_1 and $\Omega_1 \circ \Gamma_2$, respectively. Recall that we use the local chart $[t, w] \mapsto [1, t, t^2, w]$ on $Q - L_\infty$ and we treat $[X, Y, Z, 0]$, $Y^2 = XZ$ as the base section of F_2. The action of Γ_2 on the lines L_0 and L_∞ is by multiplication by -1, hence Γ_2 induces an order 4 automorphism on the elliptic surface obtained from the double cover of F_2 branched over such a trisection and the minimal section. Ω_2 fixes the lines $L_{\pm 1}$, and $\Omega_1 \circ \Gamma_2$ fixes the lines $L_{\pm i}$, hence these two automorphisms lift to automorphisms of order 2 on the elliptic surface. Thus, for the elliptic surfaces B obtained from such trisections, we have $Aut_\sigma(B) = D_4$. The list of the configurations for which a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is predicted as $Aut_B(\mathbb{P}^1)$ is given in Proposition 4.2.3. From this list, we do not consider the configurations $II^2 I_4^2$ and I_3^4 for this action since they do not have $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = ord(\phi(\alpha)) = 2$ as shown in Table 7. We also exclude the configurations $II^2 I_3^2 I_1^2$, $I_5^2 I_1^2$ and $I_3^2 I_1^6$ since the I_3 and the I_5 fibers must be over the fixed points of $\mathbb{P}^1$ to admit $Aut_B(\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but this is impossible since the order of the induced automorphism on $\mathbb{P}^1$ is 2 (see Table 6 and Table 9). For the other configurations, we show the existence of the trisections corresponding to each below.

Since the trisections we consider 4trisections are preserved by Γ_2, the conditions discussed in the part "Trisections for Γ_2"' apply here. We have:

$$
\begin{aligned}
&c = 0 \text{ iff } T \cap L_\infty \text{ is a single point iff } T \cap L_0 \text{ is a single point.}\\
&c = f = 0 \text{ iff } T \text{ is singular at } T \cap L_\infty \text{iff } T \text{ is singular at } T \cap L_0.
\end{aligned} \tag{42}
$$

• $IV^2I_2^2$, $IV^2I_1^4$ **:** Since Γ_2 fixes $t = 0, \infty$ on $\mathbb{P}^1$ and it lifts to an order 4 automorphism, we must have I_0 fibers over $t = 0, \infty$, hence $c \neq 0$. The other fixed points of $\mathbb{P}^1$ by the other two automorphisms are $t = \pm 1$ and $t = \pm i$. Without loss of generality, we may assume that there are IV fibers over $t = \pm 1$. We can write

$$
\begin{aligned}
\Delta(t) &= 4(c + dt^2 + ct^4)^3 + 27t^2(f + gt^2 + ft^4)^2\\
&= \Delta^*(S) = r[(1 + KS + S^2)^3 + LS(1 + MS + S^2)^2]
\end{aligned} \tag{43}
$$

for some constants r, K, L and M where $S = t^2$. There are IV fibers over $t = \pm 1$ iff $S = 1$ is a root of $\Delta^*(S)$ with multiplicity 4 and both $1 + KS + S^2$ and $1 + MS + S^2$ vanish at $S = 1$. This holds iff $K = M = -2$. Then the other factor of $\Delta^*(S)$ is $(S - 1)^2 + LS$. If $L = 4$, then $S = -1$ is a double root and this corresponds to two I_2 fibers over $t = \pm i$. For other values of L there are two distinct roots corresponding to four I_1 fibers. Thus, $Aut_\sigma(B) = D_4$ exists for the configurations $IV^2I_2^2$ and $IV^2I_1^4$.

• $II^4I_2^2$, $II^4I_1^4$ **:** If $\Delta^*(S)$ is as above, there are four II fibers iff $1+KS+S^2$ and $1+MS+S^2$ have two distinct common roots and these are roots of $\Delta^*(S)$ with multiplicity 2. This holds iff $K = M \neq \pm 2$. Then the remaining component of $\Delta^*(S)$ is $(1 + KS + S^2) + LS$. If $K = 0$ and $L = \pm 2$, this component has a double root corresponding to two I_2 fibers over either $t = \pm 1$ or $t = \pm i$. If $K = 0$ and $L \neq \pm 2$, then there are four I_1 fibers. Thus, $Aut_\sigma(B) = D_4$ exists for both configurations.

• $II^2I_2^4$, $II^2I_2^2I_1^4$, $II^2I_1^8$ **:** There are II fibers over $t = 0, \infty$ iff $c = 0$ and $f \neq 0$. Then

$$
\Delta(t) = 4d^3t^6 + 27t^2(f + gt^2 + ft^4)^2 = \Delta^*(S) = S[KS^2 + L(1 + MS + S^2)^2]
$$

where $S = t^2$. K and L are nonzero, hence the multiplicities of the roots of $KS^2+L(1+MS+S^2)^2$ determine the I_n fibers in the configuration. $(K, L, M) = (-4, 1, 0)$ corresponds to the configuration $II^2I_2^4$ since there are two double roots. Similarly $(-1, 1, 1)$ gives $II^2I_2^2I_1^4$ since there are one double root and two distinct roots with multiplicity 1. To get $II^2I_1^8$, we can take $(K, L, M) = (-1, 1, 2)$ for which there are four distinct roots with multiplicity 1.

• $I_4^2I_2^2$, $I_4^2I_1^4$, I_2^6, $I_2^4I_1^4$, $I_2^2I_1^8$, I_1^{12} **:** There are I_0 fibers over $t = 0, \infty$ iff $c \neq 0$.

With this condition, $\Delta(t)$ is a constant multiple of

$$\Delta^*(S) = (1 + KS + S^2)^3 + S(L + MS + LS^2)^2$$

where $S = t^2$. Such a trisection corresponds to a configuration with only I_n fibers iff $1 + KS + S^2$ and $L + MS + LS^2$ do not have a common root. The multiplicities of the roots of $\Delta^*(S)$ determine the I_n fibers. The reader can check that the following polynomial identities hold for some K, L and M with that desired condition, each giving rise to the indicated configuration:
$\Delta^*(S) = (S-1)^4(S+1)^2$ corresponds to $I_4^2 I_2^2$ (a solution is given by $K = 10/3$, $L = 2i\sqrt{3}$ and $M = 28i\sqrt{3}/9$).
$\Delta^*(S) = (S-1)^4(S^2+1)$ corresponds to $I_4^2 I_1^4$.
$\Delta^*(S) = (S-1)^2(S^2+1)^2$ corresponds to I_2^6.
$\Delta^*(S) = (S^2-1)^2(S^2+1)$ corresponds to $I_2^4 I_1^4$.
$\Delta^*(S) = (S-1)^2(S^4+1)$ corresponds to $I_2^2 I_1^8$.
When the discriminant of the polynomial $\Delta^*(S)$ is not zero, it corresponds to I_1^{12}.

We have shown the existence of $Aut_\sigma(B) = D_4$ above. If instead of the automorphism Ω_1, we take the automorphism

$$\Omega_2 \, : [X, Y, Z, W] \mapsto [Z, Y, X, -W] \tag{44}$$

on the quadric cone Q and consider the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on Q generated by Γ_2 and Ω_2, then Γ_2, Ω_2 and $\Gamma_2 \circ \Omega_2$ act as multiplication by -1 on the lines L_0, $L_{\pm 1}$ and $L_{\pm i}$, respectively. Thus, all induce order 4 automorphisms when lifted to the elliptic surface. Then $Aut_\sigma(B) = Q_8$ for the configurations corresponding to the trisections on Q preserved by this action. Such trisections (not passing through the vertex of the cone) are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + cX^2W + dXZW + cZ^2W + fX^2Y - fZ^2Y = 0. \end{aligned} \tag{45}$$

From the list of the possible configurations for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in Proposition 4.2.3, the only ones we should consider are $II^2 I_2^4$, $II^2 I_1^8$, I_3^4, $I_2^4 I_1^4$ and I_1^{12}. All the other configurations have either IV fibers, I_n fibers ($n > 0$), or I_0 fibers over $J \neq 1$ over the fixed points of $\mathbb{P}^1$, and by Lemma 5.2.1 and Lemma 5.2.2, such configurations cannot have $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 4$, hence $Aut_\sigma(B) = Q_8$ is not possible for these configurations. Similar to the above cases, we show the existence of the trisections corresponding to the relevant configurations below.

• **$II^2 I_2^4$, $II^2 I_1^8$:** There are II fibers over $t = 0, \infty$ iff $c = 0$ and $f \neq 0$. If this holds, then

$$\Delta(t) = 4d^3t^6 + 27t^2 f^2(1 - t^4)^2 = \Delta^*(S) = S[4d^3S^2 + 27f^2(1 - S^2)^2]$$

where $S = t^2$. If $27f^2 = 1$, then for $d^3 = 1$, there are two double roots of $\Delta^*(S)$, and for $d^3 \neq 1$ and $d \neq 0$, $\Delta^*(S)$ has four nonzero roots with multiplicity 1. These cases correspond to the configurations $II^2 I_2^4$ and $II^2 I_1^8$, respectively. Hence, $Aut_\sigma(B) = Q_8$ exists for these two configurations.

• I_3^4**,** $I_2^4 I_1^4$**,** I_1^{12} **:** There are I_0 fibers over $t = 0, \infty$ iff $c \neq 0$. If this holds, then $\Delta(t)$ is a constant multiple of

$$\Delta^*(S) = (1 + KS + S^2)^3 + LS(1 - S^2)^2$$

where $S = t^2$ and the multiplicities of the roots of $\Delta^*(S)$ determine the I_n fibers provided $1 + KS + S^2$ and $(1 - S^2)$ do not have a common root (i.e. $K \neq \pm 2$).
If $K = 2i\sqrt{3}$ and $L = 12i\sqrt{3}$ then $\Delta^*(S) = (1 - 2i\sqrt{3}S + S^2)^3$, which has two distinct roots with multiplicity 3, hence corresponds to I_3^4.
If $K = 1/2$ and $L = 25/8$, then $\Delta^*(S) = (S^2 + 3S + 1)^2(S^2 + (11/8)S + 1)$, which has two roots with multiplicity 2 and two roots with multiplicity 1, hence corresponds to $I_2^4 I_1^4$.
If the discriminant of the degree 6 polynomial $\Delta^*(S)$ is nonzero, there are six distinct roots and this corresponds to I_1^{12}.
Thus, $Aut_\sigma(B) = Q_8$ exists for these three configurations.

When considering $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ actions on F_2 (practically on the quadric cone Q), without loss of generality we can assume that the induced action on the base $\mathbb{P}^1$ is generated by the two maps $[X, Y, Z] \mapsto [X, -Y, Z]$ and $[X, Y, Z] \mapsto [Z, Y, X]$. Then it can be shown that the action on Q acts as multiplication by -1 on either one or three of the pairs of lines $\{L_0, L_\infty\}$, $\{L_{\pm 1}\}$ and $\{L_{\pm i}\}$, which are the lines over the fixed points of the induced automorphisms on $\mathbb{P}^1$. This shows that either one or three of the non-identity elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ lift to order 4 automorphisms on the elliptic surface. Hence, $Aut_\sigma(B)$ which is a $\mathbb{Z}/2\mathbb{Z}$ extension of $Aut_B(\mathbb{P}^1)$ (by Lemma 5.0.4) is either D_4 or Q_8.

5.3.2 $Aut_B(\mathbb{P}^1) = D_3$ case

If $Aut_B(\mathbb{P}^1) = D_3$, then $Aut_\sigma(B)$ is an extension of D_3 by $\mathbb{Z}/2\mathbb{Z}$. It can be checked consulting the table of non-abelian groups of order less than or equal to 32 in [3] (p.134) that the only such extensions of D_3 are the groups D_6 and Dic_3 (the Dicyclic group of order 12). A presentation for Dic_3 is $\langle a, b | \, a^6 = 1 \, , \, b^2 = a^3 \, , \, b^{-1}ab = a^{-1} \rangle$. We will show that $Aut_\sigma(B)$ can be both and we will give the configurations for which each case occurs.

First, consider the D_3 action on the quadric cone Q generated by the automorphisms

$$\begin{aligned}\Theta_1 &: [X,Y,Z,W] \mapsto [X,\mu Y,\mu^2 Z,\mu W]\\ \Omega_1 &: [X,Y,Z,W] \mapsto [Z,Y,X,W]\end{aligned} \tag{46}$$

where μ is a third root of 1. Note that the induced action on $\mathbb{P}^1$ is also a D_3 action. Ω_1 lifts to an order 2 automorphism on the elliptic surface while Θ_1 lifts to two automorphisms, one with order 3, other with order 6. Then $Aut_\sigma(B) = D_6$ for the elliptic surfaces B corresponding to the trisections of Q preserved by this D_3 action. From the list of the configurations given for D_3 in Proposition 4.2.3, we do not consider $I_3^3 I_1^3$ since it does not admit an induced automorphism of order 2 (see Table 6 and Table 9). Below we show the existence of trisections corresponding to the other configurations.

The trisections preserved by the action of Θ_1 and Ω_1 are given in the form:

$$\begin{aligned}&Y^2 = XZ\\ &W^3 + aYW^2 + bXZW + cX^3 + dXYZ + cZ^3 = 0.\end{aligned} \tag{47}$$

We may assume $a = 0$. Then using the local chart $(t, w) \mapsto [1, t, t^2, w]$ on $Q - L_\infty$, the equation of the trisection is:

$$w^3 + bt^2w + (c + dt^3 + ct^6) = 0.$$

Hence,

$$\Delta(t) = 4b^3t^6 + 27(c + dt^3 + ct^6)^2.$$

If $b \neq 0$ and $c \neq 0$, then bt^2 and $c + dt^3 + ct^6$ do not have a common root, hence there are only I_n fibers in the configuration corresponding to that trisection. Note that if $S = t^3$, we can write

$$\Delta(t) = \Delta^*(S) = KS^2 + L(1 + MS + S^2)^2$$

for some constants K, L and M. A root of $\Delta^*(S)$ with multiplicity n gives three copies of I_n in the configuration since $S = t^3$.
If $K = -4$, $L = 1$ and $M = 0$, then $\Delta^*(S) = (S^2 - 1)^2$ which has two roots of multiplicity two. This corresponds to the configuration I_2^6.
f $K = -1$ and $L = M = 1$, then $\Delta^*(S) = (S + 1)^2(S^2 + 1)$ which corresponds to $I_2^3 I_1^6$.
If the discriminant of the degree 4 polynomial $\Delta^*(S)$ is nonzero, there are four roots of multiplicity 1, which corresponds to I_1^{12}.
Hence, $Aut_\sigma(B) = D_6$ exists for I_2^6, $I_2^3 I_1^6$ and I_1^{12}.

Second, we consider the D_3 action on Q generated by the automorphisms

$$\begin{aligned}\Theta_1 &: [X,Y,Z,W] \mapsto [X,,\mu^2 Z,]\\ \Omega_2 &: [X,Y,Z,W] \mapsto [Z,Y,X,-W].\end{aligned} \tag{48}$$

Since Ω_2 lifts to an automorphism of order 4 on the elliptic surface (Ω_2 acts on $L_{\pm 1}$ by multiplication by -1), we get $Aut_\sigma(B) = Dic_3$ whose presentation is given above. The trisections preserved by this action are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aXZW + bX^3 - bZ^3 = 0. \end{aligned} \tag{49}$$

Apart from $I_3^3 I_1^3$, we do not consider the configuration $I_2^3 I_1^6$ from the list in Proposition 4.2.3 since it does not have any $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 4$ (see Table 9).

In the local chart, the above trisection is given by

$$w^3 + at^2 w + b(1 - t^6) = 0.$$

Hence,

$$\Delta(t) = 4a^3 t^6 + 27b^2(1 - t^6)^2 = \Delta^*(S) = KS + L(1 - S)^2$$

where $S = t^6$. If $K = 4$ and $L = 1$, then $\Delta^*(S) = (S + 1)^2$ which corresponds to I_2^6. If the discriminant of the degree two polynomial $\Delta^*(S)$ is nonzero, then this corresponds to I_1^{12}. Thus, $Aut_\sigma(B) = Dic_3$ exists for I_2^6 and I_1^{12}.

5.3.3 $Aut_B(\mathbb{P}^1) = D_4$ case

The only configurations listed in Proposition 4.2.3 for this case are $II^2 I_2^4$ and $II^2 I_1^8$. Note that if these configurationns admit an order 4 induced automorphism, then the order of the automorphism on the elliptic surface is 8 (Table 7 and table 9). $Aut_\sigma(B)$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of $Aut_B(\mathbb{P}^1) = D_4$, and it can be checked consulting Table 1 in [3] (p.134) that the only $\mathbb{Z}/2\mathbb{Z}$ extensions of D_4 which have order 8 elements are D_8, Dic_4 (the Dicyclic group of order 16) and Qd_4 (the order 16 Quasidihedral group). A presentation for Dic_4 is $\langle a, b | \, a^8 = 1 \, , \, a^4 = b^2 \, , \, b^{-1}ab = a^{-1} \rangle$. A presentation for Qd_4 is $\langle a, b | \, a^8 = b^2 = 1 \, , \, bab = a^3 \rangle$. Since the order 4 generator of D_4 lifts to an order 8 automorphism on the elliptic surface, we may assume by Lemma 5.2.5 that the order 4 generator of the D_4 action is either Δ_1 or Δ_2. Without loss of generality we may also assume that the D_4 action on the base $\mathbb{P}^1$ is generated by $[1, t] \mapsto [1, it]$ and $[1, t] \mapsto [1, 1/t]$.

If we consider the D_4 action on the quadric cone Q generated by

$$\begin{aligned} \Delta_2 \, &: \, [X, Y, Z, W] \mapsto [X, iY, -Z, -iW] \\ \Omega_1 \, &: \, [X, Y, Z, W] \mapsto [Z, Y, X, W], \end{aligned} \tag{50}$$

then Δ_2 lifts to an order 8 automorphism, Ω_1 lifts to an order 2 automorphism (Ω_1 fixes the lines $L_{\pm 1}$), and $\Delta_2 \circ \Omega_1$, which has order 2, lifts to an order 4 automorphism (the action on the lines $L_{\pm\sqrt{i}}$ are by multiplication by -1 where

$\pm\sqrt{i}$ are the fixed points on $\mathbb{P}^1$). Hence, the extension of D_4 we get by this action is $Aut_\sigma(B) = Qd_4$, the Quasidihedral group of order 16. The trisections preserved by the D_4 action are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aXZW + bX^2Y + bZ^2Y = 0. \end{aligned} \tag{51}$$

In the local chart, we get the equation

$$w^3 + at^2w + b(t + t^5) = 0. \tag{52}$$

Hence,

$$\Delta(t) = 4a^3t^6 + 27b^2t^2(1 + t^4)^2 = \Delta^*(S) = KS[LS^2 + (1 + S^2)^2] \tag{53}$$

where $S = t^2$. The root $S = 0$ of $\Delta^*(S)$ corresponds to the II fiber over $t = 0$ (there is another II fiber over $t = \infty$). The roots of $LS^2 + (1 + S^2)^2$ correspond to I_n fibers where n is the multiplicity of the root. If $L = -4$, there are two double roots which gives $II^2I_2^4$. If $L \neq -4$ and $L \neq 0$, there are 4 distinct roots which gives $II^2I_1^8$. Thus, $Aut_\sigma(B) = Qd_4$ exists for both configurations.

If we now consider the D_4 action on Q generated by

$$\begin{aligned} \Delta_2 &: [X, Y, Z, W] \mapsto [X, iY, -Z, -iW] \\ \Omega_2 &: [X, Y, Z, W] \mapsto [Z, Y, X, -W], \end{aligned} \tag{54}$$

then since Δ_2 lifts to an order 8 automorphism, and Ω_2, which has order 2, lifts to an order 4 automorphism; the extension of D_4 we get from this action is $Aut_\sigma(B) = Dic_4$, the Dicyclic group of order 16. The trisections preserved by this D_4 action are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aXZW + bX^2Y - bZ^2Y = 0. \end{aligned} \tag{55}$$

Then, in the local chart, the equation of the trisection is

$$w^3 + at^2w + b(t - t^5) = 0.$$

Hence,

$$\Delta(t) = 4a^3t^6 + 27b^2t^2(1 - t^4)^2 = \Delta^*(S) = KS[LS^2 + (1 - S^2)^2] \tag{56}$$

where $S = t^2$. The root $S = 0$ gives a II fiber over $t = 0$ (there is another II fiber over $t = \infty$). If $L = 4$, then $LS^2 + (1 - S^2)^2$ has two double roots which gives $II^2I_2^4$. If $L \neq 4$ and $L \neq 0$, there are 4 distinct roots which gives $II^2I_1^8$. Thus, $Aut_\sigma(B) = Dic_4$ exists for both configurations.

To get $Aut_\sigma(B) = D_8$, without loss of generality we can consider the D_4 action on Q generated by

$$\begin{aligned} \Delta_1 &: [X, Y, Z, W] \mapsto [X, iY, -Z, iW] \\ \Omega_1 &: [X, Y, Z, W] \mapsto [Z, Y, X, W] \end{aligned} \tag{57}$$

where all the order 2 automorphisms lift to order 2 automorphisms. But in this case, the trisections preserved by this action are given in the form

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aYW^2 + bXZW + cXZY = 0 \end{aligned} \tag{58}$$

and in local coordinates;

$$w^3 + atw + bt^2w + ct^3 = 0,$$

then $\Delta(t) = kt^6$. Hence, all such trisections correspond to the configuration $I_0^* I_o^*$ which has constant J-map. Thus, $Aut_(B) = D_8$ does not exist for $II^2 I_2^4$ and $II^2 I_1^8$.

5.3.4 $Aut_B(\mathbb{P}^1) = D_6$ case

The only configurations listed in Proposition 4.2.3 for this case are I_2^6 and I_1^{12}. Note that none of these admit an automorphism $\alpha \in Aut_\sigma(B)$ with $ord(\alpha) = 2 \cdot ord(\phi(\alpha)) = 24$ (Table 9) hence the order 6 elements of D_6 lift to order 6 automorphisms. We may assume that the action on $\mathbb{P}^1$ is generated by $[1, t] \mapsto [1, \mu t]$ and $[1, t] \mapsto [1, 1/t]$, where μ is a primitive 6th root of 1. Without loss of generality, we can assume that the order 6 generator of the D_6 action on Q is given by

$$\Theta_3 : [X, Y, Z, W] \mapsto [X, \mu Y, \mu^2 Z, kW]$$

for some k such that $k^6 = 1$. Here, we cannot have k a primitive 6th root of 1 since otherwise Θ_3 extends to an order 12 automorphism ($\Theta_3^3 = \Gamma_2$). Then to generate a D_6 action on Q with the specified action on $\mathbb{P}^1$, the only automorphisms we can consider as the second generator are Ω_1 and Ω_2 from the previous cases provided $k = -\mu$.

For the D_6 action on Q generated by

$$\begin{aligned} \Theta_3 &: [X, Y, Z, W] \mapsto [X, \mu Y, \mu^2 Z, -\mu W] \\ \Omega_1 &: [X, Y, Z, W] \mapsto [Z, Y, X, W], \end{aligned} \tag{59}$$

the automorphism Θ_3 of Q lifts to an order 6 automorphism of the elliptic surface, Ω_1 lifts to an order 2 automorphism (the lines $L_{\pm 1}$ are fixed), and $\Omega_1 \circ \Theta_3$ lifts to an order 4 automorphism (the action on the lines $l_{\pm\sqrt{\mu}}$ is by multiplication by -1). This information suffices to determine $Aut_\sigma(B)$ for the surfaces B obtained from trisections of Q preserved under this D_6 action. Consulting the Table 1 in [3] (p.134), the reader can verify that the only extension of D_6 with these properties is $\mathbb{Z}/4\mathbb{Z} \times D_3$. Below, we show that $Aut_\sigma(B) = \mathbb{Z}/4\mathbb{Z} \times D_3$ exists for the configurations I_2^6 and I_1^{12}.

The trisections preserved under the action of Θ_3 and Ω_1 are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aXZW + b(X^3 + Z^3) = 0. \end{aligned} \tag{60}$$

Then in the local chart

$$w^3 + at^2 + b(1 + t^6) = 0.$$

Hence,

$$\Delta(t) = 4a^3t^6 + 27b^2(1 + t^6)^2.$$

If $a^3 = -1$ and $27b^2 = 1$, then $\Delta(t) = (1 - t^6)^2$ which gives the configuration I_2^6. If $a^3 \neq -27b^2$, then there are 12 distinct roots which gives the configuration I_1^{12}.

The other D_6 action we consider is the action generated by

$$\begin{aligned} \Theta_3 &: [X, Y, Z, W] \mapsto [X, \mu Y, \mu^2 Z, -\mu W] \\ \Omega_2 &: [X, Y, Z, W] \mapsto [Z, Y, X, -W]. \end{aligned} \tag{61}$$

Here Θ_3 lifts to an order 6 automorphism on the elliptic surface, Ω_2 lifts to an order 4 automorphism (the action on the lines $L_{\pm 1}$ is by multiplication by -1) and $\Omega_2 \circ \Theta_3$ lifts to an order 2 automorphism (the lines $L_{\pm\sqrt{\mu}}$ are fixed). From this information and the Table 1 in [3], it can be shown that the only $\mathbb{Z}/2\mathbb{Z}$ extension of D_6 with these properties is the group G_1 given by the presentation

$$G_1 = \langle a, b |\, a^4 = b^6 = (ab)^2 = (ab^{-1})^2 = 1\rangle. \tag{62}$$

The trisections preserved by the above action on Q are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + aXZW + b(X^3 - Z^3) = 0. \end{aligned} \tag{63}$$

In the local chart, this becomes

$$w^3 + at^2w + b(1 - t^6) = 0$$

Hence

$$\Delta(t) = 4a^3t^6 + 27b^2(1 - t^6)^2. \tag{64}$$

If $a^3 = 27b^2$, then $\Delta(t) = 27b^2(1 + t^6)^2$ which has six double roots, hence corresponds to the configuration I_2^6. If $a^3 \neq 27b^2$, then there are 12 distinct roots of $\Delta(t)$ which corresponds to I_1^{12}. Thus, $Aut_\sigma(B) = G_1$ exists for I_2^6 and I_1^{12}.

5.3.5 $Aut_B(\mathbb{P}^1) = A_4$ case

Without loss of generality, we may assume that the A_4 action on $\mathbb{P}^1$ is given by the Moebius transformations generated by

$$\begin{aligned} f_1 &: z \mapsto \mu z \\ f_2 &: z \mapsto \frac{z+2}{z-1} \end{aligned} \tag{65}$$

where μ is a third root of 1. Considering $\{[X,Y,Z,0] | Y^2 = XZ\}$ as the base section of the quadric cone $Q : Y^2 = XZ$, f_1 and f_2 are induced by $[X,Y,Z] \mapsto [X, \mu Y, \mu^2 Z]$ and $[X,Y,Z] \mapsto [X-2Y+Z, -2X+Y+Z, 4X+4Y+Z]$, respectively (since $[1,z] \mapsto [1,z,z^2]$ is the embedding of $\mathbb{P}^1$ to the curve $Y^2 = XZ$ in $\mathbb{P}^2$). Without loss of generality, we may assume that the order 3 generator of the A_4 action on the quadric cone Q is given by the automorphism

$$\Theta : [X,Y,Z,W] \mapsto [X, \mu Y, \mu^2 Z, kW] \tag{66}$$

where $k^3 = 1$. Then, if

$$\Sigma : [X,Y,Z,W] \mapsto [X - 2Y + Z, -2X + Y + Z, 4X + 4Y + Z, rW] \tag{67}$$

is the order 2 generator of the A_4 action on Q, we get $r^2 = 9$ in order for Σ to have order 2, and $r = -3$ in order to have $\Theta \circ \Sigma$ to have order 3 (this should hold since A_4 is given by the presentation $\langle a, b | \, a^3 = b^2 = (ab)^3 = 1 \rangle$). Then Σ acts on the lines L_{t_0} as multiplication by -1, where $t_0 = 1 \pm i\sqrt{3}$. Hence, Σ lifts to an order 4 automorphism of the elliptic surface. (Note here that if we had considered a more general map for Σ where we had $rW + aX + bY + cZ$ instead of just rW above, then Σ would act as a reflection on the same lines L_{t_0}, hence it would again lift to an order 4 automorphism on the elliptic surface). There are only two $\mathbb{Z}/2\mathbb{Z}$ extensions of A_4 as the reader can check using Table 1 in [3] (p.134), and all the order 2 elements of A_4 lifts to order 2 elements in the extension $A_4 \times \mathbb{Z}/2\mathbb{Z}$. Then in our case, since Σ lifts to an order 4 automorphism, we should get $Aut_\sigma(B) = G_2$ for the elliptic surfaces B which are obtained from the trisections preserved under the A_4 action on Q generated by Θ and Σ. Here, G_2 is the other $\mathbb{Z}/2\mathbb{Z}$ extension of A_4 which is the Binary Tetrahedral group given by the presentation

$$G_2 = \langle a, b | \, a^3 = b^2 \, , \, (a^{-1}b)^3 = 1 \rangle \tag{68}$$

If we take $k = 1$, it can be checked that the trisections preserved by Θ and Σ are given in the form:

$$\begin{aligned} &Y^2 = XZ \\ &W^3 + a(X^2 - YZ)W + b(8X^3 + 20XYZ - Z^3) = 0. \end{aligned} \tag{69}$$

Then, in the local chart it becomes

$$w^3 + a(1 - t^3)w + b(8 + 20t^3 - t^6) = 0.$$

Hence,

$$\Delta(t) = 4a^3(1-t^3)^3+27b^2(8+20t^3-t^6)^2 = \Delta^*(S) = A(1-S)^2+B(8+20S-S^2)^2. \tag{70}$$

where $S = t^3$. If $A = -64B$ then $S = -8$ is a triple root of $\Delta^*(S)$ and $S = 0$ is a simple root. Then $\Delta(t) = ct^3(t^3 + 8)^3$ for some constant c. There are four distinct roots of multiplicity 3, hence such a trisection corresponds to the configuration I_3^4. If $A \neq -64B$ and $AB \neq 0$, then there are four distinct roots of $\Delta^*(S)$, hence twelve distinct roots of $\Delta(t)$ which corresponds to the configuration I_1^{12}. Note that the only configurations listed in Proposition 4.2.3 for A_4 are I_3^4 and I_1^{12}. Therefore, $Aut_\sigma(B) = G_2$ exists for I_3^4 and I_1^{12}.

This concludes our analysis of which configurations admit non-cyclic $Aut_B(\mathbb{P}^1)$ groups and what the corresponding $Aut_\sigma(B)$ groups for each such configuration are.

6 Results

Combining the results obtained in the previous sections, we present all possible groups $Aut_\sigma(B)$ and the corresponding configurations of singular fibers to each group.
Note that by the results of the subsections 5.1 and 5.2, we know the orders of all automorphisms in $Aut_\sigma(B)$ and in $Aut_B(\mathbb{P}^1)$ if the J-map of B is not constant. In the case of cyclic $Aut_B\mathbb{P}^1$ groups, this information suffices to determine $Aut_\sigma(B)$. The existence of the non-cyclic $Aut_B(\mathbb{P}^1)$ groups is proved and the corresponding $Aut_\sigma(B)$ groups are calculated in subsection 5.3.

Theorem 6.0.1. *Let B be a relatively minimal rational elliptic surface with section. Then*

$$Aut(B) = MW(B) \rtimes Aut_\sigma(B).$$

If the J-map of B is not constant, then Table 11 lists all the groups $Aut_\sigma(B)$ and the configurations of singular fibers of B corresponding to each group. All of the cases in Table 11 exist.

Remark : The groups $Aut_B\mathbb{P}^1$ and $Aut_\sigma(B)$ are not determined uniquely by the configuration of singular fibers on the surface B. Many configurations appear several times in Table 11. If a configuration appears in Table 11, then there is a relatively minimal rational elliptic surface B with section which has this configuration of singular fibers such that the groups $Aut_B(\mathbb{P}^1)$ and $Aut_\sigma(B)$ are as indicated in the same row of the table.

Remark : If a configuration of singular fibers (with non-constant J-map) does not appear in Table 11, then $Aut_B(\mathbb{P}^1) = 0$ and $Aut_\sigma(B) = \mathbb{Z}/2\mathbb{Z} = \langle -\mathbb{I} \rangle$ for all surfaces B with that configuration. Table 3 lists most of such configurations, except for a few for which $Aut_B(\mathbb{P}^1) = 0$ was proved later in the dissertation.

$Aut_B(\mathbb{P}^1)$	$Aut_\sigma(B)$	d	Configurations of Singular Fibers
$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	12	I_1^{12}
D_6	$D_3\times\mathbb{Z}/4\mathbb{Z}$	12	I_2^6 , I_1^{12}
	G_1	12	I_2^6 , I_1^{12}
A_4	G_2	12	I_3^4 , I_1^{12}
$\mathbb{Z}/10\mathbb{Z}$	$\mathbb{Z}/20\mathbb{Z}$	10	$II\ I_1^{10}$
$\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/18\mathbb{Z}$	9	$III\ I_1^9$
$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	8	$IV I_1^8$
D_4	Dic_4	8	$II^2I_2^4$, $II^2I_1^8$
	Qd_4	8	$II^2I_2^4$, $II^2I_1^8$
$\mathbb{Z}/7\mathbb{Z}$	$\mathbb{Z}/14\mathbb{Z}$	7	$III\ II\ I_1^7$
$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	6	$I_0^*I_1^6$
	$\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	12	$I_6I_1^6$, I_2^6 , I_1^{12}
		6	$I_0^*I_1^6$
D_3	D_6	12	I_2^6 , $I_2^3I_1^6$, I_1^{12}
	Dic_3	12	I_2^6 , I_1^{12}
$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/10\mathbb{Z}$	10	$II\ I_5I_1^5$, $II\ I_1^{10}$
		5	$IVIII\ I_1^5$
$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$	8	$II^2I_2^4$, $II^2I_1^8$
		4	$I_0^*II\ I_1^4$
	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	12	$I_8I_1^4$, $I_4I_2^4$, $I_4I_1^8$, $I_2^4I_1^4$
		8	$IVI_4I_1^4$, IVI_1^8
		4	$IV^*I_1^4$, II^4I_4 , $II^4I_1^4$
$(\mathbb{Z}/2\mathbb{Z})^2$	D_4	12	$I_4^2I_2^2$, $I_4^2I_1^4$, I_2^6 , $I_2^4I_1^4$, $I_2^2I_1^8$, I_1312
		8	$II^2I_2^4$, $II^2I_2^2I_1^4$, $II^2I_1^8$
		4	$IV^2I_2^2$, $IV^2I_1^4$, $II^4I_2^2$, $II^4I_1^4$
	Q_8	12	I_3^4 , $I_2^4I_1^4$, I_1^{12}
		8	$II^2I_2^4$, $II^2I_1^8$
$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	12	$I_9I_1^3$, $I_6I_1^6$, I_3^4 , $I_3^3I_1^3$, $I_3I_2^3I_1^3$, $I_3I_1^9$, I_2^6 , $I_2^3I_1^6$
		9	$III\ I_6I_1^3$, $III\ I_3I_2^3$, $III\ I_3I_1^6$, $III\ I_2^3I_1^3$, $III\ I_1^9$
		6	$I_3^*I_1^3$, $I_0^*I_3I_1^3$, $I_0^*I_2^3$, $I_0^*I_1^6$ II^3I_6 , $II^3I_3I_1^3$, $II^3I_2^3$, $II^3I_1^6$
		3	$I_0^*III\ I_1^3$, $III^*I_1^3$, III^3I_3 , $III^3I_1^3$, $III\ II^3I_3$, $III\ II^3I_1^3$

$Aut_B(\mathbb{P}^1)$	$Aut_\sigma(B)$	d	Configurations of Singular Fibers
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	12	$I_5^2 I_1^2$, $I_4^2 I_2^2$, $I_4^2 I_1^4$, I_3^4 , $I_3^2 I_2^2 I_1^2$ $I_3^2 I_1^6$, I_2^6 , $I_2^4 I_1^4$, $I_2^2 I_1^8$, I_1^{12}
		10	$II\ I_4^2 I_1^2$, $II\ I_3^2 I_2^2$, $II\ I_3^2 I_1^4$, $II\ I_2^4 I_1^2$, $II\ I_2^2 I_1^6$, $II\ I_1^{10}$
		8	$II^2 I_4^2$, $II^2 I_3^2 I_1^2$, $II^2 I_2^4$, $II^2 I_2^2 I_1^4$, $II^2 I_1^8$
		6	$I_4^* I_1^2$, $I_2^* I_2^2$, $I_2^* I_1^4$, $I_0^* I_2^2 I_1^2$, $I_0^* I_1^6$, $III^2 I_3^2$, $III^2 I_2^2 I_1^2$, $III^2 I_1^6$, $II^3 I_3^2$, $II^3 I_2^2 I_1^2$, $II^3 I_1^6$
		4	$I_2^* II\ I_1^2$, $I_0^* II\ I_1^4$, $IV^2 I_2^2$, $IV^2 I_1^4$, $III^2 II\ I_2^2$, $III^2 II\ I_1^4$, $II^4 I_2^2$, $II^4 I_1^4$
		2	$I_2^* II^2$, $I_0^* II^2 I_1^2$, $II^* I_1^2$
	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	12	$I_8 I_2 I_1^2$, $I_8 I_1^4$, $I_6 I_2 I_1^4$, $I_6 I_1^6$, $I_4^2 I_2^2$, $I_4^2 I_2 I_1^2$, $I_4^2 I_1^4$, $I_4 I_2^4$, $I_4 I_2^3 I_1^2$, $I_4 I_2^2 I_1^4$, $I_4 I_2 I_1^6$, $I_4 I_1^8$, $I_3^2 I_2^2 I_1^2$, $I_3^2 I_2 I_1^4$, $I_3^2 I_1^6$, I_2^6 , $I_2^5 I_1^2$, $I_2^4 I_1^4$, $I_2^3 I_1^6$, $I_2^2 I_1^8$, $I_2 I_1^{10}$, I_1^{12}
		8	$IV I_6 I_1^2$, $IV I_4 I_1^4$, $IV I_3^2 I_2$, $IV I_3^2 I_1^2$, $IV I_2^3 I_1^2$, $IV I_2^2 I_1^4$, $IV I_2 I_1^6$, $IV I_1^8$, $II^2 I_6 I_2$, $II^2 I_6 I_1^2$, $II^2 I_4 I_2 I_1^2$, $II^2 I_4 I_1^4$, $II^2 I_3^2 I_2$, $II^2 I_3^2 I_1^2$, $II^2 I_2^4$, $II^2 I_2^3 I_1^2$, $II^2 I_2^2 I_1^4$, $II^2 I_2 I_1^6$, $II^2 I_1^8$
		6	$I_0^* I_4 I_1^2$, $I_0^* I_2^3$, $I_0^* I_2^2 I_1^2$, $I_0^* I_2 I_1^4$, $I_0^* I_1^6$, $III^2 I_4 I_2$, $III^2 I_4 I_1^2$, $III^2 I_2^3$, $III^2 I_2^2 I_1^2$, $III^2 I_2 I_1^4$, $III^2 I_1^6$
		4	$IV^* I_2 I_1^2$, $IV^* I_1^4$, $IV^2 I_2^2$, $IV^2 I_2 I_1^2$, $IV^2 I_1^4$, $IV II^2 I_4$, $IV II^2 I_2^2$, $IV II^2 I_2 I_1^2$, $IV II^2 I_1^4$, $II^4 I_4$, $II^4 I_2^2$, $II^4 I_2 I_1^2$, $II^4 I_1^4$
		2	$I_0^* IV I_1^2$, $I_0^* II^2 I_2$, $I_0^* II^2 I_1^2$, $IV III^2 I_2$, $IV III^2 I_1^2$, $III^2 II^2 I_2$, $III^2 II^2 I_1^2$

Table 11: $Aut_\sigma(B)$ and configurations of singular fibers for B with non-constant J-map.
$G_1 = \langle S, T \,|\, S^4 = T^6 = (ST)^2 = (ST^{-1})^2 = 1 \rangle$.
$G_2 = \langle S, T \,|\, S^3 = T^2 \,,\, (S^{-1}T)^3 = 1 \rangle$ the Binary Tetrahedral group.
$Qd_4 = \langle S, T \,|\, S^8 = T^2 = 1 \,,\, TST = S^3 \rangle$, the Quasidihedral group of order 16.
$Dic_n = \langle S, T \,|\, S^{2n} = 1 \,,\, S^n = T^2 \,,\, T^{-1}ST = S^{-1} \rangle$, the Dicyclic group of order $4n$.
$D_n = \langle S, T \,|\, S^n = T^2 = 1 \,,\, TST = S^{-1} \rangle$, the Dihedral group of order $2n$.
Q_8: the Quaternionic group of order 8.
A_4: the Alternating group of order 12.

7 Appendix

Proof of Lemma 5.2.4:

Since E_∞ is the unique curve on F_2 that has self intersection (-2), every automorphism of F_2 preserves E_∞. If F is a fiber of F_2, then the linear system $|E_\infty + 2F|$ induces $\phi : F_2 \to \mathbb{P}^3$ which maps F_2 onto the quadric cone Q in $\mathbb{P}^3$ given by $Y^2 = XZ$ (p.419 in [4]). This map collapses E_∞ to the vertex $\nu = [0,0,0,1]$ of Q. The blow up of Q at ν gives F_2 back. ϕ gives an isomorphism $F_2 - E_\infty \to Q - \nu$ (p.424 in [4]). In fact, $Q - \nu$ is the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ since we have the two local charts

$$([1,t],w) \mapsto [1,t,t^2,w] \tag{71}$$

and

$$([s,1],u) \mapsto [s^2,s,1,u] \tag{72}$$

where

$$s = \frac{1}{t} \quad \text{and} \quad u = \frac{w}{t^2}. \tag{73}$$

Here $[s,t] \mapsto [s^2, st, t^2, 0]$ gives the base section of this line bundle.

Any automorphism of F_2 sends a fiber F to another fiber and preserves E_∞. Giving an automorphism of F_2 is equivalent to (using the above identification of $F_2 - E_\infty$ with $Q - \nu$) giving an automorphism of Q sending each line $[X_0, Y_0, Z_0, W] \quad W \in \mathbb{C}$ of the cone to another line of the cone. The base section of the line bundle above need not be preserved.
Note first that any automorphism of the base section $\mathbb{P}^1$ gives rise to an automorphism of Q, hence if we mark two points (take $[0,0,1,0]$ and $[1,0,0,0]$) on $\mathbb{P}^1$, then any automorphism of Q is conjugate to an automorphism whose induced automorphism on the base section fixes these two points. We are concerned with the order 2 automorphisms of Q inducing order 2 automorphisms on $\mathbb{P}^1$, hence we may assume the induced map on the base is given by $[s,t] \mapsto [s,-t]$ which fixes the specified points. Such an automorphism of Q will be given as follows in the local charts of $Q - \nu$:

$$\gamma : \begin{cases} ([1,t],w) \mapsto ([1,-t], a(t)w + b_0 + b_1 t + b_2 t^2) \\ ([s,1],v) \mapsto ([-s,1], c(s)v + b_0 s^2 + b_1 s + b_2) \end{cases} \tag{74}$$

Here, the base section maps by $b_0 + b_1 t + b_2 t^2$ to another section. $a(t) = c(s)$ if $s = 1/t$ and this gives a holomorphic map on $\mathbb{P}^1$, thus it is a constant, let's say a.
We want to have $ord(\gamma) = 2$;

$$\gamma^2 : ([1,t],w) \mapsto ([1,t], a^2 w + (a+1)b_0 + (a-1)b_1 t + (a+1)b_2 t^2). \tag{75}$$

If γ^2 is the identity; $a = -1$ and $b_1 = 0$, or $a = 1$ and $b_0 = b_2 = 0$.

If

$$\gamma_b : ([1,t],w) \mapsto ([1,-t], w+bt)$$

$$\gamma_{(b,c)} : ([1,t],w) \mapsto ([1,-t], -w+b+ct^2)$$

$$\theta_b : ([1,t],w) \mapsto ([1,t], w+\frac{b}{2}t)$$

$$\theta_{(b,c)} : ([1,t],w) \mapsto ([1,t], w-\frac{b}{2}-\frac{c}{2}t^2),$$

then we have;

$$\theta_b \circ \gamma_b \circ \theta_b^{-1} = \gamma_0 : ([1,t],w) \mapsto ([1,-t],w) \tag{76}$$

$$\theta_{(b,c)} \circ \gamma_{(b,c)} \circ \theta_{(b,c)}^{-1} = \gamma_{(0,0)} : ([1,t],w) \mapsto ([1,-t],-w). \tag{77}$$

Hence, any order two automorphism which also induces an order two automorphism on $\mathbb{P}^1$ is conjugate to one of γ_0 or $\gamma_{(0,0)}$. Note that these automorphisms on Q are induced by the automorphisms

$$[X,Y,Z,W] \mapsto [X,-Y,Z,W]$$

and

$$[X,Y,Z,W] \mapsto [X,-Y,Z,-W]$$

of $\mathbb{P}^3$ proving the lemma.

Proof of Lemma 5.2.5

If $\delta \in Aut(F_2)$ and $\theta \circ \delta^2 \circ \theta^{-1} = \Gamma_2$, $(\theta \circ \delta \circ \theta^{-1})^2 = \Gamma_2$. Since we are considering conjugacy classes, we may assume $\delta^2 = \Gamma_2$. Also the automorphism induced on $\mathbb{P}^1$ by δ has order 4 since Γ_2 induces an order 2 automorphism, moreover δ is conjugate to an automorphism which induces the automorphism $z \mapsto iz$ on $\mathbb{P}^1$. Using the local charts defined in the proof of lemma 5.2.4,

$$\delta : ([1,t],w) \mapsto ([1,it], aw+b_0+b_1t+b_2t^2)$$

$$\delta^2 : ([1,t],w) \mapsto ([1,-t], a^2w+(a+1)b_0+(a+i)b_1t+(a-1)b_2t^2).$$

If $\delta^2 = \gamma_2$ (note that Γ_2 is given by γ_2 in the local charts), then $a = i$ or $a = -i$. We get the two automorphisms:

$$\delta_1 : ([1,t],w) \mapsto ([1,it], iw)$$

$$\delta_{(2,b)} : ([1,t],w) \mapsto ([1,it], -iw+bt).$$

With the notation in the above proof of Lemma conjugacy2;

$$\theta_{ib} \circ \delta_{(2,b)} \circ \theta_{ib}^{-1} = \delta_{(2,0)} : ([1,t],w) \mapsto ([1,it],-iw).$$

Thus, we get only two automorphisms. Note that these automorphisms on Q are induced by the following automorphisms of $\mathbb{P}^3$ (since $([1,t],w) \mapsto [1,t,t^2,w]$

is the embedding):

$$\Delta_1 : [X, Y, Z, W] \mapsto [X, iY, -Z, iW]$$
$$\Delta_2 : [X, Y, Z, W] \mapsto [X, iY, -Z, -iW].$$

References

[1] Artin, Michael. *Algebra*. Englewood Cliffs, NJ: Prentice Hall, 1991.

[2] Bouchard, Vincent, and Ron Donagi. "On a Class of Non-simply Connected Calabi-Yau 3-folds." *Communications in Number Theory and Physics* 2, no. 1 (2008): 1–61.

[3] Coxeter, H.S.M.,W.O.J. Moser. *Generators and Relations for Discrete Groups*, 4th ed. Berlin: Springer Verlag, 1980.

[4] Fauntleroy, A. "On the Moduli of Curves on Rational Ruled Surfaces." *American Journal of Mathematics* 109, no. 3 (1987): 417–52.

[5] Kodaira, K. "On Compact Complex Analytic Surfaces II." *Annals of Mathematics* 77 (1963): 563–626.

[6] Miranda, Rick. "Persson's List of Singular Fibers For a Rational Elliptic Surface." *Mathematische Zeitschrift* vol. 205 (1990): 191–211.

[7] Miranda, Rick, and Ulf Persson. "On Extremal Rational Elliptic Surfaces." *Mathematische Zeitschrift* vol. 193 (1986): 537–58.

[8] Miranda, Rick. *The Basic Theory of Elliptic Surfaces*. Universita di Pisa Dipartimento di Matematica, 1989.

[9] Oguiso, Keiji, and Tetsuji Shioda. "The Mordell-Weil Lattice of a Rational Elliptic Surface." *Commentarii Mathematici Universitatis Sancti Pauli* 40 (1991): 83–99.

[10] Persson, Ulf. "Configurations of Kodaira Fibers on Rational Elliptic Surfaces." *Mathematische Zeitschrift* vol. 205 no.1 (1990): 1–47.