A CONSTRUCTIBLE HIGHER RIEMANN HILBERT CORRESPONDENCE

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ABSTRACT

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We introduce a notion of a constructible vector bundle with connection and establish a constructible version of the Higher Riemann-Hilbert correspondence studied by Block and Smith. We also give a construction for a super connection of a graded vector bundle, starting from a higher parallel transport and give a different proof of their result.

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Chapter 1

Introduction

The Riemann-Hilbert correspondence is a classical result in topology that relates vector bundles over a space M with a flat connection and representations of the fundamental group, $\pi_1(M)$. The correspondence is executed by the familiar notion of parallel transport. Parallel sections of a flat vector bundle define a locally constant sheaf. We are concerned with some generalisations of these notions.

In one direction, we may replace locally constant sheaves with sheaves constructible on the space M with a fixed stratification. The corresponding replacement for the fundamental group will be a category of exit paths inspired by unpublished work of MacPherson. These are paths that have an 'exit property'. i.e. that they may not re-enter a lower stratum. We introduce a notion of a constructible vector bundle with connection to take the place of ordinary vector bundles.

The parallel transport that arises from a flat connection is homotopy invariant.

That is, two homotopic paths joining a pair of points induce the same map on the fibers at the endpoints. We could ask instead, that homotopic paths give rise to homotopic transports and higher homotopies give higher homotopies and so on. This is the second direction of the generalisation. We give a construction for a vector bundle with a flat super connection starting with such a coherent transport functor.

We will make use of previous work by Block, Smith and Igusa which in turn was inspired by works of Chen on iterated integrals in the 70's, for a notion of a derived parallel transport construction.

In the second chapter we will review the classical theory and give a notion of a flat constructible vector bundle that would correspond to parallel transport for exit paths. In chapter 3 we start by recalling work of Igusa [Ig09], Block-Smith [BS09]. We then construct a graded vector bundle with a flat super connection starting with a higher transport (or ∞-local system) and show that this is inverse to the constructions of Igusa and Block Smith. In chapter 4 we'll extend the results to the constructible case, combining ideas from the previous two chapters. We end with a few comments on how this relates to equivalence with sheaves.

Chapter 2

Classical Theory

Here we will describe the 1-categorical Riemann Hilbert correspondence in a way that generalises to the ∞ case. Given a vector bundle $p : V \to M$ with a flat connection ∇ we have a notion of a parallel transport functor. This is a rule $F_\gamma(t, s)$ that gives for each path γ in M, a map $V_{\gamma(s)} \to V_{\gamma(t)}$. This can be thought of as functor $\pi_{\leq 1}(M) \to Vect$ the category of vector spaces. This is standard. We will describe below a way back, i.e. we'll construct a bundle and a connection given the data of the transport functor. We will associate a locally constant sheaf with this data. This sheaf will have as sections over an open set U the sections of the vector bundle that are parallel with respect to the connection.

In section 2 we will try to generalise all of this to the constructible case. After introducing our notion of a stratified space, we will build 'constructible vector bundles' as analogues of constructible sheaves to replace vector bundles, and we will see how to extend the above constructions.

2.1 Parallel transport

Let $p : V \to M$ be a smooth vector bundle. A connection on this bundle is a section $\nabla \in \Gamma(T^*M \otimes End(V)$ which satisfies the Liebnitz rule, $[\nabla, f] = df$. Locally we can write $\nabla = d - A$, where $A \in \Omega^1(M, End(V)$ is a matrix of 1-forms.

A section s of the bundle is called parallel with respect to the connection if it satisfies the equation $\nabla s(x) = 0$. In local co-ordinates, this is $ds(x) = As(x)$. This defines an ODE which we know have solutions.

For a path γ, we say $F(t, s) : V(s) \to V(t)$ defines a parallel transport along γ if for any $v_s \in V(s), F(t, s)v_s \in V(t)$ (as t varies) defines a parallel section over γ. Then $F(t, s)$ satisfies

$$\frac{\partial F(t,s)}{\partial t} = A(\gamma'(t))F(t,s)$$

There is a way of computing explicitly the parallel transport. It is given as an iterated integral

$$\sum_{k=0}^{\infty} \int \int_{t<t_1<t_2...<s} A(t_1)A(t_2)..A(t_k)$$

where $A(t_i) = A_{t_i}(\gamma'(t_i)$.

It can be checked that this satisfies $F(t, x)F(x, s) = F(t, s)$. The connection is called flat if $(d - A)^2 = 0$. If a connection is flat, the parallel transport it gives rise to is equal on homotopic paths.

Let $\pi_{\leq 1}(M)$ denote the category whose objects are points of M, and whose morphisms are homotopy classes of paths. Then we may think of the parallel transport (of a flat connection) as a functor $\pi_{\leq 1}(M) \to Vect$, which takes a point $x \to V_x$ and a path $x \to y$ to the transport map. If the connection is flat, this depends only on homotopy class and the equation $F(t,x)F(x,s) = F(t,s)$ gives composition.

Given the information of a functor $F : \pi_{\leq 1}(M) \to Vect$, we may recover the vector bundle and a connection on it as follows. Let $\{V = v_x\}_{v_x \in F(x), x \in M}$ as a set. There is a natural projection map $p : v_x \to x$. If (U, x_0) is a contractible open ball containing x_0, we can write

$$p^{-1}(U) \simeq F_{x_0} \times U$$

$$v_x \to (F^\gamma(x, x_0)v_{x_0}, x)$$

We use this to topologise the bundle and give it a smooth structure. Transition functions are given by the choice of path $x_0 \to x_1$. Now we can define a connection on this bundle as

$$A(t) = \frac{\partial F}{\partial t}|_{t=0}$$

Given a flat vector bundle $V|_M$ or a transport functor F, we may define a sheaf (denoted $\mathcal{F}$) by

$$\mathcal{F}(U) = \{s : U \to V | \nabla s = 0\}$$

$$\mathcal{F}(U) = \{s : U \to V | s(x) = F(x,y)s(y)\}$$

It can be checked that this defines a sheaf (because we can patch together solutions of ODEs) that is locally constant ($\mathcal{F}(U, x_0) \simeq F_{x_0}$).

2.2 Constructible constructions

A stratified space is a space built from manifolds with some measure of control over how they fit together. Standard examples are manifolds with corners filtered by dimension. A constructible sheaf is a sheaf that is locally constant when restricted to each stratum. Given a locally constant sheaf $\mathcal{F}$ we get a notion of parallel transport given by taking the stalk $\mathcal{F}_x$ over each point x and a map $\mathcal{F}_x \to \mathcal{F}_y$ for each path $x \to y$ using the local triviality. When $\mathcal{F}$ is constructible this is no longer possible as we no longer have the same notion of local triviality where two strata meet. Where two strata meet, the stalks at different nearby points $x \in S_0, y \in S_1$ are different (non isomorphic) vector spaces. However, we may salvage a notion of transport of we restrict ourselves to what are called 'exit paths'. Suppose a path γ is such that $x = \gamma(0)$ lies in a lower stratum and $\gamma(0, 1]$ is in a higher stratum. An element in the stalk $\mathcal{F}_x$ still comes from the restriction of a section that is defined in a neighbourhood of that point. We can use this fact to get a lift of our path for a small part of the path at x, and get a map $\mathcal{F}_x \to \mathcal{F}_y$ as before once we've entered the higher stratum. It is a fol theorem (by MacPherson) that there's an equivalence between constructible sheaves and representations of exit paths. We take inspiration from this and imagine vector bundles which do not have local triviality where strata meet, but have a notion of a parallel transport for exit paths. We can then construct a connection on such a 'vector bundle' as we would a connection for an ordinary vector bundle with transport.

2.2.1 Stratified spaces

A stratified space M is a filtered space $M_0 \subset M_1 \subset M_2 .... \subset M_n = M$ satisfying some conditions. In our case we may think of 'stratified manifolds' as being built out of smooth manifolds as follows:

Let $M_0 = S_0$ the 0th-stratum be a closed manifold. Let $(\bar{S}_1, \partial S_1)$ be a closed manifold with boundary. ($S_1 = \bar{S}_1 - \partial S_1$ will be a manifold that represents the 1-stratum). We will obtain the stratified space M_1 by gluing M_0 and the S_1 by a given attaching map $f_1 : \partial S_1 \to M_0$. i.e. $M_1 = M_0 \cup_{f_1} S_1$. We have $M_1 - M_0 = \bar{S}_1 - \partial S_1 = S_1$.

To get M_i we need a manifold with boundary $(\bar{S}_{i+1}, \partial S_{i+1})$ and an attaching map $f_{i+1} : \partial S_{i+1} \to M_i$. Remember M_i is not a manifold for $i \geq 1$. Then let $M_{i+1} = M_i \cup_{f_{i+1}} S_{i+1}$. Again we have $M_{i+1} - M_i = \bar{S}_{i+1} - \partial S_{i+1} = S_{i+1}$.

We would like the maps f_i be submersions. This doesn't quite make sense because M_i is singular. So instead we ask that f_i be a submersion over each stratum in M_i. i.e. $f_{i+1} : f_{i+1}^{-1}(M_j - M_{j-1}) \to M_j - M_{j-1}$ is a submersion for $j < i$

Definition 2.2.1. We define a stratified space to be the all the data specified above. i.e. $\{M_i, S_i, f_i\}$ where f_i are submersions in the above sense. We call the spaces

$S_i = M_i - M_{i-1}$ the pure strata and $\bar{S}_i$ the closed pure strata.

It is now easy to see for example, that simpicial complexes, manifolds with edges are examples of our stratified spaces. Most stratified spaces 'in nature' arise this way, though these do not consist of all (Whitney/Thom-Mather) stratified spaces. Spaces like these are considered in [BF02] and [Kr10].

Remark 2.2.2. The map $f_{n+1} : \partial S_{n+1} \to M_n$ is a submersion on strata by definition. It is also proper since ∂S_{n+1} is compact. By Ehreshmann's theorem, this means that the map is a fiber bundle over each stratum in M_n. A collar neighbourhood of $\partial S_{n+1} \subseteq S_{n+1}$ in the glued space, takes the place of tubular neighbourhoods in Thom-Mather stratified spaces.

2.2.2 Exit paths

Given a stratified space M, the category of exit paths has as objects points of space M and as morphisms paths with the exit property. i.e a path $\gamma : [0,1] \to M$ with the property that if $t_1 \leq t_2$ then $rk(\gamma(t_1)) \leq rk(\gamma(t_2))$ where $rk(x)$ is the dimension of the stratum in which x lives. In words, these are paths that only go up strata, and cannot return to a lower stratum they've exited. We denote this category $\pi_{\leq 1}^{exit}(M)$ and $\pi_{\leq 1}M - Rep$ denotes the functors from this category to $Vect$.

When M is a smooth stratified space, exit paths will be continuous paths that are smooth when restricted to each stratum.

2.2.3 Vector bundles

We want to define a notion of a smooth vector bundle over a stratified space defined above. Given the data $\{S_i, M_i, f_i\}$ of a stratified space, we'll define a constructible vector bundle as follows:

We start with an ordinary smooth vector bundle $p_0 : V_0 \to M_0$ over the 0 stratum M_0. Let $\mathbb{S}_1 \to S_1$ be a smooth vector bundle. Let $\tilde{f}_1 : \mathbb{S}_1|_{\partial S_1} \to V_0|_{M_0}$ be a bundle map covering the attaching map $f_1 : \partial S_1 \to M_0$. Define the constructible bundle V_1 to be $\mathbb{S}_1 \cup_{\tilde{f}_1} V_0$. This V_1 is naturally equipped with a projection map p_1 to M_1 which we can think of as the 0-section in it. We have that V_1 restricted to each open stratum is an ordinary smooth vector bundle.

Next, look at a vector bundle $\mathbb{S}_2 \to S_2$ and a map $\tilde{f}_2 : \mathbb{S}_2|_{\partial S_2} \to V_1|_{M_1}$, such that the map restricted to M_0 and $M_1 - M_0 = S_1$ is a map of ordinary vector bundles. Then define the constructible bundle V_2 to be $\mathbb{S}_2 \cup_{\tilde{f}_2} V_1$. Again we see that this "bundle" restricted to the open strata gives an ordinary vector bundle.

Continuing in this way, we can define a constructible vector bundle on all of M inductively.

Definition 2.2.3. Given a stratified space $\{S_i, M_i, f_i\}$, a constructible vector bundle consists of the data $\{\mathbb{S}_i, V_i, \tilde{f}_i\}$, where $\mathbb{S}_i \to S_i$ is an ordinary vector bundle, $\tilde{f}_i$ covering f_i is a vector bundle map when restricted to each stratum and V_i is inductively defined to be $\mathbb{S}_i \cup_{\tilde{f}_i} V_{i-1}$.

In the sequel, we will take the maps $\tilde{f}_i$ attaching the vector bundles to be 0.

Baues-Ferrario [BF02] study vector bundles of this form and prove a classifying theorem etc. though they do not have a notion of a connection.

We wish to endow such a vector bundle with a (flat) connection that would give rise to the parallel transport for exit paths, similar to the case of constructible sheaves we described in the introduction.

Let us begin with the 0-stratum again. Parallel transport here is the ordinary one and so, a connection on $V_0 \to M_0$ should be an ordinary connection, i.e a section in $\Gamma(End(V_0) \otimes T^*M_0)$.

Let $\gamma : [0,1] \to M_1$ be an exit path such that $\gamma(0) \in M_0$ and $\gamma((0,1]) \subseteq M_1 - M_0$. Let $F(t,s) : V_1(\gamma(s)) \to V_1(\gamma(t))$ be the transport. We then have $F(t,s)F(s,0) = F(t,0) : V_1(\gamma(0)) \to V_1(\gamma(t))$. Here $F(t,s)$ is happening in the open 1-stratum for $s > 0$ and there is again, an ordinary connection in the open 1-stratum that gives rise to it.

The remaining information is $F(t,0)$ as $t \to 0$. $F(0,0)$ as such is not defined, so instead we look at

$$\begin{array}{ccccc} V_0 & \longleftarrow & f_1^*(V_0) & \xrightarrow{\alpha^1} & \mathbb{S}_1 \\ \downarrow & & \downarrow & & \downarrow \\ M_0 & \xleftarrow{f_1} & \partial S_1 & \longrightarrow & \partial S_1 \end{array}$$

and let α^1 take the place of the map $F(0,0)$. We call α^1 an exit map.

Here we assume $\mathbb{S}_1|_{\partial S_1}$ is the extension of the bundle $\mathbb{S}_1$. It might not always be possible to extend a given vector bundle and a transport to the boundary, but we can find a bundle isomorphic to the one we have that does extend. We'll return to this in a moment.

Denote by $\{F^0, F^1\}$ the ordinary parallel transport in the 0 and 1 strata. Now given a vector $v_0 \in V_0$ over $\gamma(0) \in M_0$, we can write $F^{exit}(0,0)v_0 = F^0(0,0)v_0 = v_0$ and $F^{exit}(t,0)v_0 = F^1(t,0)(\alpha^1 v_0)$. and we had $F^{exit}(t,s)v_s = F^1(t,s)v_s$. i.e. we have written the exit transport in terms of the ordinary transports on the strata and the exit connection map α.

This is the object of this section. Given a notion of a parallel transport for exit paths we will try and break it down into ordinary transport on the strata and maps α at the 'exit'. A connection on a constructible bundle then will be the connections that give rise to the ordinary transports and the exit maps.

Now say γ is a more typical exit path with $\gamma([0,1/2]) \subseteq M_0$ and $\gamma((1/2,1]) \subseteq M_1$. We could calculate the transport along γ by making an exit as it were at $\gamma(0)$ and transporting in $M_1 - M_0$ or we could make the exit at $\gamma(1/2)$ and travel the rest of the path over M_1. These two routes have to take us to the same destination, because the paths are (trivially) homotopic. Another way of saying this is that the parallel transport in M_0 and M_1 agrees along the tangent directions common to both M_0 and M_1. This is expressed by the equation

$$F^1(1,1/2)F^1(1/2,0)\alpha^1 v_0 = F^1(1,1/2)\alpha^1 F^0_0(1/2,0)v_0$$

where F^i is the transport in the $i-th$ stratum.

In general, a homotopy of exit paths can be broken down into a homotopy in each strata and a thin homotopy of the above form where paths straddling strata. So having this condition assures any two exit paths that are homotopic have the same transport. (Assuming the transports are homotopy invariant inside each stratum).

The equation for a general t and x looks like

$$F^1(t,x)\alpha_1(\gamma(x)) = \alpha^1(\gamma(t)F^0(t,x)$$

for a curve γ. Multiplying by the invertible $F^0(x,s)$, this equation is equivalent to

$$F^1(t,x)\alpha(\gamma(x)F^0(x,s) = \alpha(\gamma(t)F^0(t,x)F^0(x,s) = \alpha(\gamma(t)F^0(t,s)$$

Now the LHS is independent of x. So we have

$$\frac{\partial}{\partial x}F^1(t,x)\alpha(\gamma(x)F^0(x,s) = F^1(t,x)[A^1(\frac{\partial}{\partial x})\alpha + \frac{\partial}{\partial x}\alpha + \alpha A^0(\frac{\partial}{\partial x})]F^0(x,s) = 0$$

Here $A(\frac{\partial}{\partial x})$ is the connection acting on the vector $\gamma'(x)$. So the condition is equivalent (since we can reverse the calculation) to

$$A^1\alpha + d\alpha + \alpha A^0$$

Remembering that $d\alpha = d\circ\alpha + \alpha\circ d$ by the product rule and $\nabla^1 = d - A^1$ etc., we get that the condition is equal to

$$\nabla^1\alpha = \alpha\nabla^0$$

So far we have that a connection on a bundle with two strata consists of a connection over each stratum and an exit map that satisfies a differential equation involving the connection. Let's now go one more stratum up. As before we can argue that the information of parallel transport for the exit paths is given by $\{F^0, F^1, \alpha^1\}$ on M_1, a transport F^2 on $\mathbb{S}_2|_{S_2}$ and an exit map α^2 given as

$$\begin{array}{ccccc} V_1 & \longleftarrow & f_2^*(V_1) & \xrightarrow{\alpha^2} & \mathbb{S}_2 \\ \downarrow & & \downarrow & & \downarrow \\ M_1 & \xleftarrow{f_2} & \partial S_2 & \longrightarrow & \partial S_2 \end{array}$$

Then the lift of an exit path starting in M_1 at 0 can again be given as $F^{exit}(0,0)v_1 = F^1(0,0)v_1 = v_1$ and $F^{exit}(t,0)v_1 = F^2(t,0)(\alpha^2 v_1)$. and so on.

We now have to address the issue of compatibility of the αs. So let's say we have a small (contractible to a point in M_2) triangle in M_2 given as $\{(x_0, x_1, x_2)|x_i \in M_i\}$. So the path $x_0 \to x_1$ is in M_1 etc. Since the triangle is contractible, we'd have that the transports along the paths $x_0 \to x_1 \to x_2$ and $x_0 \to x_2$ are the same. i.e. if v_0 is a vector over x_0, we have $F(x_2, x_1)F(x_1, x_0)v_0 = F(x_2, x_0)v_0$. Writing this in terms of the exit maps and the transports on each stratum, we get

$$F^2(x_2, x_1)\alpha^2 F^1(x_1, x_0)\alpha^1 v_0 = F^2(x_2, x_0)\alpha^2 v_0$$

Loosely speaking, we again have the condition $\alpha^2 F^1 = F^2\alpha^1$ which as before captures the idea that the transport along the directions common to both the strata is the same. Say we also have a compatibility condition for the exit maps of the sort $\alpha^2\alpha^1 = \alpha^2$.Then we have

$$F^2(x_2, x_1)\alpha^2 F^1(x_1, x_0)\alpha^1 v_0 = F^2(x_2, x_1)F^2(x_1, x_0)\alpha^2\alpha^1 v_0 = F^2(x_2, x_1)F^2(x_1, x_0)\alpha^2 v_0$$

$$= F^2(x_2, x_0)\alpha^2 v_0$$

where the last equation holds because F^2 is an ordinary transport. In words, if we have a triangle $\{(x_0, x_1, x_2)|x_i \in M_i\}$, we can pull back everything in M_0 and

M_1 to the 2-stratum using the attaching maps and calculate the ordinary parallel transport there.

We need to make sense of the equation $\alpha^2\alpha^1 = \alpha^2$. Now α^i is a bundle map over ∂S_i, so the composition $\alpha^2\alpha^1$ is not as such defined.

To fix this we will pull back the bundles to a space where we can compose the maps. Define S_{12} as the pullback

$$\begin{array}{ccc} S_{12} & \longrightarrow & \partial S_2 \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ S_1 & \xrightarrow{f_1} & M_1 \end{array}$$

Here $f_1 : S_1 \to M_1$ is the extension of the attaching map $f_1 : \partial S_1 \to M_0$ by taking it to be the identity on $S_1 - \partial S_1 \to M_1 - M_0$. We can now pull back the bundles $V_1|_{M_1}, \mathbb{S}_1|_{S_1}, \mathbb{S}_2|_{\partial S_2}$ to the space S_{12}. We continue to call them by the same names. If we have a map α^{12} that extends α^2, and the following diagram commutes, as bundles over S_{12}

$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha^1} & \mathbb{S}_1 \\ & {\scriptstyle \alpha^2}\searrow & \downarrow{\scriptstyle \alpha^{12}} \\ & & \mathbb{S}_2 \end{array}$$

Now, $\alpha^{12}\alpha^1 = \alpha^2$ gives

$$F^2(x_1, x_0)\alpha^{12}\alpha^1 v_0 = F^2(x_1, x_0)\alpha^2 v_0$$

$$\alpha^{12}F^1(x_1, x_0)\alpha^1 v_0 = F^2(x_1, x_0)\alpha^2 v_0$$

The image of this last equation over ∂S_2, remembering that off M_0, we have $\alpha^{12} = \alpha^2$ is

$$\alpha^2 F^1(x_1, x_0)\alpha^1 v_0 = F^2(x_1, x_0)\alpha^2 v_0$$

which is the compatibility condition that we wanted.

We can (and should) do this for all pairs of strata, if $i \geq j$ we ask to have the diagram

$$\begin{array}{ccc} V_j & \xrightarrow{\alpha^j} & \mathbb{S}_j \\ & \overset{\alpha^i}{\searrow} & \downarrow{\scriptstyle \alpha^{ij}} \\ & & \mathbb{S}_i \end{array}$$

commute where the bundles are the pullbacks to the space S_{ij} defined as

$$\begin{array}{ccc} S_{ij} & \longrightarrow & \partial S_i \\ \downarrow & & \downarrow{\scriptstyle f_i} \\ S_j & \xrightarrow{f_j} & M_{i-1} \end{array}$$

We will abbreviate this discussion by saying that the α^i satisfy the condition $\alpha^i \alpha^j = \alpha^i$, when $i > j$. This condition gives equations of the form

$$\alpha^i F^j \alpha^j = F^i \alpha^i$$

for all $i > j$, we then get

$$\alpha^i F^j \alpha^j F^k \alpha^k = \alpha^i F^j F^j \alpha^j = \alpha^i F^j \alpha^j = F^i \alpha^i$$

where the first equality is given by the pair j, k and the last by the pair i, j. Then a simple inductive argument gives

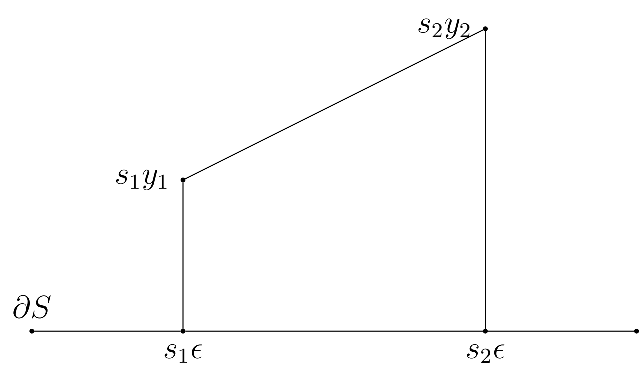
$$F^{i_n} \alpha^{i_n} ... F^{i_1} \alpha^{i_1} = F^{i_n} \alpha^{i_n}$$

i.e. we can compute the transport of a general exit path by lifting the path to the top stratum and compute the pure transport there.

We wish to say any parallel transport can be broken down like this. To do this we saw, we need to be able to extend the transport on the pure strata S_i to the boundary. This is not always true. So we modify a given F on S_i on a collar neighbourhood of the boundary to a new transport $\bar{F}$ by taking it to be the identity in the collar direction. We have

Lemma 2.2.4. $\bar{F}|_{(0,\epsilon]\times\partial S} \simeq F|_{(0,\epsilon]\times\partial S}$

Proof. Choose coordinates $(y, s) \in (0, \epsilon] \times \partial S$. (This notation conflicts slightly with our notation in the rest of this paper.)



We have

$$F(s_2y_2, s_1y_1) = F(s_2y_2, s_2\epsilon)F(s_2\epsilon, s_1\epsilon)F(s_1\epsilon, s_1y_1)$$

and

$$\bar{F}(s_2t_2, s_1y_1) = \bar{F}(s_2y_2, s_2\epsilon)\bar{F}(s_2\epsilon, s_1\epsilon)\bar{F}(s_1\epsilon, s_1y_1)$$

But from the way we defined $\bar{F}$, we know this is

$$\bar{F}(s_2y_2, s_1y_1) = Id.\bar{F}(s_2\epsilon, s_1\epsilon).Id$$

So we may write

$$F(s_2y_2, s_1y_1) = F(s_2y_2, s_2\epsilon)\bar{F}(s_2y_2, s_1y_1)F(s_1\epsilon, s_1y_1)$$

And this last equation gives the isomorphism $\bar{F} \simeq F$. $\square$

Now, we'll give an inductive definition for a transport built out of ordinary transports on the closed pure strata $\bar{S}_i$ and exit maps. Given an ordinary transport $\mathbb{F}^i$ on S_i, an exit transport F^{i-1} on M_{i-1} and a map (natural transformation) $(\alpha^i : f_i^* F^{i-1} \to \mathbb{F}^i)|_{\partial S_i}$, we can construct a transport on M_i as $F_i = \mathbb{F}^i \alpha^i F^{i-1}$, i.e. Given a path γ, such that $\gamma[0, s] \subseteq M_{i-1}$ and$\gamma(s, 1] \subseteq M_i - M_{i-1}$, we define $F^i(t, 0) = \mathbb{F}^i(t, s)\alpha^i(s)F(s, 0)$. In general, we may define $F^i = \mathbb{F}^i\alpha^i..\mathbb{F}^{i_1}\alpha^{i_1}....$

Definition 2.2.5. Denote by $\pi_{\leq 1}^{broken}(M) - Rep$ the category of the transports that arise this way. i.e. The objects of the category are given by

1. An ordinary parallel transport $\mathbb{F}^i \in \pi_{\leq 1}(\bar{S}_i) - Rep$
2. Exit maps $(\alpha^i : f_i^* F^{i-1} \to \mathbb{F}^i)|_{\partial S_i}$
3. Maps α_{ij} of transports over S_{ij}

$$\begin{array}{ccc} F^i & \xrightarrow{\alpha^j} & \mathbb{F}^i \\ & \searrow^{\alpha^i} & \downarrow{\scriptstyle \alpha^{ij}} \\ & & \mathbb{F}^j \end{array}$$

4. Such that the α_i are compatible with the transports , $\alpha^i F^i = \mathbb{F}^{i+1}\alpha^i$

5. And the α^i are compatible with each other '$\alpha^i\alpha^j = \alpha^i$'

Maps between transports are defined as natural transformations Φ^i between $\mathbb{F}^i$ and $\mathbb{G}^i$ on the closed pure strata that commute with the exit maps.

Definition 2.2.6. Denote by $\pi_{\leq 1}^{exit}(M) - Rep$ the category of transports of exit paths. i.e. we view the $\pi_{\leq 1}^{exit}(M)$ as a subcategory of the path groupoid of M that consists of exit paths.

Theorem 2.2.7. *There is an equivalence of categories* $\Theta : \pi_{\leq 1}^{broken}(M) - Rep \to \pi_{\leq 1}^{exit}(M) - Rep$

Proof. The functor Θ is defined on objects as

$$\Theta(\{\mathbb{F}^i, \alpha^i\}) = F^{exit} = \mathbb{F}^{i_n}\alpha^{i_n}..\mathbb{F}^{i_1}\alpha^{i_1}...$$

Lemma 2.2.3 says that this functor is essentially surjective. To see that this is fully faithful, we need to say that if F, G are in the image of the functor Θ, i.e. they are made from functors on the pure strata and exit maps, then any map between them is given by a natural transformation that also can be broken down. Now since a map Φ in $\pi_{\leq 1}^{exit}(M) - Rep$ is a natural transformation it satisfies $\Phi_y F(x \to y) = G(x \to y)\Phi_x$. If x is in ∂S, we can simply define $G(x \to y)^{(-1)}\Phi_y F(x \to y) = \Phi_x$ and this gives a broken down Φ. $\square$

We now return to vector bundles and define a connection on a constructible vector bundle as follows:

Definition 2.2.8. A flat connection on a constructible vector bundle $\{\mathbb{S}_i, V_i, \tilde{f}_i\}$ on a stratified space $\{S_i, M_i, f_i\}$ is given by

1. An ordinary flat connection $\nabla^i = d - A^i$ on the bundle $\mathbb{S}_i|_{S_i}$.

2. Exit maps α^i:

$$\begin{array}{ccccc} V_{i-1} & \longleftarrow & f_i^*(V_{i-1}) & \xrightarrow{\alpha_i} & \mathbb{S}_i \\ \downarrow & & \downarrow & & \downarrow \\ M_{i-1} & \xleftarrow{f_i} & \partial S_i & \longrightarrow & \partial S_i \end{array}$$

3. Maps α_{ij} of bundles over S_{ij}

$$\begin{array}{ccc} V_i & \xrightarrow{\alpha^j} & \mathbb{S}_i \\ & \searrow^{\alpha^i} & \downarrow{\scriptstyle \alpha_{ij}} \\ & & \mathbb{S}_j \end{array}$$

4. Such that the α^i are compatible with the connection, $\alpha^i \nabla^i = \nabla^{i+1} \alpha^i$

5. And the α^i are compatible with each other '$\alpha^i \alpha^j = \alpha^i$'

We can define a category of constructible vector bundles with connection, by defining maps between bundles to be maps of ordinary vector bundles over each stratum, that commute with the connection and the exit maps. Call this category $Flat^{const}(M)$.

Theorem 2.2.9. *There is an equivalence of categories*

$$\Sigma : \pi_{\leq 1}^{broken}(M) - Rep \to Flat^{const}(M)$$

Proof. Given a transport $\mathbb{F}_i$ on each closed pure stratum $\bar{S}_i$, we can build a vector bundle $\mathbb{S}_i$ as in the ordinary case in the last section. We take the attaching maps $\tilde{f}_i$ to be zero. The maps α^i of transports give bundle maps by the way we topologise the bundles. And the conditions on the transports translate to the conditions on the vector bundles.

We define the connection on the pure strata by differentiating the transport as before. We saw that the compatibility of the αs with the transport is equivalent to the compatibility of the corresponding connection. This gives an equivalence at the level of objects.

The correspondence of morphisms is entirely similar to that of the exit maps. A natural transformation of transports gives a map of vector bundles that commutes with the connection.etc.

□

Remark 2.2.10. We saw that the map $f_{n+1} : \partial S_{n+1} \to M_n$ is a fiber bundle over each stratum in M_n. So given a path γ in M_n, we may lift it to a path $\tilde{\gamma}$ in $\bar{S}_{n+1}$. Exit paths, we may lift uniquely. The equation $F^1\alpha^1 v_0 = \alpha^1 F^0 v_0$ that expresses compatibility between the transports in the 0 and 1 strata, should be read as $F^1_{\tilde{\gamma}}\alpha^1 v_0 = \alpha^1 F^0_{\gamma} v_0$.

Remark 2.2.11. A constructible vector bundle with connection has attaching maps $\tilde{f}_i$ which have to do with the topology of the bundle, and connection maps α_i which have to do with the parallel transport. A representation of exit paths, or a constructible sheaf determines and is determined by the vector bundles on the strata and the exit connection maps α_i. The attaching maps are a choice for the vector bundle that is extraneous data. We generally take the $\tilde{f}_i$ to be 0.

Remark 2.2.12. All of this has a strict analogue in sheaf theory. Let $j_i : M_i - M_{i-1} = S_i \to M_i$ be the inclusion of the open i-stratum into the i-th filtered part of the space. If F is a sheaf on M_i, the unit of the adjunction $F \to j^* j_* F$ contains the information of the exit connection. We may restrict to M_{i-1} by the inclusion and continue down to M_0 inductively.

2.2.4 Sheaves

Given the data of a constructible bundle and parallel transport F, we can define a constructible sheaf as before: For U, an open set in M, define a sheaf $\mathcal{F}$ as $\mathcal{F}(U) = \{s \in Maps(U, V) | s(y) = F(f)(s(x))\}$ where f is any exit path in U. Again, this is well defined, because of homotopy invariance.

This pre-sheaf is again a sheaf because we can patch parallel sections uniquely. If two sections agree on an open set, they agree when restricted to each stratum and so patch together because of the locally constant case. So we have patching.

This sheaf is constructible because it is locally constant when restricted to each stratum, by construction.

Example 2.2.13. Look at an annulus M closed on the inside with the stratification given by S_1 is the inner circle. And S_2 is the open annulus.

$\pi_{\leq 1}(M)$ is equivalent to a category with two objects, given by choosing a point on each stratum. and maps $\{x_0 \xrightarrow{\alpha} x_0 \xrightarrow{\beta} x_1 \xrightarrow{\gamma} x_1\}$. Here α, γ represent the loops based in $x_0 \in S_0$ etc. and the map β is an exit path joining x_0, x_1. We have that α and γ generate $\mathbb{Z}$ and they satisfy the relation $\alpha\beta = \beta\gamma$.

A representation then of $\pi_1^{exit}(M)$ is given by a vector space V, W for the circle and the open annulus. Maps $V \to V$, $W \to W$ which give representations of Z, and a map $V \to W$ which intertwines them.

The constructible sheaf corresponding to a particular representation is a locally constant sheaf on the inner circle given by V, α and on the open annulus given by W, γ. If the map β is 0, we can think of this constructible sheaf as a different local system on each stratum. If the map is the identity, the we should think of this constructible sheaf as a locally constant sheaf on the whole space which restricts to a locally constant sheaf on each stratum.

2.2.5 Constructible Sheaves on a Simplicial Complex

Let $M = (S, \Sigma)$ be a simplicial complex. i.e. S is a finite set, and Σ is a collection of subsets of S. We may think of this as a stratified space in our sense, where the strata are given by the simplices $\sigma \in \Sigma$.

A flat vector bundle on a contractible space is isomorphic to a product bundle with a trivial connection. So a constructible vector bundle is determined by specifying a vector space V_σ for each simplex σ and maps $\alpha^i : V_{\sigma_k} \to V_{\sigma_i}$ for each pair

$\sigma_k \subseteq \sigma_i$ (i.e. $i > k$) such that $\alpha^i = \alpha^i \alpha^j$ for $i > j$ as before.

A constructible sheaf restricts to a locally constant sheaf on each stratum which in our case is a simplex. Since a simplex is contractible this is just a constant sheaf, i.e. a vector space. It is perhaps not hard to believe that the remaining information is given by maps of vector bundles where a simplex lies in another. Our equivalence then reduces to a folk theorem (probably) due to MacPherson.

We define the star of a simplex σ as $Star(\sigma) = \{\cup_{\sigma \subseteq \tau} \tau - \cup_{\sigma \not\subseteq \tau} \tau\}$.

To a sheaf $\mathcal{F}$ we can associate a constructible vector bundle as $V(\sigma) = \mathcal{F}(star(\sigma))$. It is not too hard to see that this extends to an equivalence of categories. The interesting point here is that the exit maps for vector bundles correspond to the restriction maps for the sheaves.

Finally, an exit path restricted to each simplex is contractible, so the homotopy class of an exit path is completely determined by the sequence of simplicies it visits. And so a representation of exit paths is given again by the choice of a vector space for each simplex and a map of vector spaces given by incidence of simpices.

Chapter 3

The derived case

In this chapter we will attempt to generalise the results of the previous section. Remember we had that a flat connection gave rise to a parallel transport functor that was homotopy invariant. If $x \rightrightarrows y$ were two homotopic paths, we had $V_x \to V_y$ was the same map. Now instead we may ask that the transport be homotopy coherent, that is that the maps $V_x \rightrightarrows V_y$ be homotopic and that higher homotopies of maps $x \rightrightarrows y$ give higher homotopies $V_x \rightrightarrows V_y$ and so on. All of this can be neatly captured in the form of an ∞ functor $\pi_\infty(M) \to C(k)$. where $\pi_\infty(M) = Sing^\bullet(M)$ the singular complex replaces the category $\pi_{\leq 1}(M)$. Correspondingly we look at graded vector bundles with a connection with higher components, and a higher 'flatness'.

We will begin in section 1 by recalling work by Block-Smith-Igusa on higher parallel transport. This construction uses the tool of iterated integrals introduced by Chen. In section 2 we will give a describe a way to recover a flat connection from a given transport functor.

3.1 Higher Parallel Transport -Iterated Integrals

In this section we recall the notions of a (flat) graded super connection and the parallel transport differential forms. Almost all of this section is lifted verbatim from Igusa [Ig09].

Let V be a $\mathbb{Z}$-graded vector bundle over M. Then, a graded connection on V is a sequence of operators ∇_k, such that $(-1)^k\nabla_k$ is an ordinary connection on V^k.

Definition 3.1.1. A superconnection on E is a linear map $D : \Gamma V \to \bigoplus \Omega^p(M, V)$ of total degree 1 which satisfies the graded Leibnitz rule. $[D, f] = df$ for $f \in \Omega^0(M)$. i.e. $D(\Gamma V^k) \subseteq \bigoplus \Omega^p(M, V^{k+1-p})$.

Superconnections are locally given as $D = d - A_0 - A_1....$, where $A_p \in \Omega^p(M, End^{1-p}(E))$.

The Path Space: Let PM denote the space of paths $\gamma : [0, 1] \to M$ given a smooth structure by saying $\phi : U \to PM$ is a plot if the adjoint map $\tilde{(}\phi) : U \times I \to M$ is a plot of M. We may then talk of smooth vector bundles and the tangent bundle of PM.

For every $t \in I$ we have the evaluation map $ev_t : PM \to M$ that sends γ to $\gamma(t)$.

Let W_t be the pull back of E to PM along ev_t. i.e. $(W_t)_\gamma = E_{\gamma(t)}$. In this notation we can see that the ordinary parallel transport ϕ_s^t can be thought of as an element in $\Omega^0(PM, Hom^0(W_t, W_s)$. i.e it is a choice, for each γ an element in $Hom^0(W_t, W_s) = Hom^0(E_{\gamma(t)}, E_{\gamma(s)})$.

The higher parallel transports are given as a sequence of forms on PM which integrate over families of paths.

Definition 3.1.2. The *contraction* $/t : \Omega^{p+1}(M, End^q(V)) \to \Omega^p(PM, End^q(W_t))$ is the linear mapping which sends a $p+1$-form α on M with coefficients in $End^q(V)$ to the smooth p-form on PM whose value at γ is the alternating map $(\alpha/t)_\gamma$: $(T_\gamma PM)^p \to End^q(W_t)_\gamma = End^q(V_{\gamma(t)})$ given by

$$(\alpha/t)_\gamma(\eta_1, \eta_2, \cdots, \eta_p) = \alpha(\eta_1(t), \cdots, \eta_p(t), \gamma'(t)).$$

Definition 3.1.3. The parallel transport of the superconnection D is defined to be the unique family of forms $\Psi_p(t,s)$ for all $1 \geq t \geq s \geq 0$ and $p \geq 0$

$$\Psi_p(t,s) \in \Omega^p(PM, Hom^{-p}(W_s, W_t))$$

satisfying the following at each $\gamma \in PM$.

1. $\Psi_0(s,s)_\gamma$ is the identity map in

$$\Omega^0_\gamma(PM, Hom^0(W_s, W_s)) = Hom^0(V_{\gamma(s)}, V_{\gamma(s)})$$

   and $\Psi_p(s,s) = 0$ for $p > 0$.

2. For all $p \geq 0$ we have

$$\frac{\partial}{\partial t}\Psi_p(t,s)_\gamma = \sum_{i=0}^{p}(A_{i+1}/t)_\gamma \Psi_{p-i}(t,s)_\gamma$$

where $D = d - A_0 - A_1 - \cdots - A_m$ is the decomposition of D in a coordinate chart U for M in a neighborhood of $\gamma(t)$ and differentiation with respect to t is given by the chosen product structure on $V|U$.

Given the A_p, we can solve for the Ψ by induction on p using integrating factors. To do this, first note that the differential equation above has the form

$$\frac{\partial}{\partial t}\Psi_p(t,s) = (A_1(t))\Psi_p(t,s) + f(t,s)$$

where $f(t,s)$ is given in terms of $\Psi_q(t,s)$ for $q < p$:

$$f(t,s) = \sum_{q=0}^{p-1}(A_{p-q+1}/t)\Psi_q(t,s).$$

We multiply by the integrating factor $\Phi(s,t) = \Phi(t,s)^{-1}$ to get

$$\frac{\partial}{\partial t}\left(\Phi(s,t)\Psi_p(t,s)\right) = \Phi(s,t)\frac{\partial}{\partial t}\Psi_p(t,s) - \Phi(s,t)(A_1(t))\Psi_p(t,s) = \Phi(s,t)f(t,s).$$

So,

$$\Phi(s,t)\Psi_p(t,s) = \int_s^t du\,\Phi(s,u)f(u,s).$$

We may then use the iterated integrals to write, for $p \geq 0$, $\Psi_p(t,s)$ is equal to the sum over all $n \geq 0$ and all sequences of integers $k_1, \cdots, k_n \geq 1$ so that $\sum k_i = p+n$ of the following iterated integral.

$$\iint_{t\geq u_1\geq\cdots\geq u_n\geq s} du_1\cdots du_n\,(A_{k_1}/u_1)(A_{k_2}/u_2)\cdots(A_{k_n}/u_n)$$

Given a simplex $\sigma : \Delta^n \to M$, we can think of it as an $n-1$ family of paths $I^{n-1} \to PM$. We can then integrate the forms Ψ_{n-1} over the family to get $\int_\Delta \Psi_{n-1} \in Hom(V_t, V_s[n-1])$. Block-Smith in [BS09] prove that this construction gives an equivalence of dg-categories of graded vector bundles with flat super connections and parallel transport functors. We will recall their definitions in the next section and provide an inverse construction.

3.2 Higher Parallel Transport - Recovering a Super connection

Here we describe a way to get back a superconnection from the information of a parallel transport functor. First we make precise our notion of a higher parallel transport and describe it as a higher functor. We'll then show that we can recover a bundle with a flat super connection on it.

3.2.1 Parallel Transport Functor

The ordinary parallel transport along a curve γ is a linear map $F(t,s) : V(s) \to V(t)$ where $V(t)$ is the fiber of the bundle V over the point $\gamma(t)$. We ask that this map satisfy $F(r,t)F(t,s) = F(r,s)$. This is the information of a functor $F : \pi_{<1}(M) \to Vect$, where $F(\gamma(t)) = V(t)$, and the condition $F(r,t)F(t,s) = F(r,s)$ is read as the functor respecting composition of maps.

If an ordinary flat connection gives rise to the transport, homotopic paths between two points give the same transport map, in the derived case we'd want that if we have a homotopy of paths $tu \rightrightarrows su$, that there be a map between the transport of a vector along the two paths depending on the homotopy. This map is represented by an edge in $V(tu)$ which is an element in $V(tu)[1]$. i.e we'd want $F_1(tu, su) : V(su) \to V(tu)[1]$. We'd want a homotopy of homotopies to give a map of degree 2 and so on.

All of this can be captured neatly in a functor $F : \pi_\infty(M) \to C(k)$, where $\pi_\infty(M)$ is the simplicial set $Sing^\bullet(M)$. This is described precisely in [Block-Smith]. We borrow their notion:

Let $\pi_\infty(M) = Sing^\bullet(M)$ be the singular complex of M. (Denote by $\pi_n(M) = \{\sigma_n : \Delta^n \to M\}$)

Definition 3.2.1. Let $F : \pi_0(M) \to C(k)$ be a choice of a complex F_x for each point $x \in M$. Let

$$\{F_n : \pi_n(M) \to C(k)^{n-1} | F_n(\sigma_n) \in Hom^{n-1}(F_{\sigma(n)}, F_{\sigma(0)})\}$$

Define operations on these maps

$$(dF_n)(\sigma_n) = d.F_n(\sigma_n) = dF_{\sigma(0)}F_n(\sigma_n) - F_n(\sigma_n)dF_{\sigma(n)}$$

$$\hat{\delta} F_n \sigma_{n+1} = \sum_{l=1}^{n-1} (-1)^l F_n(\partial_l \sigma)$$

and for $F_m \in \{\pi_m(M) \to C(k)^{m-1}\}$,

$$F \cup F(\sigma_k) := \sum_{m+n=k} F_m(\sigma_{012..m}) F_n(\sigma_{mm+1m+2..k})$$

A higher parallel transport (or local system) is given by the choice F' and a collection F_i, such that $F = F_0 + F_1 + F_2...$ that satisfies $F \cup F + \hat{\delta} F + d.F = 0$

For $n = 2, 3$, this is

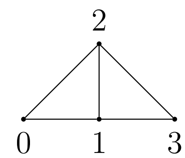
$$dF_1(\sigma_{012}) = F_0(\sigma_{01})F_0(\sigma_{12}) - F_0(\sigma_{02})$$

$$dF_2(\sigma_{0123}) = F_0(\sigma_{01})F_1(\sigma_{123}) - F_1(\sigma_{012})F_0(\sigma_{23}) - F_1(\sigma_{023} + F_1(\sigma_{013})$$

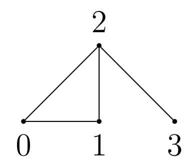
These equations express the idea that the higher $F_i(\sigma_i)$ are a homotopy between the faces of the simplex σ_i. If the higher F_i are 0, for $i \geq 1$, we see the equation for $n = 1$ says that F_0 is an ordinary local system, i.e equal on homotopic paths 01-12 and 02.

We also need the functor to respect composition of the higher dimensional simplices. i.e. we want analogues of the rule $F(r,t)F(t,s) = F(r,s)$ for simplicies of dimension ≥ 1. These are given by the so called pasting diagrams. These follow as a consequence of the coherence of F.

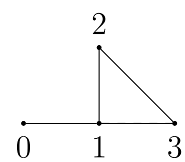
For instance, if (012) are on a line, the rule the homotopy between 01.12 and 02 is a singular (rank ≤ 1) 2-simplex and we ask that $F_1(012)$ is 0. Then we get $F_0(\sigma_{01})F_0(\sigma_{12}) = F_0(\sigma_{02})$. In dimension 2, a singular 3-simplex 0123 gives $F_1(\sigma_{013}) = F_0(\sigma_{01})F_1(\sigma_{123}) + F_1(\sigma_{012})F_0(\sigma_{23})$. In n dimensions the rule is $F_n(\sigma_{01.n}) = F_0(\sigma_{01})F_{n-1}(\sigma_{12..n}) + F_1(\sigma_{012})F_{n-2}(\sigma_{23..n}) .... + F_{n-1}(\sigma_{2..n-1})F_0(\sigma_{n-1n})$. These will be important to us in the next section. We can think of equation in dimension 2 as giving a "vertical" composition



thought of as a homotopy between 02.23 and 01.13



thought of as a homotopy between 02.23 and 01.12.23



thought of as a homotopy between 01.12.23 and 01.13. The pictures in the higher dimensions are harder to draw, but perhaps not too hard to imagine.

Continuity and smoothness: The functor $\{F_n\}$ defined algbraically has no immideate notion of continuity or smoothness. We can understand the pasting equations as expressing a notion of continuity.

For instance, the equation $F_0(t+dt,s) = F_0(t+dt,t)F_0(t,s)$ along with $F_0(t+dt,t) \to Id$ as $dt \to 0$ expresses the idea that $F_0(t+dt,s)$ is 'close' to $F_0(t,s)$. We see the higher dimensional versions below. We ask that $\{F_n\}$ are smooth functions of the co-ordinates on a smooth simplex, so that we may differentiate them as below.

The pasting laws follow as a consequence of the coherence laws and are weaker than them. The coherence laws express some thing Stokes theoremy. The pasting diagrams only represent continuity and smoothness.

They also allow us to define the F_i for piecewise smooth simplices. A peicewise smooth simplex is one that has a traingulation such that each subsimplex (of any dimension) is smooth.

3.2.2 Recovering the connection

Given a functor (or an ∞-local system $F : \pi_\infty(M) \to C(k)$. look at the set $V = \{f_x\}_{f_x \in F_x, x \in M}$. This will make the total space of a bundle, with the natural projection map p.

Let U, x_0 be an open ball in M. For each point $x \in U$, let γ_x be the line segment (shortest path, for a choice of Riemannian metric) joining x to x_0. Then we can define a map $p^{-1}(U) \to U \times F_0(x_0)$ as $f_x \to (x, F_0^\gamma(x, x_0)(f_{x_0})$. This gives a bijection since F_0 is invertible.

We may use this bijection to topologise the bundle V, and give it a smooth structure. And we can think of F as a parallel transport functor on it. i.e. for each point x a complex of vector spaces V_x, for each path $x \to y$, a chain map $F_x \to F_y$. for each homotopy of paths, a homotopy of chain maps and so on.

The p-th component of a super connection takes as input p tangent vectors at a point x and gives a map $End^{p-1}(V_x)$ whereas parallel transport takes a p simplex and returns a degree $p-1$ map in $Hom^{p-1}(V_{x_p}, V_{x_0})$, the first and last vertices of the simplex. We will define the connection in each dimension as the infinitesimal increment by parallel transport.

We may think of the higher transports as acting on families of paths and, we'll define transport forms in the sense of the previous section as variations of F. These satisfy the differential equations with respect to the components A_i of the connection as we defined. The homotopy coherence of F will impose conditions on the connection, which will say that the connection is flat.

$\mathbf{n = -1}$

The differential of V_x at each point will make the -1 component of the connection. This makes a continuous section of $End^{-1}(V)$ because of the way we topologised the bundle. In local coordinates, we denote this $A_0(t)$.

$\mathbf{n = 0}$

Let $F_0(t,s) : V(s) \to V(t)$ be the degree 0 component of the transport. Then, define

$$A_1(t) = \frac{F_0(t+dt,t) - I}{dt}$$

We have a differential equation for A_1:

$$\frac{\partial}{\partial t}F_0(t,s) = \frac{F_0(t+dt,t)F_0(t,s) - F_0(t,s)}{dt} = \frac{(F_0(t+dt,t) - I)F_0(t,s)}{dt} =$$

$$A_1(t)F_0(t,s)$$

in the limit.

This is the differential equation that defined parallel transport in terms of A_1 in the previous section.

$\mathbf{n = 1}$

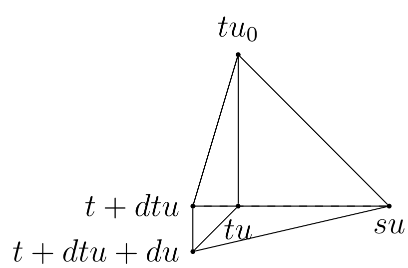
Choose local coordinates t, u and let (tu, su, tu^0) represent a 2-simplex.

Let $F_1(tu, su) : V(su) \to V(tu)[1]$ be the degree 1 component of the transport functor acted on the 2 simplex. Here we can think of u as a parameter for the family of paths with running variable t.

Again, we define

$$A_2(t) = \frac{F_1(t + dtu + du, tu)}{dtdu}$$

as the infinitesimal increment by F_1.



We have a differential equation similar to the n=0 case:

$$\frac{\partial}{\partial t} F_1(tu, su) = \frac{F_1(t + dtu, su) - F_1(tu, su)}{dt}$$

$$= \frac{F_0(t + dtu, tu)F_1(tu, su) + F_1(t + dtu, tu^0)F_0(tu^0, su) - F_1(tu, su)}{dt}$$

$$= \frac{(F_0(t+dtu,tu)-I)F_1(tu,su)+F_1(t+dtu,tu^0)F_0(tu^0,su)}{dt}$$

$$= A_1^t(tu)F_1(tu,su) + \frac{F_1(t+dtu,tu^0)}{dt}F_0(tu^0,su)$$

where in the first step we used the pasting law for F_1. Differentiating with respect to u now gives:

$$\frac{\partial}{\partial u}\frac{\partial}{\partial t}F_1(tu,su) = \frac{\partial}{\partial u}A_1^t(tu).F_1(tu,su) + A_1^t(tu).\frac{\partial}{\partial u}F_1(tu,su)$$
$$+\frac{F_1(t+dtu+du,tu)}{dtdu}F_0(tu,tu^0)F_0(tu^0,su)$$

As $u^0 \to u$, $F_1(tu,su)$ represents the trivial homotopy between tu and su, and so is 0. $F_0(tu,tu^0) \to I$ and $F_0(tu^0,su) \to F_0(tu,su)$.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial u}F_1 = A_1^t(tu).\frac{\partial}{\partial u}F_1(tu,su) + A_2(tu)F_0(tu,su)$$

Setting $F_0 = \psi_0$ and $\frac{\partial}{\partial u}F_1 = \psi_1$ we get

$$\frac{\partial}{\partial t}\psi_1 = A_1(\frac{\partial}{\partial t})\psi_1 + A_2(\frac{\partial}{\partial t},\frac{\partial}{\partial u})\psi_0$$

$\mathbf{n = 2}$

Define

$$A_3(t) = \frac{F_2(t+dtu_1+du_1u_2+du_2,tu_1u_2)}{dtdu_1du_2}$$

In the local coordinates let (tu_1u_2, su_1u_2) represent a 3-simplex and let

$(F_2(tu_1u_2, su_1u_2) : V(su_1u_2) \to V(tu_1u_2)[2]$ be the degree 2 component of the parallel transport map.

Differentiating with respect to t, we get:

$$\frac{\partial}{\partial t}F_2(tu_1u_2, su_1u_2) = \frac{F_2(t+dtu_1u_2, su_1u_2) - F_2(tu_1u_2, su_1u_2)}{dt}$$

By the pasting law for F_2 we get,

$$= F_2(t+dtu_1u_2, tu_1^0, u_2)F_0(tu_1^0u_2, su_1u_2) + F_1(t+dtu_1u_2, tu_1u_2^0)F_1(tu_1u_2^0, su_1, u_2)$$

$$+F_0(t+dtu_1u_2, tu_1, u_2)F_2(tu_1u_2, su_1, u_2) - F_2(tu_1u_2, su_1u_2)$$

We drop some cumbersome notation when it is perhaps clear

$$= A_1^t(tu_1u_2)F_2(tu_1u_2, su_1, u_2) + \frac{F_1(t+dtu_1u_2, -)}{dt}F_1(tu_1u_2^0, -) +$$

$$\frac{F_2(t+dt-, -)}{dt}F_0(tu_1^0u_2, -)$$

Differentiating with respect to u_1 now and letting $u_1^0 \to u_1$ we get:

$$= A_1^t(tu_1u_2)\frac{\partial}{\partial u_1}F_2(tu_1u_2, su_1, u_2) + \frac{F_1(t+dtu_1+du_1u_2, -)}{dtdu_1}F_1(tu_1u_2^0, -)$$

$$+\frac{F_2(t+dtu_1+du_1-, -)}{dtdu_1}F_0(tu_1u_2, su_1u_2)$$

Differentiating with respect to u_2 now and letting $u_2^0 \to u_2$ we get:

$$= A_1^t(tu_1u_2)\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_2(tu_1u_2, su_1, u_2) + \frac{F_1(t + dtu_1 + du_1u_2, -)}{dtdu_1}\frac{\partial}{\partial u_2}F_1(tu_1u_2^0, -)$$

$$+\frac{F_2(t + dtu_1 + du_1u_2 + du_2, tu_1u_2)}{dtdu_1du_2}F_0(tu_1u_2, su_1u_2)$$

Set

$$\psi_2 := \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_2(tu_1u_2, su_1, u_2)$$

and remember we have

$$A_3(t) = \frac{F_2(t + dtu_1 + du_1u_2 + du_2, tu_1u_2)}{dtdu_1du_2}.$$

Then the differential equation becomes:

$$\frac{\partial}{\partial t}\psi_2 = A_1(\frac{\partial}{\partial t})\psi_2 + A_2(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_1})\psi_1 + A_3(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2})\psi_0$$

$\mathbf{n \geq 2}$

The case for higher n is completely analogous, though harder to write down. As before we define:

$$A_n(t) = \frac{F_n(t + dtu_\bullet + du_\bullet, tu_\bullet)}{dtdu_\bullet}$$

where $du_\bullet = du_1du_2..du_{n-1}$. We also set

$$\psi_n := \frac{\partial^n}{\partial u_1u_2..u_n}F_n(tu_\bullet, su_\bullet)$$

Differentiating w.r.t. t:

$$\frac{\partial}{\partial t}F_n(tu_{1..n}, su_{1...n}) = \frac{F_n(t+dtu_{1..n}, su_{1..n}) - F_2(tu_{1..n}, su_{1..n})}{dt}$$

By the pasting law for F_n we get,

$$= \Sigma_{i+j=n} F_i(t+dtu_{1..n}, tu_{1..i}u^0u_{i+1..n})F_j(tu_{1..i}u^0u_{i+1..n}, su_{1..n}) - F_n(tu_{1..n}, su_{1..n})$$

Differentiating this k times, we get

$$\Sigma_{i+j=n,k<i}\frac{F_i(t+dtu_1+du_1..u_k+du_ku_{k+1..n}, tu_{1..i}u^0u_{i+1..n})F_j(tu_{1..i-1}u_i^0u_{i+1..n}, su_{1..n})}{dtdu_{1..k}}$$

$$+\Sigma_{i+j=n,k\geq i}F_i(t+dtu_1+du_1..u_i+du_i, tu_{1..i-1}u_i^0u_{i+1..n})\frac{\partial}{\partial u_{i..k}}F_j(tu_{1..i}u^0u_{i+1..n}, su_{1..n})$$

In the end we get, (writing $\psi_p := \frac{\partial^p}{\partial u_1u_2..u_p}F_p(tu_\bullet, su_\bullet)$)

$$\frac{\partial}{\partial t}\psi_n = A_1(\frac{\partial}{\partial t})\psi_n + A_2(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_1})\psi_{n-1} + ...A_n(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_1}, ...\frac{\partial}{\partial u_n})\psi_0$$

Flatness

Homotopy coherence of the higher transport functors gives conditions on the A_ns just as homotopy invariance in the ordinary case gave flatness of the ordinary connection. This is worked out in detail in [Ig09], so we will be very brief.

$\mathbf{n=0}$: The fact that F_0 is a chain map gives $A_0(t)F_0(t,s) = F_0(t,s)A_0(s)$. Differentiating this with respect to t gives $A_0' = A_0A_1 + A_1A_0$.

$\mathbf{n=1}$: Let (tu, tu^0, su) be a 2-simplex as before. The coherence condition for F_0, F_1 gives $A_0(tu)F_1(tu,su) - F_1(tu,su)A_0(su) = F_0(tu,su) + F_0(tu,tu^0)F_0(tu^0,su)$ Here $d.(F_1(tu,su) = A_0(tu)F_1(tu,su) - F_1(tu,su)A_0(su)$ in $Hom(V(tu), V(su))[1]$.

Differentiating this gives the relation $A_0A_2 + A_2A_0 = A_1' - A_1A_1$.

n ≥ 2 : The higher case is similar. We have $d(F_n(tu_\bullet, su_\bullet)) = A_0(tu_\bullet)F_1(tu_\bullet, su_\bullet) - F_n(tu_\bullet, su_\bullet)A_0(su_\bullet)$ in $Hom(V(tu_\bullet), V(su_\bullet)[n]$. The right hand side of the equation is gotten from the definition of the local system $F \cup F + \hat{\delta}F$. Differentiating this gives $A_0A_n + A_nA_0 = A_{n-1}' - A_1A_{n-1} - A_2A_{n-2} - ...A_{n-1}A_1$. These equations for each n can be combined into one single equation $(d - A_0 - A_1 - A_2....)^2 = 0$ which we call flatness for the super connection.

3.3 An equivalence of categories

In section 3.1 we saw, given a flat super connection on a graded vector bundle, how to construct a parallel transport functor. We just described a notion of constructing a vector bundle and a connection on it starting with a parallel transport.

These constructions are inverse to each other.

On the one hand, starting with the transport $\{F_n\}$, we defined the connection to be $A_n = \frac{F(t+dtu_\bullet+du_\bullet,tu_\bullet)}{dtdu_\bullet}$. The transport forms are defined as the unique forms that satisfy the differential equations wrt. the A_ns. We saw that $\frac{\partial F_n}{\partial u_\bullet}$ satisfy the equations, which gives us the inverse in one direction.

On the other hand, from [Ig09], Corollary 2.12, we have the estimate $\psi_n(t, s) = (A_{n+1}/x)(t - s) + o(t - s)$, where $x \in (t, s)$. This gives

$$[\int_{u_1u_2..}^{u_1+du_1,...u_n+du_n} \psi_n(t + dt, t) - A_{n+1}(tu_1u_2..)]/dtdu_1...du_n \to 0$$

So starting with a connection A_n we can construct the parallel transport $\int \psi$ and recover the connection as the infinitesimal version of the transport.

This extends to an equivalence of categories, first we recall the definitions. Block-Smith [BS09] define a category of higher parallel transport functors, ∞-local systems as follows:

Definition 3.3.1. Denote by $(\pi_\infty(M), C(k))$ or $\pi_\infty(M) - Rep$, the category of higher transport fuctors or ∞-local systems. The objects are the transport functors we defined before. $Hom(F, G)$ between two transports F, G is a complex whose degree k maps are given as

$$\sum_{i+j=k} \{\Phi : \pi_i(M) \to C^j | \Phi(\sigma) \in C^j(F(\sigma(i)), G(\sigma(0)))\}$$

The differential is given as

$$D := \delta\Phi + d\Phi + G \cup \Phi + (-1)^\Phi \Phi \cup F$$

where as before

$$d\Phi_n(\sigma_n) = dF_{\sigma(0)}\Phi_n(\sigma_n) - \Phi_n(\sigma_n)dG_{\sigma(n)}$$

$$\hat{\delta}\phi_n(\sigma_{n+1}) = \sum_{l=1}^{n-1}(-1)^l \phi_n(\partial_l \sigma)$$

$$\Phi \cup F(\sigma_k) := \Sigma_{m+n=k}\Phi_m(\sigma_{012..m})F_n(\sigma_{mm+1m+2..k})$$

$$G \cup \Phi(\sigma_k) := \Sigma_{m+n=k}G_m(\sigma_{012..m})\Phi_n(\sigma_{mm+1m+2..k})$$

We have that $D^2 = 0$, so that $\pi_\infty(M) - Rep$ is a dg-category.

A little unpacking: A degree 0 morphism between F and G is like a natural transformation of functors. For a point x, it gives a degree 0 map $F_x \to G_x$. For a 1-simplex σ joining x and y, an 'ordinary' natural transformation would give a commuting square

$$\begin{array}{ccc} F_x & \xrightarrow{F(\sigma)} & F_y \\ \downarrow{\scriptstyle \Phi_x} & & \downarrow{\scriptstyle \Phi_y} \\ G_x & \xrightarrow{G(\sigma)} & G_y \end{array}$$

We instead ask that the two compositions be homotopic (not equal), and ask that Φ specify this homotopy, i.e. $\Phi(\sigma) : F_x \to G_y[1]$. For a 2-simplex $\sigma(xyz)$ we would get $\Phi(\sigma) : F_x \to G_z[2]$ etc. We may think of a degree k morphism as a 'natural transformation' of this kind between F and G shifted by k.

The cone: Given a closed morphism Φ of degree k, we may construct a new ∞-local system as $\begin{pmatrix} F & 0 \\ \Phi & G[k] \end{pmatrix}$. That is, a point x is taken to $F_x \oplus G_x[k]$, a 1-simplex $\sigma(xy)$ to the linear map $\begin{pmatrix} F(\sigma) & 0 \\ \Phi(\sigma) & G[k](\sigma) \end{pmatrix}$ etc. The fact that Φ is closed implies that this is in fact an ∞-local system. This allows us to confuse closed morphisms and objects. This will be useful later.

We ask that the morphisms obey pasting laws just as the objects do. For instance the diagram

$$\begin{array}{ccccc} F_x & \xrightarrow{F(\sigma_1)} & F_y & \xrightarrow{F(\sigma_2)} & F_z \\ \downarrow{\scriptstyle \Phi_x} & & \downarrow{\scriptstyle \Phi_y} & & \downarrow{\scriptstyle \Phi_z} \\ G_x & \xrightarrow{G(\sigma_1)} & G_y & \xrightarrow{G(\sigma_2)} & G_z \end{array}$$

leads to the pasting law

$$\Phi_1(z,x) = \Phi_1(z,y)F_0(y,x) + G_0(z,y)\Phi_1(y,x)$$

etc. Now these again express a notion of continuity and smoothness for the morphisms. These laws can be expressed by simply saying that the cone object $\begin{pmatrix} F(\sigma) & 0 \\ \Phi(\sigma) & G[k](\sigma) \end{pmatrix}$ obeys the pasting laws. Remember that the pasting laws are weaker than the coherence laws. When the cone object obeys the coherence laws as well, we have that the morphism is closed.

Definition 3.3.2. Denote by $Flat^\infty(M)$ the dg category of vector bundles. The objects as we defined before are pairs (V, ∇), where V is a graded vector bundle over M and ∇ is a linear map $\Gamma V \to \prod_p \Omega^p(M, V)$ of total degree 1 (so, $\nabla(\Gamma V^\bullet) \subseteq \Omega^j(M, V^{\bullet+j-1})$) that satisfies the graded Liebnitz rule.

$Hom((V, \nabla_V), (W, \nabla_W))$ between two bundles V, W is a complex whose degree k elements are given by maps $\phi : \Gamma V \to \prod_p \Omega^p(M, V)$ of degree k. i.e.

$$\phi(\Gamma V^\bullet) \subseteq \Omega^j(M, V^{\bullet+j+k-1})$$

The differential is given as

$$d\phi = \phi\nabla_W - \nabla_V\phi$$

This is almost exactly as in [Bl05]

We again have a cone object associated to a (closed) morphism ϕ of degree k. We look at the bundle $V \oplus W[k]$ and a connection given as $\begin{pmatrix} \nabla_V & 0 \\ \phi & \nabla_W[k] \end{pmatrix}$ .The morphism ϕ being closed gives us that this is a flat connection.

We may now define the functor $\Sigma : \pi_\infty(M) - Rep \to Flat^\infty(M)$ on the mapping spaces as an infinitesimal increment. Given a map Φ of degree k in $\pi_\infty(M) - Rep$, the $n+1$st component of the degree k morphism in $Flat^\infty(M)$ as

$$\Sigma : \Phi \to \Phi(t + dtu_1 + du_1..u_n + du_n, tu_1u_2..)/dtdu_1..du_n$$

We can also describe this map as the 21 component of the image of the cone object $\begin{pmatrix} F(\sigma) & 0 \\ \Phi(\sigma) & G[k](\sigma) \end{pmatrix}$. In the notation for objects, this would be $\Phi \to A_n^\Phi$

Theorem 3.3.3. *The functor Σ is a dg equivalence of categories.*

Proof. It follows from [BS09] theorem 4.2, that the functor induces a chain map on morphisms. The composition of two degree 0 maps Φ, Ψ is defined as $\Phi \circ \Psi(\sigma_n) = \Phi_0(\sigma_n(0))\Psi_n(\sigma_n) + \Phi_1(\sigma_n(01))\Psi_{n-1}(\sigma_n(12..n)) + ..\Phi_n(\sigma_n(012..n)\Psi_0(\sigma_n(n))$. This is mapped by Σ to its infinitesimal version $\Sigma(\Phi \circ \Psi)_n = A_0^\Phi A_n^\Psi + A_1^\Phi A_{n-1}^\Psi + ..A_n^\Phi A_0^\Psi$ which gives that this is a dg-functor.

The functor is essentially surjective because we saw that it is inverse to the construction of Igusa etal. for objects. To say that the functor is fully faithful, we need to say that it induces a quasi isomorphism on hom-spaces.i.e. an isomorphism on homology groups. Given a represenative of a homology class, i.e. a closed morphism Φ in $\pi_\infty(M) - Rep(F, G)$ we saw that we may construct a cone object $\begin{pmatrix} F & 0 \\ \Phi & G[k] \end{pmatrix}$.

The image of this cone object is a cone $\begin{pmatrix} \Sigma(F) & 0 \\ \Sigma(\Phi) & \Sigma(G[k]) \end{pmatrix}$. Since we know that there is a correspondence of objects, we get a correspondence of closed morphisms as well. $\square$

Chapter 4

The Constructible case

Here we want to combine the results of the previous two chapters. i.e. Given a notion of a stratified space, we wish to look at graded constructible vector bundles and a higher parallel transport for exit paths. We begin by defining a higher dimensional analogue of exit paths. We will then follow the formula in chapter 2. We will look at an exit transport that is built out of pure transports on individual strata and we will work out the conditions the pure (higher) transports need to satisfy with respect to the exit maps to give a coherent exit transport. We will then understand these as another set of conditions on the super connections that give rise to them. We will also establish that every higher exit transport is isomorphic to one of our broken transports.

4.1 Differentiating a parallel transport

Definition 4.1.1. Given a stratified space $\{M_i, S_i, f_i\}$ a simplex σ_n is said to have the exit property if we have that $rk(\sigma_0) \leq rk(\sigma_{01}) \leq rk(\sigma_{012}) ... \leq rk(\sigma_{01...n})$ where rk is the rank or dimension of the stratum

The inequalities here are not strict, in fact an exit simplex can lie entirely in one stratum (i.e. we have all equalities).

We can now think of the category of exit simplicies as a subcategory of the category of all simpliies. Call this category $\pi_\infty^{exit}(M)$. A higher parallel transport functor now is a collection $F_n^{exit} : \pi_\infty^{exit}(M) \to C(k)$ where $F_n^{exit} : V(su_\bullet^0) \to V(tu_\bullet)[n]$ is a degree n chain map between two possibly non-isomorphic chain complexes. This satisfies a coherence relation that is inherited from $\pi_\infty(M) - Rep$.

Remark 4.1.2. Notation: Here we are dealing with transports of different degrees on a stratified space. Subscripts will generally refer to the degree of the transport and superscripts will index the strata.

Let's say we have a 2-simplex in M_2 given as $\{(x_0, x_1, x_2)|x_i \in M_i\}$. So the path $x_0 \to x_1$ is in M_1 etc. If v_0 is a vector over x_0, we have the coherence relation $F(x_2, x_1)F(x_1, x_0)v_0 - F(x_2, x_0)v_0 = d.F_1(x_2, x_0)$. Writing this in terms of the exit maps and the transports on each stratum, we get

$$A_0(x_2)F_1(x_2, x_0)\alpha^2 v_0 - F_1(x_2, x_0)\alpha^2 A_0(x_0)v_0 = F^2(x_2, x_1)\alpha^2 F^1(x_1, x_0)\alpha^1 v_0 -$$

$$F^2(x_2, x_0)\alpha^2 v_0$$

If we ask that the 'α's are chain maps, so that we may write for instance, $\alpha^2 A_0 = A_0\alpha^2$ and also for the condition $\alpha^2 F^1(x_1, x_0)\alpha^1 v_0 = F^2(x_1, x_0)\alpha^2 v_0$ as in the one categorical case in chapter 2. Then the equation reduces to

$$(A_0^2(x_2)F_1^2(x_2, x_0) - F_1^2(x_2, x_0)A_0^2(x_0))\alpha^2 v_0 = (F^2(x_2, x_1)F^2(x_1, x_0) - F^2(x_2, x_0))\alpha^2 v_0$$

which holds because F^2 is an ordinary transport on S_2.

A general coherence relation looks like $\Sigma_{p+q=n} F_p F_q$. In terms of the pure transports this would be $\Sigma_{p+q=n} F_p^i \alpha^i F_q^j \alpha^j$. So if we impose conditions of the form $\alpha^i F_q^j \alpha^j = F_q^i \alpha^i$, for every $p, q, j < i$, the coherence relation reduces to $\Sigma_{p+q=n} F_p^i F_q^i \alpha^i$, i.e. the coherence condition in the top stratum.

Just as in the 1-categorical case we may write the condition $\alpha^i F_q^j \alpha^j = F_q^i \alpha^i$ as the two conditions $\alpha^i F_q^j = F_q^i \alpha^i$ and the condition '$\alpha^i \alpha^j = \alpha^i$'.

These give corresponding conditions on the super connections as follows:

Lemma 4.1.3. *If $\nabla^0 = d - \Sigma_n A_n^0, \nabla^1 = d - \Sigma_n A_n^1$ are flat super connections on M_0 and S_1 that give rise to the parallel transports F_n^0, F_n^1 and α^1 is an exit map. Then the condition $\alpha^1 F_n^0(tu_\bullet, su_\bullet) = F_n^1(tu_\bullet, su_\bullet)\alpha^1$ for all n is equivalent to the condition $\alpha^1 A_n^0 = A_n^1 \alpha^1$ for $n = 0, 2, 3....$ and $d\alpha^1 = A_n^1 \alpha^1 - \alpha^1 A_n^0$ for $n = 1$.*

Proof. We can differentiate the condition $\alpha^1(tu_\bullet)F_n^0(tu_\bullet, su_\bullet) = F_n^1(tu_\bullet, su_\bullet)\alpha^1(su_\bullet)$ with respect to $u_1, u_2, ..$ etc and take the limit $u^0 \to u$ to get the condition

$$\alpha(t)\psi_n^0(t, s) = \psi_n^1(t, s)\alpha(s)$$

for each n. We can recover the condition on the Fs by integrating with respect to the variables $u_1, u_2..$ with fixed s and t.

Now remember that we have the following facts :

1.
$$\frac{\partial}{\partial x}\psi_n^0(x,s) = A_1^0\psi_n^0 + A_2^0\psi_{n-1}^0 + ...A_n^0\psi_0^0$$

2. There's a similar equation for $\psi_n^0(t,x)$:
$$\frac{\partial}{\partial x}\psi_n^0(t,x) = \psi_n^0 A_1^0 + \psi_{n-1}^0 A_2^0 + ...\psi_0^0 A_n^0$$

3.
$$\psi_n^0(t,s) = \psi_0^0(t,x)\psi_n^0(x,s) + \psi_1^0(t,x)\psi_{n-1}^0(x,s)... + \psi_n^0(t,x)\psi_0^0(x,s)$$

The case $n = 0$ follows from the fact that α^1 is a chain map. The case $n = 1$ follows from the case of the ordinary parallel transport.

If we have the conditions $n = 0, 1$, for $n = 2$, write:

$$\alpha(t)\psi_1^0(t,x) = \psi_1^1(t,x)\alpha(x)$$

Multiplying by $\psi_0^0(x,s)$ gives

$$\alpha(t)\psi_1^0(t,x)\psi_0^0(x,s) = \psi_1^1(t,x)\alpha(x)\psi_0^0(x,s)$$

Similarly we get

$$\alpha(t)\psi_0^0(t,x)\psi_1^0(x,s) = \psi_0^1(t,x)\alpha(x)\psi_1^0(x,s)$$

Adding these and using equation 3 above gives:

$$\alpha(t)\psi_1^0(t,s) = \psi_1^1(t,x)\alpha(x)\psi_0^0(x,s) + \psi_0^1(t,x)\alpha(x)\psi_1^0(x,s)$$

In other words, the RHS of the above equation doesn't depend on x. So we may differentiate with respect to x to get

$$\psi_0(\alpha' - A_1^1\alpha + \alpha A_1^0)\psi_1 + \psi_0(\alpha' - A_1^1\alpha + \alpha A_1^0)\psi_1 + \psi_0(-A_2^1\alpha + \alpha A_2^0)\psi_0 = 0$$

The case $n = 1$ gives that the first two terms here are 0. So we get

$$(A_2^1\alpha - \alpha A_2^0) = 0$$

since the ψ_0 are invertible.

Notice that this is reversible, given the conditions for $n = 0, 1, 2..$, we can say that

$$\psi_1^1(t,x)\alpha(x)\psi_0^0(x,s) + \psi_0^1(t,x)\alpha(x)\psi_1^0(x,s)$$

is independent of x and setting $x = t, s$ gives

$$\alpha(t)\psi_1^0(t,s) = \psi_1^1(t,s)\alpha(s)$$

The general case is similar: Given the conditions for $n = 0, 1, 2...n$, we can write

$$\alpha(t)\psi_k^0(t,x) = \psi_k^1(t,x)\alpha(x)$$

Multiplying by $\psi_{n-k}^0(x,s)$ and adding over k, gives

$$\alpha(t)\psi_n^0(t,s) = \sum_{0\le k\le n} \psi_k^1(t,x)\alpha(x)\psi_{n-k}^0(x,s)$$

Once again we see that the RHS is independent of x and we can differentiate w.r.t. x to get

$$\sum_{i+j+k=n} \psi_i^1(t,x)(\delta_{j1}\alpha' - A_j\alpha + w\alpha A_j)\psi_k^0(x,s) = 0$$

where δ_{1j} is the dirac delta. The α' term appears only with the A_1s.

Using the induction hypothesis for the case $n = 0, 1..n-1$, this reduces to the condition

$$A_n^1\alpha = \alpha A_n^0$$

We can again use the conditions on the A_ns to recover the conditions on the ψ_n and so the conditions on F_n. So we have concluded that the coherence conditions for F reduce to the compatibility conditions for the different A^i.

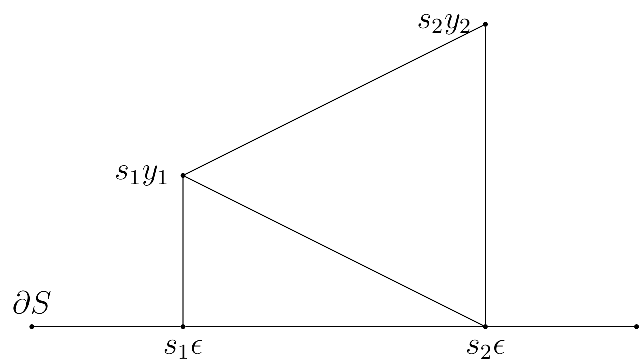
□

Remembering that $\nabla = d - A_0 - A_1 - A_2...$ and $d\alpha = d\circ\alpha + \alpha\circ d$, the condition can be written as $\alpha^1\nabla^0 = \nabla^1\alpha^1$. In general the condition would be $\alpha^i\nabla^j = \nabla^i\alpha^i$, for $i > j$.

We also need an analogue of lemma 2.2.3 that allows us to say that any parallel transport arises this way. The setup is the same. We wish to modify a given $\{F_n\}$ to $\{\bar{F}_n\}$. We take $\bar{F}_0$ to be the identity in the collar direction. For $n > 0$ we let $\bar{F}_n$ be equal to 0 if it has a vertical direction $(0,\epsilon)$ and F_n otherwise.

Lemma 4.1.4. $\bar{F}_n|_{(0,\epsilon]\times\partial S} \simeq F_n|_{(0,\epsilon]\times\partial S}$

Proof. Choose coordinates $(s,y) \in \partial S \times (0,\epsilon]$. (This conflicts slightly with our notation in the rest of this paper.)



We'll define a degree 0 morphism Φ from F to $barF$ and show that it's a homotopy equivalence. First, we define $\Phi_0 : F_{st} \to \bar{F}_{sy} = F_{s\epsilon}$ to be $F_0(s\epsilon, sy)$.

For $n > 0$, we look at the solid formed by the projection of a simplex σ onto $S \times \{\epsilon\}$. Define $\Phi_n : F_{\sigma(0)} \to \bar{F}_{\sigma(n)}[n]$ to be $F_n(Cyl(\sigma_n))$ where by $Cyl(\sigma_n)$ we mean the solid.

So for $n = 1$, this gives (as seen above) $\Phi_1 : F_{s_1y_1} \to \bar{F}_{s_2\epsilon}[1]$ is $F_1(Cyl(s_1y_1, s_2y_2))$, This we know can be written by the pasting diagram pictured as $F_1(s_2\epsilon, s_1\epsilon, s_1y_1) + F_1(s_2\epsilon, s_2y_2, s_1y_1)$.

The map is easily seen to be a homotopy equivalence. By Prop.2.13 in [BS09], we only need to check that the map $\Phi_0 : F_{sy} \to \bar{F}_{sy} = F_{s\epsilon}$ is a quasi isomorphism. And this is true because we've defined it to be $F_0(s\epsilon, sy)$ which is an isomorphism.

It remains to be said that the map Φ is a closed morphism. This is perhaps not too hard to believe. For $n = 1$, we can triangulate the cylinder as pictured and the see that the relation $d\Phi + \hat{\delta}\Phi + \Phi \cup F + (-1)^{\Phi}\bar{F} \cup \Phi$ is just the sum of the coherence relations for the two triangles, remembering that $\Phi_1(s_2y_2, s_1y_1) =$

$F_1(s_2\epsilon, s_1\epsilon, s_1y_1) - F_1(s_2\epsilon, s_2y_2, s_1y_1)$. For $n = 0, 1$,

$$(\hat{\delta}\Phi + \Phi \cup F + \bar{F} \cup \Phi)(s_2y_2, s_1y_1) = \Phi_0(s_2y_2)F_0(s_2y_2, s_1y_1) - \bar{F}_0(s_2y_2, s_1y_1)\Phi_0(s_1y_1)$$

$$= F_0(s_2\epsilon, s_2y_2)F_0(s_2t_2, s_1y_1) - F_0(s_2\epsilon, s_1\epsilon)F_0(s_1\epsilon, s_1y_1)$$

$$= (F_0(s_2\epsilon, s_2y_2)F_0(s_2y_2, s_1y_1) - F_0(s_2\epsilon, s_1y_1)) + (F_0(s_2\epsilon, s_1y_1) -$$

$$F_0(s_2\epsilon, s_1\epsilon)F_0(s_1\epsilon, s_1y_1))$$

$$= d.F_1(s_2\epsilon, s_1\epsilon, s_1y_1) - d.F_1(s_2\epsilon, s_2y_2, s_1y_1)$$

$$= d.\Phi(s_2y_2, s_1y_1)$$

In the case $n \geq 2$, we can write triangulate the cylinder formed by the projection of an n -simplex into $n+1$ $n+1$-simplices and write $\Phi_n(s_ny_n, s_1y_1)$ as the sum $\Sigma_i(-1)^iF_{n+1}(s_1\epsilon, s_2\epsilon ... s_i\epsilon, s_iy_is_{i+1}y_{i+1}...s_ny_n)$. Then the sum of the coherence relations for these $n+1$ simplicies gives the condition that Φ_n is closed.

□

The higher analogues of the theorems in chapter 2 now follow straightforwardly.

First, we'll give an inductive definition for a transport built out of ordinary transports on the closed pure strata $\bar{S}_i$ and exit maps. Given an ordinary transport $\mathbb{F}^i_n$ on S_i, a transport F^{i-1}_n on M_{i-1} and a map (degree 0 natural transformation) $(\alpha^i : f^*_iF^{i-1} \to \mathbb{F}^i)|_{\partial S_i}$, we can construct a transport on M_i as $F^i = \mathbb{F}^i\alpha^iF^{i-1}$, i.e. Given an exit n-simplex σ_n whose last vertex lies in the i-th stratum, we can define $F^i_n(\sigma_n) = \mathbb{F}^i(\sigma_n)\alpha^i(\sigma_n(0))$, we saw that this defines a higher exit transport if the α satisfy the right conditions.

Remark 4.1.5. Note that α^i is not a degree 0 morphism in $(\pi_\infty(\partial S_i) - Rep)$ as defined in the previous section. We ask that there is a strict relation $\alpha_i F^i = \mathbb{F}^{i+1}\alpha_i$ for the α^i, whereas a degree 0 morphism for the category $(\pi_\infty(\partial S_i) - Rep$ would only give us this relation upto homotopy. Generally speaking, we weaken conditions that were upto homotopy in the pure strata, but we do not weaken the constructible conditions.

Definition 4.1.6. Denote by $\pi_{\leq\infty}^{broken}(M) - Rep$ the category of the transports that arise this way. i.e. The objects of the category are given by

1. An ordinary parallel transport $\mathbb{F}^i \in \pi_{\leq\infty}(\bar{S}_i) - Rep$

2. Exit maps $(\alpha^i : f_i^* F^{i-1} \to \mathbb{F}^i)|_{\partial S_i}$

3. Maps α_{ij} of transports over S_{ij}

$$\begin{array}{ccc} F^i & \xrightarrow{\alpha^j} & \mathbb{F}^i \\ & \searrow^{\alpha^i} & \downarrow{\scriptstyle \alpha^{ij}} \\ & & \mathbb{F}^j \end{array}$$

4. Such that the α_i are compatible with the transports , $\alpha_i F^i = \mathbb{F}^{i+1}\alpha_i$

5. And the α^i are compatible with each other '$\alpha^i\alpha^j = \alpha^i$'

Maps between transports are defined as morphisms Φ_n^i between $\mathbb{F}^i$ and $\mathbb{G}^i$ on the closed pure strata that commute (strictly) with the exit maps. The differential of each pure transport given as

$$D := \delta\Phi + d\Phi + G \cup \Phi + (-1)^{\Phi}\Phi \cup F$$

commutes with the 'α's because each of the maps in the RHS of the expression does.

Theorem 4.1.7. *There is an equivalence of categories* $\Theta : \pi_{\leq\infty}^{broken}(M) - Rep \to \pi_{\leq\infty}^{exit}(M) - Rep$

Proof. The functor Θ is defined on objects as

$$\Theta(\{\mathbb{F}^i, \alpha^i\})(\sigma_n)v_{(\sigma_n(0))} = F^{exit}(\sigma_n)v_{(\sigma_n(0))} = \mathbb{F}^{i_n}(\sigma_n)\alpha^{i_n}v_{(\sigma_n(0))}$$

where i_n is the stratum $\sigma_n(n)$ belongs to.

This functor is essentially surjective. Given an exit transport F, it restricts to a transport on the pure strata. Call these restrictions $\mathbb{F}^i$, these can be modified by Lemma 4.1.4 so that they extend to the boundary of the pure strata $\bar{\mathbb{F}}^i$. We can define maps $\alpha^i = \bar{\mathbb{F}}^i(0,\epsilon)F^{exit}(\epsilon, 0)$ and $\alpha^{ij} = \bar{\mathbb{F}}^i(0,\epsilon)\bar{\mathbb{F}}^j(\epsilon, 0)$. Then, $\Theta(\{\bar{\mathbb{F}}^i, \alpha^i\}) \simeq F^{exit}$, where the ϵ is as in Lemma 4.1.4 specifies the collar neighbourhood of the boundary S_i

To see that this is fully faithful, we need to say $Hom_{\pi_\infty^{broken}(M)-Rep}(F, G) \simeq Hom_{\pi_\infty^{exit}(M)-Rep}(\Theta(F), \Theta(G))$ is a quasi isomorphism. If F, G are in the image of the functor Θ, they are made from functors on the closed pure strata and exit maps. We need to say any closed morphism between them is given by a morphism defined on the individual strata. For x_0 in the boundary we can define the map $\Phi_0(x_0) : (\lim_{x\to x_0} F_0(x)) \to (lim_{x\to x_0} G_0(x))$ as $(\lim_{x\to x_0} \Phi_0(x))$. The map in higher dimensions is similarly defined by taking a limiting sequence of simplices. (Any choice will give the same limit because of the pasting laws). ∎

We can define a constructible graded vector bundle just as in the 1-categorical case. Then a super connection on a constructible vector bundle is

Definition 4.1.8. A flat connection on a constructible graded vector bundle $\{\mathbb{S}_i, V_i, \tilde{f}_i\}$ on a stratified space $\{S_i, M_i, f_i\}$ is given by

1. An flat super connection $\nabla^i = d - A_0^i - A_1^i - A_2^i ...$ on the bundle $\mathbb{S}_i|_{S_i}$.

2. Exit maps α^i:

$$\begin{array}{ccccc} V_{i-1} & \longleftarrow & f_i^*(V_{i-1}) & \xrightarrow{\alpha_i} & \mathbb{S}_i \\ \downarrow & & \downarrow & & \downarrow \\ M_{i-1} & \xleftarrow{f_i} & \partial S_i & \longrightarrow & \partial S_i \end{array}$$

3. Maps α_{ij} of bundles over S_{ij}

$$\begin{array}{ccc} V_i & \xrightarrow{\alpha^j} & \mathbb{S}_i \\ & \searrow^{\alpha^i} & \downarrow{\scriptstyle \alpha_{ij}} \\ & & \mathbb{S}_j \end{array}$$

4. Such that the α^i are compatible with the connection, $\alpha^i \nabla^j = \nabla^i \alpha^i$

5. And the α^i are compatible with each other '$\alpha^i \alpha^j = \alpha^i$'

We can define a category of constructible vector bundles with connection, by defining a morphism of degree k to be a morphism of degree k on each pure stratum that commutes with the exit maps α. We have that the differential commutes with the αs because the morphisms and the connections do. Call this category $Flat_\infty^{const}(M)$.

Theorem 4.1.9. *There is an equivalence of categories*

$$\Sigma : \pi_{\infty}^{broken}(M) - Rep \to Flat_{\infty}^{const}(M)$$

Proof. Given a transport $\mathbb{F}_i$ on each closed pure stratum $\bar{S}_i$, we can build a vector bundle $\mathbb{S}_i$ as in the case of the last chapter. We take the attaching maps $\tilde{f}_i$ to be zero. The maps α^i of transports give bundle maps by the way we topologise the bundles. And the conditions on the transports translate to the conditions on the vector bundles.

We define the connection on the pure strata by differentiating the transport as before. We saw that the compatibility of the αs with the transport is equivalent to the compatibility of the corresponding connection. This gives an equivalence at the level of objects.

The correspondence for objects also gives an equivalence of maps by recognizing that each closed morphism Φ determines and is determined by a cone object $\begin{pmatrix} F & 0 \\ \Phi & G \end{pmatrix}$ and that the compatibility for morphisms $\alpha^i \Phi^j = \Phi^i \alpha^i$ gives the correct compatibility for the cone objects $\alpha^i \begin{pmatrix} F^j & 0 \\ \Phi^j & G^j \end{pmatrix} = \begin{pmatrix} F^i & 0 \\ \Phi^i & G^i \end{pmatrix} \alpha^i$.

□

4.2 Comments

Here we talk informally about some possible lines of future investigations. Things are not stated here precisely let alone proved.

4.2.1 Relation to the work of K.Igusa

Notice that in the last section we did not use Igusa's formulation of writing parallel transports as sections of a bundle over PM. We could do this by replacing the path space by an appropriate space of exit paths that is defined as a subspace of the space of all paths PM. This was not necessary. It was enough for our purposes to look at parallel transport forms on each individual strata that were compatible with the exit maps. But it's an interesting line to pursue in the future.

4.2.2 Relation to the work of Block-Smith

Block-Smith formulate this slightly differently. By Swan's theorem, a vector bundle is equivalent to a module over $C^\infty(M)$. The information of a (flat) super connection on a vector bundle is given by a "cohesive module" over $\Omega^*(M)$, the forms on M.(Cohesive modules are elements of $P_{\Omega^*(M)}$ and not modules over the algebra $\Omega^*(M)$. See [Bl05].)

We may adapt our constructible vector bundles to their language as follows: Let's say we have just two strata, i.e. we have:

$$M_0 \xleftarrow{f_1} \partial S_1 \xrightarrow{i_1} S_1$$

We can define an algebra A of stratified forms as the direct sum $\Omega^*(M_0) \oplus \Omega^*(\partial S_1) \oplus \Omega^*(S_1)$ plus formal elements f_1^*, i_1^*, α^* that satisfy the obvious relations

$$i_1^*.\omega_{S_1} = i_1^*(\omega_{S_1}),\ \alpha^2 = 0,\ \alpha\omega = i_1^*\omega\alpha$$

etc. Now a module for this algebra satisfies exactly the relations for the constructible vector bundles that we saw were needed. This can be easily extended to the case of many strata with relations such as $\alpha^i\alpha^j = \alpha^i$ etc.

When our space is a simplicial complex stratified by the simplices, we have $\Omega^*(S_i) \sim \mathbb{R}$. And the algebra of stratified forms reduces to just the exit maps. This returns us to the folk case of MacPherson we discussed in section 2.2.5. There are extensions of the simplicial case to include perversities. See for example [Vy99]. One line for future inquiries is to extend our results to include perversities for general (not contractible) strata.

4.2.3 Relation to the work of Jacob Lurie

Jacob Lurie [JL-A] proves a derived equivalence of sheaves and exit path representations for his ∞-categories. In the case of functors to $C(k)$ we can give explicit definitions of the sheaf obtained from a functor. We will briefly say how that story would go without proofs.

Recall that taking parallel sections of a vector bundle defined a locally constant sheaf in the ordinary case. We would like to do the same in our ∞ setting.

For a functor $F : \pi_{\leq 1}(X) \to Vect$ the sections of a sheaf $\mathcal{F}$ over an open set were given as $\mathcal{F}(U) = \{s \in Maps(U, E) | s(y) = F(\gamma)(s(x))\}$ where γ is any path in U.

This definition wouldn't work in the higher setting because homotopic paths have different transport maps and as such we shouldn't expect any sections y this definition, even when U is contractible, say. So we weaken the condition $s(y) = F(f)(s(x))$.

Let $F : \pi_\infty(X) \to C(k)$ be a functor (local system). Given an open set U a section of the sheaf s over U is a choice

1. For each point $x \in U$, an element $s_0(x) \in F_x$
2. For a 1-simplex σ joining x and y, an element $s_1(y) \in F_y[1]$, such that $d.s_1(y) = s_0(y) - F_0(\sigma)s_0(x)$.
3. For a 2-simplex σ joining x, y and z, an element $s_2(z) \in F_z[2]$, such that $d.s_2(z) = F_1(\sigma)s_0(x) - s_1(\partial\sigma)$.
4. For a $n+1$-simplex σ joining $x_0, ..x_n$, an element $s_n(x_n) \in F_{x_n}[n]$, such that $d.s_n(x_n) = F_n(\sigma)s_n(x_0) - s_n(\partial\sigma)$.

Here ∂ is the boundary of a simplex and d the differential in $F(\sigma)$.

It's easy to see that $\mathcal{F}(U)$ is a complex of vector spaces by addition, differential etc. defined pointwise. So this gives us a functor

$$(\pi_\infty(M), C(k)) \to PSh(M)$$

where $PSh(M)$ is presheaves of complexes.

We need to say $\mathcal{F}$ defined this way is actually a sheaf. This is a version of the nerve theorem in topology that says that the homotopy type of a space can be recovered from a Cech-complex. The sheaf would be locally constant (in cohomology) because over a contractible (U, x_0), we can show that the restriction $\mathcal{F}(U) \to \mathcal{F}_{x_0}$ is a quasi isomorphism. The functor going in reverse that takes a sheaf to a parallel transport can be given as a kind of Kan extension.

Bibliography

[BF02] Hans-Joachim Baues, Davide L. Ferrario, *K-theory of stratified vector bundles*, http://arxiv.org/abs/math/0208186

[Bl05] Jonathan Block, *Duality and equivalence of module categories in noncommutative geometry I*, http://arxiv.org/abs/math/0509284

[BS09] Jonathan Block, Aaron M. Smith, *A Riemann–Hilbert correspondence for infinity local systems*, Advances in Mathematics Volume 252, 15 February 2014, Pages 382405

[Che73] Kuo-tsai Chen, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. (2) **97** (1973), 217–246.

[Che75] Kuo Tsai Chen, *Connections, holonomy and path space homology*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 1, Stanford Univ., Stanford, Calif., 1973), Amer. Math. Soc., Providence, R. I., 1975, pp. 39–52.

[Che77] K.-T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), 831–879.

[Ig09] Kiyoshi Igusa, *Iterated integrals of super connections*, http://arxiv.org/abs/0912.0249

[JL-A] Jacob Lurie, *Higher Algebra*, http://www.math.harvard.edu/ lurie/

[Kr10] Matthias Kreck, *Differential Algebraic Topology From Stratifolds to Exotic Spheres*, http://www.him.uni-bonn.de/fileadmin/user_upload/kreck-DA.pdf

[Vy99] Maxim Vybornov, *Sheaves on Triangulated Spaces and Koszul Duality*, http://arxiv.org/abs/math/9910150