ON THE EXISTENCE OF SOLUTIONS TO THE MUSKAT PROBLEM WITH SURFACE TENSION

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ABSTRACT

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We consider the Muskat Problem with surface tension in two dimensions over the real line, with H^s initial data and allowing the two fluids to have different constant densities and viscosities. We take the angle between the interface and the horizontal, and derive an evolution equation for it. We use energy methods to prove that a solution θ exists locally and can be continued while $||\theta||_s$ remains bounded and the arc chord condition holds. Furthermore, the resulting solution is unique, and depends continuously on the initial data. Additionally, when both fluids have the same viscosity and the initial data is sufficiently small, we show the energy is non-increasing, and that the solution θ exists globally in time.

Contents

Chapter 1

Introduction:

We consider the dynamics of the interface between two incompressible fluids, also known as the Muskat problem. In this paper, we focus on the initial value problem in two dimensions with surface tension. We will take the angle θ between the interface and the horizontal, and prove that it satisfies the evolution equation

$$\begin{aligned}\theta_t = &\frac{\tau}{2}H(\theta_{\alpha\alpha\alpha}) - \frac{1}{2}H\big((R\cos(\theta) + 2A_\mu U)\theta_\alpha\big)\\ &- A_\mu H\left(\frac{-H(\gamma\theta_\alpha)}{2} + m\cdot\hat{t}\right) + (V - W\cdot\hat{t})\theta_\alpha + m\cdot\hat{n}\end{aligned} \tag{1.1}$$

We will use energy methods to prove that θ exists locally, obtaining a bound for the energy that is polynomial in nature, as well as continuation criteria that depend only on the Sobolev norm and the arc-chord condition. Additionally, we show that the resulting solution is unique, varying continuously with the initial data. This extends previous results of Ambrose [1], who proved the same in the periodic case. Furthermore, we show that when the viscosity remains constant, the bound can be tightened such that the lowest degree terms are strictly negative. As a consequence, when the initial data is sufficiently small, the energy is non-increasing, and we will show that this implies global existence for θ.

We begin by rigorously stating the problem. The fluids are incompressible and satisfy Darcy's law. Therefore, letting $v(x,t)$ be the velocity and $p(x,t)$ be the pressure, we have

$$\frac{\mu}{k}v = -\bigtriangledown p - (0, g\rho)$$

$$\bigtriangledown \cdot v = 0$$

Here μ is the dynamic viscosity, k is the permeability of the medium, ρ is the density of the liquid, and g is the acceleration from gravity. For this paper, we assume that the two fluids have different constant viscosities and densities (which we label μ_1, μ_2, ρ_1, and ρ_2), and that the surface tension τ is non-zero. We consider this interface as a two-dimensional parametric function $(x(\alpha,t), y(\alpha,t))$, and denote it $z(\alpha,t)$. The curve evolves according to the Birkhoff-Rott integral,

$$\Phi(z_t)^*(\alpha,t) = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{\gamma(\alpha')}{z(\alpha,t) - z(\alpha',t)} d\alpha'$$

Here Φ maps R^2 to the complex plane, $*$ is complex conjugation, and γ is the vortex sheet strength, satisfying the integral equation

$$\gamma = \tau\kappa_\alpha - (\rho_1 - \rho_2)y_\alpha - 2A_\mu s_\alpha W \cdot \hat{t}$$

where τ is the surface tension, κ is the curvature, s_α is the arc length, $A_\mu = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$ is the Atwood number, and W is the aforementioned Birkhoff-Rott integral. Next, we define our notation.

1.1 Notation:

We define the Lebesgue spaces L^2 and L^∞ in the standard way, with norms

$$||f||_{L^2} = \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2}$$

$$||f||_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|$$

We also define the Sobolev space H^s in the usual way, via

$$||f||_s = ||f||_{L^2} + ||\partial_x^s f||_{L^2}$$

We define the Hilbert transform H in the standard way,

$$H(f)(\alpha) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x)}{\alpha - x} dx$$

Furthermore, we define the commutator $[H, f]$ to be the non-singular integral operator

$$[H, f]g(\alpha) = H(fg)(\alpha) - f(\alpha)H(g)(\alpha)$$

Additionally, for any L^2 function g, we define the Fourier transform as follows,

$$\hat{g}(\zeta) = \int e^{-2\pi i \zeta x} g(x) dx$$

For any set S, we let χ_S be the characteristic function on the set S. That is, $\chi_S(x) = 1$ if $x \in S$, otherwise $\chi_S(x) = 0$.

If z is a complex number, we let z^* denote its complex conjugate.

Finally, we use the notation $A \lesssim B$ to denote $A \leq C \cdot B$ for some constant C.

1.2 Past Results:

The Muskat problem without surface tension has been widely studied. It has been shown the problem is well posed when the initial data satisfies the Rayleigh-Taylor condition [8], and that otherwise the problem is ill-posed (see [17] and [9]). Castro, Cordoba, Fefferman, and Gancedo proved that turning waves can develop in finite time [2], and that there exists analytic initial data satisfying Rayleigh-Taylor resulting in a singularity in finite time[3]. In [6], it was shown that when both fluids have the same viscosity, the interface obeys an L^2 maximum principle, and that it exists globally for small initial data. These results were later extended to three-dimensional space in [7].

The presence of surface tension makes the equation more regular, ensuring the problem is well-posed [13]. Furthermore, in the periodic case, Escher and Matioc [11],[12] proved global existence for small initial data in Holder spaces. Also in the periodic case, Ambrose [1] proved that as the surface tension coefficient approaches zero, the solutions to the Muskat problem with surface tension exist on a uniform time interval and converge to a solution of the problem without surface tension. For additional results regarding the Muskat problem and Hele-Shaw cells, we refer the reader to [4], [5], [14], [16], [17] and the discussion therein.

1.3 Main Results:

For our paper, we adapt the method of [1] to the real line. Namely, rather than work with $z(\alpha, t)$ directly, we instead focus on bounding the angle between the tangent and the horizontal, $\theta(\alpha, t)$, as it determines the interface up to a constant. Our first result is that θ exists locally in time.

Theorem 1.1. *Let $\theta_0 \in H^s$ satisfy the arc-chord condition. Then there exists some $T > 0$ and $\theta \in C([0, T], H^s)$ such that θ is a solution to (1.1) and $\theta(\cdot, 0) = \theta_0$.*

Additionally, we prove that the solution θ to (1.1) is unique.

Theorem 1.2. *Let $d_1 < \infty, d_2 > 0$. Define the set $\mathcal{O}$ by*

$$\mathcal{O} = \{\theta \in H^s \mid ||\theta||_s \leq d_1, |\frac{z_d(\alpha) - z_d(\alpha')}{\alpha - \alpha'}| > d_2 \forall \alpha, \alpha' \in R\}$$

Let $\theta_0, \phi_0 \in \mathcal{O}$ be given. Then the solution of the initial value problem (1.1) with $\theta(\cdot, 0) = \theta_0$ is unique. Furthermore, if $T > 0$ such that $\theta \in C([0, T]; \mathcal{O})$ is the solution corresponding to θ_0 and $\phi \in C([0, T]; \mathcal{O})$ is the solution corresponding to ϕ_0, then

$$\sup_{t \in [0,T]} ||\theta - \phi||_2 \lesssim ||\theta_0 - \phi_0||_2$$

Finally, we show that when both fluids have the same viscosity and the initial data is sufficiently small, θ exists globally in time.

Theorem 1.3. *Suppose the Atwood number A_μ is zero, and that $||\theta_0||_s \leq c$ for c small enough. Then there exists some $\theta \in C([0, \infty); H^s)$ such that θ is a solution to (1.1) and $\theta(\cdot, 0) = \theta_0$.*

1.4 Strategy of Proof:

In this subsection we will discuss the general strategy used throughout this paper. The preliminary work of setting up the evolution equation is almost exactly the same as in [1], with the end result being

$$\begin{aligned}\theta_t &= \frac{\tau}{2}H(\theta_{\alpha\alpha\alpha}) - \tau\left(\frac{A_\mu}{2}\theta_\alpha\theta_{\alpha\alpha} + A_\mu H\big(U^{st}\theta_\alpha\big)\right)\\ &+ \left[H(k\theta_\alpha) + \tau\big(V^{st} - W^{st}\cdot\hat{t}\big)\theta_\alpha + \tau A_\mu^2\theta_\alpha W^{st}\cdot\hat{t}\right]\\ &+ \frac{A_\mu}{2}R\sin(\theta)\theta_\alpha + A_\mu^2\theta_\alpha\tilde{W}\cdot\hat{t} + \big(\tilde{V} - \tilde{W}\cdot\hat{t}\big)\theta_\alpha + m\cdot\hat{n} - A_\mu H(m\cdot\hat{t})\end{aligned}$$

Next, we define χ_ϵ to be the standard mollifier, and let θ^ϵ denote the solution of the mollified equation

$$\begin{aligned}\theta_t^\epsilon &= \frac{\tau}{2}\chi_\epsilon^2 H(\theta_{\alpha\alpha\alpha}^\epsilon) - \tau\chi_\epsilon\left(\frac{A_\mu}{2}(\chi_\epsilon\theta_\alpha^\epsilon)(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon) + A_\mu H\big(U^{st,\epsilon}\chi_\epsilon\theta_\alpha^\epsilon\big)\right)\\ &+ \chi_\epsilon\left[H(k^\epsilon\chi_\epsilon\theta_\alpha^\epsilon) + \tau\big(V^{st,\epsilon} - W^{st,\epsilon}\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta_\alpha^\epsilon + \tau A_\mu^2\big(\chi_\epsilon\theta_\alpha^\epsilon\big)W^{st,\epsilon}\cdot\hat{t}^\epsilon\right]\\ &+ \frac{A_\mu}{2}R\chi_\epsilon\Big(\sin(\chi_\epsilon\theta^\epsilon)\chi_\epsilon\theta_\alpha^\epsilon\Big) + A_\mu^2\chi_\epsilon\Big(\big(\chi_\epsilon\theta_\alpha^\epsilon\big)\tilde{W}^\epsilon\cdot\hat{t}^\epsilon\Big)\\ &+ \chi_\epsilon\left[\big(\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta_\alpha^\epsilon\right] + \chi_\epsilon(m^\epsilon\cdot\hat{n}^\epsilon) - A_\mu\chi_\epsilon H(m^\epsilon\cdot\hat{t}^\epsilon)\end{aligned}$$

Via basic properties of mollifiers and Picard's theorem, it's easy to show that the θ^ϵ exist on some time interval $[0, T^\epsilon]$. Our main goal in the paper is to prove the necessary energy estimates to get a uniform time of existence $[0, T]$ for all θ^ϵ. Once those estimates have been obtained, we note that the θ^ϵ form an equicontinuous family, and by Arzela-Ascoli, some subsequence must converge to a limit θ. Via standard methods (the ones used in Chapter 3 of [15]), we show that this θ does indeed satisfy the original evolution equation and exists on the same time interval as the θ^ϵ.

Local Existence:

Now, the first goal is to get an energy estimate on $\frac{dE}{dt}$ independent of ϵ. Now,

$$\frac{dE}{dt} = \int \theta^\epsilon \theta^\epsilon_t + (\partial^s_\alpha \theta^\epsilon)(\partial^{s-2}_\alpha \theta^\epsilon_{\alpha\alpha,t}) d\alpha$$

Bounding θ^ϵ_t is fairly straightforwards (if tedious), so the main difficulty is the high degree term. The core idea here is that after differentiating θ^ϵ_t twice (and performing some algebraic manipulation), we get an equation of the form

$$\theta^\epsilon_{\alpha\alpha,t} = \chi_\epsilon \left[\frac{-\tau}{2} \Lambda^3 (\chi_\epsilon \theta^\epsilon_{\alpha\alpha}) + \Upsilon^\epsilon_5 \Lambda(\chi_\epsilon \theta^\epsilon_{\alpha\alpha}) + \Upsilon^\epsilon_6 \chi_\epsilon \theta^\epsilon_{\alpha\alpha\alpha} + \Upsilon^\epsilon_7 \right]$$

Here, $\Lambda = \partial_\alpha H$, the Υ^ϵ_j are L^∞, and their derivatives are H^{s-3}. (Proving this fact is nontrivial, but uninstructive for a high level understanding of our proof as a whole. We refer the reader to Section 4 and the first half of Section 5 for the details.) So, applying the $s-2$ derivatives to $\theta^\epsilon_{\alpha\alpha,t}$ via the product rule, we see that every term has a bound of the form $||\partial_\alpha \Upsilon^\epsilon_j||_{s-3} ||\theta||^2_s$, except for the following terms:

$$\int (\chi_\epsilon \partial^s_\alpha \theta^\epsilon) \left[\frac{-\tau}{2} \Lambda^3 (\chi_\epsilon \partial^s_\alpha \theta^\epsilon) + \Upsilon^\epsilon_5 \Lambda (\chi_\epsilon \partial^s_\alpha \theta^\epsilon) + \Upsilon^\epsilon_6 \chi_\epsilon \partial^{s+1}_\alpha \theta^\epsilon \right]$$

Exploiting the fact that $\Lambda^{1/2}$ is self-adjoint, we have

$$\int \frac{-\tau}{2} (\Lambda^{3/2} \chi_\epsilon \partial_\alpha \theta^\epsilon)^2 + (\chi_\epsilon \partial^s_\alpha \theta^\epsilon)(\Lambda \chi_\epsilon \partial^s_\alpha \theta^\epsilon) \Upsilon^\epsilon_5 + \frac{1}{2} \Upsilon^\epsilon_6 \partial_\alpha (\chi_\epsilon \partial^s_\alpha \theta^\epsilon)^2$$

Here the first term is strictly negative, since

$$\int \frac{-\tau}{2} (\Lambda^{3/2} \chi_\epsilon \partial_\alpha \theta^\epsilon)^2 = -\frac{\tau}{2} ||\Lambda^{3/2} \chi_\epsilon \partial_\alpha \theta^\epsilon||^2_{L^2}$$

For the Υ_5^ϵ term, we cannot dispose of the $(\Lambda\chi_\epsilon\partial_\alpha^s\theta^\epsilon)$ factor, but we can separate it via use of Young's inequality

$$\int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)(\Lambda\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\Upsilon_5^\epsilon \leq \int(\Lambda\chi_\epsilon\partial_\alpha^s\theta^\epsilon)^2 + \int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon\cdot\Upsilon_5^\epsilon)^2$$

Since $\theta^\epsilon \in H^s$ and $\Upsilon_5^\epsilon \in L^\infty$, the second term can be bounded directly. To bound the first term, we make use of the dissipative surface tension term from before, and it can be shown that

$$\int(\Lambda\chi_\epsilon\partial_\alpha^s\theta^\epsilon)^2 - \frac{\tau}{2}(\Lambda^{3/2}\chi_\epsilon\partial_\alpha\theta^\epsilon)^2 \lesssim ||\theta^\epsilon||_s^2$$

Finally, to bound the Υ_6^ϵ term, since $\theta^\epsilon \in H^s$ and $\partial_\alpha\Upsilon_6^\epsilon \in L^2$, we can integrate by parts, obtaining

$$\int\frac{-1}{2}\partial_\alpha\Upsilon_6^\epsilon\cdot(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)^2 \lesssim ||\partial_\alpha\Upsilon_6^\epsilon||_{L^2}||\theta^\epsilon||_s^2$$

Therefore, in the end we obtain an energy estimate of the form

$$\frac{dE}{dt} \lesssim e^{||\theta^\epsilon||_s}$$

Which in turn, is sufficient for local existence.

Uniqueness:

The strategy for proving uniqueness is similar to that used to prove local existence. Once again we consider an energy estimate $\frac{dE}{dt}$, but this time the energy we are concerned with is that of the difference between two solutions. Given $\theta^\epsilon, \phi^\delta$, we let $E_d = \frac{1}{2}||\theta^\epsilon - \phi^\delta||_2^2$, and aim to bound

$$\frac{dE_d}{dt} = \int_{\mathbb{R}}(\theta^\epsilon - \phi^\delta)(\theta^\epsilon - \phi^\delta)_t + (\theta_{\alpha\alpha}^\epsilon - \phi_{\alpha\alpha}^\delta)(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t}d\alpha$$

Once again we decompose $(\theta^\epsilon - \phi^\delta)_t$ into groups of terms,

$$(\theta^\epsilon - \phi^\delta)_t = \chi_\delta \left[\frac{-\tau}{2} \chi_\delta \Lambda^3 (\theta^\epsilon - \phi^\delta) + \tau \Upsilon_8 \chi_\delta (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}) \right]$$
$$+ \chi_\delta \left[\Upsilon_9 \chi_\delta \Lambda (\theta^\epsilon - \phi^\delta) + \Upsilon_{10} \chi_\delta (\theta^\epsilon_\alpha - \phi^\delta_\alpha) \right]$$
$$+ \chi_\delta \left[\Upsilon_{11} + \Upsilon_{12} + \Upsilon_{13} \right] + (\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1} \theta^\epsilon_t)$$

The Λ^3 term still functions as a dissipative term, which is necessary to bound the Υ_8 and Υ_9 terms after differentiation. The Υ_{10} term can be dealt with using an integration by parts, while the remaining Υ contain remainder terms. The main difficulty is in successfully bounding the remainder terms involved, as the Lipschitz bounds necessary prove to be very technical in nature, even if the strategy involved is similar to the lemmas proved for local existence.

Global Existence:

Now, the continuation criteria for the existence of the θ^ϵ is that their H^s norm must remain bounded, and the arc-chord condition must hold. However, it is simple to show that if $||\theta^\epsilon||_s$ is sufficiently small (the precise condition being that $||\theta^\epsilon||_{L^\infty} < c < \pi/2$ for some constant c), then the arc-chord condition will hold automatically. This implies that for small initial data, if $\frac{dE}{dt} \leq 0$, then the θ^ϵ will exist globally, and therefore so will θ. Furthermore, while an exponential bound was used for $\frac{dE}{dt}$ during the proof of local existence, it's simple to see that the vast majority of terms contain powers of $||\theta^\epsilon||_s$ of order 3 or higher. Additionally, most (though not all) of the second-order terms can be shown to be strictly negative, such as $-\frac{\tau}{2}||\Lambda^{3/2}\chi_\epsilon \partial_\alpha \theta^\epsilon||^2_{L^2}$.

Therefore, our goal is to show a bound of the form

$$\frac{dE}{dt} \leq -c_2||\theta^\epsilon||_s^2 + c_3||\theta^\epsilon||_s^3 + c_4||\theta^\epsilon||_s^4 + ...$$

and in particular, if $||\theta^\epsilon||_s$ is sufficiently small, then $\frac{dE}{dt} \leq 0$, and so the θ^ϵ and therefore θ exist globally in time.

1.5 Outline of the paper:

In chapter 2 we set up the basic equation of motion, getting the formula for θ_t. In chapter 3, we then mollify said equation, and the resulting formula for θ_t^ϵ is the starting point for the remainder of the proof. Chapter 4 is devoted to some useful Lemmas bounding the various terms in the equation of θ_t^ϵ, and the energy estimates for local existence are proved in chapter 5. In chapter 6 we tackle the problem of uniqueness, and finish the proof of local existence. Finally, chapter 7 introduces tighter bounds for some of the same terms under the more stringent conditions we require for global existence, letting us finish the proof of global existence in chapter 8.

Chapter 2

Equation of Motion:

Now, Darcy's Law gives

$$\frac{\mu}{k} v = - \bigtriangledown p - (0, g\rho) \tag{2.1}$$

Where $v(x,t)$ is the incompressible velocity, $p(x,t)$ is pressure, μ is viscosity, ρ is density, and g is gravity. Furthermore, we assume that $\mu = \mu_1, \rho = \rho_1$ for $x \in \Omega_1(t)$, and $\mu = \mu_2, \rho = \rho_2$ for $x \in \Omega_2(t)$. Finally, we assume that $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_1 \cup \Omega_2 = \mathbb{R}^2$, and $\partial\Omega_j(t) = z(\alpha, t) = \{(x(\alpha,t), y(\alpha,t)) | \alpha \in \mathbb{R}\}$. (For reference, we will let Ω_2 be the section below the curve.)

Now, define $\Phi : \mathbb{R}^2 \to \mathbb{C}$ by $\Phi(x,y) = x + iy$. In particular, $\Phi(z(\alpha,t)) = \Phi(x(\alpha,t), y(\alpha,t)) = x(\alpha,t) + iy(\alpha,t)$. The arc length is $s_\alpha = (x_\alpha^2 + y_\alpha^2)^{1/2}$, and the unit tangent and normal vectors are

$$\hat{t} = \frac{(x_\alpha, y_\alpha)}{s_\alpha}, \; \hat{n} = \frac{(-y_\alpha, x_\alpha)}{s_\alpha}$$

The angle between the curve and the horizontal is

$$\theta(\alpha,t)=\arctan\Big(\frac{y_\alpha(\alpha,t)}{x_\alpha(\alpha,t)}\Big)$$

and we have

$$\hat{t}=(\cos(\theta),\sin(\theta)),\hat{n}=(-\sin(\theta),\cos(\theta))$$

$$\hat{t}_\alpha=\theta_\alpha\hat{n},\hat{n}_\alpha=-\theta_\alpha\hat{t} \tag{2.2}$$

Now, defining U to be the normal component of the velocity and V to be the tangential component, we immediately obtain

$$(x,y)_t=U\hat{n}+V\hat{t}$$

Therefore,

$$\begin{aligned}\theta_t&=\frac{d}{dt}\arctan(\frac{y_\alpha}{x_\alpha})=\frac{1}{1+(y_\alpha/x_\alpha)^2}\cdot\frac{y_{\alpha t}x_\alpha-x_{\alpha t}y_\alpha}{x_\alpha^2}\\&=\frac{y_{\alpha t}x_\alpha-x_{\alpha t}y_\alpha}{x_\alpha^2+y_\alpha^2}=\frac{\frac{1}{s_\alpha}(x_\alpha^2+y_\alpha^2)(U_\alpha+\theta_\alpha V)}{x_\alpha^2+y_\alpha^2}=\frac{U_\alpha+V\theta_\alpha}{s_\alpha}\end{aligned} \tag{2.3}$$

since

$$\begin{aligned}(x,y)_{\alpha t}&=U_\alpha\hat{n}-\theta_\alpha U\hat{t}+V_\alpha\hat{t}+\theta_\alpha V\hat{n}\\&=\frac{1}{s_\alpha}(x_\alpha(V_\alpha-\theta_\alpha U)-y_\alpha(U_\alpha+\theta_\alpha V),y_\alpha(V_\alpha-\theta_\alpha U)+x_\alpha(U_\alpha+\theta_\alpha V))\end{aligned}$$

To simplify the equations, we choose a parametrization such that $s_\alpha=1$. Furthermore, we will choose the boundary conditions to be $\lim_{\alpha\to\pm\infty}y(\alpha,t)=0$, and $\lim_{\alpha\to\pm\infty}\big(x(\alpha,t)-\alpha\big)=c$ for some constant c. As a consequence, we have that

$$s_{\alpha t}=V_\alpha-\theta_\alpha U=0 \tag{2.4}$$

Since the interface is a vortex sheet, the normal velocity U must satisfy $U = W \cdot \hat{n}$, where the Birkhoff-Rott integral W is

$$\Phi(W)^* = \frac{1}{2\pi i} \int \frac{\gamma(\alpha')}{z(\alpha) - z(\alpha')} d\alpha' \tag{2.5}$$

where γ is the vortex sheet strength. Furthermore, note that

$$\frac{\gamma(\alpha)}{z_\alpha(\alpha)} \cdot PV \int \frac{-z_\alpha(\alpha')z_\alpha(\alpha)}{(z(\alpha) - z(\alpha'))^2} = PV\gamma(\alpha) \frac{z_\alpha(\alpha')}{z(\alpha) - z(\alpha')} \Big|_{\alpha'=-\infty}^{+\infty} = 0$$

Therefore, we have that

$$\begin{aligned}
\Phi(W)^*_\alpha &= \frac{1}{2\pi i} PV \int \frac{\gamma(\alpha')}{z_\alpha(\alpha')} \cdot \frac{-z_\alpha(\alpha')z_\alpha(\alpha)}{(z(\alpha) - z(\alpha'))^2} d\alpha' \\
&= \frac{1}{2\pi i} PV \int \left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')} - \frac{\gamma(\alpha)}{z_\alpha(\alpha)} \right) \frac{-z_\alpha(\alpha')z_\alpha(\alpha)}{(z(\alpha) - z(\alpha'))^2} d\alpha' \\
&= \frac{-z_\alpha(\alpha)}{2\pi i} PV \int \left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')} - \frac{\gamma(\alpha)}{z_\alpha(\alpha)} \right) \partial_{\alpha'} \left(\frac{1}{z(\alpha) - z(\alpha')} \right) d\alpha'
\end{aligned}$$

Now, integrate by parts to get

$$\Phi(W)^*_\alpha = \frac{z_\alpha(\alpha)}{2\pi i} PV \int \left(\frac{\gamma_\alpha(\alpha')}{z_\alpha(\alpha')} - \frac{\gamma(\alpha')z_{\alpha\alpha}(\alpha')}{z_\alpha^2(\alpha')} \right) \cdot \frac{1}{z(\alpha) - z(\alpha')} d\alpha'$$

Next, we approximate $z(\alpha) - z(\alpha')$ by $z_\alpha(\alpha')(\alpha - \alpha')$, splitting

$$\Phi(W)^*_\alpha = \Phi(A_1)^* + \Phi(A_2)^* + \Phi(R_1)^* + \Phi(R_2)^*$$

where

$$\Phi(A_1)^* = \frac{z_\alpha(\alpha)}{2\pi i} PV \int \frac{\gamma_\alpha(\alpha')}{z_\alpha(\alpha')} \cdot \frac{1}{z_\alpha(\alpha')(\alpha - \alpha')} d\alpha'$$
$$\Phi(A_2)^* = \frac{-z_\alpha(\alpha)}{2\pi i} PV \int \frac{\gamma(\alpha') z_{\alpha\alpha}(\alpha')}{z_\alpha^2(\alpha')} \cdot \frac{1}{z_\alpha(\alpha')(\alpha - \alpha')} d\alpha'$$
$$\Phi(R_1)^* = \frac{z_\alpha(\alpha)}{2\pi i} PV \int \frac{\gamma_\alpha(\alpha')}{z_\alpha(\alpha')} \cdot \left(\frac{1}{z(\alpha) - z(\alpha')} - \frac{1}{z_\alpha(\alpha')(\alpha - \alpha')} \right) d\alpha'$$
$$\Phi(R_2)^* = \frac{-z_\alpha(\alpha)}{2\pi i} PV \int \frac{\gamma(\alpha') z_{\alpha\alpha}(\alpha')}{z_\alpha^2(\alpha')} \cdot \left(\frac{1}{z(\alpha) - z(\alpha')} - \frac{1}{z_\alpha(\alpha')(\alpha - \alpha')} \right) d\alpha'$$

Separating A_1 into a Hilbert transform plus a commutator, we have

$$\Phi(A_1)^* = \frac{z_\alpha(\alpha)}{2i} H\left(\frac{\gamma_\alpha}{z_\alpha^2}\right) = \frac{1}{2iz_\alpha} H(\gamma_\alpha) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right](\gamma_\alpha)$$

Since $z_\alpha \cdot z_\alpha^* = s_\alpha^2 = 1$, we have

$$\Phi(A_1)^* = \frac{z_\alpha^*}{2i} H(\gamma_\alpha) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right](\gamma_\alpha)$$
$$\Phi(A_1) = \frac{z_\alpha i}{2} H(\gamma_\alpha) + \frac{z_\alpha^* i}{2}\left[H, \frac{1}{\left(z_\alpha^*\right)^2}\right](\gamma_\alpha)$$

Now, since $s_\alpha = 1$, therefore $|z_\alpha| = 1$, and in particular,

$$\Phi(\hat{t}) = z_\alpha, \Phi(\hat{n}) = iz_\alpha$$

And therefore, we have that

$$A_1 = \frac{H(\gamma_\alpha)}{2} \cdot \hat{n} + B_1$$

where

$$B_1 = \Phi^{-1}\left(\frac{z_\alpha^* i}{2}\left[H, \frac{1}{\left(z_\alpha^*\right)^2}\right](\gamma_\alpha)\right)$$

Similarly,

$$\Phi(A_2)^* = \frac{-z_\alpha}{2i} H\Big(\frac{\gamma z_{\alpha\alpha}}{z_\alpha^3}\Big) = \frac{-1}{2iz_\alpha} H\Big(\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\Big) - \frac{z_\alpha}{2i}\Big[H, \frac{1}{z_\alpha^2}\Big]\Big(\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\Big)$$

Since $\hat{a} \cdot \hat{b} = Re(\Phi(a)\Phi(b)^*)$, we have

$$\begin{aligned} A_2 \cdot \hat{t} &= Re\bigg(\frac{-1}{2i} H\Big(\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\Big)\bigg) + B_2 \cdot \hat{t} \\ &= \frac{-1}{2} H\bigg(\gamma Re\Big(\frac{-iz_{\alpha\alpha}}{z_\alpha}\Big)\bigg) + B_2 \cdot \hat{t} \\ &= \frac{-1}{2} H\bigg(\gamma Re\big(z_{\alpha\alpha}(iz_\alpha)^*\big)\bigg) + B_2 \cdot \hat{t} \end{aligned}$$

Where

$$B_2 = \Phi^{-1}\bigg(\frac{-z_\alpha^* i}{2}\Big[H, \frac{1}{\left(z_\alpha^*\right)^2}\Big]\Big(\frac{\gamma z_{\alpha\alpha}^*}{z_\alpha^*}\Big)\bigg)$$

However, $Re\big(z_{\alpha\alpha} \cdot (iz_\alpha)^*\big) = Re\big(\Phi(\hat{t}_\alpha)\Phi(\hat{n})^*\big) = \hat{t}_\alpha \cdot \hat{n} = \theta_\alpha$, and therefore

$$A_2 \cdot \hat{t} = \frac{-1}{2} H(\gamma\theta_\alpha) + B_2 \cdot \hat{t}$$

Similarly,

$$\begin{aligned} A_2 \cdot \hat{n} &= Re\bigg(\frac{-1}{2} H\Big(\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\Big)\bigg) + B_2 \cdot \hat{n} \\ &= \frac{-1}{2} H\bigg(\gamma Re\big(z_{\alpha\alpha} z_\alpha^*\big)\bigg) + B_2 \cdot \hat{n} \\ &= B_2 \cdot \hat{n} \end{aligned}$$

since $Re(z_{\alpha\alpha} z_\alpha^*) = \hat{t}_\alpha \cdot \hat{t} = 0$. Therefore, combining the previous results, we have

$$W_\alpha = \frac{H(\gamma_\alpha)}{2} \cdot \hat{n} - \frac{H(\gamma\theta_\alpha)}{2} \cdot \hat{t} + m \tag{2.6}$$

where $m = B_1 + B_2 + R_1 + R_2$.

Defining the integral operator $K[z]$ by

$$K[z](f(\alpha)) = \frac{1}{2\pi i}\int f(\alpha')\left(\frac{1}{z(\alpha)-z(\alpha')} - \frac{1}{z_\alpha(\alpha')(\alpha-\alpha')}\right)d\alpha' \tag{2.7}$$

we have

$$\begin{aligned}\Phi(m)^* = \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right](\gamma_\alpha) - \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\right) \\ + z_\alpha K[z]\left(\frac{\gamma_\alpha}{z_\alpha}\right) - z_\alpha K[z]\left(\frac{\gamma z_{\alpha\alpha}}{z_\alpha^2}\right)\end{aligned} \tag{2.8}$$

Now,

$$\gamma = \tau\kappa_\alpha - Ry_\alpha - 2A_\mu s_\alpha W \cdot \hat{t}$$

where $R = (\rho_1 - \rho_2)g$, and $A_\mu = \frac{\mu_1-\mu_2}{\mu_1+\mu_2}$ is the Atwood number. Substituting in $y_\alpha = s_\alpha \sin(\theta), \kappa = \theta_\alpha/s_\alpha$, and $s_\alpha = 1$, we obtain

$$\gamma = \tau\theta_{\alpha\alpha} - R\sin(\theta) - 2A_\mu W \cdot \hat{t} \tag{2.9}$$

Next we want to rewrite the equation for θ_t,

$$\begin{aligned}\theta_t &= \frac{U_\alpha + V\theta_\alpha}{s_\alpha} = U_\alpha + V\theta_\alpha \\ &= (W\cdot\hat{n})_\alpha + V\theta_\alpha = W_\alpha\cdot\hat{n} + W\cdot\hat{n}_\alpha + V\theta_\alpha \\ &= W_\alpha\cdot\hat{n} + (V - W\cdot\hat{t})\theta_\alpha \\ &= \frac{H(\gamma_\alpha)}{2} + m\cdot\hat{n} + (V - W\cdot\hat{t})\theta_\alpha\end{aligned}$$

Differentiating the equation for γ, we get

$$\gamma_\alpha = \tau\theta_{\alpha\alpha\alpha} - R\cos(\theta)\theta_\alpha - 2A_\mu(W_\alpha\cdot\hat{t} + \theta_\alpha U) \tag{2.10}$$

Substituting this in and using our equation for $W \cdot \hat{t}$, we re-derive (1.1)

$$\begin{aligned}\theta_t = \frac{\tau}{2} H(\theta_{\alpha\alpha\alpha}) - \frac{1}{2} H\big((R\cos(\theta) + 2A_\mu U)\theta_\alpha\big) \\ - A_\mu H\left(\frac{-H(\gamma\theta_\alpha)}{2} + m \cdot \hat{t}\right) + (V - W \cdot \hat{t})\theta_\alpha + m \cdot \hat{n}\end{aligned}$$

Next we'll split up into surface tension and non-surface tension terms. Namely, define

$$W = \tau W^{st} + \tilde{W}$$

$$\Phi(W^{st})^* = \frac{1}{2\pi i} \int \frac{\theta_{\alpha\alpha}(\alpha')}{z(\alpha) - z(\alpha')} d\alpha'$$

$$\tilde{W} = W - \tau W^{st}$$

Similarly,

$$U^{st} = W^{st} \cdot \hat{n},\ \tilde{U} = \tilde{W} \cdot \hat{n}$$

$$V^{st}_\alpha = U^{st}\theta_\alpha,\ \tilde{V}_\alpha = \tilde{U}\theta_\alpha$$

Furthermore, for convenience, we will define

$$k(\alpha, t) = k[\theta](\alpha, t) = -\frac{R\cos(\theta)}{2} - A_\mu \tilde{U}$$

Then, since $H\big(H(f)\big) = -f$ for $f \in L^2$, we have:

$$\begin{aligned}\theta_t &= \frac{\tau}{2} H(\theta_{\alpha\alpha\alpha}) - \tau\left(\frac{A_\mu}{2}\theta_\alpha\theta_{\alpha\alpha} + A_\mu H\big(U^{st}\theta_\alpha\big)\right) \\ &+ \left[H(k\theta_\alpha) + \tau\big(V^{st} - W^{st} \cdot \hat{t}\big)\theta_\alpha + \tau A_\mu^2 \theta_\alpha W^{st} \cdot \hat{t}\right] \\ &+ \frac{A_\mu}{2} R\sin(\theta)\theta_\alpha + A_\mu^2 \theta_\alpha \tilde{W} \cdot \hat{t} + \big(\tilde{V} - \tilde{W} \cdot \hat{t}\big)\theta_\alpha + m \cdot \hat{n} - A_\mu H(m \cdot \hat{t})\end{aligned} \tag{2.11}$$

Chapter 3

Mollified Equations:

The next step is to create a mollified version of (2.11), the solution of which we will refer to as θ^ϵ. First we define z_d^ϵ by

$$z_d^\epsilon(\alpha,t) = \int_0^\alpha \cos\big(\theta^\epsilon(\alpha',t)\big) + i\sin\big(\theta^\epsilon(\alpha',t)\big)d\alpha' \tag{3.1}$$

Note that the value $z(0)$ is irrelevant to the equation of motion (2.11), as only terms of the form $z(\alpha) - z(\alpha')$ and z_α appear. Now, define the mollified equation by

$$\begin{aligned}\theta_t^\epsilon &= \frac{\tau}{2}\chi_\epsilon^2 H(\theta_{\alpha\alpha\alpha}^\epsilon) - \tau\chi_\epsilon\Bigg(\frac{A_\mu}{2}(\chi_\epsilon\theta_\alpha^\epsilon)(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon) + A_\mu H\big(U^{st,\epsilon}\chi_\epsilon\theta_\alpha^\epsilon\big)\Bigg)\\ &+ \chi_\epsilon\Big[H(k^\epsilon\chi_\epsilon\theta_\alpha^\epsilon) + \tau\big(V^{st,\epsilon} - W^{st,\epsilon}\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta_\alpha^\epsilon + \tau A_\mu^2\big(\chi_\epsilon\theta_\alpha^\epsilon\big)W^{st,\epsilon}\cdot\hat{t}^\epsilon\Big]\\ &+ \frac{A_\mu}{2}R\chi_\epsilon\Bigg(\sin(\chi_\epsilon\theta^\epsilon)\chi_\epsilon\theta_\alpha^\epsilon\Bigg) + A_\mu^2\chi_\epsilon\Bigg(\big(\chi_\epsilon\theta_\alpha^\epsilon\big)\tilde{W}^\epsilon\cdot\hat{t}^\epsilon\Bigg)\\ &+ \chi_\epsilon\Big[\big(\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta_\alpha^\epsilon\Big] + \chi_\epsilon(m^\epsilon\cdot\hat{n}^\epsilon) - A_\mu\chi_\epsilon H(m^\epsilon\cdot\hat{t}^\epsilon)\end{aligned} \tag{3.2}$$

where χ_ϵ is a standard mollifier, and

$$\hat{t}^\epsilon = \big(\cos(\theta^\epsilon), \sin(\theta^\epsilon)\big),\ \hat{n}^\epsilon = \big(-\sin(\theta^\epsilon), \cos(\theta^\epsilon)\big) \tag{3.3}$$

$$\Phi(W^{st,\epsilon}) = \frac{1}{2\pi i} PV \int \frac{\chi_\epsilon \theta^\epsilon_{\alpha\alpha}(\alpha')}{z^\epsilon_d(\alpha) - z^\epsilon_d(\alpha')} d\alpha' \tag{3.4}$$

$$\Phi(\tilde{W}^\epsilon) = \frac{1}{2\pi i} PV \int \frac{\tilde{\gamma}[\theta^\epsilon]}{z^\epsilon_d(\alpha) - z^\epsilon_d(\alpha')} d\alpha' \tag{3.5}$$

$$U^{st,\epsilon} = W^{st,\epsilon} \cdot \hat{n}^\epsilon,\ \tilde{U}^\epsilon = \tilde{W}^\epsilon \cdot \hat{n}^\epsilon \tag{3.6}$$

$$\partial_\alpha V^{st,\epsilon} = U^{st,\epsilon}\theta^\epsilon_\alpha,\ \partial_\alpha \tilde{V}^\epsilon = \tilde{U}^\epsilon \theta^\epsilon_\alpha \tag{3.7}$$

$$k^\epsilon = \frac{-R\cos(\chi_\epsilon \theta^\epsilon)}{2} - A_\mu \tilde{U}^\epsilon \tag{3.8}$$

Note in particular that as $\epsilon \to 0$, our mollified equation (3.2) approaches the original evolution equation, (2.11). Our first goal is to use Picard's theorem to prove that a solution to (3.2) exists on some time interval $[0, T^\epsilon]$. To this end, we define an open subset $\mathcal{O}$ of H^s by

$$\mathcal{O} = \{\theta \in H^s \mid ||\theta||_s \leq d_1, |\frac{z_{d,\epsilon}(\alpha) - z_{d,\epsilon}(\alpha')}{\alpha - \alpha'}| > d_2 \forall \alpha, \alpha' \in R\} \tag{3.9}$$

To apply Picard's theorem (the specific version we're using is Theorem 3.1 in [15]), we must show the right hand side of the evolution equation maps O into H^s and is Lipschitz continuous. Proving these properties is time consuming but ultimately not difficult for the mollified equation, the details are not included here. Applying Picard's theorem then gives us the following:

Lemma 3.1. *Let $\tau, \epsilon, d_1, d_2 > 0$ be fixed. Suppose that $\theta_0 \in \mathcal{O}$. Then there exists* *some $T^\epsilon > 0$ and $\theta^\epsilon \in C^1([0,T^\epsilon);\mathcal{O})$ such that $\theta^\epsilon(\cdot,0)=\theta_0$, and for all $t\in[0,T^\epsilon)$, θ^ϵ satisfies* (3.2).

Next we wish to show that the solutions, θ^ϵ exist on a common time interval $[0,T)$, which we will do by proving an energy estimate uniform in ϵ. To this end, we differentiate (3.2) with respect to alpha. This gives

$$
\begin{aligned}
\theta^\epsilon_{\alpha,t} &= \frac{\tau}{2}\chi_\epsilon^2\big(H(\partial_\alpha^4\theta^\epsilon)\big) - \tau\chi_\epsilon\left[\frac{A_\mu}{2}(\chi_\epsilon\theta_\alpha)(\chi_\epsilon\theta_{\alpha\alpha\alpha}) + A_\mu H(U_\alpha^{st,\epsilon}\chi_\epsilon\theta^\epsilon_\alpha)\right] \\
&- \tau\chi_\epsilon\left(\frac{A_\mu}{2}(\chi_\epsilon\theta^\epsilon_{\alpha\alpha})^2\right) - \tau\chi_\epsilon\left(A_\mu H(U^{st,\epsilon}\chi_\epsilon\theta^\epsilon_{\alpha\alpha})\right) + \chi_\epsilon\big[H(k^\epsilon_\alpha\chi_\epsilon\theta^\epsilon_\alpha)\big] \\
&+ \chi_\epsilon\big[H(k^\epsilon\chi_\epsilon\theta^\epsilon_{\alpha\alpha})\big] - \tau\chi_\epsilon\big[(W_\alpha^{st,\epsilon}\cdot\hat{t}^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha\big] + \tau\chi_\epsilon\left[(V^{st,\epsilon} - W^{st,\epsilon}\cdot\hat{t}^\epsilon)\chi_\epsilon\theta^\epsilon_{\alpha\alpha}\right] \\
&+ \tau A_\mu^2\chi_\epsilon\left[(\chi_\epsilon\theta^\epsilon_{\alpha\alpha})W^{st,\epsilon}\cdot\hat{t}^\epsilon + (\chi_\epsilon\theta^\epsilon_\alpha)\partial_\alpha(W^{st,\epsilon}\cdot\hat{t}^\epsilon)\right] \\
&+ \partial_\alpha\chi_\epsilon\left[\frac{A_\mu}{2}R\sin(\chi_\epsilon\theta^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha + A_\mu^2(\chi_\epsilon\theta^\epsilon_\alpha)\tilde{W}^\epsilon\cdot\hat{t}^\epsilon\right] \\
&+ \partial_\alpha\chi_\epsilon\left[(\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha + m^\epsilon\cdot\hat{n}^\epsilon - A_\mu H(m^\epsilon\cdot\hat{t}^\epsilon)\right]
\end{aligned}
\tag{3.10}
$$

Also, we have

$$U_\alpha^{st,\epsilon} = W_\alpha^{st,\epsilon}\cdot\hat{n}^\epsilon - \theta^\epsilon_\alpha W^{st,\epsilon}\cdot\hat{t}^\epsilon$$

and as in the non-mollified case, we have that

$$W_\alpha^{st,\epsilon}\cdot\hat{n}^\epsilon = \frac{\chi_\epsilon H(\theta^\epsilon_{\alpha\alpha\alpha})}{2} + m^{st,\epsilon}\cdot\hat{n}^\epsilon \tag{3.11}$$

Now, we define the operator ∂_α^{-1} by

$$\partial_\alpha^{-1}f(\alpha) = \int_0^\alpha f(\alpha')d\alpha' \tag{3.12}$$

Note in particular that $\partial_\alpha \partial_\alpha^{-1}(f) = f$, while $\partial_\alpha^{-1}\partial_\alpha(f) = f(\alpha) - f(0)$. Therefore,

$$
\begin{aligned}
U_\alpha^{st,\epsilon} &= \frac{\chi_\epsilon H(\theta_{\alpha\alpha\alpha}^\epsilon)}{2} - \theta_\alpha^\epsilon W^{st,\epsilon} \cdot \hat{t}^\epsilon + m^{st,\epsilon} \cdot \hat{n}^\epsilon \\
U^{st,\epsilon} &= \frac{\chi_\epsilon H(\theta_{\alpha\alpha}^\epsilon)}{2} - \partial_\alpha^{-1}\left(\theta_\alpha^\epsilon W^{st,\epsilon} \cdot \hat{t}^\epsilon + m^{st,\epsilon} \cdot \hat{n}^\epsilon\right) + C
\end{aligned}
$$

for some constant C. Now,

$$
\begin{aligned}
H(U^{st,\epsilon}\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) &= \partial_\alpha^{-1}\partial_\alpha H(U^{st,\epsilon}\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + C \\
&= \partial_\alpha^{-1} H(U_\alpha^{st,\epsilon}\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \partial_\alpha^{-1} H(U^{st,\epsilon}\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) + C
\end{aligned}
$$

Expanding $U_\alpha^{st,\epsilon}$ and pulling $U^{st,\epsilon}$ through a commutator, we obtain

$$
\begin{aligned}
H(U^{st,\epsilon}\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) &= \partial_\alpha^{-1}\frac{1}{2} H\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big] \\
&\quad + \partial_\alpha^{-1} H\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(-\theta_\alpha^\epsilon W^{st,\epsilon} \cdot \hat{t}^\epsilon + m^{st,\epsilon} \cdot \hat{n}^\epsilon)\big] \\
&\quad + \partial_\alpha^{-1}\big(U^{st,\epsilon} H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big) + \partial_\alpha^{-1}[H, U^{st,\epsilon}](\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) + C
\end{aligned}
$$

Next, we pull $\chi_\epsilon \theta_{\alpha\alpha}^\epsilon$ through the commutator and use the fact $H^2 = -I$ to get

$$
\begin{aligned}
H(U^{st,\epsilon}\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) &= \frac{-1}{2}\partial_\alpha^{-1}\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big] + \frac{1}{2}\partial_\alpha^{-1}[H, \chi_\epsilon \theta_{\alpha\alpha}^\epsilon]\big(H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big) \\
&\quad + \partial_\alpha^{-1}\big(U^{st,\epsilon} H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big) + \partial_\alpha^{-1}[H, U^{st,\epsilon}](\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) \\
&\quad + \partial_\alpha^{-1} H\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(-\theta_\alpha^\epsilon W^{st,\epsilon} \cdot \hat{t}^\epsilon + m^{st,\epsilon} \cdot \hat{n}^\epsilon)\big] + C
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
H(U_\alpha^{st,\epsilon}\chi_\epsilon \theta_\alpha^\epsilon) &= \frac{1}{2} H\big(H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\chi_\epsilon \theta_\alpha^\epsilon\big) - H\big((\chi_\epsilon \theta_\alpha^\epsilon)\theta_\alpha^\epsilon W^{st,\epsilon} \cdot \hat{t}^\epsilon\big) \\
&\quad + H\big((\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon} \cdot \hat{n}^\epsilon\big) \\
&= \frac{-1}{2}(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)(\chi_\epsilon \theta_\alpha^\epsilon) + \frac{1}{2}[H, \chi_\epsilon \theta_\alpha^\epsilon]\big(H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big) \\
&\quad - H\big((\chi_\epsilon \theta_\alpha^\epsilon)\theta_\alpha^\epsilon W^{st,\epsilon} \cdot \hat{t}^\epsilon\big) + H\big((\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon} \cdot \hat{n}^\epsilon\big)
\end{aligned}
$$

And finally, applying

$$W_\alpha^{st,\epsilon} \cdot \hat{t}^\epsilon = \frac{-H(\theta_\alpha^\epsilon(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon))}{2} + m^{st,\epsilon} \cdot \hat{t}^\epsilon \tag{3.13}$$

we obtain

$$\begin{aligned} -\tau(W_\alpha^{st,\epsilon} \cdot \hat{t}^\epsilon)\chi_\epsilon \theta_\alpha^\epsilon &= -\tau\Big(\frac{-H(\theta_\alpha^\epsilon(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon))}{2} + m^{st,\epsilon} \cdot \hat{t}^\epsilon\Big)\chi_\epsilon \theta_\alpha^\epsilon \\ &= \frac{\tau}{2}\theta_\alpha^\epsilon(\chi_\epsilon \theta_\alpha^\epsilon)H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \frac{\tau}{2}(\chi_\epsilon \theta_\alpha^\epsilon)[H, \theta_\alpha^\epsilon](\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) - \tau(\chi_\epsilon \theta_\alpha^\epsilon)m^{st,\epsilon} \cdot \hat{t}^\epsilon \end{aligned}$$

Therefore, rewriting (3.10), we have

$$\theta_{\alpha,t}^\epsilon = \chi_\epsilon\left[\frac{\tau}{2}H(\chi_\epsilon \partial_\alpha^4 \theta^\epsilon) + \tau\Upsilon_1^\epsilon \chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon + \Upsilon_2^\epsilon + \Upsilon_3^\epsilon + \Upsilon_4^\epsilon + C\right] \tag{3.14}$$

Here, Υ_1^ϵ cancels out, since

$$\Upsilon_1^\epsilon = \frac{A_\mu}{2}(\chi_\epsilon \theta_\alpha^\epsilon) - \frac{A_\mu}{2}(\chi_\epsilon \theta_\alpha^\epsilon) = 0 \tag{3.15}$$

Υ_2^ϵ contains terms proportional to $H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)$, or

$$\Upsilon_2^\epsilon = k^\epsilon H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) - \tau A_\mu \partial_\alpha^{-1}\big(U^{st,\epsilon} H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big) + \frac{\tau}{2}\theta_\alpha^\epsilon(\chi_\epsilon \theta_\alpha^\epsilon)H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) \tag{3.16}$$

Υ_3^ϵ contains the terms

$$\begin{aligned} \Upsilon_3^\epsilon = &\frac{-\tau A_\mu}{2}(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)^2 + \frac{\tau A_\mu}{2}\partial_\alpha^{-1}\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\big] \\ &+ \tau(V^{st,\epsilon} - W^{st,\epsilon} \cdot \hat{t}^\epsilon)\chi_\epsilon \theta_{\alpha\alpha}^\epsilon + \tau A_\mu^2(W^{st,\epsilon} \cdot \hat{t}^\epsilon)\chi_\epsilon \theta_{\alpha\alpha}^\epsilon \\ &+ \frac{A_\mu R}{2}\sin(\chi_\epsilon \theta^\epsilon)(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + A_\mu^2(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)\tilde{W}^\epsilon \cdot \hat{t}^\epsilon + (\tilde{V}^\epsilon - \tilde{W}^\epsilon \cdot \hat{t}^\epsilon)\chi_\epsilon \theta_{\alpha\alpha}^\epsilon \end{aligned} \tag{3.17}$$

Finally, Υ_4^ϵ contains the remaining terms,

$$\begin{aligned}
\Upsilon_4^\epsilon &= A_\mu \tau H\big((\chi_\epsilon \theta_\alpha^\epsilon)\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon\big) - A_\mu \tau H\big((\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon}\cdot \hat{n}^\epsilon\big) \\
&- \frac{A_\mu \tau}{2}[H, \chi_\epsilon \theta_\alpha^\epsilon] H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) - \frac{A_\mu \tau}{2}\partial_\alpha^{-1}[H, \chi_\epsilon \theta_{\alpha\alpha}^\epsilon] H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) \\
&- A_\mu \tau \partial_\alpha^{-1} H\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(-\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon + m^{st,\epsilon}\cdot \hat{n}^\epsilon)\big] \\
&- A_\mu \tau \partial_\alpha^{-1}[H, U^{st,\epsilon}]\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon + [H, k^\epsilon]\chi_\epsilon \theta_{\alpha\alpha}^\epsilon \\
&+ H(k_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon) - \tau(\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon}\cdot \hat{t}^\epsilon + \frac{\tau}{2}(\chi_\epsilon \theta_\alpha^\epsilon)[H, \theta_\alpha^\epsilon](\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) \\
&+ \tau A_\mu^2 (\chi_\epsilon \theta_\alpha^\epsilon)\partial_\alpha(W^{st,\epsilon}\cdot \hat{t}^\epsilon) + \frac{A_\mu R}{2}(\chi_\epsilon \theta_\alpha^\epsilon)^2 \cos(\chi_\epsilon \theta^\epsilon) \\
&+ A_\mu^2 (\chi_\epsilon \theta_\alpha^\epsilon)\partial_\alpha(\tilde{W}^\epsilon \cdot \hat{t}^\epsilon) + (\chi_\epsilon \theta_\alpha^\epsilon)\partial_\alpha(\tilde{V}^\epsilon - \tilde{W}^\epsilon \cdot \hat{t}^\epsilon) \\
&+ \partial_\alpha(m^\epsilon \cdot \hat{n}^\epsilon) - A_\mu \partial_\alpha H(m^\epsilon \cdot \hat{t}^\epsilon)
\end{aligned} \tag{3.18}$$

The final equation is obtained by differentiating once more,

$$\theta_{\alpha\alpha,t}^\epsilon = \chi_\epsilon \left[\frac{-\tau}{2}\Lambda^3(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \Upsilon_5^\epsilon \Lambda(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \Upsilon_6^\epsilon \chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon + \Upsilon_7^\epsilon\right] \tag{3.19}$$

where $\Lambda(f) = H(\partial_\alpha f)$. Here, Υ_5^ϵ is derived from Υ_2^ϵ, and is

$$\Upsilon_5^\epsilon = k^\epsilon - \tau A_\mu U^{st,\epsilon} + \frac{\tau}{2}\theta_\alpha^\epsilon(\chi_\epsilon \theta_\alpha^\epsilon) \tag{3.20}$$

Similarly, Υ_6^ϵ comes from Υ_3^ϵ,

$$\Upsilon_6^\epsilon = \frac{-\tau A_\mu}{2}(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + (V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon) + A_\mu^2(W^\epsilon \cdot \hat{t}^\epsilon) + \frac{A_\mu R}{2}\sin(\chi_\epsilon \theta^\epsilon) \tag{3.21}$$

Finally, Υ_7^ϵ again contains the remainder terms,

$$\begin{aligned}
\Upsilon_7^\epsilon &= \partial_\alpha \Upsilon_4^\epsilon + k_\alpha^\epsilon H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \frac{\tau}{2}H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\theta_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon)_\alpha \\
&+ (V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)_\alpha(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + A_\mu^2(W^\epsilon \cdot \hat{t}^\epsilon)_\alpha(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) \\
&+ \frac{A_\mu R}{2}\cos(\chi_\epsilon \theta_\alpha^\epsilon)(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\chi_\epsilon \theta_\alpha^\epsilon)
\end{aligned} \tag{3.22}$$

In the following sections, (3.19) will be the starting point for our energy estimates. In particular, the next section is devoted to bounding the individual terms contained in the Υ_j^ϵ.

Chapter 4

Some Necessary Bounds:

Now, in order to estimate $\frac{\partial E}{\partial t}$, we will want to bound the Sobolev norms of the Υ_j^ϵ. However, as many of the terms in the Υ_j^ϵ depend on $U^\epsilon, V^\epsilon, W^\epsilon$, and m^ϵ, which in turn can be expressed via the nonsingular integral operators $K[z_d^\epsilon], [H, f]$, and the vortex sheet strength γ^ϵ. Therefore, our immediate goal is to find bounds for $K[z_d^\epsilon], [H, f]$, and γ^ϵ. We start by establishing the relationship between $\theta^\epsilon \in H^s$, and the derivatives of z_d.

Lemma 4.1. *Suppose that $\theta^\epsilon \in H^s$. Then, we have that $z_\alpha^\epsilon \in L^\infty$ and $z_{\alpha\alpha}^\epsilon \in H^{s-1}$, with bounds*

$$||z_\alpha^\epsilon||_{L^\infty} = 1 \tag{4.1}$$

$$||z_{\alpha\alpha}^\epsilon||_{s-1} \lesssim ||\theta_\alpha^\epsilon||_{s-1}(1 + ||\theta_\alpha^\epsilon||_{s-1}^{s-1}) \tag{4.2}$$

Proof of Lemma 4.1: Now, since $z_\alpha^\epsilon = \big(\cos(\theta^\epsilon), \sin(\theta^\epsilon)\big)$, we have that $|z_\alpha^\epsilon| = 1$, and therefore $z^\epsilon_\alpha \in L^\infty$ trivially. Furthermore, we have that

$$z^\epsilon_{\alpha\alpha} = \theta^\epsilon_\alpha\big(-\sin(\theta^\epsilon), \cos(\theta^\epsilon)\big)$$

and in general, every term of $\partial^{s-1}_\alpha z^\epsilon_{\alpha\alpha}$ will be of the form

$$\prod_{i_1+\ldots i_n=s} \partial^{i_j}_\alpha \theta^\epsilon \cdot T$$

where T, being the leftover unit vector, satisfies $|T| = 1$. Therefore, since $||\partial^{i_j}_\alpha \theta^\epsilon||_{L^\infty} \leq ||\theta^\epsilon_\alpha||_{s-1}$ for $1 \leq i_j \leq s-1$, we have that

$$\begin{aligned} || \prod_{i_1+\ldots i_n=s} \partial^{i_j}_\alpha \theta^\epsilon \cdot T||_{L^2} &\leq ||\partial^{i_1}_\alpha \theta^\epsilon||_{L^2} \cdot \big(||\partial^{i_2}_\alpha \theta^\epsilon||_{L^\infty} \cdot \ldots \cdot ||\partial^{i_n}_\alpha \theta^\epsilon||_{L^\infty}\big)||T||_{L^\infty} \\ &\leq ||\theta^\epsilon_\alpha||^n_{s-1} \end{aligned}$$

Therefore, since $1 \leq n \leq s$, we have that $||\theta^\epsilon_\alpha||^n_{s-1} \leq ||\theta^\epsilon_\alpha||_{s-1} + ||\theta^\epsilon_\alpha||^s_{s-1}$, and

$$||z^\epsilon_{\alpha\alpha}||_{s-1} \leq C||\theta^\epsilon_\alpha||_{s-1}(1 + ||\theta^\epsilon_\alpha||^{s-1}_{s-1})$$

as desired. □

Now, we will define for convenience the divided differences q_1 and q_2 as follows,

$$q_1[f] = \frac{f(\alpha) - f(\alpha')}{\alpha - \alpha'} = \int_0^1 f_\alpha(t\alpha + (1-t)\alpha')dt \tag{4.3}$$

$$q_2[f] = \frac{f(\alpha) - f(\alpha') - (\alpha - \alpha')f_\alpha(\alpha')}{(\alpha - \alpha')^2} = \int_0^1 (t-1)f_{\alpha\alpha}(t\alpha + (1-t)\alpha')dt \tag{4.4}$$

These expressions will show up frequently within the following equations, most notably inside the integral operator $K[z^\epsilon_d]$ and commutation with the Hilbert transform $[H, f]$. Furthermore, we have the following facts,

Lemma 4.2. *Let $\theta^\epsilon \in H^s$. Then, $q_1[z_d^\epsilon] \in L^\infty$, $q_2[z_d^\epsilon] \in H^{s-1}$, and $\partial_\alpha^k q_2[z_d^\epsilon] \in L^1$ for $1 \leq k \leq s-1$ with respect to both α and α'. Furthermore, the respective bounds are*

$$||q_1||_{L^\infty} \leq 1$$

$$||q_2||_{s-1} \lesssim ||z_{\alpha\alpha}^\epsilon||_{s-1}$$

$$||\partial_\alpha^k q_2||_{L^1} \lesssim (1 + ||z_{\alpha\alpha}^\epsilon||_{k-2})\sqrt{||\partial_\alpha^{k+2} z_d^\epsilon||_{L^2}}$$

Proof of Lemma 4.2 The bounds for $||q_1||_{L^\infty}$ and $||q_2||_{s-1}$ follow immediately from their integral representations and Lemma 4.1, since $z_\alpha^\epsilon \in L^\infty$ and $z_{\alpha\alpha}^\epsilon \in H^{s-1}$. To bound the L^1 norm of q_2's derivatives, we will split the L^1 integral into two parts, bounding the behavior near $\alpha = \alpha'$ via $\partial_\alpha q_2$'s integral representation, and the behavior at infinity using the fractional representation. To be specific, we have

$$\begin{aligned} \partial_\alpha q_2 &= \frac{z_\alpha^\epsilon(\alpha) - z_\alpha^\epsilon(\alpha')}{(\alpha-\alpha')^2} - \frac{2[z_d^\epsilon(\alpha) - z_d^\epsilon(\alpha') - (\alpha-\alpha')z_\alpha^\epsilon(\alpha')]}{(\alpha-\alpha')^3} \\ &= \int_0^1 t(t-1) z_{\alpha\alpha\alpha}^\epsilon(t\alpha + (1-t)\alpha')dt \end{aligned}$$

Via characteristic functions, we can rewrite this as

$$\begin{aligned} \partial_\alpha q_2 = \chi_{|\alpha-\alpha'|<1}&\left(\int_0^1 t(t-1) z_{\alpha\alpha\alpha}^\epsilon(t\alpha + (1-t)\alpha')dt\right) \\ + \chi_{|\alpha-\alpha'|>1}&\left(\frac{z_\alpha^\epsilon(\alpha) - z_\alpha^\epsilon(\alpha')}{(\alpha-\alpha')^2} - \frac{2[z_d^\epsilon(\alpha) - z_d^\epsilon(\alpha') - (\alpha-\alpha')z_\alpha^\epsilon(\alpha')]}{(\alpha-\alpha')^3}\right) \end{aligned}$$

Similarly, we have in the general case that

$$\begin{aligned} |\partial_\alpha^k q_2| &\lesssim h_1 + h_2 \\ h_1 &= \chi_{|\alpha-\alpha'|<c}\left(\int_0^1 t^k(t-1)|\partial_\alpha^{k+2} z_d^\epsilon(t\alpha + (1-t)\alpha')|dt\right) \\ h_2 &= \chi_{|\alpha-\alpha'|>c}\left(\frac{||\partial_\alpha^k z_d^\epsilon||_{L^\infty}}{|\alpha-\alpha'|^2} + \frac{||\partial_\alpha^{k-1} z_d^\epsilon||_{L^\infty}}{|\alpha-\alpha'|^3} + \dots + \frac{||\partial_\alpha z_d^\epsilon||_{L^\infty}}{|\alpha-\alpha'|^{k+1}}\right) \end{aligned} \tag{4.5}$$

Here c is a constant to be defined later. Note that

$$h_2 \le \chi_{|\alpha-\alpha'|>c} \frac{||z^\epsilon_{\alpha\alpha}||_{H^{k-2}} + ||z^\epsilon_\alpha||_{L^\infty}}{|\alpha-\alpha'|^2}$$

Now, since $||\chi_{|\alpha-\alpha'|<c}||_{L^2} \lesssim \sqrt{c}$, $||\frac{\chi_{|\alpha-\alpha'|>c}}{|\alpha-\alpha'|}||_{L^2} \lesssim \frac{1}{\sqrt{c}}$, $||z^\epsilon_\alpha||_{L^\infty} \lesssim 1$, we have that

$$||h_1||_{L^1} \lesssim \sqrt{c}||\partial^{k+2}_\alpha z^\epsilon_d||_{L^2}$$

$$||h_2||_{L^1} \lesssim \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{c}}||z^\epsilon_{\alpha\alpha}||_{H^{k-2}}$$

Choosing $c = \frac{1}{||\partial^{k+2}_\alpha z^\epsilon_d||_{L^2}}$, we have

$$||\partial^k_\alpha q_2||_{L^1} \lesssim (1 + ||z^\epsilon_{\alpha\alpha}||_{H^{k-2}})\sqrt{||\partial^{k+2}_\alpha z^\epsilon_d||_{L^2}} \tag{4.6}$$

as desired. □

With bounds on q_1 and q_2, we now have the necessary tools to bound $K[z^\epsilon_d]$,

Lemma 4.3. *Let $\theta^\epsilon \in H^s$, and suppose that the arc-chord condition is satisfied, that is, that there exists $d_2 > 0$ such that*

$$|\frac{z_{d,\epsilon}(\alpha) - z_{d,\epsilon}(\alpha')}{\alpha - \alpha'}| > d_2 \tag{4.7}$$

Then the integral operator $K[z^\epsilon_d](f)$ satisfies the following:

$$K[z^\epsilon_d] : H^0 \to L^\infty$$

$$\partial_\alpha K[z^\epsilon_d] : H^1 \to H^{s-1}$$

$$\partial_\alpha K[z^\epsilon_d] : H^0 \to H^{s-2}$$

Proof of Lemma 4.3: Now, the L^∞ bound on $K[z_d^\epsilon]$ is immediate from Lemma 4.2, since

$$\begin{aligned} K[z_d^\epsilon](f) &= \frac{1}{2\pi i}\int f(\alpha')\bigg(\frac{1}{z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')} - \frac{1}{z_\alpha^\epsilon(\alpha')(\alpha-\alpha')}\bigg)d\alpha' \\ &= \frac{1}{2\pi i}\int f(\alpha')\frac{z_\alpha^\epsilon(\alpha')(\alpha-\alpha') - \big(z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')\big)}{z_\alpha^\epsilon(\alpha')(\alpha-\alpha')\big(z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')\big)}d\alpha' \\ &= \frac{1}{2\pi i}\int \frac{-f(\alpha')}{z_\alpha^\epsilon(\alpha')}\cdot\frac{q_2[z_d^\epsilon]}{q_1[z_d^\epsilon]}d\alpha' \end{aligned}$$

As a consequence,

$$||K[z_d^\epsilon](f)||_{L^\infty} \le C||f||_{L^2}\cdot||q_2||_{L^2}\cdot||\frac{1}{q_1}||_{L^\infty}\cdot||\frac{1}{z_\alpha^\epsilon}||_{L^\infty} \tag{4.8}$$

Since $\frac{1}{q_1}$ is bounded by the arc-chord condition, therefore $K: H^0 \to L^\infty$.

Now, to bound $\partial_\alpha K[z_d^\epsilon]$, we use the fact that the integral is of the form

$$\partial_\alpha \int P_1(\alpha')\cdot P_2(\alpha,\alpha')\cdot P_3(\alpha,\alpha')$$

where $P_1 = \frac{f(\alpha')}{z_\alpha^\epsilon(\alpha')} \in L^2$, $P_2 = q_2 \in L^2$, $\partial_\alpha P_2 \in L^1$, $P_3 = \frac{1}{q_1} \in L^\infty$ so long as the arc chord condition holds, and $\partial_\alpha P_3 \in L^2$. Therefore, we have that

$$||\partial_\alpha^{k+1}K[z_d^\epsilon]||_{L^2} \lesssim ||f||_{L^2}\bigg(||q_2||_{L^2}||\partial_\alpha^{k+1}\frac{1}{q_1}||_{L^2} + \sum_{j=1}^{k+1}||\partial_\alpha^j q_2||_{L^1}||\partial_\alpha^{k+1-j}\frac{1}{q_1}||_{L^\infty}\bigg) \tag{4.9}$$

Therefore, letting $k = s-2$, we obtain $\partial_\alpha K[z_d^\epsilon] : H^0 \to H^{s-2}$. Finally, we note that if $k = s-1$, we have

$$\begin{aligned} ||\partial_\alpha^s K[z_d^\epsilon]||_{L^2} &\lesssim ||f||_{L^2}\bigg(||q_2||_{L^2}||\partial_\alpha^s\frac{1}{q_1}||_{L^2} + \sum_{j=1}^{s-1}||\partial_\alpha^j q_2||_{L^1}||\partial_\alpha^{s-j}\frac{1}{q_1}||_{L^\infty}\bigg) \\ &\quad + ||\int\frac{f(\alpha')}{z_\alpha^\epsilon(\alpha')}\cdot\frac{\partial_\alpha^s q_2[z_d^\epsilon]}{q_1[z_d^\epsilon]}d\alpha'||_{L^2} \end{aligned}$$

However, the last term can be dealt with by using the fact that $q_2[z_d^\epsilon] = \partial_{\alpha'} q_1[z_d^\epsilon]$ and integrating by parts, giving us

$$\begin{aligned} ||\partial_\alpha^s K[z_d^\epsilon]||_{L^2} \lesssim ||f||_{L^2} \left(||q_2||_{L^2} ||\partial_\alpha^s \frac{1}{q_1}||_{L^2} + \sum_{j=1}^{s-1} ||\partial_\alpha^j q_2||_{L^1} ||\partial_\alpha^{s-j} \frac{1}{q_1}||_{L^\infty} \right) \\ + (||f||_{L^2} ||z_{\alpha\alpha}^\epsilon||_{L^2} + ||f_\alpha||_{L^2}) ||\partial_\alpha^s q_1[z_d^\epsilon]||_{L^1} || \frac{1}{q_1[z_d^\epsilon]} ||_{L^\infty} \end{aligned} \tag{4.10}$$

Therefore, if $f \in H^1$, then $\partial_\alpha K[z_d^\epsilon] \in H^{s-1}$, completing the proof. □

With $K[z_d^\epsilon]$ bounded, the next task is to estimate our other integral operator, $[H, f]$.

Lemma 4.4. *Suppose that f is an L^∞ function such that $\partial_\alpha f \in H^{s-1}$ with $s \geq 6$. Then,*

$$[H, f] : H^0 \to H^{s-2}$$

with the bound

$$||[H, f](g)||_{s-2} \lesssim \sqrt{||f_\alpha||_{s-1}} (1 + ||f||_{L^\infty} + ||f_\alpha||_{s-3}) ||g||_{L^2} \tag{4.11}$$

Proof of Lemma 4.4: By definition, we have $||[H, f](g)||_{L^2} \leq 2||f||_{L^\infty} ||g||_{L^2}$. For the rest of the argument, we imitate the proof of Lemma 4.2 to obtain bounds on the L^1 norm of $\partial_\alpha^k q_1[f]$, and apply them to the equations

$$\partial_\alpha^k [H, f](g) = \int g(\alpha') \partial_\alpha^k q_1[f] d\alpha' \tag{4.12}$$

$$||\partial_\alpha^k [H, f](g)||_{L^2} \leq ||g||_{L^2} ||\partial_\alpha^k q_1[f]||_{L^1} \tag{4.13}$$

to finish the proof.

As before, we split $\partial_\alpha q_1[f]$ into the area around $\alpha = \alpha'$, and infinity.

$$\partial_\alpha q_1[f] = \frac{f_\alpha(\alpha)(\alpha-\alpha') - (f(\alpha) - f(\alpha'))}{(\alpha-\alpha')^2} = \int_0^1 t f_{\alpha\alpha}(t\alpha + (1-t)\alpha')dt$$

$$\partial_\alpha q_1[f] \leq \min(||f_{\alpha\alpha}||_{L^\infty}, \frac{f_\alpha(\alpha)}{|\alpha-\alpha'|} + \frac{2||f||_{L^\infty}}{|\alpha-\alpha'|^2})$$

In general,

$$\partial_\alpha^k q_1[f] \lesssim \min(||\partial_\alpha^{k+1} f||_{L^\infty}, \frac{\partial_\alpha^k f(\alpha)}{|\alpha-\alpha'|} + \frac{||\partial_\alpha^{k-1} f||_{L^\infty}}{|\alpha-\alpha'|^2} + ... \frac{||f||_{L^\infty}}{|\alpha-\alpha'|^{k+1}})$$

And in particular, we have

$$\begin{aligned} |\partial_\alpha^k q_1[f]| &\lesssim h_1 + h_2 + h_3 \\ h_1(\alpha,\alpha') &= ||\partial_\alpha^{k+1} f||_{L^\infty} \chi_{|\alpha-\alpha'|\leq c} \\ h_2(\alpha,\alpha') &= \frac{|\partial_\alpha^k f(\alpha)|}{|\alpha-\alpha'|} \chi_{|\alpha-\alpha'|>c} \\ h_3(\alpha,\alpha') &= \frac{||\partial_\alpha f||_{k-1} + ||f||_{L^\infty}}{|\alpha-\alpha'|^2} \chi_{|\alpha-\alpha'|>c} \end{aligned} \tag{4.14}$$

where c is a constant to be determined, and χ_S is the characteristic function on the set S. Now, clearly $h_1, h_3 \in L^1$, and making use of the fact that $||\frac{1}{|\alpha-\alpha'|}\chi_{|\alpha-\alpha'|>c}||_{L^2} \lesssim 1/\sqrt{c}$, we have

$$\begin{aligned} ||\partial_\alpha^k [H,f](g)||_{L^2} &\lesssim ||h_1||_{L^1}||g||_{L^2} + ||h_3||_{L^1}||g||_{L^2} + ||\int_{|\alpha-\alpha'|>c} \frac{\partial_\alpha^k f(\alpha)}{|\alpha-\alpha'|} g(\alpha')d\alpha'||_{L^2} \\ &\lesssim (c||\partial_\alpha^{k+1} f||_{L^\infty} + ||\partial_\alpha^k f||_{L^2} + \frac{1}{c}(||\partial_\alpha f||_{k-1} + ||f||_{L^\infty}))||g||_{L^2} \end{aligned}$$

And choosing $c = \frac{1}{\sqrt{||f_\alpha||_{k+1}}}$, we have

$$||\partial_\alpha^k [H,f](g)||_{L^2} \lesssim \sqrt{||f_\alpha||_{k+1}}(1 + ||f||_{L^\infty} + ||\partial_\alpha f||_{k-1})||g||_{L^2}$$

This completes the proof. $\square$

However, while Lemma 4.4 is a powerful tool, it's not quite sufficient for our purposes, as the Υ_j^ϵ contain terms that require additional degrees of regularity without possessing any extra structure for f. Fortunately, if g is in a higher order Sobolev space, we can work around this.

Lemma 4.5. *Suppose that $f \in L^\infty, \partial_\alpha f \in H^{n-1}, g \in H^{n-j+1}$, and $n \geq 2j$. Then,*

$$||[H,f](g)||_{H^n} \lesssim ||g||_{n-j+1}\sqrt{||\partial_\alpha f||_{n-1}}(1+||f||_{L^\infty}+||\partial_\alpha f||_{n-1}) \tag{4.15}$$

Proof of Lemma 4.5: Now, we break up ∂_α^n into $\partial_\alpha^j \partial_\alpha^{n-j}$ to obtain

$$\partial_\alpha^n [H,f](g) = \partial_\alpha^j \sum_{l=0}^{n-j} \binom{n-j}{l} \Big[H\big((\partial_\alpha^l f)(\partial_\alpha^{n-j-l} g)\big) - (\partial_\alpha^l f) H(\partial_\alpha^{n-j-l} g)\Big]$$

Next, we separate the sum into $l \geq j-1, l < j-1$ (noting that $n-j \geq j$), getting

$$\begin{aligned}\partial_\alpha^n [H,f](g) = \partial_\alpha^j &\sum_{l=0}^{j-2} \binom{n-j}{l} [H, \partial_\alpha^l f](\partial_\alpha^{n-j-l} g) \\ &+ \partial_\alpha^j \sum_{l=j-1}^{n-j} \binom{n-j}{l} \Big[H\big((\partial_\alpha^l f)(\partial_\alpha^{n-j-l} g)\big) - (\partial_\alpha^l f) H(\partial_\alpha^{n-j-l} g)\Big]\end{aligned}$$

Now, since $f \in L^\infty$ and $\partial_\alpha^l f \in H^{n-j+2}$ for $1 \leq l \leq j-2$, therefore in particular $\partial_\alpha^l f \in H^{j+2}$, and therefore by the previous lemma,

$$||[H, \partial_\alpha^l f](\partial_\alpha^{n-j-l} g)||_j \lesssim \sqrt{||\partial_\alpha^l f_\alpha||_{j+1}}(1+||\partial_\alpha^l f||_{L^\infty}+||\partial_\alpha^{l+1} f||_{j-1})||\partial_\alpha^{n-j-l} g||_{L^2} \tag{4.16}$$

Similarly, for $j-1 \le l \le n-j$, we have that $n-j-l \le n-2j+1$, and therefore $\partial_\alpha^{n-j-l} g \in H^j, \partial_\alpha^l f \in H^j$. Therefore, in particular, we have

$$||H\big((\partial_\alpha^l f)(\partial_\alpha^{n-j-l} g)\big)||_j, ||(\partial_\alpha^{l+1} f)H(\partial_\alpha^{n-j-l} g)||_j \lesssim ||g||_{n-j+1}||f_\alpha||_{n-1} \tag{4.17}$$

Combining (4.16) and (4.17), we have that

$$||[H,f](g)||_{H^n} \lesssim ||g||_{n-j+1}\sqrt{||\partial_\alpha f||_{n-1}}(1+||f||_{L^\infty}+||\partial_\alpha f||_{n-1})$$

as desired. □

With both $K[z_d^\epsilon]$ and $[H,f]$ bounded, we can turn to the last reoccuring term in γ^ϵ.

Lemma 4.6. *Suppose that $\theta^\epsilon \in H^s$ and that the arc-chord condition is satisfied. Then, $\tilde{\gamma}[\theta^\epsilon] \in H^s$.*

Proof of Lemma 4.6: Recall that

$$\tilde{\gamma} = -R\sin(\chi_\epsilon\theta^\epsilon) - 2A_\mu W^\epsilon \cdot \hat{t}^\epsilon \tag{4.18}$$

By Lemma 5 in [1], we know that $\gamma \in H^0$. Since $\sin(\chi_\epsilon\theta^\epsilon) \in H^s$, we merely need to show $W^\epsilon \cdot \hat{t}^\epsilon \in H^s$. Now,

$$\Phi(W^\epsilon)^* = \frac{1}{2i}H(\frac{\gamma}{z_\alpha^\epsilon}) + K[z_d^\epsilon](\gamma) \tag{4.19}$$

Since $\Phi(\hat{t}^\epsilon) = z_\alpha^\epsilon$, and for any vectors a, b, we have $a \cdot b = Re(\Phi(a)\Phi(b)^*)$, therefore

$$W^\epsilon \cdot \hat{t}^\epsilon = Re\big(\frac{z_\alpha^\epsilon}{2i}H(\frac{\gamma}{z_\alpha^\epsilon})\big) + Re\big(z_\alpha^\epsilon K[z_d^\epsilon](\gamma)\big)$$

Using the fact that $Re(\frac{1}{2i}H(\gamma)) = 0$, we get

$$W^\epsilon \cdot \hat{t}^\epsilon = Re\big(\frac{z^\epsilon_\alpha}{2i}[H, \frac{1}{z^\epsilon_\alpha}](\gamma)\big) + Re\big(z^\epsilon_\alpha K[z^\epsilon_d](\gamma)\big) \tag{4.20}$$

We then use Lemmas 4.4 and 4.3 to get that $W^\epsilon \cdot \hat{t} \in H^{s-2}$ and therefore $\gamma = \chi_\epsilon \theta^\epsilon_{\alpha\alpha} + \tilde{\gamma} \in H^{s-2}$. At that point, applying Lemma 4.5 and 4.3 again, we obtain that $\tilde{\gamma} \in H^s$, as desired.□

With the necessary bounds on $K[z^\epsilon_d]$, $[H, f]$, and γ, we are now ready to turn our attention to W^ϵ and m^ϵ.

Lemma 4.7. *Suppose that $\theta^\epsilon \in H^s$ and that the arc-chord condition is satisfied. Then, $\partial_\alpha(W^{st,\epsilon} \cdot \hat{t}^\epsilon), \partial_\alpha \tilde{W} \in H^{s-1}$, $\partial_\alpha(W^{st,\epsilon} \cdot \hat{n}) \in H^{s-3}$, and $\partial_\alpha k^\epsilon \in H^{s-1}$.*

Proof of Lemma 4.7: Recall from (4.19) that

$$\Phi(W^\epsilon)^* = \frac{1}{2i}H(\frac{\gamma}{z^\epsilon_\alpha}) + K[z^\epsilon_d](\gamma)$$

Now, we know from Lemma 4.3 that $\partial_\alpha K[z^\epsilon_d](\gamma) \in H^{s-1}$, so we merely need to worry about the first term. However, since $\frac{\tilde{\gamma}}{z^\epsilon_\alpha} \in H^s$ by Lemmas 4.1 and 4.6, therefore $\frac{1}{2i}H(\frac{\gamma}{z^\epsilon_\alpha}) \in H^s$, and so $\partial_\alpha \tilde{W}^\epsilon \in H^{s-1}$.

For $W^{st,\epsilon}$, note that $\gamma^{st} = \chi_\epsilon \theta^\epsilon_{\alpha\alpha} \in H^{s-2}$, and therefore $\frac{1}{2i}H(\frac{\gamma^{st}}{z^\epsilon_\alpha}) \in H^{s-2}$ and $\partial_\alpha(W^{st,\epsilon} \cdot \hat{n}) \in H^{s-3}$. Next, recall from (4.20) that

$$W^{st,\epsilon} \cdot \hat{t}^\epsilon = Re\big(\frac{z^\epsilon_\alpha}{2i}[H, \frac{1}{z^\epsilon_\alpha}](\gamma^{st})\big) + Re\big(z^\epsilon_\alpha K[z^\epsilon_d](\gamma^{st})\big)$$

and so, as in the proof of the previous lemma, we can apply Lemmas 4.5 and 4.3 to obtain that $\partial_\alpha(W^{st,\epsilon} \cdot \hat{t}^\epsilon) \in H^{s-1}$. Finally, $k^\epsilon = \frac{-R\cos(\chi_\epsilon \theta^\epsilon)}{2} - A_\mu \tilde{U}^\epsilon$, and therefore $\partial_\alpha k^\epsilon \in H^{s-1}$ is an immediate consequence of $\tilde{U}^\epsilon = \tilde{W}^\epsilon \cdot \hat{n}^\epsilon$. □

Remark 4.8. While not explicitly stated in the formulation, Lemma 4.7 gives the necessary bounds for U^ϵ and V^ϵ as well, via the relations $U^\epsilon = W^\epsilon \cdot \hat{n}^\epsilon$, and $\partial_\alpha V^\epsilon = U^\epsilon \theta^\epsilon_\alpha$.

Our final lemma of this section deals with the remainder term, m^ϵ.

Lemma 4.9. *Suppose that $\theta^\epsilon \in H^s$, and the arc-chord condition is satisfied. Then, $m^\epsilon \in H^s$.*

Proof of Lemma 4.9: This follows from Lemmas 4.3, 4.5, and the equation

$$\Phi(m^\epsilon)^* = z^\epsilon_\alpha K[z^\epsilon_d]\big(\partial_\alpha(\frac{\gamma}{z^\epsilon_\alpha})\big) + \frac{z^\epsilon_\alpha}{2i}[H, \frac{1}{(z^\epsilon_\alpha)^2}]\big(z^\epsilon_\alpha \partial_\alpha(\frac{\gamma}{z^\epsilon_\alpha})\big) \tag{4.21}$$

Chapter 5

Energy Estimate for Local Existence:

With the necessary lemmas proved, the next step is to bound Υ_4^ϵ through Υ_7^ϵ.

Lemma 5.1. *Suppose that $\theta^\epsilon \in H^s$, and the arc-chord condition is satisfied. Then $\partial_\alpha \Upsilon_4^\epsilon, \Upsilon_7^\epsilon \in H^{s-2}$, $\Upsilon_5^\epsilon, \Upsilon_6^\epsilon \in L^\infty$, and $\partial_\alpha \Upsilon_5^\epsilon, \partial_\alpha \Upsilon_6^\epsilon \in H^{s-3}$.*

Proof of Lemma 5.1: To prove this lemma, we will seperate each Υ^ϵ into its individual terms, and show that each term is a product of Sobolev functions that we've already bounded through Lemmas 4.5, 4.7, and 4.9. In essence, we're checking that the bounds we proved during the last section are sufficient to cover every term in $\theta^\epsilon_{\alpha\alpha,t}$.

Now, we shall write $\Upsilon_4^\epsilon = \sum_{i=1}^{16} \Xi_i$ in the obvious way, with each Ξ_i corresponding to the ith term in (3.18). Then,

$$||\Xi_1||_{s-1} = ||A_\mu \tau H\big((\chi_\epsilon \theta_\alpha^\epsilon)\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon\big)||_{s-1} = A_\mu \tau ||(\chi_\epsilon \theta_\alpha^\epsilon)\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon||_{s-1}$$

$$||\Xi_1||_{s-1} \leq A_\mu ||\chi_\epsilon \theta_\alpha^\epsilon||_{s-1}||\theta_\alpha^\epsilon||_{s-1}(||\partial_\alpha(W^{st,\epsilon}\cdot \hat{t}^\epsilon)||_{s-2} + ||W^{st,\epsilon}\cdot \hat{t}^\epsilon||_{L^\infty})$$

Which is bounded by Lemma 4.7. Next,

$$||\Xi_2||_{s-1} = ||A_\mu \tau H\big((\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon}\cdot \hat{n}^\epsilon\big)||_{s-1} \leq A_\mu \tau ||\chi_\epsilon \theta_\alpha^\epsilon||_{s-1}||m^{st,\epsilon}\cdot \hat{n}^\epsilon||_{s-1}$$

which is bounded by Lemma 4.9.

$$||\Xi_3||_{s-1} = ||\frac{A_\mu \tau}{2}[H, \chi_\epsilon \theta_\alpha^\epsilon]H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)||_{s-1} = \frac{A_\mu \tau}{2}||[H, \chi_\epsilon \theta_\alpha^\epsilon]H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)||_{s-1}$$

is bounded by Lemma 4.5. Similarly,

$$\begin{aligned} ||\partial_\alpha \Xi_4||_{s-2} &= ||\partial_\alpha \left(\frac{A_\mu \tau}{2}\partial_\alpha^{-1}[H, \chi_\epsilon \theta_{\alpha\alpha}^\epsilon]H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)\right)||_{s-2} \\ &\leq \frac{A_\mu}{2}||[H, \chi_\epsilon \theta_{\alpha\alpha}^\epsilon]H(\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon)||_{s-2} \end{aligned}$$

is also bounded by Lemma 4.5.

$$\begin{aligned} ||\partial_\alpha \Xi_5||_{s-2} &= ||\partial_\alpha \left(A_\mu \tau \partial_\alpha^{-1} H\big[(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(-\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon + m^{st,\epsilon}\cdot \hat{n}^\epsilon)\big]\right)||_{s-2} \\ &= A_\mu \tau ||(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(-\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon + m^{st,\epsilon}\cdot \hat{n}^\epsilon)||_{s-2} \\ &\leq A_\mu \tau ||\chi_\epsilon \theta_{\alpha\alpha}^\epsilon||_{s-2}\big(||\theta_\alpha^\epsilon W^{st,\epsilon}\cdot \hat{t}^\epsilon||_{s-2} + ||m^{st,\epsilon}\cdot \hat{n}^\epsilon||_{s-2}\big) \end{aligned}$$

is bounded through Lemmas 4.7, 4.9.

$$||\partial_\alpha \Xi_6||_{s-2} = ||\partial_\alpha\big(A_\mu \tau \partial_\alpha^{-1}[H, U^{st,\epsilon}]\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon\big)||_{s-2} = A_\mu \tau ||[H, U^{st,\epsilon}]\chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon||_{s-2}$$

which in turn is bounded through Lemmas 4.5 and 4.7, since $U = W \cdot \hat{n}$.

$$||\Xi_7||_{s-1} = ||[H, k^\epsilon]\chi_\epsilon \theta^\epsilon_{\alpha\alpha}||_{s-1}$$

is also bounded via Lemmas 4.5 and 4.7.

$$||\Xi_8||_{s-1} = ||H(k^\epsilon_\alpha \chi_\epsilon \theta^\epsilon_\alpha)||_{s-1} \leq ||k^\epsilon_\alpha||_{s-1}||\chi_\epsilon \theta^\epsilon_\alpha||_{s-1}$$

is bounded through Lemma 4.7.

$$||\Xi_9||_{s-1} = ||\tau(\chi_\epsilon \theta^\epsilon_\alpha) m^{st,\epsilon} \cdot \hat{t}^\epsilon||_{s-1} \leq \tau||\chi_\epsilon \theta^\epsilon_\alpha||_{s-1}||m^{st,\epsilon} \cdot \hat{t}^\epsilon||_{s-1}$$

is bounded because of Lemma 4.9.

$$||\Xi_{10}||_{s-1} = ||\frac{\tau}{2}(\chi_\epsilon \theta^\epsilon_\alpha)[H, \theta^\epsilon_\alpha](\chi_\epsilon \theta^\epsilon_{\alpha\alpha})||_{s-1} \leq \frac{\tau}{2}||\chi_\epsilon \theta^\epsilon_\alpha||_{s-1}||[H, \theta^\epsilon_\alpha](\chi_\epsilon \theta^\epsilon_{\alpha\alpha})||_{s-1}$$

is bounded by Lemma 4.5.

$$||\Xi_{11}||_{s-1} = ||\tau A^2_\mu (\chi_\epsilon \theta^\epsilon_\alpha)\partial_\alpha (W^{st,\epsilon} \cdot \hat{t}^\epsilon)||_{s-1} \leq \tau A^2_\mu ||\chi_\epsilon \theta^\epsilon_\alpha||_{s-1}||\partial_\alpha (W^{st,\epsilon} \cdot \hat{t}^\epsilon)||_{s-1}$$

is bounded by Lemma 4.7. The bound on the next term,

$$||\Xi_{12}||_{s-1} = ||\frac{A_\mu R}{2}(\chi_\epsilon \theta^\epsilon_\alpha)^2 \cos(\chi_\epsilon \theta^\epsilon)||_{s-1} \leq \frac{A_\mu R}{2}||\chi_\epsilon \theta^\epsilon_\alpha||^2_{s-1}(1 + ||\partial_\alpha \cos(\chi_\epsilon \theta^\epsilon)||_{s-2})$$

is immediate. The next pair of terms,

$$||\Xi_{13}||_{s-1} = ||A^2_\mu (\chi_\epsilon \theta^\epsilon_\alpha)\partial_\alpha (\tilde{W}^\epsilon \cdot \hat{t}^\epsilon)||_{s-1} \leq A^2_\mu ||\chi_\epsilon \theta^\epsilon_\alpha||_{s-1}||\partial_\alpha (\tilde{W}^\epsilon \cdot \hat{t}^\epsilon)||_{s-1}$$

$$||\Xi_{14}||_{s-1} = ||(\chi_\epsilon \theta^\epsilon_\alpha)\partial_\alpha (\tilde{V}^\epsilon - \tilde{W}^\epsilon \cdot \hat{t}^\epsilon)||_{s-1} = ||(\chi_\epsilon \theta^\epsilon_\alpha)(\tilde{W}^\epsilon_\alpha \cdot \hat{t}^\epsilon)||_{s-1}$$

are both bounded through Lemma 4.7. Finally, the last two terms,

$$||\Xi_{15}||_{s-1} = ||\partial_\alpha (m^\epsilon \cdot \hat{n}^\epsilon)||_{s-1}$$

$$||\Xi_{16}||_{s-1} = ||A_\mu \partial_\alpha H(m^\epsilon \cdot \hat{t}^\epsilon)||_{s-1} = A_\mu ||\partial_\alpha (m^\epsilon \cdot \hat{t}^\epsilon)||_{s-1}$$

are both bounded by Lemma 4.9. Therefore, $\partial_\alpha \Upsilon_4^\epsilon \in H^{s-2}$. Now, recall from (3.22) that

$$\begin{aligned}
\Upsilon_7^\epsilon &= \partial_\alpha \Upsilon_4^\epsilon + k_\alpha^\epsilon H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \frac{\tau}{2} H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\theta_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon)_\alpha \\
&+ (V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)_\alpha (\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + A_\mu^2 (W^\epsilon \cdot \hat{t}^\epsilon)_\alpha (\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) \\
&+ \frac{A_\mu R}{2} \cos(\chi_\epsilon \theta_\alpha^\epsilon)(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\chi_\epsilon \theta_\alpha^\epsilon) \\
&= \partial_\alpha \Upsilon_4^\epsilon + \sum_{i=17}^{21} \Xi_i
\end{aligned}$$

These terms can all be bounded using Lemma 4.7. Specifically,

$$||\Xi_{17}||_{s-2} = ||k_\alpha^\epsilon H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-2} \leq ||k_\alpha^\epsilon||_{s-2} ||\chi_\epsilon \theta_{\alpha\alpha}^\epsilon||_{s-2}$$

$$||\Xi_{18}||_{s-2} = ||\frac{\tau}{2} H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\theta_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon)_\alpha||_{s-2} \leq \frac{\tau}{2} ||\chi_\epsilon \theta_{\alpha\alpha}^\epsilon||_{s-2} ||\theta_\alpha^\epsilon||_{s-1} ||\chi_\epsilon \theta_\alpha^\epsilon||_{s-1}$$

$$||\Xi_{19}||_{s-2} = ||(V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)_\alpha (\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-2} \leq ||\tilde{W}_\alpha^\epsilon \cdot \hat{t}^\epsilon||_{s-2} ||\chi_\epsilon \theta_{\alpha\alpha}^\epsilon||_{s-2}$$

$$||\Xi_{20}||_{s-2} = ||A_\mu^2 (W^\epsilon \cdot \hat{t}^\epsilon)_\alpha (\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-2} \leq A_\mu^2 ||\partial_\alpha (W^\epsilon \cdot \hat{t}^\epsilon)||_{s-2} ||\chi_\epsilon \theta_{\alpha\alpha}^\epsilon||_{s-2}$$

$$\begin{aligned}
||\Xi_{21}||_{s-2} &= ||\frac{A_\mu R}{2} \cos(\chi_\epsilon \theta_\alpha^\epsilon)(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\chi_\epsilon \theta_\alpha^\epsilon)||_{s-2} \\
&\leq \frac{A_\mu R}{2} (1 + ||\partial_\alpha \cos(\chi_\epsilon \theta_\alpha^\epsilon)||_{s-3}) ||\chi_\epsilon \theta_{\alpha\alpha}^\epsilon||_{s-2} ||\chi_\epsilon \theta_\alpha^\epsilon||_{s-2}
\end{aligned}$$

Therefore, $\Upsilon_7^\epsilon \in H^{s-2}$.

The final two Υ^ϵ,

$$\Upsilon_5^\epsilon = k^\epsilon - \tau A_\mu U^{st,\epsilon} + \frac{\tau}{2} \theta_\alpha^\epsilon (\chi_\epsilon \theta_\alpha^\epsilon)$$

$$\Upsilon_6^\epsilon = \frac{-\tau A_\mu}{2}(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + (V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon) + A_\mu^2 (W^\epsilon \cdot \hat{t}^\epsilon) + \frac{A_\mu R}{2}\sin(\chi_\epsilon \theta^\epsilon)$$

can both be bounded immediately from Lemma 4.7, finishing the proof.□

Our last lemma before the proof of our first main result is needed to handle the low-degree term of $\frac{\partial E}{\partial t}$; that is, to show that θ_t^ϵ is L^2.

Lemma 5.2. *Suppose that $\theta^\epsilon \in H^s$, and the arc-chord condition is satisfied. Then, $\theta_t^\epsilon \in L^2$.*

Proof of Lemma 5.2: As in the previous lemma, we will split θ_t^ϵ into individual terms, and show that each term is a product of Sobolev functions that we've already bounded through Lemmas 4.5, 4.7, and 4.9. Recall from (3.2) that

$$\begin{aligned}
\theta_t^\epsilon = &\frac{\tau}{2}\chi_\epsilon^2 H(\theta_{\alpha\alpha\alpha}^\epsilon) - \tau\chi_\epsilon\bigg(\frac{A_\mu}{2}(\chi_\epsilon\theta_\alpha^\epsilon)(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon) + A_\mu H\big(U^{st,\epsilon}\chi_\epsilon\theta_\alpha^\epsilon\big)\bigg) \\
&+ \chi_\epsilon\Big[H(k^\epsilon\chi_\epsilon\theta_\alpha^\epsilon) + \tau\big(V^{st,\epsilon} - W^{st,\epsilon}\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta_\alpha^\epsilon + \tau A_\mu^2\big(\chi_\epsilon\theta_\alpha^\epsilon\big)W^{st,\epsilon}\cdot\hat{t}^\epsilon\Big] \\
&+ \frac{A_\mu}{2}R\chi_\epsilon\bigg(\sin(\chi_\epsilon\theta^\epsilon)\chi_\epsilon\theta_\alpha^\epsilon\bigg) + A_\mu^2\chi_\epsilon\bigg(\big(\chi_\epsilon\theta_\alpha^\epsilon\big)\tilde{W}^\epsilon\cdot\hat{t}^\epsilon\bigg) \\
&+ \chi_\epsilon\Big[\big(\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta_\alpha^\epsilon\Big] + \chi_\epsilon(m^\epsilon\cdot\hat{n}^\epsilon) - A_\mu\chi_\epsilon H(m^\epsilon\cdot\hat{t}^\epsilon)
\end{aligned}$$

As before, we will write $\theta_t^\epsilon = \sum_{i=1}^{11}\Xi_i$, with each Ξ_i corresponding to the ith term in the above equation. Bounding the first pair of terms is immediate, as

$$||\Xi_1||_{L^2} = ||\frac{\tau}{2}\chi_\epsilon^2 H(\theta_{\alpha\alpha\alpha}^\epsilon)||_{L^2} \lesssim ||\theta^\epsilon||_{H^s}$$

$$||\Xi_2||_{L^2} = ||-\tau\chi_\epsilon\bigg(\frac{A_\mu}{2}(\chi_\epsilon\theta_\alpha^\epsilon)(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon)\bigg)||_{L^2} \lesssim ||\theta^\epsilon||_{H^s}^2$$

as long as $s \geq 3$. Next,

$$||\Xi_3||_{L^2} = || - \tau\chi_\epsilon\Big(A_\mu H\big(U^{st,\epsilon}\chi_\epsilon\theta^\epsilon_\alpha\big)\Big)||_{L^2} \lesssim ||\theta^\epsilon_\alpha||_{L^2}||W^{st,\epsilon}\cdot\hat{n}^\epsilon||_{L^\infty}$$

which is bounded by Lemma 4.7.

$$||\Xi_4||_{L^2} = ||\chi_\epsilon\big(H(k^\epsilon\chi_\epsilon\theta^\epsilon_\alpha)\big)||_{L^2} \lesssim ||\theta^\epsilon_\alpha||_{L^2}||k^\epsilon||_{L^\infty}$$

Since $k^\epsilon = \frac{-R\cos(\theta^\epsilon)}{2} - A_\mu\tilde{U}^\epsilon$, this is bounded by Lemma 4.7.

$$||\Xi_5||_{L^2} = ||\chi_\epsilon\big(\tau\big(V^{st,\epsilon} - W^{st,\epsilon}\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta^\epsilon_\alpha\big)||_{L^2} \lesssim ||\theta^\epsilon_\alpha||_{L^2}||V^{st,\epsilon} - W^{st,\epsilon}\cdot\hat{t}||_{L^\infty}$$

$$||\Xi_6||_{L^2} = \chi_\epsilon\big(\tau A^2_\mu\big(\chi_\epsilon\theta^\epsilon_\alpha\big)W^{st,\epsilon}\cdot\hat{t}^\epsilon\big) \lesssim ||\theta^\epsilon_\alpha||_{L^2}||W^{st,\epsilon}\cdot\hat{t}^\epsilon||_{L^\infty}$$

$$||\Xi_7||_{L^2} = ||\frac{A_\mu}{2}R\chi_\epsilon\Big(\sin(\chi_\epsilon\theta^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha\Big)||_{L^2} \lesssim ||\theta^\epsilon||_{L^2}||\theta^\epsilon_\alpha||_{L^2}$$

$$||\Xi_8||_{L^2} = ||A^2_\mu\chi_\epsilon\Big(\big(\chi_\epsilon\theta^\epsilon_\alpha\big)\tilde{W}^\epsilon\cdot\hat{t}^\epsilon\Big)||_{L^2} \lesssim ||\theta^\epsilon_\alpha||_{L^2}||\tilde{W}^\epsilon\cdot\hat{t}^\epsilon||_{L^\infty}$$

$$||\Xi_9||_{L^2} = ||\chi_\epsilon\Big[\big(\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon\big)\chi_\epsilon\theta^\epsilon_\alpha\Big]||_{L^2} \lesssim ||\theta^\epsilon_\alpha||_{L^2}||\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon||_{L^\infty}$$

$$||\Xi_{10}||_{L^2} = ||\chi_\epsilon(m^\epsilon\cdot\hat{n}^\epsilon)||_{L^2} \lesssim ||m^\epsilon||_{L^2}$$

$$||\Xi_{11}||_{L^2} = || - A_\mu\chi_\epsilon H(m^\epsilon\cdot\hat{t}^\epsilon)||_{L^2} \lesssim ||m^\epsilon||_{L^2}$$

Therefore, combining the above equations, we have that $\theta^\epsilon_t \in L^2$, as desired. □

Finally, with bounds on θ^ϵ_t and all the Υ^ϵ_j, we are now ready to prove our first major result.

Theorem 5.3. *Let* $E(t) = \frac{1}{2}\int_{\mathbb{R}} \theta^\epsilon(\alpha,t)^2 + (\partial_\alpha^s \theta^\epsilon(\alpha,t))^2 d\alpha$*, and suppose that*

$$||\theta_\epsilon||_s < d_1,\ |\frac{z_d^\epsilon(\alpha) - z_d^\epsilon(\alpha')}{\alpha - \alpha'}| > d_2$$

Then,

$$\frac{\partial E}{\partial t} \lesssim e^{||\theta^\epsilon||_s}$$

Proof of Theorem 5.3: Now,

$$\frac{\partial E}{\partial t} = \int \theta^\epsilon(\alpha)\theta_t^\epsilon(\alpha) + \partial_\alpha^s\theta^\epsilon(\alpha)\partial_\alpha^s\theta_t^\epsilon(\alpha)d\alpha \tag{5.1}$$

To bound $\int \theta^\epsilon(\alpha)\theta_t^\epsilon(\alpha)$, we note that $\theta_t^\epsilon \in L^2$ from Lemma 5.2, immediately obtaining

$$\int \theta^\epsilon(\alpha)\theta_t^\epsilon(\alpha) \lesssim e^{||\theta^\epsilon||_s} \tag{5.2}$$

Therefore the main difficulty is bounding $\int \partial_\alpha^s\theta^\epsilon\partial_\alpha^s\theta_t^\epsilon$. We start by using (3.19) along with the fact that χ_ϵ is self-adjoint to obtain

$$\begin{aligned}\int \partial_\alpha^s\theta^\epsilon\partial_\alpha^s\theta_t^\epsilon &= \int \partial_\alpha^s\theta^\epsilon\partial_\alpha^{s-2}\theta_{t,\alpha\alpha}^\epsilon \\ &= \frac{-\tau}{2}\int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)(\partial_\alpha^s\Lambda^3\chi_\epsilon\theta^\epsilon) + \int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_5^\epsilon\Lambda(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon)) \\ &+ \int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_6^\epsilon\chi_\epsilon\theta_{\alpha\alpha\alpha}^\epsilon) + \int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\partial_\alpha^{s-2}\Upsilon_7^\epsilon\end{aligned} \tag{5.3}$$

We have the necessary bounds on the Υ_i^ϵ from Lemma (5.1), so the main difficulty is that some of the θ terms in (5.3) have more than s derivatives. The Υ_7^ϵ integral is the simplest to deal with, as it contains no such term and can be bounded directly:

$$\int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\partial_\alpha^{s-2}\Upsilon_7^\epsilon \leq ||\theta^\epsilon||_s||\Upsilon_7^\epsilon||_{s-2} \tag{5.4}$$

For the Υ_6^ϵ integral, we expand via the product rule,

$$\int (\chi_\epsilon \partial_\alpha^s \theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_6^\epsilon \chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) = \int \Upsilon_6^\epsilon(\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^{s+1}\chi_\epsilon\theta^\epsilon) + \sum_{j=1}^{s-2}\binom{s-2}{j}\int (\partial_\alpha^s \chi_\epsilon\theta^\epsilon)(\partial_\alpha^j \Upsilon_6^\epsilon)(\partial_\alpha^{s+1-j}\chi_\epsilon\theta^\epsilon)$$

For the first term, we remove the $\partial_\alpha^{s+1}\chi_\epsilon\theta^\epsilon$ through an integration by parts, obtaining

$$\int \Upsilon_6^\epsilon(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)(\partial_\alpha^{s+1}\chi_\epsilon\theta^\epsilon) = \frac{1}{2}\int \Upsilon_6^\epsilon \partial_\alpha(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 = \frac{-1}{2}\int (\partial_\alpha \Upsilon_6^\epsilon)(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2$$

To estimate the sum, we note that $j \leq s-2$, $s+1-j \leq s$, and bound them directly to obtain

$$\int \Upsilon_6^\epsilon(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)(\partial_\alpha^{s+1}\chi_\epsilon\theta^\epsilon) \lesssim ||\theta^\epsilon||_s^2 ||\partial_\alpha \Upsilon_6^\epsilon||_{s-3} \tag{5.5}$$

Next, we expand the Υ_5^ϵ integral using the product rule,

$$\int (\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_5^\epsilon \Lambda(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon)) = \int \Upsilon_5^\epsilon(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)\Lambda(\partial_\alpha^s\chi_\epsilon\theta^\epsilon) + \sum_{j=1}^{s-2}\binom{s-2}{j}\int (\partial_\alpha^s\chi_\epsilon\theta^\epsilon)(\partial_\alpha^j\Upsilon_5^\epsilon)\Lambda(\partial_\alpha^{s-j}\chi_\epsilon\theta^\epsilon)$$

As before, we have that $j \leq s-2$, $s+1-j \leq s$, letting us bound the sum directly. For the other term, we use Young's inequality to isolate the $\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon$, getting

$$\int \Upsilon_5^\epsilon(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)\Lambda(\partial_\alpha^s\chi_\epsilon\theta^\epsilon) \leq \frac{1}{2}\int (\Upsilon_5^\epsilon)^2(\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 + \frac{1}{2}\int (\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2$$

Since the first term in this inequality can also be bounded directly, we obtain

$$\int (\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_5^\epsilon\Lambda(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon)) \lesssim (||\Upsilon_5^\epsilon||_{L^\infty}^2 + ||\partial_\alpha\Upsilon_5^\epsilon||_{s-3})||\chi_\epsilon\theta^\epsilon||_{H^s}^2 + \int (\partial_\alpha^{s+1}\chi_\epsilon\theta^\epsilon)^2 \tag{5.6}$$

Finally, using the fact that $\Lambda^{3/2}$ is self-adjoint, we take the remaining term from (5.3) to get our dissipation term

$$\frac{-\tau}{2}\int(\chi_\epsilon\partial_\alpha^s\theta^\epsilon)(\partial_\alpha^s\Lambda^3\chi_\epsilon\theta^\epsilon) = \frac{-\tau}{2}\int(\Lambda^{3/2}(\partial_\alpha^s\chi_\epsilon\theta^\epsilon))^2 \tag{5.7}$$

And therefore, applying (5.4), (5.5), (5.6), and (5.7) to (5.3), we have

$$\int\partial_\alpha^s\theta^\epsilon\partial_\alpha^s\theta_t^\epsilon \leq Ce^{||\theta^\epsilon||_s} + \frac{1}{2}\int(\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 - \frac{\tau}{2}\int(\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 \tag{5.8}$$

And combining this with (5.2) and (5.1), we get

$$\frac{\partial E}{\partial t} \leq Ce^{||\theta^\epsilon||_s} + \frac{1}{2}\int(\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 - \frac{\tau}{2}\int(\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 \tag{5.9}$$

Letting $v = \partial_\alpha^s\chi_\epsilon\theta^\epsilon$, we have by Plancherel,

$$\frac{1}{2}\int(\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 - \frac{\tau}{2}\int(\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon)^2 = \int|\hat{v}(\zeta)|^2(|\frac{(2\pi\zeta)^2}{2}| - |\frac{(2\pi\zeta)^3\tau}{2}|)d\zeta$$

Since $\tau > 0$, therefore $|\frac{(2\pi\zeta)^2}{2}| - |\frac{(2\pi\zeta)^3\tau}{2}|$ is bounded above by a constant M_τ independent of ζ. Therefore, plugging this back into (5.9), we obtain

$$\frac{\partial E}{\partial t} \leq Ce^{||\theta^\epsilon||_s} + M_\tau\int|\hat{v}(\zeta)|^2 d\zeta \leq Ce^{||\theta^\epsilon||_s} + M_\tau||\theta^\epsilon||_s^2$$

Which in turn, gives us the desired bound,

$$\frac{\partial E}{\partial t} \lesssim e^{||\theta^\epsilon||_s} \tag{5.10}$$

concluding the proof! □

With the energy estimate proved, we can finally show that the θ^ϵ all exist on the same time interval.

Lemma 5.4. *Let θ^ϵ be as in Lemma 3.1. Then there exists some $T > 0$ such that for all $\epsilon > 0$, θ^ϵ is a solution to* (3.2) *on the time interval $[0, T]$, and $\theta^\epsilon \in C([0, T], \mathcal{O})$*

Proof of Lemma 5.4: Now, from the continuation theorem for autonomous differential equations on Banach spaces (the version we use is Theorem 3.3 of [15]), each θ^ϵ can be continued as long as it does not leave the set $\mathcal{O}$. We will aim to show that the θ^ϵ cannot leave the set $\mathcal{O}$ in arbitrarily small time without violating the energy bound from Theorem 5.3. Now, let T^ϵ be the maximal time of existence for each θ^ϵ. Suppose that there exists a sequence ϵ_n such that $T^{\epsilon_n} \to 0$ as $n \to \infty$. Then, (passing to a subsequence if necessary), we have that either

$$||\theta^{\epsilon_n}(\cdot, T^{\epsilon_n})||_s \geq d_1 \tag{5.11}$$

or

$$|\frac{z_d^{\epsilon_n}(\alpha, T^{\epsilon_n}) - z_d^{\epsilon_n}(\alpha', T^{\epsilon_n})}{\alpha - \alpha'}| \leq d_2 \tag{5.12}$$

for all n. Suppose that (5.11) holds. Then, by Theorem 5.3,

$$\begin{aligned} ||\theta^{\epsilon_n}(\cdot, T^{\epsilon_n})||_s^2 - ||\theta_0||_s^2 &= \int_0^{T^{\epsilon_n}} \frac{d}{dt}||\theta^{\epsilon_n}(\cdot, t)||_s^2 dt \\ &= \int_0^{T^{\epsilon_n}} 2\frac{dE^{\epsilon_n}}{dt} dt \\ &\leq 2T^{\epsilon_n}||\frac{dE^{\epsilon_n}}{dt}||_{L^\infty} \\ &\leq Ce^{d_1}T^{\epsilon_n} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. However, since $\theta_0 \in O$, therefore $||\theta_0||_s < d_1$, and in particular,

$$||\theta^{\epsilon_n}(\cdot, T^{\epsilon_n})||_s^2 - ||\theta_0||_s^2 > d_1^2 - ||\theta_0||_s^2 > 0$$

a contradiction. Therefore, (5.11) cannot hold.

Suppose (5.12) holds. Again, by theorem 5.3, we have

$$\begin{aligned}
\frac{z_d^{\epsilon_n}(\alpha, T^{\epsilon_n}) - z_d^{\epsilon_n}(\alpha', T^{\epsilon_n})}{\alpha - \alpha'} - \frac{z_d(\alpha,0) - z_d(\alpha',0)}{\alpha-\alpha'} &= \int_{\alpha'}^{\alpha} \frac{z_{\alpha,\epsilon_n}(\beta, T^{\epsilon_n}) - z_\alpha(\beta,0)}{\alpha-\alpha'} d\beta \\
&= \frac{1}{\alpha-\alpha'} \int_{\alpha'}^{\alpha} \int_0^{T^{\epsilon_n}} \frac{d}{dt} z_\alpha(\beta,t) dt d\beta \\
&\leq \frac{1}{\alpha-\alpha'} \int_{\alpha'}^{\alpha} \int_0^{T^{\epsilon_n}} \theta_{\alpha,t}(\beta,t) dt d\beta \\
&\leq \frac{1}{\alpha-\alpha'} \int_{\alpha'}^{\alpha} \int_0^{T^{\epsilon_n}} ||\theta_{\alpha,t}||_{L^\infty} dt d\beta \\
&\leq \frac{1}{\alpha-\alpha'} \int_{\alpha'}^{\alpha} \int_0^{T^{\epsilon_n}} ||\theta_t||_{H^2} dt d\beta \\
&\leq T^{\epsilon_n} ||\theta_t||_{H^2 L_t^\infty} \\
&\to 0
\end{aligned}$$

since $||\theta_t||_{H^2}$ is bounded independently of ϵ by Lemmas 5.1 and 5.2. However, once again we have

$$|\frac{z_d^{\epsilon_n}(\alpha, T^{\epsilon_n}) - z_d^{\epsilon_n}(\alpha', T^{\epsilon_n})}{\alpha - \alpha'} - \frac{z_d(\alpha,0) - z_d(\alpha',0)}{\alpha-\alpha'}| > |\frac{z_d(\alpha,0) - z_d(\alpha',0)}{\alpha-\alpha'}| - d_2 > 0$$

and therefore, (5.12) cannot hold. Therefore, no such sequence of T^{ϵ_n} can exist, and therefore there exists some $T > 0$ such that $T^\epsilon > T$ for all ϵ, and so the θ^ϵ exist on the time interval $[0, T]$, as desired.□

Chapter 6

Uniqueness and Proof of Local Existence:

So far we have proved that the θ^ϵ satisfy (3.2) and exist on a uniform time interval $[0, T]$. However, while it is simple to show the θ^ϵ converge pointwise to a limit θ, this is insufficient for proving that θ satisfies (1.1). Therefore, we will first prove that the θ^ϵ satisfying (3.2) depends continuously on both ϵ and the initial data θ_0.

Now, given two sets of initial data θ_0, ϕ_0, we let θ^ϵ and ϕ^δ denote the solutions to the mollified equation. As before, we will bound $\partial_t ||\theta^\epsilon - \phi^\delta||_2$ by bounding the energy E_d, which is defined by

$$E_d = \frac{1}{2}||\theta^\epsilon - \phi^\delta||_2^2 \tag{6.1}$$

Then, our goal is to bound

$$\frac{dE_d}{dt} = \int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)(\theta^\epsilon - \phi^\delta)_t + (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t} d\alpha \tag{6.2}$$

To be precise, most of this section will be dedicated to proving the following result:

Theorem 6.1. *Let $d_1 < \infty, d_2 > 0$. Define the set $\mathcal{O}$ by*

$$\mathcal{O} = \{\theta \in H^s \mid ||\theta||_s \leq d_1, |\frac{z_d(\alpha) - z_d(\alpha')}{\alpha - \alpha'}| > d_2 \forall \alpha, \alpha' \in R\}$$

Suppose that $\theta^\epsilon, \phi^\delta \in C([0,T]; \mathcal{O})$ satisfy (3.2) with corresponding initial data θ_0, ϕ_0. Then, there exists constants c_1, c_2 such that

$$\frac{dE_d}{dt} \leq c_1 ||\theta^\epsilon - \phi^\delta||_2^2 + c_2(\epsilon + \delta)||\theta^\epsilon - \phi^\delta||_2$$

$$||\theta^\epsilon - \phi^\delta||_2 \leq ||\theta_0 - \phi_0||_2 e^{c_1 t/2} + \frac{c_2\sqrt{2}}{c_1}(\epsilon + \delta)(e^{c_1 t/2} - 1)$$

Remark 6.2. While we work with the mollified equation of θ^ϵ for the remainder of this chapter, it is important to note that when $\epsilon = 0$, (3.2) collapses to the original evolution equation, (1.1). In fact, by defining $\chi_0 = I$, (1.1) can be considered a special case of (3.2), and the results of this chapter hold when applied to solutions θ, ϕ of the unmollified equation. This gives us the bound we'll use for uniqueness,

$$||\theta - \phi||_2 \leq e^{cT/2} ||\theta_0 - \phi_0||_2$$

Conversely, when $\theta_0 = \phi_0$, we have that $E_d(0) = 0$, giving us the bound we'll need for local existence,

$$||\theta^\epsilon - \theta^\delta||_2 \lesssim (\epsilon + \delta)$$

To accomplish this goal, we first aim to rewrite $(\theta^\epsilon - \phi^\delta)_t$ as

$$\begin{aligned}(\theta^\epsilon - \phi^\delta)_t = \chi_\delta\left[\frac{-\tau}{2}\chi_\delta\Lambda^3(\theta^\epsilon - \phi^\delta) + \tau\Upsilon_8\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\right]\\ + \chi_\delta\left[\Upsilon_9\chi_\delta\Lambda(\theta^\epsilon - \phi^\delta) + \Upsilon_{10}\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)\right]\\ + \chi_\delta\left[\Upsilon_{11} + \Upsilon_{12} + \Upsilon_{13}\right] + (\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_t)\end{aligned} \tag{6.3}$$

Here, Υ_{11} contains the remainder terms that can be dealt with immediately, Υ_{12} denotes the terms that need to be expanded after differentiation, Υ_{13} contains the remainder terms that scale with $\chi_\epsilon - \chi_\delta$, and the remaining Υ_j are bounded collections of terms. We begin by collapsing surface tension terms in (3.2), obtaining

$$\begin{aligned}\theta^\epsilon_t = \chi_\epsilon\left[\frac{\tau}{2}H(\chi_\epsilon\theta^\epsilon_{\alpha\alpha\alpha}) - \frac{1}{2}H\big((R\cos(\chi_\epsilon\theta^\epsilon) + 2A_\mu U^\epsilon)(\chi_\epsilon\theta^\epsilon_\alpha)\big)\right]\\ + \chi_\epsilon\left[-A_\mu\frac{\gamma^\epsilon(\chi_\epsilon\theta^\epsilon_\alpha)}{2} + (V^\epsilon - W^\epsilon\cdot\hat{t}^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha + m^\epsilon\cdot\hat{n}^\epsilon - A_\mu H(m^\epsilon\cdot\hat{t}^\epsilon)\right]\end{aligned} \tag{6.4}$$

Therefore, we can write

$$(\theta^\epsilon - \phi^\delta)_t = (\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_t) + \sum_{i=1}^{6}\chi_\delta B_i \tag{6.5}$$

where each of the B_i correspond to the ith term in (6.4). Now,

$$\begin{aligned}B_1 &= \frac{-\tau}{2}\Lambda^3(\chi_\epsilon\theta^\epsilon - \chi_\delta\phi^\delta)\\ &= \frac{-\tau}{2}(\chi_\epsilon - \chi_\delta)\Lambda^3(\theta^\epsilon) + \frac{-\tau}{2}\chi_\delta\Lambda^3(\theta^\epsilon - \phi^\delta)\end{aligned} \tag{6.6}$$

For B_2, we first rewrite

$$\frac{1}{2}H\big((R\cos(\chi_\epsilon\theta^\epsilon) + 2A_\mu U^\epsilon)(\chi_\epsilon\theta^\epsilon_\alpha)\big) = H(k^\epsilon(\chi_\epsilon\theta_\alpha)) - \tau A_\mu H(U^{st,\epsilon}(\chi_\epsilon\theta^\epsilon_\alpha))$$

Therefore,

$$\begin{aligned}B_2 &= H(k^{\epsilon,\theta}\chi_\epsilon\theta^\epsilon_\alpha) - H(k^{\delta,\phi}\chi_\delta\phi^\delta_\alpha) - \tau A_\mu H(U^{st,\epsilon,\theta}\chi_\epsilon\theta^\epsilon_\alpha) + \tau A_\mu H(U^{st,\delta,\phi}\chi_\delta\phi^\delta_\alpha)\\ &= H\big(k^{\epsilon,\theta}(\chi_\epsilon-\chi_\delta)\theta^\epsilon_\alpha\big) + H\big((k^{\epsilon,\theta}-k^{\delta,\phi})\chi_\delta\theta^\delta_\alpha\big) + H\big(k^{\delta,\phi}\chi_\delta(\theta^\epsilon_\alpha-\phi^\delta_\alpha)\big)\\ &- \tau A_\mu H\big(U^{st,\epsilon,\theta}(\chi_\epsilon-\chi_\delta)\theta^\epsilon_\alpha\big) - \tau A_\mu H\big((U^{st,\epsilon,\theta}-U^{st,\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big)\\ &- \tau A_\mu H\big(U^{st,\delta,\phi}\chi_\delta(\theta^\epsilon_\alpha-\phi^\delta_\alpha)\big)\end{aligned}$$

We pull $k^{\delta,\phi}$ and $U^{st,\delta,\phi}$ through the Hilbert transform, incurring commutators,

$$\begin{aligned}B_2 &= H\big(k^{\epsilon,\theta}(\chi_\epsilon-\chi_\delta)\theta^\epsilon_\alpha\big) + H\big((k^{\epsilon,\theta}-k^{\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big)\\ &+ k^{\delta,\phi}\chi_\delta H(\theta^\epsilon_\alpha-\phi^\delta_\alpha) + [H,k^{\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha-\phi^\delta_\alpha))\\ &- \tau A_\mu H\big(U^{st,\epsilon,\theta}(\chi_\epsilon-\chi_\delta)\theta^\epsilon_\alpha\big) - \tau A_\mu H\big((U^{st,\epsilon,\theta}-U^{st,\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big)\\ &- \tau A_\mu U^{st,\delta,\phi}\chi_\delta H(\theta^\epsilon_\alpha-\phi^\delta_\alpha) - \tau A_\mu[H,U^{st,\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha-\phi^\delta_\alpha))\end{aligned} \tag{6.7}$$

For B_3, recall that $\gamma^\epsilon = \tau\chi_\epsilon\theta^\epsilon_{\alpha\alpha} + \tilde{\gamma}^\epsilon$, and so

$$\frac{-A_\mu}{2}\gamma^\epsilon(\chi_\epsilon\theta^\epsilon_\alpha) = \frac{-\tau A_\mu}{2}(\chi_\epsilon\theta^\epsilon_{\alpha\alpha})(\chi_\epsilon\theta^\epsilon_\alpha) - \frac{A_\mu}{2}\tilde{\gamma}^\epsilon(\chi_\epsilon\theta^\epsilon_\alpha)$$

Therefore,

$$B_3 = \frac{-\tau A_\mu}{2}(\chi_\epsilon\theta^\epsilon_{\alpha\alpha})(\chi_\epsilon\theta^\epsilon_\alpha) + \frac{\tau A_\mu}{2}(\chi_\delta\phi^\delta_{\alpha\alpha})(\chi_\delta\phi^\delta_\alpha) - \frac{A_\mu}{2}\tilde{\gamma}^{\epsilon,\theta}(\chi_\epsilon\theta^\epsilon_\alpha) + \frac{A_\mu}{2}\tilde{\gamma}^{\delta,\phi}(\chi_\delta\phi^\delta_\alpha)$$

We begin by separating the $\chi_\epsilon - \chi_\delta$ terms,

$$\begin{aligned}B_3 = \frac{-A_\mu}{2}\gamma^{\epsilon,\theta}((\chi_\epsilon-\chi_\delta)\theta^\epsilon_\alpha) - \frac{\tau A_\mu}{2}((\chi_\epsilon-\chi_\delta)\theta^\epsilon_{\alpha\alpha})(\chi_\delta\theta^\epsilon_\alpha) - \frac{\tau A_\mu}{2}(\chi_\delta\theta^\epsilon_{\alpha\alpha})(\chi_\delta\theta^\epsilon_\alpha)\\ + \frac{\tau A_\mu}{2}(\chi_\delta\phi^\delta_{\alpha\alpha})(\chi_\delta\phi^\delta_\alpha) - \frac{A_\mu}{2}\tilde{\gamma}^{\epsilon,\theta}(\chi_\delta\theta^\epsilon_\alpha) + \frac{A_\mu}{2}\tilde{\gamma}^{\delta,\phi}(\chi_\delta\phi^\delta_\alpha)\end{aligned}$$

At this point, we expand, obtaining

$$\begin{aligned} B_3 = &\frac{-A_\mu}{2}\gamma^{\epsilon,\theta}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_\alpha) - \frac{\tau A_\mu}{2}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha})(\chi_\delta \theta^\epsilon_\alpha) \\ &- \frac{\tau A_\mu}{2}(\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha))(\chi_\delta \theta^\epsilon_{\alpha\alpha}) - \frac{\tau A_\mu}{2}(\chi_\delta \phi^\delta_\alpha)(\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})) \\ &- \frac{A_\mu}{2}(\tilde{\gamma}^{\epsilon,\theta} - \tilde{\gamma}^{\delta,\phi})(\chi_\delta \theta^\epsilon_\alpha) - \frac{A_\mu}{2}\tilde{\gamma}^{\delta,\phi}(\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)) \end{aligned} \tag{6.8}$$

The remaining B_i are straightforwards,

$$\begin{aligned} B_4 = &(V - W\cdot\hat{t})^{\epsilon,\theta}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_\alpha) + (V - W\cdot\hat{t})^{\delta,\phi}(\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)) \\ &+ \big((V - W\cdot\hat{t})^{\epsilon,\theta} - (V - W\cdot\hat{t})^{\delta,\phi}\big)(\chi_\delta \theta^\epsilon_\alpha) \end{aligned} \tag{6.9}$$

$$B_5 = m^{\epsilon,\theta}\cdot\hat{n}^{\epsilon,\theta} - m^{\delta,\phi}\cdot\hat{n}^{\delta,\phi} \tag{6.10}$$

$$B_6 = -A_\mu H(m^{\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta}) + A_\mu H(m^{\delta,\phi}\cdot\hat{t}^{\delta,\phi}) \tag{6.11}$$

Therefore, returning to (6.3), we see that Υ_8 contains one term from B_3.

$$\Upsilon_8 = \frac{-A_\mu}{2}(\chi_\delta \phi^\delta_\alpha) \tag{6.12}$$

Υ_9 contains two terms from B_2,

$$\Upsilon_9 = k^{\delta,\phi} - \tau A_\mu U^{st,\delta,\phi} \tag{6.13}$$

Υ_{10} contains two terms from B_3 and one term from B_4,

$$\Upsilon_{10} = \frac{-\tau A_\mu}{2}(\chi_\delta \theta^\epsilon_{\alpha\alpha}) - \frac{A_\mu}{2}\tilde{\gamma}^{\delta,\phi} + (V - W\cdot\hat{t})^{\delta,\phi} \tag{6.14}$$

Υ_{11} contains most of the remainder terms,

$$\begin{aligned} \Upsilon_{11} = &H\big((k^{\epsilon,\theta} - k^{\delta,\phi})\chi_\delta \theta^\epsilon_\alpha\big) + [H, k^{\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)) \\ &- \tau A_\mu [H, U^{st,\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)) - \frac{A_\mu}{2}(\tilde{\gamma}^{\epsilon,\theta} - \tilde{\gamma}^{\delta,\phi})(\chi_\delta \theta^\epsilon_\alpha) \\ &+ B_5 + B_6 \end{aligned} \tag{6.15}$$

Υ_{12} contains the set of terms that must be expanded after differentiation; one term from B_2 and one term from B_4.

$$\begin{aligned}\Upsilon_{12} &= -\tau A_\mu H\big[(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big] \\ &+ \big[(V - W\cdot\hat{t})^{\epsilon,\theta} - (V - W\cdot\hat{t})^{\delta,\phi}\big](\chi_\delta\theta^\epsilon_\alpha)\end{aligned} \tag{6.16}$$

Finally, Υ_{13} contains all the terms that scale with $(\chi_\epsilon - \chi_\delta)$.

$$\begin{aligned}\Upsilon_{13} &= \frac{-\tau}{2}(\chi_\epsilon - \chi_\delta)\Lambda^3(\theta^\epsilon) + (V - W\cdot\hat{t})^{\epsilon,\theta}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_\alpha) \\ &+ H\big(k^{\epsilon,\theta}(\chi_\epsilon - \chi_\delta)\theta^\epsilon_\alpha\big) - \tau A_\mu H\big(U^{st,\epsilon,\theta}(\chi_\epsilon - \chi_\delta)\theta^\epsilon_\alpha\big) \\ &- \frac{A_\mu}{2}\gamma^{\epsilon,\theta}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_\alpha) - \frac{\tau A_\mu}{2}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha})(\chi_\delta\theta^\epsilon_\alpha)\end{aligned} \tag{6.17}$$

Remark 6.3. As our goal is to bound $\frac{dE_d}{dt}$ in terms of ϵ, δ, and $||\theta - \phi||_2$, we only need concern ourselves with groupings of terms of the form $(Q^{\epsilon,\theta} - Q^{\delta,\phi})$. Anything else can be safely bounded by a constant and subsequently ignored. Because of this, the contents of Υ_8, Υ_9, and Υ_{10} are largely irrelevant, as the $(\theta - \phi)$ component has already been isolated. As the terms in Υ_{11}, Υ_{12}, and Υ_{13} lack such a decomposition, they must be bounded individually.

The next concern is obtaining a suitable equation for $(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t}$. We begin with the decomposition of $\Upsilon_{12,\alpha\alpha}$. Recall that

$$U^{st,\epsilon}_\alpha = \frac{1}{2}\chi_\epsilon H(\theta^\epsilon_{\alpha\alpha\alpha}) - \theta^\epsilon_\alpha(W^{st,\epsilon}\cdot\hat{t}^\epsilon) + m^{st,\epsilon}\cdot\hat{n}^\epsilon$$

and

$$(V - W\cdot\hat{t})^\epsilon_\alpha = \frac{\tau}{2}H((\chi_\epsilon\theta^\epsilon_{\alpha\alpha})\theta^\epsilon_\alpha) + \frac{1}{2}H(\tilde{\gamma}^\epsilon\theta^\epsilon_\alpha) - m^\epsilon\cdot\hat{t}^\epsilon$$

Therefore,

$$
\begin{aligned}
\Upsilon_{12,\alpha\alpha} &= -\tau A_\mu H\big[(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha}\big] \\
&- \tau A_\mu H\big[H(\chi_\epsilon \theta^\epsilon_{\alpha\alpha\alpha} - \chi_\delta \phi^\delta_{\alpha\alpha\alpha})\chi_\delta \theta^\epsilon_{\alpha\alpha}\big] \\
&- \frac{\tau}{2} A_\mu H\big[H(\chi_\epsilon \theta^\epsilon_{\alpha\alpha\alpha\alpha} - \chi_\delta \phi^\delta_{\alpha\alpha\alpha\alpha})\chi_\delta \theta^\epsilon_{\alpha}\big] \\
&+ \tau A_\mu \partial_\alpha H\bigg(\Big[\theta^\epsilon_\alpha (W^{st,\epsilon,\theta}\cdot \hat{t}^{\epsilon,\theta}) - \phi^\delta_\alpha (W^{st,\delta,\phi}\cdot \hat{t}^{\delta,\phi})\Big]\chi_\delta \theta^\epsilon_\alpha\bigg) \\
&+ \tau A_\mu \partial_\alpha H\bigg(\Big[- m^{st,\epsilon,\theta}\cdot \hat{n}^{\epsilon,\theta} + m^{st,\delta,\phi}\cdot \hat{n}^{\delta,\phi}\Big]\chi_\delta \theta^\epsilon_\alpha\bigg) \\
&+ (\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha})\big[(V - W\cdot \hat{t})^{\epsilon,\theta} - (V - W\cdot \hat{t})^{\delta,\phi}\big] \\
&+ \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})\big[H((\chi_\epsilon \theta^\epsilon_{\alpha\alpha})\theta^\epsilon_\alpha) - H((\chi_\delta \phi^\delta_{\alpha\alpha})\phi^\delta_\alpha)\big] \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_{\alpha})\big[H((\chi_\epsilon \theta^\epsilon_{\alpha\alpha})\theta^\epsilon_{\alpha\alpha}) - H((\chi_\delta \phi^\delta_{\alpha\alpha})\phi^\delta_{\alpha\alpha})\big] \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_{\alpha})\big[H((\chi_\epsilon \theta^\epsilon_{\alpha\alpha\alpha})\theta^\epsilon_{\alpha}) - H((\chi_\delta \phi^\delta_{\alpha\alpha\alpha})\phi^\delta_{\alpha})\big] \\
&+ \partial_\alpha\bigg((\chi_\delta \theta^\epsilon_\alpha)\Big[(\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - m^{\epsilon,\theta}\cdot \hat{t}^{\epsilon,\theta}) - (\frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha - m^{\delta,\phi}\cdot \hat{t}^{\delta,\phi})\Big]\bigg)
\end{aligned}
\tag{6.18}
$$

After expanding and pulling various terms through commutators, we obtain

$$\begin{aligned}
\Upsilon_{12,\alpha\alpha} &= -\tau A_\mu H\big[H((\chi_\epsilon-\chi_\delta)\theta^\epsilon_{\alpha\alpha\alpha})\chi_\delta\theta^\epsilon_{\alpha\alpha}\big] - \frac{\tau}{2}A_\mu H\big[H((\chi_\epsilon-\chi_\delta)\theta^\epsilon_{\alpha\alpha\alpha\alpha})\chi_\delta\theta^\epsilon_{\alpha}\big] \\
&\quad + \tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})H\big(\theta^\epsilon_\alpha((\chi_\epsilon-\chi_\delta)\theta^\epsilon_{\alpha\alpha})\big) + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)H\big(\theta^\epsilon_{\alpha\alpha}((\chi_\epsilon-\chi_\delta)\theta^\epsilon_{\alpha\alpha})\big) \\
&\quad + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)H\big(\theta^\epsilon_\alpha((\chi_\epsilon-\chi_\delta)\theta^\epsilon_{\alpha\alpha\alpha})\big) - \tau A_\mu H\big[(U^{st,\epsilon,\theta}-U^{st,\delta,\phi})\chi_\delta\theta^\epsilon_{\alpha\alpha\alpha}\big] \\
&\quad + \tau A_\mu(\chi_\delta\theta^\epsilon_{\alpha\alpha})(\chi_\delta(\theta^\epsilon_{\alpha\alpha\alpha}-\phi^\delta_{\alpha\alpha\alpha})) - \tau A_\mu[H,\chi_\delta\theta^\epsilon_{\alpha\alpha}]\big(\chi_\delta H(\theta_{\alpha\alpha\alpha}-\phi_{\alpha\alpha\alpha})\big) \\
&\quad + \frac{\tau}{2}A_\mu(\chi_\delta\theta^\epsilon_\alpha)(\chi_\delta(\theta^\epsilon_{\alpha\alpha\alpha\alpha}-\phi^\delta_{\alpha\alpha\alpha\alpha})) - \frac{\tau}{2}A_\mu[H,\chi_\delta\theta^\epsilon_\alpha]\big(\chi_\delta H(\theta^\epsilon_{\alpha\alpha\alpha\alpha}-\phi^\delta_{\alpha\alpha\alpha\alpha})\big) \\
&\quad + \tau A_\mu\partial_\alpha H\Bigg(\Big[\theta^\epsilon_\alpha(W^{st,\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta}) - \phi^\delta_\alpha(W^{st,\delta,\phi}\cdot\hat{t}^{\delta,\phi})\Big]\chi_\delta\theta^\epsilon_\alpha\Bigg) \\
&\quad + \tau A_\mu\partial_\alpha H\Bigg(\Big[-m^{st,\epsilon,\theta}\cdot\hat{n}^{\epsilon,\theta} + m^{st,\delta,\phi}\cdot\hat{n}^{\delta,\phi}\Big]\chi_\delta\theta^\epsilon_\alpha\Bigg) \\
&\quad + (\chi_\delta\theta^\epsilon_{\alpha\alpha\alpha})\big[(V-W\cdot\hat{t})^{\epsilon,\theta} - (V-W\cdot\hat{t})^{\delta,\phi}\big] \\
&\quad + \tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})\theta^\epsilon_\alpha\chi_\delta H(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}) + \tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})[H,\theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})) \\
&\quad + \tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})(\chi_\delta\phi^\delta_{\alpha\alpha})H(\theta^\epsilon_\alpha-\phi^\delta_\alpha) + \tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})[H,\chi_\delta\phi^\delta_{\alpha\alpha}](\theta^\epsilon_\alpha-\phi^\delta_\alpha) \\
&\quad + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)\theta^\epsilon_{\alpha\alpha}\chi_\delta H(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}) + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\theta^\epsilon_{\alpha\alpha}](\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})) \\
&\quad + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)(\chi_\delta\phi^\delta_{\alpha\alpha})H(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}) + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\chi_\delta\phi^\delta_{\alpha\alpha}](\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}) \\
&\quad + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)\theta^\epsilon_\alpha\chi_\delta H(\theta^\epsilon_{\alpha\alpha\alpha}-\phi^\delta_{\alpha\alpha\alpha}) + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha\alpha}-\phi^\delta_{\alpha\alpha\alpha})) \\
&\quad + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)(\chi_\delta\phi^\delta_{\alpha\alpha\alpha})H(\theta^\epsilon_\alpha-\phi^\delta_\alpha) + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\chi_\delta\phi^\delta_{\alpha\alpha\alpha}](\theta^\epsilon_\alpha-\phi^\delta_\alpha) \\
&\quad + \partial_\alpha\Bigg((\chi_\delta\theta^\epsilon_\alpha)\Big[(\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - m^{\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta}) - (\frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha - m^{\delta,\phi}\cdot\hat{t}^{\delta,\phi})\Big]\Bigg)
\end{aligned}$$

Therefore, our equation is

$$
\begin{aligned}
(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t} = \chi_\delta & \left[\frac{-\tau}{2}\chi_\delta\Lambda^3(\theta_{\alpha\alpha} - \phi_{\alpha\alpha}) + \tau\Upsilon_{14}\chi_\delta(\theta_{\alpha\alpha\alpha\alpha} - \phi_{\alpha\alpha\alpha\alpha})\right] \\
& + \chi_\delta\left[\Upsilon_{15}\chi_\delta\Lambda(\theta_{\alpha\alpha} - \phi_{\alpha\alpha}) + \Upsilon_{16}\chi_\delta(\theta_{\alpha\alpha\alpha} - \phi_{\alpha\alpha\alpha})\right] \\
& + \chi_\delta\left[\Upsilon_{17}\chi_\delta\Lambda(\theta_\alpha - \phi_\alpha) + \Upsilon_{18}\chi_\delta(\theta_{\alpha\alpha} - \phi_{\alpha\alpha}) + \Upsilon_{19}\chi_\delta\Lambda(\theta - \phi)\right] \\
& + \chi_\delta\left[\Upsilon_{20}\chi_\delta(\theta_\alpha - \phi_\alpha) + \Upsilon_{21} + \Upsilon_{22}\right] + (\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_{\alpha\alpha,t})
\end{aligned}
\tag{6.19}
$$

Where

$$
\begin{aligned}
\Upsilon_{14} &= \Upsilon_8 + \frac{1}{2}A_\mu\chi_\delta\theta^\epsilon_\alpha \\
\Upsilon_{15} &= \Upsilon_9 + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)\theta^\epsilon_\alpha \\
\Upsilon_{16} &= 2\tau\Upsilon_{8,\alpha} + \Upsilon_{10} + \tau A_\mu\chi_\delta\theta^\epsilon_{\alpha\alpha} \\
\Upsilon_{17} &= 2\Upsilon_{9,\alpha} + \tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})\theta^\epsilon_\alpha + \frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)\theta^\epsilon_{\alpha\alpha} \\
\Upsilon_{18} &= \tau\Upsilon_{8,\alpha\alpha} + 2\Upsilon_{10,\alpha} \\
\Upsilon_{19} &= \Upsilon_{9,\alpha\alpha} \\
\Upsilon_{20} &= \Upsilon_{10,\alpha\alpha}
\end{aligned}
\tag{6.20}
$$

Υ_{21} contains the majority of the remainder terms,

$$
\begin{aligned}
\Upsilon_{21} = &-\tau A_\mu H\big[(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha}\big] \\
&- \tau A_\mu [H, \chi_\delta \theta^\epsilon_{\alpha\alpha}]\big(\chi_\delta H(\theta_{\alpha\alpha\alpha} - \phi_{\alpha\alpha\alpha})\big) \\
&- \frac{\tau}{2} A_\mu [H, \chi_\delta \theta^\epsilon_\alpha]\big(\chi_\delta H(\theta^\epsilon_{\alpha\alpha\alpha\alpha} - \phi^\delta_{\alpha\alpha\alpha\alpha})\big) \\
&+ \tau A_\mu \partial_\alpha H\bigg(\Big[\theta^\epsilon_\alpha (W^{st,\epsilon,\theta}\cdot \hat{t}^{\epsilon,\theta}) - \phi^\delta_\alpha (W^{st,\delta,\phi}\cdot \hat{t}^{\delta,\phi})\Big]\chi_\delta \theta^\epsilon_\alpha\bigg) \\
&+ \tau A_\mu \partial_\alpha H\bigg(\Big[-m^{st,\epsilon,\theta}\cdot \hat{n}^{\epsilon,\theta} + m^{st,\delta,\phi}\cdot \hat{n}^{\delta,\phi}\Big]\chi_\delta \theta^\epsilon_\alpha\bigg) \\
&+ (\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha})\big[(V - W\cdot \hat{t})^{\epsilon,\theta} - (V - W\cdot \hat{t})^{\delta,\phi}\big] \\
&+ \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})[H, \theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})) \\
&+ \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})(\chi_\delta \phi^\delta_{\alpha\alpha})H(\theta^\epsilon_\alpha - \phi^\delta_\alpha) + \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})[H, \chi_\delta \phi^\delta_{\alpha\alpha}](\theta^\epsilon_\alpha - \phi^\delta_\alpha) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \theta^\epsilon_{\alpha\alpha}](\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)(\chi_\delta \phi^\delta_{\alpha\alpha})H(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}) + \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \chi_\delta \phi^\delta_{\alpha\alpha}](\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha\alpha} - \phi^\delta_{\alpha\alpha\alpha})) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)(\chi_\delta \phi^\delta_{\alpha\alpha\alpha})H(\theta^\epsilon_\alpha - \phi^\delta_\alpha) + \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \chi_\delta \phi^\delta_{\alpha\alpha\alpha}](\theta^\epsilon_\alpha - \phi^\delta_\alpha) \\
&+ \partial_\alpha\bigg((\chi_\delta \theta^\epsilon_\alpha)\Big[(\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - m^{\epsilon,\theta}\cdot \hat{t}^{\epsilon,\theta}) - (\frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha - m^{\delta,\phi}\cdot \hat{t}^{\delta,\phi})\Big]\bigg) + \Upsilon_{11,\alpha\alpha}
\end{aligned}
\tag{6.21}
$$

Finally, Υ_{22} contains those terms that scale with $\chi_\epsilon - \chi_\delta$.

$$
\begin{aligned}
\Upsilon_{22} = &-\tau A_\mu H\big[H((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha\alpha})\chi_\delta \theta^\epsilon_{\alpha\alpha}\big] - \frac{\tau}{2} A_\mu H\big[H((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha\alpha\alpha})\chi_\delta \theta^\epsilon_\alpha\big] \\
&+ \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})H\big(\theta^\epsilon_\alpha((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha})\big) + \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)H\big(\theta^\epsilon_{\alpha\alpha}((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha})\big) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)H\big(\theta^\epsilon_\alpha((\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha\alpha})\big) + \Upsilon_{13,\alpha\alpha}
\end{aligned}
\tag{6.22}
$$

With the equations fully constructed, we now begin proving the lemmas necessary to bound Υ_{21}. We begin with a Lipschitz estimate for mollifiers,

Lemma 6.4. *Let $n \geq 0$ and suppose that $f \in H^n$. Then,*

$$||(\chi_\epsilon - \chi_\delta)f||_{n-1} \lesssim (\epsilon + \delta)||f||_n$$

Proof of Lemma 6.4: We begin by proving that $||(\chi_\epsilon f) - f||_{L^2} \lesssim \epsilon ||f||_1$. Now, by the definition of a mollifier, we have

$$\begin{aligned}
||(\chi_\epsilon f) - f||_{L^2} &= \left(\int_{\mathbb{R}} \big(\int_{\mathbb{R}} \chi_\epsilon(y)(f(x) - f(x-y))dy \big)^2 dx \right)^{1/2} \\
&= \left(\int_{\mathbb{R}} \big(\int_{\mathbb{R}} y\chi_\epsilon(y) \frac{f(x) - f(x-y)}{y} dy \big)^2 dx \right)^{1/2} \\
&= \left(\int_{\mathbb{R}} \big(\int_{\mathbb{R}} y\chi_\epsilon(y) q_1[f] dy \big)^2 dx \right)^{1/2} \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \big(y\chi_\epsilon(y) q_1[f] \big)^2 dx \right)^{1/2} dy \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (q_1[f])^2 dx \right)^{1/2} |y|\chi_\epsilon(y) dy \\
&\leq ||q_1[f]||_{L^2} \int_{\mathbb{R}} |y|\chi_\epsilon(y) dy \\
&\leq \epsilon ||f||_1
\end{aligned}$$

Now, note that

$$||(\chi_\epsilon - \chi_\delta)f||_{L^2} \leq ||\chi_\epsilon f - f||_{L^2} + ||\chi_\delta f - f||_{L^2} \leq (\epsilon + \delta)||f||_1$$

Finally, since

$$||\partial_\alpha^{n-1}\big((\chi_\epsilon - \chi_\delta)f\big)||_{L^2} = ||(\chi_\epsilon - \chi_\delta)(\partial_\alpha^{n-1} f)||_{L^2} \leq (\epsilon + \delta)||\partial_\alpha^{n-1} f||_1$$

we have that

$$||(\chi_\epsilon - \chi_\delta)f||_{n-1} \lesssim (\epsilon + \delta)||f||_n$$

as desired. □

Lemma 6.5. *Let $f, g \in H^3$. Then,*

$$||\sin(f) - \sin(g)||_2 \lesssim ||f - g||_2$$

$$||\cos(f) - \cos(g)||_2 \lesssim ||f - g||_2$$

As a consequence,

$$||\hat{t}^{\epsilon,\theta} - \hat{t}^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

$$||\hat{n}^{\epsilon,\theta} - \hat{n}^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

$$||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

Proof of Lemma 6.5: Now, for any values a, b, we know that $|\sin(a) - \sin(b)|, |\cos(a) - \cos(b)| \leq |a - b|$. Therefore, it is immediate that

$$||\sin(f) - \sin(g)||_{L^2} \leq ||f - g||_{L^2}$$

$$||\cos(f) - \cos(g)||_{L^2} \leq ||f - g||_{L^2}$$

Now,

$$\begin{aligned}\partial_{\alpha\alpha}\big(\sin(f) - \sin(g)\big) &= f_{\alpha\alpha}\cos(f) - (f_\alpha)^2\sin(f) - g_{\alpha\alpha}\cos(g) + (g_\alpha)^2\sin(g) \\ &= (f_{\alpha\alpha} - g_{\alpha\alpha})\cos(f) + g_{\alpha\alpha}\big(\cos(f) - \cos(g)\big) \\ &- (f_\alpha^2 - g_\alpha^2)\sin(f) - g_\alpha^2\big(\sin(\theta^\epsilon) - \sin(\phi^\delta)\big)\end{aligned}$$

Therefore,

$$\begin{aligned}||\partial_{\alpha\alpha}\big(\sin(f)-\sin(g)\big)||_{L^2} \leq ||f-g||_2||\cos(f)||_{L^\infty} + ||\cos(f)-\cos(g)||_{L^2}||g_{\alpha\alpha}||_{L^\infty} \\ + ||f-g||_{H^1}||f_\alpha + g_\alpha||_{L^\infty}||\sin(f)||_{L^\infty} \\ + ||\sin(f)-\sin(g)||_{L^2}||g_\alpha||_{L^\infty}^2\end{aligned}$$

And, using the fact that $f, g \in H^3$, we know that $f_\alpha, g_\alpha, g_{\alpha\alpha} \in L^\infty$. Therefore, as $||\sin(f)-\sin(g)||_{L^2}$ was bounded previously, we have that

$$||\sin(f)-\sin(g)||_2 \lesssim ||f-g||_2 \tag{6.23}$$

as desired. Similarly,

$$\begin{aligned}\partial_{\alpha\alpha}\big(\cos(f)-\cos(g)\big) &= -f_{\alpha\alpha}\sin(f) - (f_\alpha)^2\cos(f) + g_{\alpha\alpha}\sin(g) + (g_\alpha)^2\cos(g) \\ &= -(f_{\alpha\alpha}-g_{\alpha\alpha})\sin(f) - g_{\alpha\alpha}\big(\sin(f)-\sin(g)\big) \\ &- (f_\alpha^2 - g_\alpha^2)\cos(f) - g_\alpha^2\big(\cos(f)-\cos(g)\big)\end{aligned}$$

Once again, we write

$$\begin{aligned}||\partial_{\alpha\alpha}\big(\cos(f)-\cos(g)\big)||_{L^2} \leq ||f-g||_2||\sin(f)||_{L^\infty} + ||\sin(f)-\sin(g)||_{L^2}||g_{\alpha\alpha}||_{L^\infty} \\ + ||f-g||_{H^1}||f_\alpha + g_\alpha||_{L^\infty}||\cos(f)||_{L^\infty} \\ + ||\cos(f)-\cos(g)||_{L^2}||g_\alpha||_{L^\infty}^2\end{aligned}$$

And as before, since $||\cos(f)-\cos(g)||_{L^2}$ was bounded previously, we obtain

$$||\cos(f)-\cos(g)||_2 \lesssim ||f-g||_2 \tag{6.24}$$

Finally, regarding $\hat{t}, \hat{n}$, and z_α, we let $f = \theta^\epsilon, g = \phi^\delta$, and note those bounds are an immediate consequence of the facts $\Phi(\hat{t}^{\epsilon,\theta}) = z_\alpha^{\epsilon,\theta}$, $\Phi(\hat{n}^{\epsilon,\theta}) = iz_\alpha^{\epsilon,\theta}$, and $z_\alpha^{\epsilon,\theta} = \cos(\theta^\epsilon) + i\sin(\theta^\epsilon)$. Therefore, the lemma is proved. □

With the trigonometric functions bounded, we next look at the commutator, $[H, f]$.

Lemma 6.6. *Suppose that $f \in L^\infty$, $\partial_\alpha f \in H^{s-1}$, and $g^\theta, g^\phi \in L^2$. If $1 \leq j+k \leq s-2$, then*

$$||\partial_\alpha^j [H, f]\big(\partial_\alpha^k (g^\theta - g^\phi)\big)||_{L^2} \lesssim ||g^\theta - g^\phi||_{L^2}$$

Proof of Lemma 6.6:

Now, via repeated integration by parts, we have

$$\begin{aligned} \partial_\alpha^j [H, f]\big(\partial_\alpha^k (g^\theta - g^\phi)\big) &= \int (\partial_\alpha^j q_1[f]) \partial_{\alpha'}^k (g^\theta - g^\phi) d\alpha' \\ &= (-1)^k \int (g^\theta - g^\phi) \cdot \partial_{\alpha'}^k \partial_\alpha^j q_1[f] d\alpha' \end{aligned}$$

And so, applying Lemma 4.2, we have that

$$\begin{aligned} ||\partial_\alpha^j [H, f]\big(\partial_\alpha^k (g^\theta - g^\phi)\big)||_{L^2} &\leq ||g^\theta - g^\phi||_{L^2} \cdot ||\partial_{\alpha'}^k \partial_\alpha^j q_1[f]||_{L^1} \\ &\lesssim ||g^\theta - g^\phi||_{L^2} \end{aligned} \tag{6.25}$$

as desired. □

Our next goal is to get some Lipschitz bounds for the divided differences analogous to Lemma 4.2.

Lemma 6.7. *Suppose that $\theta^\epsilon, \phi^\delta \in H^s$, both satisfying the arc-chord condition. Then, we have that*

$$||q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_2 \leq ||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.26}$$

$$||q_2[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^1} \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.27}$$

$$||\partial_\alpha q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^1} \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.28}$$

$$||\partial_\alpha^2 q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^1} \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.29}$$

Proof of Lemma 6.7: Now, (6.26) follows immediately from the integral representation of q_1,

$$q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}] = \int_0^1 (z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})(t\alpha + (1-t)\alpha')dt$$

Applying Lemma 6.5, we see that

$$||q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_2 \lesssim ||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

For (6.27), recall from (4.4) that

$$\begin{aligned} q_2[f] &= \int_0^1 (t-1) f_{\alpha\alpha}(t\alpha + (1-t)\alpha')dt \\ &= \frac{f(\alpha) - f(\alpha') - (\alpha - \alpha')f_\alpha(\alpha')}{(\alpha-\alpha')^2} \\ &= \frac{q_1[f] - f_\alpha(\alpha')}{\alpha - \alpha'} \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \partial_\alpha q_2[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}] &= \chi_{|\alpha-\alpha'|<1}\left(\int_0^1 (t-1)(z_{\alpha\alpha}^{\epsilon,\theta} - z_{\alpha\alpha}^{\delta,\phi})(t\alpha - (1-t)\alpha')dt\right) \\ &+ \chi_{|\alpha-\alpha'|>1}\frac{q_1[z_d^{\epsilon,\theta}] - q_1[z_d^{\delta,\phi}] - (z_\alpha^{\epsilon,\theta}(\alpha') - z_\alpha^{\delta,\phi}(\alpha'))}{\alpha - \alpha'} \end{aligned}$$

This gives us the estimate

$$
\begin{aligned}
||\partial_\alpha q_2[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^1} &\leq ||\chi_{|\alpha-\alpha'|<1}\bigg(\int_0^1 (t-1)(z_{\alpha\alpha}^{\epsilon,\theta} - z_{\alpha\alpha}^{\delta,\phi})(t\alpha - (1-t)\alpha')dt\bigg)||_{L^1} \\
&+ ||\chi_{|\alpha-\alpha'|>1}\frac{q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}] - (z_\alpha^{\epsilon,\theta}(\alpha') - z_\alpha^{\delta,\phi}(\alpha'))}{\alpha - \alpha'}||_{L^1} \\
&\lesssim ||z_{\alpha\alpha}^{\epsilon,\theta} - z_{\alpha\alpha}^{\delta,\phi}||_{L^\infty} + ||q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^2}||\frac{\chi_{|\alpha-\alpha'|>1}}{\alpha - \alpha'}||_{L^2} \\
&+ ||(z_\alpha^{\epsilon,\theta}(\alpha') - z_\alpha^{\delta,\phi}(\alpha'))||_{L^2}||\frac{\chi_{|\alpha-\alpha'|>1}}{\alpha - \alpha'}||_{L^2} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2
\end{aligned}
$$

as desired. The proof for $\partial_\alpha q_1$ proceeds exactly as that of $q_2 = \partial_{\alpha'} q_1$, with the only difference being α substituted for α' in some places. As such, it has not been included here. Finally, for $\partial_\alpha^2 q_1[f]$, we write

$$
\begin{aligned}
\partial_\alpha^2 q_1[f] &= \frac{(\alpha-\alpha')^2 f_{\alpha\alpha}(\alpha) - 2(\alpha-\alpha')f_\alpha(\alpha) + 2(f(\alpha) - f(\alpha'))}{(\alpha-\alpha')^3} \\
&= \int_0^1 t^2 f_{\alpha\alpha\alpha}(t\alpha + (1-t)\alpha')dt
\end{aligned}
$$

Once again, we plug in $f = z_d^{\epsilon,\theta} - z_d^{\delta,\phi}$ and separate into the regions $|\alpha - \alpha'| < 1$ and $|\alpha - \alpha'| > 1$, obtaining

$$
\begin{aligned}
\partial_\alpha^2 q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}] &= \chi_{|\alpha-\alpha'|<1}\bigg(\int_0^1 t^2(z_{\alpha\alpha\alpha}^{\epsilon,\theta} - z_{\alpha\alpha\alpha}^{\delta,\phi})(t\alpha - (1-t)\alpha')dt\bigg) \\
&+ \chi_{|\alpha-\alpha'|>1}\bigg(\frac{z_{\alpha\alpha}^{\epsilon,\theta}(\alpha) - z_{\alpha\alpha}^{\delta,\phi}(\alpha)}{\alpha-\alpha'} - \frac{2\partial_\alpha q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]}{\alpha - \alpha'}\bigg)
\end{aligned}
$$

Taking the L^1 norm and applying (6.28), we get

$$\begin{aligned}||\partial_\alpha^2 q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^1} &\leq ||\chi_{|\alpha-\alpha'|<1}\Big(\int_0^1 t^2(z_{\alpha\alpha\alpha}^{\epsilon,\theta} - z_{\alpha\alpha\alpha}^{\delta,\phi})(t\alpha-(1-t)\alpha')dt\Big)||_{L^1} \\ &+ ||\chi_{|\alpha-\alpha'|>1}\Big(\frac{z_{\alpha\alpha}^{\epsilon,\theta}(\alpha) - z_{\alpha\alpha}^{\delta,\phi}(\alpha)}{\alpha-\alpha'} - \frac{2\partial_\alpha q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]}{\alpha-\alpha'}\Big)||_{L^1} \\ &\lesssim ||\chi_{|\alpha-\alpha'|<1}||_{L^2}||z_{\alpha\alpha\alpha}^{\epsilon,\theta} - z_{\alpha\alpha\alpha}^{\delta,\phi}||_{L^2} \\ &+ ||\frac{\chi_{|\alpha-\alpha'|>1}}{\alpha-\alpha'}||_{L^2}(||q_1[z_d^{\epsilon,\theta} - z_d^{\delta,\phi}]||_{L^2} + ||z_{\alpha\alpha}^{\epsilon,\theta}(\alpha) - z_{\alpha\alpha}^{\delta,\phi}(\alpha)||_{L^2}) \\ &\lesssim ||\theta^\epsilon - \phi^\delta||_2\end{aligned}$$

as desired. □

Lemma 6.8. *Suppose that $\theta, \phi \in H^s$, both satisfying the arc-chord condition. If $f \in H^2$, then*

$$||K[z_d^{\epsilon,\theta}](f) - K[z_d^{\delta,\phi}](f)||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.30}$$

Proof of Lemma 6.8:

Now, by definition,

$$K[z_d^{\epsilon,\theta}](f) - K[z_d^{\delta,\phi}](f) = \frac{1}{2\pi i}\int -f(\alpha')\Big(\frac{q_2[z_d^{\epsilon,\theta}]}{z_\alpha^{\epsilon,\theta}(\alpha')q_1[z_d^{\epsilon,\theta}]} - \frac{q_2[z_d^{\delta,\phi}]}{z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]}\Big)d\alpha'$$

Combining terms and expanding the denominator gives us

$$K[z_d^{\epsilon,\theta}](f) - K[z_d^{\delta,\phi}](f) = \int \frac{-f(\alpha')\big(q_2[z_d^{\epsilon,\theta}]z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}] - q_2[z_d^{\delta,\phi}]z_\alpha^{\epsilon,\theta}(\alpha')q_1[z_d^{\epsilon,\theta}]\big)}{2\pi i z_\alpha^{\epsilon,\theta}(\alpha')z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\epsilon,\theta}]q_1[z_d^{\delta,\phi}]}d\alpha'$$

For ease of notation, we denote $\frac{-f(\alpha')}{2\pi i z_\alpha^{\epsilon,\theta}(\alpha')z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\epsilon,\theta}]q_1[z_d^{\delta,\phi}]}$ by $F[\alpha,\alpha']$. Since $\theta^\epsilon, \phi^\delta$ satisfy the arc-chord condition, the denominator is bounded away from 0, and so

$$||\partial_\alpha^i F[\alpha,\alpha']||_{L^2} \leq ||f||_{L^2}||\partial_\alpha^i \frac{1}{2\pi i z_\alpha^{\epsilon,\theta}(\alpha')z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\epsilon,\theta}]q_1[z_d^{\delta,\phi}]}||_{L^\infty} \lesssim 1$$

Next, we expand the term within parenthesis, obtaining

$$\int F[\alpha,\alpha']\Big((q_2[z_d^{\epsilon,\theta}]-q_2[z_d^{\delta,\phi}])z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]+q_2[z_d^{\delta,\phi}](z_\alpha^{\delta,\phi}(\alpha')-z_\alpha^{\epsilon,\theta}(\alpha'))q_1[z_d^{\delta,\phi}]$$
$$+q_2[z_d^{\delta,\phi}]z_\alpha^{\epsilon,\theta}(\alpha')(q_1[z_d^{\delta,\phi}]-q_1[z_d^{\epsilon,\theta}])\Big)d\alpha'$$

We shall denote the three terms by T_1, T_2, and T_3 respectively. To be specific, let

$$T_1=\int F[\alpha,\alpha'](q_2[z_d^{\epsilon,\theta}]-q_2[z_d^{\delta,\phi}])z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]$$
$$T_2=\int F[\alpha,\alpha']q_2[z_d^{\delta,\phi}](z_\alpha^{\delta,\phi}(\alpha')-z_\alpha^{\epsilon,\theta}(\alpha'))q_1[z_d^{\delta,\phi}]$$
$$T_3=\int F[\alpha,\alpha']q_2[z_d^{\delta,\phi}]z_\alpha^{\epsilon,\theta}(\alpha')(q_1[z_d^{\delta,\phi}]-q_1[z_d^{\epsilon,\theta}])$$

T_2 and T_3 can both be bounded immediately from Lemma 6.7,

$$||T_2||_2\le ||F||_2||q_2[z_d^{\delta,\phi}]||_2||(z_\alpha^{\delta,\phi}-z_\alpha^{\epsilon,\theta})(\alpha')q_1[z_d^{\delta,\phi}]||_2\lesssim ||\theta^\epsilon-\phi^\delta||_2 \tag{6.31}$$

$$||T_3||_2\le ||F||_2||q_2[z_d^{\delta,\phi}]||_2||z_\alpha^{\epsilon,\theta}(\alpha')(q_1[z_d^{\delta,\phi}]-q_1[z_d^{\epsilon,\theta}])||_2\lesssim ||\theta^\epsilon-\phi^\delta||_2 \tag{6.32}$$

T_1 requires a little more effort. First off, the L^2 bound again follows from Lemma 6.7,

$$||T_1||_{L^2}\le ||F||_{L^2}||(q_2[z_d^{\epsilon,\theta}]-q_2[z_d^{\delta,\phi}])||_{L^1}||z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]||_{L^\infty}\lesssim ||\theta^\epsilon-\phi^\delta||_2 \tag{6.33}$$

To bound $\partial_\alpha^2 T_1$, we first expand,

$$\partial_\alpha^2 T_1=\int (q_2[z_d^{\epsilon,\theta}]-q_2[z_d^{\delta,\phi}])\partial_\alpha^2\Big(\int F[\alpha,\alpha']z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]\Big)d\alpha'$$
$$+2\int \partial_\alpha(q_2[z_d^{\epsilon,\theta}]-q_2[z_d^{\delta,\phi}])\partial_\alpha\Big(\int F[\alpha,\alpha']z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]\Big)d\alpha'$$
$$+\int \partial_\alpha^2(q_2[z_d^{\epsilon,\theta}]-q_2[z_d^{\delta,\phi}])\Big(\int F[\alpha,\alpha']z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]\Big)d\alpha'$$

Using the fact that $q_2 = \partial_{\alpha'} q_1$, we integrate by parts, obtaining

$$\begin{aligned}
\partial_\alpha^2 T_1 = & \int (q_2[z_d^{\epsilon,\theta}] - q_2[z_d^{\delta,\phi}])\partial_\alpha^2\Big(\int F[\alpha,\alpha'] z_\alpha^{\delta,\phi}(\alpha') q_1[z_d^{\delta,\phi}]\Big) d\alpha' \\
& - 2\int \partial_\alpha(q_1[z_d^{\epsilon,\theta}] - q_1[z_d^{\delta,\phi}])\partial_\alpha\partial_{\alpha'}\Big(\int F[\alpha,\alpha'] z_\alpha^{\delta,\phi}(\alpha') q_1[z_d^{\delta,\phi}]\Big) d\alpha' \\
& - \int \partial_\alpha^2(q_1[z_d^{\epsilon,\theta}] - q_1[z_d^{\delta,\phi}])\partial_{\alpha'}\Big(\int F[\alpha,\alpha'] z_\alpha^{\delta,\phi}(\alpha') q_1[z_d^{\delta,\phi}]\Big) d\alpha'
\end{aligned}$$

And once again applying Lemma 6.7, we have

$$\begin{aligned}
||\partial_\alpha^2 T_1||_{L^2} \leq & ||(q_2[z_d^{\epsilon,\theta}] - q_2[z_d^{\delta,\phi}])||_{L^1} ||\partial_\alpha^2\Big(\int F[\alpha,\alpha'] z_\alpha^{\delta,\phi}(\alpha') q_1[z_d^{\delta,\phi}]\Big)||_{L^\infty} \\
& + 2||\partial_\alpha(q_1[z_d^{\epsilon,\theta}] - q_1[z_d^{\delta,\phi}])||_{L^1} ||\partial_\alpha\partial_{\alpha'}\Big(\int F[\alpha,\alpha'] z_\alpha^{\delta,\phi}(\alpha') q_1[z_d^{\delta,\phi}]\Big)||_{L^\infty} \\
& + ||\partial_\alpha^2(q_1[z_d^{\epsilon,\theta}] - q_1[z_d^{\delta,\phi}])||_{L^1} ||\partial_{\alpha'}\Big(\int F[\alpha,\alpha'] z_\alpha^{\delta,\phi}(\alpha') q_1[z_d^{\delta,\phi}]\Big)||_{L^\infty} \\
& \lesssim ||\theta^\epsilon - \phi^\delta||_2
\end{aligned}$$

And so, combining the above with(6.31), (6.32), and (6.33), we see that

$$||K[z_d^{\epsilon,\theta}](f) - K[z_d^{\delta,\phi}](f)||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

as desired. □

We next turn our attention to the vortex sheet strength,

Lemma 6.9. *Let $\theta^\epsilon, \phi^\delta \in H^s$ with $s \geq 6$, and each satisfying the arc-chord condition. Then,*

$$||\gamma^{\epsilon,\theta} - \gamma^{\delta,\phi}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

Proof of Lemma 6.9: Now, recall that $\gamma^{\epsilon,\theta}$ is defined by the integral equation,

$$\gamma^{\epsilon,\theta} = \tau\chi_\epsilon\theta^\epsilon_{\alpha\alpha} - R\sin(\chi_\epsilon\theta^\epsilon) - Re\big(\frac{A_\mu z^{\epsilon,\theta}_\alpha}{\pi i}\int\frac{\gamma(\alpha')}{z^{\epsilon,\theta}_\alpha(\alpha) - z^{\epsilon,\theta}_d(\alpha')}d\alpha'\big)$$

We define the integral operator $J[z_d]$ by

$$J[z_d]f = Re\big(\frac{z_\alpha}{i}\int\frac{f(\alpha')}{z_d(\alpha) - z_d(\alpha')}d\alpha'\big)$$

And so, we can rewrite the equation for $\gamma^{\epsilon,\theta}$ as

$$\big(I + \frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d]\big)\gamma^{\epsilon,\theta} = \tau\chi_\epsilon\theta^\epsilon_{\alpha\alpha} - R\sin(\chi_\epsilon\theta^\epsilon)$$

This in turn lets us rewrite the equation for $\gamma^{\epsilon,\theta} - \gamma^{\delta,\phi}$,

$$\big(I+\frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d]\big)\gamma^{\epsilon,\theta} - \big(I+\frac{A_\mu}{\pi}J[z^{\delta,\phi}_d]\big)\gamma^{\delta,\phi} = \tau\chi_\epsilon\theta^\epsilon_{\alpha\alpha} - R\sin(\chi_\epsilon\theta^\epsilon) - \tau\chi_\delta\phi^\delta_{\alpha\alpha} + R\sin(\chi_\delta\phi^\delta)$$

Expanding, we get

$$\begin{aligned}&\big(I + \frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d]\big)(\gamma^{\epsilon,\theta} - \gamma^{\delta,\phi}) + \big(\frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d] - \frac{A_\mu}{\pi}J[z^{\delta,\phi}_d]\big)(\gamma^{\delta,\phi})\\ &= \tau(\chi_\epsilon\theta^\epsilon_{\alpha\alpha} - \chi_\delta\phi^\delta_{\alpha\alpha}) - R(\sin(\chi_\epsilon\theta^\epsilon) - \sin(\chi_\delta\phi^\delta))\end{aligned}$$

Rearranging the terms, we see that

$$\begin{aligned}\big(I + \frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d]\big)(\gamma^{\epsilon,\theta} - \gamma^{\delta,\phi}) &= \tau(\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha} + \tau\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\\ &\quad - R(\sin(\chi_\epsilon\theta^\epsilon) - \sin(\chi_\delta\phi^\delta))\\ &\quad - \big(\frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d] - \frac{A_\mu}{\pi}J[z^{\delta,\phi}_d]\big)(\gamma^{\delta,\phi})\end{aligned} \tag{6.34}$$

By Lemma 5 in [1], we know that $\big(I + \frac{A_\mu}{\pi}J[z^{\epsilon,\theta}_d]\big)^{-1}$ is a bounded operator from L^2 to L^2, so it is sufficient to bound the right-hand side of (6.34). Furthermore, via

Lemmas 6.4 and 6.5, we have that

$$||\tau(\chi_\epsilon - \chi_\delta)\theta^\epsilon_{\alpha\alpha}||_{L^2} \lesssim (\epsilon + \delta)$$

$$||\tau\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

$$||R(\sin(\chi_\epsilon\theta^\epsilon) - \sin(\chi_\delta\phi^\delta))||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

Therefore, all that remains is to bound $||\big(\frac{A_\mu}{\pi}J[z_d^{\epsilon,\theta}] - \frac{A_\mu}{\pi}J[z_d^{\delta,\phi}]\big)(\gamma^{\delta,\phi})||_{L^2}$. However, we can decompose

$$\frac{1}{z_d(\alpha) - z_d(\alpha')} = \frac{1}{z_\alpha(\alpha')(\alpha - \alpha')} + \left(\frac{1}{z_d(\alpha) - z_d(\alpha')} - \frac{1}{z_\alpha(\alpha')(\alpha - \alpha')}\right)$$

Applying this to $J[z_d]$ and recalling the definition of $K[z_d]$, we have

$$J[z_d](f) = Re\big(\frac{\pi z_\alpha}{i}H(\frac{f}{z_\alpha}) + 2\pi z_\alpha K[z_d](f)\big)$$

Pulling the first term through a commutator and noting that for any real function f, $Re(\frac{\pi}{i}H(f)) = 0$, we obtain

$$J[z_d](f) = Re\big(\frac{\pi z_\alpha}{i}[H, \frac{1}{z_\alpha}](f) + 2\pi z_\alpha K[z_d](f)\big)$$

and in particular,

$$\begin{aligned}
(J[z_d^{\epsilon,\theta}] - J[z_d^{\delta,\phi}])(\gamma^{\delta,\phi}) &= Re\big(\frac{\pi z_\alpha^{\epsilon,\theta}}{i}[H, \frac{1}{z_\alpha^{\epsilon,\theta}}](\gamma^{\delta,\phi}) + 2\pi z_\alpha^{\epsilon,\theta}K[z_d^{\epsilon,\theta}](\gamma^{\delta,\phi})\big) \\
&- Re\big(\frac{\pi z_\alpha^{\delta,\phi}}{i}[H, \frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\delta,\phi}) + 2\pi z_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\delta,\phi})\big) \\
&\leq |\pi(z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})[H, \frac{1}{z_\alpha^{\epsilon,\theta}}](\gamma^{\delta,\phi})| + |\pi z_\alpha^{\delta,\phi}[H, \frac{1}{z_\alpha^{\epsilon,\theta}} - \frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\delta,\phi})| \\
&+ |2\pi(z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})K[z_d^{\epsilon,\theta}](\gamma^{\delta,\phi})| \\
&+ |2\pi z_\alpha^{\delta,\phi}(K[z_d^{\epsilon,\theta}] - K[z_d^{\delta,\phi}])(\gamma^{\delta,\phi})|
\end{aligned}$$

Taking the L^2 norm and applying Lemmas 6.6 and 6.8, we get

$$\begin{aligned}||(J[z_d^{\epsilon,\theta}] - J[z_d^{\delta,\phi}])(\gamma^{\delta,\phi})||_{L^2} &\lesssim ||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_{L^2} + ||[H, \frac{1}{z_\alpha^{\epsilon,\theta}} - \frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\delta,\phi})||_{L^2} \\ &+ ||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_{L^2} + ||(K[z_d^{\epsilon,\theta}] - K[z_d^{\delta,\phi}])(\gamma^{\delta,\phi})||_{L^2} \\ &\lesssim ||\theta^\epsilon - \phi^\delta||_2\end{aligned}$$

Therefore, $||\big(\frac{A_\mu}{\pi}J[z_d^{\epsilon,\theta}] - \frac{A_\mu}{\pi}J[z_d^{\delta,\phi}]\big)(\gamma^{\delta,\phi})||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2$, as desired. □

With Lipschitz bounds on $\gamma, K[z_d]$, and $[H, \cdot]$, we now have the necessary tools for bounding W and m.

Lemma 6.10. *Let $\theta^\epsilon, \phi^\delta \in H^s$ with $s \geq 6$, and each satisfying the arc-chord condition. Then,*

$$||m^{\epsilon,\theta} - m^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

Proof of Lemma 6.10: We begin with $m^{\epsilon,\theta} - m^{\delta,\phi}$. Now,

$$\begin{aligned}\Phi(m^{\epsilon,\theta} - m^{\delta,\phi})^* &= (z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})K[z_d^{\epsilon,\theta}]\big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &+ z_\alpha^{\delta,\phi}\big(K[z_d^{\epsilon,\theta}] - K[z_d^{\delta,\phi}]\big)\big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &+ z_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}]\big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha - (\frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})_\alpha\big) \\ &+ \frac{z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\epsilon,\theta})^2}]\big(z_\alpha^{\epsilon,\theta}(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &+ \frac{z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\epsilon,\theta})^2} - \frac{1}{(z_\alpha^{\delta,\phi})^2}]\big(z_\alpha^{\epsilon,\theta}(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &+ \frac{z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\delta,\phi})^2}]\big((z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &+ \frac{z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\delta,\phi})^2}]\big(z_\alpha^{\delta,\phi}((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha - (\frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})_\alpha)\big)\end{aligned} \tag{6.35}$$

For simplicity of notation, we write

$$\Phi(m^{\epsilon,\theta} - m^{\delta,\phi})^* = \sum_{i=1}^{7} T_i$$

where T_i represents the ith term in (6.35). Now, T_1 can be bounded via Lemma 6.5,

$$||T_1||_2 \leq ||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_2 ||K[z_d^{\epsilon,\theta}]\big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big)||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.36}$$

T_2 follows from Lemma 6.8,

$$||T_2||_2 \leq ||z_\alpha^{\delta,\phi}\big(K[z_d^{\epsilon,\theta}] - K[z_d^{\delta,\phi}]\big)\big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big)||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 \tag{6.37}$$

For T_3, we expand the integral and integrate by parts,

$$\begin{aligned} T_3 &= \frac{-z_\alpha^{\epsilon,\theta}(\alpha)}{2\pi i}\int \big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}(\alpha')})_{\alpha'} - (\frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}(\alpha')})_{\alpha'}\big)\frac{q_2[z_d^{\delta,\phi}]}{z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]}d\alpha' \\ &= \frac{z_\alpha^{\epsilon,\theta}(\alpha)}{2\pi i}\int \big(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}(\alpha')} - \frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}(\alpha')}\big)\partial_{\alpha'}\frac{q_2[z_d^{\delta,\phi}]}{z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]}d\alpha' \end{aligned}$$

However, as in the proof of local existence, for $0 \leq j \leq 2$, $\partial_\alpha^j \partial_{\alpha'} q_2[z_d^{\delta,\phi}] \in L^1$ and $\partial_{\alpha'}(z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]) \in H^2$, therefore $||\partial_\alpha^j \frac{q_2[z_d^{\delta,\phi}]}{z_\alpha^{\delta,\phi}(\alpha')q_1[z_d^{\delta,\phi}]}||_{L^1}$ is bounded, and therefore

$$\begin{aligned} ||T_3||_2 &\lesssim ||z_\alpha^{\epsilon,\theta}||_2 ||\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}} - \frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}}||_{L^2} \\ &\lesssim ||\frac{z_\alpha^{\delta,\phi}\gamma^{\epsilon,\theta} - z_\alpha^{\epsilon,\theta}\gamma^{\delta,\phi}}{z_\alpha^{\epsilon,\theta} z_\alpha^{\delta,\phi}}||_{L^2} \\ &\lesssim ||z_\alpha^{\delta,\phi}\gamma^{\epsilon,\theta} - z_\alpha^{\epsilon,\theta}\gamma^{\delta,\phi}||_{L^2} \\ &\lesssim ||(z_\alpha^{\delta,\phi} - z_\alpha^{\epsilon,\theta})\gamma^{\epsilon,\theta} + z_\alpha^{\epsilon,\theta}(\gamma^{\epsilon,\theta} - \gamma^{\delta,\phi})||_{L^2} \end{aligned}$$

And so, applying Lemmas 6.5 and 6.9, we have

$$||T_3||_2 \lesssim ||z_\alpha^{\delta,\phi} - z_\alpha^{\epsilon,\theta}||_{L^2} + ||\gamma^{\epsilon,\theta} - \gamma^{\delta,\phi}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta) \tag{6.38}$$

T_4 can be immediately bounded via Lemma 6.5,

$$||T_4||_2 = ||\frac{z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\epsilon,\theta})^2}]\big(z_\alpha^{\epsilon,\theta}(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big)||_2 \lesssim ||z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 \quad (6.39)$$

For T_5, we note that $[H, f](g) = H(fg) - fH(g)$ and write

$$\begin{aligned} T_5 &= \frac{z_\alpha^{\delta,\phi}}{2i} H\big((\frac{1}{(z_\alpha^{\epsilon,\theta})^2} - \frac{1}{(z_\alpha^{\delta,\phi})^2}) z_\alpha^{\epsilon,\theta}(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &\quad - \frac{z_\alpha^{\delta,\phi}}{2i}(\frac{1}{(z_\alpha^{\epsilon,\theta})^2} - \frac{1}{(z_\alpha^{\delta,\phi})^2}) H\big(z_\alpha^{\epsilon,\theta}(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big) \\ &\lesssim ||\frac{1}{(z_\alpha^{\epsilon,\theta})^2} - \frac{1}{(z_\alpha^{\delta,\phi})^2}||_2 \\ &\lesssim ||\frac{(z_\alpha^{\delta,\phi})^2 - (z_\alpha^{\epsilon,\theta})^2}{(z_\alpha^{\delta,\phi})^2 (z_\alpha^{\epsilon,\theta})^2}||_2 \\ &\lesssim ||\frac{(z_\alpha^{\delta,\phi} + z_\alpha^{\epsilon,\theta})(z_\alpha^{\delta,\phi} - z_\alpha^{\epsilon,\theta})}{(z_\alpha^{\delta,\phi})^2 (z_\alpha^{\epsilon,\theta})^2}||_2 \\ &\lesssim ||z_\alpha^{\delta,\phi} - z_\alpha^{\epsilon,\theta}||_2 \\ &\lesssim ||\theta^\epsilon - \phi^\delta||_2 \end{aligned} \quad (6.40)$$

T_6 is an immediate consequence of Lemma 6.6,

$$\begin{aligned} ||T_6||_2 &= ||\frac{z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\delta,\phi})^2}]\big((z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha\big)||_2 \\ &\lesssim ||(z_\alpha^{\epsilon,\theta} - z_\alpha^{\delta,\phi})(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha||_{L^2} \\ &\lesssim ||\theta^\epsilon - \phi^\delta||_2 \end{aligned} \quad (6.41)$$

Finally, T_7 is dealt with similarly to T_3, via expanding the integral and integrating by parts.

$$T_7 = \frac{z_\alpha^{\delta,\phi}}{2i}[H, \frac{1}{(z_\alpha^{\delta,\phi})^2}]\big(z_\alpha^{\delta,\phi}((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_\alpha - (\frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})_\alpha)\big)$$

$$= \frac{z_\alpha^{\delta,\phi}(\alpha)}{2\pi i}\int q_1[\frac{1}{(z_\alpha^{\delta,\phi})^2}]z_\alpha^{\delta,\phi}(\alpha')\big((\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})_{\alpha'} - (\frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})_{\alpha'}\big)d\alpha'$$

$$= \frac{z_\alpha^{\delta,\phi}(\alpha)}{2\pi i}\int (\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}} - \frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})\partial_{\alpha'}\big(q_1[\frac{1}{(z_\alpha^{\delta,\phi})^2}]z_\alpha^{\delta,\phi}(\alpha')\big)d\alpha'$$

Once again, we use the fact that for $0 \leq i \leq 2$, $\partial_\alpha^i \partial_{\alpha'} q_1[\frac{1}{(z_\alpha^{\delta,\phi})^2}] \in L^1$, and $\partial_{\alpha'} z_\alpha^{\delta,\phi} \in H^2$, and therefore

$$T_7 \lesssim ||\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}} - \frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta) \tag{6.42}$$

And so, combining equations (6.36) through (6.42), we have that

$$||m^{\epsilon,\theta} - m^{\delta,\phi}||_2 \lesssim ||\theta - \phi||_2 + (\epsilon + \delta) \tag{6.43}$$

as desired. □

Lemma 6.11. *Let $\theta^\epsilon, \phi^\delta \in H^s$ with $s \geq 6$, and each satisfying the arc-chord condition. Then,*

$$||W^{\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta} - W^{\delta,\phi} \cdot \hat{t}^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

$$||\tilde{W}^{\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta} - \tilde{W}^{\delta,\phi} \cdot \hat{n}^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

$$||W^{\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta} - W^{\delta,\phi} \cdot \hat{n}^{\delta,\phi}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

Proof of Lemma 6.11: Now, recall that

$$W^{\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta} = Re\big(\frac{z_\alpha^{\epsilon,\theta}}{2i}H(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})\big) + Re(z_\alpha^{\epsilon,\theta}K[z_d^{\epsilon,\theta}](\gamma^{\epsilon,\theta}))$$

$$= Re\big(\frac{z_\alpha^{\epsilon,\theta}}{2i}[H, \frac{1}{z_\alpha^{\epsilon,\theta}}](\gamma^{\epsilon,\theta})\big) + Re(z_\alpha^{\epsilon,\theta}K[z_d^{\epsilon,\theta}](\gamma^{\epsilon,\theta}))$$

Therefore, we have that

$$
\begin{aligned}
(W\cdot\hat{t})^{\epsilon,\theta}-(W\cdot\hat{t})^{\delta,\phi} &= Re\big(\frac{z_\alpha^{\epsilon,\theta}}{2i}[H,\frac{1}{z_\alpha^{\epsilon,\theta}}](\gamma^{\epsilon,\theta})\big)+Re(z_\alpha^{\epsilon,\theta}K[z_d^{\epsilon,\theta}](\gamma^{\epsilon,\theta}))\\
&-Re\big(\frac{z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\delta,\phi})\big)-Re(z_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\delta,\phi}))\\
&= Re\big(\frac{z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\epsilon,\theta}}](\gamma^{\epsilon,\theta})\big)+Re\big(\frac{z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\epsilon,\theta}}-\frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\epsilon,\theta})\big)\\
&+Re\big(\frac{z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi})\big)+Re((z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi})K[z_d](\gamma^{\epsilon,\theta}))\\
&+Re(z_\alpha^{\delta,\phi}K[z_d^{\epsilon,\theta}-z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}))+Re(z_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}))\\
&=\sum_{i=1}^{6}T_i
\end{aligned}
$$

Where T_i denotes the ith term of $(W\cdot\hat{t})^{\epsilon,\theta}-(W\cdot\hat{t})^{\delta,\phi}$. Now,

$$
\begin{aligned}
||T_1||_2 &= ||Re\big(\frac{z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\epsilon,\theta}}](\gamma^{\epsilon,\theta})\big)||_2\lesssim||z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2\\
||T_2||_2 &= ||Re\big(\frac{z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\epsilon,\theta}}-\frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\epsilon,\theta})\big)||_2\lesssim||\frac{1}{z_\alpha^{\epsilon,\theta}}-\frac{1}{z_\alpha^{\delta,\phi}}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2\\
||T_3||_2 &= ||Re\big(\frac{z_\alpha^{\delta,\phi}}{2i}[H,\frac{1}{z_\alpha^{\delta,\phi}}](\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi})\big)||_2\lesssim||\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}||_{L^2}\lesssim||\theta^\epsilon-\phi^\delta||_2+(\epsilon+\delta)\\
||T_4||_2 &= ||Re((z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi})K[z_d](\gamma^{\epsilon,\theta}))||_2\lesssim||z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2\\
||T_5||_2 &= ||Re(z_\alpha^{\delta,\phi}K[z_d^{\epsilon,\theta}-z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}))||_2\lesssim||K[z_d^{\epsilon,\theta}-z_d^{\delta,\phi}](\gamma^{\epsilon,\theta})||_2\lesssim||\theta^\epsilon-\phi^\delta||_2\\
||T_6||_2 &= ||Re(z_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}))||_2\lesssim||\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}||_{L^2}\lesssim||\theta^\epsilon-\phi^\delta||_2+(\epsilon+\delta)
\end{aligned}
$$

And combining these bounds, we see that

$$
||W^{\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta}-W^{\delta,\phi}\cdot\hat{t}^{\delta,\phi}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2+(\epsilon+\delta) \tag{6.44}
$$

For $W\cdot\hat{n}$, we have

$$
W\cdot\hat{n}=Re\big(\frac{z_\alpha}{2}H(\frac{\gamma}{z_\alpha})\big)+Re(iz_\alpha K[z_d](\gamma))
$$

And so

$$\begin{aligned}
(W\cdot\hat{n})^{\epsilon,\theta}-(W\cdot\hat{n})^{\delta,\phi} &= Re\big(\frac{z_\alpha^{\epsilon,\theta}}{2}H(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})\big)+Re(iz_\alpha^{\epsilon,\theta}K[z_d^{\epsilon,\theta}](\gamma^{\epsilon,\theta}))\\
&- Re\big(\frac{z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})\big)-Re(iz_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\delta,\phi}))\\
&= Re\big(\frac{z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})\big)+Re\big(\frac{z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}}-\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\delta,\phi}})\big)\\
&+ Re\big(\frac{z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})\big)+Re(i(z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi})K[z_d](\gamma^{\epsilon,\theta}))\\
&+ Re(iz_\alpha^{\delta,\phi}K[z_d^{\epsilon,\theta}-z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}))+Re(iz_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}))\\
&= \sum_{i=7}^{12}T_i
\end{aligned}$$

Note that we can obtain the equation for $W^{st}\cdot\hat{n}$ or $\tilde{W}\cdot\hat{n}$ simply by replacing γ with γ^{st} or $\tilde{\gamma}$ respectively. Now, T_{10},T_{11}, and T_{12} can be bounded in exactly the same way as T_4,T_5, and T_6 were, as

$$||T_{10}||_2=||Re(i(z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi})K[z_d](\gamma^{\epsilon,\theta}))||_2\lesssim||z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2$$

$$||T_{11}||_2=||Re(iz_\alpha^{\delta,\phi}K[z_d^{\epsilon,\theta}-z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}))||_2\lesssim||K[z_d^{\epsilon,\theta}-z_d^{\delta,\phi}](\gamma^{\epsilon,\theta})||_2\lesssim||\theta^\epsilon-\phi^\delta||_2$$

$$||T_{12}||_2=||Re(iz_\alpha^{\delta,\phi}K[z_d^{\delta,\phi}](\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}))||_2\lesssim||\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}||_{L^2}\lesssim||\theta^\epsilon-\phi^\delta||_2+(\epsilon+\delta)$$

As it turns out, T_7 and T_8 can also be bounded analogously to T_1 and T_2,

$$||T_7||_2=||Re\big(\frac{z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}})\big)||_2\lesssim||z_\alpha^{\epsilon,\theta}-z_\alpha^{\delta,\phi}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2$$
$$||T_8||_2=||Re\big(\frac{z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\epsilon,\theta}}-\frac{\gamma^{\epsilon,\theta}}{z_\alpha^{\delta,\phi}})\big)||_2\lesssim||\frac{1}{z_\alpha^{\epsilon,\theta}}-\frac{1}{z_\alpha^{\delta,\phi}}||_2\lesssim||\theta^\epsilon-\phi^\delta||_2$$

Finally, the Sobolev norm of T_9 cannot be bounded directly due to the $\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}$ term without a non-singular integral to block derivatives. For the L^2 norm, we obtain

$$||T_9||_{L^2}=||Re\big(\frac{z_\alpha^{\delta,\phi}}{2}H(\frac{\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}}{z_\alpha^{\delta,\phi}})\big)||_{L^2}\lesssim||\gamma^{\epsilon,\theta}-\gamma^{\delta,\phi}||_{L^2}\lesssim||\theta^\epsilon-\phi^\delta||_2+(\epsilon+\delta)$$

and combining this with bounds on the other T_i, we get

$$||W^{\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta} - W^{\delta,\phi} \cdot \hat{n}^{\delta,\phi}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta) \tag{6.45}$$

Furthermore, if we consider the non-surface tension part of T_9, we obtain

$$||\tilde{T}_9||_2 = ||Re\big(\frac{z_\alpha^\phi}{2} H(\frac{\tilde{\gamma}^\theta - \tilde{\gamma}^\phi}{z_\alpha^\phi})\big)||_2 \lesssim ||\tilde{\gamma}^\theta - \tilde{\gamma}^\phi||_2 \lesssim ||\theta - \phi||_2 + (\epsilon + \delta)$$

and again, combining this with bounds on the other $\tilde{T}_i$ (which can be obtained in the exact same way as the bounds on the original T_i), we get our final bound of

$$||\tilde{W}^{\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta} - \tilde{W}^{\delta,\phi} \cdot \hat{n}^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta) \tag{6.46}$$

concluding the proof. □

With these lemmas complete, we are now prepared to bound the remainder terms in $\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}, \Upsilon_{21}$, and Υ_{22}. To be specific,

Lemma 6.12. *Let $\theta, \phi \in H^s$, both satisfying the arc-chord condition. Then,*

$$||\Upsilon_{11}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

$$||\Upsilon_{12}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

$$||\Upsilon_{21}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

$$||\Upsilon_{13}||_2 \lesssim (\epsilon + \delta)$$

$$||\Upsilon_{22}||_{L^2} \lesssim (\epsilon + \delta)$$

Proof of Lemma 6.12:

We will begin with Υ_{11}. Recall from (6.15) that

$$\begin{aligned}\Upsilon_{11} &= H\big((k^{\epsilon,\theta} - k^{\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big) + [H, k^{\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha))\\ &- \tau A_\mu[H, U^{st,\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)) - \frac{A_\mu}{2}(\tilde{\gamma}^{\epsilon,\theta} - \tilde{\gamma}^{\delta,\phi})(\chi_\delta\theta^\epsilon_\alpha)\\ &+ B_5 + B_6\\ &= \sum_{i=1}^{6} \Xi_i\end{aligned}$$

where Ξ_i represents the ith term in (6.15). Now, $k^{\epsilon,\theta} = \frac{-R\cos(\chi_\epsilon\theta^\epsilon)}{2} - A_\mu\tilde{W}^{\epsilon,\theta}\cdot\hat{n}^{\epsilon,\theta}$, and so by Lemmas 6.5 and 6.11, $||k^{\epsilon,\theta} - k^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon+\delta)$. Therefore,

$$||\Xi_1||_2 = ||H\big((k^{\epsilon,\theta} - k^{\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big)||_2 \le ||k^{\epsilon,\theta} - k^{\delta,\phi}||_2||\chi_\delta\theta^\epsilon_\alpha||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon+\delta)$$

Ξ_2 and Ξ_3 can be dealt with via Lemma 6.6,

$$||\Xi_2||_2 = ||[H, k^{\delta,\phi}](\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha))||_2 \lesssim ||\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

$$||\Xi_3||_2 = ||\tau A_\mu[H, U^{st,\delta,\phi}](\theta^\epsilon_\alpha - \phi^\delta_\alpha)||_2 \lesssim ||\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

Ξ_4 is immediate from Lemma 6.11

$$||\Xi_4||_2 = ||\frac{A_\mu}{2}(\tilde{\gamma}^{\epsilon,\theta} - \tilde{\gamma}^{\delta,\phi})(\chi_\delta\theta^\epsilon_\alpha)||_2 \lesssim ||\tilde{\gamma}^{\epsilon,\theta} - \tilde{\gamma}^{\delta,\phi}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon+\delta)$$

Finally, Ξ_5 and Ξ_6 both follow from Lemma 6.10.

$$\begin{aligned}||\Xi_5||_2 &= ||m^{\epsilon,\theta}\cdot\hat{n}^{\epsilon,\theta} - m^{\delta,\phi}\cdot\hat{n}^{\delta,\phi}||_2\\ &\le ||(m^{\epsilon,\theta} - m^{\delta,\phi})\cdot\hat{n}^{\epsilon,\theta}||_2 + ||m^{\delta,\phi}\cdot(\hat{n}^{\epsilon,\theta} - \hat{n}^{\delta,\phi})||_2\\ &\le ||(m^{\epsilon,\theta} - m^{\delta,\phi})||_2 + ||m^{\delta,\phi}||_2||\hat{n}^{\epsilon,\theta} - \hat{n}^{\delta,\phi}||_2\\ &\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon+\delta)\end{aligned}$$

$$
\begin{aligned}
||\Xi_6||_2 &= || - A_\mu H(m^{\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta} - m^{\delta,\phi} \cdot \hat{t}^{\delta,\phi})||_2 \\
&\leq A_\mu||(m^{\epsilon,\theta} - m^{\delta,\phi}) \cdot \hat{t}^{\epsilon,\theta}||_2 + A_\mu||m^{\delta,\phi} \cdot (\hat{t}^{\epsilon,\theta} - \hat{t}^{\delta,\phi})||_2 \\
&\leq A_\mu||(m^{\epsilon,\theta} - m^{\delta,\phi})||_2 + A_\mu||m^{\delta,\phi}||_2||\hat{t}^{\epsilon,\theta} - \hat{t}^{\delta,\phi}||_2 \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)
\end{aligned}
$$

And therefore, combining the above equations, we see that

$$
||\Upsilon_{11}||_2 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta) \tag{6.47}
$$

Next we consider Υ_{12}. Recall from (6.16) that

$$
\begin{aligned}
\Upsilon_{12} = -\tau A_\mu H\big[&(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big] \\
&+ \big[(V - W \cdot \hat{t})^{\epsilon,\theta} - (V - W \cdot \hat{t})^{\delta,\phi}\big](\chi_\delta\theta^\epsilon_\alpha)
\end{aligned}
$$

Regarding the first term, we can apply Lemma 6.11

$$
\begin{aligned}
||\tau A_\mu H\big[(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta\theta^\epsilon_\alpha\big]||_{L^2} &\lesssim ||U^{st,\epsilon,\theta} - U^{st,\delta,\phi}||_{L^2}||\chi_\delta\theta^\epsilon_\alpha||_{L^\infty} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)
\end{aligned} \tag{6.48}
$$

And for the second, recall that

$$
V^{\epsilon,\theta} = \int \theta^\epsilon_\alpha(W^{\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta})
$$

$$
V^{\epsilon,\theta} - V^{\delta,\phi} = \int (\theta^\epsilon_\alpha - \phi^\delta_\alpha)(W \cdot \hat{n})^{\epsilon,\theta} + \phi^\delta_\alpha\big((W \cdot \hat{n})^{\epsilon,\theta} - (W \cdot \hat{n})^{\delta,\phi}\big)
$$

Therefore, we have that

$$
\begin{aligned}
||V^{\epsilon,\theta} - V^{\delta,\phi}||_{L^\infty} &\lesssim ||\theta^\epsilon_\alpha - \phi^\delta_\alpha||_{L^2}||(W \cdot \hat{n})^{\epsilon,\theta}||_{L^2} \\
&\quad + ||\phi^\delta_\alpha||_{L^2}||(W \cdot \hat{n})^{\epsilon,\theta} - (W \cdot \hat{n})^{\delta,\phi}||_{L^2} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)
\end{aligned} \tag{6.49}
$$

This gives us the desired bound,

$$
\begin{aligned}
&||\big[(V - W \cdot \hat{t})^{\epsilon,\theta} - (V - W \cdot \hat{t})^{\delta,\phi}\big](\chi_\delta \theta^\epsilon_\alpha)||_{L^2} \\
&\lesssim ||(V - W \cdot \hat{t})^{\epsilon,\theta} - (V - W \cdot \hat{t})^{\delta,\phi}||_{L^\infty} ||\chi_\delta \theta^\epsilon_\alpha||_{L^2} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)
\end{aligned}
\tag{6.50}
$$

And again, combining (6.48) and (6.50), we obtain

$$
||\Upsilon_{12}||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta) \tag{6.51}
$$

Finally, recall from (6.21) that

$$
\begin{aligned}
\Upsilon_{21} &= -\tau A_\mu H\big[(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha}\big] \\
&- \tau A_\mu [H, \chi_\delta \theta^\epsilon_{\alpha\alpha}]\big(\chi_\delta H(\theta_{\alpha\alpha\alpha} - \phi_{\alpha\alpha\alpha})\big) \\
&- \frac{\tau}{2} A_\mu [H, \chi_\delta \theta^\epsilon_\alpha]\big(\chi_\delta H(\theta^\epsilon_{\alpha\alpha\alpha\alpha} - \phi^\delta_{\alpha\alpha\alpha\alpha})\big) \\
&+ \tau A_\mu \partial_\alpha H\bigg(\Big[\theta^\epsilon_\alpha (W^{st,\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta}) - \phi^\delta_\alpha (W^{st,\delta,\phi} \cdot \hat{t}^{\delta,\phi})\Big]\chi_\delta \theta^\epsilon_\alpha\bigg) \\
&+ \tau A_\mu \partial_\alpha H\bigg(\Big[- m^{st,\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta} + m^{st,\delta,\phi} \cdot \hat{n}^{\delta,\phi}\Big]\chi_\delta \theta^\epsilon_\alpha\bigg) \\
&+ (\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha})\big[(V - W \cdot \hat{t})^{\epsilon,\theta} - (V - W \cdot \hat{t})^{\delta,\phi}\big] \\
&+ \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})[H, \theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})) \\
&+ \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})(\chi_\delta \phi^\delta_{\alpha\alpha}) H(\theta^\epsilon_\alpha - \phi^\delta_\alpha) + \tau(\chi_\delta \theta^\epsilon_{\alpha\alpha})[H, \chi_\delta \phi^\delta_{\alpha\alpha}](\theta^\epsilon_\alpha - \phi^\delta_\alpha) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \theta^\epsilon_{\alpha\alpha}](\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)(\chi_\delta \phi^\delta_{\alpha\alpha}) H(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}) + \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \chi_\delta \phi^\delta_{\alpha\alpha}](\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha\alpha} - \phi^\delta_{\alpha\alpha\alpha})) \\
&+ \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)(\chi_\delta \phi^\delta_{\alpha\alpha\alpha}) H(\theta^\epsilon_\alpha - \phi^\delta_\alpha) + \frac{\tau}{2}(\chi_\delta \theta^\epsilon_\alpha)[H, \chi_\delta \phi^\delta_{\alpha\alpha\alpha}](\theta^\epsilon_\alpha - \phi^\delta_\alpha) \\
&+ \partial_\alpha\bigg((\chi_\delta \theta^\epsilon_\alpha)\Big[(\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - m^{\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta}) - (\frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha - m^{\delta,\phi} \cdot \hat{t}^{\delta,\phi})\Big]\bigg) + \Upsilon_{11,\alpha\alpha} \\
&= \Upsilon_{11,\alpha\alpha} + \sum_{i=7}^{18} \Xi_i
\end{aligned}
$$

Where again, each Ξ_i corresponds to a single term in the equation for Υ_{21}. Now,

Ξ_7 is immediate from Lemma 6.11,

$$\begin{aligned}
||\Xi_7||_{L^2} &= || - \tau A_\mu H\big[(U^{st,\epsilon,\theta} - U^{st,\delta,\phi})\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha}\big]||_{L^2} \\
&\lesssim ||U^{st,\epsilon,\theta} - U^{st,\epsilon,\phi}||_{L^2} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)
\end{aligned}$$

Both Ξ_8 and Ξ_9 require Lemma 6.6,

$$\begin{aligned}
||\Xi_8||_{L^2} &= || - \tau A_\mu [H, \chi_\delta \theta^\epsilon_{\alpha\alpha}]\big(\partial_\alpha \chi_\delta H(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\big)||_{L^2} \\
&\lesssim ||\chi_\delta H(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})||_{L^2} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 \\
||\Xi_9||_{L^2} &= || - \tau A_\mu [H, \chi_\delta \theta^\epsilon_{\alpha}]\big(\partial^2_\alpha \chi_\delta H(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\big)||_{L^2} \\
&\lesssim ||\chi_\delta H(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})||_{L^2} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2
\end{aligned}$$

For Ξ_{10}, we have

$$\begin{aligned}
||\Xi_{10}||_{L^2} &\lesssim ||\Big[\theta^\epsilon_\alpha (W^{st,\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta}) - \phi^\delta_\alpha (W^{st,\delta,\phi} \cdot \hat{t}^{\delta,\phi})\Big]\chi_\delta \theta^\epsilon_\alpha||_1 \\
&\lesssim ||\chi_\delta \theta^\epsilon_\alpha||_1 \big(||\theta^\epsilon_\alpha - \phi^\delta_\alpha||_1 ||W^{st,\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta}||_1 + ||\phi^\delta_\alpha||_1 ||W^{st,\epsilon,\theta} \cdot \hat{t}^{\epsilon,\theta} - W^{st,\delta,\phi} \cdot \hat{t}^{\delta,\phi}||_1\big) \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)
\end{aligned}$$

For Ξ_{11}, we use Lemma 6.10,

$$||\Xi_{11}||_{L^2} \lesssim ||\Big[- m^{st,\epsilon,\theta} \cdot \hat{n}^{\epsilon,\theta} + m^{st,\delta,\phi} \cdot \hat{n}^{\delta,\phi}\Big]\chi_\delta \theta^\epsilon_\alpha||_1 \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

For Ξ_{12}, we apply Lemma 6.11 and (6.49), obtaining

$$||\Xi_{12}||_{L^2} \lesssim ||\chi_\delta \theta^\epsilon_{\alpha\alpha\alpha}||_{L^2} ||(V - W \cdot \hat{t})^{\epsilon,\theta} - (V - W \cdot \hat{t})^{\delta,\phi}||_{L^\infty} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

The next nine Ξ_i are either consequences of Lemma 6.6 or can be bounded directly,

$$||\Xi_{13}||_{L^2} = ||\tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})[H,\theta^\epsilon_\alpha](\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}))||_{L^2} \lesssim ||\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{14}||_{L^2} = ||\tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})(\chi_\delta\phi^\delta_{\alpha\alpha})H(\theta^\epsilon_\alpha-\phi^\delta_\alpha)||_{L^2} \lesssim ||\theta^\epsilon_\alpha-\phi^\delta_\alpha||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{15}||_{L^2} = ||\tau(\chi_\delta\theta^\epsilon_{\alpha\alpha})[H,\chi_\delta\phi^\delta_{\alpha\alpha}](\theta^\epsilon_\alpha-\phi^\delta_\alpha)||_{L^2} \lesssim ||\theta^\epsilon_\alpha-\phi^\delta_\alpha||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{16}||_{L^2} = ||\frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\theta^\epsilon_{\alpha\alpha}](\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}))||_{L^2} \lesssim ||\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{17}||_{L^2} = ||\frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)(\chi_\delta\phi^\delta_{\alpha\alpha})H(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})||_{L^2} \lesssim ||\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{18}||_{L^2} = ||\frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\chi_\delta\phi^\delta_{\alpha\alpha}](\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})||_{L^2} \lesssim ||\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{19}||_{L^2} = ||\frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\theta^\epsilon_\alpha](\partial_\alpha\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha}))||_{L^2} \lesssim ||\chi_\delta(\theta^\epsilon_{\alpha\alpha}-\phi^\delta_{\alpha\alpha})||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{20}||_{L^2} = ||\frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)(\chi_\delta\phi^\delta_{\alpha\alpha\alpha})H(\theta^\epsilon_\alpha-\phi^\delta_\alpha)||_{L^2} \lesssim ||\theta^\epsilon_\alpha-\phi^\delta_\alpha||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

$$||\Xi_{21}||_{L^2} = ||\frac{\tau}{2}(\chi_\delta\theta^\epsilon_\alpha)[H,\chi_\delta\phi^\delta_{\alpha\alpha\alpha}](\theta^\epsilon_\alpha-\phi^\delta_\alpha)||_{L^2} \lesssim ||\theta^\epsilon_\alpha-\phi^\delta_\alpha||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2$$

Finally, Ξ_{22} is a consequence of Lemmas 6.9 and 6.10,

$$\begin{aligned}
||\Xi_{22}||_{L^2} &\lesssim ||(\chi_\delta\theta^\epsilon_\alpha)\Big[(\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - m^{\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta}) - (\frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha - m^{\delta,\phi}\cdot\hat{t}^{\delta,\phi})\Big]||_1 \\
&\lesssim ||\chi_\delta\theta^\epsilon_\alpha||_1\big(||\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - \frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha||_1 + ||m^{\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta} - m^{\delta,\phi}\cdot\hat{t}^{\delta,\phi}||_1\big) \\
&\lesssim ||\frac{1}{2}\tilde{\gamma}^{\epsilon,\theta}\theta^\epsilon_\alpha - \frac{1}{2}\tilde{\gamma}^{\delta,\phi}\phi^\delta_\alpha||_1 + ||m^{\epsilon,\theta}\cdot\hat{t}^{\epsilon,\theta} - m^{\delta,\phi}\cdot\hat{t}^{\delta,\phi}||_1 \\
&\lesssim ||\tilde{\gamma}^{\epsilon,\theta}-\tilde{\gamma}^{\delta,\phi}||_1||\theta^\epsilon_\alpha||_1 + ||\tilde{\gamma}^{\delta,\phi}||_1||\theta^\epsilon_\alpha-\phi^\delta_\alpha||_1 \\
&+ ||(m^{\epsilon,\theta}-m^{\delta,\phi})\cdot\hat{t}^{\epsilon,\theta}||_1 + ||m^{\delta,\phi}\cdot(\hat{t}^{\epsilon,\theta}-\hat{t}^{\delta,\phi})||_1 \\
&\lesssim ||\theta^\epsilon-\phi^\delta||_2 + (\epsilon+\delta)
\end{aligned}$$

Therefore, combining the bounds on all the Ξ_i with (6.47), we get

$$||\Upsilon_{21}||_{L^2} \lesssim ||\theta^\epsilon-\phi^\delta||_2 + (\epsilon+\delta) \tag{6.52}$$

as desired. Finally, the bounds for Υ_{13} and Υ_{22} are immediate consequences of Lemma 6.4, as every term contains a $(\chi_\epsilon - \chi_\delta)$ factor. □

Our last lemma before the proof of Theorem 1.2 is mainly for convenience, proving that every non-dissipative term in $(\theta-\phi)_t$'s L^2 norm can be conveniently bounded by $||\theta-\phi||_2$.

Lemma 6.13. *Let $\theta, \phi \in H^s$, both satisfying the arc-chord condition. Then,*

$$||(\theta^\epsilon - \phi^\delta)_t + \frac{\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta)||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon+\delta)$$

Proof of Lemma 6.13:

Now, recall from (6.3) that

$$\begin{aligned}(\theta^\epsilon - \phi^\delta)_t + \frac{\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta) &= \chi_\delta\Big[\tau\Upsilon_8\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\Big] \\ &+ \chi_\delta\Big[\Upsilon_9\chi_\delta\Lambda(\theta^\epsilon - \phi^\delta) + \Upsilon_{10}\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)\Big] \\ &+ \chi_\delta\Big[\Upsilon_{11} + \Upsilon_{12} + \Upsilon_{13}\Big] + (\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_t)\end{aligned}$$

Then, since Υ_8, Υ_9, and Υ_{10} are bounded by a constant, it's clear that

$$||\chi_\delta\big[\tau\Upsilon_8\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}) + \Upsilon_9\chi_\delta\Lambda(\theta^\epsilon - \phi^\delta) + \Upsilon_{10}\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)\big]||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2$$

And by Lemma 6.12, we know that

$$||\chi_\delta\big[\Upsilon_{11} + \Upsilon_{12} + \Upsilon_{13}\big]||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon+\delta)$$

Finally, by Lemma 6.4, and the fact that $\chi_\epsilon^{-1}\theta_t^\epsilon$ is H^1, we have

$$||(\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta_t^\epsilon)||_{L^2} \lesssim (\epsilon + \delta)$$

Therefore,

$$||(\theta^\epsilon - \phi^\delta)_t + \frac{\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta)||_{L^2} \lesssim ||\theta^\epsilon - \phi^\delta||_2 + (\epsilon + \delta)$$

as desired. □

Proof of Theorem 6.1:

First off, from (6.2), we have

$$\frac{dE_d}{dt} = \int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)(\theta^\epsilon - \phi^\delta)_t + (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t}$$

We will begin with the simpler term in $\int_{\mathbb{R}}(\theta^\epsilon - \phi^\delta)(\theta^\epsilon - \phi^\delta)_t$. Now, plugging in (6.3), we write

$$\begin{aligned} \int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)(\theta^\epsilon - \phi^\delta)_t = &\int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)\frac{-\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta) \\ &+ \int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)\big((\theta^\epsilon - \phi^\delta)_t + \frac{\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta)\big) \end{aligned} \tag{6.53}$$

Now, the first term is strictly dissipative, as

$$\int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)\frac{-\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta) = \frac{-\tau}{2}\int_{\mathbb{R}} \big(\Lambda^{3/2}\chi_\delta(\theta^\epsilon - \phi^\delta)\big)^2 \le 0 \tag{6.54}$$

Furthermore, by Lemma 6.13, we know that

$$\begin{aligned} &\int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)\big((\theta^\epsilon - \phi^\delta)_t + \frac{\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta)\big) \\ &\le ||\theta^\epsilon - \phi^\delta||_{L^2}||(\theta^\epsilon - \phi^\delta)_t + \frac{\tau}{2}\chi_\delta^2\Lambda^3(\theta^\epsilon - \phi^\delta)||_{L^2} \\ &\lesssim ||\theta^\epsilon - \phi^\delta||_2^2 + (\epsilon + \delta)||\theta^\epsilon - \phi^\delta||_2 \end{aligned} \tag{6.55}$$

Therefore, applying (6.54) and (6.55) to (6.53), we obtain

$$\int_{\mathbb{R}} (\theta^\epsilon - \phi^\delta)(\theta^\epsilon - \phi^\delta)_t \lesssim ||\theta^\epsilon - \phi^\delta||_2^2 + (\epsilon + \delta)||\theta^\epsilon - \phi^\delta||_2 \tag{6.56}$$

For $\int_{\mathbb{R}} (\theta_{\alpha\alpha} - \phi_{\alpha\alpha})(\theta - \phi)_{\alpha\alpha,t}$, recall from (6.19) that

$$\begin{aligned}(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t} = \chi_\delta &\left[\frac{-\tau}{2}\chi_\delta\Lambda^3(\theta_{\alpha\alpha} - \phi_{\alpha\alpha}) + \tau\Upsilon_{14}\chi_\delta(\theta_{\alpha\alpha\alpha\alpha} - \phi_{\alpha\alpha\alpha\alpha})\right] \\ &+ \chi_\delta\left[\Upsilon_{15}\chi_\delta\Lambda(\theta_{\alpha\alpha} - \phi_{\alpha\alpha}) + \Upsilon_{16}\chi_\delta(\theta_{\alpha\alpha\alpha} - \phi_{\alpha\alpha\alpha})\right] \\ &+ \chi_\delta\left[\Upsilon_{17}\chi_\delta\Lambda(\theta_\alpha - \phi_\alpha) + \Upsilon_{18}\chi_\delta(\theta_{\alpha\alpha} - \phi_{\alpha\alpha}) + \Upsilon_{19}\chi_\delta\Lambda(\theta - \phi)\right] \\ &+ \chi_\delta\left[\Upsilon_{20}\chi_\delta(\theta_\alpha - \phi_\alpha) + \Upsilon_{21} + \Upsilon_{22}\right] + (\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_{\alpha\alpha,t})\end{aligned}$$

Therefore, we denote

$$\int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})(\theta^\epsilon - \phi^\delta)_{\alpha\alpha,t} d\alpha = \sum_{i=1}^{11} Z_i \tag{6.57}$$

where each Z_i corresponds to the ith term in (6.19). Now, Z_5 through Z_{11} are all immediate from Lemma 6.12, since

$$\begin{aligned}Z_{11} &= \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})(\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_{\alpha\alpha,t}) \\ &\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}||(\chi_\epsilon - \chi_\delta)(\chi_\epsilon^{-1}\theta^\epsilon_{\alpha\alpha,t})||_{L^2} \\ &\lesssim (\epsilon + \delta)||\theta^\epsilon - \phi^\delta||_2\end{aligned} \tag{6.58}$$

$$\begin{aligned}Z_{10} = \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\Upsilon_{22} &\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}||\Upsilon_{22}||_{L^2} \\ \lesssim (\epsilon + \delta)||\theta^\epsilon - \phi^\delta||_2&\end{aligned} \tag{6.59}$$

$$\begin{aligned}Z_9 = \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\Upsilon_{21} &\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}||\Upsilon_{21}||_{L^2} \\ \lesssim ||\theta^\epsilon - \phi^\delta||_2^2 + (\epsilon + \delta)||\theta^\epsilon - \phi^\delta||_2&\end{aligned} \tag{6.60}$$

$$
\begin{aligned}
Z_8 &= \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\chi_\delta\big(\Upsilon_{20}\chi_\delta(\theta^\epsilon_\alpha - \phi^\delta_\alpha)\big) \\
&\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}||\theta^\epsilon_\alpha - \phi^\delta_\alpha||_{L^2}||\Upsilon_{20}||_{L^\infty} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2^2
\end{aligned} \tag{6.61}
$$

$$
\begin{aligned}
Z_7 &= \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\chi_\delta\big(\Upsilon_{19}\chi_\delta\Lambda(\theta^\epsilon - \phi^\delta)\big) \\
&\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}||\theta^\epsilon_\alpha - \phi^\delta_\alpha||_{L^2}||\Upsilon_{19}||_{L^\infty} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2^2
\end{aligned} \tag{6.62}
$$

$$
\begin{aligned}
Z_6 &= \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\chi_\delta\big(\Upsilon_{18}\chi_\delta(\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\big) \\
&\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}^2||\Upsilon_{18}||_{L^\infty} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2^2
\end{aligned} \tag{6.63}
$$

$$
\begin{aligned}
Z_5 &= \int_{\mathbb{R}} (\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha})\chi_\delta\big(\Upsilon_{17}\chi_\delta\Lambda(\theta^\epsilon_\alpha - \phi^\delta_\alpha)\big) \\
&\leq ||\theta^\epsilon_{\alpha\alpha} - \phi^\delta_{\alpha\alpha}||_{L^2}^2||\Upsilon_{17}||_{L^\infty} \\
&\lesssim ||\theta^\epsilon - \phi^\delta||_2^2
\end{aligned} \tag{6.64}
$$

Z_4 can be dealt with via an integration by parts,

$$
\begin{aligned}
Z_4 &= \int_{\mathbb{R}} (\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})\chi_{\delta}\big(\Upsilon_{16}\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha})\big) \\
&= \int_{\mathbb{R}} (\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))(\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha}))\Upsilon_{16} \\
&= \int_{\mathbb{R}} \frac{1}{2}\big[\partial_{\alpha}(\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))^2\big]\Upsilon_{16} \\
&= -\int_{\mathbb{R}} \frac{1}{2}(\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))^2\Upsilon_{16,\alpha} \\
&\leq \frac{1}{2}||\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}||^2_{L^2}||\Upsilon_{16,\alpha}||_{L^\infty} \\
&\lesssim ||\theta^{\epsilon} - \phi^{\delta}||^2_2
\end{aligned}
\tag{6.65}
$$

To bound Z_2 and Z_3, we must take advantage of the fact that Z_1 is a dissipative term.

$$
Z_1 = \int_{\mathbb{R}} (\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})\frac{-\tau}{2}\chi^2_{\delta}\Lambda^3(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}) = \frac{-\tau}{2}\int_{\mathbb{R}}\big[\Lambda^{3/2}\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})\big]^2 \tag{6.66}
$$

For Z_2, we integrate by parts twice,

$$
\begin{aligned}
Z_2 &= \int_{\mathbb{R}} (\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})\chi_{\delta}\big(\Upsilon_{14}\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha\alpha})\big) \\
&= \int_{\mathbb{R}} \tau(\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))(\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha\alpha}))\Upsilon_{14} \\
&= -\tau\int_{\mathbb{R}} (\chi_{\delta}(\theta^{\delta}_{\alpha\alpha\alpha} - \phi^{\epsilon}_{\alpha\alpha\alpha}))^2\Upsilon_{14} - (\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))(\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha}))\Upsilon_{14,\alpha} \\
&= -\tau\int_{\mathbb{R}} (\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha}))^2\Upsilon_{14} + \frac{\tau}{2}\int_{\mathbb{R}} (\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))^2\Upsilon_{14,\alpha\alpha} \\
&\leq \tau||\chi_{\delta}(\theta^{\epsilon}_{\alpha\alpha\alpha} - \phi^{\delta}_{\alpha\alpha\alpha})||^2_{L^2}||\Upsilon_{14}||_{L^\infty} + \frac{\tau}{2}||\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}||^2_{L^2}||\Upsilon_{14,\alpha\alpha}||_{L^\infty} \\
&\leq \tau||\chi_{\delta}\Lambda(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})||^2_{L^2}||\Upsilon_{14}||_{L^\infty} + \frac{\tau}{2}||\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}||^2_{L^2}||\Upsilon_{14,\alpha\alpha}||_{L^\infty}
\end{aligned}
\tag{6.67}
$$

For Z_3, we apply Young's inequality,

$$\begin{aligned} Z_3 &= \int_{\mathbb{R}} (\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})\chi_\delta(\Upsilon_{15}\chi_\delta\Lambda(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})) \\ &= \int_{\mathbb{R}} (\chi_\delta(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))(\chi_\delta\Lambda(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))\Upsilon_{15} \\ &\leq \frac{1}{2}\int_{\mathbb{R}} (\chi_\delta(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}))^2\Upsilon_{15}^2 + \frac{1}{2}\int_{\mathbb{R}} \left[\chi_\delta\Lambda(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})\right]^2 \\ &\leq \frac{1}{2}||\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha}||_{L^2}^2||\Upsilon_{15}||_{L^\infty}^2 + \frac{1}{2}||\chi_\delta\Lambda(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})||_{L^2}^2 \end{aligned} \tag{6.68}$$

As in the proof of local existence, we will let $v = \chi_\delta(\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})$ and apply Plancherel to let the dissipative term absorb the troublesome parts of Z_2 and Z_3. Combining (6.66), (6.67), and (6.68), we see that

$$\begin{aligned} Z_1 + Z_2 + Z_3 &\leq C||\theta^{\epsilon} - \phi^{\delta}||_2^2 + (\tau||\Upsilon_{14}||_{L^\infty} + \frac{1}{2})||\Lambda v||_{L^2}^2 - \frac{\tau}{2}||\Lambda^{3/2}v||_{L^2}^2 \\ &\leq C||\theta^{\epsilon} - \phi^{\delta}||_2^2 + \int_{\mathbb{R}} |\hat{v}(\zeta)|^2\big[(\frac{1}{2} + \tau||\Upsilon_{13}||_{L^\infty})|2\pi\zeta|^2 - \frac{\tau}{2}|2\pi\zeta|^3\big]d\zeta \\ &\leq C||\theta^{\epsilon} - \phi^{\delta}||_2^2 + \int_{\mathbb{R}} |\hat{v}(\zeta)|^2 \cdot C' \\ &\lesssim ||\theta^{\epsilon} - \phi^{\delta}||_2^2 \end{aligned}$$

And therefore, combining the bounds of the various Z_i, we see that

$$\int_{\mathbb{R}} (\theta^{\epsilon}_{\alpha\alpha} - \phi^{\delta}_{\alpha\alpha})(\theta^{\epsilon} - \phi^{\delta})_{\alpha\alpha,t}d\alpha \lesssim ||\theta^{\epsilon} - \phi^{\delta}||_2^2 + (\epsilon + \delta)||\theta^{\epsilon} - \phi^{\delta}||_2 \tag{6.69}$$

And combining this with (6.56), we have that

$$\frac{dE}{dt} \lesssim ||\theta^{\epsilon} - \phi^{\delta}||_2^2 + (\epsilon + \delta)||\theta^{\epsilon} - \phi^{\delta}||_2 \tag{6.70}$$

This in turn implies

$$\begin{aligned}
\frac{dE_d}{dt} &\le c_1 E_d + c_2(\epsilon+\delta)E_d^{1/2} \\
e^{-c_1 t}\frac{dE_d}{dt} &\le c_1 e^{-c_1 t}E_d + c_2(\epsilon+\delta)e^{-c_1 t}E_d^{1/2} \\
\frac{d}{dt}(e^{-c_1 t}E_d) &\le c_2(\epsilon+\delta)e^{-c_1 t}E_d^{1/2} \\
\frac{\frac{d}{dt}(e^{-c_1 t}E_d)}{e^{-c_1 t/2}E_d^{1/2}} &\le c_2(\epsilon+\delta)e^{-c_1 t/2} \\
\frac{d}{dt}\sqrt{e^{-c_1 t}E_d} &\le \frac{c_2}{2}(\epsilon+\delta)e^{-c_1 t/2}
\end{aligned}$$

Integrating with respect to time, we get

$$\begin{aligned}
\int_0^t \frac{d}{dt}\sqrt{e^{-c_1 s}E_d(s)}ds &\le \int_0^t \frac{c_2}{2}(\epsilon+\delta)e^{-c_1 s/2}ds \\
\sqrt{e^{-c_1 t}E_d(t)} - \sqrt{E_d(0)} &\le \frac{c_2}{c_1}(\epsilon+\delta)(1-e^{-c_1 t/2}) \\
\sqrt{E_d(t)} &\le \sqrt{E_d(0)}e^{c_1 t/2} + \frac{c_2}{c_1}(\epsilon+\delta)(e^{c_1 t/2}-1)
\end{aligned}$$

Since $\sqrt{E_d(t)} = \frac{1}{\sqrt{2}}||\theta^\epsilon - \phi^\delta||_2$, this in turn implies our desired result,

$$||\theta^\epsilon - \phi^\delta||_2 \le ||\theta_0 - \phi_0||_2 e^{c_1 t/2} + \frac{c_2\sqrt{2}}{c_1}(\epsilon+\delta)(e^{c_1 t/2}-1) \tag{6.71}$$

concluding the proof. □

With everything else ready, we can now finish the proof of local existence.

Proof of Theorem 1.1: This argument proceeds in several steps. First we use Theorem 6.1 to show the θ^ϵ must converge to an $C([0,T],H^2)$ function θ. Then we will use the uniform bound on $||\theta^\epsilon||_s$ along with interpolation to prove that $\theta \in H^{s'}$ for any

$s' < s$. After that we will prove that this θ does indeed satisfy the equation (2.11), and finally conclude that $\theta \in C([0,T], H^s)$.

Let T be as in Lemma 5.4, and let $\epsilon, \epsilon' > 0$. Then, by applying Theorem 6.1 to $\theta^\epsilon, \theta^{\epsilon'}$, we have that

$$\begin{aligned} ||\theta^\epsilon - \theta^{\epsilon'}||_2 &\leq \frac{c_2\sqrt{2}}{c_1}(\epsilon + \epsilon')(e^{c_1 t/2} - 1) \\ &\leq \frac{c_2\sqrt{2}}{c_1} e^{c_1 T/2}(\epsilon + \epsilon') \\ &\lesssim (\epsilon + \epsilon') \end{aligned}$$

and so, it's clear that the θ^ϵ form a Cauchy sequence in H^2. Therefore, as $\epsilon \to 0$, the θ^ϵ converge to a limit θ in $C([0,T], H^2)$.

Next, we use the interpolation inequality from Lemma 3.8 in [15], namely that for any $0 < s' < s$, there exists a constant C_s such that

$$||v||_{s'} \leq C_s ||v||_0^{1-s'/s} ||v||_s^{s'/s} \tag{6.72}$$

for all $v \in H^s$. We apply (6.72) to the subsequence of θ^ϵ. In particular,

$$||\theta^\epsilon - \theta^{\epsilon'}||_{s'} \leq C_s ||\theta^\epsilon - \theta^{\epsilon'}||_0^{1-s'/s} (2d_1)^{s'/s} \to 0$$

Therefore, the θ^ϵ form a Cauchy sequence in $H^{s'}$, and $\theta \in C([0,T], H^{s'})$.

Now we show that θ satisfies the evolution equation (2.11). Now, by definition, we have

$$\theta^\epsilon(\alpha,t) = \theta_0(\alpha) + \int_0^t \theta_t^\epsilon(\alpha,s)ds = \theta_0^\alpha \int_0^t B^\epsilon(\alpha,s)$$

where B^ϵ denotes the right hand side of (3.2). Since we've established convergence in $H^{s'}$ for sufficiently large s', we can pass to the limit, obtaining

$$\theta(\alpha,t) = \theta_0(\alpha) + \int_0^t \lim_{\epsilon\to 0} B^\epsilon(\alpha,s)ds = \theta_0(\alpha) + \int_0^t B(\alpha,s)$$

where B is the right hand side of (2.11). Therefore, taking the derivative, θ does indeed satisfy (2.11).

Finally, we look at the problem of the highest regularity. We start by fixing t and noting that the $\theta^\epsilon(\cdot,t)$ are uniformly bounded in H^s. Therefore, by the Banach-Alaoglu theorem, there exists a subsequence that converges weakly to some limit in H^s. Since the $\theta^\epsilon(\cdot,t)$ converge to $\theta(\cdot,t)$ in $H^{s'}$, therefore this subsequence must converge to $\theta(\cdot,t)$. Therefore, $\theta \in H^s$ pointwise in time.

It remains to show that $\theta \in C([0,T],H^s)$. We start by showing weak continuity. For $\phi \in H^{-s}$, we let $[\phi,\theta]$ denote the dual pairing of H^{-s} and H^s through the L^2 inner product. Since $s' < s$, $H^{-s'}$ is dense in H^{-s}, and for any $\phi \in H^{-s}$, there exists a sequence $\phi_n \in H^{-s'}$ that converges to ϕ in H^{-s}. Now, since $\theta^\epsilon \to \theta$ in $C([0,T];H^{s'})$, therefore

$$[\phi_n, \theta^\epsilon(\cdot,t)] \to [\phi_n, \theta(\cdot,t)]$$

uniformly on $[0, T]$ for any $\phi_n \in H^{-s'}$. Now, suppose that $\delta > 0$. Then, we have

$$|[\phi, \theta(\cdot, t)] - [\phi, \theta^\epsilon(\cdot, t)]| \leq |[\phi, \theta(\cdot, t)] - [\phi_n, \theta(\cdot, t)]| + |[\phi_n, \theta(\cdot, t)] - [\phi_n, \theta^\epsilon(\cdot, t)]|$$
$$+ |[\phi_n, \theta^\epsilon(\cdot, t)] - [\phi, \theta^\epsilon(\cdot, t)]|$$

Since $\phi_n \to \phi$ in H^{-s} and θ, θ^ϵ are uniformly bounded in H^s, therefore by selecting n large, we have

$$|[\phi, \theta(\cdot, t)] - [\phi_n, \theta(\cdot, t)]| < \frac{\delta}{3}$$
$$|[\phi, \theta^\epsilon(\cdot, t)] - [\phi_n, \theta^\epsilon(\cdot, t)]| < \frac{\delta}{3}$$

And since $\phi_n \in H^{s'}$, by picking ϵ small, we then have

$$|[\phi_n, \theta(\cdot, t)] - [\phi_n, \theta^\epsilon(\cdot, t)]| < \frac{\delta}{3}$$

And so, combining these equations, we get

$$|[\phi, \theta(\cdot, t)] - [\phi, \theta^\epsilon(\cdot, t)]| < \delta$$

and so $\theta^\epsilon \to \theta$ in H^s uniformly in time. To prove weak continuity, we use a similar argument. Once again, let $\delta > 0$, and consider

$$|[\phi, \theta(\cdot, t)] - [\phi, \theta(\cdot, t')]| \leq |[\phi, \theta(\cdot, t)] - [\phi, \theta^\epsilon(\cdot, t)]| + |[\phi, \theta^\epsilon(\cdot, t)] - [\phi, \theta^\epsilon(\cdot, t')]|$$
$$+ |[\phi, \theta(\cdot, t')] - [\phi, \theta^\epsilon(\cdot, t')]|$$

Because $\theta^\epsilon \to \theta$ in H^s uniformly in time, by choosing ϵ small, we again have

$$|[\phi, \theta(\cdot, t)] - [\phi, \theta^\epsilon(\cdot, t)]| < \frac{\delta}{3}$$
$$|[\phi, \theta(\cdot, t')] - [\phi, \theta^\epsilon(\cdot, t')]| < \frac{\delta}{3}$$

And since $\theta^\epsilon \in C((0,T];H^s)$, we can bound

$$|[\phi, \theta^\epsilon(\cdot,t)] - [\phi, \theta^\epsilon(\cdot,t')]| < \frac{\delta}{3}$$

for $|t-t'|$ small enough. Therefore, combining these equations, we see that

$$|[\phi, \theta(\cdot,t)] - [\phi, \theta(\cdot,t')]| < \delta$$

and θ is weakly continuous in H^s. To finish the argument, it is sufficient to show that $||\theta(t)||_s$ is continuous with respect to time. First we will show that θ is right-continuous in H^s at $t=0$. Now, for fixed t, we know that

$$||\theta(t)||_s \leq \limsup_{\epsilon\to 0} ||\theta^\epsilon(t)||_s$$

Subtracting $||\theta_0||_s$ from both sides and applying Theorem 5.3, we know

$$||\theta(t)||_s - ||\theta_0||_s \leq \limsup_{\epsilon\to 0} ||\theta^\epsilon(t)||_s - ||\theta_0||_s \leq \limsup_{\epsilon\to 0} \int_0^t \frac{dE^\epsilon}{dt} \leq t\cdot Ce^{d_1}$$

And sending $t\to 0$, we know that $\limsup_{t\to 0+} ||\theta(\cdot,t)||_s \leq ||\theta_0||_s$. However, since $\theta \in C_W([0,T];H^s)$, we have that $\liminf_{t\to 0+} ||\theta(\cdot,t)||_s \geq ||\theta_0||_s$. Therefore, θ is right-continuous at $t=0$.

To finish the argument, note that since $\tau > 0$, integrating equation (5.8) with respect to time implies a bound on $\int_0^T ||\Lambda^{3/2}\chi_\epsilon\theta^\epsilon||_s^2 dt$ that is independent of ϵ. This implies that the limit θ is in $L^2([0,T],H^{s+1})$. In particular, for almost every $T_0 \in [0,T]$, we have that $v(\cdot,T_0) \in H^{s+1}$. However, by taking $v(\cdot,T_0)$ as our new initial data and repeating the above construction with $s+1$ replacing s, we have that

$\theta \in C([T_0, T'], H^{\bar{s}})$ for $\bar{s} < s + 1$. Since T_0 is arbitrary, in particular, this implies $\theta \in C((0, T]; H^s)$, and combined with right continuity at zero, we have

$$\theta \in C([0, T]; H^s)$$

as desired. $\square$

Proof of Theorem 1.2: Now, by Theorem 1.1, we know that a solution θ to (1.1) with initial data $\theta(t, 0)$ exists. Furthermore, given two solutions, $\theta, \phi \in C([0, T]; \mathcal{O})$, by applying Theorem 6.1 with $\epsilon = \delta = 0$, we obtain

$$\begin{aligned} ||\theta - \phi||_2 &\leq ||\theta_0 - \phi_0||_2 e^{c_1 t/2} \\ &\leq ||\theta_0 - \phi_0||_2 e^{c_1 T/2} \\ &\lesssim ||\theta_0 - \phi_0||_2 \end{aligned}$$

Taking the supremum, we obtain

$$\sup_{t \in [0,T]} ||\theta - \phi||_2 \lesssim ||\theta_0 - \phi_0||_2 \tag{6.73}$$

And in particular, when $\theta_0 = \phi_0$, then $\theta = \phi$, and the solution θ is unique. $\square$

Chapter 7

Bounds for Global Existence:

Now, from Theorem 5.3 and Picard's theorem, we know that a solution θ^ϵ will exist until either $||\theta^\epsilon||_s \to \infty$, or the arc-chord condition is violated. Furthermore, while bounding $\frac{\partial E}{\partial t}$, all but a few of the terms can be shown to scale with $||\theta^\epsilon||_s^3$ or a higher power, with most of the exceptions being dissipation terms.

This inspires the assumption $||\theta^\epsilon||_s < c \ll 1$ for some small positive constant c. Note in particular that this bound additionally implies the arc-chord condition. Then, the lowest powers of $||\theta^\epsilon||_s^k$ dominate, as $\sum_{k\geq 1} c_k ||\theta^\epsilon||_s^k \lesssim ||\theta^\epsilon||_s$. Our goal is to show

$$\frac{\partial E}{\partial t} \leq ||\theta^\epsilon||_s^2(-c_0 + \sum_{k\geq 1} c_k ||\theta^\epsilon||_s^k) \lesssim ||\theta^\epsilon||_s^2 \cdot (-1) \leq 0$$

This will bound $||\theta^\epsilon||_s < c$ for all time, which in turn will give us the global existence for θ.

However, while most of the terms in $\int_{\mathbb{R}} \partial_\alpha^s \theta_t^\epsilon \partial_\alpha^s \theta^\epsilon$ are ultimately bounded with the same techniques as before, bounding $\int_{\mathbb{R}} \theta_t^\epsilon \theta^\epsilon$ is more difficult. Because of this, for the duration of the proof of global existence (Chapters 6 and 7), we will additionally assume that the Atwood number A_μ is zero. With this simplification, our equation for θ_t^ϵ changes to

$$\begin{aligned}\frac{d}{dt}\theta^\epsilon = \frac{\tau}{2}\chi_\epsilon^2 H(\theta_{\alpha\alpha\alpha}^\epsilon) + \chi_\epsilon H\big(\frac{-R\cos(\chi_\epsilon\theta^\epsilon)}{2}\chi_\epsilon\theta_\alpha^\epsilon\big) \\ + \chi_\epsilon\big[(V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)\chi_\epsilon\theta_\alpha^\epsilon\big] + \chi_\epsilon[m^\epsilon \cdot \hat{n}^\epsilon]\end{aligned} \tag{7.1}$$

First, we take note of a technical Lemma that will be useful later.

Lemma 7.1. *Suppose that $f \in L^1$ and $g \in L^2$. Then we have the bounds,*

$$||\int\Big(\int_0^1 (1-t)f(t\alpha + (1-t)\alpha')dt\Big)g(\alpha')d\alpha'||_{L^2} \lesssim ||f||_{L^1}||g||_{L^2} \tag{7.2}$$

$$||\int\Big(\int_0^1 tf(t\alpha + (1-t)\alpha')dt\Big)g(\alpha')d\alpha'||_{L^2} \lesssim ||f||_{L^1}||g||_{L^2} \tag{7.3}$$

$$||\int\Big(\int_0^1 f(t\alpha + (1-t)\alpha')dt\Big)g(\alpha')d\alpha'||_{L^2} \lesssim ||f||_{L^1}||g||_{L^2} \tag{7.4}$$

Proof of Lemma 7.1: For (7.4), we use the u-substitution $\beta = \alpha' - \frac{t}{1-t}\alpha$, and consider for any $h \in L^2$,

$$\begin{aligned}
&\int h(\alpha)\int\Big(\int f(t\alpha+(1-t)\alpha')dt\Big)g(\alpha')d\alpha' d\alpha \\
&= \int h(\alpha)\int\Big(\int f((1-t)\beta)dt\Big)g(\frac{t}{1-t}\alpha+\beta)d\beta d\alpha \\
&\leq \int\int\Big(\int h(\alpha)g(\frac{t}{1-t}\alpha+\beta)d\alpha\Big)f((1-t)\beta)d\beta dt \\
&\leq ||h||_{L^2}||g||_{L^2}\int\int\sqrt{\frac{1-t}{t}}\cdot f((1-t)\beta)d\beta dt \\
&\leq ||h||_{L^2}||g||_{L^2}||f||_{L^1}\int\frac{1}{\sqrt{(1-t)t}}dt \\
&\lesssim ||h||_{L^2}||g||_{L^2}||f||_{L^1}
\end{aligned}$$

Therefore, since h was an arbitrary L^2 function, we have proven (7.4). The inequalities (7.2) and (7.3) can be proved with the same change of variables, as the extra constant in the equation causes no complications. □

Now, the first term from θ_t^ϵ we bound is $m^\epsilon \cdot \hat{n}^\epsilon$.

Lemma 7.2. *Suppose that $\theta^\epsilon \in H^s$, and that there exists a small constant c such that $||\theta^\epsilon||_{H^s} < c \ll 1$. Then, we have the estimate*

$$||m^\epsilon \cdot \hat{n}^\epsilon||_{L^2} \lesssim ||\theta_\alpha^\epsilon||_{H^2}^2 ||\theta^\epsilon||_{H^2}$$

Proof of Lemma 7.2: Now, recall from (4.21) that

$$\Phi(m^\epsilon)^* = z_\alpha^\epsilon K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big) + \frac{z_\alpha^\epsilon}{2i}[H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon \partial_\alpha^\epsilon(\frac{\gamma}{z_\alpha^\epsilon})\big)$$

Using the facts

$$\Phi(\hat{n}^\epsilon) = iz_\alpha^\epsilon$$

$$m^\epsilon \cdot \hat{n}^\epsilon = Re(\Phi(\hat{n}^\epsilon)\Phi(m^\epsilon)^*)$$

we obtain

$$m^\epsilon \cdot \hat{n}^\epsilon = Re\Big[\frac{z_\alpha^\epsilon(\alpha)^2}{2\pi}\int \partial_{\alpha'}\big(\frac{\gamma[\theta^\epsilon](\alpha')}{z_\alpha^\epsilon(\alpha')}\big)\big(\frac{-q_2[z_d^\epsilon]}{z_\alpha^\epsilon(\alpha')q_1[z_d^\epsilon]} + \frac{q_1[z_\alpha^\epsilon](z_\alpha^\epsilon(\alpha)+z_\alpha^\epsilon(\alpha'))}{z_\alpha^\epsilon(\alpha)^2 z_\alpha^\epsilon(\alpha')}\big)\Big]$$

Now,

$$Re(z_\alpha^\epsilon) = \cos(\theta^\epsilon)$$

$$Re(z_{\alpha\alpha}^\epsilon) = -\theta_\alpha^\epsilon \sin(\theta^\epsilon)$$

$$Im(z_\alpha^\epsilon) = \sin(\theta^\epsilon)$$

$$Im(z_{\alpha\alpha}^\epsilon) = -\theta_\alpha^\epsilon \cos(\theta^\epsilon)$$

In particular,

$$||Re(z_{\alpha\alpha}^\epsilon)||_{L^1} \leq ||\theta_\alpha^\epsilon||_{L^2}||\theta^\epsilon||_{L^2}$$

$$||Im(z_\alpha^\epsilon)||_{L^2} \leq ||\theta^\epsilon||_{L^2}$$

Using the integral representations for q_2 and q_1, we also obtain

$$Re(q_2[z_d^\epsilon]) = -\int_0^1 (t-1)Re(z_{\alpha\alpha}^\epsilon(t\alpha + (1-t)\alpha'))dt \tag{7.5}$$

$$Re(q_1[z_\alpha^\epsilon]) = -\int_0^1 Re(z_{\alpha\alpha}^\epsilon(t\alpha + (1-t)\alpha'))dt \tag{7.6}$$

Now, using conjugates to make the denominators real, we have

$$m^\epsilon \cdot \hat{n}^\epsilon = Re\Big[\frac{z_\alpha^\epsilon(\alpha)^2}{2\pi}\int \partial_{\alpha'}\big(\frac{\gamma[\theta^\epsilon](\alpha')}{z_\alpha^\epsilon(\alpha')}\big)\Big(\frac{-q_2[z_d^\epsilon](z_\alpha^\epsilon(\alpha')q_1[z_d^\epsilon])^*}{|z_\alpha^\epsilon(\alpha')q_1[z_d^\epsilon]|^2} + \frac{q_1[z_\alpha^\epsilon](z_\alpha^\epsilon(\alpha)+z_\alpha^\epsilon(\alpha'))(z_\alpha^\epsilon(\alpha)^2 z_\alpha^\epsilon(\alpha'))^*}{|z_\alpha^\epsilon(\alpha)^2 z_\alpha^\epsilon(\alpha')|^2}\Big)\Big]$$

Now, each term in $m^\epsilon \cdot \hat{n}^\epsilon$ will contain either one of $Re(q_2[z_d^\epsilon])$, $Re(q_1[z_\alpha^\epsilon])$, or one of $Im(q_1[z_d^\epsilon])$, $Im(z_\alpha^\epsilon)$. Since $|z_\alpha^\epsilon| = 1$, and $|q_1[z_d^\epsilon]|$ is bounded above and below (by one and the arc-chord condition respectively), we have

$$||m^\epsilon \cdot \hat{n}^\epsilon||_{L^2} \lesssim (||\gamma_\alpha^\epsilon||_{L^2}||\theta^\epsilon||_{L^2} + ||\gamma^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{L^2})||\theta_\alpha^\epsilon||_{L^2}||\theta^\epsilon||_{L^2} \lesssim ||\theta_\alpha^\epsilon||_{H^2}^2||\theta^\epsilon||_{H^2}$$

This concludes the proof. □

Remark 7.3. It's worth noting that this proof fails to bound $m^\epsilon \cdot \hat{t}^\epsilon$, due to its reliance on real and imaginary parts. This is because while we can find one L^2 term in γ^ϵ, and a second in either $q_2[z_d^\epsilon]$ or $q_1[z_\alpha^\epsilon]$, obtaining the third L^2 term necessary for the bound requires finding a copy of $\sin(\theta^\epsilon)$ via taking the correct real or imaginary part. However, in $m^\epsilon \cdot \hat{t}^\epsilon$, this term is not guaranteed, which derails the argument. This in turn is the reason the $A_\mu = 0$ assumption is needed, as it removes the troublesome $H(m^\epsilon \cdot \hat{t}^\epsilon)$ term in θ_t^ϵ.

The next term we turn our attention to is $(V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)\chi_\epsilon\theta_\alpha^\epsilon$.

Lemma 7.4. *Suppose that $\theta^\epsilon \in H^s$ and that there exists a small constant c such that $||\theta^\epsilon||_{H^s} < c \ll 1$. Then, we have the estimate*

$$||V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} \lesssim ||\gamma^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{L^2}$$

Proof of Lemma 7.4: We will bound $||V^\epsilon||_{L^\infty}$ and $||W^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty}$ seperately. Recall that

$$\partial_\alpha V^\epsilon = (W^\epsilon \cdot \hat{n}^\epsilon)\theta_\alpha^\epsilon$$

Therefore, we know that

$$||V^\epsilon||_{L^\infty} \leq \int_{\mathbb{R}} |(W^\epsilon \cdot \hat{n}^\epsilon)\theta^\epsilon_\alpha| \leq ||W^\epsilon \cdot \hat{n}^\epsilon||_{L^2}||\theta^\epsilon_\alpha||_{L^2} \tag{7.7}$$

Therefore, we will start by proving

$$||W^\epsilon \cdot \hat{n}^\epsilon||_{L^2} \lesssim ||\gamma^\epsilon||_{L^2} \tag{7.8}$$

Now, recall from (4.19) that

$$\Phi(W^\epsilon)^* = \frac{1}{2i}H(\frac{\gamma^\epsilon}{z^\epsilon_\alpha}) + \frac{1}{2}K[z^\epsilon_d](\gamma^\epsilon)$$

Therefore, we have that

$$W^\epsilon \cdot \hat{n}^\epsilon = Re\big(\frac{z^\epsilon_\alpha}{2}H(\frac{\gamma^\epsilon}{z^\epsilon_\alpha}) + iz^\epsilon_\alpha K[z^\epsilon_d](\gamma^\epsilon)\big) \tag{7.9}$$

Now,

$$||Re\big(\frac{z^\epsilon_\alpha}{2}H(\frac{\gamma^\epsilon}{z^\epsilon_\alpha})\big)||_{L^2} \leq ||\frac{z^\epsilon_\alpha}{2}H(\frac{\gamma^\epsilon}{z^\epsilon_\alpha})||_{L^2} \lesssim ||\gamma^\epsilon||_{L^2} \tag{7.10}$$

Therefore, we only need to worry about the second term, in which case,

$$Re\big(iz^\epsilon_\alpha K[z^\epsilon_d](\gamma^\epsilon)\big) = Re\big(\frac{z^\epsilon_\alpha(\alpha)}{2\pi}\int \frac{-\gamma^\epsilon(\alpha')}{z_\alpha(\alpha')} \cdot \frac{q_2[z^\epsilon_d]}{q_1[z^\epsilon_d]}d\alpha'\big)$$

As in the previous lemma, we use complex conjugates to put all complex terms in the numerator, obtaining

$$Re\big(\frac{z^\epsilon_\alpha(\alpha)}{2\pi}\int \frac{-\gamma^\epsilon(\alpha')z^*_\alpha(\alpha')}{|z_\alpha(\alpha')|^2} \cdot \frac{q_2[z^\epsilon_d]q^*_1[z^\epsilon_d]}{|q_1[z^\epsilon_d]|^2}d\alpha'\big) \tag{7.11}$$

Finally, after multiplying things out, each term will contain $\gamma^\epsilon(\alpha') \in L^2$, along with either $Re(q_2[z^\epsilon_d]) \in L^1$, or $Im(q_2[z^\epsilon_d]) \in L^2$ and at least one of $Im(q_1[z^\epsilon_d]), Im(z_\alpha(\alpha')) \in$

L^2. Furthermore, the denominator is bounded away from zero due to the arc chord condition, therefore

$$||Re\big(iz_\alpha^\epsilon K[z_d^\epsilon](\gamma^\epsilon)\big)||_{L^2} \lesssim ||\gamma^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{L^2}||\theta^\epsilon||_{L^2} \tag{7.12}$$

And combining (7.10) and (7.12) with (7.9), we get

$$||W^\epsilon \cdot \hat{n}^\epsilon||_{L^2} \lesssim ||\gamma^\epsilon||_{L^2}(||\theta_\alpha^\epsilon||_{L^2}||\theta^\epsilon||_{L^2}+1) \lesssim ||\gamma^\epsilon||_{L^2}$$

proving (7.8). Combining this with (7.7), we have

$$||V^\epsilon||_{L^\infty} \lesssim ||\gamma^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{L^2} \tag{7.13}$$

as desired. It remains to bound $||W^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty}$. For this, we consider

$$\begin{aligned}
\Phi(W^\epsilon)^* &= \frac{1}{2\pi i}\int \frac{\gamma[\theta^\epsilon](\alpha')}{z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')}d\alpha' \\
&= \frac{1}{2\pi i}\int \frac{\gamma[\theta^\epsilon](\alpha')}{z_\alpha^\epsilon(\alpha)(\alpha-\alpha')} + \gamma[\theta^\epsilon](\alpha')\Big(\frac{1}{z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')} - \frac{1}{z_\alpha^\epsilon(\alpha)(\alpha-\alpha')}\Big)d\alpha'
\end{aligned}$$

Now, using the fact that $\hat{t}^\epsilon = z_\alpha^\epsilon$, we have that

$$\begin{aligned}
W^\epsilon \cdot \hat{t}^\epsilon &= Re\bigg(\frac{z_\alpha^\epsilon(\alpha)}{2\pi i}\int \frac{\gamma[\theta^\epsilon](\alpha')}{z_\alpha^\epsilon(\alpha)(\alpha-\alpha')} + \Big(\frac{\gamma[\theta^\epsilon](\alpha')}{z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')} - \frac{\gamma[\theta^\epsilon](\alpha')}{z_\alpha^\epsilon(\alpha)(\alpha-\alpha')}\Big)d\alpha'\bigg) \\
&= Re\bigg(\frac{1}{2\pi i}\int \gamma[\theta^\epsilon](\alpha')\Big(\frac{z_\alpha^\epsilon(\alpha)}{z_d^\epsilon(\alpha)-z_d^\epsilon(\alpha')} - \frac{1}{(\alpha-\alpha')}\Big)d\alpha'\bigg) \\
&= Re\bigg(\frac{1}{2\pi i}\int \gamma[\theta^\epsilon]\frac{z_\alpha^\epsilon(\alpha)(\alpha-\alpha') - z_d^\epsilon(\alpha) + z_d^\epsilon(\alpha')}{(\alpha-\alpha')^2 q_1[z_d^\epsilon]}d\alpha'\bigg) \\
&= Re\bigg(\frac{1}{2\pi i}\int \frac{\gamma[\theta^\epsilon]}{q_1[z_d^\epsilon]}\int_0^1 t z_{\alpha\alpha}^\epsilon(t\alpha+(1-t)\alpha')dtd\alpha'\bigg)
\end{aligned}$$

$$
\begin{aligned}
||W^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} &\lesssim ||\gamma^\epsilon||_{L^2}||z^\epsilon_{\alpha\alpha}||_{L^2}||\frac{1}{q_1}||_{L^\infty} \\
&\lesssim ||\gamma^\epsilon||_{L^2}||\theta^\epsilon_\alpha||_{L^2}
\end{aligned} \tag{7.14}
$$

And so, combining (7.13) and (7.14), we obtain

$$
||V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} \lesssim ||\gamma^\epsilon||_{L^2}||\theta^\epsilon_\alpha||_{L^2}
$$

as desired. □

The next lemma focuses on bounding the Taylor series of the $\cos(\chi_\epsilon\theta^\epsilon)$ term.

Lemma 7.5. *Suppose that $\theta^\epsilon \in H^1$. Then for $n \geq 1$, we have the estimates*

$$
\int \chi_\epsilon\theta^\epsilon\Lambda\big(\chi_\epsilon\theta^\epsilon\big)^{2n+1} \lesssim ||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}||\chi_\epsilon\theta^\epsilon_\alpha||^2_{L^2}||\chi_\epsilon\theta^\epsilon||^2_{L^2}
$$

Proof of Lemma 7.5: We rewrite

$$
\begin{aligned}
\int \chi_\epsilon\theta^\epsilon\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big) &= -\int \chi_\epsilon\theta^\epsilon_\alpha H\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big) \\
&= -\int \chi_\epsilon\theta^\epsilon_\alpha[H,(\chi_\epsilon\theta^\epsilon)^n](\chi_\epsilon\theta^\epsilon)^{n+1} \\
&\quad - \int \chi_\epsilon\theta^\epsilon_\alpha(\chi_\epsilon\theta^\epsilon)^n[H,(\chi_\epsilon\theta^\epsilon)^n](\chi_\epsilon\theta^\epsilon) \\
&\quad - \int \chi_\epsilon\theta^\epsilon_\alpha(\chi_\epsilon\theta^\epsilon)^{2n}H(\chi_\epsilon\theta^\epsilon)
\end{aligned} \tag{7.15}
$$

For the first term, we expand,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \chi_\epsilon \theta^\epsilon_\alpha [H, (\chi_\epsilon\theta^\epsilon)^n](\chi_\epsilon\theta^\epsilon)^{n+1} d\alpha\right| \\
&= \Bigg|\int_{\mathbb{R}} \chi_\epsilon\theta^\epsilon_\alpha(\alpha)\int_{\mathbb{R}}\Bigg(\int_0^1 n(\chi_\epsilon\theta^\epsilon(t\alpha+(1-t)\alpha'))^{n-1} \\
&\qquad\qquad \times\, \chi_\epsilon\theta^\epsilon_\alpha(t\alpha+(1-t)\alpha')dt\Bigg)\big(\chi_\epsilon\theta^\epsilon(\alpha')\big)^{n+1} d\alpha' d\alpha\Bigg| \\
&= \Bigg| n\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 \chi_\epsilon\theta^\epsilon_\alpha(\alpha)\chi_\epsilon\theta^\epsilon_\alpha(t\alpha+(1-t)\alpha')\big(\chi_\epsilon\theta^\epsilon(\alpha')\big)^{n+1} \\
&\qquad\qquad \times\, \big(\chi_\epsilon\theta^\epsilon(t\alpha+(1-t)\alpha')\big)^{n-1} dt d\alpha d\alpha'\Bigg| \\
&\le n||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}\int_0^1\int_{\mathbb{R}}|\chi_\epsilon\theta^\epsilon(\alpha')|^2\int|\chi_\epsilon\theta^\epsilon_\alpha(\alpha)||\chi_\epsilon\theta^\epsilon_\alpha(t\alpha+(1-t)\alpha')|d\alpha d\alpha' dt \\
&\le n||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}||\chi_\epsilon\theta^\epsilon_\alpha||^2_{L^2}\int_0^1\frac{1}{\sqrt{t}}\int_{\mathbb{R}}|\chi_\epsilon\theta^\epsilon(\alpha')|^2 d\alpha' dt \\
&\lesssim ||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}||\chi_\epsilon\theta^\epsilon_\alpha||^2_{L^2}||\chi_\epsilon\theta^\epsilon||^2_{L^2}
\end{aligned}
$$

The second term can be dealt with similarly,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \chi_\epsilon\theta^\epsilon_\alpha(\chi_\epsilon\theta^\epsilon)^n[H,(\chi_\epsilon\theta^\epsilon)^n](\chi_\epsilon\theta^\epsilon)d\alpha\right| \\
&= \Bigg| n\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 \chi_\epsilon\theta^\epsilon_\alpha(\alpha)\big(\chi_\epsilon\theta^\epsilon(\alpha)\big)^n\chi_\epsilon\theta^\epsilon_\alpha(t\alpha+(1-t)\alpha') \\
&\qquad\qquad \times\, \chi_\epsilon\theta^\epsilon(\alpha')\big(\chi_\epsilon\theta^\epsilon(t\alpha+(1-t)\alpha')\big)^{n-1} dt d\alpha d\alpha'\Bigg| \\
&\le n||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}\int_0^1\int_{\mathbb{R}}|\chi_\epsilon\theta^\epsilon(\alpha)||\chi_\epsilon\theta^\epsilon_\alpha(\alpha)|\int|\chi_\epsilon\theta^\epsilon(\alpha')||\chi_\epsilon\theta^\epsilon_\alpha(t\alpha+(1-t)\alpha')|d\alpha' d\alpha dt \\
&\le n||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}||\chi_\epsilon\theta^\epsilon_\alpha||_{L^2}||\chi_\epsilon\theta^\epsilon||_{L^2}\int_0^1\frac{1}{\sqrt{1-t}}\int_{\mathbb{R}}|\chi_\epsilon\theta^\epsilon(\alpha)||\chi_\epsilon\theta^\epsilon_\alpha(\alpha)|d\alpha dt \\
&\lesssim ||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}||\chi_\epsilon\theta^\epsilon_\alpha||^2_{L^2}||\chi_\epsilon\theta^\epsilon||^2_{L^2}
\end{aligned}
$$

Now, for the last term, note that

$$-\int \chi_\epsilon\theta^\epsilon_\alpha(\chi_\epsilon\theta^\epsilon)^{2n}H(\chi_\epsilon\theta^\epsilon) = \int \frac{-1}{2n+1}\partial_\alpha\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big)H(\chi_\epsilon\theta^\epsilon)$$
$$= \frac{-1}{2n+1}\int \chi_\epsilon\theta^\epsilon\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big)$$

Therefore, we have that

$$\frac{2n+2}{2n+1}\int \chi_\epsilon\theta^\epsilon\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big) = -\int \chi_\epsilon\theta^\epsilon_\alpha[H,(\chi_\epsilon\theta^\epsilon)^n](\chi_\epsilon\theta^\epsilon)^{n+1}$$
$$-\int \chi_\epsilon\theta^\epsilon_\alpha(\chi_\epsilon\theta^\epsilon)^n[H,(\chi_\epsilon\theta^\epsilon)^n](\chi_\epsilon\theta^\epsilon)$$

And using these bounds in (7.15), we obtain

$$\int \chi_\epsilon\theta^\epsilon\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big) \lesssim ||\chi_\epsilon\theta^\epsilon||^{2n-2}_{L^\infty}||\chi_\epsilon\theta^\epsilon_\alpha||^2_{L^2}||\chi_\epsilon\theta^\epsilon||^2_{L^2}$$

as desired. □

Remark 7.6. We use cosine's Taylor series because the lowest degree term in $\int \theta^\epsilon \cdot \chi_\epsilon H\big(\frac{-R\cos(\chi_\epsilon\theta^\epsilon)}{2}\chi_\epsilon\theta^\epsilon_\alpha\big)$ becomes

$$\int \frac{-R}{2}\chi_\epsilon\theta^\epsilon\Lambda(\chi_\epsilon\theta^\epsilon) = \frac{-R}{2}\int(\Lambda^{1/2}\chi_\epsilon\theta^\epsilon)^2 = \frac{-R}{2}||\Lambda^{1/2}\chi_\epsilon\theta^\epsilon||^2_{L^2}$$

which is entirely negative and an important dissipative term in the final equation.

Our final lemma switches focus from θ^ϵ_t to $\theta^\epsilon_{t,\alpha\alpha}$, and updates the various bounds we used during the proof of local existence.

Lemma 7.7. *Suppose that $\theta^\epsilon \in H^s$ and that there exists a small constant c such that $||\theta^\epsilon||_{H^s} < c \ll 1$. Then the following estimates hold:*

$$||K[z_d^\epsilon](f)||_{L^\infty} \lesssim ||f||_{L^2}||\theta_\alpha^\epsilon||_{L^2}$$

$$||\partial_\alpha K[z_d^\epsilon(f)||_{s-1} \lesssim ||f||_1||\theta_\alpha^\epsilon||_{s-1}^{1/2}$$

$$||\tilde{\gamma}[\theta^\epsilon]||_s \lesssim ||\theta^\epsilon||_s$$

$$||W_\alpha^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} \lesssim ||\theta_\alpha^\epsilon||_{H^1}$$

$$||\partial_\alpha((\partial_\alpha W) \cdot \hat{t})||_{s-3} \lesssim ||\theta^\epsilon||_s\sqrt{||\theta_\alpha^\epsilon||_{s-1}}$$

$$||\partial_\alpha((\partial_\alpha \tilde{W}) \cdot \hat{t})||_{s-2} \lesssim ||\theta^\epsilon||_s\sqrt{||\theta_\alpha^\epsilon||_{s-1}}$$

$$||m^\epsilon||_{L^\infty} \lesssim ||\theta_\alpha^\epsilon||_{L^2}$$

$$||\partial_\alpha m^\epsilon||_{s-1} \lesssim ||\theta_\alpha^\epsilon||_{s-1}^{3/2}$$

Proof of Lemma 7.7: The basic idea behind proving all of these is to simply take the bounds we found during the proofs of the local existence lemmas, and reduce to the lowest power of θ^ϵ. We'll start with (4.8), getting

$$\begin{aligned} ||K[z_d^\epsilon](f)||_{L^\infty} &\lesssim ||f||_{L^2} \cdot ||q_2||_{L^2} \cdot ||\frac{1}{q_1}||_{L^\infty} \cdot ||\frac{1}{z_\alpha^\epsilon}||_{L^\infty} \\ &\lesssim ||f||_{L^2}||q_2||_{L^2} \\ &\lesssim ||f||_{L^2}||\theta_\alpha^\epsilon||_{L^2} \end{aligned} \tag{7.16}$$

Now, recall from (4.9) that

$$||\partial_\alpha K[z_d^\epsilon]||_{L^2} \lesssim ||f||_{L^2}\Big(||q_2||_{L^2}||\partial_\alpha \frac{1}{q_1}||_{L^2} + ||\partial_\alpha q_2||_{L^1}||\frac{1}{q_1}||_{L^\infty}\Big)$$

$$\lesssim ||f||_{L^2}(||z_{\alpha\alpha}^\epsilon||_{L^2}^2 + \sqrt{||z_{\alpha\alpha\alpha}^\epsilon||_{L^2}})$$

$$\lesssim ||f||_{L^2}(||\theta_\alpha^\epsilon||_{L^2}^2 + \sqrt{||\theta_\alpha^\epsilon||_{H^1}})$$

Similarly, from (4.10) we have

$$||\partial_\alpha^s K[z_d^\epsilon]||_{L^2} \lesssim ||f||_{L^2}\Big(||q_2||_{L^2}||\partial_\alpha^s \frac{1}{q_1}||_{L^2} + \sum_{j=1}^{s-1}||\partial_\alpha^j q_2||_{L^1}||\partial_\alpha^{s-j}\frac{1}{q_1}||_{L^\infty}\Big)$$

$$+ (||f||_{L^2}||z_{\alpha\alpha}^\epsilon||_{L^2} + ||f_\alpha||_{L^2})||\partial_\alpha^s q_1[z_d^\epsilon]||_{L^1}||\frac{1}{q_1[z_d^\epsilon]}||_{L^\infty}$$

$$\lesssim ||f||_{H^1}\Big(||q_2||_{L^2} + \sum_{j=1}^{s-1}||\partial_\alpha^j q_2||_{L^1} + ||\partial_\alpha^s q_1[z_d^\epsilon]||_{L^1}\Big)$$

$$\lesssim ||f||_{H^1}\sqrt{||\theta_\alpha^\epsilon||_{s-1}}$$

And therefore,

$$||\partial_\alpha K[z_d^\epsilon(f)||_{s-1} \lesssim ||f||_1||\theta_\alpha^\epsilon||_{s-1}^{1/2} \tag{7.17}$$

as desired. Now, the bound on $\tilde{\gamma}^\epsilon$ is immediate from (4.18), since

$$||\tilde{\gamma}^\epsilon||_s = || - R\sin(\theta^\epsilon)||_s \lesssim ||\theta^\epsilon||_s \tag{7.18}$$

Next we'll bound m^ϵ. Recall from (4.21) that

$$\Phi(m^\epsilon)^* = z_\alpha^\epsilon K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big) + \frac{z_\alpha^\epsilon}{2i}[H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)$$

Therefore,

$$
\begin{aligned}
||m^\epsilon||_{L^\infty} &\lesssim ||K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^\infty} + ||[H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^\infty} \\
&\lesssim ||\theta_\alpha^\epsilon||_{L^2}||\partial_\alpha\frac{\gamma}{z_\alpha^\epsilon}||_{L^2} + ||q_1[\frac{1}{(z_\alpha^\epsilon)^2}]||_{L^2}||\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})||_{L^2} \\
&\lesssim ||\theta_\alpha^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{H^2}
\end{aligned} \tag{7.19}
$$

Also,

$$
\begin{aligned}
||\partial_\alpha^k m^\epsilon||_{L^2} \lesssim\ & ||\partial_\alpha^k z_\alpha^\epsilon||_{L^2}||K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^\infty} + ||\partial_\alpha^k z_\alpha^\epsilon||_{L^2}||[H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^\infty} \\
&+ \sum_{j=1}^{k} ||\partial_\alpha^{k-j} z_\alpha^\epsilon||_{L^\infty}||\partial_\alpha^j K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^2} \\
&+ ||\partial_\alpha^{k-j} z_\alpha^\epsilon||_{L^\infty}||\partial_\alpha^j [H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^2} \\
&\lesssim ||K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^\infty} + ||[H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^\infty} \\
&+ \sum_{j=1}^{k} ||\partial_\alpha^j K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^2} + ||\partial_\alpha^j [H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{L^2} \\
&\lesssim ||\theta_\alpha^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{H^2} + ||\partial_\alpha K[z_d^\epsilon]\big(\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{k-1} + ||\partial_\alpha [H, \frac{1}{(z_\alpha^\epsilon)^2}]\big(z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})\big)||_{k-1} \\
&\lesssim ||\theta_\alpha^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{H^2} + ||\frac{\gamma}{z_\alpha^\epsilon}||_{H^2}||\theta_\alpha^\epsilon||_{k-1}^{1/2} + ||z_\alpha^\epsilon\partial_\alpha(\frac{\gamma}{z_\alpha^\epsilon})||_{k-3}\sqrt{||\partial_\alpha(\frac{1}{(z_\alpha^\epsilon)^2})||_{k-1}}
\end{aligned}
$$

However, since $||\gamma||_j \lesssim ||\theta_\alpha^\epsilon||_{j+1}$ for $j \geq 1$, and $||z_\alpha^\epsilon||_{L^\infty} = 1, ||\partial_\alpha \frac{1}{z_\alpha^\epsilon}||_j \lesssim ||z_{\alpha\alpha}^\epsilon||_j \lesssim ||\theta_\alpha^\epsilon||_j$, we have

$$
||\partial_\alpha^k m^\epsilon||_{L^2} \lesssim ||\theta_\alpha^\epsilon||_{k-1}^{3/2}
$$

And in particular,

$$
||\partial_\alpha m^\epsilon||_{s-1} \lesssim ||\theta_\alpha^\epsilon||_{s-1}^{3/2} \tag{7.20}
$$

Finally, recall from (2.6) that

$$W_\alpha^\epsilon \cdot \hat{t}^\epsilon = \frac{H(\gamma^\epsilon \theta_\alpha^\epsilon)}{2} + m^\epsilon \cdot \hat{t}^\epsilon$$

Now,

$$\begin{aligned} ||W_\alpha^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} &\leq ||H(\gamma^\epsilon \theta_\alpha^\epsilon)||_{L^\infty} + ||m^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} \\ &\leq ||H(\gamma^\epsilon \theta_\alpha^\epsilon)||_{H^1} + ||\theta_\alpha^\epsilon||_{L^2} \qquad (7.21) \\ &\lesssim ||\theta_\alpha^\epsilon||_{H^1} \end{aligned}$$

Additionally,

$$\begin{aligned} ||\partial_\alpha(W_\alpha^\epsilon \cdot \hat{t}^\epsilon)||_{s-3} &\leq ||H(\gamma^\epsilon \theta_\alpha^\epsilon)||_{s-2} + ||\partial_\alpha(m^\epsilon)||_{s-3} \\ &\leq ||\gamma^\epsilon||_{s-2}||\theta_\alpha^\epsilon)||_{s-2} + \sqrt{||\theta_\alpha^\epsilon||_{s-1}}||\theta^\epsilon||_s \qquad (7.22) \\ &\lesssim ||\theta^\epsilon||_s \sqrt{||\theta_\alpha^\epsilon||_{s-1}} \end{aligned}$$

Similarly,

$$\begin{aligned} ||\partial_\alpha(\tilde{W}_\alpha^\epsilon \cdot \hat{t}^\epsilon)||_{s-2} &\leq ||H(\tilde{\gamma}^\epsilon \theta_\alpha^\epsilon)||_{s-1} + ||\partial_\alpha(m^\epsilon)||_{s-2} \\ &\leq ||\tilde{\gamma}^\epsilon||_{s-1}||\theta_\alpha^\epsilon)||_{s-1} + \sqrt{||\theta_\alpha^\epsilon||_{s-1}}||\theta^\epsilon||_s \qquad (7.23) \\ &\lesssim ||\theta^\epsilon||_s \sqrt{||\theta_\alpha^\epsilon||_{s-1}} \end{aligned}$$

finishing the proof. □

Chapter 8

Proof of Global Existence:

Theorem 8.1. *Suppose that the Atwood number* A_μ *is zero and* $||\theta^\epsilon||_{H^s} \leq c \ll 1$ *for a small enough constant c. Let* $E = \frac{1}{2}\int (\theta^\epsilon)^2 + (\partial_\alpha^s \theta^\epsilon)^2$*. Then,*

$$\frac{dE}{dt} \leq 0 \tag{8.1}$$

Proof of Theorem 8.1: Now,

$$\frac{dE}{dt} = \int \theta^\epsilon \theta_t^\epsilon + (\partial_\alpha^s \theta^\epsilon)(\partial_\alpha^s \theta_t^\epsilon) \tag{8.2}$$

Our goal is to show that the negative sceond order terms in $\frac{dE}{dt}$ dominate the equation, and that every other term is either of at least third order (and therefore negligible), or can otherwise be absorbed by the dissipative terms. We shall begin with $\int \theta^\epsilon \theta_t^\epsilon$. Recall from (7.1) that

$$\frac{d}{dt}\theta^\epsilon = \frac{\tau}{2}\chi_\epsilon^2 H(\theta_{\alpha\alpha\alpha}^\epsilon) + \chi_\epsilon H\big(\frac{-R\cos(\chi_\epsilon\theta^\epsilon)}{2}\chi_\epsilon\theta_\alpha^\epsilon\big)$$
$$+ \chi_\epsilon\big[(V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)\chi_\epsilon\theta_\alpha^\epsilon\big] + \chi_\epsilon[m^\epsilon \cdot \hat{n}^\epsilon]$$

Now, using the fact that $\Lambda^{1/2}$ is self-adjoint, we get

$$\begin{aligned}\int \theta^\epsilon \frac{\tau}{2}\chi_\epsilon^2 H(\theta^\epsilon_{\alpha\alpha\alpha}) &= \int \frac{-\tau}{2}(\chi_\epsilon\theta^\epsilon)\Lambda^3(\chi_\epsilon\theta^\epsilon) \\ &= \frac{-\tau}{2}\int (\Lambda^{3/2}\chi_\epsilon\theta^\epsilon)^2 \\ &= \frac{-\tau}{2}||\Lambda^{3/2}\chi_\epsilon\theta^\epsilon||_{L^2}^2 \\ &\leq 0\end{aligned} \tag{8.3}$$

Next, we note that $\cos(\chi_\epsilon\theta^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha = \partial_\alpha \sin(\chi_\epsilon\theta^\epsilon)$ and expand via Taylor series to obtain

$$\begin{aligned}\int \theta^\epsilon\chi_\epsilon H\big(\frac{-R\cos(\chi_\epsilon\theta^\epsilon)}{2}\chi_\epsilon\theta^\epsilon_\alpha\big) &= \frac{-R}{2}\int(\chi_\epsilon\theta^\epsilon)\Lambda(\sin(\chi_\epsilon\theta^\epsilon)) \\ &= \frac{-R}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\int(\chi_\epsilon\theta^\epsilon)\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big)\end{aligned} \tag{8.4}$$

Now, for the $n = 0$ term, we again use the fact that $\Lambda^{1/2}$ is self-adjoint, getting

$$\frac{-R}{2}\int(\chi_\epsilon\theta^\epsilon)\Lambda(\chi_\epsilon\theta^\epsilon) = \frac{-R}{2}\int(\Lambda^{1/2}\chi_\epsilon\theta^\epsilon)^2 = \frac{-R}{2}||\Lambda^{1/2}\chi_\epsilon\theta^\epsilon||_{L^2}^2$$

For the rest of the sum, we apply Lemma 7.5,

$$\begin{aligned}\frac{-R}{2}\sum_{n=1}^\infty \frac{(-1)^n}{(2n+1)!}\int(\chi_\epsilon\theta^\epsilon)\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big) &\leq \sum_{n=1}^\infty \frac{\mathbb{R}}{2(2n+1)!}\left|\int(\chi_\epsilon\theta^\epsilon)\Lambda\big((\chi_\epsilon\theta^\epsilon)^{2n+1}\big)\right| \\ &\lesssim \sum_{n=1}^\infty ||\chi_\epsilon\theta^\epsilon||_{L^\infty}^{2n-2}||\chi_\epsilon\theta^\epsilon_\alpha||_{L^2}^2||\chi_\epsilon\theta^\epsilon||_{L^2}^2 \\ &\lesssim ||\chi_\epsilon\theta^\epsilon_\alpha||_{L^2}^2||\chi_\epsilon\theta^\epsilon||_{L^2}^2\end{aligned}$$

And therefore, we have (for some constant C),

$$\int \theta^\epsilon\chi_\epsilon H\big(\frac{-R\cos(\chi_\epsilon\theta^\epsilon)}{2}\chi_\epsilon\theta^\epsilon_\alpha\big) \leq \frac{-R}{2}||\Lambda^{1/2}\chi_\epsilon\theta^\epsilon||_{L^2}^2 + C||\chi_\epsilon\theta^\epsilon_\alpha||_{L^2}^2||\chi_\epsilon\theta^\epsilon||_{L^2}^2 \tag{8.5}$$

For the next term, we estimate directly and apply Lemma 7.4,

$$\begin{aligned}\int \theta^\epsilon \chi_\epsilon\big[(V^\epsilon - W^\epsilon\cdot \hat{t}^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha\big] &= \int (\chi_\epsilon\theta^\epsilon)\big[(V^\epsilon - W^\epsilon\cdot \hat{t}^\epsilon)\chi_\epsilon\theta^\epsilon_\alpha\big] \\ &\le ||\chi_\epsilon\theta^\epsilon||_{L^2}||\chi_\epsilon\theta^\epsilon_\alpha||_{L^2}||V^\epsilon - W^\epsilon\cdot\hat{t}^\epsilon||_{L^\infty} \\ &\lesssim ||\theta^\epsilon||_{L^2}||\theta^\epsilon_\alpha||^2_{L^2}||\gamma^\epsilon||_{L^2}\end{aligned} \tag{8.6}$$

Finally, using Lemma 7.2, we know

$$\int \theta^\epsilon\chi_\epsilon(m^\epsilon\cdot\hat{n}^\epsilon) \le ||\theta^\epsilon||_{L^2}||m^\epsilon\cdot\hat{n}^\epsilon||_{L^2} \lesssim ||\theta^\epsilon_\alpha||^2_{H^2}||\theta^\epsilon||^2_{H^2} \tag{8.7}$$

Therefore, combining (8.3), (8.5), (8.6), and (8.7), our final result for θ^ϵ_t is

$$\int \theta^\epsilon_t\theta^\epsilon \lesssim -||\Lambda^{1/2}\theta^\epsilon||^2_{L^2} + ||\theta^\epsilon_\alpha||^2_{H^2}||\theta^\epsilon||^2_{H^2} \tag{8.8}$$

Next we need to bound $\int(\partial^s_\alpha\theta^\epsilon)(\partial^s_\alpha\theta^\epsilon_t)$. As before, we have

$$\int(\partial^s_\alpha\theta^\epsilon)(\partial^s_\alpha\theta^\epsilon_t) = \int(\partial^s_\alpha\theta^\epsilon)(\partial^{s-2}_\alpha\theta^\epsilon_{\alpha\alpha,t}) \tag{8.9}$$

$$\theta^\epsilon_{\alpha\alpha,t} = \chi_\epsilon\big[\frac{-\tau}{2}\Lambda^3(\chi_\epsilon\theta^\epsilon_{\alpha\alpha}) + \Upsilon^\epsilon_5\Lambda(\chi_\epsilon\theta^\epsilon_{\alpha\alpha}) + \Upsilon^\epsilon_6(\chi_\epsilon\theta^\epsilon_{\alpha\alpha\alpha}) + \Upsilon^\epsilon_7\big]$$

Substituting $A_\mu = 0$ into (3.20), (3.21), (3.22), and (3.18), we have

$$\Upsilon^\epsilon_5 = k^\epsilon + \frac{\tau}{2}\theta^\epsilon_\alpha(\chi_\epsilon\theta^\epsilon_\alpha) \tag{8.10}$$

$$\Upsilon^\epsilon_6 = (V^\epsilon - W^\epsilon\cdot\hat{t}^\epsilon) \tag{8.11}$$

$$\Upsilon^\epsilon_7 = \partial_\alpha\Upsilon^\epsilon_4 + k^\epsilon_\alpha H(\chi_\epsilon\theta^\epsilon_{\alpha\alpha}) + \frac{\tau}{2}H(\chi_\epsilon\theta^\epsilon_{\alpha\alpha})(\theta^\epsilon_\alpha\chi_\epsilon\theta^\epsilon_\alpha)_\alpha + (V^\epsilon - W^\epsilon\cdot\hat{t}^\epsilon)_\alpha(\chi_\epsilon\theta^\epsilon_{\alpha\alpha}) \tag{8.12}$$

$$\begin{aligned}\Upsilon^\epsilon_4 = {}& [H,k^\epsilon](\chi_\epsilon\theta^\epsilon_{\alpha\alpha}) + H(k^\epsilon_\alpha\chi_\epsilon\theta^\epsilon_\alpha) - \tau(\chi_\epsilon\theta^\epsilon_\alpha)m^{st,\epsilon}\cdot\hat{t}^\epsilon \\ &+ \frac{\tau}{2}(\chi_\epsilon\theta^\epsilon_\alpha)[H,\theta^\epsilon_\alpha](\chi_\epsilon\theta^\epsilon_{\alpha\alpha}) + (\chi_\epsilon\theta^\epsilon_\alpha)(\tilde{V}^\epsilon - \tilde{W}^\epsilon\cdot\hat{t}^\epsilon)_\alpha + (m^\epsilon\cdot\hat{n}^\epsilon)_\alpha\end{aligned} \tag{8.13}$$

First we will bound the Υ_5^ϵ term. Since $k^\epsilon = \frac{-R}{2}\cos(\theta^\epsilon)$ when $A_\mu = 0$, it's simple to see that

$$||\partial_\alpha \Upsilon_5^\epsilon||_{s-3} \lesssim ||\theta^\epsilon||_{s-3}||\theta_\alpha^\epsilon||_{s-3} + ||\theta_\alpha^\epsilon||_{s-2}^2 \tag{8.14}$$

$$||\Upsilon_5^\epsilon + \frac{R}{2}||_{L^\infty} \lesssim ||\theta^\epsilon||_{L^\infty}^2 + ||\theta_\alpha^\epsilon||_{L^\infty}^2 \lesssim ||\theta^\epsilon||_{H^2}^2 \tag{8.15}$$

$$||\Upsilon_5^\epsilon + \frac{R}{2}||_{L^2} \lesssim ||\theta^\epsilon||_{L^\infty}||\theta^\epsilon||_{L^2} + ||\theta_\alpha^\epsilon||_{L^\infty}||\theta_\alpha^\epsilon||_{L^2} \lesssim ||\theta^\epsilon||_{H^2}^2 \tag{8.16}$$

Now,

$$\begin{aligned}\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_5^\epsilon \Lambda(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)) = \frac{-R}{2}&\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\Lambda(\partial_\alpha^s \chi_\epsilon \theta^\epsilon) \\ &+ \int (\Upsilon_5^\epsilon + \frac{R}{2})(\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\Lambda(\partial_\alpha^s \chi_\epsilon \theta^\epsilon) \\ &+ \sum_{j=1}^{s-2} \binom{s-2}{j} \int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^j \Upsilon_5^\epsilon)\Lambda(\partial_\alpha^{s-j} \chi_\epsilon \theta^\epsilon)\end{aligned} \tag{8.17}$$

Using the fact that $\Lambda^{1/2}$ is self-adjoint, we have

$$\frac{-R}{2}\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon \Lambda(\partial_\alpha^s \chi_\epsilon \theta^\epsilon) = \frac{-R}{2}\int (\Lambda^{1/2}(\partial_\alpha^s \chi_\epsilon \theta^\epsilon))^2 = \frac{-R}{2}||\partial_\alpha^s \chi_\epsilon \theta^\epsilon||_{L^2} \le 0 \tag{8.18}$$

The sum can be bounded directly, since for $1 \le j \le s-2$, we have

$$\begin{aligned}\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^j \Upsilon_5^\epsilon)\Lambda(\partial_\alpha^{s-j} \chi_\epsilon \theta^\epsilon) &\lesssim ||\partial_\alpha \Upsilon_5^\epsilon||_{s-3}||\partial_\alpha \theta^\epsilon||_{s-1}^2 \\ &\lesssim ||\partial_\alpha \theta^\epsilon||_{s-1}^2(||\theta^\epsilon||_{s-3}||\theta_\alpha^\epsilon||_{s-3} + ||\theta_\alpha^\epsilon||_{s-2}^2) \\ &\lesssim ||\theta_\alpha^\epsilon||_{s-1}^3\end{aligned} \tag{8.19}$$

And finally, using Young's inequality, we have (for any constant c),

$$\int (\Upsilon_5^\epsilon + \frac{R}{2})(\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\Lambda(\partial_\alpha^s \chi_\epsilon \theta^\epsilon) \le \frac{1}{2c}\int (\Upsilon_5^\epsilon + \frac{R}{2})^2(\partial_\alpha^s \chi_\epsilon \theta^\epsilon) + \frac{c}{2}\int (\Lambda(\partial_\alpha^s \chi_\epsilon \theta^\epsilon))^2$$

Therefore, choosing $c = ||\theta^\epsilon||_{H^s}$, we have

$$\int (\Upsilon_5^\epsilon + \frac{R}{2})(\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\Lambda(\partial_\alpha^s \chi_\epsilon \theta^\epsilon) \lesssim \frac{||\theta^\epsilon||_{H^2}^4 ||\partial_\alpha^s \theta^\epsilon||_{L^2}^2}{||\theta^\epsilon||_{H^s}} + ||\theta^\epsilon||_{H^s} ||\Lambda \partial_\alpha^s \chi_\epsilon \theta^\epsilon||_{L^2}^2$$
$$\lesssim ||\theta_\alpha^\epsilon||_{H^{s-1}}^3 + ||\theta^\epsilon||_{H^s} ||\Lambda \partial_\alpha^s \chi_\epsilon \theta^\epsilon||_{L^2}^2 \tag{8.20}$$

And so, combining (8.18), (8.19), and (8.20), we have

$$\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_5^\epsilon \Lambda(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)) \lesssim ||\theta_\alpha^\epsilon||_{s-1}^3 + ||\theta^\epsilon||_{H^s} ||\Lambda \partial_\alpha^s \chi_\epsilon \theta^\epsilon||_{L^2}^2 \tag{8.21}$$

Now, the Υ_6^ϵ term in $\frac{dE}{dt}$ is

$$\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_6^\epsilon \chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) = \int \Upsilon_6^\epsilon (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^{s+1} \chi_\epsilon \theta^\epsilon)$$
$$+ \sum_{j=1}^{s-2} \binom{s-2}{j} \int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^j \Upsilon_6^\epsilon)(\partial_\alpha^{s+1-j} \chi_\epsilon \theta^\epsilon)$$

As in the local existence proof, we integrate by parts in the first term, getting

$$\int \Upsilon_6^\epsilon (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^{s+1} \chi_\epsilon \theta^\epsilon) = \frac{-1}{2} \int (\partial_\alpha \Upsilon_6^\epsilon)(\partial_\alpha^s \chi_\epsilon \theta^\epsilon)^2 \lesssim ||\partial_\alpha \Upsilon_6^\epsilon||_{L^\infty} ||\partial_\alpha^s \theta^\epsilon||_{L^2}^2 \tag{8.22}$$

For the sum, we have

$$\sum_{j=1}^{s-2} \binom{s-2}{j} \int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)(\partial_\alpha^j \Upsilon_6^\epsilon)(\partial_\alpha^{s+1-j} \chi_\epsilon \theta^\epsilon) \lesssim ||\partial_\alpha^s \theta^\epsilon||_{L^2} ||\theta_{\alpha\alpha\alpha}||_{s-3} ||\partial_\alpha \Upsilon_6^\epsilon||_{s-3} \tag{8.23}$$

Since $\partial_\alpha \Upsilon_6^\epsilon = (V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)_\alpha = W_\alpha^\epsilon \cdot \hat{t}^\epsilon$, therefore applying Lemma 7.7 to (8.22) and (8.23), we have

$$\int (\partial_\alpha^s \chi_\epsilon \theta^\epsilon)\partial_\alpha^{s-2}(\Upsilon_6^\epsilon \chi_\epsilon \theta_{\alpha\alpha\alpha}^\epsilon) \lesssim ||\partial_\alpha^s \theta^\epsilon||_{L^2} ||\theta_{\alpha\alpha\alpha}^\epsilon||_{s-3} ||\theta^\epsilon||_s \sqrt{||\theta_\alpha^\epsilon||_{s-1}} \tag{8.24}$$

Finally, there's Υ_7,

$$\begin{aligned}
\Upsilon_7^\epsilon &= k_\alpha^\epsilon H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + \frac{\tau}{2} H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\theta_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon)_\alpha + (V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)_\alpha (\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) \\
&+ \partial_\alpha \Big([H, k^\epsilon](\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) + H(k_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon) - \tau(\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon} \cdot \hat{t}^\epsilon + \frac{\tau}{2}(\chi_\epsilon \theta_\alpha^\epsilon)[H, \theta_\alpha^\epsilon](\chi_\epsilon \theta_{\alpha\alpha}^\epsilon) \Big) \\
&+ \partial_\alpha \Big((\chi_\epsilon \theta_\alpha^\epsilon)(\tilde{V}^\epsilon - \tilde{W}^\epsilon \cdot \hat{t}^\epsilon)_\alpha + (m^\epsilon \cdot \hat{n}^\epsilon)_\alpha \Big) \\
&= \sum_{j=1}^{9} \Xi_j
\end{aligned}$$

Now, through repeated applications of Lemmas 4.5 and 7.7, we obtain

$$\begin{aligned}
||\Xi_1||_{s-2} &= ||\frac{R}{2}\sin(\theta^\epsilon)\theta_\alpha^\epsilon H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-2} \\
&\lesssim ||\sin(\theta^\epsilon)||_{s-2} ||\theta_\alpha^\epsilon||_{s-2} ||\theta_{\alpha\alpha}^\epsilon||_{s-2}
\end{aligned}$$

$$\begin{aligned}
||\Xi_2||_{s-2} &= ||\frac{\tau}{2} H(\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)(\theta_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon)_\alpha||_{s-2} \\
&\lesssim ||\theta_{\alpha\alpha}^\epsilon||_{s-2} ||\theta_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon||_{s-1} \\
&\lesssim ||\theta_{\alpha\alpha}^\epsilon||_{s-2} ||\theta_\alpha^\epsilon||_{s-1}^2
\end{aligned}$$

$$\begin{aligned}
||\Xi_3||_{s-2} &= ||(V^\epsilon - W^\epsilon \cdot \hat{t}^\epsilon)_\alpha (\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-2} \\
&\lesssim (||W_\alpha^\epsilon \cdot \hat{t}^\epsilon||_{L^\infty} + ||\partial_\alpha (W_\alpha^\epsilon \cdot \hat{t}^\epsilon)||_{s-3}) ||\theta_{\alpha\alpha}^\epsilon||_{s-2} \\
&\lesssim ||\theta^\epsilon||_s \sqrt{||\theta_\alpha^\epsilon||_{s-1}} ||\theta_{\alpha\alpha}^\epsilon||_{s-2}
\end{aligned}$$

$$\begin{aligned}
||\Xi_4||_{s-2} &\leq ||[H, k^\epsilon](\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-1} \\
&\lesssim ||\theta_{\alpha\alpha}^\epsilon||_{s-2} \sqrt{||\partial_\alpha k^\epsilon||_{s-2}} \\
&\lesssim ||\theta_{\alpha\alpha}^\epsilon||_{s-2} \sqrt{||\theta_\alpha^\epsilon||_{s-2}}
\end{aligned}$$

$$
\begin{aligned}
||\Xi_5||_{s-2} &\leq ||H(k_\alpha^\epsilon \chi_\epsilon \theta_\alpha^\epsilon)||_{s-1} \\
&\lesssim ||k_\alpha^\epsilon||_{s-1}||\theta_\alpha^\epsilon||_{s-1} \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}^2
\end{aligned}
$$

$$
\begin{aligned}
||\Xi_6||_{s-2} &\leq ||\tau(\chi_\epsilon \theta_\alpha^\epsilon) m^{st,\epsilon} \cdot \hat{t}^\epsilon||_{s-1} \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}(||m^{st,\epsilon}||_{L^\infty} + ||\partial_\alpha m^{st,\epsilon}||_{s-2}) \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}^{3/2}
\end{aligned}
$$

$$
\begin{aligned}
||\Xi_7||_{s-2} &\leq ||\frac{\tau}{2}(\chi_\epsilon \theta_\alpha^\epsilon)[H, \theta_\alpha^\epsilon](\chi_\epsilon \theta_{\alpha\alpha}^\epsilon)||_{s-1} \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}||\theta_{\alpha\alpha}^\epsilon||_{s-2}\sqrt{||\theta_\alpha^\epsilon||_{s-1}}
\end{aligned}
$$

$$
\begin{aligned}
||\Xi_8||_{s-2} &\leq ||(\chi_\epsilon \theta_\alpha^\epsilon)(\tilde{V}^\epsilon - \tilde{W}^\epsilon \cdot \hat{t}^\epsilon)_\alpha||_{s-1} \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}(||\tilde{W}_\alpha^\epsilon \cdot \hat{t}||_{L^\infty} + ||\partial_\alpha(\tilde{W}_\alpha^\epsilon \cdot \hat{t})||_{s-2}) \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}||\theta^\epsilon||_s\sqrt{||\theta_\alpha^\epsilon||_{s-1}}
\end{aligned}
$$

$$
\begin{aligned}
||\Xi_9||_{s-2} &\leq ||\partial_\alpha(m^\epsilon \cdot \hat{n}^\epsilon)||_{s-1} \\
&\lesssim ||m_\alpha^\epsilon||_{s-1} + ||m^\epsilon||_{L^\infty}||\partial_\alpha^{s-1}\hat{n}^\epsilon||_{L^2} \\
&\lesssim ||\theta_\alpha^\epsilon||_{s-1}^{3/2} + ||\theta_\alpha^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{s-1}
\end{aligned}
$$

Therefore, combining the bounds on all the Ξ_i, we have

$$
||\Upsilon_7^\epsilon||_{s-2} \lesssim ||\theta_\alpha^\epsilon||_{s-1}^{3/2} \tag{8.25}
$$

And therefore,

$$\int \partial_\alpha^s(\chi_\epsilon\theta^\epsilon)\partial_\alpha^{s-2}\Upsilon_7^\epsilon \lesssim ||\partial_\alpha^s\theta^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{s-1}^{3/2} \tag{8.26}$$

Finally, as in the local existence case, the $\frac{-\tau}{2}\Lambda^3(\chi_\epsilon\theta_{\alpha\alpha}^\epsilon)$ term in $\theta_{\alpha\alpha,t}^\epsilon$ gives us

$$\int (\chi_\epsilon\partial_\alpha^s\theta^\epsilon)\frac{-\tau}{2}\Lambda^3(\chi_\epsilon\partial_\alpha^s\theta^\epsilon) = \frac{-\tau}{2}\int(\Lambda^{3/2}\chi_\epsilon\partial_\alpha^s\theta^\epsilon)^2 = \frac{-\tau}{2}||\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2 \tag{8.27}$$

Therefore, combining (8.21), (8.24), (8.26), (8.27), and (8.8), we have

$$\begin{aligned}\frac{dE}{dt} &\lesssim -||\Lambda^{1/2}\theta^\epsilon||_{L^2}^2 - ||\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2 + ||\theta_\alpha^\epsilon||_{H^2}^2||\theta^\epsilon||_{H^2}^2 \\ &+ ||\theta_\alpha^\epsilon||_{s-1}^3 + ||\theta^\epsilon||_s||\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2 \\ &+ ||\partial_\alpha^s\theta^\epsilon||_{L^2}||\theta_{\alpha\alpha\alpha}^\epsilon||_{s-3}||\theta^\epsilon||_s\sqrt{||\theta_\alpha^\epsilon||_{s-1}} \\ &+ ||\partial_\alpha^s\theta^\epsilon||_{L^2}||\theta_\alpha^\epsilon||_{s-1}^{3/2}\end{aligned} \tag{8.28}$$

Simplifying, this becomes

$$\begin{aligned}\frac{dE}{dt} &\lesssim -||\Lambda^{1/2}\theta^\epsilon||_{L^2}^2 - ||\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2 + ||\theta_\alpha^\epsilon||_{s-1}^2||\theta^\epsilon||_s^{1/2} \\ &+ ||\theta_\alpha^\epsilon||_{s-1}^{5/2} + ||\theta^\epsilon||_s^{1/2}||\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2 \\ &+ ||\partial_\alpha^s\theta^\epsilon||_{L^2}||\theta_{\alpha\alpha\alpha}^\epsilon||_{s-3}\sqrt{||\theta_\alpha^\epsilon||_{s-1}} \\ &+ ||\theta_\alpha^\epsilon||_{s-1}^{5/2}\end{aligned}$$

Next, we collapse the positive terms to get

$$\frac{dE}{dt} \lesssim -||\Lambda^{1/2}\theta^\epsilon||_{L^2}^2 - ||\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2 + (||\theta_\alpha^\epsilon||_{s-1}^2 + ||\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2)||\theta^\epsilon||_s^{1/2}$$

However, since $||\theta_\alpha^\epsilon||_{s-1}, ||\Lambda\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2} \lesssim ||\Lambda^{1/2}\theta^\epsilon||_{L^2} + ||\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}$, we obtain

$$\frac{dE}{dt} \lesssim (||\Lambda^{1/2}\theta^\epsilon||_{L^2}^2 + ||\Lambda^{3/2}\partial_\alpha^s\chi_\epsilon\theta^\epsilon||_{L^2}^2)(-1 + ||\theta^\epsilon||_s^{1/2}) \leq 0 \tag{8.29}$$

Therefore, $\frac{d}{dt}||\theta^\epsilon||_s \leq 0$ for all time t, and therefore θ^ϵ exists globally in time. □

Proof of Theorem 1.3: By Theorem 8.1, we know that $||\theta^\epsilon||_s$ is non-increasing, and therefore θ^ϵ can never leave the set O. Therefore, the maximal time T in Lemma 5.4 is infinity, and so applying Theorem 1.1, we know that $\theta \in C([0, \infty); H^s)$, as desired. □

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