ISOLATED HYPERSURFACE SINGULARITIES AS NONCOMMUTATIVE SPACES

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A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2010

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Acknowledgments

First and foremost, I would like to thank my advisor Tony Pantev for his continuous support and encouragement throughout my time at Penn. Tony has been an excellent advisor. His enthusiasm for my work and for mathematics in general was always a great source of motivation. I am deeply grateful for all the insights which he shared with me while giving me confidence to explore my own ideas.

There are many others without whom this work would not have been possible and it is a pleasure to thank them. I begin by thanking Pranav Pandit for being my mathematical companion and a great friend. Much of my current mathematical perspective has been shaped in our countless mathematical discussions.

I thank the Penn faculty for creating such an inspiring atmosphere in the department. Special thanks to Jonathan Block, David Harbater, Florian Pop and Jim Stasheff for their interest in my work which was highly motivating. Further, I thank Matthew Ballard and Patrick Clarke for many inspiring conversations. Thanks also to Ron Donagi and Jonathan Block for serving on my thesis committee.

Thanks to SAS for the Dissertation Completion Fellowship which supported my last year at Penn.

The department's daily survival depends entirely on the hard work of Janet, Monica, Paula and Robin. Many thanks to them for being so helpful and friendly, and for never losing their patience with us absentminded grad students.

I would like to thank Bertrand Toën for his suggestions on 2-periodic dg categories and valuable comments in general. Daniel Murfet has made various valuable comments and suggestions for which I am very grateful. I also thank him for being a coauthor of a paper building on this thesis and his own work. Further, I thank Chris Brav, Ragnar Buchweitz, Andrei Caldararu and Mikhail Khovanov for their invitations to give talks on this work, and for many inspiring discussions on the contents. Many thanks to Denis Auroux, Ludmil Katzarkov and Paul Seidel for giving me the opportunity to speak at the Workshop on Homological Mirror Symmetry in Miami.

I would also like to express gratitude to my teachers and advisors before my time at Penn. I thank Manfred Schwall for lending me the high school telescope and arousing my interest in physics, Heinrich Matzat and Julia Hartmann for guiding me during my Diplom studies in Heidelberg.

My friends at the department have made life in DRL thoroughly enjoyable and I thank everyone for that. In particular, I thank Armin, Umut, Alberto, Dragos, Shuvra, Pranav, Aaron, John, Calder, Andrew, David, Colin, Ting, Ricardo and Sohrab for the many fun times. Special thanks to Shuvra for being "our only true friend". Thanks to Armin, Dragos and Pranav for being great officemates.

I thank my parents, Christine and Jürgen, and my sister Kathrin for their support and encouragement in all my endeavors.

Finally, I thank Cisca for being the most supportive and loving girlfriend I could possibly hope for.

ABSTRACT

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We study the category of matrix factorizations associated to the germ of an isolated hypersurface singularity. This category is shown to admit a compact generator which is given by the stabilization of the residue field. We deduce a quasi-equivalence between the category of matrix factorizations and the dg derived category of an explicitly computable dg algebra. Building on this result, we employ a variant of Toën's derived Morita theory to identify continuous functors between matrix factorization categories as integral transforms. This enables us to calculate the Hochschild chain and cochain complexes of these categories. Finally, we give interpretations of the results of this thesis in terms of noncommutative geometry based on dg categories.

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Chapter 1

Introduction

Let k be an arbitrary field and let $(R, \mathfrak{m})$ be a regular local k-algebra with residue field k. We fix an element $w \in \mathfrak{m}$ and introduce the corresponding hypersurface algebra $S = R/w$. We are interested in the case when the hypersurface $\mathrm{Spec}(S)$ has an isolated singularity at $\mathfrak{m}$. Singularities of this kind have been a classical object of study for centuries. The perspective on hypersurface singularities we take in this thesis is one motivated by noncommutative algebraic geometry based on differential graded categories in the sense of [KKP08]. Namely, we study a dg category which we want to think of as the category of complexes of sheaves on a hypothetical noncommutative space $\mathcal{X}$ attached to the singularity: the category of matrix factorizations of w. In this thesis, we establish various properties of this category and discuss the geometric implications for $\mathcal{X}$. Specifically, we show that $\mathcal{X}$ is a dg affine, homologically smooth and proper noncommutative Calabi-Yau space over k.

We calculate the noncommutative analogues of Hodge and de Rham cohomology and show that the Hodge-to-de Rham spectral sequence degenerates. Furthermore, we study the derived Morita theory of matrix factorization categories which enables us to determine the Hochschild cohomology and thus the deformation theory of $\mathcal{X}$.

A matrix factorization of w is defined to be a $\mathbb{Z}/2$-graded finite free R-module X together with an R-linear endomorphism d of odd degree satisfying $d^2 = w\,\mathrm{id}_X$. The collection of all matrix factorizations naturally forms a differential $\mathbb{Z}/2$-graded category which we denote by $\mathrm{MF}(R, w)$. The associated homotopy category is denoted by $[\mathrm{MF}(R, w)]$. Matrix factorizations first appeared in Eisenbud's work [Eis80] on the homological algebra of complete intersections. Since then, they have been used extensively in singularity theory. We refer the reader to [Yos90] for a survey as well as further references. In the unpublished work [Buc86], Buchweitz introduced the notion of the stabilized derived category, giving a new conceptual perspective on Eisenbud's work and extending it to a more general context. More recently, matrix factorizations were proposed by Kontsevich as descriptions of B-branes in Landau-Ginzburg models in topological string theory. As such they appear in the framework of mirror symmetry as for example explained in [Sei08]. Orlov [Orl03] introduced the singularity category, generalizing Buchweitz's categorical construction to a global setup, and established various important results in [Orl05a, Orl05b, Orl09].

In Chapter 2, we survey some important aspects of the inspiring articles [Eis80] and [Buc86], which lead to the intuitive insight that the category of matrix factorizations describes the stable homological features of the algebra S. The main purpose is to introduce notation and to formulate the results in the form needed later on.

Section 3 addresses the question of the existence of generators in matrix factorization categories. We construct a compact generator of the category $\mathrm{MF}^{\infty}(R, w)$ consisting of factorizations of possibly infinite rank. Our argument utilizes Bousfield localization to reduce the problem to a statement which we call homological Nakayama lemma for infinitely generated maximal Cohen-Macaulay modules. This lemma seems to be an interesting result in its own right since the Nakayama lemma obviously fails for general infinitely generated modules. We use a method of Eisenbud to explicitly construct the generator as a matrix factorization corresponding to the stabilization of the residue field.

Section 4 contains some first applications of the results on compact generation.

We start by introducing a homotopy theoretic framework for 2-periodic dg categories, analogous to [Toë07], which will serve as a natural context to study matrix factorization categories.

Using a method due to Keller [Kel94], we obtain a quasi-equivalence between the category $\mathrm{MF}^{\infty}(R, w)$ and the dg derived category of modules over a dg algebra A. This algebra A is given as the endomorphism algebra of the compact generator. Our concrete description of this generator as a stabilized residue field allows us to determine A explicitly. As an immediate corollary, we obtain that the idempotent completion of $[\mathrm{MF}(R, w)]$ coincides with $[\mathrm{MF}(\widehat{R}, w)]$.

In addition, we illustrate how to compute a minimal A_∞-model for A. The transfer method we use originates from the work of Gugenheim-Stasheff [GS86] and Merkulov [Mer99], the elegant description in terms of trees is due to Kontsevich-Soibelman [KS01]. In the case of a quadratic hypersurface the A_∞-structure turns out to be formal and we recover a variant of a result of Buchweitz, Eisenbud and Herzog [BEH87] describing matrix factorizations as modules over a certain Clifford algebra. In the general case, we are able to give partial formulas for the higher multiplications which are neatly related to the higher coefficients of w.

In Chapter 5, we use Toën's derived Morita theory for dg categories [Toë07] to describe functors between categories of matrix factorizations. It turns out that every continuous functor can be represented by an integral transform. We describe the identity functor as an integral transform with kernel given by the stabilized diagonal. This allows us to calculate the Hochschild cochain complex of matrix factorization categories as the derived endomorphism complex of the identity functor.

Furthermore, we compute the Hochschild chain complex, proving along the way that it is quasi-isomorphic to the derived homomorphism complex between the inverse Serre functor and the identity functor.

In the last chapter, we give interpretations of our results in terms of noncommutative geometry in the sense of [KKP08].

Finally, I would like to point out relations to previous work. I thank Daniel Murfet for informing me that Corollary 4.3 was first proven by Schoutens in [Sch03]. An independent proof by Murfet will be contained as an appendix in [KVdB08]. It is also possible to deduce the statement using results from [Orl09] as explained in [Sei08, 11.1]. However, Theorem 3.1 is stronger since it also implies that the category $[\mathrm{MF}^{\infty}(R, w)]$ is compactly generated which is essential to obtain Theorem 4.2.

I thank Paul Seidel for drawing my attention to his work [Sei08]. The dg algebra $A = \mathrm{End}(k^{\mathrm{stab}})$ which we construct in Section 4 already appears in Section 10 of loc. cit. and is interpreted as a deformed Koszul dual. In fact, one may expect that the algebra A is in fact the Koszul dual, in the sense of [Pos09], of the curved dg algebra R with zero differential and curvature w. Within this framework, Theorem 4.2 could be interpreted as an equivalence of appropriate module categories over the curved algebra R and its Koszul dual A. The homological perturbation techniques which we apply in Section 4 were already used in Section 10 of [Sei08].

The idea of describing functors between matrix factorization categories as integral transforms appeared in [KR08]. Furthermore, our Theorem 3.6 is inspired by Proposition 7 in loc. cit. As the authors informed me, the argument in loc. cit. is only valid for bounded below $\mathbb{Z}$-graded matrix factorizations of isolated singularities, which is sufficient for the purposes of loc. cit. However, in this thesis we are specifically interested in the $\mathbb{Z}/2$-graded case so we have to use the alternative argument given in the proof of Theorem 3.6. As an application, we then also prove a $\mathbb{Z}/2$-graded version of [KR08, Proposition 8] in the form of Corollary 4.4.

The relation between idempotent completion and formal completion is studied in a more general context on the level of triangulated categories in [Orl09].

A heuristic calculation of the Hochschild cohomology in the one-variable case with $w = x^n$ was carried out in [KR04]. This article already contains the essential idea to represent the identity functor by a matrix factorization.

There are two alternative approaches to the calculation of Hochschild invariants of matrix factorization categories. In [Seg09], the bar complex of the category is used to calculate Hochschild homology. However, the author uses the product total complex and it is not clear how this notion of homology is related to the usual Hochschild homology. Therefore, some additional reasoning is required to make this argument work. In forthcoming work by Caldararu-Tu [CT09], the category of matrix factorizations is considered as a category of modules over the above mentioned curved dg algebra given by R in even degree, zero differential and curvature w.

From this curved dg algebra the authors construct an explicit bar complex, generalizing the bar complex for dg algebras. The Hochschild homology of the curved dg algebra is then defined to be the cohomology of this complex. The relation between the Hochschild homology of the curved dg algebra and the Hochschild homology of the category of modules over it is stated as a conjecture.

We should also mention that Theorem 5.7 has been anticipated for some time. For example, it is stated without proof in [KKP08]. However, to my knowledge, no complete proof has previously appeared in the literature.

Chapter 2

Homological algebra of matrix factorizations

We fix some notation which we will use throughout the thesis. Let k be an arbitrary field and let $(R, \mathfrak{m})$ be a regular local k-algebra with residue field k. Let $x_1, \ldots, x_n$ be a minimal system of generators of the maximal ideal $\mathfrak{m} \subset R$ and denote the derivations on R corresponding to these generators by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. We fix $w \in \mathfrak{m}$ which we call the *superpotential* and say w has an isolated singularity if the *Tyurina algebra* $\Omega_w := R/(w, \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n})$ is a finite dimensional k-vector space.

Definition 2.1. The *category of matrix factorizations* $\mathrm{MF}(R, w)$ of the superpotential w on R is defined to be the differential $\mathbb{Z}/2$-graded category specified by the following data:

- The objects of $\mathrm{MF}(R, w)$ are pairs (X, d) where $X = X^0 \oplus X^1$ is a free $\mathbb{Z}/2$-graded R-module of finite rank equipped with an R-linear map d of odd degree satisfying $d^2 = w \,\mathrm{id}_X$.

- The morphisms $\mathrm{Hom}(X, X')$ are given by the $\mathbb{Z}/2$-graded module of R-linear maps from X to X' provided with the differential given by

$$d(f) = d_{X'} \circ f - (-1)^{|f|} f \circ d_X.$$

One easily verifies that $\mathrm{Hom}(X, X')$ is a complex. The *homotopy category of matrix factorizations* $[\mathrm{MF}(R, w)]$ is obtained by applying $\mathrm{H}^0(-)$ to the morphism complexes of $\mathrm{MF}(R, w)$.

After choosing bases for X^0 and X^1, we obtain a pair

$$X^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X^0$$

of matrices (φ, ψ) such that

$$\varphi \circ \psi = \psi \circ \varphi = w \,\mathrm{id}\,.$$

This immediately implies that the ranks of X^0 and X^1 agree, so φ and ψ are in fact quadratic matrices.

Example 2.2. Consider $R = \mathbb{C}[[x]]$ and $w = x^n$. Then we have factorizations

$$R \underset{x^{n-k}}{\overset{x^k}{\rightleftarrows}} R$$

and these are in fact the only indecomposable objects in $\mathrm{Z}^0(\mathrm{MF}(R, w))$ (cf. [Yos90], [KL04]).

Example 2.3. Consider $R = \mathbb{C}[[x, y, z]]$ and $w = x^3 + y^3 + z^3 - 3xyz$. Suppose that $(a, b, c) \in (\mathbb{C}^*)^3$ is a zero of w. The matrix

$$\varphi = \begin{pmatrix} ax & cy & bz \\ cz & bx & ay \\ by & az & cx \end{pmatrix}$$

satisfies $\det(\varphi) = abcw$. Thus setting $\psi = \frac{1}{abc}\varphi^{\#}$, where $-^{\#}$ denotes the matrix of cofactors, we obtain a family of rank 3 factorizations parameterized by $\{w = 0\} \subset (\mathbb{C}^*)^3$ (cf. [BHLW06]).

We will now explain the relevance of the category of matrix factorizations in terms of homological algebra. To this end, we recall how matrix factorizations naturally arise in Eisenbud's work [Eis80].

2.1 Eisenbud's matrix factorizations

We start with a prelude on the homological algebra of regular local rings which will put us in the right context. Let $(R, \mathfrak{m})$ be a regular local ring and let M be a finitely generated R-module. A sequence $x_1, \ldots, x_r \in \mathfrak{m}$ is called an M-sequence if x_i is a nonzerodivisor in $M/(x_1, \ldots, x_{i-1})$ for all $1 \leq i \leq r$. The *depth* of M is the length of a maximal M-sequence. The *projective dimension* $\mathrm{pd}(M)$ of M is the length of a minimal free resolution of M. The Auslander-Buchsbaum formula (see e.g. [Eis95]) relates these notions via

$$\mathrm{pd}(M) = \dim(R) - \mathrm{depth}(M).$$

This yields a rather precise understanding of free resolutions over regular local rings. Let us point out two immediate important consequences. Firstly, the length of minimal free resolutions is bounded by the Krull dimension of R. Secondly, if the depth of a module M equals the Krull dimension of R, then M is free.

A natural problem is to try and obtain a similar understanding of free resolutions over singular rings. An example of such a ring is a hypersurface singularity defined by $S = R/w$, where w is singular at the maximal ideal. It turns out that in contrast to the regular case, the condition

$$\mathrm{depth}(M) = \dim(S) \tag{2.1}$$

does not imply that M is free. A finitely generated S-module satisfying (2.1) is called a *maximal Cohen-Macaulay module*.

Let M be a maximal Cohen-Macaulay module over S. We may consider M as an R-module which is annihilated by w and use the Auslander-Buchsbaum formula

$$\mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}(M)$$

to deduce that M admits an R-free resolution of length 1. Hence, we obtain an exact sequence

$$0 \longrightarrow X^1 \xrightarrow{\varphi} X^0 \longrightarrow M \to 0$$

where X^0 and X^1 are free R-modules. Since multiplication by w annihilates M, there exists a homotopy ψ such that the diagram

$$\begin{array}{ccc} X^1 & \xrightarrow{\varphi} & X^0 \\ {\scriptstyle w}\downarrow & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle w} \\ X^1 & \xrightarrow{\varphi} & X^0 \end{array}$$

commutes. Thus, the pair (φ, ψ) is a matrix factorization of w, such that the original maximal Cohen-Macaulay module M is isomorphic to $\operatorname{coker}(\varphi)$.

We come back to the question about properties of S-free resolutions of M. Curiously, every maximal Cohen-Macaulay module over S admits a 2-periodic S-free resolution. It is obtained by reducing the corresponding matrix factorization modulo w and extending 2-periodically:

$$\cdots \longrightarrow \overline{X^1} \xrightarrow{\overline{\varphi}} \overline{X^0} \xrightarrow{\overline{\psi}} \overline{X^1} \xrightarrow{\overline{\varphi}} \overline{X^0} \longrightarrow M \longrightarrow 0.$$

We illustrate the consequences of this construction from a categorical point of view. Consider the *stable category of maximal Cohen-Macaulay S-modules* $\underline{\mathrm{MCM}}(S)$ which is defined as follows. The objects are maximal Cohen-Macaulay modules, the morphisms are defined by

$$\underline{\mathrm{Hom}}_S(M, M') = \mathrm{Hom}_S(M, M')/P,$$

where P denotes the set of S-linear homomorphisms factoring through some free S-module. Reversing the above construction we can associate the maximal Cohen-

Macaulay module coker(φ) to a matrix factorization given by

$$X^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X^0.$$

In fact, this assignment extends to a functor

$$\text{coker}: \ [\text{MF}(R,w)] \to \underline{\text{MCM}}(S)$$

establishing an equivalence between the homotopy category of matrix factorizations and the stable category of maximal Cohen-Macaulay modules.

2.2 Buchweitz's stabilized derived category

We now want to focus on resolutions of arbitrary finitely generated S-modules. To this end, we are lucky as it turns out that high enough syzygies of any such a module are maximal Cohen-Macaulay. This follows from the homological characterization of maximal Cohen-Macaulay modules as being $\text{Hom}_S(-,S)$-acyclic combined with the fact that S has finite injective dimension. An immediate implication is the following striking result.

Theorem 2.4 (Eisenbud)**.** *Every finitely generated S-module admits a free resolution which will eventually become 2-periodic.*

In other words, the resolution "stabilizes" leading to the slogan that the category $\underline{\text{MCM}}(S)$ describes the stable homological algebra of S.

While the category $\underline{\mathrm{MCM}}(S)$ restricts attention to maximal Cohen-Macaulay modules, Buchweitz's stabilized derived category is designed to capture the fact that *arbitrary* finitely generated S-modules stabilize. Let $\mathrm{D}^{\mathrm{b}}(S)$ denote the derived category of all complexes of S-modules with finitely generated total cohomology. Such a complex is called *perfect* if it is isomorphic in $\mathrm{D}^{\mathrm{b}}(S)$ to a bounded complex of free S-modules. The full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(S)$ formed by the perfect complexes is denoted by $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$. It is easy to see that $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$ forms a thick subcategory of $\mathrm{D}^{\mathrm{b}}(S)$. The *stabilized derived category of* S is then defined to be the Verdier quotient

$$\underline{\mathrm{D}}^{\mathrm{b}}(S) := \mathrm{D}^{\mathrm{b}}(S)/\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(S).$$

There exists an obvious functor

$$\underline{\mathrm{MCM}}(S) \to \underline{\mathrm{D}}^{\mathrm{b}}(S)$$

which Buchweitz proves to be an equivalence of categories. Observe that $\underline{\mathrm{D}}^{\mathrm{b}}(S)$ as well as $[\mathrm{MF}(R, w)]$ are naturally triangulated categories which via the above equivalences induce two triangulated structures on $\underline{\mathrm{MCM}}(S)$. Those structures turn out to be isomorphic (via the identity functor). The triangulated structure can also be constructed directly using the fact that $\underline{\mathrm{MCM}}(S)$ is the stable category associated to the Frobenius category $\mathrm{MCM}(S)$ (cf. [Kel90]).

It is interesting to describe the morphisms in the category $\underline{\mathrm{D}}^{\mathrm{b}}(S)$.

Proposition 2.5 (Buchweitz)**.** *Let X, Y be complexes in $\underline{\mathrm{D}}^{\mathrm{b}}(S)$. Then there exists* *a natural number $i(X,Y)$ such that*

$$\mathrm{Hom}_{\underline{\mathrm{D}}^{\mathrm{b}}(S)}(X, Y[i]) \cong \mathrm{Hom}_{\mathrm{D}^{b}(S)}(X, Y[i])$$

for $i \geq i(X,Y)$.

The proposition explains the nomenclature for $\underline{\mathrm{D}}^{\mathrm{b}}(S)$. The Ext-groups in the derived category "stabilize" in high degrees. After this stabilization has taken place, the Ext-groups and the $\underline{\mathrm{Ext}}$-groups coincide. Buchweitz introduces $\underline{\mathrm{D}}^{\mathrm{b}}(S)$ more generally for Gorenstein algebras. In our specific situation of a local hypersurface algebra S the phenomenon of stabilization just translates into the above mentioned fact that resolutions over S eventually become 2-periodic.

Combining the equivalences of categories explained in this section, we conclude that a finite S-module interpreted as an object of $\underline{\mathrm{D}}^{\mathrm{b}}(S)$ functorially corresponds to a matrix factorization. If L is an S-module, we call the corresponding matrix factorization L^{stab} the *stabilization of* L. The objects of our interest tend to naturally arise as objects of $\underline{\mathrm{D}}^{\mathrm{b}}(S)$ and we will analyze them computationally by studying their stabilization.

2.3 Stabilization

Let L be an S-module. In [Eis80, Section 7], Eisenbud gives a method for explicitly constructing L^{stab} in terms of an R-free resolution of L. We apply his construction in the case when L is a module of the form $L = R/I$ such that the ideal I is generated by a regular sequence and $w \in I$. Since we use the construction throughout the article, we give a detailed description of this special case.

Let $f_1, \ldots, f_m$ be a regular sequence generating I. Consider the corresponding Koszul complex

$$K^\bullet = (\bigwedge^\bullet V,\ s_0),$$

where $V = R^m$ and s_0 denotes contraction with $(f_1, \ldots, f_m) \in \mathrm{Hom}_R(V, R)$. The complex $K^\bullet$ is an R-free resolution of L. Since w annihilates the R-module L, multiplication by w on $K^\bullet$ is homotopic to zero. In fact, we can explicitly construct a contracting homotopy. Since $w \in I$, we can write $w = \sum_i f_i w_i$ for some elements $w_i \in R$. Exterior multiplication with the element $(w_1, \ldots, w_m) \in V$ defines a contracting homotopy which we denote by s_1.

Since both s_0 and s_1 square to 0, the $\mathbb{Z}/2$-graded object

$$(\bigoplus_{i=0}^{m} \bigwedge^i V,\ s_0 + s_1)$$

defines a matrix factorization of w. We claim that it represents the stabilization of L. To see this we will construct an explicit S-free resolution of L. Define $Z^\bullet$ to be the total complex of the double complex

$$
\begin{array}{ccccccccccccc}
\ddots & & & & \ddots & & & & \ddots & & & & \\
\overline{K^0} & \xrightarrow{s_0} & \overline{K^1} & \xrightarrow{s_0} & \overline{K^2} & \longrightarrow & \cdots & \longrightarrow & \overline{K^m} & & & & \\
 & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} & & & & \downarrow{\scriptstyle s_1} & & & & \\
 & & \overline{K^0} & \xrightarrow{s_0} & \overline{K^1} & \longrightarrow & \cdots & \longrightarrow & \overline{K^{m-1}} & \xrightarrow{s_0} & \overline{K^m} & & \\
 & & & & \downarrow{\scriptstyle s_1} & & & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} & & \\
 & & & & \overline{K^0} & \longrightarrow & \cdots & \longrightarrow & \overline{K^{m-2}} & \xrightarrow{s_0} & \overline{K^{m-1}} & \xrightarrow{s_0} & \overline{K^m}
\end{array}
$$

where $\overline{}$ denotes the functor $- \otimes_R S$.

Lemma 2.6 (Eisenbud)**.** *The complex $Z^\bullet$ is an S-free resolution of L.*

Proof. We use the spectral sequence arising from the horizontal filtration to compute the cohomology of $Z^\bullet$. On the first page we obtain

$$
\begin{array}{ccc}
\ddots & L & \\
 & \downarrow{\scriptstyle g} & \\
 & L & L \\
 & & \downarrow{\scriptstyle g} \\
 & & L \quad\quad L
\end{array}
$$

since the complex $K^\bullet \otimes_R S$ is isomorphic in $\mathrm{D^b}(R)$ to the complex $L \otimes_R (R \xrightarrow{w} R)$ and L is annihilated by w. To determine the map g we use the roof establishing the just mentioned isomorphism.

$$
\begin{array}{ccc}
 & (\ldots K^2 \xrightarrow{s_0} K^1 \xrightarrow{s_0} K^0) \otimes_R (R \xrightarrow{w} R) & \\
 \swarrow{\scriptstyle p_1,\ \simeq} & & \searrow{\scriptstyle p_2,\ \simeq} \\
(\ldots K^2 \xrightarrow{s_0} K^1 \xrightarrow{s_0} K^0) \otimes_R S & & L \otimes_R (R \xrightarrow{w} R)
\end{array}
$$

We introduce the homotopy

$$R \underset{t}{\overset{w}{\rightleftarrows}} R$$

which is simply the identity map on R. Then the map $s_1 \otimes 1 + 1 \otimes t$ is a map of degree -1 on the complex forming the apex of the roof. The map induced on $K^\bullet \otimes_R S$ via p_1 is $s_1 \otimes 1$ while the one induced on $L \otimes_R (R \to R)$ via p_2 is $1 \otimes t$. This proves that the vertical maps g on the first page of the above spectral sequence are in fact given by the identity on L. Passing to the second page of the spectral sequence we immediately obtain the result. ☐

Corollary 2.7. *The stabilization L^{stab} of L is given by the matrix factorization*

$$(\bigoplus_{i=0}^{m} \bigwedge^i V,\ s_0 + s_1).$$

Proof. This simply follows by inspecting the explicit form of the constructed S-free resolution of L. It becomes 2-periodic after m steps where the 2-periodic part is exactly the reduction modulo w of the given matrix factorization. ☐

It is convenient to formulate this construction in the language of supergeometry as it is often done in the physics literature. The underlying space of L^{stab} can be interpreted as the superalgebra $R\langle\theta_1, \ldots, \theta_m\rangle$ where θ_i are odd supercommuting variables and R has degree 0. The twisted differential defining the factorization corresponds to the odd differential operator $\delta = \sum \delta_i$ with

$$\delta_i = f_i \frac{\partial}{\partial\theta_i} + w_i\theta_i.$$

This interpretation is useful, since every R-linear endomorphism of the super polynomial ring $R\langle\theta_1,\ldots,\theta_m\rangle$ is represented by a differential operator, as can be easily seen by a dimension count. Thus, denoting the $\mathbb{Z}/2$-graded R-module of all polynomial differential operators on $R\langle\theta_1,\ldots,\theta_m\rangle$ by A, we obtain the explicit description

$$\mathrm{Hom}_{\mathrm{MF}(R,w)}(L^{\mathrm{stab}}, L^{\mathrm{stab}}) \cong (A,\ [\delta,-])$$

of the dg algebra of endomorphisms of L^{stab}.

Chapter 3

Generators in matrix factorization categories

We use the same notation as in the previous chapter. In order to obtain a setup in which we can talk about compactness and apply the technique of Bousfield localization (cf. [BN93]), we are forced to enlarge the category of matrix factorizations to admit arbitrary coproducts. To this end, we use the category $\mathrm{MF}^{\infty}(R, w)$ of matrix factorizations of possibly infinite rank. By the existence of a compact generator this category will simply turn out to be an explicit model for the dg derived category of unbounded modules over $\mathrm{MF}(R, w)$ (see Theorem 4.2).

We introduce some general notions. Let $\mathcal{T}$ be a triangulated category admitting infinite coproducts. Let X be an object of $\mathcal{T}$. We call X *compact* if the functor $\mathrm{Hom}(X, -)$ commutes with infinite coproducts. The object X is a *generator* of

$\mathcal{T}$ if the smallest triangulated subcategory of $\mathcal{T}$ containing X and closed under coproducts and isomorphisms is $\mathcal{T}$ itself. The full subcategory of $\mathcal{T}$ consisting of all objects Y satisfying $\mathrm{Hom}(X[i], Y) = 0$ for all integers i is called the *right orthogonal complement of* X. Using Bousfield localization, one shows that for a compact object X the right orthogonal complement of X is equivalent to 0 if and only if X is a generator of $\mathcal{T}$ ([SS03, Lemma 2.2.1]). With this terminology, we can state the result which we prove in this chapter.

Theorem 3.1. *Assume that w has an isolated singularity and consider the residue field k as an S-module. Then k^{stab} is a compact generator of the triangulated category* $[\mathrm{MF}^{\infty}(R, w)]$.

For the proof we need some preparation.

3.1 The stabilized residue field

We start by computing the matrix factorization k^{stab} and its endomorphism dg algebra explicitly, using Eisenbud's method. It can be applied in the form presented in 2.3 since the maximal ideal $\mathfrak{m}$ is generated by the regular sequence $x_1, \ldots, x_n$. Writing $w = \sum x_i w_i$, we obtain the factorization k^{stab} as

$$(\bigoplus_{i=0}^{n} \bigwedge^i V,\ s_0 + s_1).$$

where s_0 denotes contraction with $(x_1, \ldots, x_n)$ and s_1 is given by exterior multiplication by $(w_1, \ldots, w_n)$.

In supergeometric terms, k^{stab} is given by the superalgebra $R\langle\theta_1,\dots,\theta_n\rangle$ with odd differential operator $\delta = \sum \delta_i$ with

$$\delta_i = x_i \frac{\partial}{\partial\theta_i} + w_i\theta_i.$$

We denote the $\mathbb{Z}/2$-graded R-module of all polynomial differential operators on $R\langle\theta_1,\dots,\theta_n\rangle$ by A and obtain the description

$$\mathrm{Hom}_{\mathrm{MF}(R,w)}(k^{\mathrm{stab}}, k^{\mathrm{stab}}) \cong (A,\ [\delta, -]).$$

We will now reduce the proof of Theorem 3.1 to a statement in homological algebra.

3.2 Duality in $\mathrm{MCM}(S)$

Let $\underline{\mathrm{Mod}}(S)$ be the stable category of S-modules. The objects are arbitrary S-modules whereas the morphisms $\underline{\mathrm{Hom}}_S(M,N)$ are defined to be the quotient of $\mathrm{Hom}_S(M,N)$ by the two-sided ideal of morphisms factoring through some free S-module. Analogously to the case of finitely generated modules, the category $[\mathrm{MF}^\infty(R,w)]$ is equivalent to a full subcategory of $\underline{\mathrm{Mod}}(S)$ which we denote by $\underline{\mathrm{MCM}}^\infty(S)$. For our purposes, it is not necessary to give a characterization of the class of S-modules which form the objects of $\underline{\mathrm{MCM}}^\infty(S)$. We may think of them as generalized maximal Cohen-Macaulay modules.

The category $\underline{\mathrm{MCM}}(S)$ admits a dualization functor. For an object M we define the dual to be $\mathrm{D}(M) = \mathrm{Hom}_S(M, S)$. Maximal Cohen-Macaulay modules are characterized by their property of being $\mathrm{D}(-)$-acyclic (cf. [Buc86]). If M corresponds to the matrix factorization

$$X^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X^0,$$

i.e. $M = \mathrm{coker}(\varphi)$, then $\mathrm{D}(M)$ corresponds to the factorization

$$\mathrm{Hom}_R(X^0, R) \underset{\psi^*}{\overset{\varphi^*}{\rightleftarrows}} \mathrm{Hom}_R(X^1, R),$$

where $-^*$ denotes the R-linear dualization functor $\mathrm{Hom}_R(-, R)$. In other words, $\mathrm{D}(M) = \mathrm{coker}(\varphi^*)$, which proves that $\mathrm{D}(M)$ is in fact a maximal Cohen-Macaulay module.

The following proposition is proved in [Yos90, Lemma 3.9] for finitely generated modules.

Proposition 3.2. *Let M and N objects in $\underline{\mathrm{MCM}}^\infty(S)$ and assume M is finitely generated. Then we have a natural isomorphism*

$$\underline{\mathrm{Hom}}_S(M, N) \cong \mathrm{Tor}_1^S(\mathrm{D}(M), N).$$

Proof. Applying $\mathrm{Hom}_S(-, S)$ to the exact sequence

$$\cdots \longrightarrow \overline{X^0} \xrightarrow{\overline{\psi}} \overline{X^1} \xrightarrow{\overline{\varphi}} \overline{X^0} \longrightarrow M \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \mathrm{D}(M) \longrightarrow \mathrm{Hom}_S(\overline{X^0}, S) \xrightarrow{\overline{\varphi}^*} \mathrm{Hom}_S(\overline{X^1}, S) \xrightarrow{\overline{\psi}^*} \mathrm{Hom}_S(\overline{X^0}, S) \longrightarrow \cdots.$$

Using the 2-periodicity we can reflect this sequence to obtain the exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_S(\overline{X^1}, S) \xrightarrow{\overline{\psi}^*} \mathrm{Hom}_S(\overline{X^0}, S) \xrightarrow{\overline{\varphi}^*} \mathrm{Hom}_S(\overline{X^1}, S) \longrightarrow \mathrm{D}(M) \longrightarrow 0,$$

which is consistent with the above statement about the matrix factorization for $\mathrm{D}(M)$.

Therefore, $\mathrm{Tor}_1^S(\mathrm{D}(M), N)$ is given by the middle cohomology of the complex

$$\mathrm{Hom}_S(\overline{X^1}, N) \xrightarrow{\overline{\psi}^*} \mathrm{Hom}_S(\overline{X^0}, N) \xrightarrow{\overline{\varphi}^*} \mathrm{Hom}_S(\overline{X^1}, N).$$

It is immediate that the kernel of $\overline{\varphi}^*$ is isomorphic to $\mathrm{Hom}_S(M, N)$. Note that we have a commutative diagramm

$$\begin{array}{ccc} \mathrm{Hom}_S(\overline{X^1}, S) & \xrightarrow{\quad\overline{\psi}^*\quad} & \mathrm{Hom}_S(\overline{X^0}, S) \\ & \searrow \qquad \nearrow & \\ & \mathrm{Hom}_S(M, S) & \end{array}$$

which after applying $- \otimes_S N$ implies that we have an exact sequence

$$\mathrm{Hom}_S(M, S) \otimes_S N \xrightarrow{f} \mathrm{Hom}_S(M, N) \longrightarrow \mathrm{Tor}_1^S(\mathrm{D}(M), N) \longrightarrow 0.$$

Using the fact that M is finitely generated it is easy to see that the image of f consists of exactly those homomorphisms which factor through some free S-module. By the definition of $\underline{\mathrm{Hom}}_S(M, N)$ this implies the claim. □

Corollary 3.3. *For an object N in $\underline{\mathrm{MCM}}^\infty(S)$, we have an isomorphism*

$$\bigoplus_{i \in \mathbb{Z}/2} \underline{\mathrm{Hom}}_S(k^{\mathrm{stab}}[i], N) \cong \bigoplus_{i \in \{1,2\}} \mathrm{Tor}_i^S(k, N).$$

Remark. As analyzed more precisely in the proof the isomorphism does not necessarily preserve the parity of the grading.

Proof. We slightly abuse notation and denote both the matrix factorization as well as the corresponding object of $\underline{\mathrm{MCM}}(S)$ by k^{stab}. Using the fact that the Koszul complex is self-dual one verifies by inspecting the construction of k^{stab} that $\mathrm{D}(k^{\mathrm{stab}}) \cong k^{stab}[\epsilon]$, where ϵ is the parity of the Krull dimension of R. In other words, the stabilized residue field is, up to translation, self-dual. By Proposition 3.2 we conclude that

$$\underline{\mathrm{Hom}}_S(k^{\mathrm{stab}}, N) \cong \mathrm{Tor}^S_{1+\epsilon}(k^{stab}, N).$$

Next, we note that $\mathrm{Tor}^S_i(k, N)$ is 2-periodic, since N has by assumption a 2-periodic S-free resolution. On the other hand, $\mathrm{Tor}^S_i(k, N)$ agrees with $\mathrm{Tor}^S_i(k^{\mathrm{stab}}, N)$ for $i >> 0$. This implies

$$\mathrm{Tor}^S_i(k^{\mathrm{stab}}, N) \cong \mathrm{Tor}^S_i(k, N)$$

for all $i > 0$. □

Note that, since $\mathrm{Tor}^S_i(k, N)$ is 2-periodic, vanishing for $i \in \{1, 2\}$ implies vanishing for all $i > 0$. Therefore, we reduced Theorem 3.1 to a statement which we might call the *homological Nakayama lemma*: a module N in $\underline{\mathrm{MCM}}^{\infty}(S)$ is free if and only if $\mathrm{Tor}^S_i(k, N) = 0$ for all $i > 0$. For general infinitely generated S-modules this statement is certainly false, but it turns out to be true for modules in $\underline{\mathrm{MCM}}^{\infty}(S)$

if we assume that w has an isolated singularity. We will give the proof in the next section.

3.3 The homological Nakayama lemma

Consider a matrix factorization

$$X^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X^0$$

where X^0 and X^1 are free R-modules of possibly infinite rank and let $M = \mathrm{coker}(\varphi)$ be the corresponding S-module. As already explained, applying $- \otimes_R S$ to $X^\bullet$ and extending periodically one obtains an S-free resolution

$$\cdots \longrightarrow \overline{X^0} \xrightarrow{d} \overline{X^1} \xrightarrow{d} \overline{X^0} \longrightarrow M \longrightarrow 0$$

of M. We introduce the abbreviation $\partial_k = \frac{\partial}{\partial x_k}$. The formula $\partial_k w = \partial_k(\varphi\psi) = \partial_k(\varphi)\psi + \varphi\partial_k(\psi)$ establishes that multiplication by $\partial_k w$ is homotopic to zero on the endomorphism complex of X. Interpreting this fact in terms of the resolution $\overline{X}$ of M one easily deduces the following fundamental observation.

Lemma 3.4. *For any S-module N, multiplication by $\partial_k w$ annihilates the S-module $\mathrm{Tor}_i^S(N, M)$ for all $i > 0$.*

We will use the following result from [GJ81].

Theorem 3.5 (Gruson, Jensen)**.** *The projective dimension of an arbitrary flat S-module is at most $n - 1$.*

Recall that the *Tyurina algebra* is defined to be $\Omega_w = S/(\partial_1 w, \ldots, \partial_n w)$.

Theorem 3.6. *Let X be a matrix factorization of possibly infinite rank and let $M = \operatorname{coker}(\varphi)$. Then the following are equivalent:*

(1) M is a free S-module.

(2) M is a flat S-module.

(3) $\operatorname{Tor}_i^S(N, M) = 0$ for every finitely generated S-module N and $i > 0$.

(4) $\operatorname{Tor}_i^S(N, M) = 0$ for every finitely generated Ω_w-module N and $i > 0$.

If w has an isolated singularity then the above are equivalent to

(5) $\operatorname{Tor}_i^S(k, M) = 0$ for $i > 0$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(2) $\Rightarrow$ (1): The module M is flat and has therefore finite projective dimension by Theorem 3.5. This implies that $\operatorname{syz}^k(M)$ is projective for $k >> 0$. Since M has a 2-periodic resolution, we have $M \cong \operatorname{syz}^k(M)$ for every even natural number k. So M is projective and since S is local Kaplansky's theorem implies that M is free.

(3) $\Rightarrow$ (2): This is the standard homological criterion for flatness.

(4) ⇒ (3): Let us fix a partial derivative $\partial w = \partial_k w$ and let N be a finitely generated S-module. We compute the cohomology of the complex

$$S/\partial w \otimes_S^{\mathrm{L}} M \otimes_S^{\mathrm{L}} N$$

in two different ways. Namely, this complex is quasi-isomorphic to the total complex of the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\overline{X^0} \otimes_S N & \xleftarrow{\partial w} & \overline{X^0} \otimes_S N & \longleftarrow & 0 & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow & & \\
\overline{X^1} \otimes_S N & \xleftarrow{-\partial w} & \overline{X^1} \otimes_S N & \longleftarrow & 0 & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow & & \\
\overline{X^0} \otimes_S N & \xleftarrow{\partial w} & \overline{X^0} \otimes_S N & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}$$

which we may filter horizontally as well as vertically. Both filtrations lead to spectral sequences converging strongly to the target $\mathrm{H}^*(S/\partial w \otimes_S^{\mathrm{L}} M \otimes_S^{\mathrm{L}} N)$. The vertical filtration leads to a spectral sequence with E^1 given by

$$\begin{array}{cccccc}
\vdots & & \vdots & & \vdots & \\
\mathrm{Tor}_2^S(M,N) & \xleftarrow{\partial w} & \mathrm{Tor}_2^S(M,N) & \longleftarrow & 0 & \cdots \\
\mathrm{Tor}_1^S(M,N) & \xleftarrow{-\partial w} & \mathrm{Tor}_1^S(M,N) & \longleftarrow & 0 & \cdots \\
M \otimes_S N & \xleftarrow{\partial w} & M \otimes_S N & \longleftarrow & 0 & \cdots
\end{array}$$

It degenerates at E^2 which, using Lemma 3.4, is given by

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathrm{Tor}_2^S(M,N) & \mathrm{Tor}_2^S(M,N) & 0 & \cdots \\
\mathrm{Tor}_1^S(M,N) & \mathrm{Tor}_1^S(M,N) & 0 & \cdots \\
M \otimes_S N/\partial w & \mathrm{Tor}_1^S(M \otimes_S N, S/\partial w) & 0 & \cdots
\end{array}$$

This implies that $\mathrm{Tor}_i^S(M,N) = 0$ for $i > 0$ if and only if $\mathrm{H}^j(S/\partial w \otimes_S^{\mathrm{L}} M \otimes_S^{\mathrm{L}} N) = 0$ for $j \geq 2$.

Using the horizontal filtration of the above double complex we obtain a spectral sequence with first page

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
\overline{X^0} \otimes_S N/\partial w & \overline{X^0} \otimes_S \mathrm{Tor}_1^S(N, S/\partial w) & 0 & \cdots \\
\downarrow d & \downarrow d & \downarrow & \\
\overline{X^1} \otimes_S N/\partial w & \overline{X^1} \otimes_S \mathrm{Tor}_1^S(N, S/\partial w) & 0 & \cdots \\
\downarrow d & \downarrow d & \downarrow & \\
\overline{X^0} \otimes_S N/\partial w & \overline{X^0} \otimes_S \mathrm{Tor}_1^S(N, S/\partial w) & 0 & \cdots
\end{array}$$

and E^2 given by

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathrm{Tor}_2^S(M, N/\partial w) & \mathrm{Tor}_2^S(M, \mathrm{Tor}_1^S(N, S/\partial w)) & 0 & \cdots \\
\mathrm{Tor}_1^S(M, N/\partial w) & \mathrm{Tor}_1^S(M, \mathrm{Tor}_1^S(N, S/\partial w)) & 0 & \cdots \\
M \otimes_S N/\partial w & M \otimes_S \mathrm{Tor}_1^S(N, S/\partial w) & 0 & \cdots
\end{array}
$$

Observe that both S-modules $N/\partial w$ and $\mathrm{Tor}_1^S(N, S/\partial w)$ are finitely generated and annihilated by ∂w. This allows us to conclude that $\mathrm{H}^j(S/\partial w \otimes_S^{\mathrm{L}} M \otimes_S^{\mathrm{L}} N) = 0$ for $j \geq 2$ if $\mathrm{Tor}_i^S(M, -)$ vanishes on all finitely generated $S/\partial w$-modules for $i > 0$.

Applying this construction iteratively to all partial derivatives of w yields the implication.

(5) $\Rightarrow$ (4): These are standard arguments for modules over Artinian rings. Let N be a finitely generated Ω_w-module. It admits a finite filtration with successive quotients isomorphic to cyclic Ω_w-modules. If Ω_w/I is such a cyclic module then we obtain a short exact sequence

$$0 \longrightarrow K \longrightarrow \Omega_w/I \longrightarrow \Omega_w/\mathfrak{m} \longrightarrow 0$$

where by assumption $\Omega_w/\mathfrak{m} \cong k$. The claim follows inductively from inspection of the associated long exact Tor-sequence. $\square$

3.4 A counterexample

An example of a non-isolated singularity for which the homological Nakayama lemma does not hold can be constructed as follows. We consider the local k-algebra $R = k[[x, y]]$ with superpotential $w = xy^2$. The hypersurface algebra is given by $S = R/xy^2$ and we consider the S-module of formal Laurent series $k((x))$. Consider the matrix factorization with underlying $\mathbb{Z}/2$-graded R-module

$$X^1 = \bigoplus_{i\in\mathbb{Z}} Rh_i \oplus \bigoplus_{i\in\mathbb{Z}} Re_i$$

$$X^0 = \bigoplus_{i\in\mathbb{Z}} Rf_i \oplus \bigoplus_{i\in\mathbb{Z}} Rg_i$$

with φ given by the assigments

$$h_i \mapsto yf_i - xg_i + g_{i+1}$$

$$e_i \mapsto xyg_i$$

and ψ defined by

$$f_i \mapsto xyh_i + xe_i - e_{i+1}$$

$$g_i \mapsto ye_i.$$

The cokernel M of φ is a first syzygy module of the S-module $k((x))$. In fact, the factorization was found by imitating Eisenbud's method which in general only works for finitely generated modules. Using the factorization and the resulting 2-periodic S-free resolution of M one checks that $\mathrm{Tor}_i^S(k, M) = 0$ for all $i > 0$. But M is not free and in fact, consistently with Theorem 3.6, we have $\operatorname{Tor}_i^S(\Omega_w, M) \neq 0$ for both $i = 1$ and $i = 2$.

Chapter 4

First applications

4.1 The homotopy theory of 2-periodic dg categories

Before giving applications of Theorem 3.1, we introduce a homotopical framework for 2-periodic dg categories. Most statements in this section are immediate consequences or variants of well-known results. We define model structures in the 2-periodic context which allow us to obtain a homotopy theory analogous to the one developed in [Tab05, Toë07]. All dg categories are assumed to be small by virtue of choosing small quasi-equivalent dg categories. We refer to loc. cit. for details on how to take the necessary set-theoretic precautions. Our model category terminology is the one used in [Hov99] which also contains the standard results we need.

For a field k consider the dg algebra $k[u, u^{-1}]$ where the variable u has degree 2 and the differential is the zero map. Let $\mathrm{C}(k)$ be the dg category of unbounded complexes of k-modules and define $\mathrm{C}(k[u, u^{-1}])$ to be the dg category of functors from $k[u, u^{-1}]$, considered as a dg category with one object, to $\mathrm{C}(k)$. There is an obvious enriched equivalence between the dg category of $\mathbb{Z}/2$-graded complexes over k and the category $\mathrm{C}(k[u, u^{-1}])$.

There is an adjunction

$$\mathrm{C}(k) \xrightarrow{-\otimes_k k[u,u^{-1}]} \mathrm{C}(k[u, u^{-1}]) \quad , \quad \mathrm{C}(k) \xleftarrow{\quad F \quad} \mathrm{C}(k[u, u^{-1}])$$

where F denotes the forgetful functor. The image under $- \otimes_k k[u, u^{-1}]$ of the generating (trivial) cofibrations for the projective model structure on $\mathrm{C}(k)$, as defined in [Hov99], form the generating (trivial) cofibrations for a model structure on $\mathrm{C}(k[u, u^{-1}])$. Therefore, $\mathrm{C}(k[u, u^{-1}])$ admits a cofibrantly generated model structure such that the above adjunction is a Quillen adjunction. As for the category $\mathrm{C}(k)$, weak equivalences in $\mathrm{C}(k[u, u^{-1}])$ are defined to be quasi-isomorphisms and fibrations are levelwise surjective maps. Since k is a field, every object in $\mathrm{C}(k)$ and $\mathrm{C}(k[u, u^{-1}])$ is cofibrant. Note that $\mathrm{C}(k[u, u^{-1}])$ has a monoidal structure given by the tensor product over $k[u, u^{-1}]$. Under the equivalence with $\mathbb{Z}/2$-graded complexes, this tensor product translates into the $\mathbb{Z}/2$-graded tensor product.

Let dgcat_k denote the category of small dg categories over k. We introduce the category $\mathrm{dgcat}_{k[u,u^{-1}]}$ of 2-periodic dg categories where the objects are small categories enriched over $\mathrm{C}(k[u,u^{-1}])$ and the morphisms are dg functors. The above adjunction on the level of complexes induces an adjunction

$$\mathrm{dgcat}_k \xrightarrow{-\otimes_k k[u,u^{-1}]} \mathrm{dgcat}_{k[u,u^{-1}]} \quad , \quad \mathrm{dgcat}_k \xleftarrow{\quad F \quad} \mathrm{dgcat}_{k[u,u^{-1}]} \, .$$

We claim that $\mathrm{dgcat}_{k[u,u^{-1}]}$ admits the structure of a cofibrantly generated model category such that this adjunction is a Quillen adjunction. The category dgcat_k admits a cofibrantly generated model structure which is described in [Tab05]. Again, we can take the generating (trivial) cofibrations in $\mathrm{dgcat}_{k[u,u^{-1}]}$ to be the image under $-\otimes k[u,u^{-1}]$ of the generating (trivial) cofibrations for dgcat_k. The only property needed to apply [Hov99, Theorem 2.1.19] which does not formally follow from the above adjunction is that relative J-cell complexes are weak equivalences. However, the proof of this fact for dgcat_k given in [Tab05] implies the statement for $\mathrm{dgcat}_{k[u,u^{-1}]}$ in complete analogy. As for dgcat_k, the weak equivalences in $\mathrm{dgcat}_{k[u,u^{-1}]}$ are quasi-equivalences.

The existence of this model structure allows for a description of the mapping spaces in the category $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$ as done for $\mathrm{Ho}(\mathrm{dgcat}_k)$ in [Toë07].

Note that $\mathrm{dgcat}_{k[u,u^{-1}]}$ admits a closed monoidal structure where the tensor product of two dg categories is given by the product on objects and the tensor product over $k[u,u^{-1}]$ on morphism complexes. Thus, we have an adjunction

$$\mathrm{Hom}(T \otimes T', T'') = \mathrm{Hom}(T, \underline{\mathrm{Hom}}(T', T''))$$

where $\underline{\mathrm{Hom}}$ denotes the dg category of $\mathrm{C}(k[u,u^{-1}])$-enriched functors. For a category T in $\mathrm{dgcat}_{k[u,u^{-1}]}$, we define the dg category of modules over T to be

$$T\text{-}\mathrm{mod} = \underline{\mathrm{Hom}}(T, \mathrm{C}(k[u,u^{-1}])).$$

For an object x in T we define $\underline{h}_x$ to be the T^{op}-module given by $\underline{h}_x(y) = T(y,x)$. The dg Yoneda functor

$$\underline{h}_- : T \longrightarrow T^{\mathrm{op}}\text{-}\mathrm{mod},\ x \mapsto \underline{h}_x$$

is $\mathrm{C}(k[u,u^{-1}])$-fully faithful (i.e. induces an isomorphism of morphism complexes). Dually we have a $\mathrm{C}(k[u,u^{-1}])$-fully faithful functor

$$\underline{h}^- : T^{\mathrm{op}} \longrightarrow T\text{-}\mathrm{mod},\ x \mapsto T(x,-).$$

Functors of the form $\underline{h}_x$ are called *representable*, the ones of the form $\underline{h}^x$ *corepresentable*.

There exists a $\mathrm{C}(k[u,u^{-1}])$-model structure on T-mod where the fibrations and weak equivalences are defined levelwise using the model structure on $\mathrm{C}(k[u,u^{-1}])$. This model structure is cofibrantly generated where generating (trivial) cofibrations are obtained by applying the functors $\underline{h}^x \otimes -$ to generating (trivial) cofibrations in $\mathrm{C}(k[u,u^{-1}])$ for all $x \in T$. The homotopy category $\mathrm{Ho}(T\text{-}\mathrm{mod})$ yields the *derived category of* T which we denote by $\mathrm{D}(T)$. Note that due to the existence of the model structure we can define $\mathrm{Int}(T\text{-}\mathrm{mod})$ to be the full dg subcategory of T consisting of objects which are both fibrant and cofibrant. As in the last section we denote by $[T]$ the category obtained from a dg category T by applying the functor $\mathrm{H}^0(-)$ to all morphism complexes. This yields a functor $[-]$ from $\mathrm{dgcat}_{k[u,u^{-1}]}$ to cat. We have a natural equivalence of categories $[\mathrm{Int}(T\text{-}\mathrm{mod})] \simeq \mathrm{D}(T)$.

Since the representable T^{op}-modules $\underline{h}_x$ are cofibrant and fibrant, the Yoneda embedding yields a functor

$$T \longrightarrow \mathrm{Int}(T^{\mathrm{op}}\text{-}\mathrm{mod}).$$

To simplify notation we introduce $\widehat{T} = \mathrm{Int}(T^{\mathrm{op}}\text{-}\mathrm{mod})$.

A T^{op}-module M is called *compact* or *perfect* if $[M, -]$ commutes with coproducts. It is easy to see that all representable modules are perfect. Therefore, the Yoneda embedding provides a functor

$$T \longrightarrow \widehat{T}_{\mathrm{pe}},$$

where the subscript indicates the full dg subcategory of perfect modules. The category $\widehat{T}_{\mathrm{pe}}$ is called the *triangulated hull* of T and the dg category T is called *triangulated* if the Yoneda embedding into its triangulated hull is a quasi-equivalence.

We conclude by observing that the homotopical framework which we defined for 2-periodic categories is in exact analogy to the one defined in [Toë07]. All properties pointed out in sections 2 and 3 of loc. cit. hold mutatis mutandis in the 2-periodic context. Therefore all proofs in sections 4,5,6 and 7 can be repeated more or less verbatim to obtain identical results over $k[u, u^{-1}]$. In fact, as the author of loc. cit. points out in the introduction, he expects the results to generalize to M-enriched categories for certain very general monoidal model categories M. The 2-periodic version would then correspond to the choice $M = \mathrm{C}(k[u,u^{-1}])$. We will therefore cite results in loc. cit. without further comments, if the 2-periodic reformulation is obvious.

4.2 Equivalences of categories

Let T be a 2-periodic dg category and consider a set W of morphisms in T. By [Toë07, 8.7] there exists a dg category $\mathrm{L}_W(T)$ and a morphism $l : T \to \mathrm{L}_W(T)$ in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$ which is called the localization of T with respect to W. It enjoys the following universal property which determines it uniquely up to isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$. For every dg category T' the pullback map

$$l^* : \ [\mathrm{L}_W(T), T'] \longrightarrow [T, T']$$

is injective and the image consists of morphisms $f : T \to T'$ such that $[f]$ maps morphisms in W to isomorphisms in $[T']$.

For a 2-periodic dg category T we introduce the *dg derived category of* T as the localization $\mathrm{L}_W(T\text{-}\mathrm{mod})$ with respect to the set of weak equivalences. The category $[\mathrm{L}_W(T\text{-}\mathrm{mod})]$ is equivalent to the derived category $\mathrm{D}(T)$ of T which we introduced in Section 4.1.

The essential arguments in the proof of the following theorem are due to Keller [Kel94, 4.3].

Theorem 4.1. *Let T be a triangulated 2-periodic dg category which admits coproducts. Let S be a full dg subcategory of T whose objects are compact in $[T]$. Assume that the smallest triangulated subcategory of $[T]$ which contains the objects of $[S]$ and is closed under coproducts is $[T]$ itself. Then the map*

$$f: \; T \to \mathrm{L}_W(S^{\mathrm{op}}\text{-}\mathrm{mod}), \; x \mapsto l(T(-,x)|_S)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$. *Furthermore, f induces an isomorphism*

$$T_{\mathrm{pe}} \simeq \mathrm{L}_W(S^{\mathrm{op}}\text{-}\mathrm{mod})_{\mathrm{pe}}$$

between the full dg subcategories of compact objects.

Proof. Note that, since both T and $\mathrm{L}_W(S^{\mathrm{op}}\text{-}\mathrm{mod})$ are triangulated dg categories, the induced functor $[f] : [T] \to \mathrm{D}(S^{\mathrm{op}})$ is an exact functor of triangulated categories.

We claim that $[f]$ commutes with coproducts. Indeed, if $\{x_i\}$ are objects in T then the natural map

$$\coprod T(-,x_i)|_S \to T(-,\coprod x_i)|_S$$

is a weak equivalence since the objects in S are compact in $[T]$. It therefore becomes an isomorphism in $\mathrm{D}(S^{\mathrm{op}})$ which proves the claim.

The restriction of f to the dg subcategory S factors over the weak equivalence $\mathrm{Int}(S^{\mathrm{op}}\text{-}\mathrm{mod}) \to \mathrm{L}_W(S^{\mathrm{op}}\text{-}\mathrm{mod})$ since representable modules are cofibrant in $S^{\mathrm{op}}\text{-}\mathrm{mod}$. Therefore, by the dg Yoneda lemma the restriction of f to S is quasi-fully faithful.

Consider the full subcategory A of $[T]$ consisting of objects x such that the map

$$[T](s,x) \to \mathrm{D}(S^{\mathrm{op}})(f(s),f(x))$$

is an isomorphism for all objects s of S. By the five-lemma the category A is triangulated and since $[f]$ commutes with coproducts, A contains coproducts. However, since we just saw that A contains the objects of S, we have $A = [T]$ by assumption. Fixing an object y in T and applying the same argument to the subcategory formed by objects x such that the map

$$[T](x,y) \to \mathrm{D}(S^{\mathrm{op}})(f(x),f(y))$$

is an isomorphism, we deduce that $[f]$ is fully faithful. This clearly implies that f is quasi-fully faithful since f is a map of triangulated dg categories.

It remains to show that $[f]$ is essentially surjective. Since $[f]$ commutes with coproducts, the essential image of $[f]$ contains the quasi-representable functors and is closed under coproducts. Using the Bousfield localization argument [SS03, Lemma 2.2.1] which we already used in Section 3, we conclude that the essential image must be all of $\mathrm{D}(S^{\mathrm{op}})$.

The statement about the dg subcategories of compact objects can be obtained as follows. Because $[f]$ commutes with coproducts, it preserves compactness. The essential image of the restriction of $[f]$ to $[T_{\mathrm{pe}}]$ contains the quasi-representable S^{op}-modules, is triangulated and closed under summands. By [Kel94, 5.3], it follows that the restriction of $[f]$ is essentially surjective. □

Recall the notation $\widehat{T} = \mathrm{Int}(T^{\mathrm{op}}\text{-}\mathrm{mod})$ for a 2-periodic dg category T. The natural map $\widehat{T} \to \mathrm{L}_W(T^{\mathrm{op}}\text{-}\mathrm{mod})$ is an isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$. Hence, we may use the dg category $\widehat{T}$ as an explicit model for the dg derived category of T^{op}.

Recall the description of the dg algebra of endomorphisms of k^{stab} as the $\mathbb{Z}/2$-graded algebra

$$A = R\left\langle \theta_1, \ldots, \theta_n, \frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_n}\right\rangle$$

of polynomial differential operators on $R\langle\theta_1, \ldots, \theta_n\rangle$ equipped with the differential $[\delta, -]$ where

$$\delta = \sum_{i=1}^{n} x_i \frac{\partial}{\partial\theta_i} + w_i\theta_i.$$

We reserve the letter A for this dg algebra throughout this section. We slightly abuse notation and also use the symbol A to refer to the corresponding 2-periodic dg category with a single object.

Theorem 4.2. *Let (R, w) be an isolated hypersurface singularity. Then there exist the following isomorphisms in* $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$.

1. $\mathrm{MF}^{\infty}(R, w) \xrightarrow{\simeq} \widehat{\mathrm{MF}(R, w)}$
2. $\mathrm{MF}^{\infty}(R, w) \xrightarrow{\simeq} \widehat{A}$
3. $\widehat{\mathrm{MF}(R, w)}_{\mathrm{pe}} \xrightarrow{\simeq} \widehat{A}_{\mathrm{pe}}$

Proof. By Theorem 3.1 the object k^{stab} in $\mathrm{MF}^{\infty}(R, w)$ is a compact generator and the corresponding full dg subcategory is isomorphic to A. In particular, the objects in the full dg subcategory $\mathrm{MF}(R,w)$ generate $\mathrm{MF}^\infty(R,w)$. Since $\mathrm{MF}(R,w)$ also consists of compact objects we deduce the first two isomorphisms from Theorem 4.1. The last statement follows immediately from the second part of Theorem 4.1. □

We comment on the relevance of the previous theorem. The first isomorphism gives a natural interpretation of the dg category $\mathrm{MF}^\infty(R,w)$ which we defined in a somewhat ad hoc way. Namely, it is an explicit model for the dg derived category of $\mathrm{MF}(R,w)$. The second isomorphism will turn out to be useful since we have an explicit description of the dg algebra A. Finally, the last isomorphism identifies the triangulated hull of the dg category $\mathrm{MF}(R,w)$. Note that the natural map $\mathrm{MF}(R,w) \to \widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}}$ is not necessarily an isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$ since the triangulated category $[\mathrm{MF}(R,w)]$ is not necessarily idempotent complete. We will determine an explicit model for $\widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}}$ in Section 4.3.

Corollary 4.3. *The triangulated category* $[\widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}}]$ *is equivalent to the smallest triangulated subcategory of* $[\mathrm{MF}^\infty(R,w)]$ *which contains* k^{stab} *and is closed under summands.*

The next corollary introduces a notion of quasi-isomorphism between matrix factorizations. The category $[\mathrm{MF}^\infty(R,w)]$ is obtained as a localization from $\mathrm{MF}^\infty(R,w)$ by inverting all such quasi-isomorphisms. One may therefore think of the category $[\mathrm{MF}^\infty(R,w)]$ as a derived category of twisted complexes of R-modules.

Corollary 4.4. *Let X, Y be objects in $\mathrm{MF}^\infty(R, w)$. A 0-cycle $f \in \mathrm{Hom}(X, Y)$ induces an isomorphism in $[\mathrm{MF}^\infty(R, w)]$ if and only if f induces a quasi-isomorphism of the complexes of k-modules $k \otimes_R X$ and $k \otimes_R Y$.*

Proof. By Theorem 4.2, f induces an isomorphism if and only if $\mathrm{Hom}(k^{\mathrm{stab}}, f)$ is a quasi-isomorphism. However, by Corollary 3.3 the cohomology of the complex $\mathrm{Hom}(k^{\mathrm{stab}}, X)$ is up to shift naturally isomorphic to the cohomology of the complex $k \otimes_R X$ and the analogous statement is true for Y. □

4.3 Formal completion

We address the question of describing the triangulated hull $\widehat{\mathrm{MF}(R, w)}_{\mathrm{pe}}$ of $\mathrm{MF}(R, w)$ explicitly. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R and consider the category $\mathrm{MF}(\widehat{R}, w)$.

Lemma 4.5. *The Yoneda embedding*

$$\mathrm{MF}(\widehat{R}, w) \to \widehat{\mathrm{MF}(\widehat{R}, w)}_{\mathrm{pe}}$$

is an isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$. In other words, the dg category $\mathrm{MF}(\widehat{R}, w)$ is triangulated and, in particular, the triangulated category $[\mathrm{MF}(\widehat{R}, w)]$ is idempotent complete.

Proof. By [Kel94, 5.3], the category $[\widehat{\mathrm{MF}(\widehat{R}, w)}_{\mathrm{pe}}]$ is the smallest triangulated subcategory of $[\widehat{\mathrm{MF}(\widehat{R}, w)}]$ which is closed under summands and contains the Yoneda image of $[\mathrm{MF}(\widehat{R}, w)]$. It therefore suffices to show that $[\mathrm{MF}(\widehat{R}, w)]$ is idempotent complete.

As explained in 2.1, the category $[\mathrm{MF}(\widehat{R}, w)]$ is equivalent to the stable category associated to the Frobenius category $\mathrm{MCM}(\widehat{R}/w)$ of maximal Cohen-Macaulay modules over $\widehat{R}/w$. By the classical result [Swa60, Remark on page 566], the endomorphism algebra $\mathrm{End}(M)$ of an indecomposable module M over a complete local ring is local, i.e. the sum of two non-units is a non-unit. Now let e be an element in $\mathrm{End}(M)$ whose image $\overline{e}$ in the stable endomorphism algebra $\underline{\mathrm{End}}(M)$ is a non-trivial idempotent. In particular, $\overline{e}$ and $1-\overline{e}$ are non-units. This certainly implies that e and $1-e$ are non-units in $\mathrm{End}(M)$ and therefore $1 = e+(1-e)$ is a non-unit which is a contradiction. So $\underline{\mathrm{End}}(M)$ does not contain non-trivial idempotents which implies the statement. □

The lemma enables us to describe the category $\widehat{\mathrm{MF}(R, w)}_{\mathrm{pe}}$ explicitly.

Theorem 4.6. *There exists an isomorphism in* $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$

$$\widehat{\mathrm{MF}(R, w)}_{\mathrm{pe}} \simeq \mathrm{MF}(\widehat{R}, w).$$

In particular, the idempotent completion of $[\mathrm{MF}(R, w)]$ *in* $[\widehat{\mathrm{MF}(R, w)}]$ *is equivalent to* $[\mathrm{MF}(\widehat{R}, w)]$.

Proof. Combining Lemma 4.5 with Theorem 4.2, we obtain an isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$

$$\mathrm{MF}(\widehat{R}, w) \xrightarrow{\simeq} \widehat{A_{(\widehat{R},w)}}_{\mathrm{pe}},$$

where $A_{(\widehat{R},w)}$ denotes the endomorphism dg algebra of the stabilized residue field in the category $\mathrm{MF}(\widehat{R}, w)$. On the other hand denoting the analogous dg algebra for $\mathrm{MF}(R, w)$ by $A_{(R,w)}$ we have an isomorphism

$$\widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}} \xrightarrow{\simeq} \widehat{A_{(R,w)}}_{\mathrm{pe}},$$

by Theorem 4.2. There is a natural inclusion map of dg algebras $A_{(R,w)} \to A_{(\widehat{R},w)}$ which, using our explicit description of both algebras, is clearly a quasi-isomorphism. This weak equivalence implies an isomorphism in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$ between $\widehat{A_{(R,w)}}_{\mathrm{pe}}$ and $\widehat{A_{(\widehat{R},w)}}_{\mathrm{pe}}$. □

In [Orl09], the relation between idempotent completion and formal completion is studied in a more general context on the level of triangulated categories.

4.4 Quadratic superpotential

Let $R = k[[x_1, \dots, x_n]]$ and let $w \in R$ be a quadratic form. Assuming $\mathrm{char}(k) \neq 2$ we may diagonalize w, so after a change of coordinates we have $w = \sum_{i=1}^{n} a_i x_i^2$ with $a_i \in k$. We write $w = x_i w_i$ setting $w_i = a_i x_i$. With above notation we have

$$A = R \langle \theta_1, \dots, \theta_n, \partial_1, \dots, \partial_n \rangle$$

with differential d given by

$$\theta_i \mapsto x_i$$

$$\partial_i \mapsto a_i x_i.$$

The elements $\overline{\partial_i} = \partial_i - a_i\theta_i$ are cycles in A and generate the cohomology $\mathrm{H}^*(A)$ as a k-algebra. Note that the relations

$$\overline{\partial_i}^2 = -a_i$$
$$\overline{\partial_i}\,\overline{\partial_j} = -\overline{\partial_j}\,\overline{\partial_i} \quad \text{for } i \neq j$$

imply that the k-subalgebra of A^{op} generated by $\{\overline{\partial_i}\}$ is isomorphic to the Clifford algebra $\mathrm{Cl}(w)$ corresponding to the quadratic form w. Furthermore, the inclusion

$$(\mathrm{Cl}(w), 0) \subset (A^{\mathrm{op}}, d)$$

is a quasi-isomorphism establishing the formality of the dg algebra A^{op}. Therefore, in the quadratic case Theorem 4.2 reproduces a variant of the results in [BEH87] describing matrix factorizations as modules over the Clifford algebra $\mathrm{Cl}(w)$ (see also [Yos90, Chapter 14]).

4.5 The minimal A_∞ model

If the superpotential is of degree greater than 2, the algebra A will not be formal. However, there is a well-known structure which allows us nevertheless to pass to the cohomology algebra of A: the structure of an A_∞ algebra. For the basic theory we refer the reader to [Sta70, Che77, Kel99, KS]. The relevance to our situation is the following. In addition to the usual multiplication on the cohomology algebra of A there exist higher multiplications. In a precise sense, they measure the failure of being able to choose a multiplicatively closed set of representatives of $\mathrm{H}^*(A)$ in A. The system of higher multiplications forms an A_∞-algebra and as such $\mathrm{H}^*(A)$ will be quasi-isomorphic to A.

One of the motivations for passing from A to $\mathrm{H}^*(A)$ is that the latter algebra is finite dimensional over k; it is referred to as a *minimal model of* A. By the general theory in [LH03] we obtain a description of the category of matrix factorizations as a category of modules over the A_∞ algebra $\mathrm{H}^*(A)$. We do not spell out a precise formulation of this equivalence but restrict ourselves to the description of the A_∞ structure in some special cases.

We use the method described in [KS01, 6.4] (also cf. [GS86, Mer99]) to compute the A_∞ structure in terms of trees. The setup is as follows. Let X be an A_∞ algebra. This structure can be described by the data of a coderivation q of degree 1 on the coalgebra $\mathrm{T}X[1]$ such that $[q,q]=0$. The Taylor coefficients of $q = q_1 + q_2 + q_3 + \dots$ are maps

$$q_k : X[1]^{\otimes k} \to X[1]$$

which describe, after introducing the sign shifts accounting for the transfer from $X[1]$ to X, the higher multiplications m_k on X. Now assume, that we are given an idempotent $p : X \to X$ of degree 0 commuting with d. The image of p is therefore a subcomplex of X which we denote by Y. Let i denote the inclusion of Y into X.

We also assume, that a homotopy $h : X \to X[1]$ between id and p is given, i.e.

$$dh + hd = \operatorname{id} - p$$

identifying p as a homotopy equivalence between X and Y. In this situation, the A_∞-structure induced on Y can be calculated explicitly in terms of trees as for example described in [KS01, 6.4]. This method can in principle be used to determine the A_∞ structure on $\mathrm{H}^*(A)$ for $R = k[[x_1, \dots, x_n]]$. However the computation is rather tedious and the author has not been able to find closed formulas for all higher multiplications.

Since we will not use the results of this calculation elsewhere, we will define the contracting homotopy h and only state the results of the tree calculation.

We start with the one-dimensional case $R = k[[x]]$ which will serve as a guideline for what to do in the higher dimensional case. For an element a in $k[[x]]$ we define $\overline{a}$ to be the unique element in $xk[[x]]$ defined by

$$a = a_0 + \overline{a}$$

with $a_0 \in k$. The algebra A is of the form

$$A \cong k[[x]] \otimes_k k \langle \theta, \partial \rangle$$

with differential d given by

$$\theta \mapsto x$$

$$\partial \mapsto \frac{w}{x}.$$

We define a homotopy h contracting A onto its cohomology. The assignment $h(a) = \frac{\overline{a}}{x}\theta$ for $a \in k[[x]]$ extends to a unique $k\langle\theta,\partial\rangle$-linear homotopy of A. A simple calculation shows that the map

$$p = \mathrm{id} - [d,h]$$

is a projection and we have maps of complexes

$$\begin{array}{c} k[[x]] \otimes_k k\langle\theta,\partial\rangle \\ \iota \Big\uparrow \Big\downarrow p \\ k\left\langle\overline{\partial}\right\rangle \end{array}$$

where the differential on $k\left\langle\overline{\partial}\right\rangle$ is 0 and $\iota(\overline{\partial}) = \partial - \frac{w}{x^2}\theta$.

We have determined all the data needed for the tree formula. It leads to the following higher multiplications.

Theorem 4.7. *Let $R = k[[x]]$ with superpotential $w = \sum_{i=2}^{\infty} r_i x^i$. Then the unital A_∞-structure on $\mathrm{H}^*(A) \cong k1 \oplus k\overline{\partial}$ induced by the above homotopy is uniquely determined by the formulas*

$$m_i(\overline{\partial},\ldots,\overline{\partial}) = \pm r_i.$$

In the more general case $R = k[[x_1,\ldots,x_n]]$ one can still construct an explicit contracting homotopy, however the resulting formulas get more complicated. The following result does not determine the A_∞ structure completely but at least shows some interesting properties.

Theorem 4.8. *Let* $R = k[[x_1, \ldots, x_n]]$ *with superpotential* $w = \sum_{\underline{i}} r_{\underline{i}} x^{\underline{i}}$ *where* $\underline{i} = (i_1, \ldots, i_n)$ *denotes a multi-index. Then there exists a contracting homotopy of* A *such that the induced* A_∞ *structure on* $\mathrm{H}^*(A)$ *has the following properties*

- *The underlying associative algebra on* $\mathrm{H}^*(A)$ *is given by the Clifford algebra corresponding to the quadratic term of* w *as described in the previous section.*
- *For the generators* $\overline{\partial_1}, \ldots, \overline{\partial_n}$ *we have*

$$m_i(\underbrace{\overline{\partial_1}, \ldots, \overline{\partial_1}}_{i_1}, \underbrace{\overline{\partial_2}, \ldots, \overline{\partial_2}}_{i_2}, \ldots, \underbrace{\overline{\partial_n}, \ldots, \overline{\partial_n}}_{i_n}) = \pm r_{\underline{i}}$$

  where $i = |\underline{i}|$.

Note that the formulas on the generators $\overline{\partial}_i$ do note determine the A_∞-structure. One instructive feature is the direct relation to the coefficients of w. Formality can thereafter only be expected if the superpotential is quadratic.

We illustrate the determination of the contracting homotopy for $R = k[[x_1, x_2]]$. Note that it is possible to write $w = x_1 w_1 + x_2 w_2$ with $w_1 \in k[[x_1]]$ and $w_2 \in k[[x_1, x_2]]$. Then the endomorphism algebra A of the stabilized residue field is of the form

$$k[[x_1]] \langle \theta_1, \partial_1 \rangle \otimes_{k[[x_1]]} k[[x_1, x_2]] \langle \theta_2, \partial_2 \rangle$$

with differential d given by

$$\theta_i \mapsto x_i$$

$$\partial_i \mapsto w_i$$

Applying the construction from the one dimensional case twice, we obtain maps of complexes

$$
\begin{array}{cc}
k[[x_1]]\,\langle\theta_1,\partial_1\rangle \otimes_{k[[x_1]]} k[[x_1,x_2]]\,\langle\theta_2,\partial_2\rangle & \\
\iota_2 \big\uparrow\big\downarrow p_2 & \iota_2\circ p_2 = id - [d,h_2] \\
k[[x_1]]\,\langle\theta_1,\partial_1\rangle \otimes_{k[[x_1]]} k[[x_1]]\left\langle \widetilde{\partial_2}\right\rangle & \\
\iota_1 \big\uparrow\big\downarrow p_1 & \iota_1\circ p_1 = id - [d,h_1] \\
k < \overline{\partial_1}, \overline{\partial_2} > &
\end{array}
$$

where $h_2(a) = \frac{a}{x_2}\theta_2$ for $a \in k[[x_1,x_2]]$ and $h_1(b) = \frac{b}{x_1}\theta_1$ for $b \in k[[x_1]]$. Here, the symbol $\frac{}{y}$ denotes division by y, discarding the remainder. As in the one dimensional case, h_1 and h_2 are extended linearly to yield the homotopies in the above diagram. In fact, we can collapse the sequence of homotopy equivalences to a single one given by

$$\iota = \iota_2\circ\iota_1$$

$$p = p_1\circ p_2$$

$$h = h_2 + \iota_2\circ h_1\circ p_2$$

The inclusions into A are given by

$$\overline{\partial_1} = \partial_1 - \frac{w_1}{x_1}\theta_1$$

$$\overline{\partial_2} = \partial_2 - \frac{w_2}{x_2}\theta_2 - \frac{r_2}{x_1}\theta_1$$

where $r_2 \in k[[x_1]]$ is the remainder of the division of w_2 by x_2.

An application of the tree formula yields the formulas described in Theorem 4.8. This calculation generalizes to an arbitrary number of variables.

Chapter 5

Derived Morita theory

As proved in [Toë07, Section 6] the category $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$ admits internal homomorphism categories satisfying the usual adjunction

$$[U \otimes^{\mathrm{L}} T, T'] \cong [U, \mathrm{R}\underline{\mathrm{Hom}}(T, T')].$$

We will use Morita theory [Toë07, Section 7] to determine the dg category of functors between two matrix factorization categories. As an application, we calculate the Hochschild chain and cochain complexes of these categories. In this chapter, we assume all singularities to be isolated.

5.1 Internal homomorphism categories

Given two categories $\mathrm{MF}^{\infty}(R, w)$ and $\mathrm{MF}^{\infty}(R', w')$ there is a natural class of dg functors between them. Namely, every object T in the category $\mathrm{MF}^{\infty}(R \otimes_k R', -w \otimes$

$1 + 1 \otimes w')$ defines a dg functor via the association

$$\mathrm{MF}^\infty(R, w) \to \mathrm{MF}^\infty(R', w'),\ X \mapsto X \otimes_R T,$$

where the tensor product is $\mathbb{Z}/2$-graded. In other words, the object T acts as the kernel of an integral transform. We will show that every continuous functor between matrix factorization categories is isomorphic to an integral transform.

By Theorem 4.2, there is an isomorphism

$$\mathrm{MF}^\infty(R, w) \xrightarrow{\simeq} \widehat{A}$$

where A is the 2-periodic endomorphism dg algebra of the compact generator $E = k^{\mathrm{stab}}$. The matrix factorization $E^\vee = \mathrm{Hom}_R(E, R)$ is a compact generator of the category $\mathrm{MF}^\infty(R, -w)$. Indeed, using the self-duality of Koszul complexes, one explicitly verifies that $E^\vee$ stabilizes a shift of the residue field. Therefore, Theorem 4.2 gives a natural isomorphism

$$\mathrm{MF}^\infty(R, -w) \xrightarrow{\simeq} \widehat{A^{\mathrm{op}}}.$$

Consider a second hypersurface singularity (R', w') with corresponding category $\mathrm{MF}^\infty(R', w')$ and stabilized residue field E'. One immediately identifies the object $E \otimes_k E'$ of the category $\mathrm{MF}^\infty(R \otimes_k R', w \otimes 1 + 1 \otimes w')$ as a stabilization of the residue field. Therefore, we obtain an isomorphism of dg algebras

$$A_{(R \otimes_k R', w \otimes 1 + 1 \otimes w')} \xrightarrow{\cong} A_{(R,w)} \otimes A_{(R',w')}.$$

Combining Theorem 1.4 in [Toë07] with the above observations and Theorem 4.2, we obtain the following result.

Theorem 5.1. *There exists a natural isomorphism in* $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$

$$\mathrm{R}\underline{\mathrm{Hom}}_c(\mathrm{MF}^\infty(R,w),\mathrm{MF}^\infty(R',w')) \cong \mathrm{MF}^\infty(R\otimes_k R', -w\otimes 1 + 1\otimes w').$$

We conclude the section with a compatibility statement.

Proposition 5.2. *Let T be an object in $\mathrm{MF}^\infty(R\otimes_k R', -w\otimes 1 + 1\otimes w')$, let E resp. E' be the compact generators of $\mathrm{MF}^\infty(R,w)$ resp. $\mathrm{MF}^\infty(R',w')$ as constructed above. Then the diagram of functors*

$$\begin{array}{ccc} [\mathrm{MF}^\infty(R,w)] & \xrightarrow{-\otimes_R T} & [\mathrm{MF}^\infty(R',w')] \\ \downarrow{\scriptstyle \mathrm{Hom}(E,-)} & & \downarrow{\scriptstyle \mathrm{Hom}(E',-)} \\ \mathrm{D}(A^{\mathrm{op}}) & \xrightarrow{-\otimes^{\mathrm{L}}_A \mathrm{Hom}(E^\vee\otimes_k E',T)} & \mathrm{D}(A'^{\mathrm{op}}) \end{array}$$

commutes up to a natural equivalence.

Proof. Using the natural isomorphism of complexes

$$\mathrm{Hom}_{R\otimes_k R'}(E^\vee\otimes_k E',T) \cong \mathrm{Hom}_{R'}(E', E\otimes_R T)$$

we obtain a natural transformation

$$\mathrm{Hom}_R(E,-)\otimes^{\mathrm{L}}_A \mathrm{Hom}_{R'}(E^\vee\otimes_k E',T) \to \mathrm{Hom}(E', -\otimes_R T)$$

via composition. Both functors respect the triangulated structure and commute with infinite coproducts. Evaluated on the compact generator E, the above transformation yields an isomorphism in $\mathrm{D}(A'^{\mathrm{op}})$. Therefore, it must be an equivalence of functors on $[\mathrm{MF}^\infty(R,w)]$. □

5.2 Hochschild cohomology

One of the many neat applications of the homotopy theory developed in [Toë07] is the description of the Hochschild cochain complex of a dg category as the endomorphism complex of the identity functor. This result carries over to the 2-periodic case and we will use it to determine the Hochschild cohomology of the dg category $\mathrm{MF}^{\infty}(R, w)$.

Consider the matrix factorization category $\mathrm{MF}^{\infty}(R, w)$ corresponding to an isolated hypersurface singularity. We choose the stabilized residue field as a compact generator which yields an isomorphism

$$\mathrm{MF}^{\infty}(R, w) \xrightarrow{\simeq} \widehat{A}$$

in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$ as explained above. Let us introduce the notation $\widetilde{w} = -w \otimes 1 + 1 \otimes w$. We have to identify an object in $\mathrm{MF}^{\infty}(R \otimes_k R, \widetilde{w})$ which induces the identity functor on $\mathrm{MF}^{\infty}(R, w)$. Equivalently, we have to find an object whose image under the equivalence

$$[\mathrm{MF}^{\infty}(R \otimes_k R, \widetilde{w})] \xrightarrow{\simeq} \mathrm{D}(A \otimes A^{\mathrm{op}})$$

given by $\mathrm{Hom}(E^{\vee} \otimes_k E, -)$ is isomorphic to the $A \otimes A^{\mathrm{op}}$-module A. There is an obvious candidate for the integral kernel which induces the identity functor: the *stabilized diagonal* Δ^{stab}. Analogously to the stabilized residue field, it can be constructed as the stabilization of R considered as an $R \otimes_k R/\widetilde{w}$-module using the method described in section 2.3.

We define $\Delta_i = x_i \otimes 1 - 1 \otimes x_i$ for a minimal system of generators $x_1, \ldots, x_n \in \mathfrak{m} \subset R$. Because $\widetilde{w}$ vanishes along the diagonal in $R \otimes_k R$, we can find an expression of the form $\widetilde{w} = \sum_{i=1}^n \Delta_i \widetilde{w}_i$. Since R is regular, the sequence $\Delta_1, \ldots, \Delta_n$ is regular. By Corollary 2.7, the matrix factorization Δ^{stab} is of the form

$$(\bigoplus_{i=0}^{n} \bigwedge^i V,\ s_0 + s_1),$$

with s_0 given by contraction with $(\Delta_1, \ldots, \Delta_n)$ and s_1 by exterior multiplication with $(\widetilde{w}_1, \ldots, \widetilde{w}_n)^{\mathrm{tr}}$.

To prove that Δ^{stab} is actually isomorphic to the identity functor we will use the following lemma.

Lemma 5.3. *Let (R, w) be a hypersurface singularity and let $X^\bullet$ and $Y^\bullet$ be objects in $\mathrm{MF}(R, w)$. Let A be the endomorphism dg algebra of $X^\bullet$ and assume that $Y^\bullet$ is the stabilization of the R/w-module L. Then there exists a natural isomorphism*

$$\mathrm{Hom}_R^{\mathbb{Z}/2}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_R^{\mathbb{Z}/2}(X^\bullet, L)$$

in the category $\mathrm{D}(A^{\mathrm{op}})$.

Proof. As above, we use the notation $S = R/w$ and abbreviate the functor $- \otimes_R S$

by $\overline{}$. Let $Z^\bullet$ be the total product complex of the double complex

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Hom}(\overline{X^1}, \overline{Y^0}) & \longrightarrow & \operatorname{Hom}(\overline{X^0}, \overline{Y^0}) & \longrightarrow & \operatorname{Hom}(\overline{X^1}, \overline{Y^0}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Hom}(\overline{X^1}, \overline{Y^1}) & \longrightarrow & \operatorname{Hom}(\overline{X^0}, \overline{Y^1}) & \longrightarrow & \operatorname{Hom}(\overline{X^1}, \overline{Y^1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Hom}(\overline{X^1}, \overline{Y^0}) & \longrightarrow & \operatorname{Hom}(\overline{X^0}, \overline{Y^0}) & \longrightarrow & \operatorname{Hom}(\overline{X^1}, \overline{Y^0}) & \longrightarrow & \cdots
\end{array}
$$

Since the complex $Z^\bullet$ is 2-periodic, we may think of it as a $\mathbb{Z}/2$-graded complex. Consider the natural map

$$\operatorname{Hom}_R^{\mathbb{Z}/2}(X^\bullet, Y^\bullet) \longrightarrow Z^\bullet$$

which is given by reducing modulo w and extending 2-periodically. We claim that this map is a quasi-isomorphism. We prove the surjectivity on H^0. Assume an element of $\mathrm{H}^0(Z^\bullet)$ is represented by the map of complexes

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \overline{X^0} & \longrightarrow & \overline{X^1} & \longrightarrow & \overline{X^0} & \\
 & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} & \\
\cdots \longrightarrow & \overline{Y^0} & \longrightarrow & \overline{Y^1} & \longrightarrow & \overline{Y^0}. &
\end{array}
$$

The collection $\{f^i\}$ induces a map f between the maximal Cohen-Macaulay modules M and N which are resolved by $\overline{X^\bullet}$ resp. $\overline{Y^\bullet}$. Alternatively, we can lift f to a map of complexes

$$
\begin{array}{ccc}
X^1 & \xrightarrow{\varphi} & X^0 \\
{\scriptstyle \widetilde{f^1}}\downarrow & & \downarrow{\scriptstyle \widetilde{f^0}} \\
Y^1 & \xrightarrow{\varphi} & Y^0.
\end{array}
$$

Using the relation $\varphi \circ \psi = w$ one immediately checks that the maps $\widetilde{f}^i$ also commute with ψ. Reducing $\widetilde{f}^i$ modulo w and extending periodically, we obtain a map of complexes representing f which therefore has to be homotopic to $\{f^i\}$. A similar argument shows injectivity on H^0.

Next, denote by $W^\bullet$ the total product complex of the double complex

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^1}) & \longrightarrow & \mathrm{Hom}(\overline{X^0}, \overline{Y^1}) & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^0}) & \longrightarrow & \mathrm{Hom}(\overline{X^0}, \overline{Y^0}) & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^0}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^{-1}}) & \longrightarrow & \mathrm{Hom}(\overline{X^0}, \overline{Y^{-1}}) & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^{-1}}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^{-r}}) & \longrightarrow & \mathrm{Hom}(\overline{X^0}, \overline{Y^{-r}}) & \longrightarrow & \mathrm{Hom}(\overline{X^1}, \overline{Y^{-r}}) & \longrightarrow & \cdots
\end{array}
$$

where the complex

$$
\cdots \longrightarrow \overline{Y^0} \longrightarrow \overline{Y^1} \longrightarrow \overline{Y^0} \longrightarrow \overline{Y^{-1}} \longrightarrow \cdots \longrightarrow \overline{Y^{-r}}
$$

is a resolution of the S-module L. Since by assumption N is an even syzygy of L, such a resolution exists and r is an even number. Again, due to 2-periodicity, we may think of $W^\bullet$ as a $\mathbb{Z}/2$-graded complex. There is an obvious projection map

$$
p : \; W^\bullet \to Z^\bullet
$$

which we claim to be a quasi-isomorphism. We have isomorphisms of $\mathbb{Z}/2$-graded vector spaces

$$\mathrm{H}^*(Z^\bullet) \cong \mathrm{Ext}^1(M, N) \oplus \mathrm{Ext}^2(M, N)$$

$$\mathrm{H}^*(W^\bullet) \cong \mathrm{Ext}^1(M, L) \oplus \mathrm{Ext}^2(M, L)$$

Since M is maximal Cohen-Macaulay, the functor $\mathrm{Ext}^i(M, -)$ annihilates free S-modules for $i > 0$. Using long exact sequences for Ext^* we conclude that $\mathrm{Ext}^i(M, L) \cong \mathrm{Ext}^i(M, N)$ for $i > 0$ and therefore $\mathrm{H}^*(Z^\bullet) \cong \mathrm{H}^*(W^\bullet)$. Therefore, it suffices to show that $\mathrm{H}^*(p)$ is surjective. This amounts to showing that any map of complexes

$$\begin{CD}
\cdots @>>> \overline{X^0} @>>> \overline{X^1} @>>> \overline{X^0} \\
@. @VV{f^2}V @VV{f^1}V @VV{f^0}V \\
\cdots @>>> \overline{Y^0} @>>> \overline{Y^1} @>>> \overline{Y^0}
\end{CD}$$

can be extended to a map of complexes

$$\begin{CD}
\cdots @>>> \overline{X^0} @>>> \overline{X^1} @>>> \overline{X^0} @>>> \overline{X^1} @>>> \cdots @>>> \overline{X^0} \\
@. @VV{f^2}V @VV{f^1}V @VV{f^0}V @VV{f^{-1}}V @. @VV{f^{-r}}V \\
\cdots @>>> \overline{Y^0} @>>> \overline{Y^1} @>>> \overline{Y^0} @>>> \overline{Y^{-1}} @>>> \cdots @>>> \overline{Y^{-r}}.
\end{CD}$$

Using the short exact sequence

$$0 \longrightarrow M \longrightarrow \overline{X^1} \longrightarrow \mathrm{syz}(M) \longrightarrow 0$$

one shows that the obstruction for the existence of the map f^{-1} lies in the space $\mathrm{Ext}^1(\mathrm{syz}(M), \overline{Y^{-1}})$ which vanishes since $\mathrm{syz}(M)$ is maximal Cohen-Macaulay. An iteration of this argument leads to the desired extension.

Finally, the complex $W^\bullet$ admits an augmentation map to the complex $\mathrm{Hom}^{\mathbb{Z}/2}(X^\bullet, L)$ which is a quasi-isomorphism by a standard spectral sequence argument.

We conclude by noting that we have constructed quasi-isomorphisms of the form

$$\mathrm{Hom}_R^{\mathbb{Z}/2}(X^\bullet, Y^\bullet) \xrightarrow{\simeq} Z^\bullet \xleftarrow{\simeq} W^\bullet \xrightarrow{\simeq} \mathrm{Hom}_R^{\mathbb{Z}/2}(X^\bullet, L),$$

inducing the claimed isomorphism in the category $\mathrm{D}(A^{\mathrm{op}})$. □

Corollary 5.4. *The stabilized diagonal* Δ^{stab} *is isomorphic to the identity functor on* $\mathrm{MF}^\infty(R, w)$.

Proof. We apply the lemma with

$$X^\bullet = E^\vee \otimes_k E$$

where E is the stabilized residue field in $\mathrm{MF}(R, w)$ and $Y^\bullet = \Delta^{\mathrm{stab}}$ which is the stabilization of the diagonal R as an $R \otimes_k R/\widetilde{w}$-module. We obtain an isomorphism

$$\mathrm{Hom}_{R\otimes_k R}^{\mathbb{Z}/2}(E^\vee \otimes_k E, \Delta^{\mathrm{stab}}) \cong \mathrm{Hom}_{R\otimes_k R}^{\mathbb{Z}/2}(E^\vee \otimes_k E, R)$$

and further

$$\begin{aligned}\mathrm{Hom}_{R\otimes_k R}^{\mathbb{Z}/2}(E^\vee \otimes_k E, R) &\cong \mathrm{Hom}_R^{\mathbb{Z}/2}(E^\vee \otimes_R E, R)\\ &\cong \mathrm{Hom}_R^{\mathbb{Z}/2}(E, E).\end{aligned}$$

But the latter complex is by definition A and all the maps respect the $A \otimes A^{\mathrm{op}}$-module structure. □

Corollary 5.5. *The Hochschild cochain complex of* $\mathrm{MF}^{\infty}(R, w)$ *is quasi-isomorphic to the* $\mathbb{Z}/2$*-folded Koszul complex of the regular sequence* $\partial_1 w, \ldots, \partial_n w$ *in* R*. In particular, the Hochschild cohomology is isomorphic to the Jacobian algebra*

$$\mathrm{HH}^*(\mathrm{MF}^{\infty}(R, w)) \cong R/(\partial_1 w, \ldots, \partial_n w)$$

concentrated in even degree.

Proof. By [Toë07, Corollary 8.1] and Corollary 5.4 the Hochschild cochain complex is quasi-isomorphic to

$$\mathrm{MF}^{\infty}(R \otimes_k R, \widetilde{w})(\Delta^{\mathrm{stab}}, \Delta^{\mathrm{stab}}).$$

We apply the lemma with $X^\bullet = Y^\bullet = \Delta^{\mathrm{stab}}$. Since $Y^\bullet$ stabilizes the $R \otimes_k R/\widetilde{w}$-module R, we obtain an isomorphism

$$\mathrm{Hom}^{\mathbb{Z}/2}_{R\otimes_k R}(\Delta^{\mathrm{stab}}, \Delta^{\mathrm{stab}}) \cong \mathrm{Hom}^{\mathbb{Z}/2}_{R\otimes_k R}(\Delta^{\mathrm{stab}}, R).$$

The latter complex is isomorphic to the Koszul complex of the sequence formed by the reduction of the elements $\widetilde{w_1}, \ldots, \widetilde{w_n}$ modulo the ideal $(\Delta_1, \ldots, \Delta_n)$. We only have to observe that $\widetilde{w}_i$ is congruent to $\partial_i w$ modulo $(\Delta_1, \ldots, \Delta_n)$. Indeed,

$$\begin{aligned}
\partial_i w(x) &= \lim_{\Delta_i \to 0} \frac{w(x + \Delta_i) - w(x)}{\Delta_i} \\
&= \lim_{\Delta_i \to 0} \frac{\widetilde{w} \mod (\Delta_1, \ldots, \widehat{\Delta_i}, \ldots, \Delta_n)}{\Delta_i} \\
&= \lim_{\Delta_i \to 0} \widetilde{w}_i \mod (\Delta_1, \ldots, \widehat{\Delta_i}, \ldots, \Delta_n) \\
&= \widetilde{w}_i \mod (\Delta_1, \ldots, \Delta_n)
\end{aligned}$$

□

From this calculation we can also obtain the Hochschild cohomology of the category $\mathrm{MF}(R,w)$ for any isolated hypersurface singularity (R,w).

Corollary 5.6. *The Hochschild cohomology of the* 2*-periodic dg category* $\mathrm{MF}(R,w)$ *is isomorphic to the Jacobian algebra*

$$\mathrm{HH}^*(\mathrm{MF}(R,w)) \cong R/(\partial_1 w, \ldots, \partial_n w)$$

concentrated in even degree.

Proof. By Theorem 4.2 we have an isomorphism $\mathrm{MF}^\infty(R,w) \simeq \widehat{\mathrm{MF}(R,w)}$ in the category $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$. Therefore, the statement follows immediately from [Toë07, Corollary 8.2]. □

Note that, by the same argument, $\mathrm{HH}^*(\widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}})$ is isomorphic to the Jacobian algebra.

5.3 Hochschild homology

We draw attention to the well-known fact that the category $[\mathrm{MF}(R,w)]$ is a Calabi-Yau category (cf. [Buc86, 10.1.5]). A lift of this result to a statement about dg categories would therefore show that Hochschild cochain and chain complex are in duality via the trace pairing. According to the above computation we would then expect the following result to hold for the Hochschild homology of the category $\mathrm{MF}(R, w)$.

Theorem 5.7. *The Hochschild homology of the* 2*-periodic dg category* $\mathrm{MF}(R, w)$ *is given by*

$$\mathrm{HH}_*(\mathrm{MF}(R, w)) \cong R/(\partial_1 w, \ldots, \partial_n w)$$

concentrated in the degree given by the parity of the Krull dimension of R*.*

We give a proof of this theorem which does not refer to the trace pairing. Along the way, we actually prove that matrix factorization categories are Calabi-Yau in the sense of [KKP08, 4.28].

The following definition of Hochschild homology is due to Toën and we reformulate it in the 2-periodic situation. Let T be a 2-periodic dg category. Let $\mathbf{1}$ denote $k[u, u^{-1}]$ considered as a dg category with a single object. Applying [Toë07, Lemma 6.2] we obtain an isomorphism

$$[T \otimes T^{\mathrm{op}}, \widehat{\mathbf{1}}] \cong \mathrm{Iso}(\mathrm{Ho}(T \otimes T^{\mathrm{op}}\text{- mod}))$$

On the other hand, by [Toë07, Theorem 7.2] we have a natural isomorphism

$$[\widehat{T \otimes T^{\mathrm{op}}}, \widehat{\mathbf{1}}]_c \xrightarrow{\simeq} [T \otimes T^{\mathrm{op}}, \widehat{\mathbf{1}}]$$

given by the pullback functor. Therefore T, considered as an object in $T \otimes T^{\mathrm{op}}$- mod gives rise to a continuous functor $\widehat{T \otimes T^{\mathrm{op}}} \to \widehat{\mathbf{1}}$. Passing to homotopy categories we obtain a map of derived categories

$$\mathrm{tr}:\ \mathrm{D}(T^{\mathrm{op}} \otimes T) \to \mathrm{D}(k[u, u^{-1}]).$$

The Hochschild chain complex of T is then defined to be the image of T under this map which, in Morita theoretic terms, coincides with the trace of the identity functor on T.

If A is a 2-periodic dg algebra, then there is an alternative description of the Hochschild chain complex of A. Let us introduce the notation $A^e = A \otimes A^{\mathrm{op}}$. Consider the tensor product

$$(A^e)^{\mathrm{op}}\text{-}\operatorname{mod} \times A^e\text{-}\operatorname{mod} \to \mathrm{C}(k[u, u^{-1}]),\ (M, N) \mapsto M \otimes_{A^e} N$$

which is defined to be the coequalizer of the two natural maps

$$M \otimes A^e \otimes N \rightrightarrows M \otimes N.$$

The tensor product is a Quillen bifunctor and can thus be derived. In this situation, one checks directly from the definition that the map

$$\mathrm{tr}:\ \mathrm{D}(A^{\mathrm{op}} \otimes A) \to \mathrm{D}(k[u, u^{-1}])$$

is given by the functor $- \otimes^{\mathrm{L}}_{A^e} A$. Therefore, the Hochschild chain complex of A admits the familiar description

$$\mathrm{C}_*(A) = A \otimes^{\mathrm{L}}_{A^e} A.$$

The following lemma is well-known.

Lemma 5.8. *The Hochschild chain complex of a 2-periodic dg category T and its triangulated hull $\widehat{T}_{\mathrm{pe}}$ are isomorphic in $\mathrm{D}(k[u, u^{-1}])$.*

Proof. Consider the natural functor $f: T \otimes T^{\mathrm{op}} \to \widehat{T}_{\mathrm{pe}} \otimes (\widehat{T}_{\mathrm{pe}})^{\mathrm{op}}$. By the dg Yoneda lemma, the restriction of the $\widehat{T}_{\mathrm{pe}} \otimes (\widehat{T}_{\mathrm{pe}})^{\mathrm{op}}$-module $\widehat{T}_{\mathrm{pe}}$ along f coincides with T. From this, we obtain a commutative diagram

$$\begin{array}{ccccc}
\widehat{T}_{\mathrm{pe}} \otimes (\widehat{T}_{\mathrm{pe}})^{\mathrm{op}} & \longrightarrow & \widehat{\widehat{T}_{\mathrm{pe}} \otimes (\widehat{T}_{\mathrm{pe}})^{\mathrm{op}}} & \longrightarrow & \widehat{\mathbf{1}} \\
{\scriptstyle f}\uparrow & & {\scriptstyle f_!}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
T \otimes T^{\mathrm{op}} & \longrightarrow & \widehat{T \otimes T^{\mathrm{op}}} & \longrightarrow & \widehat{\mathbf{1}}
\end{array}$$

in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$, where the horizontal functors are the ones constructed in the definition of the Hochschild chain complex above. To obtain the result, we have to show that the functor

$$[f_!]: \ \mathrm{D}(T \otimes T^{\mathrm{op}}) \to \mathrm{D}(\widehat{T}_{\mathrm{pe}} \otimes (\widehat{T}_{\mathrm{pe}})^{\mathrm{op}})$$

maps T to $\widehat{T}_{\mathrm{pe}}$. Since, by iterated application of [Toë07, Lemma 7.5], we have a Quillen equivalence

$$T \otimes T^{\mathrm{op}}\text{-}\,\mathrm{mod} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \widehat{T}_{\mathrm{pe}} \otimes (\widehat{T}_{\mathrm{pe}})^{\mathrm{op}}\text{-}\,\mathrm{mod}$$

it suffices to show that f^* maps $\widehat{T}_{\mathrm{pe}}$ to T. This follows from the dg Yoneda lemma. $\square$

Let E be the stabilized residue field in the category $\mathrm{MF}(R, w)$ and denote $\mathrm{Hom}(E, E)$ by A. We have morphisms in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$

$$\mathrm{MF}(R, w) \longrightarrow \widehat{\mathrm{MF}(R, w)}_{\mathrm{pe}} \xrightarrow{\simeq} \widehat{A}_{\mathrm{pe}} \longleftarrow A$$

where the middle morphism is the isomorphism from Theorem 4.2. Applying Lemma 5.8, we conclude that the Hochschild chain complexes of the categories $\mathrm{MF}(R, w)$ and A are isomorphic in $\mathrm{Ho}(\mathrm{C}(k[u, u^{-1}]))$.

Then, using that A is a perfect A^e-module by Lemma 5.4, we have an isomorphism

$$\mathrm{C}_*(A) \simeq A \otimes^{\mathrm{L}}_{A^e} A \simeq \mathrm{RHom}_{A^e}(A^!, A)$$

in $\mathrm{Ho}(\mathrm{dgcat}_{k[u,u^{-1}]})$. Here, RHom_{A^e} denotes the $\mathrm{Ho}(\mathrm{C}(k[u, u^{-1}]))$-enriched derived Hom functor with respect to the natural $\mathrm{C}(k[u, u^{-1}])$-module structure on the model category A^e- mod and we define

$$A^! = \mathrm{RHom}_{(A^e)^{\mathrm{op}}}(A, A^e).$$

Note that $A^!$ admits a natural A^e-module structure.

Via the compact generator $E^\vee \otimes_k E$ we obtain an isomorphism

$$\mathrm{MF}^\infty(R \otimes_k R, -\widetilde{w}) \xrightarrow{\simeq} \widehat{(A^e)^{\mathrm{op}}} = \mathrm{Int}(A^e\text{- mod}).$$

Therefore, we can calculate $\mathrm{C}_*(A)$ as a morphism complex in $\mathrm{MF}^\infty(R \otimes_k R, -\widetilde{w})$, provided we determine the matrix factorization corresponding to $A^!$.

Lemma 5.9. *The matrix factorization corresponding to $A^!$ is the stabilized diagonal shifted by the parity of the dimension of R.*

Proof. Let $\widetilde{E} = E^\vee \otimes_k E$. We have to find a matrix factorization X in $\mathrm{MF}(R \otimes_k R, -\widetilde{w})$ such that

$$\mathrm{Hom}(\widetilde{E}, X) \simeq \mathrm{RHom}_{(A^e)^{\mathrm{op}}}(A, A^e)$$

For any factorization X, we have

$$\operatorname{Hom}(\widetilde{E}, X) \cong \operatorname{Hom}(X^\vee, \widetilde{E}^\vee),$$

where the right-hand side is a morphism complex in the category $\mathrm{MF}(R \otimes_k R, \widetilde{w})$. Now $\widetilde{E}^\vee$ is a compact generator of $\mathrm{MF}^\infty(R \otimes_k R, \widetilde{w})$ with endomorphism dg algebra A^e. By Theorem 4.2, we obtain a quasi-isomorphism

$$\operatorname{Hom}(X^\vee, \widetilde{E}^\vee) \simeq \operatorname{RHom}_{(A^e)^{\mathrm{op}}}(\operatorname{Hom}(\widetilde{E}^\vee, X^\vee), A^e).$$

We choose X to be the stabilized diagonal shifted by the parity of the dimension of R. Then $X^\vee$ is isomorphic to the stabilized diagonal in the category $\mathrm{MF}(R \otimes_k R, \widetilde{w})$ and the argument of Corollary 5.4 yields a quasi-isomorphism

$$\operatorname{Hom}(\widetilde{E}^\vee, X^\vee) \simeq A$$

completing the proof. □

Note that the lemma in combination with Corollary 5.5 immediately implies Theorem 5.7. We also remark that, in light of derived Morita theory, the bimodule $A^!$ determines an endofunctor on the category $\mathrm{MF}(R, w)$. It corresponds to the inverse Serre functor and the fact that it is isomorphic to a shift of the identity expresses the Calabi Yau property of $\mathrm{MF}(R, w)$.

Chapter 6

Noncommutative geometry

We conclude with some remarks on the geometric implications of our results. In the introduction, we pointed out that we want to think of the dg category of matrix factorizations as the derived category of sheaves on a noncommutative space. Let $\mathcal{X}$ be a hypothetical noncommutative space associated to an isolated hypersurface singularity (R, w), where R is a regular local k-algebra with residue field k. Without explicitly knowing how to think of $\mathcal{X}$ itself, we postulate

$$\mathrm{D}^{\mathrm{qcoh}}_{\mathrm{dg}}(\mathcal{X}) \simeq \mathrm{MF}^{\infty}(R, w).$$

The symbol $\mathcal{X}$ is thus merely of linguistical character, the defining mathematical structure attached to it is $\mathrm{D}^{\mathrm{qcoh}}_{\mathrm{dg}}(\mathcal{X})$. We show how to establish several important properties of the space $\mathcal{X}$, using the results of the previous chapters. Details on the terminology which we use can be found in [KS06] and [KKP08].

$\mathcal{X}$ is dg affine. A noncommutative space $\mathcal{X}$ is called dg affine if $\mathrm{D}^{\mathrm{qcoh}}_{\mathrm{dg}}(\mathcal{X})$ is quasi-equivalent to the dg derived category of some dg algebra A. In our case, this is expressed by Theorem 4.2 where A is explicitly given as the endomorphism algebra of the stabilized residue field.

Perfect complexes on $\mathcal{X}$. A general noncommutative space $\mathcal{X}$ is given by the dg category $\mathrm{D}^{\mathrm{qcoh}}_{\mathrm{dg}}(\mathcal{X})$ of quasi-coherent sheaves on $\mathcal{X}$. The category $\mathrm{D}^{\mathrm{perf}}_{\mathrm{dg}}(\mathcal{X})$ is then defined to be the full dg subcategory of compact objects. In the case at hand, we have

$$\mathrm{D}^{\mathrm{perf}}_{\mathrm{dg}}(\mathcal{X}) \simeq \widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}}$$

by Corollary 4.3. Note that the explicit description

$$\widehat{\mathrm{MF}(R,w)}_{\mathrm{pe}} \simeq \mathrm{MF}(\widehat{R},w)$$

given in Theorem 4.6 implies that $\mathcal{X}$ in fact only depends on the formal germ of (R,w).

$\mathcal{X}$ is proper over k. By definition, a dg affine noncommutative space is proper over k if the cohomology of its defining dg algebra is finite dimensional over k. In our case, the algebra $\mathrm{H}^*(A)$ is isomorphic to a finitely generated Clifford algebra and thus finite dimensional. Note that, in our 2-periodic situation, the finiteness condition refers to the $\mathbb{Z}/2$-graded object A. To be more precise, we should call $\mathcal{X}$ proper over $k[u, u^{-1}]$.

$\mathcal{X}$ is homologically smooth over k. A dg affine noncommutative space is defined to be homologically smooth if A is a perfect $A \otimes A^{\mathrm{op}}$-module. For $\mathcal{X}$ this follows from the proof of Corollary 5.4: the stabilized diagonal is a compact object in $[\mathrm{MF}^\infty(R \otimes_k R, \widetilde{w})]$ which maps to A under the coproduct preserving equivalence

$$[\mathrm{MF}^\infty(R \otimes_k R, \widetilde{w})] \xrightarrow{\simeq} \mathrm{D}(A \otimes A^{\mathrm{op}}).$$

The homological smoothness of $\mathcal{X}$ suggests that we may as well think of the category $\mathrm{D}^{\mathrm{perf}}_{\mathrm{dg}}(\mathcal{X})$ as an analogue of the bounded derived category of coherent sheaves on $\mathcal{X}$.

Deformation theory of $\mathcal{X}$. A deformation of $\mathcal{X}$ is defined to be a deformation of the dg category $\mathrm{D}^{\mathrm{qcoh}}_{\mathrm{dg}}(\mathcal{X})$. All deformations of $\mathcal{X}$ are obtained by deforming the germ of the singularity of (R, w). This follows from our calculation of the Hochschild cochain complex which governs the deformation theory of the dg category $\mathrm{MF}^\infty(R, w)$. By comparison, the Jacobian ring parameterizes deformations of the germ (R, w) as explained in [AGZV85]. Since the Hochschild cohomology is concentrated in even degree, we conclude that all deformations are unobstructed. This means that $\mathcal{X}$ defines a smooth point in an appropriate moduli space of dg categories.

Hodge-to-de Rham degeneration for $\mathcal{X}$. The Hochschild homology of the category $\mathrm{D}^{\mathrm{perf}}_{\mathrm{dg}}(\mathcal{X})$ should be thought of as the Hodge cohomology of the space $\mathcal{X}$. The periodic cyclic homology of $\mathrm{D}^{\mathrm{perf}}_{\mathrm{dg}}(\mathcal{X})$ plays the role of the de Rham cohomology of $\mathcal{X}$. Generalizing the case of a commutative scheme, there is a spectral sequence from Hochschild homology to periodic cyclic homology. For the space $\mathcal{X}$ this spectral sequence degenerates, confirming the general degeneration conjecture in the case of matrix factorization categories. Indeed, this immediately follows from the fact that the Hochschild homology is concentrated in a single degree. Therefore, Connes' B operator must vanish on all higher pages of the Hodge-to-de Rham spectral sequence since it has degree 1. In particular, we obtain

$$\mathrm{HP}_*(\mathrm{MF}(R,w)) \cong \mathrm{HH}_*(\mathrm{MF}(R,w))$$

where HP_* denotes periodic cyclic homology.

$\mathcal{X}$ is a Calabi-Yau space. Lemma 5.9 implies the existence of an isomorphism

$$A^! \simeq A[n]$$

in $\mathrm{D}(A \otimes A^{\mathrm{op}})$ where n is the dimension of R. Thus, $\mathcal{X}$ is a Calabi-Yau space in the sense of [KKP08, 4.28]. In view of [DM10, 5.2] this gives, in the case of matrix factorization categories, an affirmative answer to a general conjecture by

Kontsevich-Soibelman [KS06, 11.2.8].

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