Introduction

Introductions are the hardest part, I'll tell you that much for free.

This thesis is an investigation into shifted symplectic structures and shifted Poisson structures, and their relation to certain spaces of framed maps. Shifted symplectic structures could pithily be described via a pullback square of "symplectic geometry" and "derived algebraic geometry" over "algebraic geometry", although maybe not if you want anyone to understand what you're saying.

0.1 Shifted Symplectic Structures

Shifted symplectic structures, first described in [PTVV], are the natural extension of ordinary (algebraic) symplectic structures to the land of derived algebraic geometry. Promoting everything to the level of derived stacks has the notable effect of replacing the cotangent sheaf L_X with the cotangent *complex* $\mathbb{L}_X$, and dually the tangent sheaf with the tangent complex. This has the expected "homotopic" effect of replacing isomorphisms with quasi-isomorphisms, equalities with equalities up to homotopy, et cetera. More interestingly, the usual nondegeneracy requirement that the induced map $T_X \to L_X$ is an isomorphism seems to become a requirement that $\mathbb{T}_X \to \mathbb{L}_X$ is a quasi-isomorphism. This is generally impossible, unless $\mathbb{L}_X$ is concentrated in cohomological degrees $[-a, a]$ for some a.

More generally, we might look for a map $\mathbb{T}_X \to \mathbb{L}_X[n]$, which would correspond to a 2-form of degree n, and which would ultimately correspond to an n-shifted symplectic structure[1]. Assuming $\mathbb{L}_X$ is concentrated in degrees $[a, b]$ (with $|a|, |b| < \infty$), an n-shifted symplectic structure may be possible for $n = a + b$.

This is a wide generalization of ordinary symplectic structures. First, it really is a generalization: 0-shifted symplectic structures on smooth varieties are symplectic structures in the ordinary sense. Second, there are constructions which will provide new shifted symplectic spaces from old ones, often with different shifts. Thus, someone uninterested in derived algebraic geometry might still use this machinery to end up with a 0-shifted structure on a smooth variety. Finally, a lot of derived stacks—like the classifying stack BG—have shifted symplectic structures.

0.2 Maps and Framed Maps

One particular example, shown in [PTVV], is as follows. Let Y have an n-shifted symplectic structure, and let X be $\mathcal{O}$-compact oriented in dimension d. This latter criterion is nontrivial; it is, for example, satisfied for X a smooth compact Calabi-Yau variety. Then the mapping stack $\mathrm{Map}(X, Y)$ has an $(n-d)$-shifted symplectic structure.

This is a powerful theorem that also recreates some examples of known classical symplectic structures. For example, if X is a K3 surface, then the symplectic structure on $\mathrm{Map}(X, BG)$ was described by Mukai.

[1]This is misleading; in the classical case, the form is equivalent to the map $T \to L$, but in the derived case more structure is needed.

However, there are also a number of cases this does not cover. Let G be a reductive group. Let $\mathrm{Map}(\mathbb{P}^2, L, BG)$ be the space of principal G-bundles on $\mathbb{P}^2$ with a trivial framing along a line L. This space has a symplectic structure as described in [Bo]. For another example, let $\mathrm{Map}(\mathbb{P}^1, p, G/B)$ be the space of maps from $\mathbb{P}^1$ to a flag variety G/B sending a marked point in the source to a marked point in the target. This space also has a symplectic structure, as described in [FKMM].

Both of these examples detail spaces of *framed maps*. Specifically, fix maps $i : D \to X$ and $f : D \to Y$, and look at the homotopy fiber of $\mathrm{Map}(X, Y) \to \mathrm{Map}(D, Y)$ above f. The resulting space, $\mathrm{Map}(X, D, Y)$ parametrizes maps $g : X \to Y$ with homotopies $g \circ i \sim f$ on D. As the above examples show, these spaces will have shifted symplectic structures under certain circumstances. Looking at things in a different perspective, in the above cases the source X does not have a d-orientation, and $\mathrm{Map}(X, Y)$ does not a have a symplectic structure; $\mathrm{Map}(X, D, Y)$ is the "correct" space for symplectic structures.

The main result for this is

Theorem 0.1. *Let X be a d-dimensional proper smooth scheme and D an effective divisor. Suppose E is an effective divisor of X such that $\tilde{D} = 2D + E$ is anticanonical. Let Y be a derived Artin stack such that* $\mathrm{Map}(X, Y)$*,* $\mathrm{Map}(\tilde{D}, Y)$*,* $\mathrm{Map}(D, Y)$*, and* $\mathrm{Map}(D + E, Y)$ *are themselves derived Artin stacks of locally finite presentation over k. Fix a base map $f : D \to Y$. Suppose Y is n-shifted symplectic and the projection* $\mathrm{Map}(D + E, Y) \to \mathrm{Map}(D, Y)$ *is etale over f. Then* $\mathrm{Map}(X, D, Y)$ *has an $(n - d)$-shifted symplectic structure.*

This theorem will provide, for example, the symplectic structure on $\mathrm{Map}(\mathbb{P}^2, L, BG)$. On the other hand, it is a bit fragile; varieties with effective anticanonical divisor are common enough, but the cohomological condition that $\mathrm{Map}(D + E, Y) \to \mathrm{Map}(D, Y)$ is etale over f is not guaranteed. On a more conceptual level, we would like to know why $\mathrm{Map}(X, Y)$ isn't symplectic in this scenario (and $\mathrm{Map}(X, D, Y)$ whenever the etaleness condition is not satisfied).

0.3 Shifted Poisson Structures

To justify this sudden change of topics, we note that in the above examples, even when the mapping space doesn't have a symplectic structure, it still has a Poisson structure. To give two examples, if G is a semisimple group and P is a parabolic subgroup, $\mathrm{Map}(\mathbb{P}^1, p, G/P)$ will have a Poisson structure. And the space $\mathrm{Map}(\mathbb{P}^2, L, BG)$ will not have a symplectic structure if we choose a nontrivial framing on L, but it will still have a Poisson structure.

To motivate the definition of a shifted Poisson structure, we note two things. First, if we let $\bullet_{n+1}$ denote a point with the trivial $(n + 1)$-shifted symplectic structure, then an n-shifted symplectic structure on a stack X is exactly the same as a Lagrangian structure on $X \to \bullet_{n+1}$. Second, if X is a smooth underived scheme, then a Poisson structure on X can be used to construct a 1-shifted symplectic space Y and a morphism $X \to Y$ with Lagrangian structure; conversely, given such a map to a 1-shifted symplectic Y, we can construct a Poisson structure on X.

With this in mind, we take an n-shifted Poisson structure on X to be a formal derived stack Y with an $(n+1)$-shifted symplectic structure, and morphism $X \to Y$ with Lagrangian structure. The preceding paragraph tells us that any n-shifted symplectic structure is n-shifted Poisson, and that a 0-shifted Poisson structure on a smooth scheme is a Poisson structure in the usual sense.

With this definition, we prove a number of results about Poisson structures generalizing those about symplectic structures. For framed mapping spaces we have the following theorem:

Theorem 0.2. *Let X be a d-dimensional proper smooth scheme and D an effective divisor. Suppose E is an effective divisor of X such that $\tilde{D} = 2D + E$ is anticanonical. Let Y be a derived Artin stack such that* $\mathrm{Map}(X, Y)$*,* $\mathrm{Map}(\tilde{D}, Y)$*,* $\mathrm{Map}(D, Y)$*, and* $\mathrm{Map}(D + E, Y)$ *are themselves derived Artin stacks of locally finite presentation over k. Fix a base map $f : D \to Y$. Suppose Y is n-shifted Poisson. Then* $\mathrm{Map}(X, D, Y)$ *has an $(n - d)$-shifted Poisson structure.*

We have, notably, discarded the etaleness assumption. This theorem is pretty broadly applicable. For example, we obtain the Poisson structure on the remaining cases of $\mathrm{Map}(\mathbb{P}^2, L, BG)$. The case of $\mathrm{Map}(\mathbb{P}^1, p, G/P)$ requires a more roundabout approach, but ultimately is understood via these tools.

Even in the case that Y is symplectic, this theorem is illuminating. For example, it tells us "why", if X has nonzero effective anticanonical divisor, the space $\mathrm{Map}(X, Y)$ does not have a symplectic structure; it has a (nonsymplectic) Poisson structure. In fact, the same is true for any $\mathrm{Map}(X, D, Y)$ for which the etaleness condition of Theorem 0.1 does not hold.

0.4 Organization

Chapter 1 is an overview of shifted symplectic structures. It collects some definitions and results but is not a detailed reference.

Chapter 2 is about shifted Poisson structures. Poisson structures and coisotropic morphisms are defined. 0-shifted Poisson structures on smooth schemes are shown to be Poisson structures in the ordinary sense. Results for symplectic structures are generalized to Poisson structures.

Chapter 3 concerns framed mapping spaces. It contains the main results of this thesis, particularly those given in this introduction.

Chapter 4 is about the spaces $\mathrm{Map}(\mathbb{P}^1, p, G/P)$. The Poisson structure on this space is constructed.

Chapter 1

Shifted Symplectic Structures

In the following, k will be the base field, of characteristic 0.

Shifted symplectic structures are first defined in [PTVV]. We will recall some definitions and results.

Let X be a derived Artin stack. We can form the de Rham algebra $\Omega_X^* = \mathrm{Sym}_{\mathcal{O}_X}^*(\mathbb{L}_X[1])$. This is a weighted sheaf whose weight p piece is $\Omega_X^p = \mathrm{Sym}_{\mathcal{O}_X}^p(\mathbb{L}_X[1]) = \wedge^p \mathbb{L}_X[p]$.

Definition 1.1. The *space*[1] *of p-forms of degree n on X* is

$$\mathcal{A}^p(X,n) = \tau_{\leq 0} \operatorname{Hom}_{L_{QCoh(X)}}(\mathcal{O}_X, \wedge^p \mathbb{L}_X[n]).$$

Here $L_{QCoh(X)}$ is the ∞-category of chain complexes of quasicoherent $\mathcal{O}_X$-modules.

Let $d_{\mathbb{L}_X}$ denote the differential on $\mathbb{L}_X$ or the induced differential on $\bigwedge^* \mathbb{L}_X$. Let d_{dR} be the de Rham differential on $\bigwedge^* \mathbb{L}_X$. Then we construct the weighted negative cyclic chain complex NC^w, whose degree n, weight p part is

$$NC^n(\Omega_X)(p) = (\bigoplus_{i \geq 0} \wedge^{p+i} \mathbb{L}_X[n-i], d_{\mathbb{L}_X} + d_{dR}).$$

Definition 1.2. The *space of closed p-forms of degree n* is

$$\mathcal{A}^{p,cl}(X,n) = \tau_{\leq 0} \operatorname{Hom}_{L_{QCoh(X)}}(\mathcal{O}_X, NC^n(\Omega_X)(p)).$$

There is a natural "underlying form" map $\mathcal{A}^{p,cl}(X,n) \to \mathcal{A}^p(X,n)$ corresponding to the projection $\bigoplus_{i \geq 0} \wedge^{p+i} \mathbb{L}_X[n-i] \to \wedge^p \mathbb{L}_X[n]$.

Intuitively, a "closed p-form" ω of degree n consists of forms $\omega_{p+i} \in \wedge^{p+i}(\mathbb{L}_X)_{n-i}$ for $i \geq 0$ such that $(d_{\mathbb{L}_X} + d_{dR})(\omega_p + \omega_{p+1} + \cdots) = 0$. This has the following interpretation. The underlying form of ω is ω_p; we require $d_{\mathbb{L}_X}\omega_p = 0$ so that ω_p defines a class in cohomology. For de Rham closedness, we do not require that $d_{dR}\omega_p$ equals zero, but that it is homotopic to zero with specified homotopy: $d_{dR}\omega_p = -d_{\mathbb{L}_X}\omega_{p+1}$ for some ω_{p+1}. We then require that ω_{p+1} be de Rham closed, again in the sense that $d_{dR}\omega_{p+1} = -d_{\mathbb{L}_X}\omega_{p+2}$, et cetera. Note that a closed p-form is not a condition on a p-form, but an extra structure consisting of the forms ω_{p+i} for $i \geq 1$. In the case of a (0-shifted) p-form on an ordinary variety, $\mathbb{L}_X$ is concentrated in degree 0, so we must have $\omega_{p+i} = 0$ for $i \geq 1$. Thus the structure reduces to a condition in this case.

Now we can define symplectic structures:

[1] Here I use "space", "simplicial set", and "connective chain complex" interchangably.

Definition 1.3. A 2-form $\omega : \mathcal{O}_X \to \wedge^2 \mathbb{L}_X[n]$ of degree n is *nondegenerate* if the adjoint map $\mathbb{T}_X \to \mathbb{L}_X[n]$ is a quasi-isomorphism. Let $\mathcal{A}^2(X,n)^{nd}$ denote the non-degenerate 2-forms of degree n on X.

An *n-shifted symplectic form* on X is a closed 2-form whose underlying form is nondegenerate. The space of n-shifted symplectic forms is the (homotopy[2]) product

$$\mathrm{Symp}(X,n) = \mathcal{A}^{2,cl}(X,n) \times_{\mathcal{A}^2(X,n)} \mathcal{A}^2(X,n)^{nd}.$$

Let us give a few examples of spaces with shifted symplectic structures.

- If X is an ordinary (underived) smooth scheme, then a 0-shifted symplectic structure on X is precisely the same as a symplectic structure in the ordinary sense.
- Let G be a reductive affine smooth group scheme over k. Then the classifying stack BG has a 2-shifted symplectic structure. A 2-shifted symplectic form on G is the same as a nondegenerate G-invariant quadratic form on $\mathfrak{g}$.
- Let X be a derived Deligne-Mumford stack locally of finite presentation over k. Then we can define the n-shifted cotangent stack $T^\vee X[n] = \mathbb{R}\,\mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{T}_X[-n])$. Then $T^\vee X[n]$ has a natural n-shifted symplectic form defined analogously to the canonical symplectic form on $T^\vee X$ for a smooth scheme X.

  In particular, applying this to $T^\vee(BG)[1]$ yields a 1-shifted symplectic form on $\mathfrak{g}^\vee/G$.

1.1 Lagrangian Structures

Let Y be a derived Artin stack with an n-shifted symplectic form ω and let $f : X \to Y$ be a morphism.

Definition 1.4. The *space of isotropic structures* on f (with respect to ω) is

$$\mathrm{Isot}(f,\omega) = \mathrm{Path}_{0,f^*(\omega)}(\mathcal{A}^{2,cl}(X,n))$$

the space of paths from 0 to $f^*(\omega)$ in $\mathcal{A}^{2,cl}(X,n)$.

Let $\mathbb{T}_f$ be the relative tangent complex of f, so that we have a distinguished triangle

$$\mathbb{T}_f \to \mathbb{T}_X \to f^*(\mathbb{T}_Y).$$

An isotropic structure $h \in \mathrm{Isot}(f,\omega)$ provides a homotopy between the morphism

$$\mathbb{T}_X \wedge \mathbb{T}_X \to f^*(\mathbb{T}_Y) \wedge f^*(\mathbb{T}_Y) \to \mathcal{O}_X[n]$$

and 0. Then we also get a homotopy from

$$\mathbb{T}_f \otimes \mathbb{T}_X \to \mathbb{T}_X \wedge \mathbb{T}_X \to f^*(\mathbb{T}_Y) \wedge f^*(\mathbb{T}_Y) \to \mathcal{O}_X[n]$$

to 0. There is another homotopy between this morphism and 0, coming from the canonical homotopy from $\mathbb{T}_f \to f^*\mathbb{T}_Y$ to 0. Composing these yields a loop in $\mathrm{Hom}_{L_{QCoh(X)}}(\mathbb{T}_f \otimes \mathbb{T}_X, \mathcal{O}_X[n])$, which is a point of $\mathrm{Hom}_{L_{QCoh(X)}}(\mathbb{T}_f \otimes \mathbb{T}_X, \mathcal{O}_X[n-1])$. By adjunction, we get a map $\Theta_h : \mathbb{T}_f \to \mathbb{L}_X[n-1]$.

Definition 1.5. We say h is *Lagrangian* if Θ_h is a quasi-isomorphism of complexes.

Note that an isotropic or Lagrangian structure is indeed a structure on a map, rather than a property as in the ordinary case. As with the closedness structure, the structure reduces to a condition when we restrict to ordinary varieties.

[2] $\mathcal{A}^2(X,n)^{nd}$ is a union of connected components of $\mathcal{A}^2(X,n)$, so the ordinary product is the homotopy product.

Remark 1.5.1. Note that if $\bullet_n$ denotes the point with trivial n-shifted symplectic structure, then a Lagrangian structure on $X \to \bullet_n$ is just an $(n-1)$-shifted symplectic structure on X.

Lagrangian structures are useful in generating new symplectic spaces:

Theorem 1.6 ([PTVV], Theorem 2.9)**.** *Let X, L_1, L_2 be derived Artin stacks, $\omega \in \mathrm{Symp}(X, n)$ an n-shifted symplectic structure on X, and $f_i : L_i \to X$ a morphism with Lagrangian structure h_i for $i = 1, 2$. Then the product $L_1 \times_X^h L_2$ has a natural $(n-1)$-shifted symplectic structure, which we denote by $R(\omega, h_1, h_2)$.*

Proof. We briefly show the construction of $R(\omega, h_1, h_2)$; the complete proof is given in [PTVV]. Let $Z = L_1 \times_X^h L_2$, and $\pi_i : Z \to L_i$ the projection for $i = 1, 2$. Let $u : f_1 \circ \pi_1 \Rightarrow f_2 \circ \pi_2$ be the natural homotopy.

The Lagrangian structure h_i yields a homotopy $0 \sim f_i^* \omega$ in $\mathcal{A}^{2,cl}(L_i, n)$. Pulling back by π_i gives homotopies

$$\pi_i^* h_i : 0 \sim h_i^* f_i^* \omega,$$

for $i = 1, 2$. The homotopy u gives a homotopy

$$u^* \omega : h_1^* f_1^* \omega \sim h_2^* f_2^* \omega.$$

Concatenating these paths yields a loop in $\mathcal{A}^{2,cl}(L_i, n)$; for concreteness we take $\pi_1^* h_1 + u^* \omega - \pi_2^* h_2$. This defines an element

$$R(\omega, h_1, h_2) \in \pi_1(\mathcal{A}^{2,cl}(Z, n)) \simeq \pi_0(\mathcal{A}^{2,cl}(Z, n-1)).$$

□

Remark 1.6.1. In particular, let X_1 and X_2 be derived Artin stacks with n-shifted symplectic structures ω_1 and ω_2, respectively. Considering ω_1 and ω_2 as Lagrangian structures on the maps $X_1, X_2 \to \bullet_{n+1}$, this theorem provides the symplectic structure $\pi_1^* \omega_1 - \pi_2^* \omega_2$ on $X_1 \times X_2$.

1.2 Symplectic Structures on Mapping Stacks

Let X and Y be derived Artin stacks. The evaluation map $ev : X \times \mathrm{Map}(X, Y) \to Y$ yields a pullback map

$$ev^* : \mathbb{R}\Gamma(Y, \Omega_Y^*) \to \mathbb{R}\Gamma(X \times \mathrm{Map}(X, Y), \Omega^*_{X \times \mathrm{Map}(X,Y)}).$$

If X is $\mathcal{O}$-compact (see [PTVV], Definition 2.1), we have a map

$$\mathbb{R}\Gamma(X \times \mathrm{Map}(X, Y), \Omega^*_{X \times \mathrm{Map}(X,Y)}) \to \mathbb{R}\Gamma(X, \mathcal{O}_X) \otimes \mathbb{R}\Gamma(\mathrm{Map}(X, Y), \Omega^*_{\mathrm{Map}(X,Y)}),$$

which is the Künneth formula $\mathbb{R}\Gamma(X \times \mathrm{Map}(X, Y), \Omega^*_{\mathrm{Map}(X,Y)}) \simeq \mathbb{R}\Gamma(X, \Omega_X^*) \otimes \mathbb{R}\Gamma(\mathrm{Map}(X, Y), \Omega^*_{\mathrm{Map}(X,Y)})$ followed by projection $\mathbb{R}\Gamma(X, \Omega^*(X)) \to \mathbb{R}\Gamma(X, \mathcal{O}_X)$ onto the 0-forms.

Given a "fundamental class" $[X] : \mathbb{R}\Gamma(X, \mathcal{O}_X) \to k[-d]$, we can compose these morphisms to obtain

$$\mathbb{R}\Gamma(Y, \Omega_Y^*) \to \mathbb{R}\Gamma(\mathrm{Map}(X, Y), \Omega^*_{\mathrm{Map}(X,Y)})[-d],$$

which will induce a map on (closed) p-forms:

$$\mathcal{A}^{p(,cl)}(Y, n) \to \mathcal{A}^{p(,cl)}(\mathrm{Map}(X, Y), n-d).$$

If $[X]$ satisfies a certain nondegeneracy condition, the above map will preserve nondegeneracy of forms. We now describe the condition.

For any perfect complex E on X, we let $E^\vee = \mathbb{R}\operatorname{Hom}(E, \mathcal{O}_X)$, and we have a natural pairing

$$\mathbb{R}\Gamma(X, E) \otimes \mathbb{R}\Gamma(X, E^\vee) \xrightarrow{\cup} \mathbb{R}\Gamma(X, \mathcal{O}_X) \xrightarrow{[X]} k[-d] \ ,$$

which is adjoint to a map

$$\mathbb{R}\Gamma(X, E) \xrightarrow{-\cap[X]} \mathbb{R}\Gamma(X, E^\vee)^\vee[-d] \ .$$

More generally, for any $A \in \mathbf{cdga}_k^{\leq 0}$, we let $X_A = X \times \operatorname{Spec} A$, and for any perfect complex E on X_A we have a map

$$\mathbb{R}\Gamma(X_A, E) \xrightarrow{-\cap[X]_A} \mathbb{R}\Gamma(X_A, E^\vee)^\vee[-d] \ .$$

Definition 1.7. We say $[X]$ is a *d-orientation* if for every $A \in \mathbf{cdga}_k^{\leq 0}$ and perfect complex E on X_A, the map $- \cap [X]_A$ is a quasi-isomorphism.

Then we have

Theorem 1.8 ([PTVV], Theorem 2.5)**.** *Let Y be a derived Artin stack, and let X be an $\mathcal{O}$-compact derived stack with a d-orientation $[X]$. Assume the derived mapping stack $\operatorname{Map}(X, Y)$ is a derived Artin stack locally of finite presentation over k. Then we have a map*

$$\int_{[X]} ev^*(-) : \operatorname{Symp}(Y, n) \to \operatorname{Symp}(\operatorname{Map}(X, Y), n - d).$$

Fix $\omega \in \operatorname{Symp}(Y, n)$. Let us describe the induced structure $\int_{[X]} ev^*(\omega)$ now. For any $f \in \operatorname{Map}(X, Y)$, the tangent complex at f is $\mathbb{T}_f \operatorname{Map}(X, Y) \simeq \mathbb{R}\Gamma(X, f^*\mathbb{T}_Y)$. Then the pairing $\bigwedge^2 \mathbb{T}_f \operatorname{Map}(X, Y) \to k[n - d]$ is given by

$$\begin{array}{ccc} \mathbb{R}\Gamma(X, f^*\mathbb{T}_Y) \wedge \mathbb{R}\Gamma(X, f^*\mathbb{T}_Y) & \xrightarrow{\cup} & \mathbb{R}\Gamma(X, f^*\mathbb{T}_Y \wedge f^*\mathbb{T}_Y) \\ & \swarrow_{f^*(\omega)} & \\ \mathbb{R}\Gamma(X, \mathcal{O}_X[n]) & \xrightarrow{[X]} & k[n-d] \end{array}$$

Several examples of orientations are given in [PTVV], following Theorem 2.5. One particular example is the case that X is Calabi-Yau. If X has dimension d and we have an isomorphism $\omega_X \simeq \mathcal{O}_X$, then projection of $\mathbb{R}\Gamma(X, \mathcal{O}_X)$ onto the degree d cohomology $H^d(X, \mathcal{O}_X)[-d]$, followed by the isomorphism

$$H^d(X, \mathcal{O}_X) \simeq H^d(X, \omega_X) \simeq k$$

provides a map $[X] : \mathbb{R}\Gamma(X, \mathcal{O}_X) \to k[-d]$. This is an orientation by Serre duality.

1.2.1 Boundary Structures

The following is due to [Ca]. Let X, Y, and Z be derived Artin stacks. Given a map $f : Z \to X$, we have a pullback map $(- \circ f) : \operatorname{Map}(X, Y) \to \operatorname{Map}(Z, Y)$. Assume that X and Z are $\mathcal{O}$-compact and Z has a d-orientation $[Z]$, that Y has an n-shifted symplectic form ω, and that both mapping spaces are Artin stacks. Then $\operatorname{Map}(Z, Y)$ will have an $(n - d)$-shifted symplectic structure. It is natural to ask when the pullback map $(- \circ f)$ has an isotropic or Lagrangian structure.

Definition 1.9. The *space of boundary structures* on f (with respect to $[Z]$) is

$$\operatorname{Bnd}(f, [Z]) = \operatorname{Path}_{0, f_*[Z]}(\operatorname{Hom}_k(\mathbb{R}\Gamma(X, \mathcal{O}_X), k[-d]))$$

the space of paths from 0 to $f_*[Z]$ in $\operatorname{Hom}_k(\mathbb{R}\Gamma(X, \mathcal{O}_X), k[-d])$.

This definition is dual to the definition of isotropic structures, and it is clear that a boundary structure on f will yield an isotropic structure on $-\circ f$ with respect to $\int_{[Z]} ev_Z^*\omega$, via the identity

$$(-\circ f)^* \int_{[Z]} ev_Z^*\omega = \int_{f_*[Z]} ev_X^*\omega.$$

This can be extended to a dual notion of nondegeneracy (see [Ca], definition 2.8) which guarantees that the isotropic structure is Lagrangian:

Theorem 1.10 ([Ca], Theorem 2.9)**.** *Let X, Y, Z be as above. Then we have a map* $\mathrm{Bnd}(f, [Z]) \to \mathrm{Isot}(f^*, \int_{[Z]} ev_Z^*\omega)$ *sending nondegenerate boundary structures to Lagrangian structures.*

In particular we are interested in the following case. Let X be a geometrically connected smooth proper algebraic variety of dimension $d+1$, and say it has a smooth anticanonical effective divisor D. Then D is a d-dimensional Calabi-Yau variety by the adjunction formula, and so has a d-orientation $[D] : \mathbb{R}\Gamma(D, \mathcal{O}_D) \to k[-d]$. Similarly, using $K_X \simeq \mathcal{O}_\Sigma(-D)$, we get a map $[X] : \mathbb{R}\Gamma(X, \mathcal{O}_X(-D)) \to k[-d-1]$. Then the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \iota^*\mathcal{O}_D \longrightarrow 0$$

gives us a commutative diagram

$$\begin{array}{ccccc} \mathbb{R}\Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}\Gamma(D, \mathcal{O}_D) & \longrightarrow & \mathbb{R}\Gamma(X, \mathcal{O}_X(-D))[1] \\ & & \big\downarrow{\scriptstyle [D]} & & \big\downarrow{\scriptstyle [X]} \\ & & k[-d] & = & k[-d] \end{array} .$$

Since the top row is naturally homotopic to 0, this provides a path between 0 and $\iota_*[D]$, that is, a boundary structure on ι. In fact:

Lemma 1.11 ([Ca], Claim 3.3)**.** *This is a nondegenerate structure.*

And so, using Theorem 1.10, we get

Corollary 1.12. *Let X be a geometrically connected smooth proper algebraic variety of dimension $d+1$, and let D be a smooth anticanonical effective divisor. Let Y have an n-shifted symplectic form $\omega \in$* $\mathrm{Symp}(Y, n)$*. Assume* $\mathrm{Map}(X, Y)$ *and* $\mathrm{Map}(D, Y)$ *are derived Artin stacks.*

Then there exist a natural $(n-d)$-shifted symplectic form on $\mathrm{Map}(D, Y)$ *and Lagrangian structure on* $\mathrm{Map}(X, Y) \to \mathrm{Map}(D, Y)$*.*

Chapter 2

Shifted Poisson Structures

To start, consider a smooth (underived) scheme X with Poisson bivector field π. The Poisson structure is encoded by the sheaf map $\pi^\sharp : T_X^\vee \to T_X$. This map extends to a map $\wedge^p \pi^\sharp : \wedge^p T_X^\vee \to \wedge^p T_X$ such that the square

$$\begin{array}{ccc} \wedge^p T_X^\vee & \xrightarrow{\wedge^p \pi^\sharp} & \wedge^p T_X \\ \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle [\pi,-]} \\ \wedge^{p+1} T_X^\vee & \xrightarrow{\wedge^{p+1} \pi^\sharp} & \wedge^{p+1} T_X \end{array}$$

commutes, where $[-,-]$ is the Schouten bracket. To see this, first note that $[\pi,-]$ has square 0: for $a \in \Gamma(\wedge^p TX, U)$, we have $[\pi,[\pi,a]] = \frac{1}{2}[[\pi,\pi],a]$, but $[\pi,\pi] = 0$ is exactly equivalent to the Jacobi identity for π. Additionally, $[\pi,-]$ is a derivation: $[\pi, ab] = [\pi,a]b + (-1)^a a[\pi,b]$. The claim holds almost by definition for $p = 0$: for $f \in \Gamma(\mathcal{O}_X, U)$, we have

$$[\pi, a] = \iota_{df}\pi = \pi^\sharp(df).$$

Assuming the claim for $p-1$, note that $\Gamma(\wedge^p T_X^\vee, U)$ is generated k-linearly by sections of the form $f d\alpha$, for $\alpha \in \Gamma(\wedge^{p-1} T_X^\vee, U)$. Then

$$\begin{aligned} \wedge^{p+1}\pi^\sharp(d(f d\alpha)) &= \wedge^{p+1}\pi^\sharp(df \wedge d\alpha) \\ &= \pi^\sharp(df) \wedge (\wedge^p \pi^\sharp(d\alpha)), \end{aligned}$$

and

$$\begin{aligned} [\pi, \wedge^p \pi^\sharp(f d\alpha)] &= [\pi, f \wedge^p \pi^\sharp(d\alpha)] \\ &= [\pi, f] \wedge [\pi, \wedge^{p-1}\pi^\sharp(\alpha)] + f[\pi,[\pi,\wedge^{p-1}\pi^\sharp(\alpha)]] \\ &= \pi^\sharp(df) \wedge (\wedge^p \pi^\sharp(d\alpha)). \end{aligned}$$

Thus the map $T^\vee X \to TX$ induces a morphism

$$\mathrm{Sym}^\bullet(\pi^\sharp[-1]) : (\mathrm{Sym}^\bullet(T^\vee X[-1]), d) \to (\mathrm{Sym}^\bullet(TX[-1]), [\pi,-])$$

of graded mixed cdga.

We can then form the derived quotient $[X/\pi^\sharp]$. This is a formal stack equipped with a map $q : X \to [X/\pi^\sharp]$. It satisfies the universal property that a map $f : X \to F$ to a formal derived stack F descends to $\varphi : [X/\pi^\sharp] \to F$ iff the map $\mathrm{Sym}^\bullet(\pi^\sharp[1])$ factors through

$$\psi : (\mathrm{Sym}^\bullet(\mathbb{L}_{f,big}[-1]), d) \to (\mathrm{Sym}^\bullet(TX[-1]), [\pi,-]),$$

and a map ψ of mixed graded cdgas uniquely determines φ.

The structure sheaf of this stack is $(\mathrm{Sym}^\bullet_{\mathcal{O}_X}(TX[-1]), [\pi, -])$. Its tangent complex is

$$\mathbb{T}_{[X/\pi^\sharp]} \simeq \{\ T^\vee X \xrightarrow{\pi^\sharp} TX\ \},$$

with TX sitting in degree 0. Looking at the 2-forms, we have

$$\wedge^2 \mathbb{L}_{[X/\pi^\sharp]} \simeq \{\wedge^2 T^\vee X \to T^\vee X \otimes TX \to \mathrm{Sym}^2\, TX\}.$$

The degree 1 component, $T^\vee X \otimes TX$, contains a canonical section ω corresponding to the identity $TX \to TX$. For this to define a 2-form we need $d\omega = 0$. To see this, note that the image of ω via

$$T^\vee X \otimes TX \to TX \otimes TX$$

is precisely the bivector field π; that this disappears in $\mathrm{Sym}^2\, TX$ is precisely the fact that π is antisymmetric. Nondegeneracy is clear, as the map $\mathbb{T}_{[X/\pi^\sharp]} \to \mathbb{L}_{[X/\pi^\sharp]}[1]$ is literally the identity using the above representatives for $\mathbb{T}_{[X/\pi^\sharp]}$ and $\mathbb{L}_{[X/\pi^\sharp]}$. For closedness, let

$$\zeta \in (\mathcal{O}_{[X/\pi^\sharp]})_1 \otimes (\mathbb{L}_{[X/\pi^\sharp]})_0 \cong TX \otimes T^\vee X$$

be the section corresponding to the identity on TX. Then $d_{dR}\zeta = \omega$, so we have $d_{dR}\omega = 0$. Thus we can take 0 as a closedness structure for ω. (Note that ζ does not define a form on $[X/\pi^\sharp]$, as generally $d\zeta \neq 0$; thus ω is not necessarily exact.)

Thus $[X/\pi^\sharp]$ has a canonical 1-shifted symplectic structure. Looking at $q : X \to [X/\pi^\sharp]$, we see that $q^*\omega$ is a form of degree 1, so it is zero in $\wedge^2 \mathbb{L}_X \simeq \wedge^2 T^\vee X$. Thus q is isotropic with isotropic structure 0. Further, $\mathbb{T}_q \simeq T^\vee X$, and the induced map $\mathbb{T}_q \to \mathbb{L}_X$ is the identity. So in fact, q has a Lagrangian structure. Finally, note that we can recover $\pi^\sharp$ via $T^\vee X \simeq \mathbb{T}_q \to TX$. Thus a Poisson structure on X lets us construct a 1-symplectic formal derived stack Y and $q : X \to Y$ with Lagrangian structure.

Conversely, suppose X is a smooth variety, Y a formal derived stack with 1-shifted symplectic structure ω, and $q : X \to Y$ a map with Lagrangian structure. Then the quasi-isomorphism $\mathbb{T}_q \to \mathbb{L}_X$ gives us a map $\pi^\sharp : T^\vee X \simeq \mathbb{T}_q \to TX$. Using the fiber sequence

$$T^\vee X \xrightarrow{\pi^\sharp} TX \longrightarrow q^*\mathbb{T}_Y\ ,$$

we have

$$q^*\mathbb{T}_Y \simeq \{\ T^\vee X \xrightarrow{\pi^\sharp} TX\ \},$$

with TX sitting in degree 0. Under this identification, we have

$$q^*(\wedge^2 \mathbb{L}_Y) \simeq \{\wedge^2 T^\vee X \to T^\vee X \otimes TX \to \mathrm{Sym}^2\, TX\},$$

with $q^*\omega$ corresponding to the identity in $T^\vee X \otimes TX$. As before, antisymmetry of $\pi^\sharp$ is exactly the fact that $d(q^*\omega) = 0$. In particular, $\pi^\sharp$ corresponds to a bivector field π. For the Jacobi identity, look at the second infinitesimal neighborhood $X_{q,2}$ of X along q. Its structure sheaf is given by $\mathcal{O}_{X_{q,2}} \simeq (\mathrm{Sym}^{\leq 2}_{\mathcal{O}_X}(TX[-1]), [\pi, -])$. For this to be a dg-algebra, we need $[\pi, [\pi, -]] = \frac{1}{2}[[\pi, \pi], -] = 0$ on $\mathcal{O}_X$, so $[\pi, \pi] = 0$, which is the Jacobi identity.

Thus a 1-shifted symplectic formal stack Y and $q : X \to Y$ with Lagrangian structure gives a Poisson structure on X. Note that the actual Poisson structure on X only depended on a formal neighborhood of X in Y; in particular, all the relevant structures involved $q^*\mathbb{L}_Y$ and its various byproducts.

With this in mind, we define:

Definition 2.1. An n-shifted Poisson structure on X is (Y, ω, q, h), where Y is a formal derived stack with an $(n+1)$-shifted symplectic structure $\omega \in \mathrm{Symp}(Y, n+1)$, and $q : X \to Y$ is a map with Lagrangian structure h. Y is called the *Poisson base* of X.

An equivalence of Poisson structures $(Y, \omega, q, h) \to (Y', \omega', q', h')$ is a pair (g, γ) consisting of a map $g : (Y, q) \to (Y', q')$ (in the category of formal derived stacks under X), and a homotopy $\gamma : q^*\omega \sim q^*g^*\omega'$ in $|NC^2(\mathrm{Sym}^\bullet q^*\mathbb{L}_Y[1])|$ such that

$$(q')^*\mathbb{L}_{Y'} \simeq q^*g^*\mathbb{L}_{Y'} \to q^*\mathbb{L}_Y$$

is a quasi-isomorphism, and the image of γ in $\mathcal{A}^{2,cl}(X, n)$ intertwines h and h'.

Remark 2.1.1. As per the remark following Definition 1.5, if $\bullet_{(n+1)}$ denotes the point with trivial $(n+1)$-shifted symplectic structure, then a symplectic structure on a derived Artin stack X is the same as a Lagrangian structure on the map $X \to \bullet_{(n+1)}$. Thus every symplectic structure is naturally Poisson.

Now consider a smooth variety X with Poisson structure π, which we consider in terms of the 1-shifted symplectic structure ω on $[X/\pi^\sharp]$ and the Lagrangian structure on $q : X \to [X/\pi^\sharp]$. Let $s : W \to X$ be a coisotropic subvariety. That is, W is also a smooth variety, and the Poisson structure restricted to the conormal bundle $N^\vee_{W|X} \to T^X \to TX$ factors through the tangent space TW of W. What relation does this have to $[X/\pi^\sharp]$? Let the adjoint of $N^\vee_{W|X} \to TW$ be $\pi^\sharp_W : T^\vee W \to N_{W|X}$; one can show that the morphism of mixed graded cdgas induced by $\pi^\sharp$ descends to $\pi^\sharp_W$, so we have a formal quotient $[W/\pi^\sharp_W]$. In addition, there is a natural map $q' : [W/\pi^\sharp_W] \to [X/\pi^\sharp]$ descending from $W \to X \to [X/\pi^\sharp]$.

We can write

$$\mathbb{T}_{[W/\pi^\sharp_W]} \simeq \{N^\vee_{W|X} \to TW\},$$

with TW in degree 0; thus,

$$\wedge^2\mathbb{L}_{[W/\pi^\sharp_W]} \simeq \{\wedge^2 T^\vee W \to T^\vee W \otimes N_{W|X} \to \mathrm{Sym}^2 N_{W|X}\}.$$

I claim that $q^*\omega = 0$. To see this, note that $\mathrm{Hom}(TX, TX) \to \mathrm{Hom}(TW, N_{W|X})$ sends the identity to the composition $TW \to TX \to N_{W|X}$, which is 0 by definition. Thus q has isotropic structure 0. Further, we have

$$\mathbb{T}_q \simeq \{T^\vee W \to N_{W|X}\},$$

with $T^\vee W$ sitting in degree 0. The map $\mathbb{T}_q \to \mathbb{L}^\vee_{[W/\pi^\sharp_W]}$ is clearly an isomorphism, so the isotropic structure on q is Lagrangian. Then $P = [W/\pi^\sharp_W] \times_{[X/\pi^\sharp]} X$ is a Lagrangian intersection, so it has a 0-shifted symplectic structure. One can check that $\mathbb{T}_P$ is an extension

$$0 \to TW \to \mathbb{T}_P \to T^\vee W \to 0.$$

Let $a : W \to P$ be the natural map; if ω_P is the symplectic form on P, then $a^*\omega_P = 0$, so a is isotropic (with isotropic structure 0). However, we have $\mathbb{T}_a \simeq T^\vee W[-1]$, and $\mathbb{T}_a \to \mathbb{L}_W[-1]$ is the identity. So in fact $a : W \to P$ is isotropic.

Conversely, suppose $s : W \to X$ is a smooth subvariety, that $q' : X' \to Y$ has a Lagrangian structure for some X', and that $a : W \to P := X' \times_{[X/\pi^\sharp]} X$ has a Lagrangian structure. Let $pr_1 : P \to X'$ be the projection and $s' = pr_1 \circ a : W \to X'$. Then we have an exact sequence

$$\mathbb{T}_a \to \mathbb{T}_s \to a^*\mathbb{T}_{pr_1}.$$

Using $\mathbb{T}_a \simeq \mathbb{L}_W[-1]$ from the Lagrangian structure, and $\mathbb{T}_{pr_1} \simeq pr_2^*\mathbb{T}_q \simeq pr_2^* T^\vee X$, we have

$$T^\vee W[-1] \to \mathbb{T}_s \to s^*T^\vee X,$$

so that $\mathbb{T}_s \simeq N^\vee_{W|X}$. Further, the diagram

$$\begin{array}{ccc} \mathbb{T}_s & \longrightarrow & TW \\ \big\downarrow & & \big\downarrow \\ s^*\mathbb{T}_q & \longrightarrow & s^*TX \end{array}$$

commutes, that is, the Poisson map $N^\vee_{W|X} \to T^\vee X \to TX$ factors through $N^\vee_{W|X} \to TW$. So W is coisotropic in the usual sense.

This leads us to define coisotropic structures in general:

Definition 2.2. Let X be a derived Artin stack with n-shifted Poisson structure given by $f : X \to Y$. Let W be a derived Artin stack with a map $g : W \to X$. A *coisotropic structure* on g consists of the following data:

- X' a formal derived stack
- $f' : X' \to Y$
- $g' : W \to X'$
- An homotopy $\eta : f \circ g \simeq f' \circ g'$
- A Lagrangian structure α on f'

Note that the above data define a map $X' \to Y' \times_Y X$, and that $Y' \times_Y X$ has an n-shifted symplectic form by Theorem 1.6. We finally require:

- A Lagrangian structure β on the map $a : X' \to Y' \times_Y X$.

We refer to the map $f' : X' \to Y$ as the *coisotropic base* of $W \to X$.

Let $(X'_i, f'_i, g'_i, \eta_i, \alpha_i, \beta_i)$ be coisotropic structures for $i = 1, 2$. An equivalence of coisotropic structures is a pair (h, γ), where $h : (X'_1, f'_1, g'_1, \eta'_1) \to (X'_2, f'_2, g'_2, \eta'_2)$ is a morphism in the appropriate slice category, and $\gamma : (g'_1)^*\alpha_1 \sim (g'_1)^*h^*\alpha_2$ is a homotopy in $|NC^2(\mathrm{Sym}^\bullet(g'_1)^*\mathbb{L}_{X'_1}[1])|$, such that

$$(g'_2)^*\mathbb{L}_{X'_2} \simeq (g'_1)^*h^*\mathbb{L}_{X'_2} \to (g'_1)^*\mathbb{L}_{X'_1}$$

is a quasi-isomorphism, and if ω_i is the symplectic form on $X'_i \times_Y X$ for $i = 1, 2$, then the homotopy $a_1^*\omega_1 \sim a_2^*\omega_2$ induced by γ intertwines the homotopies β_1, β_2.

Remark 2.2.1. It is clear from the definition that if X has an n-shifted symplectic structure, considered as an n-shifted Poisson structure via $X \to \bullet_{(n+1)}$, then any Lagrangian morphism $Y \to X$ is also coisotropic over $\bullet_n \to \bullet_{(n+1)}$.

Conversely, the classical fact that a morphism which is both coisotropic and isotropic must be Lagrangian is not true. For example, the morphism $\mathbb{A}^1 \to \bullet_1$ is clearly isotropic (with isotropic structure 0), and coisotropic over $T^\vee\mathbb{A}^1[1] \to \bullet_2$. However, a Lagrangian structure on $\mathbb{A}^1 \to \bullet_1$ would be a symplectic structure on $\mathbb{A}^1$ (in the classical sense), which clearly does not exist.

Definition 2.3. Let X be a derived Artin stack with Poisson structure given by $\mathcal{P}_2 = (Y_2, \omega_2, f_2, h_2)$, and $g : W \to X$ with coisotropic structure $(X', f', g', \eta, \alpha, \beta)$. Let $\mathcal{P}_1 = (Y_1, \omega_1, f_1, h_1)$ be another Poisson structure and $(k, \gamma) : \mathcal{P}_1 \to \mathcal{P}_2$ an equivalence. The pullback of the coisotropic structure via h is as follows. Let $\tilde{Y}_1$ be a formal neighborhood of X in Y_1.

- Let $X'_1 = X' \times_{Y_2} \tilde{Y}_1$.
- The map $f'_1 : X'_1 \to Y_1$ is $X'_1 \to \tilde{Y}_1 \to Y_1$.

- $g_1' : W \to X_1'$ is induced by $W \to X'$ and $W \to X \to \tilde{Y_1}$.
- $\eta_1 = id : f_1 \circ g \sim f_1' \circ g_1'$.
- Let $pr : X_1' \to X'$ be the projection. The homotopy γ yields a homotopy

$$(f_1')^* \omega_1 \sim (f_1')^* k^* \omega_2 \sim pr^* (f')^* \omega_2.$$

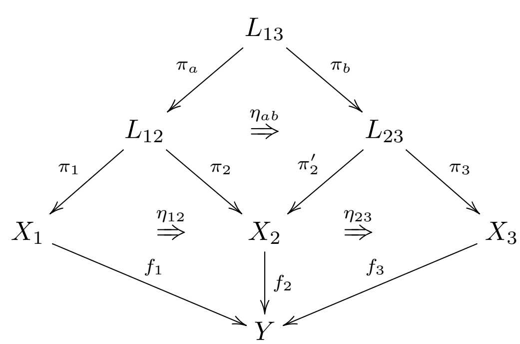
  Then α_1 is this homotopy followed by $pr^* \alpha$. The Lagrangian condition follows from the Lagrangian condition for α, using the fact that $\tilde{Y_1} \to Y_2$ and pr are etale.

- Let ω_2 be the induced symplectic structure on $P_2 := X' \times_{Y_2} X$. Let ω_1 be the induced symplectic structure on $P_1 := X_1' \times_{Y_1} X$. Let $r : P_1 \to P_2$ be the natural map. The homotopy induced by γ in the previous point also gives a homotopy $\omega_1 \sim r^* \omega_2$. Following this with $r^* \beta$ yields β_1. As in the previous point, the Lagrangian condition follows from etaleness of all relevant maps and Lagrangianness of β.

Lemma 2.4. *Let X_1, X_2, X_3, Y be derived Artin stacks, and let $\omega \in \mathrm{Symp}(Y, n)$ be an n-shifted symplectic structure on Y. For $i = 1, 2, 3$, let $f_i : X_i \to Y$ be a morphism with Lagrangian structure h_i. Note that any product $X_i \times_Y X_j$ has a canonical $(n-1)$-shifted symplectic structure. Let $g_{12} : L_{12} \to X_1 \times_Y X_2$ and $g_{23} : L_{23} \to X_2 \times_Y X_3$ be morphisms with Lagrangian structures k_{12}, k_{23} respectively.*

Then $g_{13} : L_{13} := L_{12} \times_{X_2} L_{23} \to X_1 \times_Y X_3$ has a canonical Lagrangian structure.

Proof. Let $\pi_1 : L_{12} \to X_1$ and $\pi_2 : L_{12} \to X_2$ be the projections, and let $\eta_{12} : f_1 \circ \pi_1 \to f_2 \circ \pi_2$ be the natural equivalence of morphisms. If $\omega_{12} \in \mathrm{Symp}(X_1 \times_Y X_2, n-1)$ is the symplectic form given by Theorem 1.6, then $g_{12}^* \omega_{12} = \pi_1^* h_1 + \eta_{12}^* \omega - \pi_2^* h_2$. Then k_{12} gives a path from 0 to this form. Similarly if $\pi_2' : L_{23} \to X_2$ and $\pi_3 : L_{23} \to X_3$ are the projections and $\eta_{23} : f_2 \circ \pi_2' \to f_3 \circ \pi_3$ the equivalence of morphisms, then k_{23} is a path from 0 to $(\pi_2')^* h_2 + \eta_{23}^* \omega - \pi_3^* h_3$ in $\mathcal{A}^{2,cl}(L_{23}, n-1)$.



Now let $\pi_a : L_{13} \to L_{12}$ and $\pi_b : L_{13} \to L_{23}$ be the projections and $\eta_{ab} : \pi_2 \circ \pi_a \to \pi_2' \circ \pi_b$ the natural equivalence. Then in $\mathcal{A}^{2,cl}(L_{13}, n-1)$ we have paths

$$\begin{aligned}
\pi_a^* k_{12} &: 0 \sim \pi_a^* \pi_1^* h_1 + \pi_a^* \eta_{12}^* \omega - \pi_a^* \pi_2^* h_2 \\
\pi_b^* k_{23} &: 0 \sim \pi_b^* (\pi_2')^* h_2 + \pi_b^* \eta_{23}^* \omega - \pi_b^* \pi_3^* h_3 \\
\eta_{ab}^* h_2 &: 0 \sim \pi_a^* \pi_2^* h_2 + \eta_{ab}^* f_2^* \omega - \pi_b^* (\pi_2')^* h_2.
\end{aligned}$$

Composing these gives

$$0 \sim \pi_a^* \pi_1^* h_1 + \eta_{13}^* \omega - \pi_b^* \pi_3^* h_3, \qquad (*)$$

where

$$\eta_{13} = (\pi_b \eta_{23}) \circ (\eta_{ab} f_2) \circ (\pi_a \eta_{12}) : \pi_b \circ \pi_3 \circ f_3 \to \pi_a \circ \pi_1 \circ f_1$$

is the equivalence. If $\omega_{13} \in \mathrm{Symp}(X_1 \times_Y X_3, n-1)$ is the symplectic form, then (*) is exactly the isotropic structure $0 \sim g_{13}^*\omega_{13}$ we need.

For Lagrangianness, we use the diagram

$$\begin{array}{ccccccc} \mathbb{T}_{L_{13}} & \longrightarrow & \mathbb{T}_{L_{12}} \oplus \mathbb{T}_{L_{23}} & \longrightarrow & \mathbb{T}_{X_2} & \overset{+1}{\longrightarrow} & . \\ \downarrow & & \downarrow \wr & & \downarrow \wr & & \\ \mathbb{L}_{g_{13}}[n-2] & \longrightarrow & (\mathbb{L}_{g_{12}} \oplus \mathbb{L}_{g_{23}})[n-2] & \longrightarrow & \mathbb{L}_{f_2}[n-1] & \overset{+1}{\longrightarrow} & \end{array}$$

The rows are exact and two of the three vertical maps are quasi-isomorphisms, so the third is as well. ☐

Restated in Poisson language, the above is a generalization of Theorem 1.6:

Corollary 2.5. *Let X have an n-shifted Poisson structure given by $f : X \to Y$. For $i = 1, 2$, let $g_i : X_i' \to X$ be coisotropic with coisotropic base $h_i : Y_i' \to Y$. Then $X_1 \times_X X_2$ is $(n-1)$-shifted Poisson with base $Y_1 \times_Y Y_2$.*

Now let us generalize the situation of mapping spaces. It is relatively clear that Lagrangian structures descend to mapping spaces:

Theorem 2.6 ([Ca], Theorem 2.10)**.** *Let X, Y, Z be derived Artin stacks and $f : Y \to Z$ a map. Assume X is $\mathcal{O}$-compact with d-orientation $[X]$. Assume the stacks $\mathrm{Map}(X, Y)$ and $\mathrm{Map}(X, Z)$ are derived Artin stacks locally of finite presentation over k. Then we have a map*

$$\int_{[X]} ev^*(-) : \mathrm{Lagr}(f, \omega) \to \mathrm{Lagr}(f \circ -, \int_{[X]} ev^*(\omega)),$$

that is, from Lagrangian structures on f to Lagrangian structures on $(f \circ -)$.

Again, using the language of Poisson structures, we have

Corollary 2.7. *Let Y have an n-shifted Poisson structure with base Z. Let X be $\mathcal{O}$-compact with d-orientation $[X]$. Assume the stacks $\mathrm{Map}(X, Y)$ and $\mathrm{Map}(X, Z)$ are derived Artin stacks locally of finite presentation over k. Then $\mathrm{Map}(X, Y)$ has an $(n-d)$-shifted Poisson structure with base $\mathrm{Map}(X, Z)$.*

We also have a variant of Theorem 2.6 to the coisotropic case:

Theorem 2.8. *Let Y be n-shifted Posison with base Z, and let $g : Y' \to Y$ be coisotropic with base $h : Z' \to Z$. Let X be $\mathcal{O}$-compact with d-orientation $[X]$. Assume the stacks $\mathrm{Map}(X, Y)$, $\mathrm{Map}(X, Z)$, $\mathrm{Map}(X, Y')$, and $\mathrm{Map}(X, Z')$ are derived Artin stacks locally of finite presentation over k. Then $\mathrm{Map}(X, Y') \to \mathrm{Map}(X, Y)$ is coisotropic with base $\mathrm{Map}(X, Z') \to \mathrm{Map}(X, Z)$.*

Proof. By Theorem 2.6, the maps $\mathrm{Map}(X, Z') \to \mathrm{Map}(X, Z)$, $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$, and $\mathrm{Map}(X, Y') \to \mathrm{Map}(X, Z' \times_Z Y)$ have natural Lagrangian structures. But

$$\mathrm{Map}(X, Z' \times_Z Y) \cong \mathrm{Map}(X, Z') \times_{\mathrm{Map}(X,Z)} \mathrm{Map}(X, Y)$$

as symplectic spaces. ☐

And similarly of Theorem 1.10:

Theorem 2.9. *Let Y be n-shifted Poisson given by $f : Y \to Z$ and Lagrangian structure $h : 0 \sim \omega$. Let $g : W \to X$ be a map of $\mathcal{O}$-compact derived Artin stacks, and let $[W]$ be a d-orientation on W and γ a boundary structure on g.*

Then $\mathrm{Map}(X, Y) \to \mathrm{Map}(W, Y) \times_{\mathrm{Map}(W,Z)} \mathrm{Map}(X, Z)$ has a natural Lagrangian structure. Equivalently, $\mathrm{Map}(X, Y) \to \mathrm{Map}(W, Y)$ has a coisotropic structure over $\mathrm{Map}(X, Z) \to \mathrm{Map}(W, Z)$.

Proof. $\mathrm{Map}(W,Z)$ has symplectic structure $\int_{[W]} ev_W^*\omega$. The Lagrangian structure on $\mathrm{Map}(W,Y) \to \mathrm{Map}(W,Z)$ is given by

$$\int_{[W]} ev_W^* h : 0 \sim \int_{[W]} ev_W^* g^*\omega = (g \circ -)^* \int_{[W]} ev_W^*\omega.$$

The Lagrangian structure on $\mathrm{Map}(X,Z) \to \mathrm{Map}(W,Z)$ is given by

$$\int_\gamma ev_X^*\omega : 0 \sim \int_{f_*[W]} ev_X^*\omega = (- \circ f)^* \int_{[W]} ev_W^*\omega.$$

Let $\tilde{\omega}$ be the induced symplectic structure on $\mathrm{Map}(W,Y) \times_{\mathrm{Map}(W,Z)} \mathrm{Map}(X,Z)$, and let $r : \mathrm{Map}(X,Y) \to \mathrm{Map}(W,Y) \times_{\mathrm{Map}(W,Z)} \mathrm{Map}(X,Z)$ be the natural map. Then

$$r^*\tilde{\omega} = \int_{f_*[W]} ev_X^* h - \int_\gamma ev_W^* g^*\omega,$$

and the isotropy is given by

$$\int_\gamma ev_X^* h : 0 \sim r^*\tilde{\omega}.$$

For the Lagrangian condition, fix a dga A and $\sigma : \mathrm{Spec}\, A \to \mathrm{Map}(X,Y)$ corresponding to $\tilde{\sigma} : X \times \mathrm{Spec}\, A \to Y$. Let $\pi_2 : X \times \mathrm{Spec}\, A \to \mathrm{Spec}\, A$ be the projection. Then

$$\sigma^*\mathbb{T}_r \simeq (\pi_2)_* \,\mathrm{HoFib}(\tilde{\sigma}^*\mathbb{T}_g \to (f \times 1_{\mathrm{Spec}\, A})_*(f \times 1_{\mathrm{Spec}\, A})^*\tilde{\sigma}^*\mathbb{T}_g)$$

and

$$\sigma^*\mathbb{L}_{\mathrm{Map}(X,Y)} \simeq ((\pi_2)_*\tilde{\sigma}^*\mathbb{T}_Y)^\vee = \mathrm{Hom}\left((\pi_2)_*\tilde{\sigma}^*\mathbb{T}_Y, \mathcal{O}_{\mathrm{Spec}\, A}\right).$$

The map $\sigma^*\mathbb{T}_r \to \sigma^*\mathbb{L}_{\mathrm{Map}(X,Y)}[n-d-1]$ is induced by the maps $\mathbb{T}_g \to \mathbb{L}_Y[n]$, a quasi-isomorphism given by the Lagrangian structure h, and

$$(\pi_2)_* \,\mathrm{HoFib}(\tilde{\sigma}^*\mathbb{L}_Y \to (f \times 1_{\mathrm{Spec}\, A})_*(f \times 1_{\mathrm{Spec}\, A})^*\tilde{\sigma}^*\mathbb{L}_Y) \to ((\pi_2)_*\tilde{\sigma}^*\mathbb{T}_Y)^\vee[d-1],$$

a quasi-isomorphism given by the nondegenerate boundary structure. □

Specifically, we want to generalize the case of 1.12:

Corollary 2.10. *Let X be a geometrically connected smooth proper algebraic variety of dimension $d+1$, and let D be a smooth anticanonical effective divisor. Let Y have an n-shifted Poisson structure given by $Y \to Z$. Assume $\mathrm{Map}(X,Y)$, $\mathrm{Map}(D,Y)$,$\mathrm{Map}(X,Z)$, and $\mathrm{Map}(D,Z)$ are derived Artin stacks.*

Then there exist a natural $(n-d)$-shifted Poisson structure on $\mathrm{Map}(D,Y)$ (over $\mathrm{Map}(D,Z)$) and coisotropic structure on $\mathrm{Map}(X,Y) \to \mathrm{Map}(D,Y)$ (over $\mathrm{Map}(X,Z) \to \mathrm{Map}(D,Z)$).

Finally, we need more technical "Poisson generalizations" of some results. The following is a generalization of the first statement of Theorem 2.9:

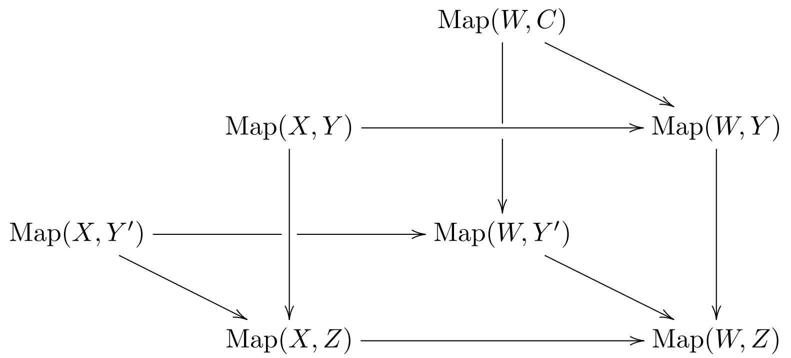
Corollary 2.11. *Let Y be n-shifted Poisson given by $f : Y \to Z$. Let $C \to Y$ be coisotropic over $Y' \to Z$. Let $g : W \to X$ be a map of $\mathcal{O}$-compact derived Artin stacks, and let $[W]$ be a d-orientation on W and γ a boundary structure on g.*

Then $\mathrm{Map}(X,C) \to \mathrm{Map}(W,C) \times_{\mathrm{Map}(W,Y)} \mathrm{Map}(X,Y)$ has a natural coisotropic structure.

Proof. Recall that the Poisson structure on $\Psi := \mathrm{Map}(W, C) \times_{\mathrm{Map}(W,Y)} \mathrm{Map}(X, Y)$ is given by

$$\Psi \to \Xi := \mathrm{Map}(X, Z) \times_{\mathrm{Map}(W,Z)} \mathrm{Map}(W, Y'),$$

with a natural Lagrangian structure, as per Corollary 2.5. I claim that$\mathrm{Map}(X, C) \to \Psi$ is coisotropic over $\mathrm{Map}(X, Y') \to \Psi$. First, the Lagrangian structure on $\mathrm{Map}(X, Y') \to \Psi$ is exactly the one given by Theorem 2.9. Next, note that $\mathrm{Map}(X, Y') \times_\Psi \Xi$ is exactly the limit of the diagram



which we can rewrite as $\mathrm{Map}(X, Y \times_Z Y') \times_{\mathrm{Map}(W,Y\times_Z Y')} \mathrm{Map}(W, C)$. This space has a symplectic form arising from the form on $Y \times_Z Y'$, which will agree with the structure on $\mathrm{Map}(X, Y') \times_\Psi \Xi$ up to sign. But the fact that

$$\mathrm{Map}(X, C) \to \mathrm{Map}(X, Y \times_Z Y') \times_{\mathrm{Map}(W,Y\times_Z Y')} \mathrm{Map}(W, C)$$

has a Lagrangian structure is precisely Theorem 2.9. □

This is a generalization of Lemma 2.4:

Corollary 2.12. *Let X_1, X_2, X_3, Y be derived Artin stacks, and let Y have an n-shifted Poisson structure given by $Y \to Z$ for some $(n+1)$-shifted symplectic Z. For $i = 1, 2, 3$, let $f_i : X_i \to Y$ be a morphism coisotropic over $Y_i' \to Z$. Note that any product $X_i \times_Y X_j$ has a canonical $(n-1)$-shifted Poisson structure over $Y_i' \times_Y Y_j'$. Let $g_{12} : C_{12} \to X_1 \times_Y X_2$ and $g_{23} : C_{23} \to X_2 \times_Y X_3$ be morphisms coisotropic over $L_{12} \to Y_1' \times_Y Y_2'$ and $L_{23} \to Y_2' \times_Y Y_3'$, respectively.*

Then $C_{13} := C_{12} \times_{X_2} C_{23} \to X_1 \times_Y X_3$ has a canonical coisotropic structure over $L_{12} \times_{Y_2'} L_{23} \to Y_1' \times_Z Y_3'$.

Proof. We need to show that

$$C_{13} \to T := (X_1 \times_Y X_3) \times_{Y_1'\times_Z Y_3'} (L_{12} \times_{Y_2'} L_{23})$$

has a Lagrangian structure. As in the previous proof, writing T as a limit gives us

$$T \cong (X_1 \times_{Y_1'} L_{12}) \times_{Y\times_Z Y_2'} (X_3 \times_{Y_3'} L_{23}).$$

The base is again symplectic, and the maps

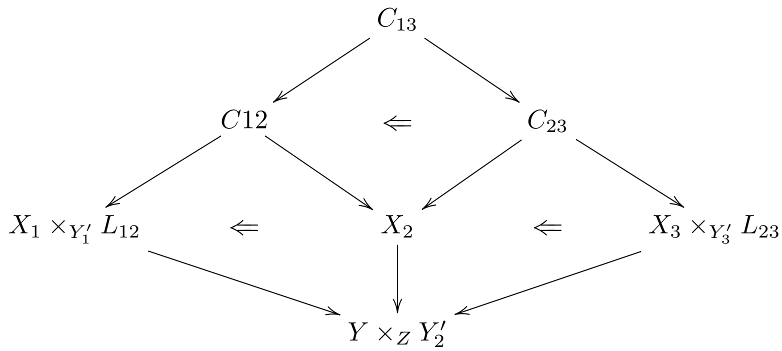
$$X_1 \times_{Y_1'} L_{12} \to Y \times_Z Y_2' \leftarrow X_3 \times_{Y_3'} L_{23}$$

have Lagrangian structures provided by Lemma 2.4. Thus this expresses T as a Lagrangian intersection, which again has a symplectic structure that agrees with the original structure on T up to sign. Now, $X_2 \to Y \times_Z Y_2'$ has a Lagrangian structure by assumption, and further rearrangement gives a Lagrangian structure on

$$C_{12} \to (X_1 \times_Y X_2) \times_{Y_1'\times_Z Y_2'} L_{12} \cong (X_1 \times_{Y_1'} L_{12}) \times_{Y\times_Z Y_2'} X_2,$$

and similarly for C_{23}.

But then we can apply Lemma 2.4 to get the Lagrangian structure on $C_{13} \to T$.



$\square$

Chapter 3

Framed Mapping Spaces

Definition 3.1. Let D, X, and Y be derived Artin stacks, and fix maps $i : D \to X$ and $f : D \to Y$. We define the *framed mapping space* $\mathrm{Map}(X, D, Y, f) = \mathrm{HoFib}_f(\mathrm{Map}(X,Y) \to \mathrm{Map}(D,Y))$, the homotopy fiber of $\mathrm{Map}(X,Y)$ over $f \in \mathrm{Map}(D,Y)$. Where f is understood we will write $\mathrm{Map}(X, D, Y)$.

In the following, X will generally be a smooth scheme and $i : D \to X$ the inclusion of a divisor; or X and D will both be divisors in some smooth scheme.

Now, for any $g : X \to Y$ framed along D, let us consider $(\mathbb{T}_{\mathrm{Map}(X,D,Y)})_g$. We have an exact sequence

$$\begin{array}{ccccc} (\mathbb{T}_{\mathrm{Map}(X,D,Y)})_g & \longrightarrow & (\mathbb{T}_{\mathrm{Map}(X,Y)})_g & \longrightarrow & (\mathbb{T}_{\mathrm{Map}(D,Y)})_(i \circ g)\ , \\ & & \downarrow \wr & & \downarrow \wr \\ & & \mathbb{R}\Gamma(g^*\mathbb{T}_Y, X) & & \mathbb{R}\Gamma(i^*g^*\mathbb{T}_Y, D) \end{array}$$

so we can identify $(\mathbb{T}_{\mathrm{Map}(X,D,Y)})_g \simeq \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, X)$, where $(g^*\mathbb{T}_Y)_{-D}$ is the subsheaf of $g^*\mathbb{T}_Y$ vanishing on D. In our cases we will be able to write $D = V(a)$ locally, so $(g^*\mathbb{T}_Y)_{-D} \simeq a(g^*\mathbb{T}_Y)$.

More globally, let $ev : X \times \mathrm{Map}(X, D, Y) \to Y$ be the evaluation map and $\pi : X \times \mathrm{Map}(X, D, Y) \to \mathrm{Map}(X, D, Y)$ the projection. Then $\mathbb{T}_{\mathrm{Map}(X,D,Y)} \simeq \pi_* \left((ev^*\mathbb{T}_Y)_{-(D\times \mathrm{Map}(X,D,Y))}\right)$.

For $p \geq 0$ we have a cup product map

$$\wedge^p \mathbb{T}_{\mathrm{Map}(X,Y)} \sim \wedge^p(\pi_* ev^*\mathbb{T}_Y) \to \pi_* \wedge^p (ev^*\mathbb{T}_Y).$$

This induces a map

$$\pi_*(\wedge^p ev^*\mathbb{L}_Y) \to (\pi_*(\wedge^p ev^*\mathbb{T}_Y))^\vee \to \wedge^p \mathbb{L}_{\mathrm{Map}(X,Y)}.$$

This map is compatible with the mixed structure on both sides, so descends to the level of negative cyclic complexes:

$$\pi_* ev^*(NC(Y)) \to NC(\mathrm{Map}(X,Y)).$$

With this in mind, we define a special class of forms on $\mathrm{Map}(X,Y)$:

Definition 3.2. A p-form on $\mathrm{Map}(X,Y)$ (resp. closed p-form) is *multiplicative* if the corresponding map $\mathcal{O}_{\mathrm{Map}(X,Y)} \to \wedge^p \mathbb{L}_{\mathrm{Map}(X,Y)}$ factors through $\pi_*(\wedge^p ev^*\mathbb{L}_Y)$ (resp. factors through $\pi_* ev^*(NC(Y))$).

Note that all forms obtained from the $\int_{[X]} ev^*(-)$ map of Theorem 1.8 are multiplicative.

The importance of multiplicative forms is as follows. Suppose $\mathcal{E}_1, \mathcal{E}_2 \to ev^*\mathbb{T}_Y$ are two sheaves on $\mathrm{Map}(X,Y)$ which are orthogonal in the sense that the multiplication map $\mathcal{E}_1 \otimes \mathcal{E}_2 \to \wedge^2 ev^*\mathbb{T}_Y$ is 0. Then for any 2-form ω, we have a pullback via

$$\wedge^2 \mathbb{L}_{\mathrm{Map}(X,Y)} \to (\pi_*\mathcal{E}_1)^\vee \otimes (\pi_*\mathcal{E}_2)^\vee.$$

If ω is multiplicative, then we can lift the pullback through $\pi_*(\wedge^2 ev^*\mathbb{L}_Y) \to \pi_*(\mathcal{E}_1^\vee \otimes \mathcal{E}_2^\vee)$, which is the 0 map. Thus the pullback will be 0.

We want to generalize the case of Theorem 1.8 and Corollary 2.7 to spaces $\mathrm{Map}(X, D, Y)$. The main theorem of this section is

Theorem 3.3. *Let X be a d-dimensional proper smooth scheme and D an effective divisor. Suppose E is an effective divisor of X such that $\tilde{D} = 2D+E$ is anticanonical. Let Y be a derived Artin stack such that $\mathrm{Map}(X,Y)$, $\mathrm{Map}(\tilde{D},Y)$, $\mathrm{Map}(D,Y)$, and $\mathrm{Map}(D+E,Y)$ are themselves derived Artin stacks of locally finite presentation over k. Fix a base map $f : D \to Y$.*

1. *Suppose Y is n-shifted symplectic and the projection $\mathrm{Map}(D+E,Y) \to \mathrm{Map}(D,Y)$ is etale over f. Then $\mathrm{Map}(X,D,Y)$ has an $(n-d)$-shifted symplectic structure.*
2. *Suppose Y is n-shifted Poisson. Then $\mathrm{Map}(X,D,Y)$ has an $(n-d)$-shifted Poisson structure.*

Proof. Consider the fiber diagram

$$\begin{array}{ccccc} \mathrm{Map}(X,D,Y) & \longrightarrow & \mathrm{Map}(\tilde{D},D,Y) & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(X,Y) & \longrightarrow & \mathrm{Map}(\tilde{D},Y) & \longrightarrow & \mathrm{Map}(D,Y) \end{array},$$

where both squares and the larger rectangle are Cartesian. Then $\mathrm{Map}(\tilde{D},Y)$ is $(n-d+1)$-shifted symplectic (resp. Poisson) by Theorem 1.8 (Corollary 2.7), and $\mathrm{Map}(X,Y) \to \mathrm{Map}(\tilde{D},Y)$ has a canonical Lagrangian structure (coisotropic structure) by Corollary 1.12 (Corollary 2.10). If we can show that $\mathrm{Map}(\tilde{D},D,Y) \to \mathrm{Map}(\tilde{D},Y)$ has a Lagrangian structure (coisotropic structure) as well, we will be done by Theorem 1.6 (Corollary 2.5). We state this as a separate lemma:

Lemma 3.4. *Let $X, Y, D, \tilde{D}$ be as in the theorem. Then*

1. *Suppose Y is n-shifted symplectic and the projection $\mathrm{Map}(D+E,Y) \to \mathrm{Map}(D,Y)$ is etale over f. Then $\varphi : \mathrm{Map}(\tilde{D},D,Y) \to \mathrm{Map}(\tilde{D},Y)$ has an canonical Lagrangian structure.*
2. *Suppose Y is n-shifted Poisson. Then $\mathrm{Map}(X,D,Y)$ has a canonical coisotropic structure.*

Proof. Let $i : D \to \tilde{D}$ be the inclusion, and let $g : \tilde{D} \to Y$ be a map such that $f = g \circ i$. Then $\mathbb{T}_g \mathrm{Map}(\tilde{D},D,Y) \simeq \mathbb{R}\Gamma(\mathrm{HoFib}(g^*\mathbb{T}_Y \to i_*i^*g^*\mathbb{T}_Y), \tilde{D})$. Let us write this as $\mathbb{T}_g \mathrm{Map}(\tilde{D},D,Y) \simeq \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})$. Similarly, for any extension of $f : D \to Y$ to $\tilde{f} : D+E \to Y$, we have $\mathbb{T}_g \mathrm{Map}(\tilde{D}, D+E, Y, \tilde{f}) \simeq \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-(D+E)}, \tilde{D})$.

Let us consider (1). The multiplication

$$(g^*\mathbb{T}_Y)_{-D} \otimes (g^*\mathbb{T}_Y)_{-(D+E)} \to \wedge^2 g^*\mathbb{T}_Y$$

is zero; in an affine local patch of X, if $D = V(a)$ and $E = V(b)$, then the map on sheaves is

$$a(g^*\mathbb{T}_Y) \otimes ab(g^*\mathbb{T}_Y) \to \wedge^2 g^*\mathbb{T}_Y$$

and $a^2b = 0$ on $2D+E$.

Now, the symplectic structure on $\mathrm{Map}(\tilde{D},Y)$ is a multiplicative form. Thus, $\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})$ and $\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-(D+E)}, \tilde{D})$ are orthogonal in $\mathbb{T}_g \mathrm{Map}(\tilde{D},Y) \simeq \mathbb{R}\Gamma(g^*\mathbb{T}_Y, \tilde{D})$ under this structure. Thus the map

$$\begin{aligned} \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) &\to \mathbb{T}_g \mathrm{Map}(\tilde{D},Y) \to \mathbb{L}_g \mathrm{Map}(\tilde{D},Y)[n-d+1] \\ &\to \left(\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-(D+E)}, \tilde{D})\right)^\vee [n-d+1] \end{aligned}$$

is 0, so

$$\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) \to \mathbb{L}_g \operatorname{Map}(\tilde{D}, Y)[n-d+1] \simeq \Big(\mathbb{R}\Gamma(g^*\mathbb{T}_Y, \tilde{D})\Big)^\vee [n-d+1]$$

factors through a map

$$\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) \to \Big(\mathbb{R}\Gamma(g^*\mathbb{T}_Y \otimes \mathcal{O}_{D+E}, \tilde{D})\Big)^\vee [n-d+1]. \qquad (*)$$

In fact, extending g to a map $\tilde{g} : X \to Y$, consider the diagram

$$\begin{array}{ccccc} \tilde{g}^*\mathbb{T}_Y(-2D-E) & \longrightarrow & \tilde{g}^*\mathbb{T}_Y(-2D-E) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{g}^*\mathbb{T}_Y(-D) & \longrightarrow & \tilde{g}^*\mathbb{T}_Y & \longrightarrow & \tilde{g}^*\mathbb{T}_Y \otimes \mathcal{O}_D \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{g}^*\mathbb{T}_Y(-D) \otimes \mathcal{O}_{D+E} & \longrightarrow & \tilde{g}^*\mathbb{T}_Y \otimes \mathcal{O}_{2D+E} & \longrightarrow & \tilde{g}^*\mathbb{T}_Y \otimes \mathcal{O}_D \end{array}.$$

As all columns and the first two rows are distinguished triangles, so is the last row; restricting back to $\tilde{D}$, we get a quasi-isomorphism

$$(g^*\mathbb{T}_Y)_{-D} \simeq g^*\mathbb{T}_Y \otimes \mathcal{O}_{D+E}(-D) \simeq g^*\mathbb{T}_Y \otimes j_* K_{D+E},$$

where $j : D+E \to \tilde{D}$ is the inclusion.

Then the map (*) can be rewritten as a map

$$\mathbb{R}\Gamma(j^*g^*\mathbb{T}_Y \otimes K_{D+E}, D+E) \to (\mathbb{R}\Gamma(j^*g^*\mathbb{T}_Y, D+E))^\vee [n-d+1].$$

This is just the quasi-isomorphism

$$\mathbb{R}\Gamma(j^*g^*\mathbb{T}_Y \otimes K_{D+E}, D+E) \to \mathbb{R}\Gamma(j^*g^*\mathbb{L}_Y \otimes K_{D+E}, D+E)[n]$$

given by the symplectic structure on Y, followed by the Serre duality quasi-isomorphism

$$\mathbb{R}\Gamma((j^*g^*\mathbb{T}_Y)^\vee \otimes K_{D+E}, D+E)[n] \to \mathbb{R}\Gamma(j^*g^*\mathbb{T}_Y, D+E)^\vee[n-d+1].$$

In particular, (*) is a quasi-isomorphism.

Now let us consider case (1). The etaleness assumption gives us that

$$\mathbb{R}\Gamma(g^*\mathbb{T}_Y \otimes \mathcal{O}_{D+E}, \tilde{D}) \to \mathbb{R}\Gamma(g^*\mathbb{T}_Y \otimes \mathcal{O}_D, \tilde{D})$$

is a quasi-isomorphism, and so

$$\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-(D+E)}, \tilde{D}) \to \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})$$

is as well. Thus the map

$$\wedge^2\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) \to \wedge^2\mathbb{R}\Gamma(g^*\mathbb{T}_Y, \tilde{D})$$

is 0, so 0 is an isotropic structure on φ. In addition, the map $\mathbb{T}_\varphi \to \mathbb{L}_{\operatorname{Map}(\tilde{D},D,Y)}[n-d]$ is precisely the map (*) shifted by 1. This is a quasi-isomorphism, so we have a Lagrangian structure on φ.

Now consider case (2). Suppose Y has a Poisson structure given by $p : Y \to Z$, where Z has an $(n+1)$-shifted symplectic structure ω and p has a Lagrangian structure γ. Recall that the $(n-d+1)$-shifted Poisson structure on $\operatorname{Map}(\tilde{D}, Y)$ is given by $\operatorname{Map}(\tilde{D}, Y) \to \operatorname{Map}(\tilde{D}, Z)$, with the symplectic and Lagrangian structures induced from ω and γ. I claim that $\operatorname{Map}(\tilde{D}, D, Y) \to$

$\mathrm{Map}(\tilde{D}, Y)$ is coisotropic. The base B will be the formal neighborhood of $\mathrm{Map}(\tilde{D}, D, Y)$ in $\mathrm{Map}(D+E, D, Y) \times_{\mathrm{Map}(D+E,D,Z)} \mathrm{Map}(\tilde{D}, D, Z)$, and $q : B \to \mathrm{Map}(\tilde{D}, D, Z)$ comes from the projection $\mathrm{Map}(D+E, D, Y) \times_{\mathrm{Map}(D+E,D,Z)} \mathrm{Map}(\tilde{D}, D, Z)$ followed by $\mathrm{Map}(\tilde{D}, D, Z) \to \mathrm{Map}(\tilde{D}, Z)$.

First let's find a convenient representation of $\mathbb{T}_B$. Let $g \in \mathrm{Map}(\tilde{D}, Y)$. In the diagram

$$\begin{array}{ccccc}
\mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)}, \tilde{D}) & \longrightarrow & \mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-D}, \tilde{D}) & \longrightarrow & \mathbb{R}\Gamma((j^*g^*\mathbb{T}_p)_{-D}, D+E) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) & \longrightarrow & \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) \oplus \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, D+E) & \longrightarrow & \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, D+E) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{T}_B)_g & \longrightarrow & \mathbb{R}\Gamma((g^*p^*\mathbb{T}_Z)_{-D}, \tilde{D}) \oplus \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, D+E) & \longrightarrow & \mathbb{R}\Gamma((j^*g^*p^*\mathbb{T}_Z)_{-D}, D+E)
\end{array}$$

the last two columns and all rows are triangles, so the first column is as well. Thus we have

$$(\mathbb{T}_B)_g \simeq \mathrm{HoCofib}(\mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)}, \tilde{D}) \to \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})). \qquad (**)$$

For the Lagrangian structure on q, let us identify $q^*\Omega$, where Ω is the symplectic structure on $\mathrm{Map}(\tilde{D}, Z)$. For $\ell \geq 2$, we have by (**):

$$\begin{aligned}
(\wedge^\ell \mathbb{L}_B)_g \simeq \{ \wedge^\ell\, & \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})^\vee \\
& \to \mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)}, \tilde{D})^\vee \otimes \wedge^{\ell-1}\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})^\vee \\
& \to \cdots \}.
\end{aligned}$$

(That is, the two are equivalent as dg-objects). Now, Ω on $\mathrm{Map}(\tilde{D}, Z)$ is multiplicative, and pulling back a multiplicative form to

$$\mathrm{Sym}^s\, \mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)}, \tilde{D})^\vee \otimes \wedge^{\ell-s}\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})^\vee$$

with $s > 0$ yields 0. The weight ℓ part of $q^*\Omega$ corresponds to a map $k \to (\wedge^\ell \mathbb{L}_B)_g$, which in turn decomposes to a nonzero map $k \to \wedge^\ell \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})^\vee$ and a 0 map to all later terms in the sequence. A homotopy from this to 0 is given by restricting the isotropic structure $\int_{[\tilde{D}]} ev^*\gamma$ from $\wedge^\ell \mathbb{R}\Gamma(g^*\mathbb{T}_Y, \tilde{D})^\vee$ to $\wedge^\ell \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D})^\vee$, and taking the 0 homotopy on all later terms.

For the Lagrangian condition, using (**), consider the diagram

$$\begin{array}{ccccc}
\mathbb{R}\Gamma((g^*\mathbb{T}_p) \otimes \mathcal{O}_{D+E}, \tilde{D})[-1] & \longrightarrow & \mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)}, \tilde{D}) & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_p, \tilde{D}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_Y) \otimes \mathcal{O}_D, \tilde{D})[-1] & \longrightarrow & \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D}, \tilde{D}) & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_Y, \tilde{D}) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{T}_q)_g & \longrightarrow & (\mathbb{T}_B)_g & \longrightarrow & \mathbb{R}\Gamma(g^*p^*\mathbb{T}_Z, \tilde{D})
\end{array}\ .$$

The second two columns and all rows are triangles, so the first column is too. Thus we have

$$(\mathbb{T}_q)_g \simeq \mathrm{HoCofib}(\mathbb{R}\Gamma((g^*\mathbb{T}_p) \otimes \mathcal{O}_{D+E}, \tilde{D})[-1] \to \mathbb{R}\Gamma((g^*\mathbb{T}_Y) \otimes \mathcal{O}_D, \tilde{D})[-1]). \qquad (***)$$

Similarly to the map (*) above, we obtain a quasi-isomorphism

$$\mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-D}, \tilde{D}) \to \Big(\mathbb{R}\Gamma(g^*\mathbb{T}_Y \otimes \mathcal{O}_{D+E}, \tilde{D})\Big)^\vee [n-d+1],$$

namely

$$\mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-D},\tilde{D}) \simeq \mathbb{R}\Gamma(j^*g^*\mathbb{T}_Y\otimes K_{D+E}, D+E) \to \mathbb{R}\Gamma(j^*g^*\mathbb{L}_Y\otimes K_{D+E}, D+E)[n],$$

where $\mathbb{T}_p \to \mathbb{L}_Y[n]$ is a quasi-isomorphism coming from the Lagrangian structure on p, followed by the Serre duality quasi-isomorphism

$$\mathbb{R}\Gamma((j^*g^*\mathbb{T}_Y)^\vee\otimes K_{D+E}, D+E)[n] \to \mathbb{R}\Gamma(j^*g^*\mathbb{T}_Y, D+E)^\vee[n-d+1].$$

Similarly, there is a natural quasi-isomorphism

$$\mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)},\tilde{D}) \to \Big(\mathbb{R}\Gamma(g^*\mathbb{T}_Y\otimes\mathcal{O}_D,\tilde{D})\Big)^\vee[n-d+1].$$

Then the map $\mathbb{T}_q \to \mathbb{L}_B[n-d]$ is given by the diagram

$$\begin{array}{ccc}
(\mathbb{T}_q)_g & \longrightarrow & (\mathbb{L}_B)_g[n-d+1] \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_p)\otimes\mathcal{O}_{D+E},\tilde{D}) & \xrightarrow{\sim} & \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D},\tilde{D})^\vee[n-d+1] \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_Y)\otimes\mathcal{O}_D,\tilde{D}) & \xrightarrow{\sim} & \mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)},\tilde{D})^\vee[n-d+1]
\end{array};$$

the columns are triangles by (**) and (***). Then this map is a quasi-isomorphism, so the isotropic structure is Lagrangian.

Let $Q = \mathrm{Map}(\tilde{D},Y)\times_{\mathrm{Map}(\tilde{D},Z)} B$ be the product, and $r : \mathrm{Map}(\tilde{D},D,Y) \to Q$ the map. For any $g \in \mathrm{Map}(\tilde{D},D,Y)$, consider the diagram

$$\begin{array}{ccccc}
\mathbb{R}\Gamma((g^*\mathbb{T}_p)\otimes\mathcal{O}_{D+E},\tilde{D})[-1] & \longrightarrow & \mathbb{R}\Gamma((g^*\mathbb{T}_p)_{-(D+E)},\tilde{D}) & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_p,\tilde{D}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D},\tilde{D}) & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_Y,\tilde{D})\oplus\mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D},\tilde{D}) & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_Y,\tilde{D}) \\
\downarrow & & \downarrow & & \downarrow \\
(r^*\mathbb{T}_Q)_g & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_Y,\tilde{D})\oplus(r^*\pi_2^*\mathbb{T}_B) & \longrightarrow & \mathbb{R}\Gamma(g^*\mathbb{T}_Z,\tilde{D});
\end{array}$$

again, everything but the first column is a triangle, so the first column is too.

Then

$$(r^*\mathbb{T}_Q)_g \simeq \mathrm{HoCofib}(\mathbb{R}\Gamma((g^*\mathbb{T}_p)\otimes\mathcal{O}_{D+E},\tilde{D})[-1] \to \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D},\tilde{D})).$$

The map

$$(\mathbb{T}_{\mathrm{Map}(\tilde{D},D,Y)})_g \simeq \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D},\tilde{D}) \to (r^*\mathbb{T}_Q)_g$$

is precisely the structure morphism of the above cofiber. Letting ω_Q be the symplectic structure on Q, we get $r^*\omega_Q = 0$, so r has 0 as isotropic structure. It is easy to check that $(\mathbb{T}_r)_q \simeq \mathbb{R}\Gamma((g^*\mathbb{T}_p)\otimes\mathcal{O}_{D+E},\tilde{D})[-2]$, and the map $(\mathbb{T}_r)_q \to (\mathbb{L}_{\mathrm{Map}(\tilde{D},D,Y)})_q[n-d-1]$ is the quasi-isomorphism

$$\mathbb{R}\Gamma((g^*\mathbb{T}_p)\otimes\mathcal{O}_{D+E},\tilde{D}) \to \mathbb{R}\Gamma((g^*\mathbb{T}_Y)_{-D},\tilde{D})^\vee[n-d+1]$$

shifted by -2. □

□

Remark 3.4.1. Using similar methods, one can show that if Y is (pre)symplectic, then $\mathrm{Map}(X, D+E, Y)$ has a natural presymplectic structure.

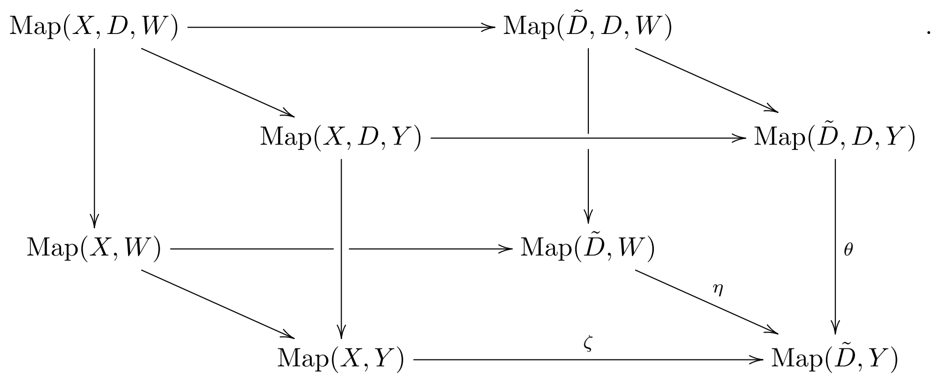
Analogously to Theorems 2.6 and 2.8, we have:

Theorem 3.5. *Let X be a d-dimensional proper smooth scheme and D an effective divisor. Suppose E is an effective divisor of X such that $\tilde{D} = 2D+E$ is anticanonical. Let Y be a derived Artin stack such that $\mathrm{Map}(X,Y)$, $\mathrm{Map}(\tilde{D},Y)$, $\mathrm{Map}(D,Y)$, and $\mathrm{Map}(D+E,Y)$ are themselves derived Artin stacks of locally finite presentation over k. Fix a base map $f : D \to Y$. Let W be a derived Artin stack and pick a map $s : W \to Y$.*

1. *Suppose Y is n-shifted symplectic, that the projection $\mathrm{Map}(D+E,Y) \to \mathrm{Map}(D,Y)$ is etale over f, and that $\mathrm{Map}(D+E,W) \to \mathrm{Map}(D,W)$ is etale over any lift $\tilde{f}$ of f. Suppose $s : W \to Y$ has a Lagrangian structure. Then $\mathrm{Map}(X,D,W) \to \mathrm{Map}(X,D,Y)$ has a natural Lagrangian structure.*

2. *Suppose Y is n-shifted Poisson, and $s : W \to Y$ has a coisotropic structure. Then $\mathrm{Map}(X,D,W) \to \mathrm{Map}(X,D,Y)$ has a natural coisotropic structure.*

Proof. Similarly to the previous theorem, we use the fiber diagram



The front and back faces are Cartesian squares. We have a Lagrangian (resp. coisotropic) structure on ζ by Corollary 1.12 (Corollary 2.10), on η by Theorem 1.6 (Corollary 2.5), and on θ by Lemma 3.4. We also have a Lagrangian structure on $\mathrm{Map}(X,W) \to \mathrm{Map}(X,Y) \times_{\mathrm{Map}(\tilde{D},Y)} \mathrm{Map}(\tilde{D},W)$ by Theorem 2.9 (Corollary 2.11). If we show that $\mathrm{Map}(\tilde{D},D,W) \to \mathrm{Map}(\tilde{D},D,Y) \times_{\mathrm{Map}(\tilde{D},Y)} \mathrm{Map}(\tilde{D},W)$ has a Lagrangian (Poisson) structure, then we will be done by Lemma 2.4 (Corolllary 2.12). As before, we put this in a separate lemma:

Lemma 3.6. *Let $X, Y, W, D, \tilde{D}$ be as in the theorem. Then*

1. *Suppose Y is n-shifted symplectic, that the projection $\mathrm{Map}(D+E,Y) \to \mathrm{Map}(D,Y)$ is etale over f, and that $\mathrm{Map}(D+E,W) \to \mathrm{Map}(D,W)$ is etale over any lift $\tilde{f}$ of f. Suppose $s : W \to Y$ has a Lagrangian structure. Then $r : \mathrm{Map}(\tilde{D},D,W) \to \mathrm{Map}(\tilde{D},D,Y) \times_{\mathrm{Map}(\tilde{D},Y)} \mathrm{Map}(\tilde{D},W)$ has a canonical Lagrangian structure.*

2. *Suppose Y is n-shifted Poisson and that $s : W \to Y$ has a coisotropic structure. Then $r : \mathrm{Map}(\tilde{D},D,W) \to \mathrm{Map}(\tilde{D},D,Y) \times_{\mathrm{Map}(\tilde{D},Y)} \mathrm{Map}(\tilde{D},W)$ has a canonical coisotropic structure.*

Proof. For (1), let γ be the Lagrangian structure on s. If Ω is the induced symplectic structure on $\mathrm{Map}(\tilde{D},D,Y) \times_{\mathrm{Map}(\tilde{D},Y)} \mathrm{Map}(\tilde{D},W)$, one can check that $r^*\Omega = -\int_{[\tilde{D}]} ev^*\gamma$. But this is a multiplicative form, so is already 0 on

$$(\wedge^{\ell}\mathbb{T}_{\mathrm{Map}(\tilde{D},D,W)})_g \simeq \wedge^{\ell}\mathbb{R}\Gamma((g^*\mathbb{T}_W)_{-D}) \simeq \wedge^{\ell}\mathbb{R}\Gamma((g^*\mathbb{T}_W)_{-(D+E)})(\ell \geq 2)$$

for any $g \in \mathrm{Map}(\tilde{D}, D, W)$; here the second quasi-isomorphism comes from the etaleness condition. Thus 0 is an isotropic structure. For the Lagrangian condition, we have

$$(\mathbb{T}_r)_g \simeq \mathbb{R}\Gamma(g^*\mathbb{T}_s \otimes \mathcal{O}_D, \tilde{D}) \simeq \mathbb{R}\Gamma(g^*\mathbb{T}_s \otimes \mathcal{O}_{D+E}, \tilde{D}),$$

and the map $\mathbb{T}_r \to \mathbb{L}_{\mathrm{Map}(\tilde{D},D,W)}[n-d]$ is the quasi-isomorphism

$$\mathbb{R}\Gamma(g^*\mathbb{T}_s, D+E) \to \mathbb{R}\Gamma(g^*\mathbb{L}_W, D+E)[n-1]$$

from the Lagrangian condition on s, followed by the Serre quasi-isomorphism

$$\mathbb{R}\Gamma(g^*\mathbb{L}_W, D+E) \to \mathbb{R}\Gamma(g^*\mathbb{T}_W \otimes K_{D+E}, D+E)^\vee[1-d] \simeq \mathbb{R}\Gamma((g^*\mathbb{T}_W)_{-D}, \tilde{D})^\vee[1-d],$$

as in the proof of the previous theorem.

For (2), let the Poisson structure on Y be given by $p : Y \to Z$ with Lagrangian structure γ, and the coisotropic structure on $W \to Y$ given by $u : W \to Y \times_Z Y'$ with Lagrangian structure ϵ, where $p' : Y' \to Z$ has Lagrangian structure γ'.

Letting B be a formal neighborhood of $\mathrm{Map}(\tilde{D}, D, Y)$ in $\mathrm{Map}(D+E, D, Y) \times_{\mathrm{Map}(D+E,D,Z)} \mathrm{Map}(\tilde{D}, D, Z)$, recall that the coisotropic structure on $\mathrm{Map}(\tilde{D}, D, Y) \to \mathrm{Map}(\tilde{D}, Y)$ came from the map $B \to \mathrm{Map}(\tilde{D}, Z)$. In our present case, the Poisson structure on $\Psi := \mathrm{Map}(\tilde{D}, D, Y) \times_{\mathrm{Map}(\tilde{D},Y)} \mathrm{Map}(\tilde{D}, W)$ comes from

$$\Psi \to \Xi := B \times_{\mathrm{Map}(\tilde{D},Z)} \mathrm{Map}(\tilde{D}, Y').$$

Let B' be a formal neighborhood of $\mathrm{Map}(\tilde{D}, D, W)$ in $\mathrm{Map}(D+E, D, W) \times_{\mathrm{Map}(D+E,D,Y')} \mathrm{Map}(\tilde{D}, D, Y')$. The maps

$$\times_{\mathrm{Map}(D+E,D,W)\,\mathrm{Map}(D+E,D,Y')} \mathrm{Map}(\tilde{D}, D, Y') \xrightarrow{\pi_2} \mathrm{Map}(\tilde{D}, D, Y') \longrightarrow \mathrm{Map}(\tilde{D}, Y')$$

and

$$\mathrm{Map}(D+E, D, W) \times_{\mathrm{Map}(D+E,D,Y')} \mathrm{Map}(\tilde{D}, D, Y') \to \mathrm{Map}(D+E, D, Y) \times_{\mathrm{Map}(D+E,D,Z)} \mathrm{Map}(\tilde{D}, D, Z)$$

give a map $q' : B' \to \Xi$. As in the previous theorem, one can give an isotropic structure on q' basically arising from the isotropic structure on $\mathrm{Map}(\tilde{D}, W) \to \mathrm{Map}(\tilde{D}, Y \times_Z Y')$, and Lagrangianness comes from

$$\begin{array}{ccc}
(\mathbb{T}_{q'})_g & \longrightarrow & (\mathbb{L}_{B'})_g[n-d+2] \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_u) \otimes \mathcal{O}_{D+E}, \tilde{D}) & \xrightarrow{\sim} & \mathbb{R}\Gamma((g^*\mathbb{T}_W)_{-D}, \tilde{D})^\vee[n-d+2] \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma((g^*\mathbb{T}_s) \otimes \mathcal{O}_D, \tilde{D}) & \xrightarrow{\sim} & \mathbb{R}\Gamma((g^*\mathbb{T}_{s'})_{-(D+E)}, \tilde{D})^\vee[n-d+2]
\end{array} .$$

Again similarly to the previous theorem, letting $\rho : \mathrm{Map}(\tilde{D}, D, W) \to R := B' \times_\Xi \Psi$, if Ω_R is the induced structure on R, then $\rho^*\Omega_R$ is already 0, so 0 is an isotropic structure. For the Lagrangian condition, at any $g \in \mathrm{Map}(\tilde{D}, D, W)$, the corresponding map $(\mathbb{T}_\rho)_g \to (\mathbb{L}_{\mathrm{Map}(\tilde{D},D,W)})_g[n-d-1]$ is just

$$\mathbb{R}\Gamma(g^*\mathbb{T}_u \otimes \mathcal{O}_{D+E}, \tilde{D})[-1] \to \mathbb{R}\Gamma((g^*\mathbb{T}_W)_{-D}, \tilde{D})[n-d-1],$$

analogous to previous maps. This gives the coisotropic structure on $\mathrm{Map}(\tilde{D}, D, W) \to \Psi$ we needed. $\square$

$\square$

3.1 Framed Vector Bundles on Surfaces

As an example of these theorems, let X be a smooth surface with effective anticanonical bundle, and take effective divisors D and E such that $2D+E$ is anticanonical. Let G be a reductive group. Choose a map $D \to BG$, that is, a G-bundle $\mathcal{G} \to D$. The space $\mathrm{Map}(X, D, BG, \mathcal{G})$ has, by Theorem 3.3, a 0-shifted Poisson structure. This structure will be symplectic if $\mathrm{Map}(D+E, BG) \to \mathrm{Map}(D, BG)$ is etale over $\mathcal{G}$. That is, if for every extension $\tilde{\mathcal{G}} \to D+E$ of $\mathcal{G}$, the map

$$H^*(Lie(\tilde{\mathcal{G}}), D+E) \to H^*(Lie(\mathcal{G}), D)$$

is an isomorphism in all degrees. Assuming the moduli space is a smooth variety (or looking at a semistable locus), this will be an ordinary Poisson or symplectic structure. Taking $\zeta \in H^0(X, E)$ to be a section vanishing on E, this is precisely Theorem 4.3 of [Bo].

In particular, let us consider the case where $X = \mathbb{P}^2$, $D = E$ is a line L, $G = SL_n$, and $\mathcal{G}$ is the trivial bundle. The space $\mathrm{Map}(\mathbb{P}^2, L, BSL_n, \mathcal{G})$ may be identified with the framed $SU(n)$-instantons on S^4 ([Do]). In this case, the only extension of $\mathcal{G}$ to $2L$ is the trivial bundle again, and the failure of

$$H^*(\mathfrak{sl}_n \otimes \mathcal{O}_{2L}, 2L) \to H^*(\mathfrak{sl}_n \otimes \mathcal{O}_L, L)$$

is given by

$$H^*(\mathfrak{sl}_n \otimes \mathcal{O}_L(-1), L) = 0,$$

so we have a symplectic structure.

Chapter 4

Monopoles

Let G be a semisimple complex Lie group and B a Borel. Let $Y = G/B$ be the flag variety of G. Fix a point $p \in \mathbb{P}^1$. The space $\mathrm{Map}(\mathbb{P}^1, p, Y)$ is the space of framed G-monopoles on $\mathbb{R}^3$ [Ja]. In [FKMM] the authors show that this space has a symplectic structure. More generally, let P be a parabolic subgroup of G and $Y = G/P$ the partial flag variety; then it is shown that $\mathrm{Map}(\mathbb{P}^1, p, Y)$ has a Poisson structure. Here I will show that the Poisson and symplectic structures arise from the machinery of shifted structures on framed mapping spaces.

4.1 Classical Construction of the Symplectic Structure

The following construction is described in [FKMM].

Let $\mathfrak{g}_Y = \mathfrak{g} \otimes \mathcal{O}_Y$ denote the trivial $\mathfrak{g}$-bundle on Y. Let $\mathfrak{p}_Y \subset \mathfrak{g}_Y$ be subbundle whose fiber over a parabolic P is its Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$; similarly let $\mathfrak{r}_Y \subset \mathfrak{p}_Y$ be the subbundle whose fiber over P is the nilpotent radical $\mathfrak{r}$. Let $\mathfrak{l}_Y = \mathfrak{p}_Y/\mathfrak{r}_Y$ be the bundle of abstract Levi algebras.

Recall that TY is canonically isomorphic to $\mathfrak{g}_Y/\mathfrak{p}_Y$, and a G-invariant symmetric nondegenerate bilinear form on $\mathfrak{g}$ will give an isomorphism $T^\vee Y \cong \mathfrak{r}_Y$. In the Borel case, we note that $\mathfrak{l}_Y$ is trivial.

The Poisson structure on $\mathrm{Map}(\mathbb{P}^1, p, Y)$ is defined as follows. First note that at any $f \in \mathrm{Map}(\mathbb{P}^1, p, Y)$, we have

$$T_f \,\mathrm{Map}(\mathbb{P}^1, p, Y) \cong H^0(f^*TY(-1), \mathbb{P}^1) \cong H^0(f^*(\mathfrak{g}/\mathfrak{p})(-1), \mathbb{P}^1)$$

and

$$T_f^\vee \,\mathrm{Map}(\mathbb{P}^1, p, Y) \cong H^1(f^*T^\vee Y(-1), \mathbb{P}^1) \cong H^1(f^*(\mathfrak{r})(-1), \mathbb{P}^1).$$

Now consider the complex

$$\mathfrak{r}_Y \to \mathfrak{g}_Y \to \mathfrak{g}_Y/\mathfrak{p}_Y$$

on Y. Pulling back by f and twisting by -1 yields

$$f^*(\mathfrak{r}_Y)(-1) \to f^*(\mathfrak{g}_Y)(-1) \to f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1).$$

Now we take the hypercohomology spectral sequence of this complex. At page 0, we get the sheaf cohomology of each of the sheaves. Since $\mathfrak{g}_Y$ is trivial, $f^*(\mathfrak{g}_Y)(-1)$ is a sum of $\mathcal{O}_{\mathbb{P}^1}(-1)$s and its cohomology vanishes. Thus the d_1 differentials vanish, and at E_2 we get a differential

$$d_2 : H^1(f^*(\mathfrak{r}_Y)(-1), \mathbb{P}^1) \to H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1), \mathbb{P}^1),$$

that is,

$$T_f^\vee \,\mathrm{Map}(\mathbb{P}^1, p, Y) \to T_f \,\mathrm{Map}(\mathbb{P}^1, p, Y).$$

This is the Poisson structure. Verifying that this really is a Poisson structure is done in [FKMM], or can be derived from later results.

Remark 4.0.1. Assuming this really is a Poisson structure, let us show nondegeneracy for the case $P = B$. The complex

$$f^*(\mathfrak{r}_Y)(-1) \to f^*(\mathfrak{g}_Y)(-1) \to f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1)$$

is quasi-isomorphic to

$$0 \to f^*(\mathfrak{l}_Y)(-1) \to 0,$$

which has zero hypercohomology, as $\mathfrak{l}_Y$ is trivial. Thus in particular the differential $d_2 : H^1(f^*(\mathfrak{r}_Y)(-1), \mathbb{P}^1) \to H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1), \mathbb{P}^1)$ must be an isomorphism.

4.2 Construction via Shifted Poisson Structures

In this section we describe a construction of a Poisson structure on $\mathrm{Map}(\mathbb{P}^1, p, G/P)$ using the machinery of shifted Poisson structures.

Our first hope might be to find a 1-shifted Poisson structure on $Y = G/P$ and use Theorem 3.3 to induce a structure on $\mathrm{Map}(\mathbb{P}^1, p, Y)$. This should make us suspicious in the $P = B$ case, as we would expect a 1-shifted symplectic structure on G/B, which would yield a quasi-isomorphism between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{r}[1]$, which is clearly impossible.

In the general case, we can also show we can't get our Poisson structure this way. Suppose Z is 2-shifted symplectic, and $g : Y \to Z$ has a Lagrangian structure defining a Posson structure on Y. Using the Lagrangian condition, we get $\mathbb{T}_g \simeq \mathbb{L}_Y[1] \cong \mathfrak{r}[1]$, and in particular the map $\mathbb{T}_g \to \mathbb{T}_Y$ is the zero map. For any $f \in \mathrm{Map}(\mathbb{P}^1, p, Y)$, the map $T_f^\vee \mathrm{Map}(\mathbb{P}^1, p, Y) \to T_f \mathrm{Map}(\mathbb{P}^1, p, Y)$ will just be the map

$$H^1(f^*(\mathfrak{r})(-1), \mathbb{P}^1) \simeq \mathbb{R}\Gamma(f^*(\mathbb{T}_g)(-1), \mathbb{P}^1) \to \mathbb{R}\Gamma(f^*(\mathbb{T}_Y)(-1), \mathbb{P}^1) \simeq H^0(f^*(\mathfrak{g}/\mathfrak{p})(-1), \mathbb{P}^1),$$

so will also be zero, which we do not want.

Instead, we note that G/P is already related to an existing shifted symplectic stack, BG, via the fiber diagram

$$\begin{array}{ccc} G/P & \longrightarrow & BP \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & BG \end{array} \quad . \qquad (*)$$

Recall that the symplectic structure on BG is given by a G-invariant nondegenerate symmetric quadratic form on $\mathfrak{g}$. Fix such a form ω.

Choose an opposite parabolic P^- so that $P \cap P^- = L$ is a Levi subgroup of G. Letting $\mathfrak{l} = \mathrm{Lie}(L)$, we can then write $\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{l} \oplus \mathfrak{r}$. Since $\mathfrak{l}$ is orthogonal to $\mathfrak{r}$ and $\mathfrak{r}^-$, ω descends to $\mathfrak{l}$ and is also L-invariant and nondegenerate. Thus, BL has a symplectic structure ω_L induced from BG. Recall that the identification $L \cong P/\mathrm{rad}(P)$ gives us a map $P \to L$.

Lemma 4.1. *The map $\iota : BP \to BG \times BL$ has a Lagrangian structure given by* 0. *Thus $BP \to BG$ has a coisotropic structure.*

Proof. The claim that 0 is an isotropic structure reduces to the claim that $\iota : \mathfrak{p} \to \mathfrak{g} \oplus \mathfrak{l}$ is isotropic in the usual sense with respect to $\omega - \omega_L$. Write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{r}$ and recall that $\mathfrak{r}$ is orthogonal to itself and $\mathfrak{l}$. Then

$$(\omega - \omega_L)(\iota_*(\ell, r), \iota_*(\ell', r')) = \omega(\ell, \ell') - \omega(\ell, \ell') = 0,$$

so we have isotropy.

For the Lagrangian condition, recall that ω pairs up $\mathfrak{r}$ nondegenerately with $\mathfrak{r}^-$. Let $\Delta, \overline{\Delta} : \mathfrak{l} \to \mathfrak{g} \oplus \mathfrak{l}$ be the diagonal and antidiagonal maps, and note that $\omega - \omega_L$ pairs up $\Delta(\mathfrak{l})$ and $\overline{\Delta}(\mathfrak{l})$ nondegenerately. Then the map $\mathbb{T}_\iota \to \mathbb{L}_{BP}[1]$ is just the adjoint $\overline{\Delta}(\mathfrak{l}) \oplus \mathfrak{r}^- \to (\Delta(\mathfrak{l}) \oplus \mathfrak{r})^\vee$. □

So $BP \to BG$ has a coisotropic structure, and if $\bullet \to BG$ had one too, we would get a 1-shifted Poisson structure on G/P by Corollary 2.5. As mentioned, there is no decent shifted Poisson structure on G/P and it is also easy to check that $\bullet \to BG$ has no coisotropic structure. Instead, we apply the functor $\mathrm{Map}(\mathbb{P}^1, p, -)$ to * to get

$$\begin{array}{ccc} \mathrm{Map}(\mathbb{P}^1, p, G/P) & \longrightarrow & \mathrm{Map}(\mathbb{P}^1, p, BP) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathbb{P}^1, p, \bullet) & \longrightarrow & \mathrm{Map}(\mathbb{P}^1, p, BG) \end{array},$$

and note that now $\mathrm{Map}(\mathbb{P}^1, p, BG)$ is 1-symplectic and $\mathrm{Map}(\mathbb{P}^1, p, BP) \to \mathrm{Map}(\mathbb{P}^1, p, BG)$ is coisotropic by Theorem 3.5. Let $G_{\mathbb{P}^1}$ denote the trivial G-bundle on $\mathbb{P}^1$, framed at p. Then the map

$$\bullet = \mathrm{Map}(\mathbb{P}^1, p, \bullet) \to \mathrm{Map}(\mathbb{P}^1, p, BG)$$

is just the point $G_{\mathbb{P}^1}$. And this *is* coisotropic, as

$$\mathbb{T}_{G_{\mathbb{P}^1}} \mathrm{Map}(\mathbb{P}^1, p, BG) = \mathcal{E}xt^*(\mathcal{O}_{\mathbb{P}^1} \otimes \mathfrak{g}, \mathcal{O}_{\mathbb{P}^1} \otimes \mathfrak{g})(-1)[1] \simeq 0.$$

Then the map $\bullet \to \mathrm{Map}(\mathbb{P}^1, p, BG)$ is trivially Lagrangian, hence coisotropic. Then Corollary 2.5 gives a 0-shifted Poisson structure on $\mathrm{Map}(\mathbb{P}^1, p, G/P)$.

To be a little more specific, the coisotropic bases are the maps $\mathrm{Map}(\mathbb{P}^1, p, \overline{BL}) \to \bullet$ and $\bullet \to \bullet$ respectively, so the Poisson structure comes from a map $\mathrm{Map}(\mathbb{P}^1, p, G/P) \to \mathrm{Map}(\mathbb{P}^1, p, BL)$. In particular, in the case P is a Borel, L is a torus, and so $\mathrm{Map}(\mathbb{P}^1, p, BL) = \prod_{i=1}^r \mathrm{Map}(\mathbb{P}^1, p, B\mathbb{G}_m)$, where r is the rank of G. But $\mathrm{Map}(\mathbb{P}^1, p, B\mathbb{G}_m) \cong \mathbb{Z}$, so $\mathrm{Map}(\mathbb{P}^1, p, BL)$ is a disjoint union of points $\bullet_1$ with the trivial 1-symplectic structure. Thus, $\mathrm{Map}(\mathbb{P}^1, p, G/P)$ is in fact symplectic.

Theorem 4.2. *The Poisson structures described above coincide.*

Proof. Let us look at the construction using shifted Poisson structures first. Recall that, for X an underived smooth scheme with Poisson structure $g : X \to Z$, we recover the map $T_X^\vee \to T_X$ by using the Lagrangian structure on g to yield an isomorphism $T_g \cong L_X$; with respect to this isomorphism, the Poisson structure is given by $L_X \cong T_g \to T_X$. In the case $X = \mathrm{Map}(\mathbb{P}^1, p, G/P)$, let's look at the map $g : X \to Z = \mathrm{Map}(\mathbb{P}^1, p, BL)$. For $f \in X$, the tangent map $g_* : T_f X \to g^*(\mathbb{T}Z)_f$ is the composition

$$\mathbb{R}\Gamma(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1), \mathbb{P}^1) \xrightarrow{\partial^*} \mathbb{R}\Gamma(f^*(\mathfrak{p}_Y)(-1), \mathbb{P}^1)[1] \xrightarrow{\pi} \mathbb{R}\Gamma(f^*(\mathfrak{l}_Y)(-1, \mathbb{P}^1))[1]$$

Here ∂^* is the connecting map coming from the short exact sequence

$$0 \to f^*(\mathfrak{p}_Y)(-1) \to f^*(\mathfrak{g}_Y(-1)) \to f^*(\mathfrak{g}_Y/\mathfrak{p}_Y(-1)) \to 0;$$

since $f^*(\mathfrak{g}_Y(-1))$ is acyclic, ∂^* is a quasi-isomorphism. Then from the sequence

$$0 \to f^*(\mathfrak{r}_Y)(-1) \to f^*(\mathfrak{p}_Y)(-1) \to f^*(\mathfrak{l}_Y)(-1) \to 0$$

we see that the fiber of π (and thus of g_*) is $\mathbb{R}\Gamma(f^*(\mathfrak{r}_Y)(-1), \mathbb{P}^1)[1] \simeq H^1(f^*(\mathfrak{r}_Y)(-1), \mathbb{P}^1)$. This is identified with $T_f^\vee X$ in the canonical way, and the Poisson map is then

$$H^1(f^*(\mathfrak{r}_Y)(-1), \mathbb{P}^1) \to H^1(f^*(\mathfrak{p}_Y)(-1), \mathbb{P}^1) \cong H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y(-1)), \mathbb{P}^1).$$

For the spectral sequence, consider the map of complexes

$$\begin{array}{ccccc} \mathfrak{r}_Y & \longrightarrow & \mathfrak{g}_Y & \longrightarrow & \mathfrak{g}_Y/\mathfrak{p}_Y \\ \downarrow & & \| & & \| \\ \mathfrak{p}_Y & \longrightarrow & \mathfrak{g}_Y & \longrightarrow & \mathfrak{g}_Y/\mathfrak{p}_Y \end{array}.$$

Pull back by f, twist by -1, and look at the induced map on the E_2 page:

$$\begin{array}{ccc} H^1(f^*(\mathfrak{r}_Y)(-1),\mathbb{P}^1) & \longrightarrow & H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1),\mathbb{P}^1) \\ \downarrow & & \| \\ H^1(f^*(\mathfrak{p}_Y)(-1),\mathbb{P}^1) & \longrightarrow & H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1),\mathbb{P}^1) \end{array}.$$

The bottom map is the inverse to the connecting isomorphism $H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1),\mathbb{P}^1) \to H^1(f^*(\mathfrak{p}_Y)(-1),\mathbb{P}^1)$ from the long exact sequence. Thus the Poisson map is exactly the composition

$$H^1(f^*(\mathfrak{r}_Y)(-1),\mathbb{P}^1) \to H^1(f^*(\mathfrak{p}_Y)(-1),\mathbb{P}^1) \cong H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y(-1)),\mathbb{P}^1),$$

as above. □

Remark 4.2.1. In either case, much of the "real work" of the Poisson structure lies in the identification

$$(H^0(f^*(\mathfrak{g}_Y/\mathfrak{p}_Y)(-1),\mathbb{P}^1))^\vee \cong H^1(f^*(\mathfrak{r})(-1),\mathbb{P}^1)$$

arising from Serre duality and the isomorphism $(\mathfrak{g}_Y/\mathfrak{p}_Y)^\vee \cong \mathfrak{r}_Y$.

Remark 4.2.2. Given that G/P doesn't have a 1-shifted Poisson structure, what structure does it have? Recall that an n-shifted symplectic structure on X is equivalent to a map $X \to \bullet_{n+1}$ with a Lagrangian structure. Generalizing this by replacing $\bullet_{n+1}$ with an arbitrary $(n+1)$-shifted symplectic derived stack Z yields the notion of an n-shifted Poisson structure.

However, there is another generalization we can make: if $X \to \bullet_{n+1}$ only has an isotropic structure, we get an n-shifted presymplectic structure on X. Combining these two, we might say an n-shifted "pre-Poisson" structure on X is an $(n+1)$-shifted symplectic derived stack Z and a map $X \to Z$ with an isotropic structure. This is the structure G/P has; specifically, $G/P \to BL$ has an isotropic structure.

4.3 Other Fibers

In the fiber square

$$\begin{array}{ccc} \mathrm{Map}(\mathbb{P}^1,p,G/P) & \longrightarrow & \mathrm{Map}(\mathbb{P}^1,p,BP) \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \mathrm{Map}(\mathbb{P}^1,p,BG) \end{array},$$

the base point in $\mathrm{Map}(\mathbb{P}^1,p,BG)$ is the trivial G-bundle on $\mathbb{P}^1$ framed at p. This gives us $\mathrm{Map}(\mathbb{P}^1,p,G/P)$ as the fiber, but is also neccessary to get a Poisson structure: for general $\mathcal{G} \in \mathrm{Map}(\mathbb{P}^1,p,BG)$, the corresponding $\bullet \to \mathrm{Map}(\mathbb{P}^1,p,BG)$ is not Lagrangian or even coisotropic.

Let's be a little more precise. Let $aut(\mathcal{G})$ denote the vector bundle whose fiber at x is $\mathrm{Lie}(Aut(\mathcal{G}_x))$, and let $aut_p(\mathcal{G})$ be the sheaf of sections of $aut(\mathcal{G})$ vanishing at p. Then $\mathbb{T}_{\mathcal{G}} \simeq \mathbb{R}\Gamma(aut_p(\mathcal{G}),\mathbb{P}^1)[1]$. The automorphisms of $\mathcal{G}$ within $\mathrm{Map}(\mathbb{P}^1,p,BG)$ are $Aut_p(\mathcal{G})$, the group of automorphisms of $\mathcal{G}$ over $\mathbb{P}^1$ which are the identity above p. Then the corresponding map $B(Aut_p(\mathcal{G})) \to \mathrm{Map}(\mathbb{P}^1,p,BG)$ will be an isomorphism on $\mathbb{T}$ in degree -1, and will be Lagrangian (with Lagrangian structure 0).

Then applying Corollary 2.5, we get

Theorem 4.3. *The product $B(Aut_p(\mathcal{G}))\times_{\mathrm{Map}(\mathbb{P}^1,p,BG)}\mathrm{Map}(\mathbb{P}^1,p,BP)$ will have a Poisson structure. This structure is symplectic if $P = B$.*

The space $B(Aut_p(\mathcal{G})) \times_{\mathrm{Map}(\mathbb{P}^1,p,BG)} \mathrm{Map}(\mathbb{P}^1,p,BP)$ can be described as follows. G acts on $\mathcal{G}$ from the right and G/P on the left, so we can form the balanced product $\mathcal{G} \times_G G/P$, a G/P-bundle over $\mathbb{P}^1$. Note that $Aut_p(\mathcal{G})$ still acts on $\mathcal{G} \times_G G/P$. Let $\Gamma(\mathcal{G} \times_G G/P(-1), \mathbb{P}^1)$ denote the sections of $\mathcal{G} \times_G G/P$ sending p to a specified point of $(\mathcal{G} \times_G G/P)_x$. Then

$$B(Aut_p(\mathcal{G})) \times_{\mathrm{Map}(\mathbb{P}^1,p,BG)} \mathrm{Map}(\mathbb{P}^1,p,BP) \cong \Gamma(\mathcal{G} \times_G G/P(-1), \mathbb{P}^1)/Aut_p(\mathcal{G}).$$

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