# RIGIDITY OF MAGNETIC AND GEODESIC FLOWS

Pilar Herreros

A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania
in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2009

Christopher Croke
Supervisor of Dissertation

Tony Pantev
Graduate Group Chairperson

# Acknowledgments

This thesis could not have been written without the support of my advisor Christopher Croke. His guidance on all the aspects of research and his encouragement during these years were as important as all the geometry I learned from him.

An important role was played by all the geometry topology group at UPenn, with special mention to Wolfgang Ziller, Martin Kerin and John Olsen, who in one way or another helped me develop an understanding of geometry.

I would also like to thank all those who helped by making the department a nice place to be. This is a large number of people, including Martin, David, Mike, Chris, Jen, Paul, Armin, Shea, Wil, Andrew, the quizzo team, and all those who welcomed me to this country and their friendship.

I could not leave out Janet Burns, Monica Pallanti, Paula Scarborough and Robin Toney, that made everything possible to make the department run smoothly, and looked over us with a good word and a smile.

I must also mention my sister Francisca, Juan, Elena, the people at Goetia, and all the others that helped make these years worth living.

Last but not least, my deepest thanks to my parents and stepparents, that gave me their unconditional support from the beginning. And specially to Roberto that agreed with my plans, giving me all his support, love and infinite patience.

ABSTRACT

RIGIDITY OF MAGNETIC AND GEODESIC FLOWS

Pilar Herreros

Christipher Croke, Advisor

In this thesis we will discus the flow of particles on a manifold, with and without the presence of a magnetic field. We will address two independent rigidity problems regarding this flows. The first problem relates to scattering boundary rigidity in the presence of a magnetic field. It has been shown in [DPSU] that, under some additional assumptions, two simple domains with the same scattering data are equivalent. We show that the simplicity of a region can be read from the metric in the boundary and the scattering data. This lets us extend the results in [DPSU] to regions with the same scattering data, where only one is known apriori to be simple. We will then use this results to resolve a local version of a question by Robert Bryant. That is, we show that a surface of constant curvature can not be modified in a small region while keeping all the curves of a fixed constant geodesic curvature closed.

The second problem involves blocking properties of the geodesic flow. We say that a pair of points $x$ and $y$ is secure if there exists a finite set of blocking points such that any geodesic between $x$ and $y$ passes through one of the blocking points. The main point of this part is to exhibit new examples of blocking phenomena both in the manifold and the billiard table setting. As an approach to this, we study if the product of secure configurations (or manifolds) is also secure. We introduce the concept of *midpoint security* that imposes that the geodesic reaches a blocking point exactly at its midpoint. We prove that products of midpoint secure configurations

are midpoint secure. On the other hand, we give an example of a compact $C^1$ surface that contains secure configurations that are not midpoint secure. This surface provides the first example of an insecure product of secure configurations, as well as billiard tables with similar blocking behavior.

# Contents

# List of Figures

# Chapter 1

# Introduction

## 1.1 Scattering Boundary Rigidity in the Presence of a Magnetic Field

A magnetic field on a Riemannian manifold can be represented by a closed $2-$form $\Omega$, or equivalently by the $(1,1)$ tensor $Y : TM \to TM$ defined by $\Omega(\xi,\nu) = \langle Y(\xi), \nu\rangle$ for all $x \in M$ and $\xi, \nu \in T_xM$. The trajectory of a charged particle in such a magnetic field is then modeled by the equation

$$\nabla_{\gamma'}\gamma' = Y(\gamma'),$$

we will call such curves *magnetic geodesics.* In contrast to regular geodesics, magnetic geodesics are not reversible, and can't be rescaled, i.e. the trajectory depends on the energy $|\gamma'|^2$.

Magnetic geodesics and the magnetic flow where first considered by V.I. Arnold [Ar1] and D.V. Anosov and Y.G. Sinai [AS]. The existence of closed magnetic geodesics, and the magnetic flow in general, has been widely studied since then. Some of the approaches to this subject are, the Morse-Novikov theory for variational functionals (e.g. [No, NT, Ta]), Aubry-Mather's theory (e.g. [CMP]), the theory of dynamical systems (e.g. [Grog, Ni, PP]) and using methods from symplectic geometry (e.g. [Ar2, Gi1, Gi2]).

In the special case of surfaces, where the $2-$form has the form $\Omega = k(x)dA$, a magnetic geodesic of energy $c$ has geodesic curvature $k_g = k(x)/\sqrt{c}$. This relates magnetic geodesics with the problem of prescribing geodesic curvature, in particular with the study of curves of constant geodesic curvature. This relation was used by V.I. Arnold in [Ar3], and later by many others (see e.g [Le, Sch], REF!!), to study the existence of closed curves with prescribed geodesic curvature.

It is clear that on surfaces of constant curvature the curves of large constant geodesic curvature are circles, therefore closed. The study of these curves goes back to Darboux, who in 1894 claimed (in a footnote in his book [Da]) that the converse is true, that is, if all curves of constant (sufficiently large) geodesic curvature are closed, then the surface has to be of constant Gauss curvature.

The proof of this result depends strongly on the fact that curves of geodesic curvature are closed for all large curvature, or equivalently low energy. This raises the following question, brought to my attention by R. Bryant.

**Question 1.** *Are surfaces of constant Gauss curvature the only surfaces for which all curves of a fixed constant nonzero geodesic curvature are closed?*

For the case where the constant is 0, this question corresponds to existence of surfaces all of whose geodesics are closed. The first examples of such surfaces where given by Zoll [Zo] who, in 1903, constructed a surface of revolution with this property.

Using the relation between geodesic curvature and magnetic geodesics we can approach this question by studying magnetic geodesics on a surface, in the presence of a constant magnetic field. A first step in this direction is to determine if a surface of constant curvature can be locally changed keeping all the curves of a fixed constant geodesic curvature closed. We show in section 2.4 that the metric can not be changed in a small region without loosing this property. In fact, more generally, we show that a Riemannian surface with a magnetic flow whose orbits are closed can't be changed locally without "opening" some of its orbits.

The easier way of changing the metric in a small region without opening the orbits is to require that all magnetic geodesics that enter the region leave it at the same place and in the same direction as before, to join the outside part of the orbit. This is the *magnetic scattering data* of the region; for each point and inward direction on the boundary, it associates the exit point and direction of the corresponding unit speed magnetic geodesic.

With this problem in mind, we can ask the following boundary rigidity question for magnetic geodesics. In a Riemannian manifold with boundary, in the presence of a magnetic field, is the metric determined by the metric on the boundary and the magnetic scattering data?

In general this is not true, even for the geodesic case. For example, a round sphere with a small disk removed has the same scattering data as a round $\mathbb{R}P^2$ with a disk of the same size removed. One of the usual conditions to obtain boundary rigidity is to assume that the region is *simple*. In our setting *simple* means a compact region that is magnetically convex, and where the magnetic exponential map has no conjugate points (see section 2.3).

For simple domains, scattering rigidity for geodesics is equivalent to distance boundary rigidity (see [Cr]) and it has been widely studied. It is known to hold for simple subdomains of $\mathbb{R}^n$ [Gro] or [C4] , an open round hemisphere [Mi], hyperbolic space [BCG] [Cr], and some spaces of negative curvature [Ot, Cr3] among others. For a discussion on the subject, see [Cr].

Recently N. Dairbekov, P. Paternain, P. Stefanov and G. Uhlmann proved (in [DPSU]) magnetic boundary rigidity between two simple manifolds in several classes of metrics, including simple conformal metrics, simple analytic metrics, and all $2-$dimensional simple metrics.

To be able to apply this results to local perturbations of existing metrics we would like to be able to compare a simple domain with any other (not necessarily simple) domain with the same boundary behavior. For this we prove the following theorem. Thus, proving magnetic rigidity for the simple domains considered in [DPSU].

**Theorem 1.** *Magnetic simplicity can be read from the metric on the boundary and the scattering data.*

To change the metric in a small region without opening the orbits, it is not necessary to preserve the scattering data. It could be the case, in principle, that orbits exit the region in a different place, but after some time, came back to the region and leave it in the proper place to close up again (see figure [REF]). In section 2.4 we look at this case in 2 dimensions, and we show the following theorem.

**Theorem 2.** *Let $M$ and $\widehat{M}$ be compact surfaces with magnetic fields, all of whose magnetic geodesics are closed. If the metric and magnetic fields agree outside a small enough simple domain, then they agree everywhere.*

Applying these two theorems for a constant magnetic field on a surface of constant curvature, we conclude that:

**Corollary 3.** *A surface of constant curvature can not be perturbed in a small enough region while keeping all the curves of a fixed constant geodesic curvature closed.*

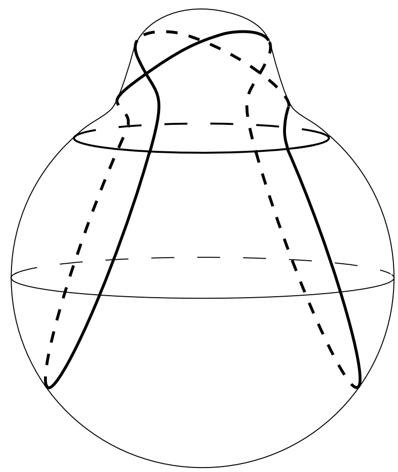


Figure 1.1: A closed orbit going through $R$ twice.

## 1.2 Blocking: new examples and properties of products

We say that a pair of points $x$ and $y$ is secure if there exists a finite set of blocking points such that any geodesic between $x$ and $y$ passes through one of the blocking points. The study of this property originated in the context of polygonal billiards and translation surfaces (see e.g. [G1], [G2], [HS], [M1], [M2] ). It has an interpretation in geometric optics, where being secure means that one point is shaded from the light emanated by the other point by finitely many point blocks. Or that they can't "see" each other. Another interpretation relates to the name security, as we can think of two points being secure if any path between them passes through one of finitely many check points.

Recently security has been studied on Riemannian manifolds, in particular for compact locally symmetric spaces (see [GS]), and its relation with entropy (see [BG], [LS]) among others. Security is a rare phenomenon. The emphasis in the area is to show that small blocking sets imply very strong conditions on the geometry of the manifold. For instance, on all previous examples of security the blocking occurs at the midpoints of geodesics. We mention some other such results and conjectures below. The main point of this section is to exhibit new examples of blocking phenomena both in the manifold and the billiard table setting. Here the manifolds will be closed $C^1$ smooth Riemannian manifolds, while the billiard tables we consider are compact $C^1$ smooth Riemannian manifolds with boundary (with and without corners).

The search for examples of secure manifolds and for its relationship with other aspects of Riemannian geometry raises the question of how this property behaves under products of manifolds. This question was brought to my attention by K. Burns and E. Gutkin, who partially solved it in [BG] where they prove that if (a configuration on) a product manifold is secure, then so is (its projection to) each factor. We study the converse of this result, more precisely whether the product of secure configurations (or manifolds) is also secure.

In section 3.1 we introduce the concept of *midpoint security* that requires a geodesic joining two points to reach a blocking point exactly at its midpoint. We discuss its relation with security and prove the following result.

**Theorem 4.** *A product manifold is midpoint secure if and only if each factor is midpoint secure.*

On the other hand, we give some cases where midpoint security is actually necessary to achieve security on a product of secure configurations (see Proposition 3.1.5). Section 3.2 is devoted to the construction of a compact $C^1$ surface $N$ that contains secure configurations that are not midpoint secure. The existence of such a manifold proves the following theorem.

**Theorem 5.** *There are secure configurations that are not midpoint secure.*

This surface also yields the first example of an insecure product of secure configurations. More generally we prove the following statement.

**Theorem 6.** *Let $N$ be the surface constructed in section 3.2. For any compact connected Riemannian manifold $M$ without boundary, any configuration in the product $N \times M$ that projects to one of the mentioned secure but not midpoint secure configurations in $N$, is insecure.*

We finish the chapter using this surface to give examples of non-planar billiard tables with secure configurations that are not midpoint secure.

# Chapter 2

# Scattering Boundary Rigidity in the Presence of a Magnetic Field

## 2.1 Magnetic Jacobi fields and conjugate points

Let $(M, g)$ be a compact Riemannian manifold with boundary, with a magnetic field given by the closed 2-form $\Omega$ on $M$. Denote by $\omega_0$ the canonical symplectic form on $TM$, that is, the pull back of the canonical symplectic form of $T^*M$ by the Riemmanian metric. The geodesic flow can be described as the Hamiltonian flow of $H$ w.r.t. $\omega_0$, where $H : TM \to \mathbb{R}$ is defined as

$$H(v) = \frac{1}{2}|v|_g^2, \ v \in TM.$$

In a similar way, the magnetic flow $\psi^t : TM \to TM$ can be described as the Hamiltonian flow of $H$ with respect to the modified symplectic form $\omega = \omega_0 + \pi^*\Omega$. This flow has orbits $t \to (\gamma(t), \gamma'(t))$, where $\gamma$ is a magnetic geodesic, i.e. $\nabla_{\gamma'}\gamma' = Y(\gamma')$. Note that when $\Omega = 0$ we recover the geodesic flow, whose orbits are geodesics.

It follows from the above definitions that the magnetic geodesics have constant speed. In fact,

$$\frac{d}{dt}\langle \gamma'(t), \gamma'(t)\rangle = 2\langle Y(\gamma'), \gamma'\rangle = 2\Omega(\gamma', \gamma') = 0.$$

Moreover, the trajectories of the magnetic geodesics depend on the energy level. Unlike geodesics, a rescaling of a magnetic geodesic is not longer a magnetic geodesic. We will restrict our attention to a single energy level, or equivalently to unit speed magnetic geodesics. Therefore, from now on, we will only consider the magnetic flow $\psi^t : SM \to SM$.

The choice of energy level is not a restriction, since we can study other energy levels by considering the form $\tilde{\Omega} = \lambda\Omega$, for any $\lambda \in \mathbb{R}$.

For $x \in M$ we define the *magnetic exponential map* at $x$ to be the partial map $exp_x^\mu : T_xM \to M$ given by

$$exp_x^\mu(t\xi) = \pi \circ \psi^t(\xi),\ t \geq 0, \xi \in S_xM.$$

This map takes a vector $t\xi \in T_xM$ to the point in $M$ that corresponds to following the magnetic geodesic with initial direction $\xi$, a time $t$. This function is $C^\infty$ on $T_xM \setminus \{0\}$ but in general only $C^1$ at 0. The lack of smoothness at the origin can be explained by the fact that magnetic geodesics are not reversible. When we pass through the origin we change from $\gamma_\xi$ to $\gamma_{-\xi}$, that in general only agree up to first order. For a proof see Appendix A in [DPSU].

We will say that a point $p \in M$ is *conjugate* to $x$ along a magnetic geodesic $\gamma$ if $p = \gamma(t_0) = exp_x^\mu(t_0\xi)$ and $v = t_0\xi$ is a critical point of $exp_x^\mu$. The *multiplicity* of the conjugate point $p$ is then the dimension of the kernel of $d_v exp_x^\mu$.

In what follows, and throughout this thesis, if $V$ is a vector field along a geodesic $\gamma(t)$, $V'$ will denote the covariant derivative $\nabla_{\gamma'} V$.

We want to give an alternative characterizations of conjugate points. For this consider a variation of $\gamma$ through magnetic geodesics. That is

$$f(t,s) = \gamma_s(t)$$

where $\gamma_s(t)$ is a magnetic geodesic for each $s \in (-\epsilon, \epsilon)$ and $t \in [0,T]$. Therefore $\frac{D}{\partial t}\frac{\partial f}{\partial t} = Y(\frac{\partial f}{\partial t})$. Using this and the definition of the curvature tensor we can write:

$$\begin{aligned} \tfrac{D}{\partial s}(Y(\tfrac{\partial f}{\partial t})) &= \tfrac{D}{\partial s}\tfrac{D}{\partial t}\tfrac{\partial f}{\partial t} = \tfrac{D}{\partial t}\tfrac{D}{\partial s}\tfrac{\partial f}{\partial t} - R(\tfrac{\partial f}{\partial s}, \tfrac{\partial f}{\partial t})\tfrac{\partial f}{\partial t} \\ &= \tfrac{D}{\partial t}\tfrac{D}{\partial t}\tfrac{\partial f}{\partial s} + R(\tfrac{\partial f}{\partial t}, \tfrac{\partial f}{\partial s})\tfrac{\partial f}{\partial t} \end{aligned}$$

If we call the variational field $J(t) = \frac{\partial f}{\partial s}(t,0)$ we get for $s=0$

$$\nabla_J(Y(\gamma')) = J'' + R(\gamma', J)\gamma'.$$

Note also that

$$\nabla_J(Y(\gamma')) = Y(\nabla_J \gamma') + (\nabla_J Y)(\gamma')$$

and

$$\nabla_J \gamma'(t) = \frac{D}{\partial s}\frac{\partial f}{\partial t}(t,0) = \frac{D}{\partial t}\frac{\partial f}{\partial s}(t,0) = J'(t) \tag{2.1.1}$$

so we can rewrite the above equation as

$$J'' + R(\gamma', J)\gamma' - Y(J') - (\nabla_J Y)(\gamma') = 0.$$

Since magnetic geodesics can't be rescaled we have $|\gamma_s'| = 1$ for all the magnetic geodesics in the variation. This equation together with equation (2.1.1) gives, for any such variational field $J$,

$$\langle J', \gamma' \rangle = \langle \nabla_J \gamma', \gamma' \rangle = J \langle \gamma', \gamma' \rangle = 0.$$

This equations characterize the variational field of variations through magnetic geodesics, in a way analogous to the characterization of Jacobi fields. We will use this equation as a definition as follows.

Given a magnetic geodesic $\gamma : [0,T] \to M$, let $\mathcal{A}$ and $\mathcal{C}$ be the operators on smooth vector fields $Z$ along $\gamma$ defined by

$$\mathcal{A}(Z) = Z'' + R(\gamma', Z)\gamma' - Y(Z') - (\nabla_Z Y)(\gamma'),$$

$$\mathcal{C}(Z) = R(\gamma', Z)\gamma' - Y(Z') - (\nabla_Z Y)(\gamma').$$

A vector field $J$ along $\gamma$ is said to be a *magnetic Jacobi field* if it satisfies the equations

$$\mathcal{A}(J) = 0 \tag{2.1.2}$$

and

$$\langle J', \gamma' \rangle = 0. \tag{2.1.3}$$

Note that from equation 2.1.2 we can see that

$$\begin{aligned} \tfrac{d}{dt} \langle J', \gamma' \rangle &= \langle J'', \gamma' \rangle + \langle J', Y(\gamma') \rangle \\ &= \langle -R(\gamma', J)\gamma' + Y(J') + (\nabla_J Y)(\gamma'), \gamma' \rangle - \langle Y(J'), \gamma' \rangle = 0 \end{aligned}$$

where we used that $\langle (\nabla_J Y)(\gamma'), \gamma' \rangle = 0$ because $Y$ is skew-symmetric. Therefore, it is enough to check condition 2.1.3 at a point.

A magnetic Jacobi field along a magnetic geodesic $\gamma$ is uniquely determined by its initial conditions $J(0)$ and $J'(0)$. To see this, consider the orthonormal basis defined by extending an orthonormal basis $e_1, \ldots, e_n$ at $\gamma(0)$ by requiring that

$$e_i' = Y(e_i) \tag{2.1.4}$$

along $\gamma$. This extension gives an orthonormal basis at each point since

$$\frac{d}{dt}\langle e_i, e_j\rangle = \langle Y(e_i), e_j\rangle + \langle e_i, Y(e_j)\rangle = 0.$$

Using this basis,

$$z = \sum_{i=1}^{n} f_i e_i$$

and we can write equation 2.1.2 as the system

$$f_j'' + \sum_{i=1}^{n} f_i' y_{ij} + \sum_{i=1}^{n} f_i a_{ij} = 0$$

where $y_{ij} = \langle Y(e_i), e_j\rangle$ and

$$a_{ij} = \langle \nabla_{\gamma'} Y(e_i) + R(\gamma', e_i)\gamma' - Y(Y(e_i)) - (\nabla_{e_i} Y)(\gamma'), e_j\rangle .$$

This is a linear second order system, and therefore it has a unique solution for each set of initial conditions.

Magnetic Jacobi fields correspond exactly to variational field of variations through magnetic geodesics. In the case of magnetic Jacobi fields $J$ along $\gamma_\xi$ that vanish at 0 this can be seen by considering

$$f(t,s) = \gamma_s(t) = exp_x^\mu(t\xi(s)) \tag{2.1.5}$$

where $\xi : (-\epsilon, \epsilon) \to S_x M$ is a curve with $\xi(0) = \xi = \gamma_\xi$ and $\xi'(0) = J'(0)$. This is clearly a variation through geodesics, and therefore its variational field $\frac{\partial f}{\partial s}(t, 0)$ satisfies 2.1.2. The variational field $\frac{\partial f}{\partial s}(t, 0)$ and the magnetic Jacobi field $J(t)$ are then solutions of 2.1.2 with the same initial conditions, therefore they must agree.

For magnetic Jacobi fields $J$ that do not vanish at 0, we can use the variation

$$f(t,s) = \gamma_s(t) = exp_{\tau(s)}^\mu(t\xi(s))$$

where $\tau(s)$ is any curve with $\tau'(0) = J(0)$ and $\xi(s)$ is a vector field along $\tau$ with $\xi(0) = \gamma'(0)$ and $\xi'(0) = J'(0)$.

It is easy to see from the definition and the equation $\gamma'' = Y(\gamma')$ that $\gamma'$ is always a magnetic Jacobi field. Unlike the case of straight geodesics, this is the only magnetic Jacobi field parallel to $\gamma'$. Another difference from the straight geodesic case is that magnetic Jacobi fields that are perpendicular to $\gamma'$ at $t = 0$ don't stay perpendicular for all $t$. For this reason we will sometimes consider instead the orthogonal projection $J^\perp = J - f\gamma'$ where $f = \langle J, \gamma' \rangle$. The component $f\gamma'$ of $J$ parallel to $\gamma'$ is uniquely determined by $J^\perp$ and $J(0)$, since

$$f' = \langle J', \gamma' \rangle + \langle J, \gamma'' \rangle = \langle J, Y(\gamma') \rangle = \left\langle J^\perp, Y(\gamma') \right\rangle .$$

We will need one more property of magnetic Jacobi fields that vanish at 0. Let $\gamma : [0, T] \to M$ be a magnetic geodesic with $\gamma(0) = x$. Let $v \in T_{\gamma'} S_x M$, or equivalently under the usual identification, $v \in T_x M$ perpendicular to $\gamma'$.

Let $f(t, s)$ be a variation through magnetic geodesics of the form 2.1.5 where $t \in [0, T]$ and $\xi : (-\epsilon, \epsilon) \to S_x M$ with $\xi(0) = \gamma'(0)$ and $\xi'(0) = v$. The variational field $J_v$ of this variation is

$$\begin{aligned} J_v(t) &= \tfrac{\partial f}{\partial s}(t,0) = \tfrac{\partial}{\partial s}[\pi \circ \psi^t(\xi(s))]\big|_{s=0} = d_{\xi(0)}[\pi \circ \psi^t](\xi'(0)) \\ &= d_{\psi^t(\xi)}\pi \circ d_\xi \psi^t(v) = d_{\gamma'(t)}\pi \circ d_{\gamma'(0)}\psi^t(v) \end{aligned} \tag{2.1.6}$$

and its derivative is given by

$$J_v'(t) = \frac{D}{\partial t}\frac{\partial f}{\partial s}(t,0) = \frac{D}{\partial s}\frac{\partial}{\partial t}[\pi \circ \psi^t(\xi(s))]\bigg|_{s=0} = \frac{D}{\partial s}[\psi^t(\xi(s))]\bigg|_{s=0} = d_{\xi(0)}\psi^t(\xi'(0)) = d_{\gamma'(0)}\psi^t(v).$$

This equations are independent of the variation $f$.

Since the magnetic flow is a Hamiltonian flow with respect to the symplectic form $\omega = \omega_0 + \pi^*\Omega$, this form is invariant under the magnetic flow $\psi^t$ [Pa, pg. 10]. Therefore for any two magnetic Jacobi fields $J_v$ and $J_w$ as above, we have that

$$\begin{aligned}\omega(d_{\gamma'}\psi^t(v), d_{\gamma'}\psi^t(w)) &= \omega_0(d_{\gamma'}\psi^t(v), d_{\gamma'}\psi^t(w)) + \pi^*\Omega(d_{\gamma'}\psi^t(v), d_{\gamma'}\psi^t(w)) \\ &= \langle J_v(t), J_w'(t)\rangle - \langle J_v'(t), J_w(t)\rangle + \Omega(J_v(t), J_w(t))\end{aligned}$$

is independent of $t$. Using also that $J_v(0) = 0$, we get

$$\langle J_v, J_w'\rangle - \langle J_v', J_w\rangle + \langle Y(J_v), J_w\rangle = 0 \tag{2.1.7}$$

for any two such Jacobi fields.

We will now relate the concepts of magnetic Jacobi fields and conjugate points.

**Proposition 2.1.1.** *Let $\gamma_\xi : [0,T] \to M$ be the magnetic geodesic with $\gamma(0) = x$ and $\gamma'(0) = \xi$. The point $p = \gamma(t_0)$ is conjugate to $x$ along $\gamma$ if and only if there exist a magnetic Jacobi field $J$ along $\gamma$, not identically zero, with $J(0) = 0$ and $J(t_0)$ parallel to $\gamma'$.*

*Moreover, the multiplicity of $p$ as a conjugate point is equal to the number of linearly independent such Jacobi fields.*

Consider a variation through magnetic geodesics as in (2.1.5), with $\xi'(0) = v$ perpendicular to $\gamma'$. Then

$$J_v(t) = \frac{\partial f}{\partial s}(t, 0) = d_{t\xi} exp_x^\mu(tv)$$

is a nontrivial magnetic Jacobi field. If there is a vector $v$ for which $J_v(t_0)$ is parallel to $\gamma'$, then $d_{t_0\xi} exp_x^\mu(t_0 v)$ and $d_{t_0\xi} exp_x^\mu(\xi) = \gamma'$ will be parallel, and $t_0\xi$ is a critical point of $exp_x^\mu$. Conversely, if $t_0\xi$ is a critical point there must be a vector $v$ such that $d_{t_0\xi} exp_x^\mu(v) = 0$. Let $v^\perp = v - \langle v, \xi\rangle \xi$, this is not 0 since $d_{t_0\xi} exp_x^\mu(\xi) = \gamma' \neq 0$, and

$$J_{v^\perp}(t_0) = d_{t_0\xi} exp_x^\mu(t_0 v) - d_{t_0\xi} exp_x^\mu(t_0 \langle v, \xi\rangle \xi) = -t_0 \langle v, \xi\rangle \gamma'(t_0).$$

To prove the second statement, note that Jacobi fields $J_{v_i}$ as above are linearly independent iff the vectors $v_i$ are. Since all $v_i$ are perpendicular to $\gamma'(0)$, the number of linearly independent vectors will be the dimension of the kernel of $d_{t_0\xi}exp^{\mu}_{x}$, that is the multiplicity of the conjugate point.

## 2.2 The Index form

Let $\Lambda$ denote the $\mathbb{R}$-vector space of piecewise smooth vector fields $Z$ along $\gamma$. Define the quadratic form $Ind : \Lambda \to \mathbb{R}$ by

$$Ind_{\gamma}(Z) = \int_0^T \{|Z'|^2 - \langle \mathcal{C}(Z), Z\rangle - \langle Y(\gamma'), Z\rangle^2\}dt.$$

Note that

$$Ind_{\gamma}(Z) = -\int_0^T \{\langle \mathcal{A}(Z), Z\rangle + \langle Y(\gamma'), Z\rangle^2\}dt + \langle Z, Z'\rangle|_0^T + \sum \left\langle Z, Z'^{-} - Z'^{+} \right\rangle\Big|_{t_i}.$$

where $Z'^{\pm}$ stands for the left and right derivatives of $Z$ at the points $t_i$ where the derivative is discontinuous.

The $Ind_{\gamma}(Z)$ generalizes the index form of a geodesic in a Riemannian manifold. It is easy to see that when $\Omega = 0$ these are the same form. We will see throughout this section that, when restricted to orthogonal vector fields, we retain some of the relations between (magnetic) Jacobi fields, index form and conjugate points.

Let $\Lambda_0$ denote the $\mathbb{R}$-vector space of piecewise smooth vector fields $Z$ along $\gamma$ such that $Z(0) = Z(T) = 0$, $\Lambda^{\perp}$ the subspace of piecewise smooth vector fields that stay orthogonal to

$\gamma'$, and $\Lambda_0^\perp = \Lambda_0 \cap \Lambda^\perp$.

For any magnetic Jacobi field $J$ along a magnetic geodesic $\gamma$, let $f = \langle J, \gamma' \rangle$ and $Z = J - f\gamma'$ the component of $J$ orthogonal to $\gamma$ , using that $\gamma'' = Y(\gamma')$ we have

$$\mathcal{A}(J) = Z'' + f''\gamma' + 2f'Y(\gamma') + f\gamma''' + R(\gamma', Z + f\gamma')\gamma' - Y(Z' + f'\gamma' + f\gamma'') - (\nabla_{Z+f\gamma'}Y)(\gamma')$$

so

$$0 = A(Z) + fA(\gamma') + f''\gamma' + f'Y(\gamma') \tag{2.2.1}$$

On the other hand, any magnetic Jacobi field satisfies $\langle J', \gamma' \rangle = 0$. Using this together with $f = \langle J, \gamma' \rangle$ and $\gamma'' = Y(\gamma')$ we see that $f' = \langle J, Y(\gamma') \rangle = \langle Z, Y(\gamma') \rangle$. Since $\gamma'$ is a Jacobi field, $A(\gamma') = 0$, and we have from (2.2.1)

$$\langle A(Z), Z \rangle = -f' \langle Y(\gamma'), Z \rangle = -\langle Y(\gamma'), Z \rangle^2$$

therefore if $J$ is a magnetic Jacobi field and its orthogonal component $Z$ is in $\Lambda_0$

$$Ind_\gamma(Z) = -\int_0^T \{\langle \mathcal{A}(Z), Z \rangle + \langle Y(\gamma'), Z \rangle^2\} dt = 0. \tag{2.2.2}$$

Also, if there are no conjugate points along $\gamma$, we can restrict our attention to a neighborhood of $\gamma$ for which the magnetic exponential is a diffeomorphism. In this case we can adapt the proof given in [DPSU] for the case of simple domains to prove the following version of the Index Lemma. We include the proof below.

**Lemma 2.2.1.** *Let $\gamma$ be a magnetic geodesic without conjugate points to $\gamma(0)$. Let $J$ be a magnetic Jacobi field, and $J^\perp$ the component orthogonal to $\gamma'$. Let $Z \in \Lambda^\perp$ be a piecewise differentiable vector field along $\gamma$, perpendicular to $\gamma'$. Suppose that*

$$J^\perp(0) = Z(0) = 0 \text{ and } J^\perp(T) = Z(T) \tag{2.2.3}$$

*Then*

$$Ind_\gamma(J^\perp) \leq Ind_\gamma(Z)$$

*and equality occurs if and only if* $Z = J^\perp$*.*

Note that in the case where the vector field $Z$ satisfies $Z(0) = Z(T) = 0$, $J = \gamma'$ is a Jacobi field that satisfies the above hypothesis, giving the following corollary.

**Corollary 2.2.2.** *If* $\gamma$ *has no conjugate points and* $Z \in \Lambda_0^\perp$*, then* $Ind_\gamma(Z) \geq 0$*, with equality if and only if* $Z$ *vanishes.*

In other words, if $\gamma$ has no conjugate points, the quadratic form $Ind_\gamma : \Lambda_0^\perp \to \mathbb{R}$ is positive definite.

*Proof of Lemma 2.2.1.* Given a vector $v \in T_{\gamma(0)}M$ orthogonal to $\gamma'$, we can define a magnetic Jacobi field $J_v$ along $\gamma$ as in (2.1.6) that has $J_v(0) = 0$ and $J_v'(0) = v$. Since there are no conjugate points to $\gamma(0)$ along $\gamma$, these Jacobi fields are never 0 nor parallel to $\gamma'$. There exist a basis $\{v_1, \ldots, v_n\}$ of $T_{\gamma(0)}M$, with $v_1 = \gamma'$ such that $J_1(t) = \gamma'(t)$ and $J_i(t) = J_{v_i}(t)$, $i = 2, \ldots, n$ form a basis for $T_{\gamma(t)}M$ for all $t \in (0, T]$.

If $Z$ is a vector field with $Z(0) = 0$, we can write, for $t \in (0, T]$

$$Z(t) = \sum_{i=1}^{n} f_i(t) J_i(t)$$

where $f_1, \ldots, f_n$ are smooth functions. This function can be smoothly extended to $t = 0$, as we now show. For $i \geq 2$, we can write $J_i(t) = tA_i(t)$ where $A_i$ are smooth vector fields with $A_i(0) = J_i'(0)$. Then each $A_i(t)$ is parallel to $J_i(t)$ and $\{\gamma'(t), A_2(t), \ldots, A_n(t)\}$ is a basis for all

$t \in [0, T]$, so

$$Z(t) = g_1\gamma' + \sum_{i=2}^{n} g_i(t) A_i(t).$$

It follows that for $t \in (0, T]$, $g_1 = f_1$ and $g_i(t) = tf_i(t)$ for $i \geq 2$, and since $Z(0) = 0$, $g_i(0) = 0$ and $f_i$ extends smoothly to $t = 0$.

Using this representation we can write

$$Ind_\gamma(Z) = -\sum_{i,j} \int_0^T \langle \mathcal{A}(f_i J_i), f_j J_j \rangle \, dt - \int_0^T \langle Y(\gamma'), Z \rangle^2 \, dt + \langle Z, Z' \rangle|_0^T + \sum \left\langle Z, Z'^- - Z'^+ \right\rangle\Big|_{t_k}.$$

where $Z'^\pm$ stands for the left and right derivatives of $Z$ at the points $t_k$ where the derivative is discontinuous.

And

$$\begin{aligned} \mathcal{A}(f_i J_i) &= f_i'' J_i + 2f_i' J_i' + f_i J_i'' + f_i R(\gamma', J_i)\gamma' - f_i' Y(J_i) - f_i Y(J_i') - f_i(\nabla_{J_i} Y)(\gamma') \\ &= f_i \mathcal{A}(J_i) + f_i'' J_i + 2f_i' J_i' - f_i' Y(J_i) \end{aligned} \tag{2.2.4}$$

where the first term is 0 since $J_i$ is a magnetic Jacobi field.

We know, moreover, from (2.1.7) that

$$\left\langle J_i, J_j' \right\rangle - \langle J_i', J_j \rangle + \langle Y(J_i), J_j \rangle = 0$$

so we can write

$$\begin{aligned} \langle \mathcal{A}(f_i J_i), J_j \rangle &= \langle f_i'' J_i, J_j \rangle + 2 \langle f_i' J_i', J_j \rangle - \langle f_i' Y(J_i), J_j \rangle \\ &= f_i'' \langle J_i, J_j \rangle + f_i' \langle J_i', J_j \rangle + f_i' \left\langle J_i, J_j' \right\rangle \\ &= \tfrac{d}{dt}(f_i' \langle J_i, J_j \rangle) \end{aligned}$$

and

$$\int_0^T \langle \mathcal{A}(f_i J_i), f_j J_j \rangle \, dt = \int_0^T f_j \frac{d}{dt}(f_i' \langle J_i, J_j \rangle) dt = \langle f_i' J_i, f_j J_j \rangle|_0^T - \int_0^T \left\langle f_i' J_i, f_j' J_j \right\rangle dt.$$

Using this, and that $Z(0) = 0$ we can write the index form as

$$Ind_\gamma(Z) = \int_0^T ||\sum_1^n f_i' J_i||^2 dt - \left\langle \sum_1^n f_i' J_i, Z \right\rangle \Bigg|_0^T - \sum \left\langle \sum_1^n (f_i'^- - f_i'^+) J_i, Z \right\rangle \Bigg|_{t_k}$$

$$- \int_0^T \langle Y(\gamma'), Z\rangle^2 dt + \langle Z, Z'\rangle \Big|_0^T + \sum \langle Z, Z'^- - Z'^+\rangle \Big|_{t_k}$$

$$= \int_0^T ||\sum_1^n f_i' J_i||^2 - \langle Y(\gamma'), Z\rangle^2 dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle + \sum \left\langle Z, \sum_1^n f_i (J_i'^- - J_i'^+) \right\rangle \Bigg|_{t_k}$$

where the last term is 0 because $J_i$ is differentiable.

Let $W = \sum_2^n f_i' J_i$, and remember that $J_1 = \gamma'$, then we get

$$||\sum_1^n f_i' J_i||^2 = \langle f_1'\gamma' + W, f_1'\gamma' + W\rangle = f_1'^2 + 2f_1' \langle \gamma', W\rangle + \langle W, W\rangle .$$

Since $Z$ is orthogonal to $\gamma'$, $\langle Z, \gamma'\rangle = 0$ and by differentiating $\langle Z', \gamma'\rangle = -\langle Z, Y(\gamma')\rangle$. Using also that for magnetic Jacobi fields $\langle J', \gamma'\rangle = 0$ we have

$$-\langle Z, Y(\gamma')\rangle = \left\langle \sum_1^n f_i' J_i + f_i J_i', \gamma' \right\rangle = f_1' + \langle W, \gamma'\rangle$$

and

$$\langle Z, Y(\gamma')\rangle^2 = f_1'^2 + 2f_1' \langle W, \gamma'\rangle + \langle W, \gamma'\rangle^2 .$$

so

$$\begin{aligned} Ind_\gamma(Z) &= \int_0^T ||\sum_1^n f_i' J_i||^2 - \langle Y(\gamma'), Z\rangle^2 dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \\ &= \int_0^T \langle W, W\rangle - \langle W, \gamma'\rangle^2 dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \\ &= \int_0^T ||W^\perp|| dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \qquad (2.2.5) \end{aligned}$$

where $W^\perp$ is the component of $W$ orthogonal to $\gamma'$.

Let $J$ be the magnetic Jacobi field $J = \sum_2^n f_i(T)J_i$. The fact that it is a magnetic Jacobi field is easy to see since $\mathcal{A}$ is linear over $\mathbb{R}$, it is also clear that $J(0) = 0$ and $J^\perp(T) = Z(T)$. Since

$$J^\perp = \sum_2^n f_i(T)J_i - \sum_2^n f_i(T) \langle J_i, \gamma' \rangle \gamma'$$

the corresponding functions $f_i(T)$ are constant for $i \geq 2$, so $W = 0$ and we can see from equation (2.2.5) that

$$Ind_\gamma(J^\perp) = \left\langle J(T), \sum_1^n f_i(T)J_i'(T) \right\rangle$$

that gives

$$Ind_\gamma(Z) - Ind_\gamma(J^\perp) = \int_0^T ||W^\perp||dt \geq 0$$

with equality iff $W^\perp$ vanishes everywhere. That is

$$W^\perp = \sum_2^n f_i' J_i - \langle W, \gamma' \rangle \gamma' = 0$$

and therefore $f_i$ constant for $i \geq 2$. So $Z = f_1\gamma' + \sum_2^n f_i J_i = f_1\gamma' + J$, and since $Z$ is orthogonal to $\gamma'$ this implies that $Z = J^\perp$.

□

In what follows we will want to use the above lemma for more general vector fields, that are not 0 at 0 but are 0 at $T$. Since magnetic flows are not reversible, we can't simply reverse time. We will consider instead the associated magnetic flow $(M, g, -\Omega)$. This magnetic flow has the same magnetic geodesics, but with opposite orientation.

**Lemma 2.2.3.** *Let $(M, g, \Omega)$ be a magnetic field. Then $(M, g, -\Omega)$ is also a magnetic field.*

1. *Magnetic geodesics in both magnetic fields agree, but with opposite orientation.*

2. *Jacobi fields agree in both magnetic fields.*

3. *Index form is independent of its orientation.*

If $\gamma : [0, T] \to M$ is a magnetic geodesic in $(M, g, \Omega)$, denote by $\gamma_-$ the geodesic with opposite orientation, that is $\gamma_-(t) = \gamma(-t)$, for $t \in [-T, 0]$. Then $\gamma_-''(t) = \gamma''(-t) = Y(\gamma'(-t)) = -Y(\gamma_-'(t))$, so $\gamma_-$ is a magnetic geodesic in $(M, g, -\Omega)$. Part 2 follows from 1 and the fact that magnetic Jacobi fields are variational fields of variations through magnetic geodesics. Alternatively, we can check that for $J_-(t) = J(-t)$:

$$\begin{aligned}\mathcal{A}_-(J_-(t)) &= J_-''(t) + R(\gamma_-'(t), J_-(t))\gamma_-'(t) + Y(J_-'(t)) + (\nabla_{J_-} Y)(\gamma_-'(t)) \\ &= J''(-t) + R(\gamma'(-t), J(-t))\gamma'(-t) - Y(J'(-t)) - (\nabla_J Y)(\gamma'(-t)) \\ &= \mathcal{A}(J(-t)).\end{aligned}$$

and

$$\left\langle J_-'(t), \gamma_-'(t) \right\rangle = \langle -J'(-t), -\gamma'(-t) \rangle = \langle J'(-t), \gamma'(-t) \rangle .$$

From the above computation follows also that $\mathcal{C}_-(Z_-(t)) = \mathcal{C}(Z(t))$. So

$$\begin{aligned}Ind_{\gamma_-}(Z_-) &= \int_{-T}^{0} \{|Z_-'(t)|^2 - \langle \mathcal{C}_-(Z_-(t)), Z_-(t) \rangle - \left\langle Y_-(\gamma_-'(t)), Z_-(t) \right\rangle^2\} dt \\ &= \int_{-T}^{0} \{|Z'(-t)|^2 - \langle \mathcal{C}(Z(-t)), Z(-t) \rangle - \langle Y(\gamma'(-t)), Z(-t) \rangle^2\} dt \\ &= \int_{0}^{T} \{|Z'(t)|^2 - \langle \mathcal{C}(Z(t)), Z(t) \rangle - \langle Y(\gamma'(t)), Z(t) \rangle^2\} dt = Ind_\gamma(Z).\end{aligned}$$

**Corollary 2.2.4.** *Lemma 2.2.1 holds when we replace equation* (2.2.3) *with* $Z(0) = J^\perp(0)$ *and* $Z(T) = J^\perp(T) = 0$.

When $Z(T) = J^\perp(T) = 0$, we can consider $Z_-$ and $J_-^\perp$, this will satisfy the hypothesis of lemma 2.2.1, so we have:

$$Ind_\gamma(J^\perp) = Ind_{\gamma_-}(J_-^\perp) \le Ind_{\gamma_-}(Z_-) = Ind_\gamma(Z).$$

**Lemma 2.2.5.** *If $\gamma(t_0)$ is conjugate to $\gamma(0)$ along $\gamma$, for some $t_0 < T$, then there is a vector field $Z \in \Lambda_0^\perp$ with $Ind_\gamma(Z) < 0$.*

Let $J$ a Jacobi field along $\gamma$ with $J(0) = J(t_0) = 0$, and $\tilde{J}$ be $J^\perp$ for $t \in [0, t_0]$ and 0 for $t \in [t_0, T]$. Then $Ind_\gamma(\tilde{J}) = 0$. We can use the Index Lemma and corollary 2.2.4 to show that cutting the corner at $t_0$ by replacing $\tilde{J}\Big|_{[t_0-\epsilon, t_0+\epsilon]}$ by a Jacobi field with the same endpoints decreases the value of the Index form. So this new vector field has $Ind_\gamma(Z) < 0$.

**Lemma 2.2.6.** *If $\gamma(T)$ is the first conjugate point to $\gamma(0)$ along $\gamma$ and $Z \in \Lambda_0^\perp$, then $Ind_\gamma(Z) \geq 0$, with equality if and only if $Z$ vanishes or $Z$ is the perpendicular component of a Jacobi field.*

This is an extension of lemma 2.2.1, and it is proved by a similar argument.We will follow the proof of lemma 2.2.1, using the same notation. Suppose $\gamma(T)$ is a first conjugate point and has multiplicity $k$, then we can find a basis $\{v_0, \ldots, v_{n-1}\}$ of $T_{\gamma(0)}M$, with $v_0 = \gamma'$ such that $J_{v_i}(T)$ are parallel to $\gamma'(T)$ for $i = 1, \ldots, k$ and are not parallel to $\gamma'(T)$ for $i = k+1, \ldots, n-1$. Then $J_0(t) = \gamma'(t)$ and $J_i(t) = J_{v_i}(t)$, $i = 1, \ldots, n-1$ form a basis for $T_{\gamma(t)}M$ for all $t \in (0, T)$.

If $Z$ is a vector field in $\Lambda_0^\perp$, we can write, for $t \in (0, T)$

$$Z(t) = \sum_{i=0}^{n-1} f_i(t) J_i(t)$$

where $f_0, \ldots, f_{n-1}$ are smooth functions. We can extend these functions to $t = 0$ as before. To extend $f_i$ to $t = T$ we can write $J_i = (t - T)A_i + J_i(T)$, for $i = 1, \ldots, k$. Then $A_i$ are smooth vector fields with $A_i(T) = J_i'(T)$ that is orthogonal to $\gamma'$ since $\langle J', \gamma' \rangle = 0$ for all Jacobi fields. Then $\{\gamma', A_1, \ldots, A_k, J_{k+1}, \ldots, J_{n-1}\}$ are a basis for all $t \in (0, T]$, and

$$Z(t) = g_0 \gamma' + \sum_{i=1}^{k} g_i(t) A_i(t) + \sum_{i=k+1}^{n-1} g_i(t) J_i(t).$$

It follows that for $t \in (0,T)$, $g_i = f_i$ for $i > k$, $g_i(t) = (t-T)f_i(t)$ for $0 < i \leq k$, and $g_0 = f_0 + \sum_{i=1}^{k} f_i(t) \langle J_i(T), \gamma' \rangle$. Since $Z(T) = 0$, $g_i(T) = 0$ and $f_i$ extends smoothly to $t = T$.

Following the proof of lemma 2.2.1, we get from (2.2.5)

$$\begin{aligned} Ind_\gamma(Z) &= \int_0^T ||W^\perp|| dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \\ &= \int_0^T ||W^\perp|| dt \geq 0 \end{aligned}$$

with equality iff $W^\perp$ vanishes everywhere. That is when $f_i$ constant for $i > 0$. So $Z = f_1\gamma' + J$ for some Jacobi field $J$, and since $Z$ is orthogonal to $\gamma'$ this implies that $Z = J^\perp$.

**Corollary 2.2.7.** *$Ind_\gamma(Z)$ restricted to $\Lambda_0^\perp$ is positive definite if and only if $\gamma$ has no conjugate points.*

When $\gamma$ has no conjugate points, it follows directly from corollary 2.2.2 that $Ind_\gamma$ is positive definite. In the case that the endpoints are conjugate to each other $Ind_\gamma$ has nontrivial kernel, as can be seen from equation 2.2.2. If $\gamma$ has conjugate points, we saw on lemma 2.2.5 that there is a vector field in $\Lambda_0^\perp$ with $Ind_\gamma < 0$, therefore it is not positive definite.

We will be interested in the dependence of the index form on its parameters. For this consider a continuous (possibly constant) family of vectors $\xi(s) \in S_xM$ and the correspondent family of magnetic geodesics $\gamma_s(t) = exp_x^\mu(t\xi(s))$. Let $T_s$, the length of each geodesic, be continuous on $s$. Let $\Lambda_s$ denote the vector space of piecewise smooth vector fields $Z_s$ along $\gamma_s$, perpendicular to $\gamma_s'$ and such that $Z(0) = Z(T_s) = 0$.

Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis with $v_1 = \gamma_0'(0)$, and extend it to a continuous family

$\{v_1(s), \ldots, v_n(s)\}$ of orthonormal basis for each $s$ with $v_1(s) = \xi(s)$. This can be done by defining

$$v_i(s) = \rho_s(v_i)$$

where $\rho_s$ is a rotation of $S^n$ with $\rho_s(\xi(0)) = \xi(s)$. We extend this for all $t$ by requiring that

$$\nabla_{\gamma_s'} e_i = Y(e_i) \tag{2.2.6}$$

along each magnetic geodesic. As in 2.1.4 this gives an orthonormal basis for each point.

Using this basis, we can extend any vector field $Z = \sum_2^n a_i(t)e_i(t)$ in $\Lambda_0$ to a vector field over the family of geodesics by

$$Z(s,t) = \sum_2^n a_i(t\frac{T_0}{T_s})e_i(s,t),$$

that belongs to $\Lambda_s$ when restricted to each $\gamma_s$. We will denote the set of such vector fields by $\Lambda_{[0,1]}$.

Since everything depends continuously on $s$, so does

$$Ind_{\gamma_s}(Z) = \int_0^{T_s} \{|Z'|^2 - \langle \mathcal{C}(Z), Z\rangle - \langle Y(\gamma_s'), Z\rangle^2\}dt.$$

We will be mostly interested on whether the Index form is positive definite. For such a family of curves, the fact that the Index form is positive definite (and therefore the non existence of conjugate points) depends continuously on $s$ in the following sense. If the index form is positive definite for some $s_0$, and has a negative value for some $s_1$ there must be some $s \in (s_0, s_1)$ where it has non-trivial kernel. Moreover, the first such $s$ will be when $\gamma_s$ has conjugate endpoints and no conjugate points in the interior.

## 2.3 Simple metrics and boundary data

Consider a manifold $M_1$ such that $M \subset int(M_1)$, extend $g$ and $\Omega$ smoothly. We say that $M$ is *magnetic convex* at $x \in \partial M$ if there is a neighborhood $U$ of $x$ in $M_1$ such that all unit speed magnetic geodesics in $U$, passing through $x$ and tangent to $\partial M$ at $x$, lie in $U \setminus int(M)$. It is not hard to see that this definition depend neither on the choice of $M_1$ nor on the way we extend $g$ and $\Omega$ to $M_1$.

Let II stand for the second fundamental form of $\partial M$ and $\nu(x)$ for the inward pointing normal. Then if $M$ is magnetic convex

$$\mathrm{II}(x,\xi) \geq \langle Y_x(\xi), \nu(x)\rangle$$

for all $(x,\xi) \in TM$. [DPSU, Lemma A.6].

We say that $\partial M$ is *strictly magnetic convex* if

$$\mathrm{II}(x,\xi) > \langle Y_x(\xi), \nu(x)\rangle$$

for all $(x,\xi) \in TM$.

This condition implies that the tangent geodesics do not intersect $M$ except for $x$, as shown in [DPSU, Lemma A.6].

We say that $M$ is *simple*(w.r.t.$(g,\Omega)$) if $\partial M$ is strictly magnetic convex and the magnetic exponential map $exp^{\mu}_x : (exp^{\mu}_x)^{-1}(M) \to M$ is a diffeomorphism for every $x \in M$.

For $(x,\xi) \in SM$, let $\gamma_\xi : [l^-(x,\xi), l(x,\xi)] \to M$ be the magnetic geodesic such that $\gamma_\xi(0) = x$, $\gamma'_\xi(0) = \xi$, and $\gamma_\xi(l^-(x,\xi)), \gamma_\xi(l(x,\xi)) \in \partial M$. Where $l-$ and $l$ can take the values $\pm\infty$ if the magnetic geodesic $\gamma_\xi$ stays in the interior of $M$ for all time in the corresponding direction.

Let $\partial_+ SM$ and $\partial_- SM$ denote the bundles of inward and outward unit vectors over $\partial M$:

$$\partial_+ SM = \{(x, \xi) \in SM : x \in \partial M, \langle \xi, \nu(x) \rangle \geq 0\},$$

$$\partial_- SM = \{(x, \xi) \in SM : x \in \partial M, \langle \xi, \nu(x) \rangle \leq 0\},$$

where $\nu$ is the inward unit normal to $\partial M$. Note that $\partial(SM) = \partial_+ SM \cup \partial_- SM$ and $\partial_+ SM \cap \partial_- SM = S(\partial M)$.

In the case that $M$ is simple, is clear that the functions $l^-(x, \xi)$ and $l(x, \xi)$ are continuous and, on using the implicit function theorem, they are easily seen to be smooth near a point $(x, \xi)$ such that the magnetic geodesic $\gamma_\xi(t)$ meets $\partial M$ transversely at $t = l^-(x, \xi)$ and $t = l(x, \xi)$ respectively. By the definition of strict magnetic convexity, $\gamma_\xi(t)$ meets $\partial M$ transversely for all $(x, \xi) \in SM \setminus S(\partial M)$. In fact, these functions are smooth everywhere, as was shown by Dairbekov, Paternain, Stefanov and Uhlmann in the following lemma.

**Lemma 2.3.1.** *[DPSU, Lemma 2.3] For a simple magnetic system, the function $\mathbb{L} : \partial(SM) \to \mathbb{R}$, defined by*

$$\mathbb{L}(x, \xi) := \begin{cases} l(x, \xi) & if (x, \xi) \in \partial_+ SM \\ l^-(x, \xi) & if (x, \xi) \in \partial_- SM \end{cases}$$

*is smooth. In Particular, $l : \partial_+ SM \to \mathbb{R}$ is smooth. The ratio*

$$\frac{\mathbb{L}(x, \xi)}{\langle \nu(x), \xi \rangle}$$

*is uniformly bounded on $\partial(SM) \setminus S(\partial M)$.*

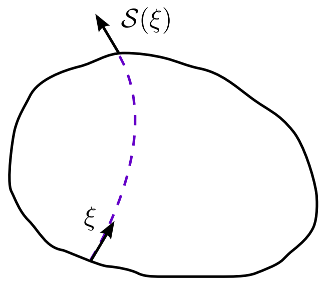



Figure 2.1: The Scattering relation.

This lemma was proved as stated, for simple magnetic systems, but the proof is a local argument using only the strong magnetic convexity of the region.

The *scattering relation* $\mathcal{S} : \partial_+SM \to \partial_-SM$ of a magnetic system $(M, g, \Omega)$ is defined as follows:

$$\mathcal{S}(x, \xi) = (\gamma_\xi(l(x, \xi)), \gamma'_\xi(l(x, \xi)))$$

when the value $l(x, \xi)$ is finite, otherwise it is not defined.

The *restricted scattering relation* $\mathtt{s} : \partial_+SM \to M$ is defined to be the postcomposition of the scattering relation with the natural projection of $\partial_-SM$ to $M$, i.e.,

$$\mathtt{s}(x, \xi) = \gamma_\xi(l(x, \xi))$$

when properly defined.

We are interested only in simple domains, and domains that have the same scattering data as a simple domain, so we will assume that $l^-$ and $l$ are finite and smooth on $\partial(SM)$. Moreover, it follows from the smoothness of $l$ and their definitions that both $\mathcal{S}$ and $\mathtt{s}$ are smooth everywhere on $\partial_+SM$.

Let $\widehat{M}$ be a compact simple domain with respect to $\widehat{\Omega}$ in the interior of a manifold $(\widehat{M_1}, \hat{g})$. Let $M$ be a compact domain in the interior of a manifold $(M_1, g)$ with $\Omega$. Related in such way that $\hat{g} = g$ and $\widehat{\Omega} = \Omega$ on $\widehat{M_1} \setminus \widehat{M} = M_1 \setminus M$, and the (restricted) scattering relations $\widehat{\mathcal{S}}, \mathcal{S}$ agree on $\partial M (= \partial \widehat{M})$. To be able to compare the magnetic flows in $M$ and $\widehat{M}$ we would like to say that $M$ is also simple, without having to impose it as a condition. The purpose of this section is to prove the following theorem.

**Theorem 2.3.2.** *Given $M$ and $\widehat{M}$ as above, then $M$ is also simple.*

To prove that the magnetic exponential is a diffeomorphism we need to show that it has no conjugate points. For this we need the following lemma.

**Lemma 2.3.3.** *If there are conjugate points in $M$, then there is a pair of points in $\partial M$ conjugate to each other.*

Suppose there is a point in the interior conjugate to $x \in \partial M$ along a geodesic $\gamma_\xi$. Let $\tau : [0,1] \to S_x M$ be a curve joining $S_x \partial M$ to $\xi$, and consider the family of magnetic geodesics $\gamma_s = \exp_x^\mu(t\tau(s))$. These geodesics exit $M$ at time $l(x, \gamma_s')$, that by the simplicity of $\widehat{M}$ is a continuous function of $s$. Close enough to $x$ the magnetic exponential is a diffeomorphism, and by lemma 2.3.1 there is a $C > 0$ such that $l(x,\eta) \leq C\langle \nu(x), \eta \rangle$ for all $\eta \in S_x M$. This implies that for $s$ small enough, the magnetic geodesic from $x$ to $\mathtt{s}(\tau(s))$ is short, and stays inside a neighborhood where the magnetic exponential is a diffeomorphism. Therefore, it has no conjugate points, and the index form is positive definite close to $x$. On the other hand, there is a perpendicular vector field along $\gamma_\xi$ for which the index form is negative. Then $Ind_{\gamma_s}$ is positive definite for $s = 0$ and not for $s = 1$. Let $s_0$ be the smallest $s$ for which $Ind_{\gamma_s}$ has non trivial kernel. Then, by the results on the previous section, $\mathtt{s}(\tau(s_0))$ is conjugate to $x$ along the magnetic geodesic $\gamma_s$ that joins them.

If there are points conjugate to each other along a magnetic geodesic $\gamma_\xi$, and both lie in the interior of $M$, there must be a point conjugate to $\gamma_\xi(0)$ along this magnetic geodesic. Therefore reducing the problem to the case above. This can be proved by a similar argument using the family of geodesics $\gamma_\xi|_{[0,sT]}$.

*Proof of Theorem 2.3.2.* It is easy to see from the definition that the domain $M$ has to be strictly magnetic convex, since the metrics and magnetic flows agree outside $M$.

To prove that the magnetic exponential map is a diffeomorphism form $(exp_x^\mu)^{-1}(M)$ to $M$ we need to show that it has no conjugate points, i.e. there are no points in $M$ that are conjugate to each other along a magnetic geodesic. For this purpose assume such points exist, then by Lemma 2.3.3 there are points $x, y \in \partial M$ conjugate to each other along a magnetic geodesic $\gamma_\xi$, where $\gamma_\xi(0) = x$ and $\gamma_\xi(t_0) = y$ for some $t_0 > 0$.

Let $J$ be a magnetic Jacobi field along $\gamma_\xi$ that vanishes at 0, and $f(s,t)$ a variation through magnetic geodesics with $f(0,t) = \gamma_\xi$ and $J$ as a variational field. We can use $f(s,t) = \gamma_s(t) = exp_x^\mu(t\xi(s))$ where $\xi : (-\epsilon, \epsilon) \to S_xM$ is a curve with $\xi(0) = \xi$, $\xi'(0) = J'(0)$. $f$ is well defined in $M$ for $(s,t) \in (-\epsilon, \epsilon) \times [0, T_s]$ where $T_s = l(x, \xi(s))$. Consider $c(s) = f(s, T_s) \in \partial M$ the curve of the exit points in $\partial M$. Then

$$\begin{aligned} \frac{dc}{ds}(0) &= \frac{df}{ds}(0, T_s) + \frac{df}{dt}\frac{dT_s}{ds}(0, T_s) \\ &= J(T_s) + \frac{dT_s}{ds}(0, T_s)\gamma_\xi'. \end{aligned}$$

If $\gamma_\xi(l(x,\xi))$ is conjugate to $\gamma_\xi(0)$ along $\gamma_\xi$, there is a Jacobi field $J$ that is 0 at $t = 0$ and parallel to $\gamma_\xi$ at $T_s$, then $\frac{dc}{ds}(0)$ is parallel to $\gamma_\xi'$. On the other hand, if $\frac{dc}{ds}(0)$ is parallel to $\gamma_\xi'$ for any Jacobi field with $J(0) = 0$ then $J(T_s)$ is parallel to $\gamma_\xi'$. Therefore $\gamma_\xi(T_s)$ is conjugate to $\gamma_\xi(0)$

along $\gamma_\xi$.

Note that we can write $c(s) = \mathtt{s}(\xi(s))$, that depends only on the scattering data, so the scattering relation detects conjugate points in the boundary. Since there are no conjugate points in the boundary of $\widehat{M}$, there can be none in $M$. Therefore the magnetic exponential is a local diffeomorphism.

We will now see that $exp_x^\mu$ is a global diffeomorphism from $(exp_x^\mu)^{-1}(M)$ to $M$. To see that it is surjective let $x \in \partial M$, and $y$ any point in $M$. Let $c : [0,1] \to M$ be a path from $x$ to $y$, and consider the set $A \subset [0,1]$ of points such that $c(s)$ is in the image of $exp_x^\mu$. This set is open, since $exp_x^\mu$ is a local diffeomorphism. To see that it is closed, choose a sequence $s_n \in A$ converging to $s_0$. Then $c(s_n) = exp_x^\mu(t(s_n)\xi(s_n))$, and there is a subsequence such that $t(s_n)$ and $\xi(s_n)$ converge to $t_0$ and $\xi_0$ respectively. If $t_0\xi_0 \notin (exp_x^\mu)^{-1}(M)$, there must be a first $t_1 < t_0$ such that $exp_x^\mu(t_1\xi_0) \in \partial M$. Then $exp_x^\mu(t\xi_0)$ must be tangent to $\partial M$ and inside $M$ for $t < t_1$, which contradicts the magnetic convexity of $M$. Then, $A$ is both open and closed, therefore $A = [0,1]$ and $y$ is in the image of $exp_x^\mu$.

To see that $exp_x^\mu$ is injective for $x \in \partial M$, note that it is a covering map. The point $x$ has only one preimage, since by the simplicity of $\widehat{M}$ there are no magnetic geodesics form $x$ to $x$. Therefore $exp_x^\mu$ is a covering map of degree 1.

To prove this for $x \notin \partial M$, we need to see that there are no trapped magnetic geodesics, that is, that there are no magnetic geodesics that stay inside $M$ for an infinite time. Note that, since any magnetic geodesic that enters the region at $\xi$ has to exit at $\mathtt{s}(\xi)$, it is enough to see that all geodesics enter the region at a finite time. Let $\gamma$ be a magnetic geodesic. We know that we can reach the point $\gamma(0)$ from the boundary, so there is a variation through magnetic geodesics $\gamma_s(t)$ with $\gamma_0 = \gamma$, $\gamma_s(0) = \gamma(0)$ for all $s \in [0,1]$, and $\gamma_1(t_1) \in \partial M$ for some $t_1 < 0$. If $\gamma_{s_0}$ intersects $\partial M$, by the magnetic convexity of $M$ it has to be a transverse intersection, therefore intersecting $\partial M$

is an open condition on $[0,1]$. It is also a closed condition, by continuity of the geodesic flow and compactness of $\partial M$. Therefore, since $\gamma_1$ intersects $\partial M$, so does $\gamma_s$ for all $s$, and $\gamma$ is not trapped.

Now we see that $exp^\mu_x$ is a global diffeomorphism from $(exp^\mu_x)^{-1}(M)$ to $M$ for $x \notin \partial M$. Since $exp^\mu_x$ is injective for $x \in \partial M$, and all geodesics come from some point $x$ in $\partial M$, magnetic geodesics in $M$ have no self intersections. In particular,any $x \in M$ has only one preimage under $exp^\mu_x$. We can then follow the same argument as for $x \in \partial M$ to show that $exp^\mu_x$ is a global diffeomorphism from $(exp^\mu_x)^{-1}(M)$ to $M$, for all $x \in M$. □

## 2.4 Rigidity for Surfaces

Consider a magnetic field on a surface $\widehat{M}$ all of whose orbits are closed, and consider a magnetically simple region $R$ on it. We want to prove that there is no way of changing the metric and magnetic field in this region in such a way that all orbits are still closed.

In the previous section we saw that such a region is magnetically rigid, therefore it can't be changed on the region preserving the scattering data. Here we will look at the general behavior of such a magnetic flow to ensure that there are no other metrics with all its orbits closed. We want to rule out the case where a magnetic geodesic that passes through the region, after coming out at a different spot and following the corresponding orbit, goes back into the region and exits at the exit point and direction of the original first magnetic geodesic, therefore forming a closed orbit out of two (or more) segments of the original orbits.

To show this, assume that we have such a magnetic field. Assume, moreover, that the region $R$ is small enough compared to $\widehat{M}$ that every magnetic geodesic passes through $R$ at most once.

**Theorem 2.4.1.** *Let $M$ and $\widehat{M}$ be compact surfaces with magnetic fields, all of whose magnetic geodesics are closed. Let $R \subset \widehat{M}$ be a strictly magnetically convex region, such that every magnetic* *geodesic passes through $R$ at most once.*

*If the metric and magnetic fields of $M$ and $\widehat{M}$ agree outside $R$, then they have the same scattering data.*

Consider the unit tangent bundle $SM$, and the magnetic geodesic vector field $G$ i.e. the vector field that generates the magnetic flow on the unit tangent bundle. For the sake of simplicity of the exposition we will assume first that $SM$ is oriented. Then $SM$ is a compact orientable 3-dimensional manifold, and $G$ is a smooth vector field that foliates $SM$ by circles. By a theorem of Epstein [Ep], this foliation is $C^\infty$ diffeomorphic to a Seifert fibration. In particular, any orbit has a neighborhood diffeomorphic to a standard fibered torus.

Note that to each orbit on $SM$ we can uniquely associate a magnetic geodesic, by projecting the orbit back to $M$. We will use this correspondence freely. As a Seifert fibration, the base $B$ or space of orbits of $SM$ is a 2-dimensional orbifold.

Let $SR$ be the subset of $SM$ that corresponds to the region $R$, and $S\partial R$ the subset of $SM$ corresponding to vectors tangent to the boundary of $R$. The orbit of a point in $S\partial R$ corresponds to a magnetic geodesic that is tangent to $R$, and since $R$ is strictly magnetically convex, it is tangent only at one point. This magnetic geodesic corresponds exactly to one in $\widehat{M}$, and therefore stays away from $R$ thereafter. This means that each orbit contains at most one point of $S\partial R$, so the set of orbits passing through it forms a 1-dimensional submanifold on $B$, we will denote it by $R_0$.

Let $m : B \to \mathbb{N}$ be a function that counts the number of times the orbit pases through $SR$ in a common period. For regular orbits this is the number of times it passes through $SR$. For a singular orbit, a neighborhood is diffeomorphic to an $(a, b)$ torus, that is a torus obtained by gluing two faces of a cylinder with a rotation by an angle of $2\pi b/a$. In this case the common period is $a$ times the period of the singular orbit. Therefore, $m$ will be $a$ times the number of times the orbit passes through $SR$. Since the other orbits in the neighborhood will be completed when the singular orbit is travelled $a$ times, $m$ will be, in general, continuous at such points. In fact, if a magnetic geodesic intersects $\partial R$ transversally (or not at all), we can chose a neighborhood small enough that all intersections are transverse, and therefore $m$ will be constant. The only discontinuities occur when a magnetic geodesic is tangent to $\partial R$, that is exactly at the orbits in $R_0$.

We will now look at these discontinuities. If a magnetic geodesic $\gamma$ corresponds to an orbit $b$ in $R_0$, it is tangent to $R$ at a point $\gamma(0)$. It agrees with a magnetic geodesic in $\widehat{M}$, so it never reaches $R$ again and $m(b) = 1$. By the magnetic convexity of $M$, there is a $\delta$ small enough that each magnetic geodesic in the ball $B_\delta(\gamma(0))$ goes through $R$ at most once. Moreover, since the magnetic geodesic is compact, we can find an $\varepsilon$ neighborhood $N_\varepsilon(\gamma)$ such that it only intersects $R$ close to $\gamma(0)$, i.e. $N_\varepsilon(\gamma) \cap R = B_\delta(\gamma(0)) \cap R$.

If the orbit $b$ corresponding to $\gamma$ is regular, orbits in a small enough neighborhood will correspond to nearby magnetic geodesics, completely contained in $N_\varepsilon(\gamma)$. These magnetic geodesis will then intersect $R$ only inside $B_\delta(\gamma(0))$, and therefore at most once. Thus, these orbits will have $m$ equal to 0 or 1.

We have that $B$ is a 2-dimensional orbifold, and $R_0$ is a continuous curve on it. The function $m$ is constant on each connected component of $B \setminus R_0$, and takes values 0 or 1. On $R_0$, the function $m = 1$, except maybe at isolated singular orbits. Since on regular orbits $m = 1$, we can say that magnetic geodesics go through $R$ at most once, except maybe at a finite number of singular ones. Any singular magnetic geodesics that is tangent must go through $R$ only once. If a singular magnetic geodesics cuts $\partial R$ transversely, we know that $m = 1$. But the corresponding orbit pases through $SR$ exactly $m/a$ times, so $a = m = 1$ and the geodesic is not singular.

In the case where $SM$ is non orientable, consider instead its orientable double cover $\widetilde{SM}$, and the associated vector field $\widetilde{G}$. Then $\widetilde{SM}$ is a compact orientable 3-dimensional manifold, and $\widetilde{G}$ is a smooth vector field that foliates $\widetilde{SM}$ by circles. We can follow the same arguments with a few modifications.

The correspondence between orbit on $\widetilde{SM}$ and magnetic geodesics is not a $1-1$ correspondence, a magnetic geodesic lifts either to an orbit that covers it twice, or two disjoint orbits. Nonetheless, will use this correspondence freely, keeping in mind this possible duplicity.

Let $\widetilde{SR}$ be the subset of $\widetilde{SM}$ that corresponds to the region $R$, and $\widetilde{S\partial R}$ the subset of $\widetilde{SM}$ corresponding to vectors tangent to the boundary of $R$. Let $\widetilde{B}$ be space of orbits of $\widetilde{SM}$ and $\widetilde{R_0}$ the set of orbits passing through $\widetilde{S\partial R}$. The counting function $m : \widetilde{B} \to \mathbb{N}$ can then take value 2, since an orbit that covers a magnetic geodesic twice will pas through $\widetilde{SR}$ twice. In fact, when $m(b) \neq 0$, it will be 1 if the magnetic geodesic corresponds to two disjoint orbits, and 2 when it corresponds to an orbit that covers it twice.

Since on regular orbits $m = 2$ only on orbits that cover a magnetic geodesic twice, we can say that magnetic geodesics go through $R$ at most once, except maybe for a finite number of singular ones. Any singular one that is tangent must go through $R$ only once, by assumption. If a singular orbit cuts $\partial R$ transversely, we know that $m$ is at most 2. But the orbit pases through $\widetilde{SR}$ $m/a$ times, so if it is singular $a = m = 2$ and the geodesic goes through $R$ only once.

Every magnetic geodesic goes through $R$ at most once, and outside $R$ they agree with the magnetic geodesics from $\widehat{M}$. For the magnetic geodesics to close, they have to exit $R$ in the same place and direction, therefore preserving the scattering data.

If the region $R$ is simple, we can use this result together with theorem 2.3.2 to get rigidity. For surfaces of constant curvature is easy to see that any circular disk that is strictly smaller than one of the orbit circles is a simple domain.

**Theorem 2.4.2.** *Let $M$ be a surface of constant curvature, and $k > 0$ big enough such that any pair of circles of curvature $k$ intersect at most twice. Let $R$ be a compact region contained in the interior of a circle of curvature $k$. Then the region $R$ can't be perturbed while keeping all the circles of curvature $k$ closed.*

Consider $M$ with the constant magnetic field that has circles of curvature $k$ as magnetic geodesics. If $R$ is contained in the interior of a circle $c$ of curvature $k$, then by compactness there is a disk $D$ such that $R \subset D$ and $D$ is contained in the interior of $c$. This disk $D$ is simple, and we can consider any perturbation $\widetilde{R}$ of $R$ as a perturbation $\widetilde{D}$ of $D$. Since any circle of curvature $k$ intersects $c$ at most twice, it will go through its interior at most once. Therefore any magnetic geodesic will go through $D$ at most once. We can then use theorem 2.4.1 to show that $D$ and $\widetilde{D}$ have the same scattering data. Since $D$ is simple, and they have the same scattering data, by theorem 2.3.2 $\widetilde{D}$ is also simple. But in [DPSU, Theorem 7.1] N. Dairbekov, P. Paternain, P. Stefanov and G. Uhlmann proved that two 2-dimensional simple magnetic systems with the same scattering data are gauge equivalent.

If $M$ is not compact, consider instead of $M$ a compact quotient that contains all the magnetic geodesics that pass through $D$. This can be achieved since all this magnetic geodesics are inside a disk of radius $4r$, where $r$ is the radius of a circle of curvature $k$.

# Chapter 3

# Blocking: new examples and properties of products

## 3.1 Security and Midpoint Security

Throughout this chapter geodesics are assumed to have positive finite length and to be parametrized by $[0,1]$ proportional to arclength. The length 0 case, when the geodesic is a point, will not be considered a geodesic and when needed will be refered to as a *constant path*.

If $\gamma : [0,1] \to M$ is a geodesic, the *endpoints* of $\gamma$ are its *initial point* $\gamma(0)$ and *final point* $\gamma(1)$. The points $\gamma(t)$ with $t \in (0,1)$ are *interior points* of $\gamma$. We will say that $\gamma$ *passes through* a point $x \in M$ if $x$ is an interior point of $\gamma$.

A *configuration* in $M$ is an ordered pair of points in $M$, these points may coincide. For a configuration $(x,y)$ in $M$ we say that a geodesic $\gamma$ *joins* $x$ to $y$ if it has initial point $x$ and final point $y$. We will denote by $G(x,y)$ the set of all geodesics joining $x$ to $y$.

A set $B$ is a *blocking set* for a collection of geodesics if every geodesic in the collection passes through a point in $B$.

**Definition 3.1.1.** A configuration $(x, y)$ is *secure* if the collection $G(x, y)$ of geodesics joining $x$ and $y$ has a finite blocking set. Otherwise the configuration is *insecure*.
A Riemannian manifold is *secure* if all configurations in it are secure.

Note that any geodesic in $G(x, y)$ has to contain a segment that joins $x$ to $y$ without passing through either endpoint, and this segment is also contained in $G(x, y)$. Blocking this segment will also block the original geodesic, so we can (and will) always choose the blocking set to be in $M \setminus \{x, y\}$. This definition is then equivalent to the one used in [BG], where they only consider geodesics that don't pass through the endpoints.

The first to consider the relation between security and product manifolds were E. Gutkin and V. Schröder. In [GS], while studying security of locally symmetric spaces they proved that if a product manifold is secure, then so are its factors. In [BG] K. Burns and E. Gutkin proved the following lemma and used it to give examples of totally insecure manifolds.

**Lemma 3.1.2.** *If a configuration in a product manifold is secure, then the projection to each factor is secure.*

*Moreover; if the configuration has blocking set $B$, then the projection of $B$ minus the endpoints is a blocking set for the projection in each factor.*

To study the converse of this lemma we introduce a related but stronger property. We will say that a set $B$ is a *midpoint blocking set* for a collection of geodesics if every geodesic in the collection has its midpoint in $B$.

**Definition 3.1.3.** A configuration $(x, y)$ is *midpoint secure* if the collection $G(x, y)$ of joining geodesics has a finite midpoint blocking set $B$.

A Riemannian manifold is *midpoint secure* if all configurations in it are midpoint secure.

Since the set of midpoints of geodesics joining $x$ to $y$ has to be contained in any midpoint blocking set, it will itself be the smallest midpoint blocking set for $G(x, y)$. So it follows that a configuration $(x, y)$ is midpoint secure if and only if the set of midpoints of the geodesics joining $x$ to $y$ is finite.

Unlike the security case, we will allow the endpoints $x$ and $y$ to be in $B$. This is actually necessary in some cases since, for instance, a simple closed geodesic travelled twice has midpoint equal to the endpoints.

Note that any midpoint secure configuration is also secure, with the same blocking set minus the endpoints if necessary. On the other hand, all previously known examples of secure configurations on Riemannian manifolds are midpoint secure. In particular, this can be seen for all configurations in locally symmetric spaces of euclidean type in [GS], where the blocking set given is actually midpoint blocking. This is also true for arithmetic polygonal billiards and translation surfaces (see [G2],[M1]).

One of the main goals of this area is to characterize certain manifolds by their blocking properties. The only known secure compact manifolds are flat, and it has been conjectured that these are in fact the only ones among closed smooth Riemannian manifolds. For more details see K. Burns and E. Gutkin [BG] and J.-F. Lafont and B. Schmidt [LS], where they give the conjecture together with some partial results in this direction.

Another case is that of compact rank one symmetric spaces (CROSS). These manifolds have the property that any pair of distinct points that are not at distance equal to the diameter are secure, with a blocking set consisting of only two points. It was conjectured in [LS] that the CROSSes are the only compact Riemannian manifolds with this property. Part of the evidence they provide is proving that all Blaschke manifolds have this property. They also conjecture that the only such manifold with the additional property that, for any point $x$, $G(x, x)$ can be blocked by a single point is the round sphere. This conjecture was later proved by B. Schmidt and J. Souto in [SS].

All the manifolds mentioned above satisfy the corresponding blocking properties if we replace security by midpoint security. This can be seen in the proofs of security of each case. Moreover, if all secure compact manifolds (without singularities) are flat as conjectured, then they also are all midpoint secure. Proving any of these conjectures using midpoint security would be a significant progress.

This also raises the question whether these conditions are actually equivalent. This is not true, as we will see in section 3.2 where we give an example of a surface with secure configurations that are not midpoint secure.

**Lemma 3.1.4.** *A configuration in a product manifold is midpoint secure if and only if the projection to each factor is midpoint secure.*

*Proof.* (The first half follows Burns and Gutkin's argument for Lemma 3.1.2, we will include it here for completeness.) Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points in the product manifold $M_1 \times M_2$. Any geodesic $\gamma$ joining $x$ and $y$ projects to geodesics $\gamma_i = \pi_i(\gamma)$ in $M_i$ joining $x_i$ and $y_i$, we will denote this by $\gamma = (\gamma_1, \gamma_2)$. Note that one $\gamma_i$ might be a constant path, but the argument works regardless.

Suppose that the configuration $(x, y)$ is midpoint secure, and let $B$ be its midpoint blocking set. Let $B_1 = \pi_1(B)$ be the projection of this set to $M_1$, we claim that this is a blocking set for $(x_1, y_1)$. To see this take any geodesic $\sigma$ in $M_2$ from $x_2$ to $y_2$. For any geodesic $\gamma_1 \in G(x_1, y_1)$ joining $x_1$ and $y_1$, the geodesic $(\gamma_1, \sigma)$ joins $x$ and $y$ in $M_1 \times M_2$ and therefore is midpoint blocked by $B$. This means that $(\gamma_1(1/2), \sigma(1/2)) \in B$, so $\gamma_1(1/2) \in B_1$, and since $\gamma_1$ is arbitrary this proves that the configuration $(x_1, y_1)$ is midpoint secure in $M_1$. Reversing the roles of $M_1$ and $M_2$ we see that the configuration $(x_2, y_2)$ is midpoint secure in $M_2$.

To prove the converse let us assume that $(x_1, y_1)$ and $(x_2, y_2)$ are midpoint secure in $M_1$ and $M_2$ respectively, and let $B_1$ and $B_2$ be the corresponding blocking sets. If $x_i = y_i$ add this point to $B_i$, so that the constant path joining them is also midpoint blocked by $B_i$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in the product manifold $M_1 \times M_2$, we will see that the set $B = B_1 \times B_2$ is a midpoint blocking set for $G(x, y)$. For this take any $\gamma \in G(x, y)$ and write it in the form $\gamma = (\gamma_1, \gamma_2)$ as above. Since $\gamma_i$ joins $x_i$ and $y_i$, it is midpoint blocked by $B_i$, that is, $\gamma_i(1/2) \in B_i$ and therefore $(\gamma_1(1/2), \gamma_2(1/2)) \in B_1 \times B_2 = B$, completing the proof. □

Theorem 4 immediately follows from this Lemma.

Regarding the security on product manifolds this result says that the product of midpoint secure configurations (or manifolds) is secure. But what if one, or both, is not midpoint secure? We will analyze the case when one of the factors is a round $S^2$ to show that in certain cases the condition on the midpoints is needed.

**Proposition 3.1.5.** *Given a Riemannian manifold $M$ the following statements are equivalent:*

1. *A configuration in the product manifold $M \times S^2$ is secure if and only if the projection to each factor is secure.*

2. *All secure configurations in $M$ are midpoint secure.*

*Proof.* Assume (1) holds and let $(x, y)$ be a secure configuration on M. Choose a point $p \in S^2$ in the 2-sphere, and let $q$ be its antipodal point. Then $(p, p)$ is (midpoint) secure with blocking set $\{p, q\}$ and by (1) we have that $((x, p), (y, p))$ is secure in $M \times S^2$. Let $B$ be the blocking set for $((x, p), (y, p))$, and let $B_M = \pi_1(B)$ and $B_S = \pi_2(B)$ be its projection to each factor. By Lemma 3.1.2 $B_M$ and $B_S$ are blocking sets for $(x, y)$ and $(p, p)$ respectively. Note that there are infinitely many simple geodesics in $G(p, p)$ and any pair of them only intersect in $q$, so any finite blocking set for $(p, p)$ has to contain $q$ and there must be a geodesic in $G(p, p)$ that is only blocked by $q$. Let $\sigma \in G(p, p)$ be one such simple great circle that is only blocked by $q$.

For any $\gamma \in G(x, y)$ the geodesic $(\gamma, \sigma)$ is in $G((x, p), (y, p))$ so it has to pass through a blocking point $b \in B$ at some time $t_0 \in (0, 1)$. By the definition of $B_S$ we have $\sigma(t_0) = \pi_2(b) \in B_S$, but $\sigma$ is only blocked by $q$ at time $1/2$. Therefore $\pi_2(b) = q$, $t_0 = 1/2$ and $\gamma(1/2) = \pi_1(b) \in B_M$, so $(x, y)$ is midpoint secure in M.

Conversely if (2) holds all secure configurations of $M$ and $S^2$ are midpoint secure, and (1) follows from Lemma 3.1.2 and Lemma 3.1.4. □

Note that this is a statement about $S^2$ as much as about the manifold $M$. The crucial property being that there is a configuration (in this case $(p, p)$) where a particular midpoint blocking point is needed to block $G(p, p)$. There are many manifolds that contain such configurations, for example $S^n$ with any metric of revolution, however these are not secure manifolds. Indeed, if a point $b$ in the midpoint blocking set $B$ is needed to block $G(x, y)$, the configuration $(x, b)$ would not be secure. If it were, with blocking set $C$, the set $(B \cup C) \setminus \{b\}$ would block $G(x,y)$, and it does not contain $b$.

## 3.2 Example

We will construct an example of a manifold that contains secure configurations that are not midpoint secure.

Let $C$ be a cylinder of length $l$ and radius 1, write it as a product of an interval $[0,l]$ and a circle $S^1$ of radius 1. Let $H_0$ and $H_l$ be a lower and an upper hemisphere, and attach them to $C$ by identifying the equators with the the curves $0 \times S^1$ and $l \times S^1$ respectively. We get $N = C \cup H_0 \cup H_l / \sim$ where $\sim$ is the identification above (see Figure 3.1).

First we need to understand some of the geodesics on $N$. Observe that any geodesic in the cylinder that reaches $l \times S^1$, forming an angle $\alpha$ with it, goes into $H_l$ where it is a half great circle that leaves $H_l$ again at its antipodal point, forming the same angle $\alpha$. From the point of view of the cylinder, any geodesic that leaves it through a point $(l,\theta)$ comes back at the point $(l,\theta+\pi)$ with the same angle.

Let $\hat{C} = [-l,-0] \times S^1$ be a reflection of $C$, where we denote 0 by $-0$ to distinguish the points in $\hat{C}$ from those in $C$. Let $T = C \cup \hat{C} / \sim$ be the torus formed by gluing both cylinders with a 1/2 twist, i.e. $(l,\theta) \sim (-l,\theta+\pi)$ and $(0,\theta) \sim (-0,\theta+\pi)$. Let $p : T \to C$ be the projection $p(t,\theta) = (|t|,\theta)$ for $t \neq -0,-l$. By the argument above, the restriction to $C$ of a geodesic in $N$ that goes through a hemisphere and back to $C$ is the projection under $p$ of a geodesic in $T$ that goes from $C$ to $\hat{C}$.

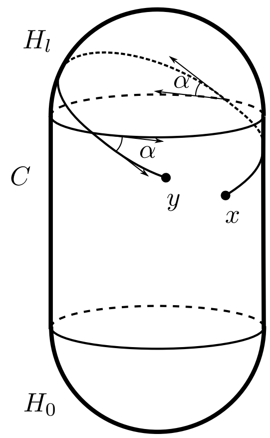



Figure 3.1: The surface $N$.

The geodesics in $N$ with endpoints in the interior of $C$ corresponds to the projections to $C$ of the geodesics in $T$, although the parametrizations usually do not agree.

**Lemma 3.2.1.** *Any pair of points in the interior of $C$ form a secure configuration in $N$.*

*Proof.* For any pair of points $x_1 = (t_1, \theta_1)$ and $x_2 = (t_2, \theta_2)$ in the interior of $C$ and any geodesic $\gamma$ between them in $N$, let $\tilde{\gamma}$ be the geodesic in $T$ that projects to $\gamma$ with $\tilde{\gamma}(0) = x_1$. By the arguments above the projection of $\tilde{\gamma}$ is $\gamma \cap C$ so any point that blocks $\tilde{\gamma}$ in $T$ will project to a point that blocks $\gamma$.

On the other hand, since $\tilde{\gamma}(1)$ projects to $\gamma(1) = x_2$ it is either $x_2$ or $-x_2 = (-t_2, \theta_2)$ and we can identify $G(x_1, x_2)$ with the union of the sets $G_T(x_1, x_2)$ and $G_T(x_1, -x_2)$ of geodesics in $T$. Since $T$ is a flat torus it is known that it is uniformly secure with security threshold 4 (see e.g. [G1]). Therefore the set $B$ given by the projection of the 8 blocking points of $G_T(x_1, \pm x_2)$ in $T$ will block $G(x_1, x_2)$. $\square$

**Lemma 3.2.2.** *Any pair of points in the interior of $C$ form a configuration that is not midpoint secure in $N$.*

*Proof.* For any pair of points $x_1 = (t_1, \theta_1)$ and $x_2 = (t_2, \theta_2)$ in the interior of $C$ let $\tilde{\gamma}_i$ be the geodesic in $T$ from $x_1$ to $-x_2 = (-t_2, \theta_2)$ that goes $i$ times around in the $S^1$ direction and doesn't cross $l \times S^1$, it has length $\tilde{l}_i = \sqrt{(t_1 + t_2)^2 + (2\pi i + \theta_2 + \pi - \theta_1)^2}$. Let $\gamma_i$ be the corresponding geodesic in $N$, it begins and ends in $C$ and goes through $H_0$ once so it has length $l_i = \tilde{l}_i + \pi$.

If $t_1 \neq t_2$ we can assume that $t_1 > t_2$. The time that $\gamma_i$ spends in $H_0$, when $\gamma_i$ is parametrized by $[0, 1]$, is an interval of length $\pi/l_i$ beginning at $(1 - \frac{\pi}{l_i})\frac{t_1}{t_1+t_2}$. Since $t_1/(t_1 + t_2) > 1/2$ and the length $l_i$ grows to infinity with $i$, for $i$ big enough $(1 - \frac{\pi}{l_i})\frac{t_1}{t_1+t_2} > \frac{1}{2}$ so $\gamma_i$ spends more than half the time before reaching $H_0$. The midpoint is then $\gamma_i(1/2) = p(\tilde{\gamma}_i(\tilde{l}_i/2l_i))$ which has $t$ coordinate $(t_1+t_2)\tilde{l}_i/2l_i = (t_1+t_2)(1/2-\pi/2l_i)$, clearly different for each $i$. If this configuration is midpoint secure all these points have to be in the blocking set, making it infinite.

In the case that $t_1 = t_2$ the midpoints $\gamma_i(1/2)$ are the midpoints of the restriction of $\gamma_i$ to $H_0$. We can see that the distance between $\gamma_i(1/2)$ and the equator depends directly on the angle that $\gamma_i$ makes with it. This angle gets smaller as $i$ increases, showing that the points $\gamma_i(1/2)$ are all distinct. □

**Remark 3.2.3.** This manifold is not secure, for example the configuration $((0, 0), (0, \pi))$ is insecure since there are infinitely many disjoint paths joining them in $H_0$. This leaves open whether all secure manifolds are also midpoint secure.

Also, as mentioned above, it is only a $C^1$ manifold. It would be interesting to find a smooth example.

From the discussion above we see that the statement (2) of Proposition 3.1.5 doesn't hold.

Therefore there are insecure configurations in $N \times S^2$ that project to secure configurations in both factors. As pointed out previously, this gives an explicit example of an insecure product of secure configurations.

For this particular manifold $N$ we can prove the stronger statement in Theorem 6 that can be restated as follows.

**Theorem 3.2.4.** *For any compact connected Riemannian manifold $M$ without boundary, any configuration in the product $N \times M$ that projects to a configuration in the interior of $C$ is insecure.*

*Proof.* Suppose not, then there is a secure configuration $(x_1, y_1), (x_2, y_2)$ where $x_1 = (t_1, \theta_1)$ and $x_2 = (t_2, \theta_2)$ are in the interior of $C$. Let $B$ be a blocking set for this configuration, and $B_M$, $B_N$ its projection to each factor. Let $\gamma_i$ be as before, since it never goes through $H_l$ it can't be blocked by a point in $H_l$. When $\gamma_i$ is restricted to $H_0$ the maximum distance between $\gamma_i$ and the equator tends to 0 as $i$ goes to $\infty$, therefore any point in $H_0$ can only block finitely many of the $\gamma_i$. By considering only big enough $i$'s we can asume that they are all blocked by points in $B_N \cap C$, let $b = (t, \theta)$ be one of these points.

By a similar analysis as in the proof of Lemma 3.2.2 (and using the same notation) we can see that $\gamma_i$ reaches $t \times S^1$ at most twice, at times $(1 - \frac{\pi}{l_i})\frac{t_1 - t}{t_1 + t_2}$ and $1 - (1 - \frac{\pi}{l_i})\frac{t_2 - t}{t_1 + t_2}$. Therefore $\gamma_i$ can be blocked by $b$ only at these times. When we vary $i$ these are all different times, and they form sequences that converge to $\frac{t_1 - t}{t_1 + t_2}$ and $1 - \frac{t_2 - t}{t_1 + t_2}$ as $i$ goes to $\infty$. Let $\sigma$ be a non-constant geodesic in $M$ joining $y_1$ and $y_2$, and let $s_1, \ldots s_n$ be all the times where $\sigma(s_i) \in B_M$. By considering all the points $b \in B_N \cap C$ simultaneously we see that the set of times when $\gamma_i(t) \in B_N$ is different for each $i$, and we can find an $i$ for which these times are all different from $s_1, \ldots s_n$. Then the geodesic $(\gamma_i, \sigma)$ is not blocked by $B$, giving a contradiction. ∎

## 3.3 Billiard Tables

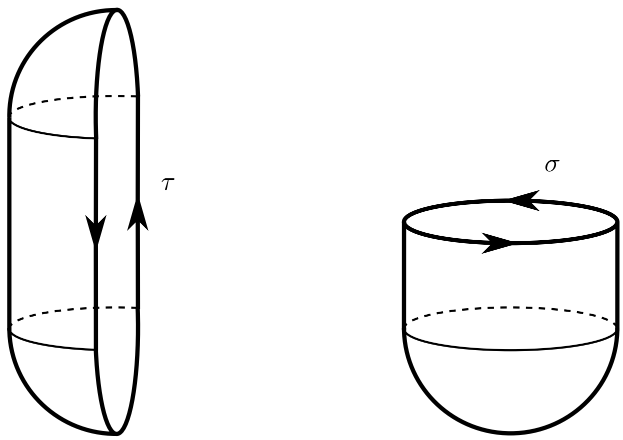
The concept of security originated in the study of billiards. In this subject a billiard table is a manifold with boundary, and the trajectories of billiard balls follow geodesics and bounce off the boundary according to the usual reflection laws. It is particularly related to the illumination problem that studies which points a light source illuminates, and which are shaded, when the boundary is considered as a perfect mirror.

Security, also called finite blocking property, has been studied mainly on flat tables, with smooth or polygonal boundaries. The examples of secure billiard tables are as limited as the examples of secure manifolds. For lattice polygonal billiards security is equivalent to being arithmetic (see [G1]), and therefore the only secure regular polygons are the equilateral triangle, square and hexagon. Moreover, it was recently proved by S. Tabachnikov in [Ta] that any planar billiard with smooth boundary is insecure.

Rational tables are secure if they are covered by a secure translation surface. For example, tables covered by a torus branched covering (like the ones presented in [Ch]) are secure (see [M1]). In [M2] T. Monteil studies security for translation surfaces, proving that security implies pure periodicity of the directional flow, and asks if this is equivalent to uniform midpoint security. Some of these ideas can be also seen in [Sc] and [HST].

In this respect, we can consider half of the surface $N$ (with boundary) as a billiard table, cutting it in either of the two natural ways. This gives examples of billiard tables with many secure configurations. These tables are not planar and they have geodesic boundary, although the surfaces are only $C^1$.

Let $\tau$ be the closed geodesic that agrees with $(t, 0)$ and $(t, \pi)$ in the cylinder and the corresponding half circles in $H_0$ and $H_l$ (see Figure 3.2 (a)). And let $M_1$ be the half of $N$ bounded by




(a) $M_1$ (b) $M_2$

Figure 3.2: Billiard tables.

it (a half cylinder with quarter spheres attached at the ends).

We can build $N$ from $M_1$ by gluing it with a reflection $\hat{M}_1$ of itself along the boundary, so we can say that $N$ is a double cover of $M_1$ in a similar way that $T$ is a double cover of $C$. This cover preserves geodesics, since a geodesic crossing from $M_1$ to $\hat{M}_1$ projects to a geodesic that bounces off $\tau$. Conversely, any geodesic in $M_1$ can be lifted to $N$, and if it bounces off the boundary the lift crosses from $M_1$ to its reflection. Therefore this table has blocking properties similar to $N$, any geodesic blocked in $N$ projects to a geodesic in $M_1$ that is blocked by the projection of the original blocking point.

Let $\sigma$ be the circle $l/2 \times S^1$, and $M_2 = \{(t,\theta), t \leq l/2\} \cup H_0$ a cylinder with a half sphere at one end, and boundary $\sigma$ (Figure 3.2(b)). The same argument we used for $M_1$ shows that $N$ is a geodesic double cover of $M_2$, and thus has similar blocking properties. In particular we have proved the following result.

**Proposition 3.3.1.** *Any pair of points in the flat part of $M_1$ or $M_2$ is secure, and not midpoint secure.*

*Moreover, any pair of points in the boundary of $M_2$ form a secure configuration.*

Unlike $M_1$ the boundary of $M_2$ is smooth, therefore we have a billiard table with smooth boundary such that all pairs of points in the boundary can be blocked. This can't be achieved by smooth planar billiards, as shown in [Ta] where insecurity is proved for points in the boundary.

If we allow manifolds with corners in the boundary, we can consider a quarter of $N$ cutting it along both $\tau$ and $\sigma$. Then $N$ is a geodesic 4-fold cover of this region, and therefore they share blocking properties as in the cases of $M_1$ and $M_2$. The same is true for the region bounded by $\tau$ and $\tau_n$, or by $\tau$, $\tau_n$ and $\sigma$, where $\tau_n$ (for $n \geq 2$) is the geodesic between the north and south poles that agrees with $(t, \pi/n)$ in the cylinder.

# Bibliography


[AS] D.V. Anosov and Y.G. Sinai, *Certain smooth ergodic systems*, Uspekhi Mat. Nauk, **22:5** (1967), 107 – 172; MR, 37, # 370; Russian Math. Surveys, **22:5** (1967), 103 – 167

[Ar1] V.I. Arnold, *Some remarks on flows of line elements and frames*, Soviet Math. Dokl., **2** (1961), 562 – 564.

[Ar2] V. I. Arnol'd, *The first steps of symplectic topology*, RUSS MATH SURV,**41**(6) (1986), 1–21.

[Ar3] V.I. Arnold, *On some problems in symplectic topology*, in Topology and Geometry Rochlin Seminar, O.Ya. Viro (Editor), Lect. Notes in Math., vol. 1346, Springer, 1988.

[BaG] V. Bangert and E. Gutkin, *Secure two-dimensional tori are flat*, arXiv:math.DS/0806.3572

[BCG] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictment négative*, Geom. Funct. Anal. **5** (1995), 731 – 799.

[BG] K. Burns and E. Gutkin, *Growth of the number of geodesics between points and insecurity for Riemannian manifolds*, arXiv:math.DS/0701579

[Ch] Y. Cheung *Hausdorff dimension of the set of nonergodic directions*, Annals of Mathematics **158** (2003), 661 – 678.

[CMP] G. Contreras, L. Macarini, and G.P. Paternain. *Periodic orbits for exact magnetic flows on surfaces*, Int. Math. Res. Not., **8** (2004), 361 – 387.

[Cr3] C. Croke, *Rigidity for surfaces of non-positive curvature*, Comm. Math. Helv. **65** (1990) no.1, 150-169.

[C4] C. Croke *Rigidity and the distance between boundary points* J. Diff. Geom. **33** (1991), 445 – 464.

[Cr] C. B. Croke, *Rigidity theorems in Riemannian geometry*, Geometric methods in inverse problems and PDE control, 47 72, IMA Vol. Math. Appl., 137, Springer, New York, 2004.

[DPSU] N. Dairbekov, P. Paternain, P. Stefanov and G. Uhlmann, *The boundary rigidity problem in the presence of a magnetic field*, Adv. Math. **216** (2007), 535 – 609.

[Da] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Vol. 3, Gauthier-Villars, Paris, 1894.

[Ep] D.B.A. Epstein,*Periodic flows on 3-manifolds*, Ann. of Math. **95** (1972), 68 – 82

[Gi1] V.L. Ginzburg, *New generalizations of Poincare s geometric theorem*, Functional Anal. Appl., **21** (2) (1987), 100 – 106.

[Gi2] V.L. Ginzburg, *On closed trajectories of a charge in a magnetic field. An application of symplectic geometry*, in Contact and Symplectic Geometry (Cambridge, 1994), C.B. Thomas (Editor), Publ. Newton Inst., 8, Cambridge University Press, Cambridge, 1996, p. 131 – 148.

[Grog] S. Grognet, *Flots magnétiques en courbure négative* (French), Ergodic Theory & Dynam. Systems **19** (1999) no. 2, 413 – 436.

[Gro] M. Gromov, *Filling Riemannian manifolds*, J. Diff. Geom. **18** (1983), 1 – 147.

[G1] E. Gutkin, *Blocking of billiard orbits and security for polygons and flat surfaces*, GAFA: Geom. & Funct. Anal. **15** (2005), 83 – 105.

[G2] E. Gutkin, *Insecure configurations in lattice translation surfaces, with applications to polygonal billiards*, Discr. Cont. Dyn. Sys. **A 16** (2006), 367 – 382.

[GS] E. Gutkin and V. Schröder, *Connecting geodesics and security of configurations in compact locally symmetric spaces*, Geometriae Dedicata **118** (2006), 185 – 208.

[HS] P. Hiemer and V. Snurnikov *Polygonal billiards with small obstacles*, Journal of Statistical Physics **90** (1998), nos. 1/2, 453 – 466.

[HST] P. Hubert, M. Schmoll and S. Troubetzkoy *Modular fibers and illumination problems* arXiv:math.GT/0602394

[LS] J.-F. Lafont and B. Schmidt, *Blocking light in compact Riemannian manifolds*, Geometry & Topology **11** (2007), 867 – 887

[Le] M. Levi, *On a problem by Arnold on periodic motions in magnetic fields*, Commun. Pure Appl. Math.**56** (2003) No.8, 1165 – 1177.

[Mi] R. Michel, *Sur la rigidité imposée par la longuer des géodésiques*, Inv. Math. **65** (1981), 71 – 83.

[M1] T. Monteil, *On the finite blocking property*, Annales de l'Institut Fourier **55** (2005), 1195 – 1217.

[M2] T. Monteil, *Finite blocking property vs pure periodicity*, arXiv:math.DS/0406506

[Ni] C. Niche, *On the topological entropy of an optical Hamiltonian flow*, Nonlinearity,**14** (2001), 817 – 827.

[No] S.P. Novikov, *The Hamiltonian formalism and a many-valued analogue of Morse theory*, Russian Math. Surveys, **37** (5) (1982), 1 – 56.

[NT] S.P. Novikov and I.A. Taimanov, *Periodic extremals of many-valued or not everywhere positive functionals*, Sov. Math. Dokl., **29**(1) (1984), 18 – 20.

[Ot] J.-P. Otal, *Le spectre marquédes longueurs des surfaces è courbure négative*, Ann. of Math. **131** (1990), 151 – 162.

[Pa] G.P. Paternain, *Geodesic flows*, Progress in Mathematics, 180 Birkauser 1999

[PP] G.P. Paternain and M. Paternain, *First derivative of topological entropy for Anosov geodesic flows in the presence of magnetic fields*, Nonlinearity,**10** (1997), 121 – 131.

[Sc] M. Schmoll *On the asymptotic quadratic growth rate of saddle connections and periodic orbits on marked flat tori* Geom. Funct. Anal. **12** (2002), no. 3, 622 – 649.

[SS] B. Schmidt and J. Souto, *Chords, light, and another synthetic characterization of the round sphere*, arXiv:math.GT/0704.3642

[Sch] M. Schneider. *Closed magnetic geodesics on S2*, Preprint, arXiv:0808.4038 [math.DG], 2008.

[Ta] S. Tabachnikov, *Birkhoff billiards are insecure*, to apear on Discr. Cont. Dyn. Syst.

[Ta] I.A. Taimanov, *Closed extremals on two-dimensional manifolds*, Russian Math. Surveys, **47**(2) (1992), 163 – 211.

[Zo] O. Zoll *Über Flächen mit Scharen geschlossener geodätischer Linien*, Math. Ann. **57** (1903), 108 – 133.