SOME LOCAL-GLOBAL NON-VANISHING RESULTS FOR THETA LIFTS FROM ORTHOGONAL GROUPS

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To Hideki Takeda
(1945-1994)

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ABSTRACT

SOME LOCAL-GLOBAL NON-VANISHING RESULTS

FOR THETA LIFTS FROM ORTHOGONAL GROUPS

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We, firstly, improve a theorem of B. Roberts which characterizes non-vanishing of a global theta lift from $\mathrm{O}(X)$ to $\mathrm{Sp}(n)$ in terms of non-vanishing of local theta lifts. In particular, we will remove all the archimedean conditions imposed upon his theorem. Secondly, we will apply our theorem to theta lifting of low rank similitude groups as Roberts did so. Namely we characterize the non-vanishing condition of a global theta lift from GO(4) to GSp(2) in our improved setting. Also we consider non-vanishing conditions of a global theta lift from GO(4) to GSp(1) and explicitly compute the lift when it exists.

Contents

Chapter 1

Introduction

Let X be a symmetric bilinear space over a global field F of characteristic not equal to 2 of an even dimension m, and σ a cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$ whose space of representation is V_σ. Then we consider the theta lift $\Theta_n(V_\sigma)$ of V_σ to $\mathrm{Sp}(n, \mathbb{A}_F)$ for $n = \frac{m}{2}$. As is always the case for the theory of theta correspondence, one of the major questions is to show that the lift $\Theta_n(V_\sigma)$ does not vanish. B. Roberts characterizes the global non-vanishing of the theta lift in terms of the local counterpart $\theta_n(\sigma_v)$. To be more precise, in [Rb4], Roberts proves that for $m \leq \frac{n}{2}$ the theta lift does not vanish if and only if the local theta lift of each local component σ_v does not vanish under various technical assumptions. In particular he assumes that the signature of $\mathrm{O}(X)$ at each real place is of the form $(2p, 2q)$. In this paper, first we will completely remove all the archimedean assumptions imposed upon his theorem. To be presice, we prove

Theorem 1.0.1. *Let F be any global field with* $\mathrm{char}\, F \neq 2$, *and $\sigma \cong \otimes \sigma_v$ a cuspidal automorphic representation of $O(X, \mathbb{A}_F)$ with even* $\dim X = m$. *Also let S_f be the finite set of finite places v such that either σ_v is ramified, $v|2$, or v is ramified in the quadratic extension $F(\sqrt{d})$ of F, where d is the discriminant of X. Assume:*

1. *The (incomplete) standard L-function $L^S(s, \sigma)$ of σ does not vanish at $s \in \{1, 2, \ldots, \frac{m}{2}\}$. (A pole is permitted).*
2. *σ_v is tempered for all $v \in S_f$.*
3. *The local theta lift $\theta_{\frac{m}{2}}(\sigma_v)$ to $Sp(\frac{m}{2}, F_v)$ exists for all places v.*

Then the global theta lift $\Theta_{\frac{m}{2}}(V_\sigma)$ to $Sp(\frac{m}{2}, \mathbb{A}_F)$ does not vanish.

Here the temperedness assumption for $v|2$ is due to the lack of Howe duality principle for even residual characteristic, and thus quite crucial. The other two are due to the lack of the corresponding result of [Rb2] for the non-tempered case. In [Rb4], he assumes that π_v is tempered for all non-archimedean places but we replace the temperedness condition by the L-function condition. (This is not an improvement of his theorem, but just another way of stating the theorem, although this makes a slight difference when we apply it to the similitude case.) Also in [Rb4] he *did not* assume π_v is tempered for archimedean v, but in [Rb5, p.301] he himself pointed out that this is a mistake and it must be assumed to be tempered. But in this paper, we will show that, after all, π_v does not have to be tempered for archimedean v.

Next we apply this theorem, as Roberts did, to theta lifting for groups of similitudes. Then we prove the following, which is an improvement of one of the main theorems of [Rb5].

Theorem 1.0.2. *Let X be a symmetric bilinear space of $\dim X = 4$ over F and σ a cuspidal automorphic representation of $GO(X, \mathbb{A}_F)$. Assume that σ_v is tempered for all $v \in S_f$, where S_f is defined in the same way as in Theorem 1.0.1. Then the global theta lift $\Theta_2(V_\sigma)$ to $GSp(2, \mathbb{A}_F)$ does not vanish if and only if the local theta lift $\theta_2(\sigma_v)$ to $GSp(2, F_v)$ does not vanish for all places v.*

Notice that the group $\mathrm{GO}(X)$ is disconnected and written as $\mathrm{GO}(X) \cong \mathrm{GSO}(X) \rtimes \{1, t\}$ for some $t \in \mathrm{GO}(X)$ with $t^2 = 1$ which acts on $\mathrm{GSO}(X)$ by conjugation. Each cuspidal automorphic representation σ of $\mathrm{GO}(X, \mathbb{A}_F)$ is "extended from" a cuspidal automorphic representation π of the identity component $\mathrm{GSO}(X, \mathbb{A}_F)$ in the sense explained in Appendix A. Let d be the discriminant of X. Roberts in [Rb5] has shown that for the purpose of similitude theta lifting, we may assume:

1. If $d = 1$, then a cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A})$ with central character χ is identified with a cuspidal automorphic representation $\tau_1 \otimes \tau_2$ of $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$, where D is a quaternion algebra over F, and the central characters of τ_1 and τ_2 are both χ. In this case, we write $\pi = \pi(\tau_1, \tau_2)$.

2. If $d \neq 1$, then a cuspidal automorphic representation τ of $\mathrm{GSO}(X, \mathbb{A})$ with central character χ is identified with a cuspidal automorphic representation τ of ${B_E}^\times(\mathbb{A}_F)$, where B is a quaternion algebra over the quadratic extension $E = F(\sqrt{d})$ of F, and the central character of τ is of the form $\chi \circ N_F^E$. In this case, we write $\pi = \pi(\chi, \tau)$.

Note that for $\tau_1 \otimes \tau_2$ and τ, there are Jacquet-Langlands lifts $\tau_1^{\mathrm{JL}} \otimes \tau_2^{\mathrm{JL}}$ and τ^{JL} to $\mathrm{GL}(2, \mathbb{A}_F) \times \mathrm{GL}(2, \mathbb{A}_F)$ and $\mathrm{GL}(2, \mathbb{A}_E)$, respectively. Also for each π we can consider the conjugate π^c of π, whose space V_{π^c} of representation is of the form $\{f \circ c : f \in V_\pi\}$ where $f \circ c(g) = f(tgt)$. If $\pi = \pi(\tau_1, \tau_2)$, then $\pi^c = \pi(\tau_2, \tau_1)$, and if $\pi = \pi(\chi, \tau)$, then $\pi^c = \pi(\chi, \tau^c)$ where τ^c is the Galois conjugate of τ. We will prove

Theorem 1.0.3. *Assume that the global theta lift $\Theta_2(V_\sigma)$ to $GSp(2, \mathbb{A}_F)$ does not vanish. Then*

1. *If* $d = 1$ *and* σ *is extended from* $\pi = \pi(\tau_1, \tau_2)$*, then the theta lift* $\Theta_1(V_\sigma)$ *to* $GSp(1, \mathbb{A}_F)$ *does not vanish if and only if* $\tau_1 = \tau_2$*. Moreover, if this is the case,* $\Theta_1(V_\sigma)$ *is the space of an irreducible cuspidal representation* Π *such that* $\Pi^\vee = \tau_1^{\mathrm{JL}} = \tau_2^{\mathrm{JL}}$*.*

2. *If* $d \neq 1$ *and* σ *is extended from* $\pi = \pi(\chi, \tau)$*, then the theta lift* $\Theta_1(V_\sigma)$ *to* $GSp(1, \mathbb{A}_F)$ *does not vanish if and only if* τ^{JL} *is the base change lift of a cuspidal automorphic representation* τ_0 *of* $GL(2, \mathbb{A}_F)$ *whose central character is* $\chi_{E/F}\chi$*, where* $\chi_{E/F}$ *is the quadratic character for* E/F*. Moreover, if this is the case,* $\Theta_1(V_\sigma)$ *is the space of an irreducible cuspidal representation* Π *such that* $\Pi^\vee = \tau_0$*.*

In light of those two theorems, one interesting thing to investigate is, of course, when a given π can be extended to σ so that $\Theta_2(V_\sigma) \neq 0$. For certain cases, it can be shown that the answer is "always". Namely,

Theorem 1.0.4.

1. *Assume* π *is generic. Then* π *can be extended to* σ *so that* $\Theta_2(V_\sigma) \neq 0$ *(without any temperedness assumption).*

2. *Assume* π *is such that* $\pi^c \ncong \pi$ *(but not necessarily generic). If* π *satisfies the temperedness assumption as in Theorem 1.0.2, then* π *can be extended to* σ *so that* $\Theta_2(V_\sigma) \neq 0$

We do not claim any originality for (1) of this theorem. For it almost directly follows from a theorem of Howe and Piatetski-Shapiro proven in [H-PS], although they do not explicitly state it in this way. Also (2) follows from Theorem 1.0.2 together with a theorem of Roberts in [Rb5], and he states its tempered version by using his notion of "global L-packets".

This paper is organized as follows. We first set up our notations in Section 2. In Section 3, after reviewing basics of both global and local theta lifting of groups of isometries, we give a proof of Theorem 1.0.1. In Section 4, we review basics of both global and local theta lifting of groups of similitudes. In Section 5, we will explicitly compute the unramified local theta lift from GO(4) to GSp(1). Then finally, in Section 6 and 7, we will give the proofs of Theorem 1.0.2 and 1.0.3, respectively. In Appendix A, we describe the relation between automorphic representations of $\mathrm{GSO}(X, \mathbb{A}_F)$ and those of $\mathrm{GO}(X, \mathbb{A}_F)$, and give the proof of Theorem 1.0.4.

Chapter 2

Notations

In this paper, F is a local or global field of char $F \neq 2$. If E is a quadratic extension of F, then we denote by N_F^E the norm map and by $\chi_{E/F}$ the quadratic character obtained by local or global class field theory. If F is a global field, we let r_1 and $2r_2$ be the number of real and complex embeddings of F as usual, and agree that $r_1 = r_2 = 0$ if F is a function field. Also in this case, we let S_∞ be the set of all archimedean places, and so $|S_\infty| = r_1 + r_2$.

We work with smooth representations instead of K-finite ones. Namely if G is a reductive group over a global filed F, then by a (cuspidal) automorphic form we mean a smooth (cuspidal) automorphic form on $G(\mathbb{A}_F)$ in the sense of [Co2, Definition 2.3]. A cuspidal automorphic form is occasionally called a cusp form. By a (cuspidal) automorphic representation we mean an irreducible representation of the group $G(\mathbb{A}_F)$ which is realized as a subspace of smooth (cuspidal) automorphic forms on which $G(\mathbb{A}_F)$ acts by right translation. (See Lecture 2 and 3 of [Co2] for more details about smooth automorphic forms and representations, especially differences between smooth ones and the usual K-finite ones.) If π is an automorphic representation, we denote the tensor product decomposition simply by $\otimes\pi_v$, and write $\pi \cong \otimes\pi_v$, where each π_v is a smooth representation of the group $G(F_v)$ for *all* places v, *i.e.* even if v is archimedean, π_v is a smooth representation of the whole group $G(F_v)$ rather than a Harish-Chandra module.

If π is a representation of a group, a Harish-Chandra module, or an automorphic representation, then we will denote the space of π by V_π. If π is a representation of a real Lie group then we will denote the space of smooth vectors in V_π by V_π^∞ and the space of K-finite vectors by ${}^K V_\pi$. If π is an admissible representation of a real Lie group, then we denote the underlying Harish-Chandra module by π^{H}, and thus we have $V_{\pi^{\mathrm{H}}} = {}^K V_\pi$. If F is a local field, we denote by $\mathrm{Irr}(G(F))$ the collection of equivalence classes of irreducible smooth admissible representations of $G(F)$. For each $\pi \in \mathrm{Irr}(G(F))$, we denote the contragredient by $\pi^\vee$. So $V_{\pi^\vee}$ is the space of smooth linear functionals on V_π. For each $v \in V_\pi$, $w \in V_{\pi^\vee}$, the map $f_{v,w} : G(F) \to \mathbb{C}$ given by $f_{v,w}(g) = \langle \pi(g)v, w \rangle$ is called a matrix coefficient of π, and by a coefficient of π we mean a finite linear combination of matrix coefficients of π. If π is a Harish-Chandra module, we denote by π^{CW} the Casselman-Wallach canonical completion of π. Thus ${}^{K}V_{\pi^{\mathrm{CW}}} = V_{\pi}$. (For the Casselman-Wallach canonical completion, see [Cs].)

For a finite dimensional vector space X over a global field F, we denote $X \otimes_F F_v$ by $X(F_v)$ for each place v and $X \otimes_F \mathbb{A}_F$ by $X(\mathbb{A}_F)$. For a natural number n, $\mathcal{S}(X(F_v)^n)$ denotes the space of Schwartz-Bruhat functions, namely the space of compactly supported smooth functions on $X(F_v)^n$ if v is finite, and the space of smooth rapidly decreasing functions on $X(F_v)^n$ if v is infinite. Then we define $\otimes' \mathcal{S}(X(F_v)^n)$ to be the restricted direct product over all places with respect to the characteristic function of $\mathcal{O}_v x_1 + \cdots + \mathcal{O}_v x_m$ for v finite, where $\mathcal{O}_v$ is the ring of integers of F_v and $x_1, \ldots, x_m$ is a fixed basis of $X(F_v)$. We set $\mathcal{S}(X(\mathbb{A}_F)^n) := \otimes' \mathcal{S}(X(F_v)^n)$.

The group $\mathrm{GSp}(n)$ is the algebraic group of symplectic similitudes of rank n over a field F. We realize this group as the group of $2n \times 2n$ matrices given by

$$\mathrm{GSp}(n) = \{g \in \mathrm{GL}(2n) : \; {}^{t}gJg = \nu(g)J\}, \; J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, and $\nu : \mathrm{GSp}(n) \to \mathbb{G}_m$ is a character, which is called the multiplier character. Then the kernel of ν is denoted by $\mathrm{Sp}(n)$. When we need to make clear that we are working with F rational points of the algebraic group, we write $\mathrm{Sp}(n, F)$ or $\mathrm{GSp}(n, F)$, but when there is no danger of confusion, we simply write $\mathrm{Sp}(n)$ or $\mathrm{GSp}(n)$.

Let X be an even dimensional symmetric bilinear space defined over a field F of dimension m equipped with a symmetric bilinear form $(\;,\;)$. Then we denote by $\mathrm{GO}(X)$ the group of symmetric similitudes, *i.e.*

$$\mathrm{GO}(X) = \{h \in \mathrm{GL}(X) : \; (hx, hy) = \nu(h)(x, y) \text{for all} x, y \in X\},$$

where $\nu : \mathrm{GO}(X) \to \mathbb{G}_m$ is a character, which is again called the multiplier character. The kernel of ν is denoted by $\mathrm{O}(X)$. Also there is a map

$$\mathrm{GO}(X) \to \{\pm 1\}, \quad h \mapsto \frac{\det(h)}{\nu(h)^{m/2}}.$$

We denote the kernel of this map by $\mathrm{GSO}(X)$, and denote $\mathrm{GSO}(X) \cap \mathrm{O}(X)$ by $\mathrm{SO}(X)$. Just at the symplectic case, we also write $\mathrm{GO}(X, F)$, $\mathrm{O}(X, F)$, etc when it is necessary to do so. If X is defined over a local or global field F of $\mathrm{char}\, F \neq 2$, then we denote by $\mathrm{disc} X \in F^{\times}/F^{\times 2}$ the discriminant of X when X is viewed as a quadratic form. Also in this case we let $\chi_X : F^{\times} \to \{\pm 1\}$ be the quadratic character of X, namely $\chi_X(a) = (a, (-1)^{\frac{m(m-1)}{2}} \mathrm{disc} X)_F$ for $a \in F^{\times}$, where $(\;,\;)_F$ is the Hilbert symbol of F. Thus if $\mathrm{disc} X \neq 1$ and $E = F(\sqrt{\mathrm{disc} X})$, then $\chi_X = \chi_{E/F}$.

Chapter 3

Theta lifting for isometry groups

In this chapter, we will first review the theory of both local and global theta lifting, and then give a proof of Theorem 1.0.1, which is, as we mentioned in Introduction, an improvement of the main theorem of [Rb4].

3.1 Basic theory

First let us consider local theta lifting. (A good reference for local theta lifting for isometry groups is [Kd3].) Let F be a local field of $\operatorname{char} F \neq 2$. We do not assume that F is non-archimedean. Also let $\mathbb{W}$ be a symplectic space over F with a complete polarization $\mathbb{W}_1 \oplus \mathbb{W}_2$. Then by choosing a basis of $\mathbb{W}$ respect to this polarization in such a way that the group $\mathrm{Sp}(\mathbb{W})$ is realized as a matrix group as in the notation section, we can see that $\mathrm{Sp}(\mathbb{W})$ is generated by the elements of the following forms:

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $A \in \mathrm{GL}(\mathbb{W}_1)$ and $B = {}^tB$.

Let $\mathcal{S}(\mathbb{W}_1)$ be the space of Schwartz-Bruhat functions as defined in the notation section. Fix an additive character φ of F. Then define an action r of $\mathrm{Sp}(\mathbb{W})$ on $\mathcal{S}(\mathbb{W}_1)$ by

$$r\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f(x) = |\det A|^{1/2} f({}^tAx)$$

$$r\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(x) = \varphi(\frac{{}^txBx}{2}) f(x)$$

$$r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = \gamma \hat{f}(x),$$

where γ is an 8-root of unity and $\hat{f}$ is the Fourier transform defined by

$$\hat{f}(x) = \int_{\mathbb{W}_1} f(y)\varphi({}^t xy)dy$$

with the Haar measure dy being chosen so that $\hat{\hat{f}}(x) = f(-x)$. (Here we identify $\mathbb{W}_1$ with F^N for N the dimension of $\mathbb{W}_1$ and view $x, y \in \mathbb{W}_1$ as a $1 \times N$ matrix.)

This action r does not give rise to a representation of $\mathrm{Sp}(\mathbb{W})$. Rather it is a projective representation. Thus there exists a 2-cocycle $c : \mathrm{Sp}(\mathbb{W}) \times \mathrm{Sp}(\mathbb{W}) \to \mathbb{C}$ so that

$$r(g_1)r(g_2) = c(g_1, g_2)r(g_1 g_2)$$

for $g_1, g_2 \in \mathrm{Sp}(\mathbb{W})$. This cocycle can be explicitly computed. (See [Ra], in particular Theorem 4.1 there.)

Now assume X is a symmetric bilinear space over F of dimension m equipped with a symmetric bilinear form $(\ ,\ )$, and W a symplectic space over F of dimension $2n$ equipped with a symplectic bilinear form $\langle\ ,\ \rangle$. Set $\mathbb{W} = X \otimes W$. Then $\mathbb{W}$ can be made into a symplectic space with a symplectic form $\langle\langle\ ,\ \rangle\rangle$ given by

$$\langle\langle x \otimes w, x' \otimes w' \rangle\rangle = (x, x')\langle w, w' \rangle.$$

Notice that if $W = W_1 \oplus W_2$ is a complete polarization of W, then $X \otimes W_1 \oplus W \otimes W_2$ is a complete polarization of $\mathbb{W}$. We choose a basis of W respect to this polarization in such a way that the group $\mathrm{Sp}(W)$ is realized as a matrix group as in the notation section, and then denote $\mathrm{Sp}(W)$ by $\mathrm{Sp}(n)$. Then for $X \otimes W_1$ we write X^n, or when we want to make clear the field F, we write $X(F)^n$.

We have the obvious homomorphism

$$\iota : \mathrm{Sp}(n) \times \mathrm{O}(X) \to \mathrm{Sp}(\mathbb{W}).$$

It can be shown that, since we assume $m = \dim X$ is even, the cocycle c is trivial on $\iota(\mathrm{Sp}(n) \times \mathrm{O}(X))$, *i.e.* $c(g_1, g_2) = 1$ for $g_1, g_2 \in \iota(\mathrm{Sp}(n) \times \mathrm{O}(X))$. (For this splitting issue, a detail discussion is found in [Kd2].) Hence, via ι, the projective representation r gives rise to a representation of $\mathrm{Sp}(n) \times \mathrm{O}(X)$ on the Schwartz-Bruhat space $\mathcal{S}(X(F)^n)$. We call this representation the Weil representation for the pair $(\mathrm{O}(X), \mathrm{Sp}(n))$ and denote it by $\omega_{n,X}$ or simply by ω when X and n are clear from the context.

If F is archimedean, then the Weil representation $\omega_{n,X}$ on $\mathcal{S}(X(F)^n)$ is a smooth Fréchet representation of the group $\mathrm{Sp}(n) \times \mathrm{O}(X)$ of moderate growth in the sense of [Cs]. (This is because $\mathcal{S}(X(F)^n)$ is the space of smooth vectors of the unitary representation of $\mathrm{Sp}(n) \times \mathrm{O}(X)$ on the Hilbert space $L^2(X(F)^n)$.) So in particular $(\omega^{\mathrm{H}})^{\mathrm{CW}} = \omega$.

Now assume F is non-archimedean. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$. We say that σ and Π correspond, or σ corresponds to Π if there is a non-zero

$\mathrm{Sp}(n) \times \mathrm{O}(X)$ homomorphism from $\omega_{n,X}$ to $\Pi \otimes \sigma$, *i.e.* $\mathrm{Hom}_{\mathrm{Sp}(n)\times\mathrm{O}(X)}(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$. If the residue characteristic of F is odd, it is known that the relation $\mathrm{Hom}_{\mathrm{Sp}(n)\times\mathrm{O}(X)}(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$ defines a graph of bijection between subsets of $\mathrm{Irr}(\mathrm{Sp}(n))$ and $\mathrm{Irr}(\mathrm{O}(X))$ (the Howe duality principle), and in particular if σ corresponds to Π, then such Π is unique. In this case we write $\Pi = \theta_n(\sigma)$ and call it the local theta lift of σ. If σ does not correspond to any $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$, then we say that the theta lift of σ vanishes and write $\theta_n(\sigma) = 0$. If the residue characteristic of F is even, then in general it is not known if the same holds. However, in [Rb2] Roberts has shown, among other things, that if σ is tempered and corresponds to $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$ for $n = \frac{m}{2}$, then Π is unique regardless of the residue characteristic of F. So in this case, we denote Π by $\theta_n(\sigma)$ even if the residue characteristic of F is even.

Next assume F is archimedean. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$. We say that σ and Π correspond, or σ corresponds to Π if there is a non-zero homomorphism of Harish-Chandra modules from $(\omega_{n,X})^{\mathrm{H}}$ to $(\Pi \otimes \sigma)^{\mathrm{H}} = \Pi^{\mathrm{H}} \otimes \sigma^{\mathrm{H}}$, *i.e.* $\mathrm{Hom}((\omega_{n,X})^{\mathrm{H}}, (\Pi \otimes \sigma)^{\mathrm{H}}) \neq 0$, where Hom means the set of homomorphisms of Harish-Chandra modules. It is known that the relation $\mathrm{Hom}((\omega_{X,n})^{\mathrm{H}}, (\Pi \otimes \sigma)^{\mathrm{H}}) \neq 0$ defines a graph of bijection between subsets of $\mathrm{Irr}(\mathrm{Sp}(n))$ and $\mathrm{Irr}(\mathrm{O}(X))$ up to infinitesimal equivalence (the Howe duality principle), and in particular if σ corresponds to Π, then such Π is unique up to infinitesimal equivalence, namely Π^{H} is unique, although Π might not be unique. In this case we write $(\Pi^{\mathrm{H}})^{\mathrm{CW}} = \theta_n(\sigma)$, where $(\Pi^{\mathrm{H}})^{\mathrm{CW}}$ is the Casselman-Wallach canonical completion of Π^{H} as in the notation section, and we call it the local theta lift of σ. If σ does not correspond to any $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$, then we say that the theta lift of σ vanishes and write $\theta_n(\sigma) = 0$. Notice that if σ and Π correspond, we have non-zero homomorphism $(\omega_{n,X})^{\mathrm{H}} \to \Pi^{\mathrm{H}} \otimes \sigma^{\mathrm{H}}$ of Harish-Chandra modules, which gives rise to a non-zero homomorphism $\omega_{n,X} \to (\Pi^{\mathrm{H}})^{\mathrm{CW}} \otimes (\sigma^{\mathrm{H}})^{\mathrm{CW}}$ by [Cs, Corollary 10.5]. (Here notice that $((\omega_{n,X})^{\mathrm{H}})^{\mathrm{CW}} = \omega_{n,X}$ by our choice of $\omega_{n,X}$ as a smooth representation.) Therefore, if we let $\mathrm{CW}(\mathrm{O}(X))$ and $\mathrm{CW}(\mathrm{Sp}(n))$ be the classes of isomorphism classes of Casselman-Wallach canonical completions of irreducible Harish-Chandra modules of $\mathrm{O}(X)$ and $\mathrm{Sp}(n)$, respectively, then the relation $\mathrm{Hom}_{\mathrm{Sp}(X)\times\mathrm{O}(n)}(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$ defines the graph of a bijection between subsets of $\mathrm{CW}(\mathrm{O}(X))$ and $\mathrm{CW}(\mathrm{Sp}(n))$, where $\mathrm{Hom}_{\mathrm{Sp}(X)\times\mathrm{O}(n)}$ is taken in the category of smooth representations, and if $\sigma \in \mathrm{CW}(\mathrm{O}(X))$ and $\Pi \in \mathrm{CW}(\mathrm{Sp}(n))$ correspond, then $\Pi = \theta_n(\sigma)$.

Now we will review the basic theory of global theta lifting. Good references for global theta lifting are [H-PS] and [Pr]. Assume F is a global field of characteristic not 2. Exactly in the same way as the local case, we define the Weil representation $\omega = \omega_{n,X}$ of the group $\mathrm{Sp}(n, \mathbb{A}_F) \times \mathrm{O}(X, \mathbb{A}_F)$ on $\mathcal{S}(X(\mathbb{A}_F)^n)$.

For each $(g, h) \in \mathrm{Sp}(n, \mathbb{A}_F) \times \mathrm{O}(X, \mathbb{A}_F)$ and $\varphi \in \mathcal{S}(X(\mathbb{A}_F)^n)$, we define the theta kernel by

$$\theta(g, h; \varphi) = \sum_{x \in X(F)^n} \omega(g, h)\varphi(x).$$

This sum is absolutely convergent. Now let σ be a cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$ with central character χ. Then for each cuspidal automorphic form $f \in V_\sigma$, consider the function $\theta(f;\varphi)$ on $\mathrm{Sp}(n, \mathbb{A}_F)$ defined by

$$\theta(f;\varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g,h;\varphi) f(h)\, dh$$

for each $g \in \mathrm{Sp}(n, \mathbb{A}_F)$. It can be shown that this integral is absolutely convergent, and indeed it is an automorphic form on $\mathrm{Sp}(n, \mathbb{A}_F)$. Also it is easy to see that the central character of $\theta(f;\varphi)$ is $\chi^{-1}{\chi_X}^n$, where χ_X is the quadratic character corresponding to X. Further consider the space $\Theta_n(V_\sigma) = \{\theta(f;\varphi) : f \in V_\sigma, \varphi \in \mathcal{S}(X(\mathbb{A}_F)^n)\}$. Then the group $\mathrm{Sp}(n, \mathbb{A}_F)$ acts on this space by right translation and this space is closed under this action. If $\Theta_n(V_\sigma)$ consists of non-zero cusp forms, then it is shown by Moeglin [Mo] that it is irreducible under this action and thus is a space of a cuspidal automorphic representation. In this case, we denote this cuspidal automorphic representation by $\Theta_n(\sigma)$. (Thus $V_{\Theta_n(\sigma)} = \Theta_n(V_\sigma)$.) Moreover if we write $\sigma \cong \otimes\sigma_v$ and $\Theta_n(\sigma) \cong \otimes\Pi_v$, then it can be shown that $\sigma_v^\vee$ corresponds to Π_v and in particular $\Pi_v \cong \theta_n(\sigma_v^\vee) \cong \theta_v(\sigma_v)^\vee$ for all $v \nmid 2$.

The main difficulty in the theory of global theta lifting is to show that the space $\Theta_n(V_\sigma)$ is non-zero. Indeed this is the major problem we will take up in this paper. To prove our non-vanishing result, we need the following well-known theorem due to S. Rallis [R].

Proposition 3.1.1. *Let σ be a cuspidal representation of $O(X, \mathbb{A}_F)$, then the theta lift $\Theta_n(V_\sigma)$ to $Sp(n, \mathbb{A}_F)$ is non-zero for some $n \leq m$. Moreover, if n_0 is the smallest integer such that $\Theta_{n_0}(V_\sigma) \neq 0$, then $\Theta_{n_0}(V_\sigma)$ consists of cups forms and $\Theta_n(V_\sigma) \neq 0$ for all $n \geq n_0$.*

3.2 Proof of Theorem 1.0.1

Now we give a proof of Theorem 1.0.1. The proof is essentially the ingenious argument of [Rb4], which has its origin in [B-S]. The basic idea is quite simple and clever, which can be described as follows: For the sake of obtaining a contradiction, assume $\Theta_n(V_\sigma) = 0$. Then by Proposition 3.1.1, the theta lift $\Theta_k(V_\sigma)$ is non-zero cuspidal for some k with $\frac{m}{2} = n < k \leq m$. We can compute the (incomplete) standard L-function $L^S(s, \pi^\vee, \chi_X)$ of the contragredient of this putative lift $\pi = \Theta_k(\sigma)$ twisted by χ_X in two different ways, one by using the functoriality of unramified theta lift, and the other by the zeta integral for symplectic group developed in [PS-R]. Then by looking at the order of vanishing of $L^S(s, \pi^\vee, \chi_X)$ at $s = k - n$, we will derive a contradiction. Before giving the proof, we need the following lemma, which is one of the main technical ingredients for our proof.

Lemma 3.2.1. *Let $\sigma \cong \otimes\sigma_v$ be as in Theorem 1.0.1. For each $k > n = \frac{m}{2}$ and for $v \in S$, the local zeta integral of the local theta lift $\theta_k(\sigma_v)$ of σ_v to $Sp(k, F_v)$ can be chosen so that it has a pole at $s = k - \frac{m}{2}$. To be precise, there exist a matrix coefficient f_v of $\theta_k(\sigma_v)$ and a standard K-finite χ_{V_v}-section Φ_v for $Sp(k, F_v)$ such that the local zeta integral $Z(s - \frac{1}{2}, f_v, \Phi_v)$ has a pole at $s = k - \frac{m}{2}$*

Proof (Sketch). The proof is essentially given in [Rb4], although, for the archimedean case, we need to prove some technical lemmas to improve the theorem of [Rb4]. Those lemmas are proven in the next subsection.

The idea of the proof is conceptually quite simple, though technically and computationally somehow complicated. The idea is roughly the following. If $\theta_n(\sigma_v)$ exists, then for $k > n$, $\theta_k(\sigma_v)$ is the unique quotient of $\operatorname{Ind}_P^{\mathrm{Sp}(k)}(\sigma_1 \otimes \sigma_2)$ where P is the parabolic subgroup whose Levi factor is isomorphic to $\mathrm{GL}(1) \times \mathrm{Sp}(k-1)$, $\sigma_1 = \chi_{X_v}|\cdot|^{k-n}$ and σ_2 is an irreducible admissible representation of $\mathrm{Sp}(k-1)$. Moreover, this unique quotient is isomorphic to the image of an intertwining integral. (For the archimedean case, see Proposition 3.3.2 and Lemma 3.3.5 below.) Then Roberts in [Rb4] has shown that the local zeta integral $Z(s - \frac{1}{2}, f_v, \Phi_v)$ can be factored in terms of the local zeta integrals of σ_1 and σ_2. That is, we have

$$Z(s-1/2, f_v, \Phi_v) = Z(s-1/2, f_1 \otimes \delta^{-1/2}, \Phi_1) Z(s-1/2, f_2, \Phi_2)$$

where $Z(s, f_1 \otimes \delta^{-1/2}, \Phi_1)$ is the local zeta integral for $\mathrm{GL}(1)$ as defined in, say, [Rb4, Section 4], f_1 and f_2 are some matrix coefficients of σ_1 and σ_2, respectively, and Φ_1 and Φ_2 are suitably chosen sections for $\mathrm{GL}(1)$ and $\mathrm{Sp}(k-1)$, respectively. Moreover those sections Φ_1 and Φ_2 can be chosen in such a way that $Z(s-1/2, f_1, \Phi_1)$ has a simple pole at $s = k - n$ and $Z(s - 1/2, f_2, \Phi_2) = 1$. Thus the local zeta integral $Z(s-1/2, f_v, \Phi_v)$ has a pole at $s = k - n$. (However, the real story is not that simple for the archimedean case, and for this, Roberts introduces a certain auxiliary zeta integral and uses a density argument. See [Rb4] for detail.) □

Once this lemma is obtained, we are ready to prove Theorem 1.0.1.

Proof of Theorem 1.0.1. Let $S = S_\infty \cup S_f$. For $v \notin S$, the functoriality of the theta correspondence at the unramified places gives the (standard) local L-factor of the local lift $\theta_k(\sigma_v)$. (See [Kd-R2, Corollary 7.1.4].) Then by taking the product over all $v \notin S$, we obtain

$$\begin{aligned} &L^S(s, \pi^\vee, \chi_X) \\ &\quad = \zeta_F^S(s) L^S(s, |\cdot|^{k-n}) L^S(s, |\cdot|^{-(k-n)}) L^S(s, |\cdot|^{k-n-1}) L^S(s, |\cdot|^{-(k-n-1)}) \\ &\qquad\qquad \cdots\cdots L^S(s, |\cdot|) L^S(s, |\cdot|^{-1}) L^S(s, \sigma), \end{aligned}$$

where $\zeta_F^S(s)$ is the incomplete Dedekind zeta function over F and the $L^S(s, \cdot)$'s are the incomplete abelian L-functions as in Tate's thesis. Now at $s = k - n$,

$L^S(s, \pi^\vee, \chi_X)$ has a zero of order at most $|S_f| + r_1 + r_2 - 2$, provided $L^S(s, \sigma)$ has no zero at $s = k - n$ as we have been assuming. This is because, at $s = k - n$, $L^S(s, |\cdot|^{-(k-n)})$ has a zero of order $|S_f| + r_1 + r_2 - 1$ and $L^S(s, |\cdot|^{-(k-n-1)})$ has a simple pole and no other factor other than $L^S(s, \sigma)$ has a pole or a zero. (Note that for $L^S(s, |\cdot|^{-(k-n)})$ the order of the zero can be computed by considering the well-known fact that the zeta function $\zeta_F(s - (k - n)) = L^{S_\infty}(s, |\cdot|^{-(k-n)})$ has a zero of order $r_1 + r_2 - 1$ at $s = k - n$ and each local factor $L_v(s, |\cdot|^{-(k-n)})$ has a simple pole at $s = k - n$.)

On the other hand, we can compute the (incomplete) standard L-function $L^S(s, \pi^\vee, \chi_X)$ of the contragredient of the putative lift $\pi = \Theta_k(\sigma)$ by using the zeta integral for $\mathrm{Sp}(k)$. Namely we have

$$L^S(s, \pi^\vee, \chi_X) = \frac{Z^*(s - \frac{1}{2}, f, \Phi)}{\prod_{v \in S} Z(s - \frac{1}{2}, f_v, \Phi_v)},$$

where $Z^*(s - \frac{1}{2}, f, \Phi)$ is the normalized zeta integral for a matrix coefficient $f = \otimes_v f_v$ of $\pi^\vee$ and a standard χ_X-section $\Phi = \otimes_v \Phi_v$, and $Z(s - \frac{1}{2}, f_v, \Phi_v)$ is the local zeta integral. Then at $s = k - n$, the L-function $L^S(s, \pi^\vee, \chi_X)$ computed in this way has a zero of order at least $|S| - 1$ because, first of all, a pole of $Z^*(s - \frac{1}{2}, f, \Phi)$ is at most simple, and $Z(s - \frac{1}{2}, f_v, \Phi_v)$ can be chosen to have a pole at $s = k - n$ due to Lemma 3.2.1.

Therefore the first way of computing $L^S(s, \pi^\vee, \chi_X)$ shows that it has a zero of order at most $|S_f| + r_1 + r_2 - 2$ at $s = k - n$ and the second one shows it is at least $|S| - 1 = |S_f| + r_1 + r_2 - 1$, which is a contradiction. This completes the proof of Theorem 1.0.1. □

Remark 3.2.2. That the poles of $Z^*(s - \frac{1}{2}, f, \Phi)$ are at most simple is a direct consequence of the fact that the (normalized) Eisenstein series $E^*(g, s, \Phi)$ defining the zeta integral has at most simple poles. The proof for the simplicity of poles of the Eisenstein series is proven by Ikeda [Ik] in full generality. (See Proposition 1.6 and 1.7 of [Ik], and also see Theorem 1.1 and (7.2.19) of [Kd-R2], though the base field is assumed to be totally real in [Kd-R2].)

3.3 Some technical lemmas on zeta integrals

What forced Roberts to impose the conditions on the infinite places in his theorem in [Rb4] is the unavailability of Lemma 3.2.1 for χ_{X_v} nontrivial and v real, and for v complex. In this subsection, we give proofs of several technical lemmas that allow us to prove Lemma 3.2.1 in full generality for the archimedean case. So we assume $F = \mathbb{R}$ or $\mathbb{C}$. There are basically two technical ingredients we need. The first one is the theory of the zeta integral for symplectic group at the archimedean place developed in [Kd-R1]. There, it is assumed that $F = \mathbb{R}$ and the character for the zeta integral (which corresponds to our χ_{X_v}) is trivial. The results in [Kd-R1] are used in two places in Roberts' argument in crucial ways. (See Lemma 7.5 and Theorem 7.8 of [Rb4].) So we need to extent the results of [Kd-R1] to the full generality. Although, as mentioned in [Kd-R1], there is no doubt that all the arguments there work for $F = \mathbb{C}$, it seems that the extension to the case with the non-trivial character (*i.e.* the sign character) is not completely immediate. Thus first we extend the results of [Kd-R1]. Namely,

Proposition 3.3.1. *All the results in [Kd-R1] hold even with the presence of the sign character.*

Proof. In this proof, all the notations as well as the numberings of propositions, lemmas, etc are as in [Kd-R1]. First as in the trivial character case, the line bundle $E_{s+\rho} = P\backslash(G \times \mathbb{C})$ (see (3.3.1) of p.105) where $p \cdot (g, v) = (pg, \chi(a(g))|a(g)|^{s+\rho}v)$ is trivialized by the nowhere vanishing section $g \mapsto (g, \chi(a(g))|a(g)|^{s+\rho})$. So each section $\Phi(g, s)$ is identified with a smooth function on Ω via

$$\Phi(g, s) \mapsto \chi(a(g))|a(g)|^{-s-\rho}\Phi(g, s).$$

For each Φ, let us write $\widetilde{\Phi}$ for the corresponding smooth function in $C^\infty(\Omega)$, *i.e.* $\widetilde{\Phi}(g, s) = \chi(a(g))|a(g)|^{s+\rho}\Phi(g, s)$. Then as in p.98, the restriction of $\widetilde{\Phi}|_{K_G}$ to K_G is left invariant under $P \cap K_G$ and so we may view $\widetilde{\Phi}|_{K_G}$ as a smooth function on $\Omega \cong (P \cap K_G)\backslash K_G$.

Now we first have to prove Proposition 3.1.1, which characterizes convergence of the integral in terms of the order of non-vanishing on the negligible set. Clearly all the computations until Lemma 3.1.3 (p.101) remain to be true for our case. What needs to be modified is the equation (3.1.14) in p.101. First notice that $|\Phi(k, s)| = |\widetilde{\Phi}(k, s)|$ for all $k \in K_G$. Thus the equation (3.1.14) becomes

$$\int_K \int_K \int_{A^+} |\widetilde{\Phi}(k(a)i(k_1, k_2), s)|\mu(a)^{-\sigma-\rho_n}\delta(a)da\ dk_1\ dk_2.$$

Lemma 3.1.3 itself has nothing to do with the presence of the sign character. The equation (3.1.16) must be modified as

$$f(u) = \widetilde{\Phi}(\phi(u)i(k_1, k_2), s)$$

as a function on $[0, 1]^n$. Then everything else works identically with the trivial character case. This proves Proposition 3.1.1 and Corollary 3.1.5.

Now that both Proposition 3.1.1 and Corollary 3.1.5 have been proven, Theorem 3.2.2. in p.104 can be proven if we prove Proposition 3.2.1. But the proof for Proposition 3.2.1. works for our case because of the trivialization of the line bundle $E_{s+\rho}$. □

The second ingredient we need in order to remove the archimedean conditions on the result of [Rb4] and thus to obtain Lemma 3.2.1 is the description of a coefficient of the local lift $\theta_k(\sigma)$ for $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F))$ with $\theta_n(\sigma) \neq 0$. Namely, we need to express a coefficient of $\theta_k(\sigma)$ in terms of its Langlands data. (See the remark preceding Theorem 7.7 of [Rb4].)

For this, we need to introduce some notations. Let $P_{n_1\dots n_t} = M_{n_1\dots n_t}U_{n_1\dots n_t}$ be a parabolic subgroup of $\mathrm{Sp}(n,F)$ whose Levi factor $M_{n_1\dots n_t}$ is isomorphic to $\mathrm{GL}(n_1)\times\cdots\times\mathrm{GL}(n_t)\times\mathrm{Sp}(n-(n_1+\cdots+n_t))$. (Here we do not assume that $P_{n_1\dots n_t}$ is the standard choice of the parabolic, and so it could be any parabolic subgroup with this Levi factor.) Also $\overline{P}_{n_1\dots n_t} = {}^tP_{n_1\dots n_t}$ and $\overline{U}_{n_1\dots n_t} = {}^tU_{n_1\dots n_t}$. Assume that π is an irreducible admissible representation of $\mathrm{Sp}(n)$ which is infinitesimally equivalent to the Langlands quotient of $\mathrm{Ind}\,^{\mathrm{Sp}(n)}_{P_{n_1\dots n_t}}(\tau_1\otimes\cdots\otimes\tau_t\otimes\tau_{t+1})$, where each τ_i is an essentially tempered representation of $\mathrm{GL}(n_i)$ for $1\leq i\leq t$ and τ_{t+1} is a tempered representation of $\mathrm{Sp}(n-(n_1+\cdots+n_t))$. Thus π^{H} is isomorphic to the image of the integral operator $I : \mathrm{Ind}\,^{\mathrm{Sp}(n)}_{P_{n_1\dots n_t}}(\tau_1\otimes\cdots\otimes\tau_{t+1}) \to \mathrm{Ind}\,^{\mathrm{Sp}(n)}_{\overline{P}_{n_1\dots n_t}}(\tau_1\otimes\cdots\otimes\tau_{t+1})$ given by

$$I(f)(g) = \int_{\overline{U}_{n_1\dots n_t}} f(\bar{u}g)\, d\bar{u},$$

for K-finite $f \in \mathrm{Ind}\,^{\mathrm{Sp}(n)}_{P_{n_1\dots n_t}}(\tau_1\otimes\cdots\otimes\tau_{t+1})$. (Namely we assume that $\tau_1\otimes\cdots\otimes\tau_{t+1}$ satisfies the convergent condition as in, say, [Kn, Theorem 7.24], with respect to this choice of parabolic subgroup.) Also let $\sigma_1 = \tau_1$ and σ_2 the irreducible admissible representation of $\mathrm{Sp}(n-n_1)$ that is infinitesimally equivalent to the Langlands quotient of $\mathrm{Ind}\,^{\mathrm{Sp}(n-n_1)}_{P_{n_2\dots n_t}}(\tau_2\otimes\cdots\otimes\tau_{t+1})$, where $P_{n_2\dots n_t} = M_{n_2\dots n_t}U_{n_2\dots n_t}$ is a parabolic subgroup of $\mathrm{Sp}(n-n_1,F)$ with the Levi factor $M_{n_2\dots n_t}$ isomorphic to $\mathrm{GL}(n_2)\times\cdots\times\mathrm{GL}(n_t)\times\mathrm{Sp}(n-(n_1+\cdots+n_t))$. (Here we view $P_{n_2\dots n_t}$ as a subgroup of $P_{n_1\dots n_t}$ in the obvious way. Also it is easy to see that the representation $\tau_2\otimes\cdots\otimes\tau_{t+1}$ does indeed satisfy the convergent condition with respect to this parabolic.) We also let P_{n_1} be the parabolic subgroup of $\mathrm{Sp}(n) = M_{n_1}U_{n_1}$ with the Levi factor M_{n_1} isomorphic to $\mathrm{GL}(n_1)\times\mathrm{Sp}(n-n_1)$ and the unipotent radical U_{n_1} contained in $U_{n_1\dots n_t}$ so that $P_{n_1}\supset P_{n_1\dots n_t}$ and $U_{n_1\dots n_t} = U_{n_2\dots n_t}U_{n_1}$ by viewing $U_{n_2\dots n_t}$ as a subgroup of $U_{n_1\dots n_t}$ in the obvious way. Then we can describe a K-finite coefficient of π as follows.

Proposition 3.3.2. *By keeping the above notations, let $H : Sp(n)\times Sp(n)\to\mathbb{C}$ be a function satisfying the following properties:*

- *$H(u_1mg_1, \bar{u_2}mg_2) = H(g_1,g_2)$ for $u_1\in U_{n_1}, \bar{u_2}\in\bar{U}_{n_1}, m\in M_{n_1}$;*
- *For any $g_1,g_2\in Sp(n)$ the function $m\mapsto H(mg_1,g_2)$ is a coefficient for $\sigma\otimes\delta^{1/2}_{P_{n_1}}$;*
- *H is C^∞ and $K\times K$-finite on the right,*

where $\sigma = \sigma_1 \otimes \sigma_2$, $\delta_{P_{n_1}}$ *is the module of* P_{n_1}, *and* K *is the standard maximal compact subgroup of* $Sp(n)$. *Then the function* f *defined by*

$$f(g) = \int_{M_{n_1}\backslash G} H(hg,h)dh = \int_{\overline{U}_{n_1}\times K} H(\bar{u}kg,k)dkd\bar{u}$$

is absolutely convergent and a K*-finite coefficient of* π.

This proposition is essentially due to Jacquet, and the $\mathrm{GL}(n)$ version is stated in (5.5) of [J1] without a proof. Since no proof is given there, we will provide a detail proof here using $\mathrm{Sp}(n)$ as our group. (The proof also works for the non-archimedean case.) We need two lemmas to prove this proposition. The first one is

Lemma 3.3.3. *By keeping the above notations and assumptions,* π *is infinitesimally equivalent to the quotient of* $\mathrm{Ind}_{P_{n_1}}^{Sp(n)}(\sigma_1 \otimes \sigma_2)$ *whose underlying Harish-Chandra module is identified with the image of the absolutely convergent intertwining integral* $J : \mathrm{Ind}_{P_{n_1}}^{Sp(n)}(\sigma_1 \otimes \sigma_2) \to \mathrm{Ind}_{\overline{P}_{n_1}}^{Sp(n)}(\sigma_1 \otimes \sigma_2)$ *given by*

$$J(f)(g) = \int_{\overline{U}_{n_1}} f(\bar{u}g)\ d\bar{u},$$

i.e. *the image of* I *described above is isomorphic to that of* J *as Harish-Chandra modules.*

Proof. First let us write

$$\begin{aligned} P_1 &= P_{n_1\ldots n_t}, & P_2 &= P_{n_1}, & P_3 &= P_{n_2\ldots n_t} \\ U_1 &= U_{n_1\ldots n_t}, & U_2 &= U_{n_1}, & U_3 &= U_{n_2\ldots n_t}. \end{aligned}$$

First note that, by induction in stages, we have the isomorphism

$$\begin{array}{ccc} \mathrm{Ind}_{P_1}^{\mathrm{Sp}(n)}(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_{t+1}) & \xrightarrow{\sim} & \mathrm{Ind}_{P_2}^{\mathrm{Sp}(n)}\mathrm{Ind}_{\mathrm{GL}(n_1)\times P_3}^{\mathrm{GL}(n_1)\times\mathrm{Sp}(n-n_1)}(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_{t+1}) \\ f & \mapsto & \bar{f}, \end{array}$$

where $\bar{f}(g)(h) = f(hg)$ for $g \in \mathrm{Sp}(n)$ and $f \in \mathrm{GL}(n_1) \times \mathrm{Sp}(n-n_1)$. Also since σ is the Langlands quotient of $\mathrm{Ind}_{P_3}^{\mathrm{Sp}(n-n_1)}(\tau_2 \otimes \cdots \otimes \tau_{t+1})$, we have the surjective $\mathrm{GL}(n_1) \times \mathrm{Sp}(n-n_1)$ homomorphism

$$I_1 : \mathrm{Ind}_{\mathrm{GL}(n_1)\times P_3}^{\mathrm{GL}(n_1)\times\mathrm{Sp}(n-n_1)}(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_{t+1}) \twoheadrightarrow \sigma_1 \otimes \sigma_2 \subset \mathrm{Ind}_{\mathrm{GL}(n_1)\times \overline{P}_3}^{\mathrm{GL}(n_1)\times\mathrm{Sp}(n-n_1)}(\sigma_1 \otimes \sigma_2),$$

in which, for each K-finite $F \in \mathrm{Ind}_{\mathrm{GL}(n_1)\times P_3}^{\mathrm{GL}(n_1)\times\mathrm{Sp}(n-n_1)}(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_{t+1})$, $I_1(F)$ is given by

$$I_1(F)(h) = \int_{\bar{U}_3} F(\bar{u}_3 h)\, d\bar{u}_3,$$

for $h \in \mathrm{GL}(n_1) \times \mathrm{Sp}(n-n_1)$. (Recall $\sigma_1 = \tau_1$.) Then I_1 induces the $\mathrm{Sp}(n)$ homomorphism

$$\begin{aligned}\varphi : \operatorname{Ind}_{P_2}^{\mathrm{Sp}(n)} &\operatorname{Ind}_{\mathrm{GL}(n_1)\times P_3}^{\mathrm{GL}(n_1)\times \mathrm{Sp}(n-n_1)}(\tau_1 \otimes \cdots \otimes \tau_{t+1}) \\ &\to \operatorname{Ind}_{P_2}^{\mathrm{Sp}(n)}(\sigma_1 \otimes \sigma_2) \subset \operatorname{Ind}_{P_2}^{\mathrm{Sp}(n)} \operatorname{Ind}_{\mathrm{GL}(n_1)\times \overline{P}_3}^{\mathrm{GL}(n_1)\times \mathrm{Sp}(n-n_1)}(\tau_1 \otimes \cdots \otimes \tau_{t+1})\end{aligned}$$

given by $\varphi(\bar{f})(g) = I_1(\bar{f}(g))$ for $\bar{f} \in \operatorname{Ind}_{P_2}^{\mathrm{Sp}(n)} \operatorname{Ind}_{\mathrm{GL}(n_1)\times P_3}^{\mathrm{GL}(n_1)\times \mathrm{Sp}(n-n_1)}(\tau_1 \otimes \cdots \otimes \tau_{t+1})$ and $g \in \mathrm{Sp}(n)$. Notice that φ is surjective due to the exactness of Ind.

Now for each K-finite $\varphi(\bar{f}) \in \operatorname{Ind}_{P_2}^{\mathrm{Sp}(n)}(\sigma_1 \otimes \sigma_2)$, define $J(\varphi(\bar{f})) \in \operatorname{Ind}_{\overline{P}_2}^{\mathrm{Sp}(n)}(\sigma_1 \otimes \sigma_2) \subset \operatorname{Ind}_{\overline{P}_2}^{\mathrm{Sp}(n)} \operatorname{Ind}_{\mathrm{GL}(n_1)\times \overline{P}_3}^{\mathrm{GL}(n_1)\times \mathrm{Sp}(n-n_1)}(\tau_1 \otimes \cdots \otimes \tau_{t+1})$ by

$$J(\varphi(\bar{f}))(g) = \int_{\overline{U}_2} \varphi(\bar{f})(\bar{u}_2 g)\, d\bar{u}_2,$$

for $g \in \mathrm{Sp}(n)$. We need to show the convergence of this integral. Since $\operatorname{Ind}_{\overline{P}_2}^{\mathrm{Sp}(n)}(\sigma_1 \otimes \sigma_2) \subset \operatorname{Ind}_{\mathrm{GL}(n_1)\times \overline{P}_3}^{\mathrm{GL}(n_1)\times \mathrm{Sp}(n-n_1)}(\tau_1 \otimes \cdots \otimes \tau_{t+1})$, it suffices to show the convergence of $J(\varphi(\bar{f}))(g)(h)$ for each $h \in \mathrm{GL}(n_1) \times \mathrm{Sp}(n-n_1)$. But

$$\begin{aligned}J(\varphi(\bar{f}))(g)(h) &= \int_{\overline{U}_2} \varphi(\bar{f})(\bar{u}_2 g)(h)\, d\bar{u}_2 \\ &= \int_{\overline{U}_2} I_1(\bar{f}(\bar{u}_2 g))(h)\, d\bar{u}_2 \\ &= \int_{\overline{U}_2} \int_{\overline{U}_3} \bar{f}(\bar{u}_2 g)(\bar{u}_3 h)\, d\bar{u}_3 d\bar{u}_2 \\ &= \int_{\overline{U}_2} \int_{\overline{U}_3} f(\bar{u}_3 h \bar{u}_2 g)\, d\bar{u}_3 d\bar{u}_2 \\ &= (\text{constant}) \int_{\overline{U}_2} \int_{\overline{U}_3} f(\bar{u}_3 \bar{u}_2 h g)\, d\bar{u}_3 d\bar{u}_2 \\ &= (\text{constant}) \int_{\overline{U}_1} f(\bar{u}_1 h g) d\bar{u}_1,\end{aligned}$$

where to obtain the fifth equality we used $h\bar{u}_2 = h\bar{u}_2 h^{-1} h$ and the invariance (up to constant) of the measure $d\bar{u}_2$ by the conjugation by h, and to obtain the last equality we used $\overline{U}_1 = \overline{U}_3 \overline{U}_2$. Thus the integral operator J converges for all K-finite $\varphi(\bar{f})$, because $\int_{\overline{U}_1} f(\bar{u}_1 hg) d\bar{u}_1$ converges for all K-finite f. Moreover via the isomorphism

$$\operatorname{Ind}_{\overline{P}_2}^{\mathrm{Sp}(n)} \operatorname{Ind}_{\mathrm{GL}(n_1)\times \overline{P}_3}^{\mathrm{GL}(n_1)\times \mathrm{Sp}(n-n_1)}(\tau_1 \otimes \cdots \otimes \tau_{t+1}) \to \operatorname{Ind}_{\overline{P}_1}^{\mathrm{Sp}(n)}(\tau_1 \otimes \cdots \otimes \tau_{t+1}),$$

we can see that the image of J indeed coincides with that of I for all K-finite vectors. Thus the proposition follows. □

The second lemma we need is the following. (For this lemma, the author would like to thank H. Jacquet, who kindly showed a variant of the proof via a private correspondence.)

Lemma 3.3.4. *Let H be as in Proposition 3.3.2. Then H is of the form*

$$H(g_1, g_2) = \sum_i \langle f_i(g_1), f_i'(g_2) \rangle,$$

for some $f_i \in \mathrm{Ind}_{P_{n_1}}^G(\sigma)$ and $f_i' \in \mathrm{Ind}_{\overline{P}_{n_1}}^G(\sigma^\vee)$, both of which are K-finite.

Proof. Let us simply write $M = M_{n_1}$, $P = P_{n_1}$, $\overline{P} = \overline{P}_{n_1}$, $U = U_{n_1}$, and $\overline{U} = \overline{U}_{n_1}$, and also write $\tau = \sigma \otimes \delta_P^{1/2}$. Let $\mathcal{M}$ be the space of coefficients of τ. Then the group $M \times M$ acts on $\mathcal{M}$ by

$$(m_1, m_2) \cdot \varphi(m) = \varphi(m_2^{-1} m m_1), \quad \text{for} \quad m \in M, \quad (m_1, m_2) \in M \times M.$$

Notice that $V_\tau \otimes V_{\tau^\vee} \cong \mathcal{M}$ as vector spaces via $(v, w) \mapsto f_{v,w}$. (Here $f_{v,w}$ is as in the notation section.) So via this isomorphism, $M \times M$ acts on $V_\tau \otimes V_{\tau^\vee}$. Moreover it is easy to see that the representation obtained by this action of $M \times M$ on $V_\tau \otimes V_{\tau^\vee}$ is isomorphic to $\tau|_M \otimes \tau^\vee|_M$ as a smooth admissible representation.

Now define a function $F : K \times K \to \mathcal{M}$ by

$$F(k_1, k_2)(m) = H(mk_1, k_2).$$

Then via the isomorphism $V_\tau \otimes V_{\tau^\vee} \cong \mathcal{M}$, we have

$$F \in \mathrm{Ind}_{(K\cap M)\times(K\cap M)}^{K\times K}(\tau_0 \otimes \tau_0^\vee),$$

where $\tau_0 = \tau|_{K\cap M}$ and $\tau_0^\vee = \tau^\vee|_{K\cap M}$. This can be seen as follows. First notice that for all $g_1, g_2 \in \mathrm{Sp}(n)$ and $m \in M$, we have $H(mg_1, g_2) = H(m^{-1}mg_1, m^{-1}g_2) = H(g_1, m^{-1}g_2)$. Then if we write $H(mk_1, k_2) = \sum_i \langle \tau(m)v_i, w_i \rangle$, we have, for $m_1,\ m_2 \in M \cap K$,

$$\begin{aligned} F(m_1k_1, m_2k_2)(m) &= H(mm_1k_1, m_2k_2) \\ &= H(m_2^{-1}mm_1k_1, k_2) \\ &= \sum_i \langle \tau(m_2^{-1}mm_1)v_i, w_i \rangle \\ &= (m_1, m_2) \cdot F(k_1, k_2)(m), \end{aligned}$$

and so $F(m_1k_1, m_2k_2) = (m_1, m_2) \cdot F(k_1, k_2)$.

Now notice that

$$\mathrm{Ind}_{(K\cap M)\times(K\cap M)}^{K\times K}(\tau_0 \otimes \tau_0^\vee) \cong \mathrm{Ind}_{K\cap M}^K \tau_0 \otimes \mathrm{Ind}_{K\cap M}^K \tau_0^\vee.$$

Thus we have

$$F(k_1,k_2)=\sum_i f_i(k_1)\otimes \tilde{f}_i(k_2),$$

for some $f_i\in \operatorname{Ind}_{K\cap M}^{K}\tau_0$ and $f_i'\in \operatorname{Ind}_{K\cap M}^{K}\tau_0^\vee$. By viewing $f_i(k_1)\otimes f_i'(k_2)\in V_\tau\otimes V_{\tau^\vee}$ as an element in $\mathcal{M}$ by the isomorphism defined above, we have

$$F(k_1,k_2)(m)=H(mk_1,k_2)=\sum_i\langle \tau(m)f_i(k_1),f_i'(k_2)\rangle.$$

Since H is $K\times K$-finite on the right, we see that f_i's and f_i''s are K-finite on the right. Now we can extend the domain of f_i's and f_i''s from K to the all of $\mathrm{Sp}(n)$ by

$$\begin{aligned} f_i(u_1m_1k_1)&=\tau(m_1)f_i(k_1)\\ f_i'(\bar{u}_2m_2k_2)&=\tau^\vee(m_2)\tilde{f}_i(k_2), \end{aligned}$$

for $u_1\in U$, $\bar{u}_1\in \overline{U}$ and $m_1,m_2\in M$. Those are well defined and indeed $f_i\in \operatorname{Ind}_P^G(\sigma)$ and $f_i'\in \operatorname{Ind}_{\overline{P}}^G(\sigma^\vee)$, because $\tau=\sigma\otimes\delta_P^{1/2}$ and $\tau^\vee=\sigma^\vee\otimes\delta_{\overline{P}}^{1/2}$, and also they are K-finite. □

Now we are ready to prove Proposition 3.3.2.

Proof of Proposition 3.3.2. Let us simply write $G=\mathrm{Sp}(n), P_{n_1}=P, U_{n_1}=U, \overline{U}_{n_1}=\overline{U}$, and $M_{n_1}=M$. Let us also write $\eta=\operatorname{Ind}_P^G(\sigma)$ and $\eta'=\operatorname{Ind}_{\overline{P}}^G(\sigma^\vee)$, where $\sigma^\vee$ is extended from M to $\overline{P}$ by letting $\overline{U}$ act trivially. Let J and J' be the intertwining operators for η and η', respectively, as defined in Lemma 3.3.3. Then $\pi\cong\eta/\ker J$ and $\pi^\vee\cong\eta'/\ker J'$, where $\ker J$ and $\ker J'$ are characterized by the property that, for all K-finite $f\in\ker J$ and $f'\in\ker J'$,

$$\begin{aligned} J(f)(g)&=\int_{\overline{U}}\langle f(\bar{u}g),\tilde{v}\rangle\ d\bar{u}=0\quad\text{for all}\quad \tilde{v}\in\sigma^\vee,\quad\text{and}\\ J'(f')(g)&=\int_U\langle v,f'(ug)\rangle\ du=0\quad\text{for all}\quad v\in\sigma. \end{aligned}$$

Then if we write $\bar{f}$ and $\bar{f}'$ for the images in $\eta/\ker J$ and $\eta'/\ker J'$, respectively, then the canonical pairing of π and $\pi^\vee$ is given by

$$\langle\bar{f},\bar{f}'\rangle=\int_{M\backslash G}\langle f(h),f'(h)\rangle\ dh,$$

for f,f' K-finite. This can be proven as follows. First of all, clearly the function $g\mapsto\langle f(g),f'(g)\rangle$ is M-invariant, and so the integral makes sense. Second of all, this integral absolutely converges, because

$$\begin{aligned} \int_{M\backslash G}\langle f(h),f'(h)\rangle\ dh&=\int_{\overline{P}\backslash G}\int_{\overline{U}}\langle f(\bar{u}k),f'(\bar{u}k)\rangle\ d\bar{u}dk\\ &=\int_K\int_{\overline{U}}\langle f(\bar{u}k),f'(k)\rangle\ d\bar{u}dk, \end{aligned}$$

where the integral $\int_{\overline{U}} \langle f(\bar{u}k), f'(k)\rangle \, d\bar{u}$ converges absolutely by Lemma 3.3.3. And finally, the characterizing property of $\ker J$ and $\ker J'$ guarantees that the integral is independent of the choice of the representatives of $\bar{f}$ and $\bar{f}'$. Therefore a coefficient of π is a finite $\mathbb{C}$ linear combination of functions of the form

$$g \mapsto \langle \pi(g)\bar{f}, \bar{f}'\rangle = \int_{M\backslash G} \langle f(hg), f'(h)\rangle \, dh = \int_K \int_{\overline{U}} \langle f(\bar{u}kg), f'(k)\rangle \, d\bar{u}dk,$$

where $f \in \eta$ and $f' \in \eta'$ are K-finite.

Now if H is a function satisfying all the three properties, then by Lemma 3.3.4 we have

$$H(g_1, g_2) = \sum_i \langle f_i(g_1), f_i'(g_2)\rangle$$

for some $f_i \in \eta$'s and $f_i' \in \eta'$'s, all of which are K-finite. Thus the proposition follows. □

To obtain Lemma 3.2.1, we need to combine Proposition 3.3.2 with the following.

Lemma 3.3.5. *Assume that σ is an irreducible admissible representation of $O(X, F)$ such that $\theta_n(\sigma)$ exists for $n = \frac{1}{2}\dim X$. Then for $k > n$, $\theta_k(\sigma)$ is infinitesimally equivalent to the Langlands quotient of* $\mathrm{Ind}\,^{Sp(n)}_{P_{n_1 \cdots n_t}}(\tau_1 \otimes \cdots \otimes \tau_t \otimes \tau)$ *for some parabolic $P_{n_1 \dots n_t}$ with $n_1 = 1$ and $\tau_1 = \chi_X |\cdot|^{k-n}$.*

Proof. For $F = \mathbb{R}$, this is just (a part of) Theorem 6.2(1) of [P]. The case for $F = \mathbb{C}$ is identical to the proof of [P, Theorem 6.2] by using Induction Principle and computation of LKT. (See [A].) □

Now all the necessary ingredients to remove the archimedean conditions on the result of Roberts [Rb4] for the case $F = \mathbb{R}$ are given, and thus Lemma 3.2.1 for $F = \mathbb{C}$ can be proven by exactly the same computation as in [Rb4].

Finally to apply his argument to the $F = \mathbb{C}$ case, we need the following.

Lemma 3.3.6. *Assume $F = \mathbb{C}$. The local zeta integral $Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ as in Proposition 8.7 of [Rb4] has a simple pole at $k - n$ for suitably chosen f_1 and Φ_1. (See [Rb4] for the notations.)*

Proof. Let $\phi_1 \in C_c^\infty(\mathbb{C}^\times)$ have support in the ball $B = B(1, \epsilon) \subset \mathbb{C}$ of radius $\epsilon < 1$ and center 1, and $\phi_1(1) = 0$. Let Φ_1 be the section obtained from ϕ_1. (See the proof of Proposition 8.7 of [Rb4] for the detail.) As in the real case [Rb4], for sufficiently large $\Re(s)$,

$$Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) = \int_{\mathbb{C}^\times} |z - 1|_{\mathbb{C}}^{-s} \phi_1'(\frac{z}{z-1})\delta(z)|z|_{\mathbb{C}}^{-1} \, dz,$$

where $\phi_1' = |\cdot|_{\mathbb{C}}^{-n}$ and $\delta = |\cdot|_{\mathbb{C}}^{k-n}$. Let $z \in \mathbb{C}^\times, z \neq 1$. Then $z/(z-1) \in B$ if and only if $|z-1| > \frac{1}{\epsilon}$. (Here note that $|z|$ is the usual complex norm, and $|z|_{\mathbb{C}} = z\bar{z} = |z|^2$.) Then by letting $w = 1/(z-1)$, we have

$$\begin{aligned} Z(s-1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) &= \int_{0<|w|<\epsilon} |w|_{\mathbb{C}}^{s-2} \phi_1'(1+w) \left|\frac{w+1}{w}\right|_{\mathbb{C}}^{k-n} \left|\frac{w+1}{w}\right|_{\mathbb{C}}^{-1} dw \\ &= \int_{0<|w|<\epsilon} |w|_{\mathbb{C}}^{s-(k-n)-1} \phi_1''(1+w)\ dw, \end{aligned}$$

where $\phi_1'' = |\cdot|_{\mathbb{C}}^{k-n-1}\phi_1'$. Now by letting $w = re^{i\theta}$, we have $dw = 2rdrd\theta$ and

$$\begin{aligned} \int_{0<|w|<\epsilon} |w|^{s-(k-n)-1} \phi_1''(1+w)\ dw &= \int_{0<r<\epsilon} \int_{0\leq\theta\leq 2\pi} r^{2(s-(k-n))-1} \phi_1''(1+re^{i\theta})\ 2rdrd\theta \\ &= \int_0^\epsilon g(r) r^{2(s-(k-n))-1} dr, \end{aligned}$$

where $g(r) = 2\int_0^{2\pi} \phi_1''(1+re^{i\theta})$. The function g is smooth, has a compact support, and $g(0) \neq 0$. By Lemma 8.6 of [Rb4], there exists an entire function $\tilde{g}$ on $\mathbb{C}$ such that

$$\int_0^\epsilon g(r) r^{2(s-(k-n))-1} dr = \tilde{g}(s)\Gamma(2(s-(k-n))).$$

Thus the lemma follows. □

This lemma is the complex analogue of Proposition 8.7 of [Rb4]. The rest of the proof of Lemma 3.2.1 is identical to the real case.

Chapter 4

Theta lifting for similitude groups

In this chapter, we will first review the theory of both local and global theta lifting for groups of similitudes, and then discuss some relations between the two. Main references for similitude theta lifting are [Rb5], and [H-K].

Let us keep the notations of the previous section. The theory of the Weil representation and the theta lifting can be extended to the pair $(\mathrm{GSp}(n), \mathrm{GO}(X))$ in the following way. First let ν denote the multiplier characters of $\mathrm{GO}(X)$ and $\mathrm{GSp}(n)$. (We use the same letter ν because there will not be any danger of confusion.) Now consider

$$R = \{(g, h) \in \mathrm{GSp}(n) \times \mathrm{GO}(X) : \nu(g)\nu(h) = 1\}.$$

Clearly $\mathrm{Sp}(n) \times \mathrm{O}(X) \subset R$. We have the obvious homomorphism

$$\iota : R \longrightarrow \mathrm{Sp}(\mathbb{W}),$$

whose restriction to $\mathrm{Sp}(n) \times \mathrm{O}(X)$ is the ι defined in the previous section.

First assume F is non-archimedean. Just as in the isometry case, if $m = \dim X$ is even, when restricted to the group $\iota(R)$, the projective representation of $\mathrm{Sp}(\mathbb{W})$ on the space $\mathcal{S}(\mathbb{W}_1)$ can be shown to be a representation, *i.e.* the cocycle c can be shown to be trivial, and thus gives rise to a representation of R on the space $\mathcal{S}(\mathbb{W}_1)$. We call it the extended Weil representation of R, and denote it by $\omega_{n,X}$ or simply ω. (We use the same symbol $\omega_{n,X}$ or ω as the isometry case, but this will not cause any confusion.) Clearly, the restriction of the extended Weil representation to $\mathrm{Sp}(n) \times \mathrm{O}(X)$ is the Weil representation defined in the previous section.

Let $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{GSp}(n))$. We say that σ and Π correspond, or σ corresponds to Π if there is a non-zero R homomorphism from $\omega_{n,X}$ to $\Pi \otimes \sigma$, *i.e.* $\mathrm{Hom}_R(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$. Let $\mathrm{GSp}(n)^+ = \{g \in \mathrm{GSp}(n) : \nu(g) \in \nu(\mathrm{GO}(X))\}$. If the residue characteristic of F is odd, it is known that the relation $\mathrm{Hom}_R(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$ defines a graph of bijection between subsets of $\mathrm{Irr}(\mathrm{GSp}(n)^+)$ and $\mathrm{Irr}(\mathrm{GO}(X))$ (the Howe duality principle). (This follows from Theorem 4.4 of [Rb1] together with the multiplicity one theorem of [Ad-Pr, Theorem 1.4].) Unlike the isometry case, it is still unknown if the group $\mathrm{Irr}(\mathrm{GSp}(n)^+)$ can be replaced by $\mathrm{GSp}(n)$ even for the odd residual characteristic case, although it is known to be true for certain cases. (See Theorem 1.8 of [Rb5].) For our purpose, however, the following is enough.

Lemma 4.0.7. *Let X be a four dimensional quadratic form over a non-archimedean local field F of* $\operatorname{char} F \neq 2$*. First assume the residual characteristic of F is odd. Then if $\sigma \in \operatorname{Irr}(GO(X))$ corresponds to $\Pi \in \operatorname{Irr}(GSp(2))$, then Π is unique. Next assume the residual characteristic is even. Then the same holds as long as σ is tempered, and in this case Π is also tempered.*

Proof. This just a part of Theorem 1.8 of in [Rb5], the proof of which is also valid even if $\operatorname{char} F > 2$. □

Thus for the odd residual case, if $\sigma \in \operatorname{Irr}(\mathrm{GO}(X))$ corresponds to $\Pi \in \operatorname{Irr}(\mathrm{GSp}(2))$, we denote Π by $\theta_2(\sigma)$, and for the even residual case, if σ is tempered and corresponds to Π, then we denote Π by $\theta_2(\sigma)$.

Next assume F is archimedean. Then just as in the archimedean case, the extended Weil representation $\omega_{n,X}$ on $\mathcal{S}(X(F)^n)$ is defined, which is a smooth Fréchet representation of the group R of moderate growth in the sense of [Cs]. So in particular $(\omega^{\mathrm{H}})^{\mathrm{CW}} = \omega$. Then let $\sigma \in \operatorname{Irr}(\mathrm{GO}(X))$ and $\Pi \in \operatorname{Irr}(\mathrm{GSp}(n))$. We say that σ and Π correspond, or σ corresponds to Π if there is a non-zero homomorphism of Harish-Chandra modules from $(\omega_{n,X})^{\mathrm{H}}$ to $((\Pi \otimes \sigma)|_R)^{\mathrm{H}}$, *i.e.* $\operatorname{Hom}((\omega_{n,X})^{\mathrm{H}}, ((\Pi \otimes \sigma)|_R)^{\mathrm{H}}) \neq 0$, where Hom means the set of homomorphisms of Harish-Chandra modules for smooth representations of R. Just as in the non-archimedean case, although the Howe duality for similitude groups is not known in full generality, we only need the following for our purposes.

Lemma 4.0.8. *Let X be a four dimensional quadratic form over $F = \mathbb{R}$ or $\mathbb{C}$. Then if $\sigma \in \operatorname{Irr}(GO(X))$ corresponds to $\Pi \in \operatorname{Irr}(GSp(2))$, then Π is unique up to infinitesimal equivalence.*

Proof. This is again essentially a part of Theorem 1.8 of in [Rb5]. In [Rb5], the signature of X is assumed to be of the form (p, q) with both p and q even, but this assumption is unnecessary. Also if $F = \mathbb{C}$, this is obvious because in this case we have $\mathrm{GSp}(n)^+ = \mathrm{GSp}(n)$. □

Thus just as we did for the isometry case, if $\sigma \in \operatorname{Irr}(\mathrm{GO}(X))$ corresponds to $\Pi \in \operatorname{Irr}(\mathrm{GSp}(2))$, we write $(\Pi^{\mathrm{H}})^{\mathrm{CW}} = \theta_n(\sigma)$, where $(\Pi^{\mathrm{H}})^{\mathrm{CW}}$ is the Casselman-Wallach canonical completion of Π^{H} as in the notation section, and we call it the local theta lift of σ. If σ does not correspond to any $\Pi \in \operatorname{Irr}(\mathrm{Sp}(n))$, then we say that the theta lift of σ vanishes and write $\theta_n(\sigma) = 0$. Again just for the isometry case, if σ and Π correspond, we have a non-zero homomorphism $(\omega_{n,X})^{\mathrm{H}} \to \Pi^{\mathrm{H}} \otimes \sigma^{\mathrm{H}}$ of Harish-Chandra modules, which gives rise to a non-zero R homomorphism $\omega_{n,X} \to (\Pi^{\mathrm{H}})^{\mathrm{CW}} \otimes (\sigma^{\mathrm{H}})^{\mathrm{CW}}$ of smooth representations by [Cs, Corollary 10.5]. (Here notice that $((\omega_{n,X})^{\mathrm{H}})^{\mathrm{CW}} = \omega_{n,X}$ by our choice of $\omega_{n,X}$ as a smooth representation.)

The extended Weil representation for the global case is also defined in the same way. Namely, $R(\mathbb{A}_F)$ acts on the space $\mathcal{S}(X(\mathbb{A}_F)^n)$ in such a way that the restriction of this action to $\mathrm{O}(X,\mathbb{A}_F)\times\mathrm{Sp}(n,\mathbb{A}_F)$ is the Weil representation discussed above. (See [Rb5], and [H-K] for the detail.) We also call this representation of $R(\mathbb{A}_F)$ the extended Weil representation of $R(\mathbb{A}_F)$, which we also denote by $\omega_{n,X}$ or simply by ω.

Now we develop the theory of global theta lifting for similitude groups. First, define the theta kernel by

$$\theta(g,h;\varphi)=\sum_{x\in X(F)^n}\omega(g,h)\varphi(x),$$

for $(g,h)\in R(\mathbb{A})$ and $\varphi\in\mathcal{S}(X(\mathbb{A}_F)^n)$. Then for each automorphic representation σ of $\mathrm{GO}(X,\mathbb{A}_F)$ with central character χ and for $f\in V_\sigma$, consider the integral

$$\theta(f;\varphi)(g)=\int_{\mathrm{O}(X,F)\backslash\mathrm{O}(X,\mathbb{A}_F)}\theta(g,h_1h;\varphi)f(h_1h)\,dh_1$$

where $h\in\mathrm{GO}(X,\mathbb{A}_F)$ is any element such that $\nu(g)\nu(h)=1$. For a suitable choice of the Haar measure dh_1 as in [H-K], it can be shown that this integral is absolutely convergent. Also the invariance property of the measure guarantees that this integral is independent of the choice of h. Now set $\mathrm{GSp}(n,\mathbb{A}_F)^+=\{g\in\mathrm{GSp}(n,\mathbb{A}_F):\nu(g)\in\nu(\mathrm{GO}(X,\mathbb{A})\}$. Then $\theta(f;\varphi)$ is a function on $\mathrm{GSp}(n,\mathbb{A}_F)^+$ which is left $\mathrm{GSp}(n,F)^+$ invariant, *i.e.* it is a function on $\mathrm{GSp}(n,F)^+\backslash\mathrm{GSp}(n,\mathbb{A}_F)^+$. We can extend this function to an automorphic form on $\mathrm{GSp}(n,\mathbb{A}_F)$ by insisting that it is left $\mathrm{GSp}(n,F)$ invariant and zero outside $\mathrm{GSp}(n,F)\mathrm{GSp}(n,\mathbb{A}_F)^+$. We denote this automorphic form also by $\theta(f;\varphi)$, whose central character is $\chi^{-1}\chi_V^n$. Then just as in the isometry case, we denote by $\Theta_n(V_\sigma)$ the space generated by the automorphic forms $\theta(f;\varphi)$ for all $f\in V_\sigma$ and all $\varphi\in\mathcal{S}(V(\mathbb{A}_F)^n)$. Again $\mathrm{GSp}(n,\mathbb{A}_F)$ acts on $\Theta_n(V_\sigma)$ by right translation, and if $\Theta_n(V_\sigma)$ is in the space of non-zero cusp forms, then its irreducible constituent provides a cuspidal automorphic representation of $\mathrm{GSp}(n,\mathbb{A}_F)$. Let us denote this constituent by Π. If we write $\sigma\cong\otimes\sigma_v$ and $\Pi\cong\otimes\Pi_v$, then each $\sigma_v^\vee$ corresponds to Π_v. In particular, if $\dim X=4$ and $n=2$, by Lemma 4.0.7 and 4.0.8, we can write $\Pi\cong\otimes\theta_2(\sigma_v^\vee)\cong\otimes\theta_2(\sigma_v)^\vee$, assuming σ_v is tempered for $v|2$.

To consider the non-vanishing problem for our similitude case, we first consider the restriction to the isometry case. If f is an automorphic form on $\mathrm{GO}(X,\mathbb{A}_F)$, then clearly $f|_{\mathrm{O}(X,\mathbb{A}_F)}$ is an automorphic form on $\mathrm{O}(X,\mathbb{A}_F)$. The same thing can be said to automorphic forms on $\mathrm{GSp}(n,\mathbb{A})$. If V is a space of automorphic forms on $\mathrm{GSp}(n,\mathbb{A}_F)$, then we let $V|_{\mathrm{Sp}(n)}=\{f|_{\mathrm{Sp}(n,\mathbb{A}_F)}:f\in V\}$. Then we have

Lemma 4.0.9. *Let σ be an automorphic representation of $GO(X,\mathbb{A}_F)$. Then*

1. $\theta_n(f;\varphi)|_{Sp(n,\mathbb{A}_k)}=\theta_n(f|_{O(X,\mathbb{A}_F)};\varphi)$*, where* $\theta_n(f|_{O(X,\mathbb{A}_F)};\varphi)$ *is the isometry theta lift of* $f|_{O(X,\mathbb{A}_F)}$.

2. $\Theta(V_\sigma) \neq 0$ *if and only if* $\Theta(V_\sigma)|_{Sp(n)} \neq 0$.

3. $\Theta(V_\sigma)$ *is in the space of cusp forms if and only if* $\Theta(V_\sigma)|_{Sp(n)}$ *is.*

Proof. (1). Consider, for each $g \in \mathrm{Sp}(n, \mathbb{A}_F)$,

$$\begin{aligned}\theta_n(f;\varphi)|_{\mathrm{Sp}(n,\mathbb{A}_k)}(g) &= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g,h;\varphi)f(h)\,dh \\ &= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g,h;\varphi)f|_{\mathrm{O}(X,\mathbb{A}_F)}(h)\,dh \\ &= \theta_n(f|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi).\end{aligned}$$

(2). Assume $\Theta(V_\sigma) \neq 0$. Then for some $f \in V_\sigma$, $g \in \mathrm{GSp}(n, \mathbb{A}_F)$, and $\varphi \in \mathcal{S}(X(\mathbb{A}_F)^n)$, we have $\theta_n(f;\varphi)(g) \neq 0$. By definition of $\theta_n(f;\varphi)$, we may assume $g \in \mathrm{GSp}(n,\mathbb{A}_F)^+$. Let

$$g_1 = g \begin{pmatrix} I_n & 0 \\ 0 & \nu(g)^{-1}I_n \end{pmatrix},$$

so that $g_1 \in \mathrm{Sp}(n, \mathbb{A}_F)$. Then for an $h \in \mathrm{GO}(X, \mathbb{A}_F)$ with $\nu(g)\nu(h) = 1$, we have

$$\begin{aligned}\theta(f;\varphi)(g) &= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g,h_1h;\varphi)f(h_1h)\,dh_1 \\ &= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} \omega(g,h_1h)\varphi(x)\right) f(h_1h)\,dh_1 \\ &= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} |\nu(g)|^{\frac{mn}{2}}\omega(g_1,1)\varphi(x\circ h^{-1}h_1^{-1})\right) f(h_1h)\,dh_1 \\ &= |\nu(g)|^{\frac{mn}{2}} \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} \omega(g_1,1)\varphi'(x\circ h_1^{-1})\right) (h\cdot f)(h_1)\,dh_1 \\ &= |\nu(g)|^{\frac{mn}{2}} \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} \omega(g_1,h_1)\varphi'(x)\right) f'(h_1)\,dh_1 \\ &= |\nu(g)|^{\frac{mn}{2}} \theta_n(f'|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi')(g_1)\end{aligned}$$

where $\varphi' \in \mathcal{S}(X(\mathbb{A}_F)^n)$ is given by $\varphi'(x) = \varphi(x \circ h^{-1})$ and $f' = h \cdot f \in V_\sigma$. (For the action of $\omega(g, h_1h)$, see, for example, p.261 of [Rb5] together with Remark 4.0.10 below.) Now $\theta_n(f'|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi') = \theta_n(f';\varphi')|_{\mathrm{Sp}(n,\mathbb{A}_F)} \in \Theta(V_\sigma)|_{\mathrm{Sp}(n)}$. This proves the only if part.

Conversely, if $\Theta(V_\sigma)|_{\mathrm{Sp}(n)} \neq 0$, then for some $f \in V_\sigma$, $g \in \mathrm{Sp}(n, \mathbb{A}_F)$, and $\varphi \in \mathcal{S}(X(\mathbb{A}_F)^n)$, we have $\theta_n(f;\varphi)|_{\mathrm{Sp}(n,\mathbb{A}_k)}(g) \neq 0$. But clearly $\theta_n(f;\varphi)|_{\mathrm{Sp}(n,\mathbb{A}_k)}(g) = \theta_n(f;\varphi)(g)$. This completes the proof.

(3). First notice that $\mathrm{GSp}(n) \cong \mathrm{Sp}(n) \rtimes \mathbb{G}_m$ and under the obvious inclusion $\mathrm{Sp}(n) \hookrightarrow \mathrm{GSp}(n) \cong \mathrm{Sp}(n) \rtimes \mathbb{G}_m$ given by $h \mapsto (h, 1)$, if $P \subset \mathrm{Sp}(n)$ is a parabolic subgroup, then $P \rtimes \mathbb{G}_m$ is a parabolic subgroup of $\mathrm{GSp}(n)$, and every parabolic subgroup of $\mathrm{GSp}(n)$ is of this form. Then if $N_P \subset P$ is the unipotent radical of P, then N_P is also the unipotent radical of $P \rtimes \mathbb{G}_m$.

Now assume $\Theta(V_\sigma)$ is in the space of cusp forms. So for each $f \in \Theta(V_\sigma)$ and each N_P, we have $\int_{N_P(\mathbb{A}_F)} f(ng)\, dn = 0$ for all $g \in \mathrm{GSp}(n, \mathbb{A}_F)$. Thus $\int_{N_P(\mathbb{A}_F)} f(nh)\, dn = 0$ for all $h \in \mathrm{Sp}(n, \mathbb{A}_F)$. So $f|_{\mathrm{Sp}(n,\mathbb{A}_F)}$ is a cusp form, *i.e.* $\Theta(V_\sigma)|_{\mathrm{Sp}(n)}$ is in the space of cusp forms.

Conversely, assume $\Theta(V_\sigma)|_{\mathrm{Sp}(n)}$ is in the space of cusp forms. Then for each $f \in \Theta(V_\sigma)$ and each N_P, we have $\int_{N_P(\mathbb{A}_F)} f(nh)\, dn = 0$ for all $h \in \mathrm{Sp}(n, \mathbb{A}_F)$. Now for each $g \in \mathrm{GSp}(n, \mathbb{A}_F)$, we have to show $\int_{N_P(\mathbb{A}_F)} f(ng)\, dn = 0$. Let $g_1 \in \mathrm{Sp}(n, \mathbb{A}_F)$ be as in (2), and $f' = (g_1^{-1}g)\cdot f$. Then $f'|_{\mathrm{Sp}(n,\mathbb{A}_F)} \in \Theta(V_\sigma)|_{\mathrm{Sp}(n)}$. So $\int_{N_P(\mathbb{A}_F)} f'(ng_1)\, dn = 0$. But $\int_{N_P(\mathbb{A}_F)} f'(ng_1)\, dn = \int_{N_P(\mathbb{A}_F)} f(ng_1 g_1^{-1} g)\, dn = \int_{N_P(\mathbb{A}_F)} f(ng)\, dn$. This completes the proof. □

Remark 4.0.10. we should mention a certain conventional discrepancy found in the literature. In [Rb1] and [H-K], the extended Weil representation is defined for the group

$$R' = \{(g, h) \in \mathrm{GSp}(n) \times \mathrm{GO}(V) : \nu(g) = \nu(h)\}.$$

On the other hand in [HST] it is defined for our group R. Let us denote the extended Weil representations of R' by ω'. By direct computation, it can be shown that ω' is obtained from ω via the isomorphism $R' \to R$ given by $(g, h) \mapsto (\nu(g)^{-1}g, h)$. Then for the local case if $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ corresponds to $\Pi \in \mathrm{Irr}(\mathrm{GSp}(n))$ via ω, then π corresponds to $\tilde{\Pi}$ via ω' where $\tilde{\Pi}$ is defined by $\tilde{\Pi}(g) = \chi(\nu(g))^{-1}\Pi(g)$ for χ the central character of Π.

The choice of R seems to be completely conventional, but the reader should be aware that it also affects the global theta lift. Indeed if we use R', then for the integral $\theta(f;\varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g, h_1 h; \varphi) f(h_1 h)\, dh_1$, we have to choose h to be such that $\nu(g) = \nu(h)$. (Note that the integral in p.389 of [HST] is not quite correct.) Accordingly, the central character of $\theta(f;\varphi)$ is $\chi\chi_V^n$, which is proved in [H-K, Lemma 5.1.9].

We should also mention the following, whose proof is left to the reader.

Lemma 4.0.11. *Let σ be an automorphic representation of $GO(X, \mathbb{A}_F)$, and σ_1 an irreducible constituent of $\{f|_{O(X,\mathbb{A}_F)} : f \in V_\sigma\}$ as an automorphic representation of $O(X, \mathbb{A}_F)$. If we write $\sigma \cong \otimes\sigma_v$ and $\sigma_1 \cong \otimes\sigma_{1v}$, then each σ_{1v} is an irreducible constituent of the restriction $\sigma_v|_{O(X,F_v)}$ of σ_v to $O(X, F_v)$.*

Chapter 5

Local parameters of unramified theta lifts

After some preliminaries, we will explicitly compute the local parameters of the unramified theta lifts from GO(4) to GSp(1)(= GL(2)).

In this chapter, the groups $\mathrm{GO}(X, F_v), \mathrm{GSp}(n, F_v)$, etc are all denoted simply by GO(X), GSp(n), etc, and we put $F = F_v$. Moreover we assume that v is finite. Also "Ind" always means unnormalized induction, and whenever we use normalized induction, we use the notation "n-Ind". Thus, for example, if τ is the principal series representation of $G = \mathrm{GL}(2)$ induced from the standard parabolic P by the two unramified characters η and η', we have $\tau \cong \text{n-Ind}_P^G(\eta \otimes \eta') = \mathrm{Ind}_P^G(\tilde{\eta} \otimes \tilde{\eta'})$, where $\tilde{\eta} = |\cdot|^{1/2}\eta$ and $\tilde{\eta'} = |\cdot|^{-1/2}\eta'$. Also we agree that all the representations have unitary central characters. So for $\text{n-Ind}_P^G(\eta \otimes \eta')$, η and η' are either both unitary or otherwise of the form $\eta = \eta_0|\cdot|^s$ and $\eta' = \eta_0|\cdot|^{-s}$ where η_0 is unitary and $-\frac{1}{2} < s < \frac{1}{2}$.

5.1 Preliminaries

For our computation of the local parameters, we need the Jacquet module of the Weil representation, which is done in [HST], which, in turn, comes from [Kd1]. We will repeat the essential point, partly because [HST] contains some unclarities. For this, let us decompose X as $X = Y_r \oplus W \oplus Y_r^*$, where Y_r is a totally isotropic space and Y_r^* is its complement so that $Y_r \oplus Y_r^*$ is r copies of the hyperbolic plane. We denote the standard basis of Y_r by $\{f_1, f_2, \dots, f_r\}$, and write $l = \dim W$ so that $m = 2r + l$. Now let Q_r be the parabolic subgroup of GO(X) preserving the flag $\langle f_1 \rangle \subset \langle f_1, f_2 \rangle \subset \cdots \subset \langle f_1, f_2, \dots, f_r \rangle$ so that its Levi factor is isomorphic to $\mathrm{GO}(W) \times \mathbb{G}_m^r$. Let Q be the parabolic subgroup preserving the flag $\langle f_1, f_2, \dots, f_r \rangle$, so that its Levi factor is isomorphic to $\mathrm{GO}(W) \times \mathrm{GL}(r)$. Further let $S_Q = R \cap (\mathrm{GSp}(n) \times Q)$ be the parabolic subgroup of R whose Levi factor M_Q is isomorphic to $R_{n,W} \times \mathrm{GL}(r)$, where $R_{n,W}$ is defined in the same way as R, but with respect to

GSp(n) and GO(W). We denote by N_Q its unipotent radical. Also let S_{Q_r} be the parabolic subgroup of M_Q whose Levi factor M_{Q_r} is isomorphic to $R_{n,W}\times\mathbb{G}_m^r$, *i.e.* $S_{Q_r}=M_Q\cap(\mathrm{GSp}(n)\times Q_r)$. We denote by N_{Q_r} its unipotent radical. Now let P_i be the standard parabolic subgroup of GSp(n) whose Levi factor is isomorphic to $\mathbb{G}_m^i\times\mathrm{GSp}(n-i)$. Then we define S_{P_i,Q_r} to be the parabolic subgroup of M_{Q_r} whose Levi factor M_{P_i,Q_r} is isomorphic to $\mathbb{G}_m^i\times R_{n-i,W}\times\mathbb{G}_m^r$, *i.e.* $S_{P_i,Q_r}=M_{Q_r}\cap(P_i\times Q_r)$. We write a typical element in M_{P_i,Q_r} by $(\alpha_1,\cdots,\alpha_i,(g,h),\beta_1,\dots,\beta_r)$. Notice we have the inclusions $S_{P_i,Q_r}\subset M_{Q_r}\subset S_{Q_r}\subset M_Q$. Also we can set $P_0=\mathrm{GSp}(n)$ and so $S_{P_0,Q_r}=M_{P_0,Q_r}=M_{Q_r}$. Now the unnormalized Jacquet module of $\omega_{n,X}$ is computed as follows, which is nothing but Lemma 4 of [HST] with the notations adjusted to ours.

Proposition 5.1.1. *The unnormalized Jacquet module $J=J(\omega_{n,X})_{N_Q}$ of $\omega_{n,X}$ with respect to N_Q has a filtration*

$$0=J^{(s+1)}\subset J^{(s)}\subset\cdots\subset J^{(1)}\subset J^{(0)}=J$$

of M_Q-modules, where $s=\min\{n,r\}$. Let $I^{(i)}=J^{(i)}/J^{(i+1)}$. Then the unnormalized Jacquet module $I^{(i)}_{N_{Q_r}}$ of $I^{(i)}$ with respect to N_{Q_r}, which is an M_{Q_r}-module, is given by

$$I^{(i)}_{N_{Q_r}}=\mathrm{Ind}\,^{M_{Q_r}}_{S_{P_i,Q_r}}\sigma_{i,r},$$

where $\sigma_{i,r}$ is given by the representation of M_{P_i,Q_r} which is of the form

$$\begin{aligned}(\alpha_1,\cdots,\alpha_i,(g,h),\beta_1,\dots,\beta_r)\mapsto&|\nu(g)|^{nr/2-ni-li/4}|\alpha|^{n+l/2}(\alpha,(-1)^{l/2}D_W)|\beta|^n\\&\cdot\prod_{j=1}^{i}\mu_{r-i+j}(\alpha_{i-j+1}^{-1}\beta_{r-i+j}\nu(g))\omega_{n-i,W}(g,h)\end{aligned}$$

for some characters μ_{r-i+j}, where $\alpha=\alpha_1\cdots\alpha_i$, $\beta=\beta_1\cdots\beta_{r-i}$, $D_W=\mathrm{disc}W$ and $(,)$ is the Hilbert symbol.

Remark 5.1.2. In the above notations, if one of $n-i$ and W is zero, $\omega_{n-i,W}$ is taken to be the trivial representation. If $n-i$ is zero, we write a typical element in M_{P_i,Q_r} by $(\alpha_1,\cdots,\alpha_i,(h),\beta_1,\dots,\beta_r)$ where $h\in\mathrm{GO}(W)$, and we have to replace $\nu(g)$ by $\nu(h)^{-1}$ in the above formula. If W is zero, we write a typical element in M_{P_i,Q_r} by $(\alpha_1,\cdots,\alpha_i,(g),\beta_1,\dots,\beta_r)$, where $g\in\mathrm{GSp}(n-i)$, and g acts as in the above formula. If both $n-i$ and W are zero, we have $M_{P_i,Q_r}\cong\mathbb{G}_m^i\times\mathbb{G}_m\times\mathbb{G}_m^r$ and write a typical element by $(\alpha_1,\cdots,\alpha_i,(\lambda),\beta_1,\dots,\beta_r)$ if, for the natural projection $\iota:M_{P_i,Q_r}\to P_i$, $\nu(\iota(\alpha_1,\cdots,\alpha_i,(\lambda),\beta_1,\dots,\beta_r))=\lambda$, and we have to replace $\nu(g)$ by λ.

Remark 5.1.3. Although this is a small point, the reader should notice that the choice of the parabolic $R_{P_i,Q}$ in [HST] is not quite correct and should be replaced by our S_{P_i,Q_r}. Also in [HST] there is a misprint for the index of α inside ν_{r-i+j}.

The following lemma will be necessary later.

Lemma 5.1.4. *Keeping the above notations, let μ and δ be admissible representations of Q_r and P_i, respectively. Then the natural map* $\mathrm{Ind}\,_{P_i\times Q_r}^{GSp(n)\times GO(V)}(\delta\otimes\mu)\to \mathrm{Ind}\,_{R\cap(P_i\times Q_r)}^{R}(\delta\otimes\mu)$ *is an injective R-homomorphism.*

Proof. Let $F\in \mathrm{Ind}\,_{P_i\times Q_r}^{\mathrm{GSp}(n)\times\mathrm{GO}(V)}(\delta\otimes\mu)$ and $\bar{F}$ its image under the natural map, namely $\bar{F}=F|_R$. Assume $\bar{F}=0$. Then for each $(g,h)\in\mathrm{GSp}(n)\times\mathrm{GO}(V)$, let's define

$$u=u(g,h)=\begin{pmatrix} I_n & O\\ O & \nu(g)\nu(h)I_n\end{pmatrix}\in\mathrm{GSp}(n),$$

where I_n is the $n\times n$ identity matrix. Then $(u,1)\in P_i\times Q_r$, $\nu(u)=\nu(g)\nu(h)$, and $(u^{-1}g,h)\in R$. So we have

$$F(g,h)=F(uu^{-1}g,h)=(\delta\otimes\mu)(u,1)F(u^{-1}g,h)=(\delta\otimes\mu)(u,1)\bar{F}(u^{-1}g,h)=0.$$

Thus the map is injective. □

5.2 Computation of local parameters

We will compute the local parameters of unramified theta lifts from $\mathrm{GO}(X)$ to $\mathrm{GSp}(1)(=\mathrm{GL}(2))$. Let $d=\mathrm{disc}X$, $E=F(\sqrt{d})$, and $\sigma\in\mathrm{Irr}(\mathrm{GO}(X))$ be an unramified representation of $\mathrm{GO}(X)$. Then [Rb3] tells us that we may assume the following.

1. If $d=1$, then $X=\mathrm{M}_{2\times 2}(F)$, the space of 2×2 matrices over F with the quadratic form given by $-\det$, and $\mathrm{GO}(X)=\mathrm{GSO}(X)\rtimes\{1,t\}$ where t acts on X as matrix transpose, *i.e.* $t\cdot x=x^t$.

2. If $d\neq 1$, then $X=\{x\in\mathrm{M}_{2\times 2}(E)|\ {}^c x^t=x\}$ is the space of Hermitian matrices over E with the quadratic form given by $-\det$, and $\mathrm{GO}(X)=\mathrm{GSO}(X)\rtimes\{1,t\}$ where t acts on X as matrix transpose, *i.e.* $t\cdot x=x^t$.

First consider the case $d=1$. Then we have

$$0\longrightarrow F^\times\longrightarrow\mathrm{GL}(2)\times\mathrm{GL}(2)\xrightarrow{\ \rho\ }\mathrm{GSO}(X)\longrightarrow 0,$$

where the map ρ is given $\rho(g_1,g_2)(x)=g_1xg_2^t$ for $x\in X$ and $(g_1,g_2)\in\mathrm{GL}(2)\times\mathrm{GL}(2)$ with g_2^t the matrix transpose of g_2. By this exact sequence, there is a natural bijection between $\mathrm{Irr}(\mathrm{GSO}(X))$ and the set of irreducible admissible representations $\tau_1\otimes\tau_2$ of $\mathrm{GL}(2)\times\mathrm{GL}(2)$ such that τ_1 and τ_2 have the same central character. Namely if $\pi\in\mathrm{Irr}(\mathrm{GSO}(V))$, then we can define the corresponding $\tau_1\otimes\tau_2$ by taking $V_{\tau_1\otimes\tau_2}$ to be V_π, and defining the action by $(\tau_1\otimes\tau_2)(g_1,g_2)=\pi(\rho(g_1,g_2))$, and conversely, if $\tau_1\otimes\tau_2$ is an irreducible admissible representation with τ_1 and τ_2 having the same central character, we can define the corresponding representation π of $\mathrm{GSO}(V)$ again by taking $V_\pi = V_{\tau_1\otimes\tau_2}$, and defining the action by $\pi(g) = (\tau_1 \otimes \tau_2)(g_1, g_2)$ for some $\rho(g_1, g_2) = g$. This is indeed well-defined due to the condition on the central character. So if π corresponds to $\tau_1 \otimes \tau_2$, we write $\pi = \pi(\tau_1, \tau_2)$.

Hence if $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$, then either $\sigma|_{\mathrm{GSO}(X)} \cong \pi(\tau_1, \tau_2)$ for some τ_1 and τ_2 with $\tau_1 \cong \tau_2$ or $\sigma|_{\mathrm{GSO}(X)} \cong \pi(\tau_1, \tau_2) \oplus \pi(\tau_2, \tau_1)$ for some τ_1 and τ_2 with $\tau_1 \not\cong \tau_2$. By using the notations of Appendix A, the former is always written as $\sigma = (\pi, +)$ and the latter is either $\sigma = (\pi, +)$ or $\sigma = (\pi, -)$. If π is unramified, so is $\sigma = (\pi, +)$ but not $\sigma = (\pi, -)$

Then we have the following. (Note that the following proposition is known to be true not only for unramified σ but for any σ. But we will give our proof to illustrate our method.)

Proposition 5.2.1. *If σ is unramified,* i.e. $\sigma = (\pi, +)$ *for some unramified* $\pi = \pi(\tau_1, \tau_2)$ *and corresponds to* $\Pi \in \mathrm{Irr}(GSp(1))$*, then* $\tau_1 \cong \tau_2 \cong \Pi$*. In particular, such* Π *is unique.*

Proof. Assume σ corresponds to Π. Since τ_i is unramified, we can write $\tau_i = \text{n-Ind}\,(\eta_i \otimes \eta_i')$ for unramified characters η_i and η_i', or by using unnormalized induction, $\tau_i = \mathrm{Ind}_P^{\mathrm{GL}(2)}(\tilde{\eta_i} \otimes \tilde{\eta_i}')$, where $\tilde{\eta_i} = |\cdot|^{1/2}\eta_i$ and $\tilde{\eta_i}' = |\cdot|^{-1/2}\eta_i'$. Then $\pi = \pi(\tau_1, \tau_2) = \mathrm{Ind}_{Q_2}^{\mathrm{GSO}(X)}(\mu)$, where Q_2 is the parabolic preserving the flag

$$0 \subset \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle \subset \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle \subset V,$$

and μ is defined by

$$\mu(\begin{pmatrix} \beta_1 & * & * & * \\ 0 & \beta_2 & * & * \\ 0 & 0 & \lambda\beta_1^{-1} & * \\ 0 & 0 & 0 & \lambda\beta_2^{-1} \end{pmatrix}) = \tilde{\eta_1}'(\lambda)(\tilde{\eta_2}/\tilde{\eta_1}')(\beta_1)(\tilde{\eta_2}'/\tilde{\eta_1}')(\beta_2),$$

where the matrix representation of each element in $\mathrm{GO}(X)$ is with respect the ordered basis

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

(See p.299 in [Rb5] for the proof, although his conventions are slightly different from ours. Also notice that $Q_2 \subset \mathrm{GSO}(X)$ and so $\mathrm{Ind}_{Q_2}^{\mathrm{GSO}(X)}(\mu)$ makes sense.)

Now if $\tau_1 \not\cong \tau_2$, then $\sigma = \mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi$. If $\tau_1 \cong \tau_2$, then $V_\sigma = V_\pi$. In this case, we have the inclusion $\sigma \hookrightarrow \mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi (\cong \mathrm{Ind}_{Q_2}^{\mathrm{GO}(X)}\mu)$ of $\mathrm{GO}(X)$-modules. So in either case, if Π corresponds to σ, we have a non-zero R-homomorphism

$$\omega_{1,X} \longrightarrow \Pi \otimes \mathrm{Ind}_{Q_2}^{\mathrm{GO}(X)}\mu = \mathrm{Ind}_{\mathrm{GSp}(1)\times Q_2}^{\mathrm{GSp}(1)\times\mathrm{GO}(X)}\Pi \otimes \mu.$$

Moreover by Lemma 5.1.4 we have a non-zero R-homomorphism

$$\omega_{1,X} \longrightarrow \mathrm{Ind}\,_{R\cap(GSp(1)\times Q_2)}^{R}\Pi\otimes\mu.$$

Notice that, by induction in stages,

$$\mathrm{Ind}\,_{R\cap(GSp(1)\times Q_2)}^{R}\Pi\otimes\mu \cong \mathrm{Ind}\,_{S_Q}^{R}\mathrm{Ind}\,_{S_{Q_2}}^{M_Q}\Pi\otimes\mu,$$

where Q is the parabolic preserving the flag $\langle\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\rangle$, and the notations are as in Proposition 5.1.1.

Hence if we take the Jacquet module of $\omega_{1,X}$ with respect to N_Q, the Frobenius reciprocity together with Proposition 5.1.1 gives M_Q-homomorphisms

$$0\subset J^{(1)}\subset J^{(0)} \xrightarrow{\ \ \varphi\ \ } \mathrm{Ind}\,_{S_{Q_2}}^{M_Q}\Pi\otimes\mu,$$

where φ is non-zero. Suppose $\ker\varphi\supseteq J^{(1)}$. Then there is a non-zero M_Q-homomorphism

$$I^{(0)} = J^{(0)}/J^{(1)} \longrightarrow \mathrm{Ind}\,_{S_{Q_2}}^{M_Q}\Pi\otimes\mu.$$

So if we take the Jacquet module of $I^{(0)}$ with respect to N_{Q_2}, the Frobenius reciprocity together with Proposition 5.1.1 gives M_{Q_2}-homomorphisms

$$\mathrm{Ind}\,_{S_{P_0,Q_2}}^{M_{Q_2}}\sigma_{0,2}\cong\sigma_{0,2}\longrightarrow\Pi\otimes\mu,$$

where $\sigma_{0,2}$ is as in Proposition 5.1.1. Then on $\sigma_{0,2}$, the element of the form $((1),\beta_1,\beta_2)\in M_{Q_2}$ acts by the character

$$((1),\beta_1,\beta_2)\mapsto|\beta_1||\beta_2|.$$

On the other hand, it acts on $\Pi\otimes\mu$ by the character

$$((1),\beta_1,\beta_2)\mapsto(\tilde{\eta_2}/\tilde{\eta_1}')(\beta_1)(\tilde{\eta_2}'/\tilde{\eta_1}')(\beta_2) = (|\cdot|\eta_2/\eta_1')(\beta_1)(\eta_2'/\eta_1')(\beta_2).$$

Those two characters must be the same. Thus we must have $\eta_2=\eta_1'$ and $\eta_2'=|\cdot|\eta_1'$, which implies $\eta_2'=|\cdot|\eta_2$. But this is a contradiction, since τ_2 is unramified, and so $\eta_2'\neq|\cdot|\eta_2$. Therefore we have $\ker\varphi\not\supseteq J^{(1)}$.

Then by restricting φ to $J^{(1)}$, we have a non-zero M_Q-homomorphism

$$J^{(1)}(=I^{(1)}) \xrightarrow{\ \ \varphi'\ \ } \mathrm{Ind}\,_{S_{Q_2}}^{M_Q}\Pi\otimes\mu.$$

Then if we take the Jacquet module of $I^{(1)}$ with respect to N_{Q_2}, the Frobenius reciprocity together with Proposition 5.1.1 gives a non-zero M_{Q_2}-homomorphism

$$\mathrm{Ind}\,_{S_{P_1,Q_2}}^{M_{Q_2}}\sigma_{1,2}\longrightarrow\Pi\otimes\mu.$$

Then on $\operatorname{Ind}_{S_{P_1,Q_2}}^{M_{Q_2}}\sigma_{1,2}$, the element of the form $((1),\beta_1,\beta_2)\in M_{Q_2}$ acts by the character

$$((1),\beta_1,\beta_2)\mapsto|\beta_1|\mu_2(\beta_2)$$

for some character μ_2, and on $\Pi\otimes\mu$ as before. Then by comparing the two characters, we have $(\eta_2/\eta_1')|\cdot|=|\cdot|$ and $\mu_2=\eta_2'/\eta_1'$. Notice that we also have $\eta_1\eta_1'=\eta_2\eta_2'$, since the central characters of τ_1 and τ_2 are the same. From those, it follows that $\eta_1=\eta_2'$ and $\eta_1'=\eta_2$. Thus $\tau_1\cong\tau_2$. Also notice that $\mu_2=\eta_1/\eta_1'$.

Next we will compute the local parameters of Π. First note that $M_{Q_2}\cong\mathrm{GSp}(1)\times\mathbb{G}_m^2$. Then by restricting φ' to the elements of the form $((g),1,1)$, we have a non-zero $\mathrm{GSp}(1)$-homomorphism

$$\operatorname{Ind}_{P_1}^{\mathrm{GSp}(1)}\sigma_{1,2}'\longrightarrow\Pi\otimes|\cdot|^{1/2}\eta_1'^{-1},$$

where $\sigma_{1,2}'$ is the character on P_1 given by

$$\sigma_{1,2}'(\begin{pmatrix}\alpha_1&*\\0&\lambda/\alpha_1\end{pmatrix})=(|\cdot|/\mu_2)(\alpha_1)\mu_2(\lambda)=(|\cdot|\eta_1'/\eta_1)(\alpha_1)(\eta_1/\eta_1')(\lambda).$$

Here notice that the element $((g),1,1)$ acts not only on Π but also on μ via $\nu(g)^{-1}(=\det(g)^{-1})$, and thus we have $\Pi\otimes|\cdot|^{1/2}\eta_1'^{-1}$. Also notice that $\operatorname{Ind}_{P_1}^{\mathrm{GSp}(1)}\sigma_{1,2}=\operatorname{Ind}_{P}^{\mathrm{GL}(2)}(|\cdot|\otimes(\eta_1/\eta_1'))$, where P is the standard parabolic of $\mathrm{GL}(2)$. This is irreducible. Since Π is also irreducible, we have

$$\operatorname{Ind}_{P}^{\mathrm{GL}(2)}(|\cdot|\otimes(\eta_1/\eta_1'))\cong\Pi\otimes|\cdot|^{1/2}\eta_1'^{-1},$$

i.e.

$$\operatorname{Ind}_{P}^{\mathrm{GL}(2)}(|\cdot|\otimes(\eta_1/\eta_1'))\otimes|\cdot|^{-1/2}\eta_1'\cong\Pi.$$

So we have

$$\Pi\cong\operatorname{Ind}_{P}^{\mathrm{GL}(2)}(|\cdot|^{1/2}\eta_1'\otimes|\cdot|^{-1/2}\eta_1)\cong\text{n-}\operatorname{Ind}_{P}^{\mathrm{GL}(2)}(\eta_1'\otimes\eta_1)\cong\tau_1\cong\tau_2.$$

□

Next we consider the case $d\neq 1$. In this case, we have

$$0\longrightarrow E^\times\longrightarrow F^\times\times\mathrm{GL}(2,E)\xrightarrow{\rho}\mathrm{GSO}(X)\longrightarrow 0.$$

The map ρ is given by $\rho(t,g)x=t^{-1}gx\,{}^cg^t$ for $x\in X$, where ^{c} denotes the Galois conjugation for the quadratic extension E/F and ^{t} denotes matrix transpose, and the first inclusion is given by $a\mapsto(N_F^E(a),a)$. Just as the $d=1$ case, this exact sequence gives a natural bijection between the set of irreducible admissible representations π of $\mathrm{GSO}(X)$ whose central character is χ and the set of irreducible admissible representations τ of $\mathrm{GL}(2,E)$ whose central character is of the form $\chi\circ N$. (From now on, in this section, we agree that $N=N_F^E$.) If π with central character χ corresponds to τ, we write $\pi=\pi(\chi,\tau)$. Also just as the $d=1$ case, each $\sigma\in\operatorname{Irr}(\mathrm{GO}(X))$ is written as $\sigma=(\pi,+)$ or $\sigma=(\pi,-)$.

Remark 5.2.2. Assume the extension E/F is unramified. Then it is easy to see that any unramified character η on $E^\times$ is Galois invariant and written as $\eta = \chi \circ N$ for some unramified character χ on $F^\times$. Accordingly if $\tau = \text{n-Ind}\,(\eta, \eta')$ is unramified, then $\eta = \chi_1 \circ N$ and $\eta' = \chi_2 \circ N$ for unramified characters χ_1 and χ_2 on $F^\times$. Hence τ is always Galois invariant, *i.e.* $\tau \cong \tau^c$, where τ^c is defined by $\tau^c(g) = \tau({}^c g)$. Also the central character of τ is $\chi_1\chi_2 \circ N$.

Analogously to the $d = 1$ case, we have

Proposition 5.2.3. *Assume the extension E/F is unramified, and $\sigma \in \mathrm{Irr}(GO(X))$ is unramified,* i.e. *$\sigma = (\pi, +)$ for some unramified $\pi = \pi(\chi, \tau)$. (So by Remark 5.2.2 $\pi = \pi^c$.) If σ corresponds to $\Pi \in \mathrm{Irr}(GSp(1))$ and Π is unramified, then τ is the base change lift of Π. Moreover the central character of Π is $\chi_{E/F}\chi$, where $\chi_{E/F}$ is the quadratic character corresponding to E/F.*

Proof. Assume σ corresponds to unramified Π. Since τ is unramified, we can write $\tau = \text{n-Ind}\,_P^{\mathrm{GL}(2,E)}(\eta \otimes \eta')$ for unramified characters η and η' on $E^\times$, or by using unnormalized induction, $\tau = \mathrm{Ind}\,_P^{\mathrm{GL}(2,E)}(\tilde\eta \otimes \tilde\eta')$, where $\tilde\eta = |\cdot|_E^{1/2}\eta$ and $\tilde\eta' = |\cdot|_E^{-1/2}\eta'$. (Here note that $|\cdot|_E = |\cdot| \circ N$.) By Remark 5.2.2, $\eta = \chi_1 \circ N$ and $\eta' = \chi_2 \circ N$ for unramified characters χ_1 and χ_2 on $F^\times$ with $\chi_1\chi_2 = \chi$. Then $\pi = \pi(\chi, \tau) = \mathrm{Ind}\,_Q^{\mathrm{GSO}(X)}(\mu)$, where Q is the parabolic preserving the flag

$$\langle \begin{pmatrix} 0 & \sqrt{d} \\ 0 & 0 \end{pmatrix} \rangle,$$

and μ is defined by

$$\mu(\begin{pmatrix} \beta_1 & * & * & * \\ 0 & a & b\,d & * \\ 0 & b & a & * \\ 0 & 0 & 0 & \lambda{\beta_1}^{-1} \end{pmatrix}) = (\tilde\eta'/\chi)(\lambda{\beta_1}^{-1})\tilde\eta(a+b\sqrt{d}) = (|\cdot|\chi_1/\chi_2)(\beta_1)(|\cdot|^{-1/2}\chi_2)(\lambda),$$

where the matrix representation of each element of $\mathrm{GO}(X)$ is with respect to the ordered basis

$$\begin{pmatrix} 0 & \sqrt{d} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ (2/d)\sqrt{d} & 0 \end{pmatrix}.$$

(Here we view $\tilde\eta'$ as a character on $F^\times$ via $F^\times \hookrightarrow E^\times$. See also p.279-278 in [Rb5].) The second equality can be seen as follows. First notice that the middle block

$$h = \begin{pmatrix} a & b\,d \\ b & a \end{pmatrix}$$

is identified with $E^\times$ by $h \leftrightarrow a+b\sqrt{d}$. In particular the Levi factor of Q is isomorphic to $\mathrm{GO}(W) \times \mathbb{G}_m$ where W is the quadratic space E equipped with the quadratic form $-N$. Thus $\lambda = -N(h)$ and so $\tilde{\eta}(a+b\sqrt{d}) = (|\cdot|_E^{1/2}\eta)(h) = (|\cdot|^{1/2}\chi_1)(N(h)) = (|\cdot|^{1/2}\chi_1)(\lambda)$.

Now if Π corresponds to σ, just as in the $d = 1$ case, we have a non-zero R-homomorphism

$$\omega_{1,X} \longrightarrow \mathrm{Ind}\,_{R\cap(\mathrm{GSp}(1)\times Q)}^{R}\Pi \otimes \mu.$$

Then if we take the Jacquet module of $\omega_{1,X}$ with respect to N_Q, the Frobenius reciprocity together with Proposition 5.1.1 gives M_Q-homomorphisms

$$0 \subset J^{(1)} \subset J^{(0)} \xrightarrow{\ \varphi\ } \Pi \otimes \mu.$$

(Here Π is actually $\Pi|_{\mathrm{GSp}(1)^+}$, where $\mathrm{GSp}(1)^+ = \{g \in \mathrm{GSp}(1) : \nu(g) \in \nu(\mathrm{GO}(X))\}$.) Suppose $\ker\varphi \supseteq J^{(1)}$. Then there is a non-zero M_Q-homomorphism

$$I^{(0)} = J^{(0)}/J^{(1)} \longrightarrow \Pi \otimes \mu.$$

In light of Proposition 5.1.1 together with $M_{Q_1} = M_Q$, we can see that the element of the form $((1,1),\beta_1) \in M_Q$ acts on $I^{(0)}$ by the character

$$((1,1),\beta_1) \mapsto |\beta_1|.$$

On the other hand, it acts on $\Pi \otimes \mu$ by the character

$$((1,1),\beta_1) \mapsto (\tilde{\eta}'/\chi)(\beta_1^{-1}) = (\chi/(|\cdot|\eta'))(\beta_1).$$

Those two characters must be the same. So we must have $|\cdot| = \chi/(|\cdot|\eta')$, *i.e.* $\chi = |\cdot|^2\eta'$. But this is a contradiction, since χ is unitary and η' is either unitary or of the form $\eta_0|\cdot|_E^s = \eta_0|\cdot|^{2s}$ for a unitary η_0 and $-\frac{1}{2} < s < \frac{1}{2}$. Therefore we have $\ker\varphi \not\supseteq J^{(1)}$.

Then by restricting φ to $J^{(1)}$, Proposition 5.1.1 together with $M_{Q_1} = M_Q$ gives a non-zero M_Q-homomorphism

$$J^{(1)}(= I^{(1)}) = \mathrm{Ind}\,_{S_{P_1,Q}}^{M_Q}\sigma_{1,1} \xrightarrow{\ \varphi'\ } \Pi \otimes \mu.$$

By considering the action of the elements of the form $(1,(1),\beta_1)$ on $\mathrm{Ind}\,_{S_{P_1,Q}}^{M_Q}\sigma_{1,1}$ and $\Pi \otimes \mu$, we see that

$$\mu_1 = \chi/\tilde{\eta}' = |\cdot|\chi_1/\chi_2,$$

where μ_1 is as in Proposition 5.1.1.

Assume $\Pi = \mathrm{Ind}\,_{P_1}^{\mathrm{GSp}(1)}\delta$ for the character δ on P_1 given by

$$\delta(\begin{pmatrix} \beta_1 & * \\ 0 & \lambda\beta_1^{-1} \end{pmatrix}) = \delta_1(\beta_1)\delta_2(\lambda).$$

Then $\Pi \otimes \mu \cong \operatorname{Ind}_{P_1 \times Q}^{\mathrm{GSp}(1)\times Q}(\delta \otimes \mu)$. By Lemma 5.1.4 φ' gives rise to a non-zero M_Q-homomorphism

$$\operatorname{Ind}_{S_{P_1,Q}}^{M_Q} \sigma_{1,1} \xrightarrow{\varphi'} \operatorname{Ind}_{S_{P_1,Q}}^{M_Q}(\delta \otimes \mu),$$

which we also call φ'. Let us view φ' as an R-homomorphism by restricting to the elements of the form $((g,h),1)$, and so we have a non-zero R-homomorphism

$$\operatorname{Ind}_{R_{1,W}}^{R} \sigma_{1,1} \xrightarrow{\varphi'} \operatorname{Ind}_{R_{1,W}}^{R}(\delta \otimes \mu),$$

where $\sigma_{1,1}$ and $\delta \otimes \mu$ are restricted to $R_{1,W}$. This φ' can be made into a non-zero $\mathrm{GSp}(1)$-homomorphism φ'' via the diagram

$$\begin{array}{ccccc}
\operatorname{Ind}_{R_{1,W}}^{R} \sigma_{1,1} & \xhookrightarrow{\tilde{\Phi}} & \operatorname{Ind}_{P_1^+}^{\mathrm{GSp}(1)^+} \widehat{\sigma_{1,1}} & \hookrightarrow & \operatorname{Ind}_{P_1}^{\mathrm{GSp}(1)} \widehat{\sigma_{1,1}} \\
\downarrow \varphi' & & \downarrow & & \downarrow \varphi'' \\
\operatorname{Ind}_{R_{1,W}}^{R}(\delta \otimes \mu) & \hookrightarrow & \operatorname{Ind}_{P_1^+}^{\mathrm{GSp}(1)^+}(\widehat{\delta \otimes \mu}) & \hookrightarrow & \operatorname{Ind}_{P_1}^{\mathrm{GSp}(1)}(\widehat{\delta \otimes \mu}),
\end{array}$$

where all the horizontal arrows are injective and the vertical arrows are non-zero intertwining operators for the corresponding groups. This diagram is given as follows. First, define $\iota : R \to \mathrm{GSp}(1)^+$ by $(g,h) \mapsto g$. Also let $\widehat{\sigma_{1,1}}$ be the character on $P_1^+(= \mathrm{GSp}(1) \cap P_1)$ defined by

$$\begin{pmatrix} \alpha_1 & * \\ 0 & \lambda\alpha_1^{-1} \end{pmatrix} \mapsto \sigma_{1,1}(\alpha_1,(h),1) = |\nu(h)||\alpha_1|^2(\alpha_1,-D_W)\mu(\alpha_1^{-1}\nu(h)^{-1}),$$

where $h \in \mathrm{GO}(W)$ is such that $\nu(h) = \lambda^{-1}$. This is well-defined. (Also note that $(\alpha_1,-D_W) = (\alpha_1,d)$ is the quadratic character for the quadratic extension E/F and we put $(\alpha_1,d) = \chi_{E/F}(\alpha_1)$). This gives us a natural map $\Phi : \sigma_{1,1} \to \widehat{\sigma_{1,1}}$ that respects the actions of $R_{1,W}$ and P_1^+ via ι. Then Φ can be extended to a non-zero map

$$\tilde{\Phi} : \operatorname{Ind}_{R_{1,W}}^{R} \sigma_{1,1} \longrightarrow \operatorname{Ind}_{P_1}^{R} \widehat{\sigma_{1,1}}$$

defined by $\tilde{\Phi}(F)(g) = \Phi(F((g,h),1))$, where $h \in \mathrm{GO}(W)$ is such that $\iota(g,h) = g$. This is well-defined, injective, and respects the actions of R and $\mathrm{GSp}(1)^+$ via ι. We can similarly define a non-zero injective map

$$\operatorname{Ind}_{R_{1,W}}^{R}(\delta \otimes \mu) \longrightarrow \operatorname{Ind}_{P_1^+}^{\mathrm{GSp}(1)^+}(\widehat{\delta \otimes \mu}).$$

Here $\widehat{\delta \otimes \mu}(\left(\begin{smallmatrix} \alpha_1 & * \\ 0 & \lambda\alpha_1{}^{-1} \end{smallmatrix}\right)) = \delta_1(\alpha_1)\delta_2(\lambda)(|\cdot|^{-1/2}\chi_2)(\lambda^{-1})$. To see the left square of the diagram, notice that, in general, if γ is a character on P_1, there is an injective map $\operatorname{Ind}_{P_1{}^+}^{\mathrm{GSp}(1)^+}\gamma \to \operatorname{Ind}_{P_1}^{\mathrm{GSp}(1)}\gamma$ given by $F \mapsto \tilde{F}$ where $\tilde{F}(g) = \gamma(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu(g)^{-1} \end{smallmatrix}\right))F((\left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu(g) \end{smallmatrix}\right) g)$.

This map respects the actions of $\mathrm{GSp}(1)^+$ and $\mathrm{GSp}(1)$ via the inclusion $\mathrm{GSp}(1)^+ \hookrightarrow \mathrm{GSp}(1)$. Since both $\widehat{\sigma_{1,1}}$ and $\widehat{\delta \otimes \mu}$ can be viewed as characters on P_1 in the obvious way, we have the injective maps as in the above diagram.

Now let us switch to the notation $P = \left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right)$ for the parabolic. Then

$$\widehat{\sigma_{1,1}}\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}\right) = (|\cdot|\chi_{E/F})(a)(\chi_1/\chi_2)(d)$$

and

$$\widehat{\delta \otimes \mu}\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}\right) = (\delta_1\delta_2|\cdot|^{1/2}/\chi_2)(a)(|\cdot|^{1/2}\delta_2/\chi_2)(d).$$

Note that, by twisting by $\chi_2/|\cdot|^{1/2}$, φ'' gives rise to a non-zero GL(2)-homomorphism

$$(\chi_2/|\cdot|^{1/2}) \otimes \mathrm{Ind}_P^{\mathrm{GL}(2)}(\widehat{\sigma_{1,1}}) \longrightarrow (\chi_2/|\cdot|^{1/2}) \otimes \mathrm{Ind}_P^{\mathrm{GL}(2)}(\widehat{\delta \otimes \mu}).$$

But

$$\begin{aligned}(\chi_2/|\cdot|^{1/2}) \otimes \mathrm{Ind}_P^{\mathrm{GL}(2)}(\widehat{\sigma_{1,1}}) &\cong \mathrm{Ind}_P^{\mathrm{GL}(2)}(|\cdot|^{1/2}\chi_2\chi_{E/F}) \otimes (|\cdot|^{-1/2}\chi_1) \\ &\cong \text{n-Ind}_P^{\mathrm{GL}(2)}(\chi_{E/F}\chi_2 \otimes \chi_1),\end{aligned}$$

and

$$(\chi_2/|\cdot|^{1/2}) \otimes \mathrm{Ind}_P^{\mathrm{GL}(2)}(\widehat{\delta \otimes \mu}) \cong \mathrm{Ind}_P^{\mathrm{GL}(2)}(\delta_1\delta_2 \otimes \delta_2) \cong \Pi.$$

Therefore $\Pi \cong \text{n-Ind}_P^{\mathrm{GL}(2)}(\chi_{E/F}\chi_2 \otimes \chi_1)$, and so τ is the base change lift of Π whose central character is $\chi_{E/F}\chi_1\chi_2 = \chi_{E/F}\chi$. □

Remark 5.2.4. It should be noted that, in [Rb3], the existence of a local theta lift of σ to GSp(1) is shown not only for unramified σ, though the local parameters of the lift are not explicitly computed there.

5.3 Local theta lifts to GSp(2)

Computation of the local parameters of the theta lift to GSp(2) is essentially done in [HST, Lemma 10 and 11], which we state below. The method is the same as ours. However the proofs there are quite sketchy and somehow sloppy, at least to the author. A careful reader might want to carry out the computation by following our computation for the GSp(1) case. Also we should mention that [HST, Lemma 10] contains a small mistake which is corrected in [Rb5, p.288].

Proposition 5.3.1. *Assume E/F is unramified (including the case $E = F$). Let $\sigma = (\pi, +)$ be an unramified irreducible representation of $GO(X)$. Then*

1. *If $d = 1$ and so $\pi = \pi(\tau_1 \otimes \tau_2)$, then $\theta_2(\sigma)$ exists and if it is unramified, the Langlands parameter of $\theta_2(\sigma)$ is $\mathrm{diag}(\alpha_1, \beta_1, \alpha_2, \beta_2) \in GSp(2, \mathbb{C})$, where (α_1, β_1) and (α_2, β_2) are the Langlands parameters of τ_1 and τ_2, respectively.*

2. *If* $d \neq 1$ *and so* $\pi = (\chi, \tau)$*, then* $\theta_2(\sigma)$ *exists and if it is unramified, the Langlands parameter of* $\theta_2(\sigma)$ *is* $\mathrm{diag}(\sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{\beta}, -\sqrt{\beta}) \in GSp(2, \mathbb{C})$*, where* (α, β) *is the Langlands parameter of* τ *and the square root is chosen so that* $\sqrt{\alpha}\sqrt{\beta} = \chi(v)$*.*

(Here the existence of $\theta_2(\sigma)$ is proven in [Rb5, Proposition 4.3]. Also see Lemma 4.0.7 and the comment below it for the notation $\theta_2(\sigma)$.)

Chapter 6

Proof of Theorem 1.0.2

In this chapter, we assume that X is a four dimensional quadratic space over a global field F of char $F \neq 2$, and we let d be the discriminant of X. D is a (possibly split) quaternion algebra over F made into a quadratic space in the usual way.

6.1 Proof of Theorem 1.0.2 for $d = 1$

First, let us consider the case $d = 1$. In this case, the group $\mathrm{GO}(X)$ is classified as

Lemma 6.1.1. *If $d = 1$, then $GO(X)$ is isomorphic to $GO(D)$ for some (possibly split) quaternion algebra over F made into a quadratic space in the ordinary way, so that we have*

$$0 \longrightarrow \mathbb{G}_m \longrightarrow D^\times \times D^\times \xrightarrow{\rho} GSO(D) \longrightarrow 0,$$

where each element $(g, h) \in D^\times \times D^\times$ acts on D via $x \mapsto gxh^*$ for $x \in D$, where h^* denotes the quaternion conjugate of h.

This short exact sequence gives rise to

$$0 \longrightarrow \mathbb{A}_F^\times \longrightarrow D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F) \xrightarrow{\rho} \mathrm{GSO}(D, \mathbb{A}_F) \longrightarrow 0.$$

By pulling back an automorphic representation of $\mathrm{GSO}(D, \mathbb{A}_F)$ via ρ we have

Lemma 6.1.2. *There is a bijective correspondence between a cuspidal automorphic representation $\tau_1 \otimes \tau_2$ of $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$ and a cuspidal automorphic representation $\pi = \pi(\tau_1, \tau_2)$ of $GSO(D, \mathbb{A}_F)$ such that all of τ_1, τ_2 and $\pi(\tau_1, \tau_2)$ have a same central character χ.*

Proof. For each $f \in V_\pi$, define $\tilde{f} : D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F) \to \mathbb{C}$ by $\tilde{f}(g, h) = f(\rho(g, h))$. Then it is easy to see that $\tilde{f}$ is a cusp form and the space generated by $\tilde{f}$ gives a cuspidal automorphic representation $\tau_1 \otimes \tau_2$ of $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$. On the other hand, for each $F \in V_{\tau_1\otimes\tau_2}$ define $\hat{F}: \mathrm{GSO}(D,\mathbb{A}_F) \to \mathbb{C}$ by $\hat{F}(x) = F(g,h)$ for (g,h) such that $\rho(g,h) = x$. This is well-defined due to the condition on the central character. It is easy to see that $\hat{F}$ is a cusp form and the space generated by $\hat{F}$ gives a cuspidal automorphic representation of $\mathrm{GSO}(D,\mathbb{A}_F)$. This correspondence is clearly bijective. □

Note that for each such $\tau_1 \otimes \tau_2$, we have the Jacquet-Langlands lift to a cuspidal automorphic representation $\tau_1^{\mathrm{JL}} \otimes \tau_2^{\mathrm{JL}}$ of $\mathrm{M}_{2\times 2}(\mathbb{A}_F)^\times \times \mathrm{M}_{2\times 2}(\mathbb{A}_F)^\times = \mathrm{GL}(2,\mathbb{A}_F) \times \mathrm{GL}(2,\mathbb{A}_F)$ so that for almost all places v, we have $\tau_{iv} \cong {\tau_i^{\mathrm{JL}}}_v$.

Now let $\sigma = (\pi,\delta)$ be a cuspidal automorphic representation of $\mathrm{GO}(D,\mathbb{A}_F)$ extending π. (See Appendix A for the notation.) To consider the non-vanishing question, we first consider the restriction of σ to $\mathrm{O}(D,\mathbb{A}_F)$. So let σ_1 be an irreducible component of the space $\{f|_{\mathrm{O}(D,\mathbb{A}_F)} : f \in V_\sigma\}$ viewed as an automorphic representation of $\mathrm{O}(D,\mathbb{A}_F)$. Then we have

Lemma 6.1.3. *Let $L^S(s,\sigma_1)$ be the incomplete standard L-function of σ_1. Then we have*

$$L^S(s,\sigma_1) = L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee}),$$

where $L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee})$ is the incomplete Rankin-Selberg L-function.

Proof. See [Rb5]. □

We need the following famous theorem.

Lemma 6.1.4. *$L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee})$ does not vanish for $\Re(s) \geq 1$.*

Proof. Non-vanishing for $\Re(s) = 1$ is well-known. Non-vanishing for $\Re(s) > 1$ is a consequence of convergence of the Euler product as in [J-S1, Section 5]. □

Now we can prove Theorem 1.0.2 for the $d = 1$ case easily.

Proof of Theorem 1.0.2 for $d = 1$. Assume $\sigma \cong \otimes\sigma_v$ has the property that σ_v has a local theta lift to $\mathrm{GSp}(2,F_v)$ for all v. Then as discussed in [Rb5], $\sigma_1 \cong \otimes\sigma_{1v}$ has the property that σ_{1v} has a local theta lift to $\mathrm{Sp}(2,F_v)$. Thus from Theorem 1.0.1, Lemma 6.1.3, and 6.1.4, σ_1 has a non-zero theta lift to $\mathrm{Sp}(2,\mathbb{A})$. Thus by (2) of Lemma 4.0.9, σ has a non-zero theta lift to $\mathrm{GSp}(2,\mathbb{A})$.

Conversely, suppose σ has a non-zero lift to $\mathrm{GSp}(2,\mathbb{A}_F)$. If the lift to $\mathrm{GSp}(1,\mathbb{A}_F)$ is zero, then by Proposition 3.1.1, each $\theta_2(\sigma_v)$ exists. If the lift to $\mathrm{GSp}(1,\mathbb{A}_F)$ is non zero, then by Proposition 3.1.1 the non-zero theta lift $\Theta_1(\sigma)$ has an irreducible cuspidal quotient such that $\Theta_1(\sigma) \cong \otimes\theta_1(\sigma_v)$ and so each $\theta_1(\sigma_v)$ exists, and by persistence principle of theta lift, σ_v has non-zero theta lift to $\mathrm{GSp}(2,F_v)$. This completes the proof for the $d = 1$ case. □

6.2 Proof of Theorem 1.0.2 for $d \neq 1$

Next assume $d \neq 1$. Let $E = F(\sqrt{d})$ be the quadratic extension of F, and c the non-trivial element in $\mathrm{Gal}(E/F)$. For each quaternion algebra D over F, let $B_{D,E} = D \otimes E$. Then for each $g \in B_{D,E}$, we define ${}^c g^*$ by linearly extending the operation ${}^c(x \otimes a)^* = x^* \otimes {}^c a$ where ^{c} is the Galois conjugation and * is the quaternion conjugation. Then the space $X_{D,E} = \{x \in B_{D,E} : {}^c x^*\}$ can be made into a four dimensional quadratic space over F via the reduced norm of the quaternion $B_{D,E}$.

Remark 6.2.1. Let us just write $B = B_{D,E}$ and consider $B^\times$ as an algebraic group over F, *i.e.* the Weil restriction $\mathrm{Res}_{E/F}(B^\times)$ of $B^\times$ to F. Then for almost all v, $B^\times(F_v)$ splits, *i.e.* either $B^\times(F_v) \cong \mathrm{GL}(2, F_v) \times \mathrm{GL}(2, F_v)$ if v splits in E, or $B^\times(F_v) \cong \mathrm{GL}(2, E_v)$ if v stays prime in E. Also if $B^\times(F_v)$ does not split, v necessarily splits in E and $B^\times(F_v) \cong D^\times(F_v) \times D^\times(F_v)$, because any quadratic field splits a quaternion algebra over a local field, *i.e.* if v stays prime, then $B \otimes F_v$ splits (see, for example, [Lm, p.154, Remark 2.7]).

Similarly to the $d = 1$ case, the orthogonal similitude groups are classified as follows.

Lemma 6.2.2. *Let X be a four dimensional quadratic space of discriminant $d \neq 1$ over F. Then there exists a quaternion D over F such that $GO(X) \cong GO(X_{D,E})$. Also we have*

$$0 \longrightarrow E^\times \longrightarrow F^\times \times B_{D,E}^\times \xrightarrow{\rho} GSO(X_{D,E}, F) \cong GSO(X, F) \longrightarrow 0.$$

and by taking the adelic points, we have

$$0 \longrightarrow \mathbb{A}_E^\times \longrightarrow \mathbb{A}_F^\times \times B_{D,E}^\times(\mathbb{A}_F) \xrightarrow{\rho} GSO(X_{D,E}, \mathbb{A}_F) \cong GSO(X, \mathbb{A}_F) \longrightarrow 0,$$

where the map ρ is given by $\rho(t, g)x = t^{-1} g x\, {}^c g^*$ and the first inclusion is given by $a \mapsto (N_F^E(a), a)$.

Therefore for the purpose of similitude theta lifting, we may assume that X is always of the form $X_{D,E}$, although there is a subtlety regarding theta correspondence since the Weil representation is defined in terms of the space X rather than the group $\mathrm{GO}(X)$. This subtlety as well as the above proposition is discussed in detail in [Rb5, Section 2 and 5].

As in the $d = 1$ case, we can pullback an automorphic representation of $\mathrm{GSO}(X, \mathbb{A}_F)$ via ρ and we have

Lemma 6.2.3. *There is a bijective correspondence between a cuspidal automorphic representation τ of $B_{D,E}^\times(\mathbb{A}_E)$ whose central character is of the form $\chi \circ N_F^E$ for some Hecke character χ of $\mathbb{A}_F^\times$ and a cuspidal automorphic representation $\pi = \pi(\chi, \tau)$ of $GSO(X_{D,E}, \mathbb{A}_F)$ whose central character is χ.*

Proof. Essentially identical to the $d = 1$ case. $\square$

Now for each cuspidal automorphic representation τ of $B^\times(\mathbb{A}_F)$, we have the Jacquet-Langlands lift to a cuspidal automorphic representation τ^{JL} of $\mathrm{GL}(2, \mathbb{A}_E)$ so that for almost all places v of E, we have $\tau_v \cong {\tau^{\mathrm{JL}}}_v$.

Now let $\sigma = (\pi, \delta)$ be a cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A}_F)$ extending π. Again just as in the $d = 1$ case, we first consider the restriction of σ to $\mathrm{O}(X, \mathbb{A}_F)$. So let σ_1 be an irreducible component of the space $\{f|_{\mathrm{O}(X,\mathbb{A}_F)} : f \in V_\sigma\}$ viewed as an automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$. Then we have

Lemma 6.2.4. *Let $L^S(s, \sigma_1)$ be the incomplete standard L-function of σ_1. Then we have*

$$L^S(s, \sigma_1) = L^S(s, \tau^{\mathrm{JL}}, \chi^{-1},\ \textit{Asai}),$$

where $L^S(s, \tau^{\mathrm{JL}}, \chi^{-1},\ \textit{Asai})$ is the incomplete Asai L-function twisted by χ^{-1}.

Proof. See [Rb5]. $\square$

Remark 6.2.5. The Asai L-function is also known as "twisted tensor L-function". It seems to the author that the term "Asai L-function" is used more commonly in the classical context of Hilbert modular forms than in our representation theoretic context, though we do not really know any rule for the terminology. See [F1] and [F2] for more about the Asai L-function.

For the Asai L-function, we need an analogue of Lemma 6.1.4. But to the best of our knowledge, the complete analogue of Lemma 6.1.4 is not known. For our purpose, however, we have only to show non-vanishing of the Asai L-function at $s = 1, 2$ so that we can apply Theorem 1.0.1. Non-vanishing at $s = 1$ is done in [F1], which is also discussed in [Rb5, p.302]. Non-vanishing at $s = 2$ follows from absolute convergence of the Euler product which can be proven by an elementary argument as follows.

Lemma 6.2.6. *The Euler product of the Asai L-function $L^S(s, \tau^{\mathrm{JL}}, \chi^{-1},\ \textit{Asai})$ converges absolutely for $\Re(s) > 1 + \frac{2}{9}$.*

Proof. Let v be a finite place of F which is inert in E. If τ_v^{JL} is unramified, the local factor of $L^S(s, \tau^{\mathrm{JL}}, \chi^{-1},\ \mathrm{Asai})$ is given by

$$\begin{aligned} &L_v(s, \tau_v, \chi_v^{-1},\ \mathrm{Asai}) \\ &\quad = (1 - \gamma_v \alpha_v q_v^{-s})^{-1}(1 - \gamma_v \beta_v q_v^{-s})^{-1}(1 - \gamma_v^2 \alpha_v \beta_v q_v^{-2s})^{-1} \\ &\quad = (1 - \gamma_v \alpha_v q_v^{-s})^{-1}(1 - \gamma_v \beta_v q_v^{-s})^{-1}(1 - \gamma_v \sqrt{\alpha_v \beta_v} q_v^{-s})^{-1}(1 + \gamma_v \sqrt{\alpha_v \beta_v} q_v^{-s})^{-1}, \end{aligned}$$

where (α_v, β_v) is the Langlands parameter of τ_v, q_v is the order of the residue field at v of F, and $\gamma_v = \chi_v(\varpi_v)^{-1}$ with ϖ_v the uniformizer at v in F. The square root is chosen appropriately. If v splits into v' and v'', and both $\tau_{v'}$ and $\tau_{v''}$ are unramified, then the local factor is given by

$$L_v(s,\tau_v,\chi_v^{-1},\ \mathrm{Asai}) \\ = (1-\gamma_v\alpha_{v'}\alpha_{v''}q_v^{-s})^{-1}(1-\gamma_v\alpha_{v'}\beta_{v''}q_v^{-s})^{-1}(1-\gamma_v\beta_{v'}\alpha_{v''}q_v^{-s})^{-1}(1-\gamma_v\beta_{v'}\beta_{v''}q_v^{-s})^{-1},$$

where $(\alpha_{v'},\beta_{v'})$ and $(\alpha_{v''},\beta_{v''})$ are the local Langlands parameters of $\tau_{v'}$ and $\tau_{v''}$, respectively. (See, for example, [F2, p.200].) Since τ is a cuspidal representation with a unitary central character, we have $|\gamma_v| = 1$ and $|\alpha_v|,\ |\alpha_{v'}|,\ |\alpha_{v''}|,\ |\beta_v|,\ |\beta_{v'}|,\ |\beta_{v''}| < {q_v}^{1/9}$. (See, for example, [K-S] for the bound of the Langlands parameters.) Therefore, each factor of the Euler product of the Asai L-function is a product of four factors of the form

$$(1-a_vq_v^{-s})^{-1} \quad \text{with} \quad |a_v| < q_v^{2/9}.$$

Then the absolute convergence of the Euler product follows from the following lemma. □

Lemma 6.2.7. *Let $a_v \in \mathbb{C}$ be such that $|a_v| < q_v^n$ for all finite places v, where q_v is the order of the residue field at v. Then the Euler product*

$$\prod_v (1-a_vq_v^{-s})^{-1}$$

converges absolutely for $\Re(s) > 1+n$.

Proof. This can be easily proven by using Lemma 2 in p.187 and the first paragraph of p.188 of [M]. □

Corollary 6.2.8. *The incomplete Asai L-function $L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\ \mathit{Asai})$ does not vanish at $s = 1, 2$.*

Once this non-vanishing is obtained, we have

Proof of Theorem 1.0.2 for $d \neq 1$. Identical to the $d = 1$ case, using Theorem 1.0.1 and Corollary 6.2.8. □

Chapter 7

Proof of Theorem 1.0.3

In this chapter, we prove Theorem 1.0.3. All the notations are as in the previous section.

7.1 Proof of Theorem 1.0.3 for $d = 1$

First, let us consider the case $d = 1$, and so as discussed in the previous section, we consider the group of the form $\mathrm{GO}(D)$ for a quaternion algebra D so that each cuspidal automorphic representation τ on $\mathrm{GSO}(D, \mathbb{A}_F)$ is of the form $\pi = \pi(\tau_1, \tau_2)$ for cuspidal automorphic representations τ_1 and τ_2 of $D^\times(\mathbb{A}_F)$.

Let $t \in \mathrm{GO}(D)$ be the quaternion conjugation, *i.e.* $t(x) = x^*$ for all $x \in D$. Then by direct computation, we have $t \notin \mathrm{GSO}(D)$. Then t acts on $\mathrm{GSO}(D)$ as $t \cdot g = tgt$ and so $\mathrm{GO}(D) \cong \mathrm{GSO}(D) \rtimes \{1, t\}$ via this action. Also define the isomorphism $c : \mathrm{GSO}(D) \to \mathrm{GSO}(D)$ by $c(g) = tgt$. Then $c(\rho(g_1, g_2)) = \rho(g_2, g_1)$.

Now for each cuspidal automorphic representation $\pi = \pi(\tau_1, \tau_2)$, we define V_π^c to be the space of all cusp forms of the form $f \circ c$ for all $f \in V_\pi$ where $f \circ c(g) = f(c(g))$, on which $\mathrm{GSO}(D, \mathbb{A}_F)$ acts by right translation. Then as an admissible representation, it is equivalent to the representation π^c with $V_{\pi^c} = V_\pi$ and the action defined by the conjugation $\pi^c(g) = \pi(tgt)$. Then it is easy to see that $\pi \cong \pi^c$ if and only if $\tau_1 \cong \tau_2$. The local analogue can be similarly defined and we have $\pi^c \cong \otimes \pi_v^c$.

The following is well-known.

Proposition 7.1.1. *$L^S(s, \tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee})$ has a pole at $s = 1$ if and only if $\tau_1 \cong \tau_2$.*

Then we can prove Theorem 1.0.3 for $d = 1$.

Proof Theorem 1.0.3 for $d = 1$. Assume a cuspidal automorphic representation σ of $\mathrm{GO}(D, \mathbb{A}_F)$ is extended from a cuspidal automorphic representation π of $\mathrm{GSO}(D)$ with extension index δ and so $\sigma = (\pi, \delta)$. (See Appendix A for the notation.) Also assume that σ has a non-zero theta lift to $\mathrm{GSp}(2, \mathbb{A}_F)$. Then it can be shown that the theta lift $\Theta_1(V_\sigma)$ to $\mathrm{GSp}(1,\mathbb{A}_F)$ does not vanish if $L^S(s,\tau_1^{\mathrm{JL}}\times\tau_2^{\mathrm{JL}\vee})$ has a pole at $s=1$, *i.e.* by Proposition 7.1.1, $\tau_1\cong\tau_2$, *i.e.* $\pi\cong\pi^c$. The proof for this is the same as the proof of Corollary 1.3 of [Rb4]. Namely, assume $\Theta_1(V_\sigma)=0$. Then for (one of) σ_1 we have $\Theta_1(V_{\sigma_1})=0$ and $\Theta_2(V_{\sigma_1})\neq 0$, where σ_1 is as in the previous section. Then $\Theta_2(V_{\sigma_1})$ is a space of an irreducible cuspidal representation by Proposition 3.1.1. Let us denote this cuspidal representation by Π. Then by functoriality of the unramified theta correspondence, we have

$$L^S(\Pi,s,\chi_X)=\zeta_F^S(s)L^S(\sigma_1,s).$$

It is well known that $\zeta_F^S(s)$ has a simple pole at $s=1$. So if $L^S(\sigma_1,s)$ has a simple pole at $s=1$, then $L^S(\Pi,s,\chi_X)$ would have a double pole, which contradicts to the fact that the poles of the standard (incomplete) Langlands L-function of $\mathrm{Sp}(n)$ are at most simple. (See Remark after proposition 1.7 of [Ik], and also Theorem 7.2.5 of [Kd-R2].)

Conversely, assume $\Theta_1(V_\sigma)\neq 0$, and so it gives rise to a cuspidal automorphic representation Π of $\mathrm{GL}(2,\mathbb{A}_F)$. First note that for almost all v, we have $\pi_v=\pi(\tau_{1v},\tau_{2v})$ where τ_{1v} and τ_{2v} are spherical representations of $\mathrm{GL}(2,F_v)$. Then by Proposition 5.2.1 we have $\tau_{1v}\cong\tau_{2v}$. So by strong multiplicity one theorem, $\tau_1\cong\tau_2$. Moreover in this case we know that $(\Pi_v)^\vee\cong\pi_1\cong\pi_2$ for almost all v. Thus by strong multiplicity one theorem together with Proposition 3.1.1, we have $\Pi^\vee\cong\pi_1\cong\pi_2$. Moreover since $\Theta_1(V_\sigma)$ is closed under the right action of $\mathrm{GL}(2,\mathbb{A}_F)$, multiplicity one theorem gives us $\Theta_1(V_\sigma)=V_\Pi=V_{\pi_1^\vee}=V_{\pi_2^\vee}$. This completes the proof. □

7.2 Proof of Theorem 1.0.3 for $d\neq 1$

Next, let us consider the case $d\neq 1$, and thus as discussed in the previous section, we consider the group of the form $\mathrm{GO}(X_{D,E})$ for a quaternion algebra D so that each cuspidal automorphic representation π of $\mathrm{GSO}(X_{D,E},\mathbb{A}_F)$ is of the form $\pi=\pi(\chi,\tau)$ where τ is a cuspidal automorphic representations of $B_{D,E}^\times(\mathbb{A}_E)$ and χ is a Hecke character on $\mathbb{A}_F^\times$. We simply write X for $X_{D,E}$.

Just as in the $d=1$ case, we define the conjugate representation π^c. Namely let $t\in\mathrm{GO}(X)$ be the quaternion conjugation, *i.e.* $t(x)=x^*$ for all $x\in X$. Then by direct computation, we have $t\notin\mathrm{GSO}(X)$. Then t acts on $\mathrm{GSO}(X)$ as $t\cdot g=tgt$ and so $\mathrm{GO}(X)\cong\mathrm{GSO}(X)\rtimes\{1,t\}$ via this action. Also define the isomorphism $c:\mathrm{GSO}(X)\rightarrow\mathrm{GSO}(X)$ by $c(g)=tgt$. Then $c(\rho(t,g))=\rho(t,{}^cg)$, where ${}^cg$ is the Galois conjugate of g. Now for each cuspidal automorphic representation $\pi=\pi(\chi,\tau)$, we define V_π^c to be the space of all cusp forms of the form $f\circ c$ for all $f\in V_\pi$ where $f\circ c(g)=f(c(g))$, on which $\mathrm{GSO}(X,\mathbb{A}_F)$ acts by right translation. Then as an admissible representation, it is equivalent to the representation π^c with $V_{\pi^c}=V_\pi$ and the action defined by the conjugation $\pi^c(g)=\pi(tgt)$. Then it is easy to see that $\pi\cong\pi^c$ if and only if $\tau^c\cong\tau$, where τ^c is the Galois conjugate of τ. The local analogue can be similarly defined and we have $\pi^c\cong\otimes_v\pi_v^c$.

We need the following, which is the analogue of Proposition 7.1.1.

Proposition 7.2.1. *Let E/F be a quadratic extension of global fields, and τ a cuspidal automorphic representation of $GL(2, \mathbb{A}_E)$ whose central character is $\chi \circ N_F^E$ for a Hecke character χ on $\mathbb{A}_F^\times$. Then the incomplete Asai L-function $L^S(s, \tau, \chi^{-1}, Asai)$ has a pole at $s = 1$ if and only if τ is the base change lift of a cuspidal automorphic representation τ_0 of $GL(2, \mathbb{A}_F)$ whose central character is $\chi_{E/F}\chi$.*

Proof. The case where χ is trivial is essentially done in P.311 of [F1]. (Also see [F-Z] for the archimedean assumption imposed on [F1].) We will extend his method to the case where χ is non-trivial. This can be done almost by directly carrying over the main argument of [F1] to our case. So we will give only the outline by emphasizing where in [F1] to be modified. Hence, in this proof, all the notations are essentially as in [F1], though we restrict to the case $G = \mathrm{GL}(2)$. (Indeed, otherwise our argument would not work.) Let P and Z be the standard parabolic and the center of G, respectively, and $\mathcal{S}(\mathbb{A}_F^2)$ the space of Schwartz-Bruhat functions. Then for $g \in \mathrm{GL}(2, \mathbb{A}_F)$, $s \in \mathbb{C}$, and $\epsilon = (0, 1) \in \mathbb{A}_F^2$, we define

$$f(g, s) = |g|^s \int_{\mathbb{A}_F^\times} \Phi(a\epsilon g)|a|^{2s} d^\times a,$$

where we write $|g|$ for $|\det g|$. (Note that we choose the ω in [F1] to be trivial.) This integral converges absolutely, uniformly in compact subsets for $\Re(s) > \frac{1}{2}$. Then the Eisenstein series

$$E(g, \Phi, s) = \sum_{\gamma \in Z(F)P(F)\backslash G(F)} f(\gamma g, s)$$

converges absolutely in $\Re(s) > 1$, which extends to a meromorphic function on $\Re(s) > 0$ and

$$E(g, \Phi, s) = \frac{c\hat{\Phi}(0)}{s-1} + R(g, s)$$

where $R(g, s)$ is holomorphic in $\Re(s) > 0$ and slowly decreasing.

Now for each $\phi \in V_\tau$, we define

$$I(s, \Phi, \phi, \chi) = \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} E(g, \Phi, s)\phi(g)\chi^{-1}(\det g)dg.$$

This integral is well-defined and converges to a holomorphic function of s in $\Re(s) > 0$. (Compare this with the integral in [F1, p.302].) Then the above equation for $E(g, \Phi, s)$ implies

$$I(s, \Phi, \phi, \chi) = \frac{c\hat{\Phi}(0)}{s-1} \int \phi(g)\chi^{-1}(\det g)dg + \int R(g, s)\phi(g)\chi^{-1}(\det g)dg,$$

where the integrals are over $Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$. Thus $I(s, \Phi, \phi, \chi)$ has a pole at $s = 1$ if and only if $\hat{\Phi}(0) \neq 0$ and $\int \phi(g)\chi^{-1}(\det g)dg \neq 0$.

Then consider the integral

$$\Psi(s,\Phi,W,\chi)=\int_{N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W(g)\Phi(\epsilon g)|g|^s\chi^{-1}(\det g)dg,$$

where W is the Whittaker vector as defined in [F1, p.302]. Then by the same computation as in [F1, p.303], we have

$$I(s,\Phi,\phi,\chi)=\Psi(s,\Phi,W,\chi).$$

If W and Φ are factorizable as $W(g)=\prod W_v(g_v)$ and $\Phi(x)=\prod\Phi_v(x_v)$, then the global integral $\Psi(s,\Phi,W,\chi)$ is also factorizable as $\Psi(s,\Phi,W,\chi)=\prod\Psi_v(s,\Phi_v,W_v,\chi_v)$, where

$$\Psi_v(s,\Phi_v,W_v,\chi_v)=\int_{N_v\backslash G_v} W_{v'}(g)W_{v''}(g)\Phi_v(\epsilon g)|g|^s\chi_v^{-1}(\det g)dg$$

if v splits into v' and v'', and

$$\Psi_v(s,\Phi_v,W_v,\chi_v)=\int_{N_v\backslash G_v} W_v(g)\Phi_v(\epsilon g)|g|^s\chi_v^{-1}(\det g)dg$$

otherwise.

Then by exactly the same computation as [F1], we have

$$\Psi(s,\Phi,W,\chi)=A(s,\Phi_S,W_S,\chi_S)L^S(s,\tau,\chi^{-1},\mathrm{Asai}).$$

for some function $A(s,\Phi_S,W_S,\chi_S)$ with $\hat{\Phi}(0)\neq 0$. (See [F1, p.310].) Therefore the (incomplete) Asai L-function $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ has a pole at $s=1$ if and only if $\Psi(s,\Phi,W,\chi)$ has a pole at $s=1$, *i.e.* if and only if $\int\phi(g)\chi^{-1}(\det g)dg\neq 0$.

Now assume $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ has a pole at $s=1$. Then by the above computation, the linear functional $F:V_\tau\to\mathbb{C}$ given by $F(\phi)=\int\phi(g)\chi^{-1}(\det g)dg$ is non-zero, and $L(h\cdot\phi)=\chi(\det h)L(\phi)$ for $h\in G(\mathbb{A}_F)$. Then for each v which splits into v' and v'', L induces a linear functional $L_v:V_{\tau_{v'}}\otimes V_{\tau_{v''}}\to\mathbb{C}$ with the property that $L_v(g\cdot(v_1\otimes v_2))=\chi_v(\det g)L_v(v_1\otimes v_2)$ for $g\in G(F_v)$ when $G(F_v)$ is diagonally embedded in $G(E_{v'})\times G(E_{v''})$. Then notice that the obvious map $T_v:V_{\tau_{v'}}\otimes V_{\tau_{v''}^\vee}\to V_{\tau_{v'}}\otimes V_{\tau_{v''}}$ has the property that $T_v(g\cdot(v_1\otimes v_2))=\chi_v(\det g)^{-1}g\cdot(v_1\otimes v_2)$. (Note that the central characters of both $\tau_{v'}$ and $\tau_{v''}$ are χ_v.) Thus the composite $L_v\circ T_v$ is a $G(F_v)$-invariant bilinear form on $\tau_{v'}\otimes\tau_{v''}^\vee$, and so $\tau_{v'}\cong\tau_{v''}$. Also if v stays prime in E, then Remark 5.2.2 implies that $\tau\cong\tau_v^c$ for almost all such v. Therefore we have $\tau_v\cong\tau_v^c$ for almost all v, and thus for all v, *i.e.* τ is a base-change lift. (To pass from “almost all” to “all”, see, for example, [Ln, p.22])

Thus τ is the base change lift of a cuspidal automorphic representation τ_0 of $\mathrm{GL}(2,\mathbb{A}_F)$ whose central character ω_0 is either χ or $\chi_{E/F}\chi$. Then from the explicit description of each unramified v-factor of the Asai L-function as in the proof of Lemma 6.2.6, we have

$$\begin{aligned} L^S(s,\tau,\chi^{-1},\mathrm{Asai}) &= \frac{L^S(s,\tau_0\otimes\tau_0,\chi^{-1})L^S(s,\omega_0\chi_{E/F}\chi^{-1})}{L^S(s,\omega_0\chi^{-1})} \\ &= \frac{L^S(s,\tau_0\otimes\tau_0^\vee,\omega_0\chi^{-1})L^S(s,\omega_0\chi_{E/F}\chi^{-1})}{L^S(s,\omega_0\chi^{-1})}. \end{aligned}$$

Assume $\omega_0 \neq \chi_{E/F}\chi$, *i.e.* $\omega_0 = \chi$. Then both $L^S(s,\tau_0\otimes\tau_0^\vee,\omega_0\chi^{-1})$ and $L^S(s,\omega_0\chi^{-1})$ have a simple pole at $s=1$, and $L^S(s,\omega_0\chi_{E/F}\chi^{-1})$ has neither a pole nor a zero at $s=1$. Thus $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ does not have a pole at $s=1$. But this contradicts to our assumption, and so $\omega_0 = \chi_{E/F}\chi$, which completes the proof of the only if part of the proposition.

Conversely, if τ is the base-change lift of τ_0 whose central character is $\chi_{E/F}\chi$, then as in above, both $L^S(s,\tau_0\otimes\tau_0^\vee,\omega_0\chi^{-1})$ and $L^S(s,\omega_0\chi^{-1})$ have neither a pole nor a zero at $s=1$, and $L^S(s,\omega_0\chi_{E/F}\chi^{-1})$ has a pole at $s=1$. Thus $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ has a pole at $s=1$ □

Once this is proven, we can prove Theorem 1.0.3 similarly to the the $d=1$ case.

Proof of Theorem 1.0.3 for $d\neq 1$. Assume τ^{JL} is the base change of a cuspidal automorphic representation τ_0 whose central character is $\chi_{E/F}\chi$. Then by the above proposition, the Asai L-function for τ^{JL} has a pole at $s=1$. So by the same argument as the $d=1$ case, we have $\Theta_1(V_\sigma)\neq 0$.

Conversely, assume $\Theta_1(V_\sigma)\neq 0$. Then by Proposition 5.2.1 and 5.2.3, $\tau_v^c = \tau_v$ for almost all v, and thus for all v. (See [Ln, p.22].) Thus τ is the base change lift of some cuspidal automorphic representation τ_0 of $\mathrm{GL}(2,\mathbb{A}_F)$ whose central character is either χ or $\chi_{E/F}\chi$. But by Proposition 5.2.3, the central character of τ_{0v} is $\chi_{E_v/F_v}\chi$ for almost all v. Thus by strong multiplicity one, the central character of τ_0 is $\chi_{E/F}\chi$. Moreover, by multiplicity one, we have $\Theta_1(V_\sigma) = V_\Pi = V_{\tau_0^\vee}$. □

7.3 Some remark for the $d \neq 1$ case

Our computation for the $d\neq 1$ case reveals the following interesting phenomenon. Let τ be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_E)$ that is the base change lift of a cuspidal automorphic representation τ_0 of $\mathrm{GL}(2,\mathbb{A}_F)$. (Note that such τ_0 is unique up to twist by $\chi_{E/F}$. See [Ln, p.21]). Assume the central character of τ_0 is χ, and so the central character of τ is $\chi\circ N_F^E$. Then we can identify τ with a cuspidal automorphic representation $\pi=\pi(\chi,\tau)$ of $\mathrm{GSO}(X,\mathbb{A}_F)$ whose central character is χ. Since τ is generic, π can be always extended to a cuspidal automorphic representation $\sigma=(\pi,\delta)$ of $\mathrm{GO}(X,\mathbb{A}_F)$ such that $\Theta_2(V_\sigma)\neq 0$. (See Appendix A.) In this case by Theorem 1.0.3 we have $\Theta_1(V_\sigma)=0$. Thus $\Theta_2(V_\sigma)$ is irreducible cuspidal.

However since $\chi = \chi_{E/F}\chi_{E/F}\chi$ and $\chi \circ N = \chi_{E/F}\chi \circ N_F^E$, we can also identify τ with a cuspidal automorphic representation $\pi' = \pi'(\tau, \chi_{E/F}\chi)$ of $\mathrm{GSO}(X, \mathbb{A}_F)$ whose central character is $\chi_{E/F}\chi$. Again we can extend π' to $\sigma' = (\pi', \delta')$ such that $\Theta_2(V_{\sigma'}) \neq 0$. Then this time, by Theorem 1.0.3 we have $\Theta_1(V_{\sigma'}) = V_{\tau_0^\vee} \neq 0$, and $\Theta_2(V_{\sigma'})$ is non-zero but non-cuspidal. This observation can be summarized as follows.

$$\tau_0 \xrightarrow{\substack{\text{base}\\\text{change}}} \tau \begin{cases} \nearrow \pi = \pi(\chi, \tau) \longrightarrow \sigma = (\pi, \delta) \xrightarrow{\substack{\text{theta}\\\text{lift}}} \begin{cases} \Theta_2(V_\sigma) \neq 0, & \text{cuspidal} \\ \Theta_1(V_\sigma) = 0 \end{cases} \\ \searrow \pi' = \pi'(\chi_{E/F}\chi, \tau) \longrightarrow \sigma' = (\pi', \delta') \xrightarrow{\substack{\text{theta}\\\text{lift}}} \begin{cases} \Theta_2(V_{\sigma'}) \neq 0, & \text{non-cuspidal} \\ \Theta_1(V_{\sigma'}) = V_{\tau_0^\vee} \end{cases} \end{cases}$$

(Of course this diagram makes sense only with the assumption that τ is cuspidal.)

Appendix A

From GSO(X) to GO(X)

In this appendix, we will describe the relation between automorphic representations of $\mathrm{GSO}(X, \mathbb{A}_F)$ and $\mathrm{GO}(X, \mathbb{A}_F)$, and prove Theorem 1.0.4.

A.1 Local representations

Let us consider the local case first. In this section, let us simply write $G = \mathrm{GO}(X, F_v)$, $H = \mathrm{GSO}(X, F_v)$, and $X = X \otimes F_v$. Also let $d = \mathrm{disc}X$ and $E_v = F_v(\sqrt{d})$. Then as discussed in [Rb3], there are three cases to consider. Namely, if $d = 1$, then X is either the quadratic space of 2×2 matrices $\mathrm{M}_{2\times 2}(F_v)$ over F_v with the quadratic form $-\det$, or the unique definite quaternion algebra D over F_v made into a quadratic space by the reduced norm. If $d \neq 1$, then X is the space of 2×2 Hermitian matrices over E_v with the quadratic form $-\det$. Now if $d = 1$ and $X = \mathrm{M}_{2\times 2}(F_v)$, or if $d \neq 1$, then we let $t \in G$ act on X by matrix transpose. If $d = 1$ and $X = D$, then we let $t \in G$ act on X by quaternion conjugation. Then we can write $G = H \rtimes \{1, t\}$ where t acts on H by conjugation.

Let π be an irreducible admissible representation of H. We define π^c by taking $V_{\pi^c} = V_\pi$ and by letting $\pi^c(g)f = \pi(tgt)f$ for all $g \in H$ and $f \in V_\pi$. Then we have

- If $\pi \ncong \pi^c$, then $\mathrm{Ind}_H^G \pi$ is irreducible, and we denote it by π^+.
- If $\pi \cong \pi^c$, then $\mathrm{Ind}_H^G \pi$ is reducible. Indeed, it is the sum of two irreducible representations, and we write $\mathrm{Ind}_H^G \pi \cong \pi^+ \oplus \pi^-$. Here, $\pi^+|_H \cong \pi^-|_H \cong \pi$ and t acts on $\pi^\pm$ via a linear operator $\theta^\pm$ with the property that $(\theta^\pm)^2 = \mathrm{Id}$ and $\theta^\pm \circ g = tgt \circ \theta^\pm$ for all $g \in H$.

Thanks to the classification of the group G, we can be more explicit about the irreducible components π^+ and π^-. First assume $d = 1$. Then we have either

$$0 \longrightarrow F^\times \longrightarrow \mathrm{GL}(2, F_v) \times \mathrm{GL}(2, F_v) \xrightarrow{\rho} H \longrightarrow 0,$$

where the map ρ is given $\rho(g_1, g_2)(f) = g_1 f g_2^t$ for $f \in V_\pi$ and $(g_1, g_2) \in \mathrm{GL}(2, F_v) \times \mathrm{GL}(2, F_v)$ with g_2^t the matrix transpose of g_2, or

$$0 \longrightarrow F^\times \longrightarrow D^\times \times D^\times \xrightarrow{\rho} H \longrightarrow 0,$$

where the map ρ is given $\rho(g_1, g_2)(x) = g_1 f g_2^*$ for $f \in V_\pi$ and $(g_1, g_2) \in D^\times \times D^\times$ with g_2^* the quaternion conjugate of g_2. Thus π can be identified with an admissible representation $\tau_1 \otimes \tau_2$ of $\mathrm{GL}(2, F_v) \times \mathrm{GL}(2, F_v)$ or $D^\times \times D^\times$ where τ_1 and τ_2 have a same central character. (Also see Section 5.) In this case, t acts, via ρ, on $\mathrm{GL}(2, F_v) \times \mathrm{GL}(2, F_v)$ or $D^\times \times D^\times$ by $t \cdot (g_1, g_2) = (g_2, g_1)$. Then it can be shown that we can choose $\theta^\pm$ to be such that $\theta^+(x_1 \otimes x_2) = x_2 \otimes x_1$ and $\theta^-(x_1 \otimes x_2) = -x_2 \otimes x_1$. We choose τ^+ and τ^- accordingly.

Next assume $d \neq 1$. Then we have

$$0 \longrightarrow E_v^\times \longrightarrow F_v^\times \times \mathrm{GL}(2, E_v) \xrightarrow{\rho} H \longrightarrow 0,$$

where the map ρ is given by $\rho(t, g)f = t^{-1} g f\, {}^c g^t$ for $f \in V_\pi$, were ^{c} denotes the Galois conjugation for the quadratic extension E_v/F_v and ^{t} denotes matrix transpose, and the first inclusion is given by $a \mapsto (N_{F_v}^{E_v}(a), a)$. Thus π can be identified with an admissible representation τ of $\mathrm{GL}(2, E_v)$ whose central character is of the form $\chi \circ N_{F_v}^{E_v}$. (Also see Section 5.) In this case, t acts, via ρ, on $\mathrm{GL}(2, E_v)$ in such a way that $t \cdot g = {}^c g$. Note that τ has a unique Whittaker model, namely it is realized as a space of functions $f : GL(2, E_v) \to \mathbb{C}$ such that $f\left(\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\right) = \psi_v(\mathrm{tr}\ a) f(g)$ for all $a \in E_v$ and $g \in \mathrm{GL}(2, E_v)$, where ψ_v is a fixed additive character of F_v. Then we define $\theta^\pm$ to be the linear operator that acts on this space of Whittaker functions by $f \mapsto \pm f \circ c$, and θ^+ is chosen to be the one that acts as $f \mapsto f \circ c$. We choose π^+ and π^- accordingly.

Remark A.1.1. We should note that our choice of π^+ and π^- is different from that of Roberts in [Rb5], but rather we follow [HST] for the Whittaker model case, although, if π is spherical, it turns out that our π^+ is spherical and coincides with the notation of [Rb5]. Also the reader should notice that in the above discussion the fields F_v and E_v do not have to be non-archimedean.

A.2 Global representations

Next let us consider the global case, and so assume F is a global field of char $F \neq 2$, X is a 4 dimensional quadratic space with discriminant d, and $E = F(\sqrt{d})$ if $d \neq 1$. Let π be a cuspidal automorphic representation of $\mathrm{GSO}(X, \mathbb{A}_F)$. Then as in Chapter 6, if $d = 1$, then $\pi = \pi(\tau_1, \tau_2)$ for a cuspidal automorphic representation $\tau_1 \otimes \tau_2$ of $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$, and if $d \neq 1$, then $\pi = \pi(\chi, \tau)$ for a cuspidal automorphic representation τ of $B_{D,E}^\times(\mathbb{A}_F)$. Define π^c by taking $V_{\pi^c} = \{f \circ c : f \in V_\pi\}$, where

$c : \mathrm{GSO}(X, \mathbb{A}_F) \to \mathrm{GSO}(X, \mathbb{A}_F)$ is the isomorphism given by conjugation $g \mapsto tgt$. Then clearly π^c is a cuspidal automorphic representation of $\mathrm{GSO}(X, \mathbb{A}_F)$. (Note that as an admissible representation, π^c is isomorphic to the representation π' with $V_{\pi'} = V_{\pi^c}$ and the action defined by $\pi'(g)f = \pi(tgt)f$, and so if we write $\pi \cong \otimes\pi_v$, then $\pi^c \cong \otimes\pi_v^c$.) By multiplicity one theorem, $\pi \cong \pi^c$ implies $V_\pi = V_{\pi^c}$ and in this case $f \circ c \in V_\pi$. Also let σ be a cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A}_F)$. Define $V_\sigma^\circ = \{f|_{\mathrm{GSO}(X,\mathbb{A}_F)} : f \in V_\sigma\}$. Then either $V_\sigma^\circ = V_\pi$ for some cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A}_F)$ such that $\pi = \pi^c$, or $V_\sigma^\circ = V_\pi \oplus V_{\pi^c}$ for some cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A}_F)$ such that $\pi \neq \pi^c$. (See [HST, p.381–382].)

Now define $\widehat{\pi}$ to be the sum of all the cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A}_F)$ lying above π, *i.e.* $\widehat{\pi} = \oplus_i \sigma_i$ where σ_i runs over all the cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A}_F)$ such that $\sigma_i^\circ = \pi$ if $\pi = \pi^c$, or $\pi \oplus \pi^c$ otherwise. First assume that $\pi \ncong \pi^c$. Then the following proposition is due to [HST].

Proposition A.2.1. *Assume $\pi \ncong \pi^c$. Then*

$$\widehat{\pi} \cong \bigoplus_\delta \bigotimes_v \pi_v^{\delta(v)},$$

where δ runs over all the maps from the set of all places of F to $\{\pm\}$ with the property that $\delta(v) = +$ for almost all places of F, and $\delta(v) = +$ if $\pi_v \ncong \pi_v^c$. Moreover each $\otimes\pi_v^{\delta(v)}$ is (isomorphic to) a cuspidal automorphic representation of $GO(X, \mathbb{A}_F)$.

Proof. See [HST, p.382-383]. □

Next assume that $\pi^c = \pi$. In this case, to the best of our knowledge, the analogue of the above proposition is not known in full generality. However, if π is generic, *i.e.* has a Whittaker model, we have the following, which is also due to [HST].

Proposition A.2.2. *Assume π is generic and $\pi^c = \pi$. Then*

$$\widehat{\pi} \cong \bigoplus_\delta \bigotimes_v \pi_v^{\delta(v)},$$

where δ runs over all the maps from the set of all places of F to $\{\pm\}$ with the property that $\delta(v) = +$ for almost all places of F, $\delta(v) = +$ if $\tau_v \ncong \tau_v^c$, and $\prod_v \delta(v) = +$. Moreover each $\otimes\pi_v^{\delta(v)}$ is (isomorphic to) a cuspidal automorphic representation of $GO(X, \mathbb{A}_F)$.

Proof. The proof is also given in [HST, p.382-383]. □

Remark A.2.3. For this proposition, the proof in [HST] works only for the generic case. This is because, if π is not generic, then our choice of π_v^+ and π_v^- can not be shown to be compatible with the global map $\theta^{\pm} : V_\pi \to V_\pi$ given by $f \mapsto \pm f \circ c$. Namely if $f \in V_\pi$ can be identified with $\otimes x_v \in \otimes V_{\pi_v}$ via a fixed isomorphism $\pi \cong \otimes \pi_v$, then we need that $\theta^+(f) = \otimes \theta^{\delta(v)}(x_v)$ for δ satisfying $\prod \delta(v) = +$. If π is generic, this can be shown. However for the non-generic case, we are unable to prove it, although we conjecture that this also holds. Also we should note that if $d = 1$, then π is generic if and only if the corresponding D splits, and if $d \neq 1$, then π is generic if and only if the corresponding $B_{D,E}$ splits.

Those propositions tell us that if π is a cuspidal automorphic representation of $\mathrm{GSO}(X, \mathbb{A}_F)$ (and π is generic if $\pi^c = \pi$) and δ is a map from the set of all places of F to $\{\pm\}$ having the property described in the above propositions, then there is a cuspidal automorphic representation $\sigma = (\pi, \delta)$ of $\mathrm{GO}(X, \mathbb{A}_F)$ lying above π such that $\sigma \cong \otimes \pi_v^{\delta(v)}$. We call such map δ an "extension index" of π, and (π, δ) the extension of π with an extension index δ.

A.3 Proof of Theorem 1.0.4

We will prove Theorem 1.0.4. We start with the following theorem due to [H-PS].

Proposition A.3.1. *Let $O = O(X)$ and $SO = SO(X)$, and also let $T = O/SO$. Assume σ is a generic cuspidal automorphic representation of $O(\mathbb{A}_F)$. Then there is a unitary character μ of $T(F)\backslash T(\mathbb{A}_F)$ such that the space $\mu \otimes V_\sigma$ has non-zero theta lift $\Theta_2(\mu \otimes V_\sigma)$, where we define $\mu \otimes V_\sigma = \{\mu f : f \in V_\sigma\}$.*

Proof. This is essentially Theorem 8.1 of [H-PS], which follows from Theorem 5.7 of the same paper. However, we should mention that in [H-PS] they state that μ is on $T(\mathbb{A}_F)$, but from the proof of Theorem 5.7 it is easy to see that μ is actually on $T(F)\backslash T(\mathbb{A})$. Indeed if μ is not trivial on $T(F)$, μf can not be automorphic. □

Now this can be extended to our similitude case.

Corollary A.3.2. *Let $GO = GO(X)$ and $GSO = GSO(X)$, and also let $T = GO/GSO$. Assume σ is a generic cuspidal automorphic representation of $GO(\mathbb{A})$. Then there is a unitary character μ of $T(F)\backslash T(\mathbb{A})$ such that the space $\mu \otimes V_\sigma$ has non-zero theta lift $\Theta_2(\mu \otimes V_\sigma)$, where $\mu \otimes V_\sigma = \{\mu f : f \in V_\sigma\}$.*

Proof. This immediately follows from the above proposition and lemma 4.0.9 together with the fact that O/SO ≅ GO/GSO so that we can choose the same μ as in the above proposition. □

Here we should mention that μ is trivial on the center of $\mathrm{GO}(X, \mathbb{A})$ because the center is contained in $\mathrm{GSO}(X, \mathbb{A})$, and so σ and $\mu \otimes V_\sigma$ have the same central character.

The following is due to Roberts [Rb3, Rb5].

Proposition A.3.3. *Let π_v be an admissible representation of GSO(X, F_v). Then at least one of π_v^+ and π_v^- has non-zero theta lift to GSp$(2, F_v)$. Moreover, if π_v is spherical, then π_v^+ always has non-zero theta lift.*

Proof. The first part is one of the main theorems of [Rb3]. (Note that the proof works even if $v|\infty$.) The second part is proven in [Rb5, Proposition 4.3]. □

Finally we can prove Theorem 1.0.4

Proof of Theorem 1.0.4. (1). First consider any extension $\sigma = (\pi, \delta)$ of π. Then by Corollary A.3.2, there exists a unitary character μ so that $\mu \otimes V_\sigma$ has non-zero theta lift. Now for each $f \in V_\sigma$, clearly $f|_{\mathrm{GSO}(X,\mathbb{A}_F)} = (\mu f)|_{\mathrm{GSO}(X,\mathbb{A}_F)}$, which means that $V_\sigma^\circ = (\mu \otimes V_\sigma)^\circ$, *i.e.* both V_σ and $\mu \otimes V_\sigma$ lie above V_π. Hence there is an extension index δ' of π so that $\mu \otimes V_\sigma = V_{\sigma'}$ for $\sigma' = (\pi, \delta')$, which we re-choose to be σ.

(2). This immediately follows from Proposition A.2.1, Theorem 1.0.2, and Proposition A.3.3.

□

Appendix B

Generic transfer from GL(4) to GSp(2)

It is well-known that that the connected component of the L-group ${}^L\mathrm{GSp}(2)^\circ$ of GSp(2) over a number field F is (coincidentally) $\mathrm{GSp}(2,\mathbb{C})$. It has a natural embedding into $\mathrm{GL}(4,\mathbb{C})$, which is the connected component of the L-group ${}^L\mathrm{GL}(4)^\circ$ of GL(4). Therefore from the point of view of Langlands functoriality, for each automorphic representation Π of $\mathrm{GSp}(2,\mathbb{A}_F)$, one can expect that there exits an automorphic representation π of $\mathrm{GL}(4,\mathbb{A}_F)$ which is a functorial lift of Π with respect to the embedding $\mathrm{GSp}(2,\mathbb{C}) \hookrightarrow \mathrm{GL}(4,\mathbb{C})$. Indeed, in [A-S1, A-S2] Asgari and Shahidi have obtained such lift for the cuspidal generic spectrum of $\mathrm{GSp}(2,\mathbb{A}_F)$.

In this appendix, we will give the converse of this in the following sense. Let $\pi = \otimes\pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}(4,\mathbb{A}_F)$. Then for each place v, π_v is parameterized by a Weil-Deligne representation of the Weil group W_{k_v} in $\mathrm{GL}(4,\mathbb{C})$. Now if the image of W_{k_v} is in $\mathrm{GSp}(4,\mathbb{C})$, from the point of view of functoriality, one can expect that there is an automorphic representation Π of $\mathrm{GSp}(2,\mathbb{A}_F)$ such that each local component Π_v is parameterized by the same Weil-Deligne representation. It is this lift that we will establish in this paper. Of course, since it is not known how to parameterize Π_v if it is ramified, our consideration can apply only to unramified v and archimedean v. (This is sometimes called a weak functorial lift [Co1].) Also due to the lack of global Langlands parameterization, we do not really know precisely what kind of global condition should be imposed on π in order for its parameterization to be symplectic. However, at least we will give a sufficient condition for π to transfer to Π so that the Langlands parameterization of π_v is indeed symplectic for unramified and archimedean v. To be more precise, we will prove:

Theorem B.0.4. *Let π be a cuspidal automorphic representation of $GL(4,\mathbb{A}_F)$. If the (incomplete) exterior square L-function $L^S(s,\pi,\wedge^2\otimes\chi^{-1})$ twisted by some Hecke character χ^{-1} has a pole at $s=1$, then there exists a cuspidal generic automorphic representation Π of $GSp(2,\mathbb{A}_F)$ so that its (weak) functorial lift to $GL(4,\mathbb{A}_F)$ via*

$GSp(2,\mathbb{C}) \hookrightarrow GL(4,\mathbb{C})$ is π. Moreover, if this is the case, then the central characters of π and Π are χ^2 and and χ, respectively.

The outline of the proof is the following. First we will identity π with an automorphic representation $\widetilde{\pi}$ of the split rank 3 orthogonal group $\mathrm{GSO}(X, \mathbb{A}_F)$, and then obtain a non-zero theta lift $\theta(V_{\widetilde{\pi}})$ to $\mathrm{GSp}(2, \mathbb{A}_F)$, and then show that the contragredient of an irreducible constituent of $\theta(V_{\widetilde{\pi}})$ is the lift Π we want.

It should be noted that his theorem has been known among experts for a long time, and has been quoted (and sometimes even used) in various places [BR, Pr-Rm, K-S, M]. As far as we understand, Jacquet, Piatetski-Shapiro, and Shalika were the ones who first announced this theorem and knew how to obtain this result, at least in principle. However, to the best of our knowledge, their proof remains unpublished, and thus we decided to provide a simpler proof in this appendix by using various results and ideas [A-S1, A-S2, GRS, HST, Mo, Rb5] which were not available at the time this result was originally obtained.

B.1 Exterior square L-functions

In this section, we quickly review the notion of the exterior squire L-function. Recall that ${}^L\mathrm{GL}(n)^\circ = \mathrm{GL}(n,\mathbb{C})$ acts on the space $\mathbb{C}^n \otimes \mathbb{C}^n$ by $g \cdot (x \otimes y) = g \cdot x \otimes g \cdot x$, and this representation of $\mathrm{GL}(n,\mathbb{C})$ has two irreducible components, one of which is the space $\wedge^2\mathbb{C}^n$ of skew-symmetric tensors spanned by the elements of the form $x \otimes y - y \otimes x$. The dimension of $\wedge^2\mathbb{C}^n$ is $\frac{1}{2}n(n-1)$, and so we have a homomorphism $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(\frac{1}{2}n(n-1),\mathbb{C})$. This representation is called the exterior square representation. In particular, if $n = 4$, we have the exterior square map $\wedge^2 : \mathrm{GL}(4,\mathbb{C}) \to \mathrm{GL}(6,\mathbb{C})$.

Now if π is a cuspidal automorphic representation of $\mathrm{GL}(4,\mathbb{A}_k)$ and each π_v is parameterized by a Weil-Deligne representation of the Weil group by $r_v : W_{k_v} \to \mathrm{GL}(4,\mathbb{C})$, then we can attach a local L-factor $L_v(s, \pi_1, \wedge^2)$ by postcomposing r_v with $\wedge^2$. If π_v is unramified and is the spherical component of the induced representation $\mathrm{Ind}_P^{\mathrm{GL}(4,k_v)}(\chi_1\chi_2\chi_3\chi_4)$ for χ_i's unramified, the local factor $L_v(s, \pi_1, \wedge^2)$ is the product of six abelian L-factors. More explicitly, we have

$$L_v(s, \pi_1, \wedge^2) = \\ L_v(s, \chi_1\chi_2)L_v(s, \chi_1\chi_3)L_v(s, \chi_1\chi_4)L_v(s, \chi_2\chi_3)L_v(s, \chi_2\chi_4)L_v(s, \chi_3\chi_4).$$

Also if χ is a Hecke character, we can define the twist $L_v(s, \pi_v, \wedge^2 \otimes \chi_v^{-1})$ by χ_v in the obvious way. By taking the product over all unramified v, we define the (incomplete) exterior square L-function $L^S(s, \pi, \wedge^2 \otimes \chi^{-1})$ twisted by χ^{-1}. Various properties of the exterior square L-function are proven in [J-S2], among which we need to mention the following.

Proposition B.1.1. *Let π be a cuspidal automorphic representation of $GL(4, \mathbb{A}_F)$ whose central character is ω_π. Assume that the (incomplete) exterior square L-function $L^S(s, \pi, \wedge^2 \otimes \chi^{-1})$ twisted by χ^{-1} has a pole at $s = 1$. Then $\omega_\pi = \chi^2$.*

Remark B.1.2. We *assume* that this property of the exterior L-function characterizes π being "symplectic". See [Pr-Rm]. Also the reader can find an interesting discussion from a point of view of Galois representation in [M].

B.2 Relation between GL(4) and GSO(X)

In this section, we associate a cuspidal automorphic representation π of $\mathrm{GL}(4, \mathbb{A}_F)$ to a cuspidal automorphic representation $\widetilde{\pi}$ of $\mathrm{GSO}(X, \mathbb{A}_F)$ for a certain 6 dimensional quadratic space X. First of all, we note that thanks to Proposition B.1.1, if π satisfies the condition of Theorem B.0.4, we may assume that the central character of π is χ^2. Thus from now on, we always assume this.

Now let X be the 6 dimensional vector space over F of the following form:

$$X = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} : a_{ij} \in k \right\}.$$

Then X can be made into a quadratic space (X, q) with the quadratic form q called the Phaffian which is given by

$$q(\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23},$$

with the property $q(x)^2 = \det(x)$ for $x \in X$. (See, for example, p.588-589 of [L].) Let $\mathrm{GO}(X)$ be the group of orthogonal similitudes of X with respect to q, *i.e.* $\mathrm{GO}(X) = \{f \in \mathrm{GL}(X) : q(f(x)) = \nu(x)q(x) \text{ for some } \nu(x) \in F^\times\}$. There is a map $\mathrm{GO}(X) \to \{\pm 1\}$ given by $h \mapsto \frac{\det(h)}{\nu(h)^3}$. We denote the kernel of this map by $\mathrm{GSO}(X)$. Then $\mathrm{GSO}(X)$ is the identity component of $\mathrm{GO}(X)$.

We can see that that $\mathrm{GO}(X)$ is split, *i.e.* $\mathrm{GO}(X) = \mathrm{GO}_{3,3}$, by describing the quadratic space (X, q) more concretely. Namely, first let us choose our basis to be

$\{x_{12},\ x_{13},\ x_{14},\ x_{23},\ x_{24},\ x_{34}\}$, where

$$x_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ x_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ x_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$x_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ x_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ x_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then with respect to this basis, the matrix of q is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and so the discriminant of q is -1. We see that each of $\{x_{12}, x_{34}\}$, $\{x_{13}, x_{24}\}$, and $\{x_{14}, x_{23}\}$ spans a hyperbolic plane, and (X, q) is the orthogonal product of those three hyperbolic planes. Also note that for each $(a_{ij}) \in \mathrm{GL}(2)$ the "(x_{ij}, x_{kl}) entry" of $\rho((a_{ij}))$ is given by $\det\left(\begin{smallmatrix} a_{ki} & a_{kj} \\ a_{li} & a_{lj} \end{smallmatrix}\right)$.

Now each $g \in \mathrm{GL}(4)$ acts on X as $g \cdot x = gx\ {}^tg$ for $x \in X$, where ${}^tg$ is the matrix transpose of g. Let T_g denote the linear operator corresponding to this action of g. Direct computation shows that $q(T_g x) = q(gx\ {}^tg) = \det(g)q(x)$ and $\det(T_g) = \det(g)^3$. Therefore $T_g \in \mathrm{GSO}(X)$. Then we have the following exact sequence of group schemes over k:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{GL}(4) \xrightarrow{\rho} \mathrm{GSO}(X) \longrightarrow 0,$$

where $\rho(g) = T_g$. (The surjectivity of ρ can be seen from the fact that the dimensions of $\mathrm{GL}(4)$ and $\mathrm{GSO}(X)$ are the same, both are connected as algebraic groups, and the image of ρ is Zariski closed.) By taking the adelic points, we have

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{GL}(4, \mathbb{A}_F) \xrightarrow{\rho} \mathrm{GSO}(X, \mathbb{A}_F) \longrightarrow \mathbb{A}_F^\times / \mathbb{A}_F^{\times 2} \longrightarrow 0.$$

From this exact sequence, we can see that $\mathrm{GSO}(X, \mathbb{A}_F) = \mathbb{A}_F^\times \rho(\mathrm{GL}(4, \mathbb{A}_F))$. Then we have

Proposition B.2.1. *There is a bijective correspondence between a cuspidal automorphic representation π of $GL(4, \mathbb{A}_F)$ with the central character of the form χ^2 for a Hecke character χ and a cuspidal automorphic representation $\widetilde{\pi}$ of $GSO(X, \mathbb{A}_F)$ with the central character χ.*

Proof. Let $\varphi \in V_\pi$. Then define a function $\tilde{\varphi} : \mathrm{GSO}(X, \mathbb{A}_F) \to \mathbb{C}$ by $\tilde{\varphi}(a\rho(g)) = \chi(a)\varphi(g)$ for all $a \in \mathbb{A}^\times$ and $g \in \mathrm{GL}(4, \mathbb{A}_F)$. The condition on the central character of π guarantees that this is well-defined, *i.e.* $a\rho(g) = a'\rho(g')$ implies $\tilde{\varphi}(a\rho(g)) = \tilde{\varphi}(a'\rho(g'))$. Then it is easy to see that $\tilde{\varphi}$ is a cusp form on $\mathrm{GSO}(X, \mathbb{A}_F)$. Conversely let $\tilde{\varphi} \in V_{\widetilde{\pi}}$. Then we can define $\varphi \in V_\pi$ by $\varphi(g) = \varphi \circ \rho(g)$. It is clear that this gives the desired bijection. □

Remark B.2.2. Since a cuspidal automorphic representation π of $\mathrm{GL}(4, \mathbb{A}_F)$ is always generic, *i.e.* has a Whittaker model, the cuspidal representation $\widetilde{\pi}$ of $\mathrm{GSO}(X, \mathbb{A}_F)$ is also generic.

By the same token, we have the following local analogue.

Proposition B.2.3. *There is a bijective correspondence between an irreducible admissible representation π_v of $GL(4, F_v)$ with the central character of the form χ_v^2 for some character χ_v and an irreducible admissible representation $\widetilde{\pi}_v$ of $GSO(X, F_v)$ with the central character χ_v.*

B.3 Global theta lifts from GSO$(X, \mathbb{A}_F)$ to GSp$(2, \mathbb{A}_F)$

We will obtain the desired lift to $\mathrm{GSp}(2, \mathbb{A}_F)$ as a theta lift from $\mathrm{GSO}(X, \mathbb{A}_F)$ and prove

Proposition B.3.1. *Let π be a cuspidal automorphic representation of $GL(4, \mathbb{A}_F)$ with central character χ^2, and $\widetilde{\pi}$ the corresponding cuspidal representation of $GSO(X, \mathbb{A}_F)$ with central character χ. Further assume the incomplete exterior square L-function $L^S(s, \pi, \wedge \otimes \chi^{-1})$ twisted by χ^{-1} has a pole at $s = 1$. Then the theta lift $\Theta_2(V_{\widetilde{\pi}})$ of $\widetilde{\pi}$ to $GSp(2, \mathbb{A}_F)$ is non-zero generic cuspidal.*

As we did in the main part of this thesis, to consider the non-vanishing problem, we first consider the corresponding isometry theta lifting. First we need to state the following non-vanishing result due to Ginzburg, Rallis, and Soudry [GRS].

Proposition B.3.2. *Let σ be a generic cuspidal automorphic representation of the split orthogonal group $SO_{n,n}(\mathbb{A}_F)$ of rank n. Assume that the (incomplete) standard L-function $L^S(s, \sigma)$ has a pole at $s = 1$. Then the theta lift $\theta_{n-1}(V_\sigma)$ of σ to $Sp(n-1, \mathbb{A}_F)$ is non-zero generic cuspidal.*

Proof. See [GRS, Section 3]. □

Now let σ be one of the irreducible components of $\widetilde{\pi}|_{\mathrm{SO}(X, \mathbb{A}_F)}$. If we write $\sigma \cong \otimes\sigma_v$ and $\widetilde{\pi} \cong \otimes\widetilde{\pi}_v$, then for v unramified, σ_v is the spherical constituent of $\widetilde{\pi}_v|_{\mathrm{SO}(X, F_v)}$ by Lemma 4.0.11. Also if we write $\pi \cong \otimes\pi_v$, then, for v unramified, π_v is a full induced representation $\mathrm{Ind}_{P_v}^{G_v}\chi'$, where $G_v = \mathrm{GL}(4, F_v)$, P_v is the standard parabolic subgroup of G_v, and χ' is an unramified character on P_v given by

$$\chi'(\begin{pmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{pmatrix}) = \chi_1(a)\chi_2(b)\chi_3(c)\chi_4(d)$$

for unramified characters χ_i's on $F_v^\times$. By our assumption on the central character, we have $\chi_1\chi_2\chi_3\chi_4 = \chi_v^2$ for some unramified character χ_v. Notice that, by the map ρ, we have

$$\begin{pmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{pmatrix} \mapsto \begin{pmatrix} ab & * & * & * & * & * \\ 0 & ac & * & * & * & * \\ 0 & 0 & ad & * & * & * \\ 0 & 0 & 0 & bc & * & * \\ 0 & 0 & 0 & 0 & bd & * \\ 0 & 0 & 0 & 0 & 0 & cd \end{pmatrix},$$

where the latter matrix is with respect the basis we have chosen. Let $GSO_v = \mathrm{GSO}(X, F_v)$, and Q_v its standard parabolic. Then by direct computation together with $\chi_1\chi_2\chi_3\chi_4 = \chi_v^2$, we see that $\widetilde{\pi}_v = \mathrm{Ind}_{Q_v}^{GSO_v}\mu'$ for a character μ' on P_v given by

$$\mu'(\begin{pmatrix} \beta_1 & * & * & * & * & * \\ 0 & \beta_2 & * & * & * & * \\ 0 & 0 & \beta_3 & * & * & * \\ 0 & 0 & 0 & \lambda\beta_3^{-1} & * & * \\ 0 & 0 & 0 & 0 & \lambda\beta_2^{-1} & * \\ 0 & 0 & 0 & 0 & 0 & \lambda\beta_1^{-1} \end{pmatrix}) = \mu_0(\lambda)\mu_1(\beta_1)\mu_2(\beta_2)\mu_3(\beta_3),$$

where

$$\mu_0 = \frac{\chi_v}{\chi_1},\ \mu_1 = \frac{\chi_1\chi_2}{\chi_v},\ \mu_2 = \frac{\chi_1\chi_3}{\chi_v},\ \mu_3 = \frac{\chi_1\chi_4}{\chi_v}.$$

Then we have

Proposition B.3.3. *Let $L^S(s, \sigma)$ be the (incomplete) standard L-function of σ. Then*

$$L^S(s, \sigma) = L^S(s, \pi, \wedge^2 \otimes \chi^{-1}),$$

where S is a finite set of places containing all the ramified places and archimedean places as usual.

Proof. Let $v \notin S$. Then we have $\mathrm{SO}(X, F_v)$-isomorphism

$$\widetilde{\pi}_v|_{\mathrm{SO}(X,F_v)} = (\mathrm{Ind}_{Q_v}^{GSO_v}\mu')|_{\mathrm{SO}(X,F_v)} \xrightarrow{\sim} \mathrm{Ind}_{Q_v\cap \mathrm{SO}(X,F_v)}^{SO(X,F_v)}\mu'|_{Q_v\cap\mathrm{SO}(X,F_v)}$$

given by restriction of functions. By the above computation, we have

$$\mu'|_{Q_v\cap \mathrm{SO}(X,F_v)}(\begin{pmatrix} \beta_1 & * & * & * & * & * \\ 0 & \beta_2 & * & * & * & * \\ 0 & 0 & \beta_3 & * & * & * \\ 0 & 0 & 0 & \beta_3^{-1} & * & * \\ 0 & 0 & 0 & 0 & \beta_2^{-1} & * \\ 0 & 0 & 0 & 0 & 0 & \beta_1^{-1} \end{pmatrix}) \\ = (\chi_1\chi_2/\chi_v)(\beta_1)(\chi_1\chi_3/\chi_v)(\beta_2)(\chi_1\chi_4/\chi_v)(\beta_3).$$

Then the standard local L-factor at v is

$$\begin{aligned} L_v(s,\sigma_v) &= L_v(s,\chi_1\chi_2/\chi_v)L_v(s,\chi_1\chi_3/\chi_v)L_v(s,\chi_1\chi_4/\chi_v)L_v(s,\chi_v/(\chi_1\chi_2)) \\ &\qquad \cdot L_v(s,\chi_v/(\chi_1\chi_3))L_v(s,\chi_v/(\chi_1\chi_4)) \\ &= L_v(s,\chi_1\chi_2/\chi_v)L_v(s,\chi_1\chi_3/\chi_v)L_v(s,\chi_1\chi_4/\chi_v)L_v(s,\chi_3\chi_4/\chi_v) \\ &\qquad \cdot L_v(s,\chi_2\chi_4/\chi_v))L_v(s,\chi_2\chi_3/\chi_v) \quad \text{because } \chi_v^2 = \chi_1\chi_2\chi_3\chi_4 \\ &= L_v(s,\widetilde{\pi}_v,\wedge^2\otimes\chi_v^{-1}). \end{aligned}$$

Thus the proposition follows. □

We should also mention the following, which can be easily proven in the same way as Lemma 4.0.9.

Lemma B.3.4. *Let $g = \theta_n(f;\varphi)$ and $g' = \theta_n(f|_{SO(X,\mathbb{A}_k)};\varphi)$, where $\theta_n(f|_{SO(X,\mathbb{A}_k)};\varphi)$ is the isometry theta lift of $f|_{SO(X,\mathbb{A}_k)}$ to Sp$(n,\mathbb{A}_k)$. Then g is in the space of generic cusp forms if and only if g' is.*

Then we can prove Proposition B.3.1.

Proof of Proposition B.3.1. By Proposition B.3.2, the theta lift $\Theta_2(V_\sigma)$ of σ to $\mathrm{Sp}(2,\mathbb{A}_k)$ is non-zero generic cuspidal, if the exterior square L-function $L^S(s,\pi,\wedge^2\otimes\chi^{-1})$ has a pole at $s=1$. Then by Lemma 4.0.9, the theta lift $\Theta_2(V_{\widetilde{\pi}})$ of $\widetilde{\pi}$ to $\mathrm{GSp}(2,\mathbb{A}_F)$ is non-zero generic cuspidal by Lemma 4.0.9 and B.3.4. This completes the proof. □

B.4 Local theta lift from GSO(X) to GSp(2)

In this section we will explicitly compute the local parameters of the local lift from $\mathrm{GSO}(X,F_v)$ to $\mathrm{GSp}(2,F_v)$ for the unramified case as we did in Chapter 5. As in Chapter 5, the groups $\mathrm{GSO}(X,F_v), \mathrm{GSp}(n,F_v)$, etc are all denoted simply by $\mathrm{GSO}(X)$, $\mathrm{GSp}(n)$, etc, and we just write $F=F_v$. Moreover we assume that v

is finite. Also "Ind" always means unnormalized induction, and whenever we use normalized induction, we use the notation "n-Ind". Thus, for example, if π is the principal series representation of $G = \mathrm{GL}(4)$ fully induced from the standard parabolic P by the four unramified characters $\chi_1, \chi_2, \chi_3, \chi_4$, then we have $\pi \cong \text{n-Ind}\,^G_P(\chi_1\chi_2\chi_3\chi_4) = \mathrm{Ind}\,^G_P(\tilde{\chi_1}\tilde{\chi_2}\tilde{\chi_3}\tilde{\chi_4})$, where $\tilde{\chi_1} = |\cdot|^{3/2}\chi_1$, $\tilde{\chi_2} = |\cdot|^{1/2}\chi_1$, $\tilde{\chi_3} = |\cdot|^{-1/2}\chi_2$, and $\tilde{\chi_4} = |\cdot|^{-3/2}\chi_4$.

First we need the following, which seems to be well-known by now.

Lemma B.4.1. *Let τ be the character on the standard parabolic P_2 of $GSp(2)$ which is trivial on the unipotent radical and sends*

$$\mathrm{diag}\,(\alpha_1, \alpha_2, \gamma\alpha_1^{-1}, \gamma\alpha_2^{-1}) \mapsto \tau_0(\gamma)\tau_1(\alpha_1)\tau_2(\alpha_2)$$

for unramified characters τ_0, τ_1, and τ_2. If Π is the unramified subquotient of the representation of $GSp(2)$ give by $\text{n-Ind}\,^{GSp(2)}_{P_2}\tau$, then the Langlands parameter of Π is

$$\mathrm{diag}\,(\chi_0(\varpi), \chi_0\chi_1(\varpi), \chi_0\chi_1\chi_2(\varpi), \chi_1\chi_2(\varpi)) \in GSp(2, \mathbb{C}) = {}^L GSp(2)^\circ,$$

where ϖ is the uniformizer of F.

By using this together with the techniques from Chapter 5, we prove

Proposition B.4.2. *Assume π is an unramified representation of $GL(4)$ with a unitary central character which is fully induced from four unramified characters $\chi_1, \chi_2, \chi_3, \chi_4$,* i.e. $\pi \cong \text{n-Ind}\,^{GL(4)}_P \chi_1\chi_2\chi_3\chi_4$, *and the central character of π is χ^2,* i.e. $\chi_1\chi_2\chi_3\chi_4 = \chi^2$. *Let $\widetilde{\pi}$ be the unramified representation of $GSO(X)$ that corresponds to π by Proposition B.2.3. If $\widetilde{\pi}$ corresponds to an unramified $\Pi \in \mathrm{Irr}(GSp(2))$ under theta correspondence, then the Langlands parameter of Π is*

$$\mathrm{diag}\,(\chi_1(\varpi), \chi_3(\varpi), \chi_2(\varpi), \chi_4(\varpi)) \in GSp(2, \mathbb{C}).$$

Proof. Let $\widetilde{\pi} \cong \mathrm{Ind}\,^{\mathrm{GSO}(X)}_{Q_3}\mu$ where μ acts on M_{Q_3} by

$$\mathrm{diag}\,(\beta_1, \beta_2, \beta_3, \lambda\beta_3^{-1}, \lambda\beta_2^{-1}, \lambda\beta_1^{-1}) \mapsto \mu_0(\lambda)\mu_1(\beta_1)\mu_2(\beta_2)\mu_3(\beta_3),$$

with Q_3 the parabolic subgroup preserving the flag $\langle x_{12}\rangle \subset \langle x_{12}, x_{13}\rangle \subset \langle x_{12}, x_{13}, x_{14}\rangle$, and x_{ij}'s as in Section B.2. Then we have

$$\mu_0 = \frac{\chi}{|\cdot|^{3/2}\chi_1},\ \mu_1 = \frac{|\cdot|^2\chi_1\chi_2}{\chi},\ \mu_2 = \frac{|\cdot|\chi_1\chi_3}{\chi},\ \mu_3 = \frac{\chi_1\chi_4}{\chi}.$$

There is a non-zero R-homomorphism

$$\omega_{X,2} \longrightarrow \Pi \otimes \mathrm{Ind}\,^{\mathrm{GSO}(X)}_{Q_3}\mu = \mathrm{Ind}\,^{\mathrm{GSp}(2)\times\mathrm{GSO}(X)}_{\mathrm{GSp}(2)\times Q_3}\Pi \otimes \mu.$$

Since there is a natural injection $\mathrm{Ind}\,^{\mathrm{GSp}(2)\times\mathrm{GSO}(X)}_{\mathrm{GSp}(2)\times Q_3}\Pi\otimes\mu \hookrightarrow \mathrm{Ind}\,^{R}_{R\cap(\mathrm{GSp}(2)\times Q_3)}\Pi\otimes\mu$ (see Lemma 5.1.4), we have a non-zero R-homomorphism

$$\omega_{X,2} \longrightarrow \mathrm{Ind}\,^{R}_{R\cap(\mathrm{GSp}(2)\times Q_3)}\Pi\otimes\mu.$$

Let Q be the parabolic subgroup preserving the flag $\langle x_{12}, x_{13}, x_{14}\rangle$. Then by induction in stages, we have a non-zero R-homomorphism

$$\omega_{X,2} \longrightarrow \mathrm{Ind}\,^{R}_{S_Q}\mathrm{Ind}\,^{M_Q}_{S_{Q_3}}\Pi\otimes\mu,$$

where S_Q and S_{Q_3} are as in Proposition 5.1.1.

Now if we take the Jacquet module of $\omega_{X,2}$ with respect to N_Q, the Frobenius reciprocity and Proposition 5.1.1 give non-zero R-homomorphisms

$$0 \subset J^{(2)} \subset J^{(1)} \subset J^{(0)} \xrightarrow{\ \varphi\ } \mathrm{Ind}\,^{M_Q}_{S_{Q_3}}\Pi\otimes\mu.$$

First assume that $\ker\varphi \supseteq J^{(1)}$. Then there is a non-zero M_Q-homomorphism

$$I^{(0)} = J^{(0)}/J^{(1)} \longrightarrow \mathrm{Ind}\,^{M_Q}_{S_{Q_3}}\Pi\otimes\mu.$$

Then if we take the Jacquet module of $I^{(0)}$ with respect to N_{Q_3}, the Frobenius reciprocity and Proposition 5.1.1 give a non-zero M_{Q_3}-homomorphism

$$\mathrm{Ind}\,^{M_{Q_3}}_{S_{P_0,Q_3}}\sigma_{0,3} \cong \sigma_{0,3} \longrightarrow \Pi\otimes\mu,$$

where $\sigma_{0,3}$ is as in Proposition 5.1.1. Then on the element of the form $((1),\beta_1,\beta_2,\beta_3)\in M_{Q_3}$ acts $\sigma_{0,3}$ by the character

$$((1),\beta_1,\beta_2,\beta_3) \mapsto |\beta_1|^2|\beta_2|^2|\beta_3|^2.$$

On the other hand, it acts on $\Pi\otimes\mu$ by the character

$$((1),\beta_1,\beta_2,\beta_3) \mapsto \mu_1(\beta_1)\mu_2(\beta_2)\mu_3(\beta_3).$$

Therefore we must have $\mu_3 = |\cdot|^2$. But this is impossible, because $\mu_3 = \chi_1\chi_4/\chi$ and so $|\mu_3| < |\cdot|$. Thus $\ker\varphi \not\supseteq J^{(1)}$.

Then by restricting φ to $J^{(1)}$, we have non-zero M_Q-homomorphisms

$$0 \subset J^{(2)} \subset J^{(1)} \xrightarrow{\ \varphi'\ } \mathrm{Ind}\,^{M_Q}_{S_{Q_3}}\Pi\otimes\mu.$$

Then if $\ker\varphi' \supseteq J^{(2)}$, we have a non-zero M_Q-homomorphism

$$I^{(1)} = J^{(1)}/J^{(2)} \longrightarrow \mathrm{Ind}\,^{M_Q}_{S_{Q_3}}\Pi\otimes\mu.$$

Then by exactly the same argument as above, this leads to a contradiction. Therefore $\ker \varphi' \not\supseteq J^{(2)}$.

Then by restricting φ' to $J^{(2)}$, we have a non-zero M_Q-homomorphism

$$J^{(2)}(= I^{(2)}) \longrightarrow \operatorname{Ind}_{S_{Q_3}}^{M_Q} \Pi \otimes \mu.$$

If we take the Jacquet module of $I^{(2)}$ with respect to N_{Q_2}, the Frobenius reciprocity and Proposition 5.1.1 give a non-zero M_{Q_3}-homomorphism

$$\operatorname{Ind}_{S_{P_2,Q_3}}^{M_{Q_3}} \sigma_{2,3} \xrightarrow{\varphi''} \Pi \otimes \mu.$$

Then the element of the form $((1), \beta_1, \beta_2, \beta_3) \in M_{Q_3}$ acts on $\operatorname{Ind}_{S_{P_2,Q_3}}^{M_{Q_3}} \sigma_{2,3}$ by the character

$$((1), \beta_1, \beta_2, \beta_3) \mapsto |\beta_1|^2 \tau_2(\beta_2)\tau_3(\beta_3),$$

and on $\Pi \otimes \mu$ by the character

$$((1), \beta_1, \beta_2, \beta_3) \mapsto \mu_1(\beta_1)\mu_2(\beta_2)\mu_3(\beta_3).$$

Thus we have

$$\mu_1 = |\cdot|^2,\ \mu_2 = \tau_2,\ \mu_3 = \tau_3.$$

Moreover, since $\mu_1 = |\cdot|^2 \chi_1\chi_2/\chi$ and $\chi_1\chi_2\chi_3\chi_4 = \chi^2$, we have

$$\chi = \chi_1\chi_2 = \chi_3\chi_4.$$

Next notice that $M_{Q_3} \cong \mathrm{GSp}(2) \times \mathbb{G}_m^3$. Then by restricting φ'' to the elements of the form $((g), 1, 1, 1) \in \mathrm{GSp}(2) \times \mathbb{G}_m^3 \cong M_{Q_3}$, we have a non-zero $\mathrm{GSp}(2)$-homomorphism

$$\operatorname{Ind}_{P_2}^{\mathrm{GSp}(2)} \sigma'_{2,3} \longrightarrow \Pi \otimes \mu_0^{-1},$$

where $\sigma'_{2,3}$ is the character on P_2 given by

$$\sigma'_{2,3}(\begin{pmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & 0 & \lambda\alpha_1^{-1} & * \\ 0 & 0 & 0 & \lambda\alpha_2^{-1} \end{pmatrix}) = (|\cdot|^2\tau_3^{-1})(\alpha_1)(|\cdot|^2\tau_2^{-1})(\alpha_2)(|\cdot|^{-1}\tau_2\tau_3)(\lambda),$$

and $\Pi \otimes \mu_0^{-1}$ is the twist of Π by μ_0 through the multiplier character ν, *i.e.* $(\Pi \otimes \mu_0)(g) = \mu_0(g)\Pi(g)$. Hence we have a non-zero $\mathrm{GSp}(2)$-homomorphism

$$(\operatorname{Ind}_{P_2}^{\mathrm{GSp}(2)} \sigma'_{2,3}) \otimes \mu_0 \longrightarrow \Pi,$$

where the twisting is again via ν. It is easy to see that

$$(\operatorname{Ind}_{P_2}^{\mathrm{GSp}(2)} \sigma'_{2,3}) \otimes \mu_0 \cong \operatorname{Ind}_{P_2}^{\mathrm{GSp}(2)} \tau,$$

where τ is the character on P_2 given by

$$\tau(\begin{pmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & 0 & \lambda\alpha_1^{-1} & * \\ 0 & 0 & 0 & \lambda\alpha_2^{-1} \end{pmatrix}) = (|\cdot|^2\mu_3^{-1})(\alpha_1)(|\cdot|^2\mu_2^{-1})(\alpha_2)(|\cdot|^{-1}\mu_0\mu_2\mu_3)(\lambda)$$

$$= (|\cdot|^2\chi\chi_1^{-1}\chi_4^{-1})(\alpha_1)(|\cdot|\chi\chi_1^{-1}\chi_3^{-1})(\alpha_2)(|\cdot|^{-3/2}\chi\chi_2^{-1})(\lambda)$$

$$= (|\cdot|^2\chi_3\chi_1^{-1})(\alpha_1)(|\cdot|\chi_4\chi_1^{-1})(\alpha_2)(|\cdot|^{-3/2}\chi_1)(\lambda).$$

Thus Π is an unramified quotient of n-Ind $_{P_2}^{\mathrm{GSp}(2)}\tau'$, where τ' is given by

$$\tau'(\begin{pmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & 0 & \lambda\alpha_1^{-1} & * \\ 0 & 0 & 0 & \lambda\alpha_2^{-1} \end{pmatrix}) = (\chi_3\chi_1^{-1})(\alpha_1)(\chi_4\chi_1^{-1})(\alpha_2)(\chi_1)(\lambda).$$

Thus by Lemma B.4.1, the Langlands parameter of Π is

$$\mathrm{diag}\,(\chi_1(\varpi), \chi_3(\varpi), \chi_2(\varpi), \chi_4(\varpi)) \in \mathrm{GSp}(2, \mathbb{C}).$$

□

B.5 Functoriality

By combining the results of the previous two sections, we have

Proposition B.5.1. *Let π be a cuspidal automorphic representation of $GL(4, \mathbb{A}_F)$ with central character χ^2 such that the exterior square L-function $L^S(s, \pi, \wedge^2 \otimes \chi^{-1})$ has a pole at $s = 1$. Then the theta lift $\Theta_2(V_{\tilde{\pi}})$ of $\tilde{\pi}$ to $GSp(2, \mathbb{A}_F)$ does not vanish, and if we let Π be an irreducible constituent of $\Theta_2(V_{\tilde{\pi}})$, then Π is a generic, cuspidal automorphic representation of $GSp(2, \mathbb{A}_F)$ with central character χ. Moreover for each unramified place v, π_v is the Langlands functorial lift of Π_v via $GSp(2, \mathbb{C}) \hookrightarrow GL(4, \mathbb{C})$.*

At this moment, our lift Π has not been shown to satisfy all the desired properties. Namely, we need to show the functoriality at the archimedean places. However, this easily follows from the following recent result by Asgari and Shahidi [A-S2].

Proposition B.5.2. *Let Π be a generic cuspidal automorphic representation of $GSp(2, \mathbb{A}_k)$ with central character χ. Then Π has a unique (weak) functorial lift π to $GL(4, \mathbb{A}_F)$ with central character χ^2. Moreover π is either cuspidal or an isobaric sum $\pi_1 \boxplus \pi_2$ of two inequivalent cuspidal automorphic representations of $GL(2, \mathbb{A}_F)$. The latter is the case if and only if Π is obtained as a theta lift from the split orthogonal group $GSO(4, \mathbb{A}_F)$.*

Proof. This is (part of) Theorem 2.4. of [A-S2]. Note that theta lift is called Weil lift there. Also although [A-S2] does not explicitly mention this, the orthogonal group $\mathrm{GSO}(4, \mathbb{A}_k)$ must be split. $\square$

Finally, we can prove Theorem B.0.4.

Proof of Theorem B.0.4. First, we can easily see that our lift Π is not a theta lift from the rank 2 orthogonal group $\mathrm{GSO}(4, \mathbb{A}_F)$ by using the latter part of the main theorem of [Mo]. Therefore Π has a unique cuspidal (weak) functorial lift π' to $\mathrm{GL}(4, \mathbb{A}_F)$ whose central character is χ^2. But we already know that π_v is a functorial lift of Π_v for almost all places v, and so by strong multiplicity one theorem for $\mathrm{GL}(4)$, we have $\pi = \pi'$. Thus Π indeed has the desired functorial property. This completes the proof. $\square$

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