An Overview of the Geometry and Combinatorics of the Macdonald Polynomial and q-t Catalan Number

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ABSTRACT

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We will explore the combinatorial and geometric properties related to the Macdonald polynomials and the diagonal harmonics. We have the combinatorial Macdonald polynomial formula that fits the defining conditions directly. The shuffle conjecture gives an elegant expression of the Frobenius series of the diagonal harmonics. While the geometric properties of the Hilbert scheme and schemes over it provides explanations from a different perspective. We use examples to show that these two approaches arrive at the same goal.

Contents

Chapter 1

Punctual Hilbert Scheme

This first chapter is an introduction to the important punctual Hilbert scheme that plays a central role in the study of symmetric functions. Many elegant but mysterious combinatorial formulas related to symmetric functions are related to geometric properties that we look at here. We start by looking at the definition of the general Hilbert scheme in category language, and explain the universal property in functor representation. Then we discuss an elementary way to express the Hilbert scheme in the affine case. We summarize basic geometric properties of the Hilbert scheme, the universal family and the zero fiber, with some emphasis on how proofs use commutative algebra theorems. Finally, as a demo of the algebraic combinatorial beauty, we review a q, t-Catalan number from both algebraic geometric and combinatorial perspectives.

1.1 Representation of Functors

We first review some definitions and notations of functor representation. A good source is [?]. Let $\mathcal{C}$ be a category. Let X be an object in $\mathcal{C}$. Then X induces a functor $h_X : \mathcal{C}^{op} \longrightarrow \text{Set}$. On the object level, we have

$$h_X(Y) = \text{Hom}(Y, X)$$

, and for any morphism $f \in \text{Hom}(Y, Z)$,

$$h_X(f) : h_X(Z) \longrightarrow h_X(Y)$$

sends $g \in \text{Hom}(Z, X)$ to fg.

The definition above of the induced functor gives rise to natural transformations between h_X and h_Y with X and Y objects of $\mathcal{C}$. What's more, the Yoneda lemma asserts that $\text{Hom}(X, Y)$ and $\text{Hom}(h_X, h_Y)$ are bijective with the obvious mapping. This implies that if there are two objects X and Y representing the same functor, i.e. h_X and h_Y are both isomorphic to some functor $F : \mathcal{C}^{op} \longrightarrow \text{Set}$, then $X \simeq Y$.

When X represents the functor F, we may use the Yoneda lemma to identify the universal object. Let $\theta : h_X \longrightarrow F$ be the natural transformation, so we have a bijection

$$\theta_Y : h_X(Y) = \text{Hom}(Y, X) \longrightarrow F(Y)$$

for any object Y. By definition, the universal object is $\mu = \theta_X(\text{id}_X)$, that is, the image of the identity map of X. The image of any other elements is produced by the formula

$$\theta_Y(f) = F(f)(\mu), \text{ for any } f \in h_X(Y)$$

Now we may look at the definition of the Hilbert scheme. We will focus on the category of finitely generated schemes over Spec k where k is an algebraicly closed field and of characteristic 0. Generally, let X be a scheme over k, define a functor $\mathfrak{Hilb}_X^n$ from schemes to sets as follows. On the object level, $\mathfrak{Hilb}_X^n(T)$ is the set of closed subschemes of $X \times T$ that are flat and surjective of degree n over T. As for morphisms, suppose we have $f : S \to T$. Then $\mathfrak{Hilb}_X^n(f)$ maps subschemes in $\mathfrak{Hilb}_X^n(T)$ to their pull back in $X \times S$.

$$\begin{array}{ccc} X \times S & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle f} \\ X \times T & \longrightarrow & T \\ \downarrow & & \\ X & & \end{array}$$

Since base extension preserves the properties of the subscheme, $\mathfrak{Hilb}_X^n(f)$ is well defined.

It was proved, in a more general context, that $\mathfrak{Hilb}_X^n$ is actually representable. The scheme H_X^n representing it is the punctual Hilbert scheme. So by Yoneda's lemma, there is a universal object, a special closed subscheme $U \subseteq X \times H_X^n$. Now, if we take S as a closed point $s \in T = H_X^n$, we get the pull back of U in $X \times \operatorname{Spec} k(s)$, which by definition of the functor is a closed subscheme flat and surjective of degree n over $\operatorname{Spec} k(s)$, that is, free of degree n over k. On the other hand, any such subscheme will induce a unique morphism from Spec(k) to H_X^n, due to the universal property. This establishes a bijection between degree n (as a vector space over k) closed subschemes of X and points of H_X^n. And we can speak of the two interchangeably.

1.2 Expressions of the Hilbert Scheme

We explore ways to express the Hilbert Scheme and implications in this section. First, we review some results from [?] that express the Hilbert scheme in elementary terms.

The original theorem assumes a special case that X is affine and quasi-projective over the base scheme,

$$X = \operatorname{Spec}(R) = \text{closed subscheme of } \operatorname{Spec}(\operatorname{Sym} M)$$

for some coherent sheaf M on the base scheme. It suits our needs to assume further that $R = \operatorname{Spec} k[X, Y]$, and the base scheme is $\operatorname{Spec}(k)$. We fix a vector space F of degree n over k. Fix a base T_i, $i = 1, \ldots, n$, for F, and single out $e = T_1$ for special use. Let $\beta : F \to R = k[X, Y]$ be a k morphism that maps e to 1. Then we define $\operatorname{Hom}^{\beta}(R, F)$ to be a functor from k algebras to sets. The functor maps an algebra B to morphisms

$$f : B \otimes R \to B \otimes F$$

such that the composition of maps is the identity map.

$$B \otimes F \xrightarrow{\mathrm{id} \otimes \beta} B \otimes R \xrightarrow{f} B \otimes F = \mathrm{id}_{B \otimes F} \tag{1.2.1}$$

This is an open subset of the Hilbert functor we defined earlier such that in $\mathfrak{Hilb}^n_X(T)$, $T = \mathrm{Spec}(B)$. As we restrict to the affine case, the scheme morphisms are now simply k algebra morphisms. Indeed, $\phi : B \to C$ induces a map from morphisms of $B \otimes R \to B \otimes F$ to morphisms $C \otimes B \otimes R = C \otimes R \to C \otimes B \otimes F = C \otimes F$. The theorem states that the open subset of the Hilbert scheme that represents $\mathrm{Hom}^\beta(R, F)$ is a quotient of $\mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee)$. The idea is that we would like to extend a certain morphism

$$t' : \mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee) \otimes M \to \mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee) \otimes \mathrm{End}(F) \tag{1.2.2}$$

to

$$t : \mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee) \otimes \mathrm{Sym}(M) \to \mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee) \otimes \mathrm{End}(F) \tag{1.2.3}$$

In (**??**), t' maps $1 \otimes m$ to $m \otimes \sum_{i,j=1}^{n} T_{ij}^\vee \otimes T_{ij}$, where $m \in M$, and $T_{ij}^\vee$ and T_{ij} $i, j = 1, \ldots, n$ are basis elements of $\mathrm{End}(F)^\vee$ and $\mathrm{End}(F)$, respectively. In order to extend (ft) to (**??**), we need $t'(1 \otimes m)$ commute with $t'(1 \otimes n)$ for any $m, n \in M$. In the case that M is free with basis elements X and Y – so that $\mathrm{Sym}(M) = k[X, Y]$, the commutativity is equavalent to that the following hold in the quotient ring:

$$t'(1 \otimes X)t'(1 \otimes Y) - t'(1 \otimes Y)t'(1 \otimes X) = 0$$

Let $U_{ij}^L = L \otimes \sum_{i,j=1}^{n} T_{ij}^\vee$ be the generators of $\mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee)$ as a k algebra, where we may substitute X and Y for L. Now we may express

$$t'(1 \otimes L) = L \otimes \sum_{i,j=1}^{n} T_{ij}^\vee \otimes T_{ij} = \sum_{i,j=1}^{n} U_{ij} T_{ij}^L$$

as an $n \times n$ matrix in the ring $\mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee)$. Explicitly, write (U^L_{ij}) for the matrix with U^L_{ij} at (i,j), and the multiplication corresponds to that in $\mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee) \otimes \mathrm{End}(F)$. Finally, the commutativity condition is simply

$$(U^X_{ij})(U^Y_{ij}) - (U^Y_{ij})(U^X_{ij}) = 0 \tag{1.2.4}$$

in the quotient ring, or that the ideal includes all elements in the matrix of (**??**).

There is a second condition on the quotient ring. This comes from (**??**). More precisely, let let $\beta(T_m) = f_m[X,Y]$, for $m = 1,\ldots,n$. We need the following reduce to zero in the quotient ring:

$$f_m[(U^X_{ij}),(U^Y_{ij})](T_1) - T_m \text{ for } m = 1,\ldots,n \tag{1.2.5}$$

In the above expression, we evaluate the action of the matrix $f_m[(U^X_{ij}, U^Y_{ij}]$ on $T_1 = e$ – the basis element in $\mathrm{Sym}(M \otimes \mathrm{End}(F)^\vee) \otimes F$, and we need the result to be T_m.

We want to further refine the formulas. First we review the local affine structure of the Hilbert scheme. There is a collection of covering open affine subspaces of the Hilbert scheme, indexed by partitions of n, and on each of them, we have a nice definition of β. We denote by $\mathcal{B}_\mu$ the monomials with degrees the same as the coordinates of the graph of μ. That is,

$$\mathcal{B}_\mu = \{X^h Y^k : (h,k) \in \mu\}$$

Then the covering collection is

$$U_\mu = \{I \in H^X_n : \mathcal{B}_\mu \text{ spans } k[X,Y]/I\},\ \mu \vdash n \tag{1.2.6}$$

We know that the coordinate ring on U_μ is generated by $c_{h,k}^{r,s}$, the parameter of X^rY^s at basis X^hY^k, for all pairs of (r,s), and $(h,k) \in \mu$. This is because the values of $c_{h,k}^{r,s}$ are determined by and uniquely identify and the ideal I. Since the global sections of $\mathcal{B}_\mu$ form a base set at any $I \in U_\mu$, we can construct the free sheaf F of the previous paragraph such that its basis elements correspond to elements in μ. Specifically, let T_i correspond to $(h_i, k_i) \in \mu$. We also construct the map β accordingly to get (**??**). Therefore

$$\beta(T_i) = X^{h_i}Y^{k_i} \tag{1.2.7}$$

Now we will examine the identity mapping (**??**) condition again. By (**??**),

$$f_m[(U_{ij}^X), (U_{ij}^Y)] = (U_{ij}^X)^{h_m}(U_{ij}^Y)^{k_m} \tag{1.2.8}$$

So (**??**) requires that the first column of $(U_{ij}^X)^{h_m}(U_{ij}^Y)^{k_m}$ is zero except at mth row where it is 1. We summarize the two conditions as follows.

Theorem 1.2.1. *We have an affine structure of the Hilbert scheme on U_μ such that the coordinate ring is generated by U_{ij}^X and U_{ij}^Y for $i, j = 1, \ldots, n$. The ring is a quotient of $k[U_{ij}^X, U_{ij}^Y]_{i,j=1,\ldots,n}$ by the ideal generated by elements in*

$$(U_{ij}^X)(U_{ij}^Y) - (U_{ij}^Y)(U_{ij}^X)$$

and

$$(U_{ij}^X)^{h_m}(U_{ij}^Y)^{k_m}(T_1) - T_m$$

Next, we derive an expression for the universal family. In [?], it is shown that the universal family is the quotient

$$u : k[X,Y] \otimes \mathcal{O}_{H_X^n} \longrightarrow k[X,Y] \otimes F$$

where the images of X and Y are determined by t' in (??) followed by evaluation on $T_1 = e$. Therefore, $u(L) = (U_{ij}^L)(T_1)$. On the open affine U_μ, we have these identities in the quotient.

$$L = \sum_{i=1}^{n} U_{i1}^L X^{h_i} Y^{k_i}, \text{ for } L = X, Y \tag{1.2.9}$$

These equations cut out the universal family as a closed subscheme.

Theorem 1.2.2. *The universal family over the open affine subset of U_μ is the closed subscheme of* $\operatorname{spec} k[X,Y] \otimes U_\mu$ *determined by equations in (??).*

1.3 Geometric Properties of the Hilbert Scheme

We review some elementary geometric properties of the punctual Hilbert scheme of $\mathbf{A}^2$ as described in [?]. From now on, we take $X = \mathbf{A}^2 = \operatorname{Spec} \mathbf{C}[X,Y]$, and write H^n for H_X^n.

First, recall the fact that H^n is smooth and irreducible of degree $2n$. The proof uses a torus group T^2 acting on $\mathbf{C}[X,Y]$, and thus also on H^n. The orbits of any point on H^n preserve its singularity and have an I_μ defined by

$$I_\mu := (x^h y^k : (h,k) \notin \mu)$$

in the closure. So smoothness follows from regularity at I_μ. The argument of torus action also shows H^n is connected. This together with smoothness implies it is irreducible.

Next, the universal scheme $H^n_+ \subseteq H^n \times \mathbf{A}^2$ is finite and flat over H^n as we discussed in the first section, so its local ring $O_{H^n_+,(I,p)}$ at any point (I,p) is finite and flat over the local ring $O_{H^n,I}$ of H^n at I. They are therefore of the same dimension, and regularity of $O_{H^n,I}$ implies it has $2n$ regular system parameters which is also regular in $O_{H^n_+,(I,p)}$ because of flatness. This means H^n_+ is Cohen-Macaulay.

In H^n_+ we can cut out a complete intersection by ideals defined on U_μ as

$$\mathfrak{I} = (x, y, p_{r,s} : (r,s) \in \mu \backslash (0,0))$$

This is because the closed subscheme $V(\mathfrak{I})$ defined by the ideal is isomorphic to the zero fiber H^n_0, the closed subscheme in H^n which is the fiber over the zero set in the Chow morphism. H^n_0 contains all points in H^n which represent ideals with only zero solution in $\mathbf{C}[X,Y]$. Since it is known that the induced reduced scheme on H^n_0 is of dimension $n-1$, $V(\mathfrak{I})$ is of the same dimension. So the number of generators in $\mathfrak{I}$ is the same as the codimension. This is why we have a complete intersection. Now H^n_+ is Cohen-Macaulay, the generators must be a regular sequence in its local rings, and the quotient which is the local ring of the closed subscheme is also Cohen-Macaulay. In fact, H^n_0 is irreducible, reduced and Cohen-Macaulay.

One important application of the above properties is to construct a free resolution of $O_{H^n_0}$ as an O_{H^n} module. First recall some commutative algebra from

[?] about the Koszul complex. For any A module M, and a sequence of elements $(x_1, \ldots, x_n)$ in A, define a complex $K.(\underline{x}, M)$ such that $K_p = M \otimes \wedge^p F$, where $F = Ae_1 + \ldots + Ae_n$ is a free A module with basis $\{e_i, \ldots, e_n\}$. Define

$$d(e_{i1} \wedge \ldots \wedge e_{ip}) = \sum_{r=1}^{p} (-1)^{r-1} x_{ir} e_{i1} \wedge \ldots \wedge \hat{e}_{ir} \wedge \ldots \wedge e_{ip}$$

Then a theorem states that if $x_1, \ldots, x_n$ is an M regular sequence, we have

$$H_p(K.(\underline{x}, M)) = 0 \text{ for } (p > 0),\ H_0(K.(\underline{x}, M)) = M / \sum_{1}^{n} x_i M = 0$$

Now we construct the resolution. Remember in the previous section we have B_μ as a basis of the pushdown B of the universal family on the affine open subset U_μ, which is considered a free O_{H^n} module. The generators of $\mathfrak{I}$ form a regular sequence locally. So we may apply the commutative algebra result to get

$$0 \to F_{n+1} \to \ldots \to F_1 \to F_0 \to O_{H_0^n}$$

with $F_i = B \otimes \wedge^i (B' \oplus O_t \oplus O_p)$, where B' is a free module with basis $B_\mu \backslash \{1\}$. O_t and O_p are free modules corresponding to regular sections x and y in the local ring of B. But they differ in the torus action.

1.4 q, t-Catalan Number

So far we have accumulated some basic knowledge of the Hilbert scheme. These were used in exploration of the q, t-Catalan number $C_n(q, t)$. The number was conjectured and eventually proved to be the Hilbert series of the diagonal harmonic subspace of alternating elements. Some easy application of the above mentioned properties can already establish the identity involving the Hilbert polynomial of $H^0(H_0^n, O(m))$ for large m. The expressions in the formula have many statistics from the partition $\mu \vdash n$. For any square $s \in \mu$, define $a_\mu(s)$, $l_\mu(s)$, $a'_\mu(s)$ and $l'_\mu(s)$ as the number of cells of μ that are respectively strictly east, north, west and south of s in μ [?]. Suppose $\mu = (\mu_1, \ldots, \mu_k)$, set

$$n(\mu) = \sum_{i1}^{k} (i-1)\mu_i$$
$$B_\mu(q,t) = \sum_{s \in \mu} q^{a'_\mu(s)} t^{l'_\mu(s)}$$
$$T_\mu = t^{n(\mu)} q^{n(\mu')},\ \Pi_\mu(q,t) = \prod_{s \in \mu \setminus (0,0)} (1 - q^{a'_\mu(s)} t^{l'_\mu(s)})$$
$$M = (1-t)(1-q)$$

and

$$\tilde{h}_\mu(q,t) = \prod_{s \in \mu} (q^{a'_\mu(s)} - t^{l'_\mu(s)+1}),\ \tilde{h}'_\mu(q,t) = \prod_{s \in \mu} (t^{l'_\mu(s)} - q^{a'_\mu(s)+1})$$

Now the q,t-Catalan number can be expressed as

$$C_n(q,t) = \sum_{\mu \vdash n} \frac{T_\mu^2 M \Pi_\mu(q,t) B_\mu(q,t)}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)} \tag{1.4.1}$$

The partition statistics enter into the formula as we calculate $\mathrm{Tor}_i(k(I_\mu), O_{H_0^n(m)})$. This uses the regular system parameter at I_μ and also the free resolution we constructed earlier.

A remarkable combinatorial formula of $C_n(q,t)$ using bounce and area statistics in the lattice was conjectured in [?] and proved in [?]. We introduce this result to end the chapter [?].

A Dyck path from $(0,0)$ to (n,n) is a sequence of north $(0,1)$ and east $(1,0)$ moves in the first quadrant of the x,y-plane, that never goes below the diagonal line $y=x$. Use $L^+_{n,n}$ to denote the collection of these Dyck paths. We can show $|L^+_{n,n}|=C_n$, the classic Catalan number, through the argument that both satisfy the recursive identity

$$C_n=\sum_{k=1}^{n} C_{k-1}Cn-k, \text{ for } n\geq 1$$

Since any Dyck path touches the line $y=x$ first at $(0,0)$ and next at some (k,k). Thus for fixed k, such paths split into two parts. The first part corresponds to an element in $L^+_{k-1,k-1}$, — just imagine we move the $y=x$ line up by 1 unit. The second part corresponds to an element in $L^+_{n-k,n-k}$.

We define the area(π) and bounce(π) statistics for all $\pi\in L^+_{n,n}$. area(π) is simply the number of whole squares below π and above $y=x$. For the bounces, think about a billiard ball moving north from $(0,0)$ and go east when stopped at a 'peak' of the Dyck path π. It moves north again when reaches the diagonal $y=x$... The places it strikes on the dagonal are $(0,0)$, (j_1,j_1), ... , $(j_b,j_b)=(n,n)$. Define bounce(π) $=\sum_{i=1}^{b-1} n-j_i$.

The combinatorial formula of the q,t Catalan numbers is

$$C_n(q,t)=\sum_{\pi\in L^+_{n,n}} q^{\text{area}(\pi)}t^{\text{bounce}(\pi)} \tag{1.4.2}$$

The proof of the formula involves the nabla operator ∇ and a recurrence of $L^+_{n,n}(k)$, defined as the subset of $L^+_{n,n}$ consisting of Dyck paths which starts by moving exactly k north steps and followed by an east step. Let

$$F_{n,k}(q,t) = \sum_{\pi \in L^+_{n,n}(k)} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}, \; F_{n,0} = \delta_{n,0}.$$

The recurrence, proved in [**?**], is as follows,

$$F_{n,k}(q,t) = \sum_{r=0}^{n-k} \begin{bmatrix} r+k-1 \\ r \end{bmatrix} t^{n-k} q^{\binom{k}{2}} F_{n-k,r}(q,t) \tag{1.4.3}$$

Chapter 2

Macdonald Polynomials

This chapter revolves around the Macdonald polynomials. We first review some fundamental theories of the symmetric functions in general. Macdonald polynomials are basis elements of the symmetric function. The famous positivity conjecture states that the polynomial expands in Schur functions with Kostka coefficients $K_{\lambda\mu}(q,t)$ and the coefficients are non-negative integral polynomials of q and t. The conjecture was proved by equating the Macdonald polynomial with a Frobenius series, an algebraic argument involving the isospectral Hilbert scheme and its fiber. In the end we look at the alternative proof which expresses explicitly a combinatorial construction of the polynomial.

2.1 Symmetric Functions

Let Λ_n be the set of homogeneous symmetric functions of degree n. The coefficient ring in our context will be $\mathbf{Q}$, though it is not necessary in general. Λ_n is a $\mathbf{Q}$ module and also $\mathbf{Q}$ vector space. Let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ be the set of all symmetric functions. Λ is a graded $\mathbf{Q}$ algebra, and $\mathbf{Q}$ vector space.

The partitions of all positive integers $\mathrm{Par}(n)$, $n \in \mathbf{N}$ is an index set for the bases of Λ. The dominance order $\geq$ makes a lattice structure of $\mathrm{Par}(n)$. Taking into account this ordering, we may get the transition matrices between the classical bases. In particular, we have the bases of the monomial function m_λ, the elementary function e_λ, the homogeneous function h_λ, the power sum function p_λ and the Schur function s_λ. A transition matrix $(M_{\lambda\mu})$ is upper triangular if its elements are zero unless $\lambda \geq \mu$. It is upper unitriangular if in addition $M_{\lambda\lambda} = 1$ for all λ. For example,

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

where $\lambda \vdash n$, and $K_{\lambda\mu}$ is the Kostka number, which counts the number of semistandard Young tableaux of shape λ and weight μ and therefore upper unitriangular.

The Macdonald polynomial is a base of the (q, t) symmetric functions. It is a generalization of the traditional symmetric function by introducing a (q, t) analog of the Hall inner product. Recall that

$$< p_\lambda, p_\mu >= z_\lambda \delta_{\lambda\mu} \tag{2.1.1}$$

where $z_\lambda = 1^{m_1} m_1! \cdots n^{m_n} m_n!$ for $\lambda = (1^{m_1}, \cdots, n^{m_n})$. $n! z_\lambda^{-1}$ counts the size of the conjugacy class of cycle type λ. We can define the q, t inner product by the plethystic expression

$$< f, g >_{q,t} = < f(X), g[X\frac{1-q}{1-t}] >$$

or equivalently

$$< p_\lambda, p_\mu >_{q,t} = \delta_{\lambda\mu} z_\lambda(q,t) \tag{2.1.2}$$

where $z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q_i^\lambda}{1-t_i^\lambda}$. The Macdonald polynomial $P_\lambda(X;q,t)$ are orthogonal with respect to the q, t inner product, and they are upper unitriangular with respect to the monomials. These properties also uniquely determine the Macdonald polynomials [**?**].

Defined the integral form as

$$J_\lambda(x;q,t) = \prod_{s\in\lambda}(1 - q^{a(s)}t^{l(s)+1})P_\lambda(x;q,t)$$

The Macdonald positivity conjecture says its expansion

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)S_\lambda(x;t) \tag{2.1.3}$$

has coefficients $K_{\lambda\mu}(q,t)$ that are all non-negative integral polynomials in q and t.

In [**?**], the transformed Macdonald polynomial is defined as

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)} J_\mu[\frac{X}{1-t^{-1}}; q, t^{-1}] \tag{2.1.4}$$

where $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})$. This form is convenient in expressing the characteristics of the original P_λ. We list the properties below:

1.

$$\tilde{H}_\mu[X(1-q);q,t] \in \mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu\} \tag{2.1.5}$$

2.

$$\tilde{H}_\mu[X(1-t);q,t] \in \mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\} \tag{2.1.6}$$

3.

$$< \tilde{H}_\mu, s_{(n)} >= 1 \tag{2.1.7}$$

In [**?**], a proof of the positivity conjecture was based on the transformed Macdonald polynomial, by showing that it is actually the Frobenius series of a doubly graded space.

2.2 Representation, Frobenius Series

We will discuss in detail how the representation theory is related to the study of symmetric functions. First we review some representation of the permutation group S_n. Here the irreducible character is indexed by partition or cycle type $\lambda \vdash n$. Let χ^λ denote the character of the Specht module S^λ in [**?**]. It has dimension f_λ, the number of standard Young tableaux of shape λ. A regular representation is an S_n module isomorphic to the group algebra of $\mathbf{C}S_n$. We know that each irreducible χ^λ has f_λ copies in the module, which is one explanation behind the identity $\sum_{\lambda \vdash n}(f^\lambda)^2 = n!$. In particular, the trivial representation $M^{(n)} = S^{(n)}$ has dimension 1 and thus a single copy in the group algebra, as it is clear $f_{(n)} = 1$. A realistic situation we later see is that the fiber of the isospectral Hilbert scheme X_n over $I_\mu \in H_n$ is a regular representation of S_n, and automatically satisfy the above condition (**??**).

An important example is that S_n and the torus group T^2 act on a finite dimensional space V commutatively. That is, for any $(t,q) \in T^2$, its action commutes with any $\omega \in S_n$. So an eigenspace of (t,q) is stable under S_n action, and we may get its decomposition into irreducible S_n representations. Therefore the character of an irreducible component of $S_n \times T$ is the eigenvalue times an S_n irreducible character χ^λ.

The Hilbert series is the sum of the weighted degrees in a graded space. In order to include the infinite length case, we may define a formal Hilbert series as

$$\mathcal{H}_M(q,t) = \frac{\sum_i (-1)^i \operatorname{tr}(\operatorname{Tor}_i^R(M,\mathbf{C}),\lambda)}{\det(m/m^2, 1-\lambda)}, \ \lambda = (t,q) \in T \tag{2.2.1}$$

In our setting, R is the regular local ring of a non-singular point x of a scheme, and M is a finitely generated R module. The torus group T acts on the scheme such that x is an isolated fixed point, thus gives an action on R. T also acts equivalently on M.

Similarly, we may define the Frobenius series which is a weighted characteristic map. Suppose the above objects admits an S_n action that commutes with T. Then we can talk about the characters and the corresponding symmetric functions under the characteristic map. Again, we have the following formal expression

$$\mathcal{F}_M(q,t) = \frac{\sum_i (-1)^i \Phi \operatorname{ch}(\operatorname{Tor}_i^R(M,\mathbf{C}))}{\det(m/m^2, 1-\lambda)}, \ \lambda = (t,q) \in T \tag{2.2.2}$$

Recall in section **??**, we constructed a the Koszul complex of $M/(x)M$ using the M regular sequence (x). When the ring R is a regular local ring, M is free, and the sequence is a regular system parameters, the resolution is minimal, which implies that its tensor with the residue field $\mathbf{C}$ has zero differentiations. This is the case when $M/(x)M$ is an irreducible $S_n \times T$ representation. So in the Frobenius series of an irreducible module, where M plays the role of $M/(x)M$, $\mathrm{Tor}_k^R(M, \mathbf{C})$ is simply $M \otimes \wedge^k \mathbf{C}$. T acts on the second part of the tensor in an equivalent manner as it acts on (x), so to be consistent with the differentiation, more precisely,

$$\mathrm{Tor}_k^R(M, \mathbf{C}) \simeq M \otimes \wedge^k T(x)^*$$

where $T(x)^* = m(x)/m(x)^2$ is the cotangent space at x. Now we see that the wedge product part cancels with the denominator in the Frobenius series, and we recover the usual Frobenius characteristic map when M has finite length.

We are interested in the case that the module is

$$R_\mu = \mathbf{C}[X, Y]/\{\text{annihilators of } \Delta_\mu \text{ as differential operators}\} \tag{2.2.3}$$

It's not hard to show that R_μ is isomorphic D_μ, the space of derivatives of Δ_μ. Later we will see that the isospectral Hilbert scheme X_n has fiber R_μ over the T fixed point $I_\mu \in H_n$, and the Frobenius series of R_μ satisfies conditions (**??**), (**??**), and (**??**), so is equal to the transformed Macdonald polynomial. It is the logic behind the algebraic geometric proof of the positivity conjecture.

2.3 Combinatorial formula

The previous section drew a road map to the algebraic geometric proof of the positivity conjecture. In this section, we go over some important elements in the combinatorial formula of Macdonald polynomial and its proof, which comes from [?].

We first define some statistics related to tableaus. We follow the convention that a tableau μ's bottom row has μ_1 squares and the top row has $\mu_{l(\mu)}$ squares. Given a filling of μ, which we denote by $\sigma : \mu \to \mathbf{Z}_+$, define a descent of σ to be a pair of vertically adjacent cells with descending value from the upper one to the lower one. We collect all these upper cells of descent pairs and call the set $\mathrm{Des}(\sigma)$. An attacking pair is a pair of cells u and v in μ that are either in the same row, or in adjacent rows, and in the second case, say, $u = (i, j)$, and $v = (i + 1, k)$, we have $j < k$. Give cells of μ the 'reading order', by reading from top to bottom and left to right. An inversion is an attacking pair, such that the one that precedes in reading order has larger value. The collection of inversion pairs is $\mathrm{Inv}(\sigma)$. Finally, define

$$\mathrm{maj}(\sigma) = \sum_{u \in \mathrm{Des}(\sigma)} (l(u) + 1)$$

$$\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)| - \sum_{u \in \mathrm{Des}(\sigma)} (a(u) + 1)$$

where $a()$ and $l()$ retrieve the arm and leg values of cells.

The combinatorial formula is:

$$C_\mu(x;q,t) = \sum_{\sigma:\mu\to\mathbf{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma \tag{2.3.1}$$

We sketch the key points in the proof of the theorem that $\tilde{H}_\mu(x;q,t) = C_\mu(x;q,t)$.

In order to show symmetry of C_μ, we decompose it into LLT polynomials, a class of symmetric function first studied by Lascoux, Leclerc and Thibon. The idea is as follows. For a fixed descent set D of μ, we associate with a symmetric function

$$F_{\mu,D}(x;q) = \sum_{\mathrm{Des}(\sigma)=D} q^{|\mathrm{Inv}(\sigma)|} x^\sigma$$

Then we have

$$C_\mu(x;q,t) = \sum_D q^{-a(D)} t^{\mathrm{maj}(D)} F_{\mu,D}(x;q)$$

where $a(D)$ and $\mathrm{maj}(D)$ count the arm and maj statistics associated with the cells in D. The rest of the work is to show $F_{\mu,D}(x;q)$ is an LLT polynomial. We do this by splitting μ in to its columns, and bijectively map each column into a ribbon shape (a skew shape with no $2*2$ block). The map will keep descent sets (descent in a ribbon must have a square below it), which in effect decides the mapping rule. But now the inversions in LLT and $F_{\mu,D}(x;q)$ also match, proving the case, i.e. C_μ is symmetric.

To prove the formula, we need show it meet the three conditions listed in section **??**. The third one is trivial from our formula, by observing the coefficient of $m_{(n)}$ in C_μ must be 1, and the fact that $m_{(n)}$ is dual to $s_{(n)} = h_{(n)}$.

Instead of proving (??) and (??), we prove their equivalence:

$$\tilde{H}_\mu[X(q-1);q,t] = \sum_{\rho\leq\mu'} c_{\rho\mu}(q,t)m_\rho(x) \tag{2.3.2}$$

$$\tilde{H}_\mu[X(t-1);q,t] = \sum_{\rho\leq\mu} d_{\rho\mu}(q,t)m_\rho(x) \tag{2.3.3}$$

as the transition matrix $M(s,m)$ is uni-upper-triangular.

We first describe some technique involving standardization, quasisymmetric function, and superization. Any filling of μ can be standardized by replacing the fillings with $1, 2, \ldots, n$ (in that order), starting from the smallest cell to the largest, and go by reading order when replacing cells with the same value. This gives us a permutation ξ. With help of this technique, we establish an identity

$$C_\mu(x;q,t) = \sum_{\xi} q^{\mathrm{inv}(\xi)} t^{\mathrm{maj}(\xi)} Q_{n,D(\xi)}(x)$$

where $Q_{n,D}(x)$ is the quasisymmetric function and $D(\xi)$ is the descent set of ξ^{-1}. The identity basically says that $Q_{n,D}(x)$ corresponds to fillings which standardize to ξ.

The idea can be extended to include negative numbers in our filling. Standardization and other related tableau statistics are similar. This gives us another identity involving superization

$$\tilde{C}_\mu(x,y;q,t) = \sum_{\xi} q^{\mathrm{inv}(\xi)} t^{\mathrm{maj}(\xi)} \tilde{Q}_{n,D(\xi)}(x,y)$$

where the superization of $C_\mu(x;q,t)$ is $\tilde{C}_\mu(x,y;q,t) = \omega_Y C_\mu[X+Y;q,t]$. $\tilde{C}_\mu(x,y;q,t)$ is generating function for super fillings, i.e. an extension of (??) to all super fillings.

The final proof of (**??**) and (**??**) use the above identities and construct very clever involutions of the fillings $\sigma : \mu \to \mathbf{Z}_+ \cup \mathbf{Z}_-$, which is sign-reversing and weight-prserving, and cancel out terms not wanted in (**??**) and (**??**). We skip many of the details. But stress on one key idea, the so called critical square, meaning the last attacking pair in reading order, that has the smallest possible common absolute value. Proof of (**??**) uses an involution to switch the sign of the first number in the critical square pair, while keep the filling intact if there is no critical square — such fillings are called a non attacking. Therefore, the expression of $\tilde{H}_\mu[X(q-1); q, t]$ contains only non attacking super fillings.

$$\tilde{H}_\mu[X(q-1); q, t] = \sum_{\substack{\tilde{\sigma}:\mu\to\mathbf{Z}_+\cup\mathbf{Z}_- \\ \tilde{\sigma} \text{ non attacking}}} x^{|\tilde{\sigma}|} q^{\mathrm{inv}(\tilde{\sigma})} t^{\mathrm{maj}(\tilde{\sigma})} q^{\mathrm{pos}(\tilde{\sigma})} (-1)^{\mathrm{neg}(\tilde{\sigma})} \tag{2.3.4}$$

Haglund's formula gives rise to some elegant new combinatorial proof of old theorems. For example, the specialization formula of Macdonald, the expression of $\tilde{H}_\mu[X; 0, t]$ in Schur functions, and the Jack specialization formula.

Next we look at how to get a combinatorial formula of $J_\mu(x; q, t)$ using the formula for $\tilde{H}_\mu(x; q, t)$. The Frobenius series of $D_{\mu'}$ switches t and q in that of D_μ, which follows from the property of $\Delta_\mu(X, Y) = \Delta_{\mu'}(Y, X)$. So we have

$$\tilde{H}_\mu[X; q, t] = \tilde{H}_{\mu'}[X; t, q] \tag{2.3.5}$$

Then use (**??**) we can get

$$\begin{aligned} J_\mu(X; q, t) &= t^{n(\mu)} \tilde{H}_\mu[X(1-t); q, 1/t] = t^{n(\mu)} \tilde{H}_\mu[Xt(1/t-1); q, 1/t] \\ &= t^{n(\mu)+n} \tilde{H}_\mu[Xt(1/t-1); q, 1/t] \end{aligned}$$

Now (**??**) implies

$$J_\mu(X;q,t) = \sum_{\substack{\tilde\sigma:\mu\to\mathbf{Z}_+\cup\mathbf{Z}_- \\ \tilde\sigma \text{ non attacking}}} x^{|\tilde\sigma|} q^{\mathrm{maj}(\tilde\sigma)} t^{\mathrm{coinv}(\tilde\sigma)} (-t)^{\mathrm{neg}(\tilde\sigma)} \tag{2.3.6}$$

where coinv$=n(\mu)-$inv.

For any fixed absolute values of a filling, we have 2^n corresponding super fillings. Consider how sign changes affect the statistics. We can see for example when $\sigma(u) = \sigma(\mathrm{south}(u))$, switching u to negtive value will give us one more descent, because of the order we give $\mathbf{Z}_+\cup\mathbf{Z}_-$. In non attacking fillings, switching signs does not affect inv. So we have the following expression

$$\begin{aligned} J_\mu(X;q,t) \quad &= \sum_{\substack{\sigma:\mu'\to\mathbf{Z}_+ \\ \text{non attacking}}} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \prod_{\substack{u\in\mu' \\ \sigma(u)=\sigma(\mathrm{south}(u))}} (1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}) \\ &\prod_{\substack{u\in\mu' \\ \sigma(u)\neq\sigma(\mathrm{south}(u))}} (1-t) \end{aligned} \tag{2.3.7}$$

This formula is used in deriving the Jack specialization. It's a good example of application of the Haglund combinatorial formula.

Chapter 3

Isospectral Hilbert Scheme, $n!$ theorem

In order to prove the Macdonald positivity conjecture, we need a more careful study of the geometric properties of the Hilbert scheme and other related objects. Especially important are the isospectral Hilbert scheme X_n, and the nested Hilbert scheme $H_{n-1,n}$. We will examine some techniques used in the proof of the Gorenstein property of X_n, which pave the way for an induction argument using the nested schemes. The Cohen-Macaulay and Gorenstein property of X_n is connected with the dimension of R_μ. And finally R_μ's Frobenius series turns out to be the Macdonald polynomial itself.

3.1 Isospectral Hilbert Scheme X_n

In section **??**, we looked at some properties of the Hilbert scheme H_n. We list them here for reference.

Theorem 3.1.1. *The punctual Hilbert scheme H_n of $\mathbf{C}^2$ is smooth, irreducible, and of dimension $2n$.*

Now we look at another important object, the isospectral Hilbert scheme X_n. By definition, it is the reduced scheme in the product

$$\begin{array}{ccc} X_n & \longrightarrow & \mathbf{C}^{2n} \\ \downarrow & & \downarrow \\ H_n & \longrightarrow & S^n\mathbf{C}^2 \end{array} \tag{3.1.1}$$

So, set theoretically, its points are of the form $(I, P_1, P_2, \ldots, P_n)$ where $I \in H^n$ and $(P_1, P_2, \ldots, P_n) \in \mathbf{C}^{2n}$ has each point P_i repeat the same number of times as its length in the local ring of I at P_i. At each point of X_n where no P_i has multiplicity n, we may find a neighborhood isomorphic to a corresponding neighborhood in some $X_k \times X_l$, $k + l = n$, by splitting the point of X_n into two parts. So, we may induce on n to study the structure of X_n. In particular, since X_1 is just $\mathbf{C}^2$, the open set in $\mathbf{C}^{2n}$ where no two points coincide is isomorphic to its preimage in X^n. By induction, the preimage is dense open. Therefore, we have the following property.

Theorem 3.1.2. *X_n is irreducible of dimension $2n$.*

Following the logic of the above paragraph, we can deduce a dimension formula for the closed subset G_r of H_n which contains ideals with some point of multiplicity at least r. Since X_n is finite over H_n, we may look at the preimage of G_r in X_n, which has a unique largest component of minimum codimension. It contains a neighborhood of $Z_r \times \mathbf{C}^2 \times X_{n-r}$. Z_r is the zero fiber H_0^n and has dimension $r-1$. $Z_r \times \mathbf{C}^2$ is the closed subset of X_r where all r points are the same. An induction argument shows that the codimension of G_r is $r-1$.

Let $\mathbf{C}[\mathbf{X},\mathbf{Y}] = \mathbf{C}[X_1, Y_1, \ldots, X_n, Y_n]$. We may construct H_n as a blow up of $\mathbf{C}[\mathbf{X},\mathbf{Y}]^{S_n}$ along the alternating polynomials A, and X_n as blow up of $\mathbf{C}[\mathbf{X},\mathbf{Y}]$ along $J = \mathbf{C}[\mathbf{X},\mathbf{Y}]A$. On the affine U_μ defined in (**??**), the coordinate ring of H_n is generated by c_{hk}^{rs} for pairs of coordinates (r,s) and $(h,k) \in \mu$. c_{hk}^{rs} is the coefficient of X^rY^s on the basis element X^hY^k. Then on U_μ, we have

$$X^rY^s = \sum_{(h,k)\in\mu} c_{hk}^{rs} X^hY^k$$

As a consequence of the equation, we get an identity involving the alternating polynomials Δ_D for any $D = \{(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)\} \subseteq \mathbf{N}^2$

$$\frac{\Delta_D}{\Delta_\mu} = \det(c_{h^jk^j}^{p_iq_i})_{i,j=1}^n \tag{3.1.2}$$

where (h_j,k_j) are elements of μ. Now A^2 is an ideal of $\mathbf{C}[\mathbf{X},\mathbf{Y}]^{S_n}$, whose inverse image in O_μ is invertible since it is the principle ideal generated by Δ_μ^2. The universal property of blow up says we have a morphism from H_n to the blow up $\operatorname{Proj} A^2 = \operatorname{Proj} A$. It also implies a commutative relation among the blow up map, the induced map, and the Chow morphism. We know the Chow morphism is projective and surjective. The canonical blow up map of a variety is birational, proper, and surjective. So the induced map is proper and thus surjective. It is an embedding since we may express all the c_{hk}^{rs} in term of the pullbacks of Δ_D. This shows the blow up is indeed isomorphic to H_n.

From (**??**), we see that the Proj of $\mathbf{C}[\mathbf{X}, \mathbf{Y}] \cdot T$ is actually X_n. $\mathbf{C}[\mathbf{X}, \mathbf{Y}]$ corresponds to $\mathbf{C}^{2n}$ in the product, and $\mathbf{C}[\mathbf{X}, \mathbf{Y}]^{S_n}$ is a subring of $\mathbf{C}[\mathbf{X}, \mathbf{Y}]$ and T. Denote by J the extension of A in $\mathbf{C}[\mathbf{X}, \mathbf{Y}]$. Then X_n is the blow up of $\mathbf{C}[\mathbf{X}, \mathbf{Y}]$ along J.

Theorem 3.1.3. *H_n is isomorphic to $\operatorname{Proj} \mathbf{C}[\mathbf{X}, \mathbf{Y}]^{S_n}[tA]$, and X_n is isomorphic to $\operatorname{Proj} \mathbf{C}[\mathbf{X}, \mathbf{Y}][tJ]$, where t is the indeterminate in the graded algebra.*

3.2 Nested Hilbert Scheme

In order to prove the Cohen-Macaulay and Gorenstein property of X_n, we will need an induction mechanism. The bridge is the nested Hilbert scheme $H_{n-1,n}$. It is the reduced closed subscheme in $H_{n-1} \times H_n$ whose points (I_{n-1}, I_n) are such that $I_{n-1} \subseteq I_n$ as ideals in $\mathbf{C}^2$. Regular functions in H_{n-1} and H_n pull back to $H_{n-1,n}$. In particular, the distinguished point which is the extra one in I_n has its coordinates as regular functions. This induces a map

$$H_{n-1,n} \to S^{n-1}\mathbf{C}^2 \times \mathbf{C}^2$$

we have the following properties.

Theorem 3.2.1. *$H_{n-1,n}$ is irreducible, smooth, and has dimension $2n$.*

One technique used in many proofs is to look at the curvilinear open subset first, and then extend to its small complement. The curvilinear subset refers to points in H_n which correspond to a curvilinear ideal $I \subseteq \mathbf{C}[X, Y]$, meaning that $\mathbf{C}[X, Y]/I$ at its localization has principle maximal ideal. So locally at each point (a, b) of I, a linear combination of $x - a$ and $y - b$ generates $(x - a, y - b)/(x - a, y - b)^2$ which has dimension 1. Hence by Nakayama's lemma the linear combination generates the whole maximal ideal. We may pick a common linear form z for all points of I. Then we have

$$\mathbf{C}[X, Y]/I \cong \mathbf{C}[z]/\prod_P (z - z(P))^{r_P}$$

where P is a point of I and r_P is its multiplicity. In particular, $\mathbf{C}[X, Y]/I$ has a basis $\{1, z, \ldots, z^{n-1}\}$. We denote by W_z all ideals in H_n with such a basis for its quotient ring. W_z is clearly open. So the curvilinear subset W is also open since it is equal to the union $\cup_z W_z$.

The complement of W is small. Since any ideal in the complement will have a local maximal ideal m at some point non principle. From the above argument, m/m^2 will have length at least 2 and the local ring of length at least 3. Therefore such ideal must be in G_3. But the single largest component in G_3 contains ideals with a single point of multiplicity 3 and all the rest multiplicity 1. Such ideals can be non curvilinear. Therefore the complement of W has codimension at least 3.

We use the above result to show normality of the universal family F. Recall that a equivalent condition for a ring to be normal is R1 and S2. Since F is already

Cohen-Macaulay from section **??**, we have S2. If we can show that singularity happens only inside a codimension 2 subset, we will have R1. Indeed, F is regular over the curvilinear W. Without loss of generality, we may look at W_x only. W_x is also U_{1^n}, since $\{1, x, \ldots, x^{n-1}\}$ is a basis for all the quotient rings corresponding to the ideals in W_x. Now any $I \in W_x$ is generated by

$$x^n - e_1 x^{n-1} + e_2 x^{n-2} - \cdots + (-1)^n e_n$$

and

$$y - a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where e_i and a_i are regular on W_x. They also uniquely determine I. Therefore W_x is the affine cell $\operatorname{Spec} \mathbf{C}[\mathbf{e}, \mathbf{a}]$ and F is cut out from $W_x \times \mathbf{C}^2$ by the two generators of the ideals. This means F is also an affine cell and thus regular. We summarize this and previous comments about F in a theorem.

Theorem 3.2.2. *The universal family F is finite and flat over H_n. It is Cohen-Macaulay and normal.*

Points of $H_{n-1,n}$ contain information of I_n and the distinguished point P, so there is a morphism

$$\alpha : H_{n-1,n} \to F$$

The fiber of $H_{n-1,n}$ over an $(I_n, P) \in F$ is the projective space of the socle in the local ring of $\mathbf{C}[X, Y]/I$ at P. Its dimension is maximal at I_μ by upper semicontinuity of fiber dimension. At I_μ the socle has dimension equal to the number of corners. Suppose the fiber has dimension d, then the socle has dimension $d+1$, and consequently $n \geq \binom{d+2}{2}$.

This formula is useful in checking the codimension of the preimage of the non curvilinear locus in $H_{n-1,n}$. If the codimension were 1, we must have fibers of dimension at least 2, since the non curvilinear has codimension at least 3 in H_n. Now according to the fiber formula of α, points with fibers of dimension $d \geq 2$ in $H_{n-1,n}$ has codimension at least $\binom{d+2}{2} - 1 - d \geq 3$, a contradiction. Therefore, the non curvilinear has codimension at least 2 in $H_{n,n-1}$. A more careful study can show that the complement of $W_x \cup W_y$ has codimension 2 in both H_n and $H_{n-1,n}$.

An easy consequence of the above result on codimension is that α is isomorphic outside of the non curvilinear locus of codimension 2. This is because of the Zariski's main theorem on birational transformations. F is normal, and α is bijective on the curvilinear locus, so it is locally an isomorphism.

In the proof of the main theorem in [**?**], Haiman used a nested isospectral Hilbert scheme $X_{n-1,n}$, which by definition is the reduced product of X_{n-1} and $H_{n-1,n}$ over H_{n-1}. We may use induction to assume that X_{n-1} is Gorenstein and thus has Gorenstein fiber over the smooth H_{n-1}. Lift this to the fiber of $X_{n-1,n}$ over $H_{n-1,n}$ and use derived category and duality theory to further pushdown to X_n. Another important technique is the so called polygraph theory. It shows that X_n is flat over the $\mathbf{y}$ coordinate ring. We look at some of these elements later. For now we list the major result.

Theorem 3.2.3. *X_n is normal, Cohen-Macaulay, and Gorenstein, with canonical sheaf $\omega_{X_n} \cong O(-1)$*

3.3 $n!$ Theorem

The geometric property of X_n is related to R_μ defined in (**??**). We will look at this connection and see how it solves the Macdonald positivity conjecture.

R_μ is by definition the quotient of $\mathbf{C}[X,Y]$ modulo J_μ, the ideal of annihilators of Δ_μ with partial differentiation. J_μ is characterized by the property that for any $f \in J_\mu$, the coefficient of Δ_μ is zero in $\mathrm{Alt}(gf)$ for all $g \in \mathbf{C}[X,Y]$.

There is a similar description related to the ideal sheaf of X_n as a closed subscheme of F^n. Here F^n is the n-fold product of the universal family over H_n. F consists of points (I,P) where $I \in H_n$ and P is a point on I. So F^n is I together with n points of it, not necessarily in the correct multiplicity. Now we can see that X_n is indeed a subscheme of F^n. The map $F \to H_n$ is affine, and we may consider the schemes as H_n algebras. Let B represent the H_n algebra of F, we have $B^{\otimes n}$ for F^n, and X_n is produced by an ideal sheaf of $B^{\otimes n}$. This ideal sheaf is the kernal of the following map

$$\phi : B^{\otimes n} \to \mathrm{Hom}(B^{\otimes n}, \wedge^n B) \tag{3.3.1}$$

This is mainly because X_n contains a dense open subset of I with n distinct points, and any alternating polynomial evaluated on n non distinct points will vanish.

This identity about the ideal sheaf of X_n and the previous observation of J_μ

imply that J_μ contains the fiber of the ideal sheaf of X_n at I_μ. More precisely, B has fiber of $B(I_\mu)$ at I_μ. By the universal property, $B(I_\mu) = \mathbf{C}[X,Y]/I_\mu$. Therefore $B^{\otimes n} = \mathbf{C}[\mathbf{X},\mathbf{Y}]/(I_\mu(X_1,Y_1) + \cdots + I_\mu(X_n,Y_n))$. Now compose the natural quotient map with the fiber of (**??**), we get

$$\eta : \mathbf{C}[\mathbf{X},\mathbf{Y}] \to \mathrm{Hom}(B^{\otimes n}(I_\mu), \wedge^n B(I_\mu)) \tag{3.3.2}$$

J_μ is exactly the kernel of η, and the image of J_μ in $B^{\otimes n}(I_\mu)$ contains the fiber of the ideal sheaf of X_n.

We mentioned earlier that X_n has a dense open subset which is the preimage of ideals in H_n with n distinct points. Clearly the fiber of X_n over H_n at those generic points has dimension $n!$ and under permutation group action affords the regular representation. So the rank of ϕ is $n!$ over the generic points. But the rank of a continuous map is lower semicontinuous, so at I_μ it can be no bigger. Indeed, at I_μ, the fiber of X_n and therefore R_μ must be a submodule of the regular representation. The condition of X_n being locally Gorenstein at a point over I_μ is equivalent to its fiber being locally free at I_μ with dimension $n!$. This is the $n!$ conjecture about the dimension of R_μ or D_μ.

Theorem 3.3.1. *The condition that X_n is locally Gorenstein over I_μ is equivalent to the $n!$ conjecture for μ that $\dim D_\mu = n!$.*

The $n!$ theorem and its proof implies the Macdonald positivity conjecture. As we discussed in section **??**, the Macdonald polynomials are uniquely determined bythe three conditions of (**??**), (**??**), and (**??**). This is actually true for the Frobenius series of R_μ. The first condition comes from the known result of $R_\mu/(\mathbf{y})$ whose expansion in the Schurs is upper triangular. The second condition is from the symmetry of $\mathbf{x}$ and $\mathbf{y}$ in R_μ. The last condition is because R_μ affords the regular representation.

Theorem 3.3.2. *The transformed Macdonald polynomial is the Frobenius series of R_μ. The q, t Kostka numbers are non negative integral polynomials.*

Chapter 4

Diagonal Harmonics

In this chapter, we further study the geometric properties in the context of Hilbert schemes. The goal is to deduce several character formulas, using the vanishing of certain cohomology. These formulas help solve earlier questions, for example, the dimension of the diagonal harmonic space. There are also new conjectures related to the findings. At first, we must look at the important polygraph $Z(n, l)$, which also plays a role in the proof that X_n is Gorenstein. It is also an important ingredient in the character formulas.

4.1 Polygraph

We will examine the polygraph $Z(n, l)$ in this section. It is by definition a reduced closed subscheme of $E^n \times E^l$ where E is a complex space $\mathbf{C}^d$. For our purpose d is always 2. Its underlying topological space is the union of linear graphs from E^n

to E^l. Denote by a_i and b_i the indeterminates in E^l, $i = 1, \ldots, l$, and x_i and y_i indeterminates in E^n, $i = 1, \ldots, n$. For any $f : [l] \to [n]$, we get a closed subspace $W_f \subseteq E^n \times E^l$, determined by equations $a_i = x_{f(i)}$ and $b_i = y_{f(i)}$, $i = 1, \ldots, l$. Then we have $Z(n, l) = \cup W_f$ over all f. We call its coordinate ring $R(n, l)$.

One important thing about $R(n, l)$ is that it is a free $\mathbf{C}[\mathbf{y}]$ module. This property is useful in the proof of the Gorenstein property of X_n. To show that $R(n, l)$ is $\mathbf{C}[\mathbf{y}]$ free, an induction process is used to reduce the problem to a sublattice of ideals $I(m, r, k) \subseteq R(n, l)$. $I(m, r, k)$ is the ideal of the subspace $Y(m, r, k)$ where certain x_j vanish. So $Y(m, r, k)$ is a union of forms $V(x_j : j \in T) \cap W_f$. T specifies which x_j of $j \in [n]$ vanish in W_f. The rule is that inside of $[r]$ and exclude $f[k]$, at least m of them do. These ideals share the same basis B of $R(n, l)$. That is, they are all $\mathbf{C}[\mathbf{y}]$ free and a subset of $R(n, l)$'s basis B spans each of them.

The local property of $R(n, l)$ is often easy to study, and it's possible to extend some result to the whole space. We may concentrate on the open subspace $\hat{U}_k \subseteq \operatorname{Spec} \mathbf{C}[\mathbf{y}]$ and its preimage U_k in $Z(n, l)$. $\hat{U}_k$ consists of points where at least $n - k + 1$ distinct values of y_j exist. So $\hat{U}_k$ has a complement defined as solutions to k equations of the form $y_i = y_j$. In particular, U_1 and U_2 are dense and have complement of codimension 1 and 2. $Z(n, l)$ decomposes into disjoint union of W_f on U_1, and therefore look like W_f locally. As a result, ideals like $I(m, r, k)$ in the sublattice is reduced on U_1, since on U_1, the local isomorphism of $Z(n, l)$ to $W_f \cong E^n$ implies that ideals generated by monomials are radical.

Next we describe how the freeness of $R(n,l)$ contribute to the proof of X_n's Gorenstein property. The goal is to deduce X_n is flat over $\mathbf{C}[\mathbf{y}]$. A bridge to that is a lemma which states that J^d is a free $\mathbf{C}[\mathbf{y}]$ module. Remember X_n is the blow up of $\mathbf{C}[\mathbf{x},\mathbf{y}]$ along the ideal J generated by the alternating polynomials A. So the freeness of J^d is sufficient to achieve that goal.

In fact, J^d is isomorphic as a $\mathbf{C}[\mathbf{x},\mathbf{y}]$ module to an alternating subspace of $R(n,nd)$. The alternating subspace is determined by the group action of S_n^d, the product of d copies of the permutation group S_n. Each copy permutes d consecutive indeterminates of E^{nd}, and keeps the others intact. The alternating subspace is thus a $\mathbf{C}[\mathbf{y}]$ submodule and direct summand in each x degree. So it is free as a $\mathbf{C}[\mathbf{y}]$ module because $R(n,nd)$ is free. It is possible to construct a map $\phi : R(n,nd) \to \mathbf{C}[\mathbf{x},\mathbf{y}]$ by restricting to some fixed W_{f_0} and get the desired isomorphism.

Another consequence of the polygraph theory is a theorem that

$$J^d \cong \cap_{i<j}(x_i - x_j, y_i - y_j)^d \tag{4.1.1}$$

The freeness implies that for $n \geq 3$, the depth of J^d is at least 2 in the subscheme $V(I) \subseteq \mathbf{C}[\mathbf{x},\mathbf{y}]$ where all n points are the same. That is, we may find at least two J^d regular elements from the ideal I of the closed subscheme. Now, by the local cohomology theory [**?**], $H^i_V(\mathbf{C}^{2n}, J^d) = 0$, for $i = 0, 1$. Then from the long exact sequence of local cohomology, we get an isomorphism of $H^0(\mathbf{C}^{2n}, J^d) \cong H^0(\mathbf{C}^{2n} \setminus V, J^d)$. With some more effort, we may show that (**??**) holds locally on $\mathbf{C}^{2n} \setminus V$. Then the cohomology isomorphism shows that J^d contains the restriction of the right hand side of (**??**) to the open subset $\mathbf{C}^{2n} \setminus V$, and therefore contains the right hand side. The other direction of inclusion is obvious, so we have the desired isomorphism.

4.2 Vanishing Theorems

Eventually, we want to use the Atiyah-Bott formula to express the Frobenius series and Hilbert series of certain spaces. It works well when the higher cohomologies vanish. We explain what this means in this section.

First, we mention that X_n has another interpretation in view of the G-Hilbert scheme. $\mathbf{C}^{2n} \parallel S_n$ is the closed subscheme in $\mathrm{Hilb}^{n!}(\mathbf{C}^{2n})$ where the fiber has a regular representation by S_n action. This is the closure of the set of ideals representing n distinct points in $\mathbf{C}^{2n}$. There is a natural morphism $\mathbf{C}^{2n} \parallel S_n \to H_n$, that comes from a S_{n-1} action. Now we already know that X_n is flat over H_n of degree $n!$, we get an inverse map $H_n \to \mathbf{C}^{2n} \parallel S_n$ because of the universal property of $\mathrm{Hilb}^{n!}(\mathbf{C}^{2n})$. Thus $\mathbf{C}^{2n} \parallel S_n \cong H_n$, and X_n is the universal family over $\mathbf{C}^{2n} \parallel S_n$.

Now we have two universal families over H_n, F and X_n. Both may be thought of as H_n algebras, and we denote them by B and P. The first vanishing theorem is about cohomologies on H_n.

Theorem 4.2.1.

$$H^i(H_n, P \otimes B^{\otimes l}) = 0 \tag{4.2.1}$$

for $i > 0$ and all l.

$$H^0(H_n, P \otimes B^{\otimes l}) = R(n, l) \tag{4.2.2}$$

This first vanishing theorem is the major result in [**?**] which paved the road to various character formulas. One consequence is a second vanishing theorem on the zero fiber $Z_n \subseteq H_n$ which is the preimage of zero under the Chow morphism. We had a free resolution of Z_n earlier in section **??** at the torus fixed points I_μ. Let's extend this to get a global resolution.

Key to this is the trace map of B to H_n. As B is locally free, each of its element induces a linear map of the fiber through multiplication. The trace of the map is a morphism of H_n module

$$\mathrm{tr} : B \longrightarrow H_n \tag{4.2.3}$$

Remember B is generated by $x^r y^s$ over H_n. The trace of $x^r y^s$ is $p_{r,s}(\mathbf{x}, \mathbf{y})$ on the generic points of H_n and hence on the whole scheme. Now the canonical map $H_n \to B$ has a left inverse $\frac{1}{n}\,\mathrm{tr}$. So we may decompose B into $H_n \oplus B'$, where B' is the kernel of tr. With the same notation as in section **??**, now we have a global resolution of Z_n as H_n module.

Use an argument of derived category, we will get the vanishing of higher cohomologies on Z_n. The global section can be found through a resolution which states that $R(n, l+1)$ maps surjectively onto the global sections of $P \otimes B^{\otimes l}$ on Z_n, with kernel generated by the image of B', O_t, and O_q. The quotient is $R(n, l)/mR(n, l)$, where m is generated by $p_{r,s}$ as the maximal ideal in $S_n\mathbf{C}^2$.

Theorem 4.2.2.

$$H^i(Z_n, P \otimes B^{\otimes l}) = 0 \tag{4.2.4}$$

for $i > 0$ *and all* l.

$$H^0(Z_n, P \otimes B^{\otimes l}) = R(n, l)/mR(n, l) \tag{4.2.5}$$

4.3 Diagonal Harmonics

Now we are ready to derive some character formulas. We will use the vanishing theorems and an Atiyah-Bott formula.

First, the Atiyah-Bott formula equates the Euler characteristic with a sum of expressions at the torus fixed points. For our purpose, the scheme is H_n, and E is any locally free sheaf , let

$$\chi_E(q, t) = \sum_i (-1)^i \mathcal{H}_{H^i(H_n, E)}(q, t) \tag{4.3.1}$$

be the Euler characteristic, where $\mathcal{H}_{H^i(H_n,E)}(q, t)$ is the Hilbert series of the cohomology of E. As mentioned earlier, the torus action induces a q, t grade on the modules. We have the following formula

$$\chi_E(q, t) = \sum_{\mu \vdash n} \frac{\mathcal{H}_{E(I_\mu)}(q, t)}{\prod_{x \in d(\mu)} (1 - t^{1+l(x)} q^{-a(x)})(1 - t^{-l(x)} q^{1+a(x)})} \tag{4.3.2}$$

In the equation, $d(\mu)$ is the graph of μ, l and a are leg and arm respectively. The numerator is the Hilbert series of the stalk of E at fixed point I_μ, and the denominator is actually the determinant of the transformation defined by $1 - (q, t)$ on the cotangent space at I_μ. The expression comes from an eigenvalue basis.

Similar to (**??**) we may get the Frobenius series expression

$$\chi\mathcal{F}_E(z;q,t) = \sum_i (-1)^i \mathcal{F}^i_{H^i(H_n,E)}(q,t) \tag{4.3.3}$$

and a corresponding formula

$$\chi\mathcal{F}_E(z;q,t) = \sum_{\mu\vdash n} \frac{\mathcal{F}_{E(I_\mu)}(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})} \tag{4.3.4}$$

Now we may appreciate the value of vanishing theorems. The higher cohomology groups all disappear in the Euler characteristics of (**??**) and (**??**). We are left with the degree zero cohomologies which were given. So, using theorem **??**, we get

$$\mathcal{F}_{R(n,l)}(z;q,t) = \chi\mathcal{F}_{P\otimes B^{\otimes l}}(z;q,t) \tag{4.3.5}$$

S_n acts on P, so $B^{\otimes l}$ does not contribute to the S_n representation. It does contribute to the q,t coefficient as a Hilbert series. We know that $B(I_\mu)$ has basis $x^r y^s$ for $(r,s)\in\mu$. Thus the Hilbert series is the sum $B_\mu(q,t)$ introduced in section **??**.

$$\mathcal{H}_{B(I_\mu)}(q,t) = B_\mu(q,t) = \sum_{s\in\mu} q^{a'_\mu(s)} t^{l'_\mu(s)} \tag{4.3.6}$$

As for $P(I_\mu)$, we know its Frobenius series is the transformed Macdonald polynomial by theorem **??**, as part of the $n!$ theorem result. So we may put the pieces together to express the Frobenius series of $R(n,l)$ with Atiyah-Bott formula

$$\mathcal{F}_{R(n,l)}(z;q,t) = \sum_{\mu\vdash n} \frac{B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})} \tag{4.3.7}$$

Our next result is for the diagonal harmonics D_n. By definition, D_n is the solution in $\mathbf{C}[\mathbf{x},\mathbf{y}]$ of the differential equations

$$p_{h,k}(\partial x, \partial y)f = \sum_i \partial x_i^h \partial y_i^k f = 0, \text{ for } 1 \leq h+k \leq n$$

In particular, we have $\sum_i (\partial x_i^2 + \partial y_i^2)f = 0$. So these are harmonic polynomials. The space is isomorphic as graded S_n module to the ring of coinvariants R_n, which is the quotient of $\mathbf{C}[\mathbf{x}, \mathbf{y}]$ by the max homogeneous ideal generated by $p_{r,s}$. We may use theorem **??** to get the Frobenius series of R_n. Take $l = 0$ in the theorem and note that we may use the resolution of Z_n to do the calculation. We have,

$$\mathcal{F}_{R_n}(z; q, t) = \sum_{\mu \vdash n} \frac{(1-q)(1-t)\prod_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})} \tag{4.3.8}$$

There are several interesting consequences. First, we can specialize the Macdonald polynomial to get a dimension formula for R_n, which is $(n+1)^{(n-1)}$. A second application is to carry out an inner product with $s_{1^n} = e_n$, and get the Hilbert series of the alternating subspace R_n^ϵ. The resulted Hilbert series $\mathcal{H}_{R_n^\epsilon}(q,t)$ is exactly $C_n(q,t)$, the Catalan number.

There are some conjectures related to the diagonal harmonics, and we discuss them a little bit here. We define some more statistics related to Dyck paths. Let $\pi \in L_{n,n}^+$. Let

$$dinv(\pi) = |(i,j) : 1 \leq i < j \leq n, a_i = a_j| + |(i,j) : 1 \leq i < j|len, a_i = a_j + 1|$$

where a_i is the area of the ith row. There is a bijective map on Dyck paths that switches dinv and area to area and bounce.

A parking function P for π places car numbered 1 through n in the squares just to the right of north steps of π, with strict decrease down columns. The function is related to the space of diagonal harmonics D_n. It's not hard to prove the number of parking functions on n cars is $(n+1)^{n-1}$, the dimension of D_n. In [?] Haglund and Loehr conjectured a combinatorial formula of the Hilbert series as

$$H_{D_n}(q,t) = \sum_{P} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \tag{4.3.9}$$

where P is any parking function of n. The dinv statistic of P is an extension of the previous one by adding extra conditions to the two sets appearing in the sum. To the first set, we require car in ith row be less than car in the jth row. To the second set, we require car in ith row be greater than car in the jth row.

Another conjecture in [?], proposed a formula for the Frobenious series of D_n, which generalizes the parking function formula.

$$F_{D_n}(x;q,t) = \sum_{\sigma \in \mathrm{WP}_n} x^{\sigma} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\sigma)} \tag{4.3.10}$$

Here we extend the parking function to word parking function which is a parking function but with possibly repeated cars. And the dinv statistic is also extended in a natural way.

4.4 Examples

Hilbert scheme when $n = 2$

We will look at some examples when $n = 2$. We first examine the Hilbert scheme H_2, which now consists of ideals I in $\mathbb{C}[x,y]$ such that $\mathbb{C}[x,y]/I$ has dimension 2. So the quotient ring is Artin, and as a vector space over $\mathbb{C}$, has dimension 2. If further more it is reduced, we will have two distinct maximal ideals containing I. In that case, I represents two distinct points in $\mathbb{C}^2$. For example, $I' = (x-1, y-1) \cap (x, y)$. This ideal represents points $(1,1)$ and $(0,0)$. A non reduced quotient ring will have a single maximal ideal, like $I'' = (x^2, y)$, or $I''' = (x^2, y-x)$.

We talked about the open subschemes $O_\mu \subseteq H_n$ earlier. There is a way to determine which O_μ contains I. We take I' to illustrate this. Calculate the initial ideal of I' which is generated by the initial monomials under the order $x < y$. Now $I' = (x-1, y-1) \cap (x, y) = (x-1, y-1) * (x, y) = (x^2 - x, xy - y, xy - x, y^2 - y)$. The initial monomials of these are x^2, xy, xy, and y^2. Add the initial monomial of $(xy - y) - (xy - x) = x - y$, which is y, and we derive that the initial ideal of I is (x^2, y). There are two monomials not inside of this initial ideal, 1 and x. This is represented by partition $\mu = 1^2$. The above calculation and dimension consideration show that 1 and x span $\mathbb{C}[x, y]/I'$, that is $I' \in O_{1^2}$. It is also clear that I'' and I''' are also inside O_{1^2}.

x^2	...	
x	xy	...
1	y	...

Table 4.1: $I \in O_{1^2}$ has its quotient ring spanned by monomials inside the partition

So we have a way to find a spanning set of monomials for each I. Inside O_μ, every polynomial in x and y is equal to some linear combination of monomials inside the partition μ, modular I. Take I' as an example, we can express any polynomial, and especially any monomial, in term of 1 and x, modular I'. For example, we may calculate to get $x^2 \equiv x$ and $y \equiv x$. We call these coefficients $c_{h,k}^{r,s}$, where (r,s) is the power of the left term in (x,y) and (h,k) are that of the right terms. So $x^2 \equiv c_{1,0}^{2,0}x + c_{0,0}^{2,0}1$, and $y \equiv c_{1,0}^{0,1}x + c_{0,0}^{0,1}1$. And we just showed $c_{1,0}^{2,0} = 1$, $c_{0,0}^{2,0} = 0$, $c_{1,0}^{0,1} = 1$, and $c_{0,0}^{0,1} = 0$. Inside O_{1^2}, these coefficients are uniquely determined by the point I. Or in algebraic geometry term, the regular ring on O_{1^2} is generated by $c_{h,k}^{r,s}$ where $(h,k) \in 1^2$. At $I'' = (x^2, y)$, we have $c_{h,k}^{h,k} = 1, (h,k) \in 1^2$ and all the rest $c_{h,k}^{r,s} = 0$.

Torus action of (t,q)

Now we may look at the torus action which eventually gives us the double degree in Hilbert series calculation. Let (t,q) act on $\mathbb{C}^2$ by multiplication on the coordinates. So $(t,q)(1,1) = (t,q)$, and $(t,q)(0,0) = (0,0)$. This will induce an action on the polynomials. Since $(1,1)$ corresponds to the maximal ideal $(x-1, y-1) \subseteq \mathbb{C}[x,y]$, the (t,q) action need to transform the maximal ideal to $(x-t, y-q) = (t^{-1}x, q^{-1}y)$. In effect, we multiply x by t^{-1} and y by q^{-1}. So the action is a morphism of $\mathbb{C}[x,y]$. It maps an ideal I to another ideal according to the morphism on polynomials. This is translated to morphism of the regular ring of H_n. For example, under (t,q) action, equations modular I' will be changed since I' is changed. Previously we had $x^2 \equiv x$ and $y \equiv x$ mod I'. Now we apply (t,q) action to get $t^{-2}x^2 \equiv t^{-1}x$ and $q^{-1}y \equiv t^{-1}x$, mod $(t,q)I'$. Collect terms to get $x^2 \equiv tx$, and $y \equiv t^{-1}qx$, mod $(t,q)I'$. The coefficients change to $c_{1,0}^{2,0} = t$, and $c_{1,0}^{0,1} = t^{-1}q$ at $(t,q)I'$, according to the rule $(t,q)c_{h,k}^{r,s} = t^{r-h}q^{s-k}c_{h,k}^{r,s}$.

The (t,q) action we described is a $\mathbb{C}$ vector space morphism at various levels. In particular, We will have induced isomorphism of the local rings. We also have isomorphism of the global sections which is cohomology H^0, for compatible modules. We look at the local ring isomorphism first. We are interested in the fixed points as they appear in the Atiyah-Bott formula. As the previous paragraph shows, the only fixed point in $\mathbb{C}^2$ is $(0,0)$. While the ideals fixed by (t,q) must be generated by monomials, like I''. We can index these fixed points by partitions $\mu \vdash n$, such as $I'' = I_{1^2}$, where the monomials not included in the ideal appear in the partition. It is clear that (t,q) does not change I''. Another fixed point is $I = (x, y^2)$.

The cotangent space

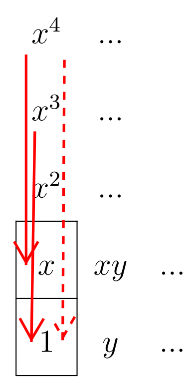
Let us look at the cotangent space at the fixed points. Remember in each O_μ, our Hilbert scheme has its regular ring generated by $c_{h,k}^{r,s}$ where $(h,k) \in \mu$. This really says that an ideal is determined by how monomials $x^r y^s$ expand in $x^h y^k$. For $I_{1^2} = (x^2, y)$, monomials except 1 and x are inside the ideal already (for example $x^3 \equiv 0$ mod I_{1^2}). Therefore $c_{h,k}^{r,s}$ is 0 unless $(r,s) = (h,k) \in \mu$. In the regular ring $\mathbb{C}[c_{h,k}^{r,s}]$ of H_2, the maximal ideal $m = (c_{h,k}^{r,s}, c_{h,k}^{h,k} - 1)$ where $(h,k) \in 1^2$ corresponds to the point $I_{1^2} \in H_2$. We want to investigate the cotangent space m/m^2 under (t,q) action. A denominator in the Atiyah-Bott formula is the determinant of the action $1 - (t,q)$. We would like to find the eigenvectors of (t,q), and get the product of 1 minus the eigenvalues. We know that the Hilbert scheme is smooth of dimension $2n$. So the task is to find $2n$ generators of m/m^2. We use I_{1^2} as an example to illustrate the basic technique. But first, we look at the open subscheme O_{1^2} in general. We may expand x^3 as

$$x^3 \equiv c_{1,0}^{3,0}x + c_{0,0}^{3,0}1$$

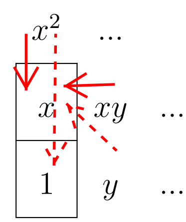
Multiply both sides by x to get

$$x^4 \equiv c_{1,0}^{3,0}x^2 + c_{0,0}^{3,0}x \equiv c_{1,0}^{3,0}(c_{1,0}^{2,0}x + c_{0,0}^{2,0}1) + c_{0,0}^{3,0}x = (c_{1,0}^{3,0}c_{1,0}^{2,0} + c_{0,0}^{3,0})x + c_{1,0}^{3,0}c_{0,0}^{2,0}1$$

At the same time we have by definition $x^4 \equiv c_{1,0}^{4,0}x + c_{0,0}^{4,0}1$. So $c_{1,0}^{4,0} = c_{1,0}^{3,0}c_{1,0}^{2,0} + c_{0,0}^{3,0}$, and $c_{0,0}^{4,0} = c_{1,0}^{3,0}c_{0,0}^{2,0}$. In the cotangent space, we have $c_{1,0}^{3,0}c_{1,0}^{2,0} \equiv c_{1,0}^{3,0}c_{0,0}^{2,0} \equiv 0$, and the result is $c_{1,0}^{4,0} \equiv c_{0,0}^{3,0}$, $c_{0,0}^{4,0} \equiv 0 \mod m^2$. We may use arrows to represent these coefficients. Let an arrow start from (r,s) and end at (h,k) on the tableau. The first result $c_{1,0}^{4,0} \equiv c_{0,0}^{3,0}$ means that the two arrows on the following graph are equivalent in the cotangent space. The second result $c_{0,0}^{4,0} \equiv 0$ means the dashed arrow in the graph is zero.



Horizontal moves are similar. Thus comes the rule that we may move arrows as much as we want, as long as we keep the root outside and the head inside the partition diagram. Once an arrow's head goes out of range, it vanishes in the cotangent space, like that dashed arrow. So, each cell in the partition diagram will give us two legitimate arrows, making up the $2n$ generators and eigenvectors of the cotangent space. What is more, we may easily write down the eigenvalues. For partition 1^2, we have the following four arrows. Two arrows correspond to cell $(1, 0)$, and two dashed arrows correspond to cell $(0, 0)$.



The $c_{h,k}^{r,s}$ corresponding to the arrows are eigenvectors with eigenvalues $t^{r-h}q^{s-k}$. Each cell gives two eigenvalues $t^{1+l}q^{-a}$ and $t^{-l}q^{1+a}$ where l and a are leg and arm statistics of the cells. In our example, cell $(1, 0)$ has $l = 0, a = 0$, and provides eigenvalues t and q. Cell $(0, 0)$ has $l = 1, a = 0$, and provides eigenvalues t^2 and $t^{-1}q$. Take the product of 1 minus the eigenvalues. We get the denominator in the Atiyah-Bott formla

$$\prod_{x\in\mu}(1 - t^{1+l(x)}q^{-a(x)})(1 - t^{-l(x)}q^{1+a(x)})$$

where x is a cell in the partition diagram of μ. We showed that when $\mu = 1^2$, the product is

$$(1 - t)(1 - q)(1 - t^2)(1 - t^{-1}q)$$

H_n module

The universal family U is a scheme over H_n, and we denote by B the induced module over H_n. We have a canonical map $O_{H_n} \to B$, and its left inverse $\frac{1}{n}\mathrm{tr} : B \to O_{H_n}$. For $n = 2$, the open subscheme O_{1^2} has its regular ring in the form $\mathbb{C}[c^{r,s}_{h,k}]$ where $(h,k) \in 1^2$. Then $B = \mathbb{C}[c^{r,s}_{h,k}, x, y]/(\text{some relation})$, where the denominator will make sure that the universal family consists of points $(I, (a,b))$ such that (a,b) solves equations in I. For example, $(I', (1,1))$ is a point of the universal family. If we take the fiber at some $I \in H_n$, which means to divide the ring by the maximal ideal of the regular ring $\mathbb{C}[c^{r,s}_{h,k}]$ corresponding to I, we are just replacing $c^{r,s}_{h,k}$ by its proper values at that I. So the relations we need are $x^r y^s = \sum_{(h,k)\in\mu} c^{r,s}_{h,k} x^h y^k$. In this way, we get a fiber of B at I as $B(I) = \mathbb{C}[x,y]/I$. We know that $x^h y^k, (h,k) \in \mu$ span the fiber. These are also eigenvectors with respect to (t,q) action. Eigenvalues are $t^h q^k$. We already explained how (t,q) acts on $c^{r,s}_{h,k}$. We need its action on x and y to be consistent, that is, keeping the relations intact, and the only way is to multiply x and y by t and q. So we may talk about the (t,q) bidegree, Hilbert series and etc too on modules.

Next, we explain a little bit cohomology of the modules. The Atiyah-Bott formula has an Euler characteristic in the form of alternating sum of cohomologies. It turned out eventually that higher cohomologies of the modules we are concerned with all vanish. So really important is H^0, the vector space of global sections. This is the section of the module that is valid at every point of H_n. Think about H_n

itself for a moment. The obvious global sections in $H^0(H_n, O_{H_n})$ are $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$, the symmetric functions. Its value at each I is obtained through replacing x_i, y_i by the coordinate values of I. The other important modules are P and Z_n of the isospectral Hilbert space and the zero fiber. P has $x_1, y_1, \ldots, x_n, y_n$ in its global section, and also has the permutation group act on them. While Z_n has a free resolution that enables us to calculate using the formula.

Assemble the Frobenius series of the diagonal coninvariants R_n

We will use the combinatorial formula for the Macdonald polynomial [?] to calculate Frobenius series out of the Atiyah-Bott formula. The shuffle conjecture [?] proposed a direct combinatorial formula for the Frobenius series. We will show through an example that these two approaches tie.

The module in the Atiyah-Bott formula is $P \otimes Z_n$. The Frobenius series of R_n is an alternating sum of the fibers' Frobenius series in Z_n's resolution times that of P. The wedge term in the resolution contribute $(1-q)(1-t)\prod_\mu(q,t)$ term. And B in the resolution gives $B_\mu(q,t)$. The Macdonald polynomial is the Frobenius series of fibers of P.

$$\mathcal{F}_{R_n}(z;q,t) = \sum_{\mu \vdash n} \frac{(1-q)(1-t)\prod_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}$$

We have the powerful combinatorial formula to expand the Macdonald polynomial in monomials. Plug into the formula and we can see this is the same as the shuffle conjecture. We may then transform the result to the Schur polynomial expansion, hence a manual calculation of the fundamental problem. This will give us the coefficient of s_{1^n} in particular, which is the q,t-Catalan number. We have a shortcut in the shuffle conjecture to calculate the q,t-Catalan number too. And finally, we may use the Fermionic formula to calculate coefficient in the shuffle conjecture which provides a shortcut in the comparison.

First, we calculate $\tilde{H}_2$. This involves fillings of $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$, which has no descent, and only has inversion in the situation of $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$. So $\tilde{H}_2 = m_2+(1+q)m_{1^2}$. Similarly, $\tilde{H}_{1^2}$ involves fillings of $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, which has no inversion, and only has descent in the situation of $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$. So $\tilde{H}_{1^2} = m_2 + (1+t)m_{1^2}$. Transform into the Schur polynomial expansion, using the facts that coefficient of $s_{(n)}$ in $\tilde{H}_\mu$ is 1, $s_{(n)} = \sum_\mu m_\mu$, and $s_{1^n} = m_{1^n}$. We get $\tilde{H}_2 = s_2 + qs_{1^2}$ and $\tilde{H}_{1^2} = s_2 + ts_{1^2}$. Plug these into the Atiyah-Bott formula. We derive

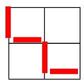
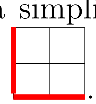
$$\begin{aligned}\mathcal{F}_{R_2}(z;q,t) &= \frac{(1-q)(1-t)(1-q)(1+q)\tilde{H}_2}{(1-tq^{-1})(1-q^2)(1-t)(1-q)} + \frac{(1-q)(1-t)(1-t)(1+t)\tilde{H}_{1^2}}{(1-t)(1-q)(1-t^2)(1-t^{-1}q)} \\ &= \frac{\tilde{H}_2}{(1-tq^{-1})} + \frac{\tilde{H}_{1^2}}{(1-t^{-1}q)}\end{aligned}$$

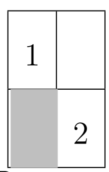
The monomial expansion is

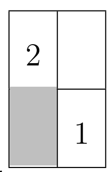
$$\begin{aligned}F_{R_2}(z;q,t) &= \frac{m_2+(1+q)m_{1^2}}{(1-tq^{-1})} + \frac{m_2+(1+t)m_{1^2}}{(1-t^{-1}q)} \\ &= m_2 + (1+q+t)m_{1^2} \qquad (4.4.1)\end{aligned}$$

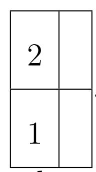
Work on this directly or plug into the Macdonald polynomial expansion the Schur expansion of $\tilde{H}_\mu$, we get

$$\mathcal{F}_{R_2}(z;q,t) = s_2 + (q+t)s_{1^2} \qquad (4.4.2)$$

We check some specializations of the shuffle conjecture. The q, t-Catalan formula [?] evaluates the coefficient of $e_n = s_{1^n}$ in D_n, which is conjectured to be $\mathcal{F}_{R_n}(z; q, t)$. Through superization, the coefficient is shown to be associated with fillings by $\bar{1}$, and resulted in a simplified definition of dinv. There are two Dyck paths when $n = 2$,  and . Here we follow the convention to start from top left and end at bottom right. The first has area 0 and dinv 1. The second has area 1 and dinv 0. So the Hilbert series of anti-symmetric diagonal harmonics is $q + t$. The same as (??).

Next, we can get the Hilbert series of R_n which is the coefficient of m_{1^n}. In our example, by (??), this is $1 + q + t$. The shuffle conjecture solves this special case by filling in standard tableaux, which are the same as parking functions in [?]. The standard tableaux are 

, 

, and 

. Each contributing 1, q, and t to the Hilbert series. Put together, we recover the term $1 + q + t$.

Finally, we look at the Fermionic formula that calculates the coefficient of m_μ. Any tableau with fillings of weight μ will standardize to a parking function f such that, $\omega(f)$ – the word read off from the diagonals, must be a μ shuffle. Reorder the numbers on each diagonal respectively to get a unique permutation $\sigma \in S_n$, which must be a μ shuffle too. Each increasing block A_i of σ contains numbers on a diagonal. This is part of the idea behind the formula

$$< D_n(z; q, t), h_\mu >= \sum_{\substack{\sigma \in S_n \\ \sigma \text{ is a } \mu \text{ shuffle}}} t^{\text{comaj}(\sigma)} \prod_{i,j} \begin{bmatrix} V_{i,j} \\ b_{i,j} \end{bmatrix}_q$$

Here $V_{i,j} = \nu(\sigma, k)$ records the contribution to inversions of the largest number in the block $A_i \cap B_j$, where B_j is the jth block in a μ partition of $\{1, ..., n\}$. $b_{i,j}$ is the size of $A_i \cap B_j$.

When $\mu = 2$, σ has to be the trivial permutation. There are only A_1 and B_1. $V_{1,1} = 2$, $b_{1,1} = 2$, and $\text{comaj}(\sigma) = 0$. Therefore, the coefficient of m_2 is 1.

This is indeed immediate once we observe that we may only fill in n 1s if the tableau has area 0. And the 1s all appear on the main diagonal, having no inversion. So the coefficient of m_n and hence s_n in the Frobenius series is always 1.

The isospectral and Macdonald polynomial

It is relatively easy to derive the fiber of the isospectral X_n when $n = 2$. First, the universal family U_n has regular ring $\mathbb{C}[x, y, c_{h,k}^{r,s}]/(x^r y^s - \sum_{(h,k)\in\mu} c_{h,k}^{r,s} x^h y^k)$ on open subscheme U_μ. This specialize to $\mathbb{C}[x, y]/(x^2, y)$ on $\mu = 1^2$, and $\mathbb{C}[x, y]/(x, y^2)$ on $\mu = 2$.

The isospectral X_n is the reduced product of H_n and $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ with base $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{S_n}$. On U_μ, the regular ring is generated by $c_{h,k}^{r,s}$, x_1, y_1,...,x_n, y_n. Modulo the relations determined by the product. The relations include $x_i^r y_i^s - \sum_{(h,k)\in\mu} c_{h,k}^{r,s} x_i^h y_i^k$, for $i = 1, \ldots, n$, similar to the universal family. In addition, the symmetric functions will have two copies in H_n and $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ respectively, which must be equal. The first copy is expressed in $c_{h,k}^{r,s}$, when specialized at the fixed point, they become zero if there is no constant term. We look at what happens when $n = 2$. At $\mu = 1^2$, we have relations $x_1^2, y_1, x_2^2, y_2, x_1 + x_2, x_1 x_2$. These are actually the generators of the annihilator ideal of $\Delta_2 = x_2 - x_1$. And the quotient ring $\mathbb{C}[x_1, x_2]/(x_1 + x_2, x_1 x_2)$ is R_{1^2}, whose Frobenius series is the Macdonald polynomial $\tilde{H}_{1^2}$. Following the same reasoning, at $\mu = 2$, the relations are generated by $x_1, y_1^2, x_2, y_2^2, y_1 + y_2, y_1 y_2$. The quotient ring is R_2, whose Frobenius series is $\tilde{H}_2$.

We avoided the complication of finding the reduced ring structure at higher n, where the explicit relations will not be enough.

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