COMBINATORIAL THEORY OF Q,T-SCHRÖDER POLYNOMIALS, PARKING FUNCTIONS AND TREES

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2004

To Athena (Lingyun),

Jie,

and my parents,

with much love and gratitude.

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ABSTRACT

COMBINATORIAL THEORY OF Q,T-SCHRÖDER POLYNOMIALS, PARKING FUNCTIONS AND TREES

Chunwei Song

James Haglund

We study various aspects of lattice path combinatorics. A new object, which has Dyck paths as its subset and is named Permutation paths, is considered and relative theories are developed. We prove a class of tree enumeration theorems and connect them to parking functions. The limit case of (q,t)-Schröder Theorem is investigated. In the end, we derive a formula for the number of m-Schröder paths and study its q and (q,t)-analogues.

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Chapter 1

Introduction

1.1 Overview

The primary focus of this dissertation is on various properties of lattice path enumeration and their (q,t)-analogues, a rapidly developing topic in the frontier of combinatorics, which as a domain of mathematics is itself going through a profound revolution.

More specifically, we study combinatorial statistics on lattice paths, multi-variable analogs of Catalan and Schröder numbers, and related more generalized issues. In introducing our work, the current chapter provides an introduction of the existing literature as well as a summary of our new theorems, while the following chapters describe our results in more details.

1.2 A Survey of the Literature

There are two noteworthy sequences of numbers that are of central interests to combinatorial mathematicians. The n^{th} *Catalan number* is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

[Sta99, Ex.6.19, pages 219–229] gives 66 combinatorial interpretations of these numbers, and updated additions are provided in [Sta].

The n^{th} *Schröder number* is

$$S_n = \sum_{d=0}^{n} \frac{1}{n-d+1}\binom{2n-d}{d, n-d, n-d}.$$

[Sta99, Ex.6.39, pages 239–240] provides 19 combinatorial interpretations of these numbers.

Among the interpretations, we are primarily interested in those associated to the lattice paths.

Definition 1.2.1. A *Dyck path* of order n is a lattice path from $(0,0)$ to (n,n) that never goes below the main diagonal $\{(i,i), 0 \leq i \leq n\}$, with steps $(0,1)$ (or NORTH, for brevity *N*) and $(1,0)$ (or EAST, for brevity *E*). Let $\mathcal{D}_n$ denote the set of all Dyck paths of order n.

Definition 1.2.2. A *Schröder path* of order n is a lattice path from $(0,0)$ to (n,n) that never goes below the main diagonal $\{(i,i), 0 \leq i \leq n\}$, with steps $(0,1)$ (or NORTH, for brevity *N*), $(1,0)$ (or EAST, for brevity *E*) and (1,1) (or Diagonal, for brevity *D*). Let $\mathcal{S}_n$ denote the set of all Schröder paths of order n.

The Catalan number $C_n = 1, 2, 5, 14, 42, ...$, counts the number of Dyck paths of order n, while the Schröder number $S_n = 2, 6, 22, 90, 394, ...$, counts the number of Schröder paths of order n. In this dissertation, we are often more concerned with the *Schröder paths with d diagonal steps*.

Definition 1.2.3. A *Schröder path* of order n and with d diagonal steps is a lattice path from $(0,0)$ to (n,n) that never goes below the main diagonal $\{(i,i), 0 \le i \le n\}$, with $(0,1)$ (or NORTH, for brevity *N*), $(1,0)$ (or EAST, for brevity *E*) and exactly d (1,1) (or Diagonal, for brevity *D*) steps. Let $\mathcal{S}_{n,d}$ denote the set of all Schröder paths of order n and with d diagonal steps.

The number of Schröder paths of order n and with d diagonal steps is counted by

$$\begin{aligned} S_{n,d} &= \binom{2n-d}{d} C_n \\ &= \frac{1}{n-d+1} \binom{2n-d}{d, n-d, n-d}. \end{aligned}$$

Clearly $S_n = \sum_{d=0}^{n} S_{n,d}$ and $C_n = S_{n,0}$.

By a *statistic* on a given set S, we mean a combinatorial rule that associates a nonnegative integer to each element of S. Efforts and progresses have been made by considering 1 variable ([CR64], [FH85], [BSS93], etc) and 2 variable ([GH96], [HL], [EHKK03], etc) generalizations of these numbers, through studying various invented statistics associated with these lattice paths. The applications of these works expand from almost every subfield of discrete mathematics to other areas such as representation theory and algebraic geometry.

Two important statistics on $\mathcal{D}_n$ are $area$ and $bounce$.

Given $\Pi \in \mathcal{D}_n$, $area(\Pi)$ is defined to be the number of complete squares between Π and the main diagonal line $y = x$. More specifically, let $a_i(\Pi)$ be the number of complete squares in the i^{th} row, from top to bottom, that are below Π and above the main diagonal. The number $a_i(\Pi)$ is also called the $length$ of the ith row of Π, and

$(a_1(\Pi), a_2(\Pi), \ldots, a_n\Pi)$ is the *area vector* of Π. Finally, $area(\Pi) = \sum_{i=1}^{n} a_i(\Pi)$. An example of a Dyck path of order 6 with area vector $(1,0,0,1,1,0)$ is illustrated in Figure 1.1.

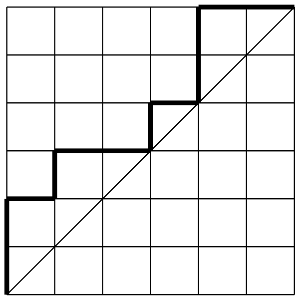


Figure 1.1: A Dyck path $\Pi \in \mathcal{D}_6$ with area(Π)=3.

Carlitz and Riordan [CR64] defined the following natural q-analogue of C_n,

$$C_n(q) = \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)},$$

and showed that

Theorem 1.2.1.

$$C_n(q) = q^{k-1} \sum_{k=1}^{n} C_{k-1}(q) C_{n-k}(q), n \geq 1.$$

The statistic $bounce$ was introduced by Haglund in [Hag03]. Here we adopt the description of [HL] to define it: start by placing a ball at the upper corner (n,n) of a Dyck path Π, then push the ball straight left. Once the ball intersects a vertical step of the path, it "ricochets" straight down until it intersects the diagonal, after which the process is iterated; the ball goes left until it hits another vertical step of the path, then follows down to the diagonal, etc. On the way from (n,n) to $(0,0)$ the ball will strike the diagonal at various points (i_j, i_j). We define $bounce(\Pi)$ to be the sum of these i_j. For convenience, we also let the Dyck path so obtained in this process be the *bounce path* of Π and denote it by $b(\Pi)$.

In addition, we say Π is *balanced* if and only if $\Pi = b(\Pi)$. In Figure 1.2, a Dyck path Π is represented by the solid line and its bounce path $b(\Pi) = B$ is the dashing line. As illustrated, $bounce(\Pi) = 2 + 6 = 8$.

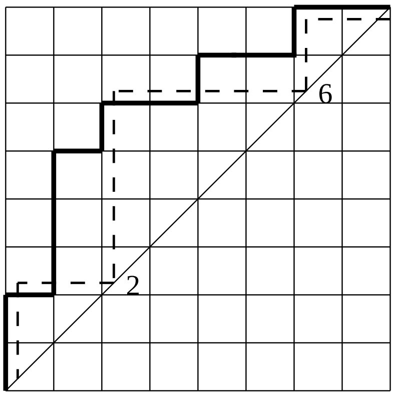


Figure 1.2: A Dyck path Π and its bounce path B.

Throughout this dissertation we use the standard notation

$$[n] := (1 - q^n)/(1 - q), [n]! := [1][2] \cdots [n], \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$$

for the q-analogue of the integer n, the q-factorial, and the q-binomial coefficient and $(a)_n := (1 - a)(1 - qa) \cdots (1 - q^{n-1}a)$ for the q-rising factorial. Sometimes it is necessary to write the base q explicitly as in $[n]_q$, $[n]!_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and $(a; q)_n$, but we often omit q if it is clear from the context. Occasionally, when $i + j + k = n$, we also use $\begin{bmatrix} n \\ i,j,k \end{bmatrix} := \frac{[n]!}{[i]![j]![k]!}$ to represent the q-trinomial coefficient.

We frequently make use of the following "q-binomial theorem" as a tool to prove identities.

Theorem 1.2.2. [And98, page 36] *The "q-binomial theorem". For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k = (-z; q)_n,$$

and

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} z^k = \frac{1}{(z;q)_n}.$$

In [GH96], Garsia and Haiman introduced a complicated rational function $C_n(q,t)$ which they proved has the following properties:

$$C_n(q,1) = \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)} = C_n(q)$$
$$q^{\binom{n}{2}} C_n(q,1/q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

In order to interpret $C_n(q,t)$, Haglund [Hag03] introduced the distribution function

$$F_n(q,t) = \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)} t^{bounce(\Pi)}$$

and conjectured that $F_n(q,t) = C_n(q,t)$. Garsia and Haglund ([GH02], [GH01]) proved this by using symmetric function methods, and as a byproduct also the conjecture in [GH96] that $C_n(q,t)$ is a polynomial with positive integer coefficients. Therefore, $C_n(q,t)$ is now called the (q,t)-*Catalan polynomial*.

There is a pair of basic statistics on the symmetric group S_n, inv and maj. In general, for any integer word or multiset permutation $w = w_1 w_2 \cdots w_n$, inv and maj are defined as

$$inv(w) = \sum_{\substack{i<j \\ w_i > w_j}} 1$$
$$maj(w) = \sum_{\substack{i \\ w_i > w_{i+1}}} i.$$

For later use, we also define the *descent set* of a word w

$$Des(w) := \{i : w_i > w_{i+1}\},$$

and the number of descents of w

$$des(w) := |Des(w)|.$$

The following result due to MacMahon [Mac60] is now classical.

Theorem 1.2.3. *For any fixed integer s and any vector $\alpha \in \mathbb{N}^s$, if M_α denotes the set of all permutations of the multiset $\{0^{\alpha_0}1^{\alpha_1}\cdots s^{\alpha_s}\}$, then*

$$\sum_{w \in M_\alpha} q^{inv(w)} = \begin{bmatrix} n \\ \alpha_1, \cdots, \alpha_s \end{bmatrix} = \sum_{w \in M_\alpha} q^{maj(w)}.$$

Accordingly we say that inv and maj are ***multiset Mahonian statistics***. If we let $s = n$, $\alpha_0 = 0$, $\alpha_1 = \cdots = \alpha_n = 1$ in the above theorem, then M_α specializes to the symmetric group S_n, $\left[\begin{smallmatrix} n \\ \alpha_1, \cdots, \alpha_s \end{smallmatrix}\right] = n!$, and therefore we say that the two statistics inv and maj on S_n are both ***Mahonian statistics***.

Given a Dyck path Π, if we encode each N step by a 0, and each E step by a 1, then from $(0,0)$ to (n,n) we obtain a word $w(\Pi)$ of n 0's and n 1's. Thus, the subset of $M_{n,n}$ each element of which has at least as many 0's as 1's in any initial segment is in bijection with $\mathcal{D}_n$. We call this special subset of 01 words the *Catalan words* of order n and denote it by CW_n. Hence we may associate with each Π the statistics of inv and maj by $inv(\Pi) = inv(w(\Pi))$ and $maj(\Pi) = maj(w(\Pi))$. It is easy to see that $\binom{n}{2} - inv(\Pi) = area(\Pi)$. The following classical result of MacMahon [Mac60, page 214] has a simple combinatorial proof in [FH85].

Theorem 1.2.4.

$$\sum_{\Pi \in \mathcal{D}_n} q^{maj(\Pi)} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Much of the theory about Dyck paths can be generalized to Schröder paths. In general for a lattice path Π that never goes below the diagonal line $x = y$, define *lower triangle* to be a triangle with vertices (i, j), $(i + 1, j)$ and $(i + 1, j + 1)$, and let the $area$ of Π, denoted by $area(\Pi)$, be the number of lower triangles between Π and the main diagonal. This new definition of $area$ agrees with the old one for Dyck paths, and is well defined for Schröder paths. Similarly, if we map $\mathcal{S}_{n,d}$ to the words of $n - d$ 0's, d 1's and $n - d$ 2's by replacing each N step by a 0, each D step by a 1 and each E step by a 2 in a Schröder path Π, then we have the maj statistic for Schröder paths. Bonin, et. al. showed that [BSS93]

Theorem 1.2.5.

$$\sum_{\Pi\in\mathcal{S}_{n,d}} q^{maj(\Pi)} = \frac{1}{[n-d+1]}\begin{bmatrix} 2n-d \\ n-d, n-d, d\end{bmatrix}.$$

In Figure 1.3 below, the Schröder path $\Pi \in S_{8,4}$ is encoded by 001221010221, which implies that $maj(\Pi) = 5 + 6 + 8 + 11 = 30$, and has area vector (0,1,1,0,0,2,1,0), which says $area(\Pi) = 1 + 1 + 2 + 1 = 5$. The length of each row, as computed from the number of lower triangles, is shown on the right.

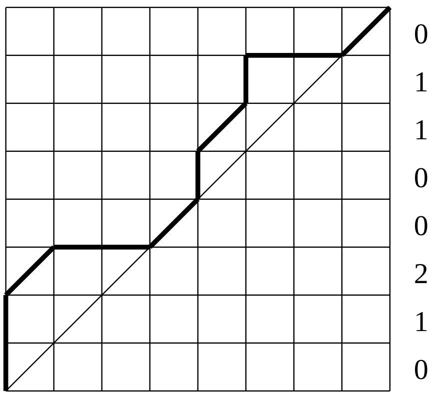


Figure 1.3: A Schröder path $\Pi \in S_{8,4}$ with $area(\Pi) = 5$ and $maj(\Pi) = 30$.

Egge, et. al [EHKK03] generalized $bounce$ to Schröder paths through a decomposition procedure and defined the (q,t)-Schröder polynomial

$$S_{n,d}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{area(\Pi)} t^{bounce(\Pi)}.$$

They generalized Garsia and Haiman's result to the following

$$q^{\binom{n}{2}-\binom{d}{2}} S_{n,d}(q,\frac{1}{q}) = \frac{1}{[n-d+1]} \begin{bmatrix} 2n-d \\ n-d, n-d, d \end{bmatrix},$$

They also conjectured that the (q,t)-Schröder polynomial is symmetric and made a stronger conjectural interpretation of $S_{n,d}(q,t)$ involving a linear operator ∇ defined on the modified Macdonald basis (for details see [EHKK03], [Hag04] or [HL]).

Conjecture 1.2.1. *For all integers n, d with $d \leq n$,*

$$S_{n,d}(q,t) =< \nabla e_n, e_{n-d} h_d > .$$

This was recently proved in [Hag04] and thus became the (q,t)-*Schröder Theorem*.

1.3 Summary of New Results

In this section, we list the main theorems in the chapters that follow.

First, in Chapter 2 we obtain some partial results about the symmetry of the (q,t)-Catalan polynomial and develop the theory of Permutation paths, which is a kind of generalized lattice path that contains Dyck paths as a subset.

Theorem 1.3.1. *The (q,t)-Catalan polynomial, $C_n(q,t)$, is equal to the following distribu-*

tion function defined on T_n, where T_n is a subset of the symmetric group S_n.

$$C_n(q,t) = \sum_{\sigma \in T_n} q^{inv(\sigma)} t^{\binom{n}{2} - maj(\sigma)}.$$

Definition 1.3.1. A Permutation path of order n is a lattice path from $(0,0)$ to (n,n), which never goes below the main diagonal (i,i), $0 \le i \le n$, or above the line $y = n$, and consists of NORTH $(0,1)$, EAST $(1,0)$ and SOUTH $(0,-1)$ steps but never repeats (i.e. no NORTH step followed or preceded by a SOUTH step). Let $\mathcal{P}_n$ denote the collection of Permutation paths of order n.

Theorem 1.3.2.

$$|\mathcal{P}_n| = n!\ .$$

Furthermore, there exists a weight-preserving bijection f between S_n and $\mathcal{P}_n$ that maps the inversion statistic to the area statistic. Namely, for any $\sigma \in S_n$, we have

$$inv(\sigma) = area(f(\sigma)).$$

Next we consider the restriction of f to $S_n(312)$, the 312-*avoiding* permutations, and call it f^*. We show that f^* is a bijective map between $S_n(312)$ and Dyck paths $\mathcal{D}_n$, a subset of the image set Permutation paths.

Theorem 1.3.3. *f^* is a weight-preserving bijection between $S_n(312)$ and Dyck paths $\mathcal{D}_n$ that maps the inversion statistic to the area statistic, and therefore*

$$\sum_{\sigma \in S_n(312)} q^{inv(\sigma)} = \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)}.$$

Theorem 1.3.3 can be generalized to the $k12\ldots(k-1)$-*avoiding* permutations in a less perfect way. For more details see Chapter 2. In the last section of Chapter 2 we introduce

Signed Permutation paths, which may be viewed as a generalization of both Permutation paths and Schröder paths. Some parallel results on Signed Permutation paths are also included.

In Chapter 3, we prove some graph theory enumeration results while investigating the parking function polynomial $R_n(q,t)$ as introduced in [HL]. We are able to show that $R_n(q,1)$ is equivalent to a group of other combinatorial statistics.

Theorem 1.3.4. ("Least-Child-Being-Monk") *Define $\mathcal{T}_{n+1,0}$ to be the set of labelled trees on $\{0,1,2,...,n+1\}$, such that the least labelled child of 0 has no children (we say such trees have the Least-Child-Being-Monk property). Then the cardinality of $\mathcal{T}_{n+1,0}$, which we denote by $t_{n+1,0}$, is equal to n^n.*

Corollary 1.3.5. *When n goes to infinity, the probability for a labelled tree to be "Least-Child-Being-Monk" is e^{-2}.*

Theorem 1.3.6. *Define $\mathcal{T}_{n+1,p}$ to be the set of labelled trees on $\{0,1,2,...,n+1\}$, such that the total number of descendants of the least labelled child of 0 is p. Then, the cardinality of $\mathcal{T}_{n+1,p}$, denoted by $t_{n+1,p}$, is equal to*

$$(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p}.$$

Corollary 1.3.7. *When n goes to infinity, the probability for a labelled tree on $\{0,1,2,\cdots,n\}$ to have the property that the least labelled child of 0 has exactly p descendants is*

$$\frac{(p+1)^{p-1}}{p\,!}\,e^{-2-p}.$$

Theorem 1.3.8. *(*Hereditary-Least-Single Trees Recurrence*) A rooted labelled tree is Hereditary-Least-Single if it has the property that every least child has no children. Let the number of Hereditary-Least-Single trees (rooted at the least labelled vertex) with n vertices be h_n.*

Then h_n satisfies the following recurrence:

$$\begin{aligned}h_n =&(n-1)h_{n-1} - 2\sum_{1\le i\le n-2} h_{n-i}h_{i+1}\binom{n-2}{i-1, n-i-1}\\ &+ \sum_{1\le i\le n-2}\sum_{1\le j\le n-i-1} ih_ih_jh_{n+1-i-j}\binom{n-2}{i-1, j-1, n-i-j}.\end{aligned}$$

The following list contains $\{h_n\}$, for n from 1 to 10, which is computed by Maple using our recurrence: 1, 1, 1, 4, 15, 96, 665, 6028, 60907, 725560 ...

Theorem 1.3.9. *Consider the exponential generating function $H(x) = \sum_{n\ge 0}\frac{h_{n+1}}{n!}x^n$. Then $H(x)$ satisfies the simple functional equation*

$$H^2(x) - H(x) + 1 = e^{xH(x)}.$$

Let

$$R_n(q,1) = \sum_{P\in\mathbb{P}_n} q^{area(P)},$$

where $\mathbb{P}_n$ is the set of parking functions [HL], and

$$M_n(q) = \sum_{\hat{s}\in\mathbb{M}_n} q^{area(\hat{s})},$$

where $\mathbb{M}_n$ is the set of major sequences [Kre80].

Theorem 1.3.10.

$$R_n(q,1) = M_n(q).$$

Corollary 1.3.11. *The 5 combinatorial statistics (see* [Bei82], [Bjö92] *and* [Ste02]*) are all equal, i.e.*

$$R_n(q,1) = R_n(1,q) = M_n(q) = T_{K_{n+1}}(1,q) = I_n(q),$$

and they all satisfy the following same recurrence:

$$Stat_1(q) = 1,$$

$$Stat_n(q) = \sum_{i=1}^{n} \binom{n-1}{i-1} [i]\ Stat_{i-1}(q)\ Stat_{n-i}(q).$$

In Chapter 4, we attack a combinatorial proof of a interesting identity derived from the limit case of the (q,t)-Schröder theorem. That is,

Theorem 1.3.12. *For* $n \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n \\ a_i>0}} q^{\sum_{i=1}^{k}\binom{a_i}{2}}\ t^{\sum_{i=1}^{k-1}(k-i)a_i}\ \frac{1}{(t^k;q)_{a_1}(q;q)_{a_k}} \\
&\times \prod_{i=1}^{k-1} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix} \frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}} \times\ (q;q)_n(t;t)_n \\
&= [z^n] \prod_{i,j\geq 0} (1+q^i t^j z) \times\ (q;q)_n(t;t)_n \\
&= \sum_{\sigma\in S_n} q^{maj(\sigma)}\ t^{\binom{n}{2}-maj(\sigma^{-1})},
\end{aligned}$$

Above we use $[z^n]f(z)$ to denote the coefficient of z^n in $f(z)$, a series in powers of z. Sometimes we also use $[z^n]\{f(z)\}$, especially when $f(z)$ is a long formula. We analyze several special cases, make parallels of some results by Carlitz [Car56] and also obtain some refined results and conjectures relating the (q,t)-Schröder polynomial statistics to the permutations whose longest increasing subsequence is of a fixed size. One of our byproducts is Theorem 1.3.13.

Definition 1.3.2. The *inverse* of a Catalan word $w \in CW_n$ is defined to be

$$w^{-1} = \overline{r(w)},$$

where r denotes the reverse operation and $^-$ denotes the complement operation that exchanges 0 and 1. We say w is an $involution$ if and only if $w = w^{-1}$.

Example 1.3.1. When n=3,

$$(000111)^{-1} = 000111,$$

$$(001011)^{-1} = 001011,$$

$$(001101)^{-1} = 010011,$$

$$(010011)^{-1} = 001101,$$

$$(010101)^{-1} = 010101.$$

So the involution set consists of 000111, 001011 and 010101.

It is easy to see that $w^{-1} \in CW_n$ if and only if $w \in CW_n$, so the inverse operation is closed on CW_n. Geometrically, given w, we may obtain w^{-1} by finding the Dyck path Π that w corresponds to under the natural map, reflecting Π over the NW-SE main diagonal to obtain a new Dyck path Π^{-1}, and then taking the Catalan word that corresponds to Π^{-1}.

Theorem 1.3.13.

$$\sum_{\substack{w \in CW_n: \\ w \text{ is an involution}}} q^{maj(w)-ndes(w)} = \sum_{\substack{\sigma \in S_n(123): \\ \sigma \text{ is an involution}}} q^{maj(\sigma)-maj(\sigma^{-1})}.$$

In Chapter 5 we turn to higher dimensional Schröder theory. That is, we study generalized Schröder paths inside a rectangle of length mn and width n. We derive a formula for the number of m-Schröder paths and study its q and (q,t)-analogues.

Definition 1.3.3. An m*-Dyck path* of order n is a lattice path from $(0,0)$ to (mn,n) which never goes below the main diagonal $\{(mi,i) : 0 \le i \le n\}$, with steps $(0,1)$ (or NORTH,

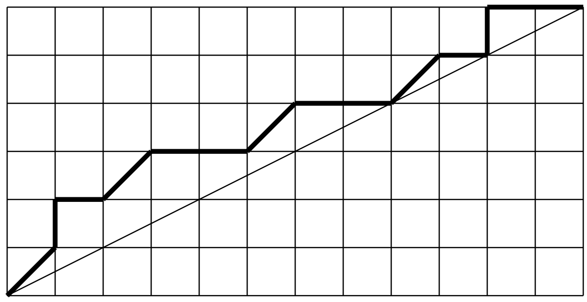


Figure 1.4: A 2-Schröder path of order 6 and with 2 diagonal steps.

for brevity *N*) and $(1,0)$ (or EAST, for brevity *E*). Let $\mathcal{D}_n^m$ denote the set of all m-Dyck paths of order n.

Definition 1.3.4. An *m-Schröder path* of order n and with d diagonal steps is a lattice path from $(0,0)$ to (mn,n), which never goes below the main diagonal $\{(mi,i) : 0 \leq i \leq n\}$, with $(0,1)$ (or NORTH, for brevity N), $(1,0)$ (or EAST, for brevity E) and exactly d (1,1) (or Diagonal, for brevity *D*) steps. Let $\mathcal{S}_{n,d}^m$ denote the set of all m-Schröder paths of order n and with d diagonal steps.

Figure 1.3 illustrates a 2-Schröder path $\Pi \in \mathcal{S}_{6,4}^2$.

Theorem 1.3.14. *The number of m-Schröder paths of order n and with d diagonal steps, denoted by $S_{n,d}^m$, is equal to*

$$\frac{1}{mn-d+1}\binom{mn+n-d}{mn-d,n-d,d}.$$

Remark 1.3.1. When m=1, the theorem above counts the ordinary Schröder paths. When $d=0$, the m-Dyck paths are counted. Actually the later result, i.e. $|\mathcal{D}_n^m| = \frac{1}{mn+1}\binom{mn+n}{n}$ is quite new [GH96] [HPW99], and not a single nice q-version seems to exist.

The following theorem generalizes a result of [BSS93].

Definition 1.3.5. Define the m-*Narayana* polynomial $d_n^m(q)$ over m-Schröder paths of order n to be

$$d_n^m(q) = \sum_{\Pi \in \mathcal{S}_n^m} q^{\mathrm{diag}(\Pi)},$$

where $\mathrm{diag}(\Pi)$ is the number of D steps in the m-Schröder path Π.

Theorem 1.3.15. *$d_n^m(q)$ has $q = -1$ as a root.*

In [FH85], there is a refined q-identity,

$$\sum_{k\geq 1} \sum_{w \in CW_{n,k}} q^{majw} = \sum_{k \geq 1} \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix},$$

where $CW_{n,k}$ is the set of Catalan words consisting of n 0's, n 1's, with k ascents (i.e. $k-1$ descents). For the generalized version, Cigler proved that there are exactly

$$\frac{1}{n}\binom{n}{k}\binom{mn}{k-1}$$

m-Dyck paths with k peaks (consecutive NE pairs) [Cig87]. In order to generalize the results of [FH85], we prove a generalized q-identity.

Theorem 1.3.16.

$$\sum_{k \geq d} \begin{bmatrix} k \\ d \end{bmatrix} \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} mn \\ k-1 \end{bmatrix} q^{(k-d)(k-1)} = \frac{1}{[mn-d+1]} \begin{bmatrix} mn+n-d \\ mn-d, n-d, d \end{bmatrix}.$$

In the last section of Chapter 5, we mention a conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov which defines the (q,t)-m-Schröder polynomial and relates it to the ∇ operator.

Chapter 2

Dyck Paths and Permutation Paths

2.1 On the Symmetry of the (q, t)-Catalan Polynomial

The (q, t)-Catalan polynomial $C_n(q, t)$, introduced in [GH96] as a rational function, is symmetric in q and t from its definition. However, the original definition is very complicated and it is only because of the fact that $F_n(q, t) = C_n(q, t)$, which is proved in [GH02] [GH01], do we know that $C_n(q, t)$ is a polynomial and has positive coefficients. Here

$$F_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)} t^{bounce(\Pi)},$$

where $area$ and $bounce$ are statistics on Dyck paths $\mathcal{D}_n$ as introduced in Chapter 1. There is no direct proof that $F_n(q, t)$ is symmetric, i.e., $F_n(q, t) = F_n(t, q)$. Therefore it is desirable to prove this combinatorially.

In this section we construct a bijection g between Dyck paths D_n and a special subgroup of S_n, which we call T_n, interchanging $area$ and inv, and $bounce$ and $\binom{n}{2} - maj$ simultaneously. Thereby we hope to prove the symmetry of the (q, t)-Catalan number combinatorially by working on the new distribution function of the statistics inv and $\binom{n}{2} - maj$

on T_n.

2.1.1 A Bijection Between $\mathcal{D}_n$ and a Special Set of Permutations

Given a Dyck path $\Pi \in \mathcal{D}_n$, we construct an injection g, which maps Π to a permutation $\sigma \in S_n$, with the properties that

$$\begin{aligned} area(\Pi) &= inv(\sigma), \\ bounce(\Pi) &= \binom{n}{2} - maj(\sigma). \end{aligned}$$

We define this map by a procedure involving two steps.

Step 1: when $\Pi \in \mathcal{D}_n$ is a balanced path.

First consider the case that Π is a balanced path. That is, $\Pi = b(\Pi)$. Suppose Π is made up of k blocks, i.e. Π has k right (from NORTH to EAST) turns and hits the diagonal exactly $k+1$ times including at $(0,0)$ and at (n,n). To better illustrate, we consider the case $k = 4$, as it will be easy to extend this to general n. As illustrated by Figure 2.1, let the sizes of the 4 blocks be a, b, c and d, respectively, from bottom to top. Notice that $n = a+b+c+d$.

The image permutation $\sigma = g(\Pi)$ is defined as follows.

$$\begin{aligned} \sigma =& a(a-1)\cdots 1(a+b)(a+b-1)\cdots(a+1)(a+b+c) \\ & (a+b+c-1)\cdots(a+b+1)n(n-1)\cdots(a+b+c+1). \end{aligned}$$

That is, σ is made up of 4 descending blocks, while any element in the j^{th} block is smaller

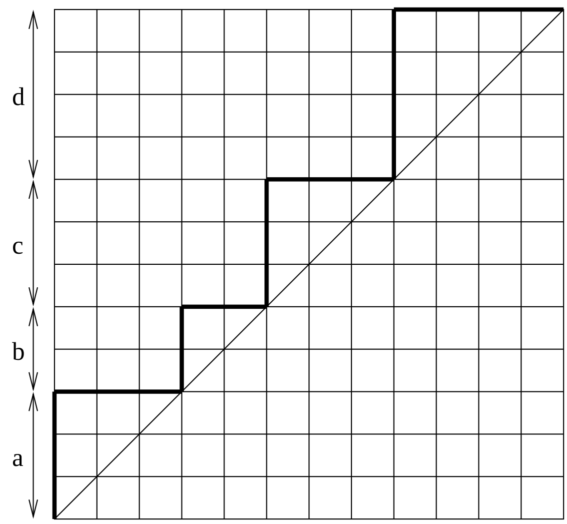


Figure 2.1: A balanced Dyck path of 4 blocks

than any element in the $(j+1)^{st}$ block, for $1 \leq j \leq 3$. Apparently,

$$area(\Pi) = \binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2},$$
$$inv(\sigma) = \binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2},$$

and therefore $area(\Pi) = inv(\sigma)$.

It is also easy to observe that $\sigma^{-1} = \sigma$. Because σ has descents everywhere except the last position of each block, we have

$$maj(\sigma^{-1}) = maj(\sigma) = \binom{n}{2} - a - (a+b) - (a+b+c).$$

Note that

$$bounce(\Pi) = a + (a+b) + (a+b+c).$$

Therefore,

$$bounce(\Pi) = \binom{n}{2} - maj(\sigma).$$

For convenience we define the set of "balanced permutations".

Definition 2.1.1. A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is said to be balanced if its one line notation can be partitioned into a number of continuously descending blocks, such that any element in a preceding block is smaller than any element in a later block, i.e., σ is of the form

$$\begin{aligned}\sigma =& \underline{a_1(a_1-1)\cdots 1}\ \underline{(a_1+a_2)(a_1+a_2-1)\cdots(a_1+1)}\ \cdots \\ & \underline{n(n-1)\cdots(a_{k-1}+\ldots+a_1+1)},\end{aligned}$$

for some integer $k \geq 1$.

Let the number of balanced permutations in S_n be b_n. Above we have established a bijection g between the balanced Dyck paths and balanced permutations. From both combinatorial structures, it is not hard to observe the following recurrence relation:

$$\begin{aligned} b_n &= \sum_{i=0}^{n} b_i, \\ b_0 &= 1. \end{aligned}$$

Therefore $b_n = 2^{n-1}$.

Step 2: for general $\Pi \in \mathcal{D}_n$.

Next we turn to the general case. It is useful to introduce the notion of *right balanced path*. Given a Dyck path Π, let $B = b(\Pi)$ be the bounce path of Π. Clearly B is a balanced path. Suppose B consists of $m+1$ blocks. That is, B has m left turns (from EAST to NORTH), which are just the hits at the main diagonal, and $m+1$ right turns (from NORTH to EAST). In general, Π has more area squares than B geometrically. For $1 \leq j \leq m$, if Π is identical with B from the j^{th} left turn to the end point (n, n), we say that Π is *j-right* *balanced*. In the case Π itself is a balanced path, i.e. B and Π are identical from the origin $(0,0)$, we say that Π is ***0-right balanced*** (so 0-right balanced means balanced). Clearly, Π being ***j-right balanced*** implies that Π is ***(j+1)-right balanced***. Furthermore we allow j to be any integer by convention.

In order to extend the map g to the entire set $\mathcal{D}_n$, we start with $B = b(\Pi)$ and let $\Pi^{(0)} = B$. According to **Step 1**, there is a permutation $\sigma^{(0)} = g(\Pi^{(0)})$ such that

$$\begin{aligned} area(\Pi^{(0)}) &= inv(\sigma^{(0)}), \\ bounce(\Pi^{(0)}) &= \binom{n}{2} - maj(\sigma^{(0)}). \end{aligned}$$

Intuitively, we obtain $g(\Pi)$ by each time adding squares to the area between two consecutive blocks of $\Pi^{(0)}$, so that it gets closer to Π, and finding the corresponding permutation that is the image of the modified path. Since $\Pi^{(0)} = B$ has $m+1$ blocks, we will reach Π together with $g(\Pi)$ after m steps. In other words, each time we modify the path obtained earlier to become "more" *right balanced*, until we get Π:

$$B = \Pi^{(0)} \to \Pi^{(1)} \to \cdots \to \Pi^{(m)} = \Pi,$$

where each $\Pi^{(j)}$ is *j-right balanced*, $0 \leq j \leq m$.

Inductively, for each $j, 1 \leq j \leq m$, we modify $\sigma^{(j-1)} = g(\Pi^{(j-1)})$ to obtain $\sigma^{(j)} = g(\Pi^{(j)})$, which satisfies

$$\begin{aligned} area(\Pi^{(j)}) &= inv(\sigma^{(j)}), \\ bounce(\Pi^{(j)}) &= \binom{n}{2} - maj(\sigma^{(j)}). \end{aligned}$$

For technical reasons, this modification is realized through a trick: given $\sigma^{(j-1)}$, we work on its inverse $(\sigma^{(j-1)})^{-1}$, instead of $\sigma^{(j-1)}$ itself, to obtain $(\sigma^{(j)})^{-1}$, and afterwards take the inverse again to obtain $\sigma^{(j)}$. We illustrate this process for the case $m+1=4$, i.e., B consists of 4 blocks, and use the same setup of Π in **Step 1** for B. W.O.L.G., we choose to show the second stage: assume for $j=2$, we have already found $\sigma^{(1)}=g(\Pi^{(1)})$ which satisfies the two statistical identities with $\Pi^{(1)}$, we go on to construct $\sigma^{(2)}=g(\Pi^{(2)})$. For technical as well as symbolic convenience, let $\rho=\left(\sigma^{(1)}\right)^{-1}$ and $\tau=\left(\sigma^{(2)}\right)^{-1}$. Also, we make the inductive assumption that ρ is of the particular form

$$\begin{aligned}\rho=&\rho_1\cdots\rho_{a+b}(a+b+c)(a+b+c-1)\cdots(a+b+1)\\&n(n-1)\cdots(a+b+c+1),\end{aligned}$$

where $\rho_1\cdots\rho_{a+b}$ could be any permutation in S_{a+b}. Finally let $\Pi^{(2)}$ be obtained from $\Pi^{(1)}$ by adding a strip of b_1 squares to the top of the second EAST segment (from the bottom) of B, adding a strip of b_2 squares to the top of the just added strip of length b_1, $\ldots$, and in the end adding a strip of b_r squares to the top of the previously added strip of length b_{r-1}. To understand this construction, be aware that

$$\begin{aligned}&1\le b_r\le\cdots\le b_1\le b,\\&1\le r\le c-1,\end{aligned}$$

and see Figure 2.2 (if $r=0$ then just let $\tau=\rho$, we are done and pass on; if $r=c$, then $\Pi^{(2)}$, and therefore $\Pi^{(1)}$ would have a different balanced path than B which is impossible).

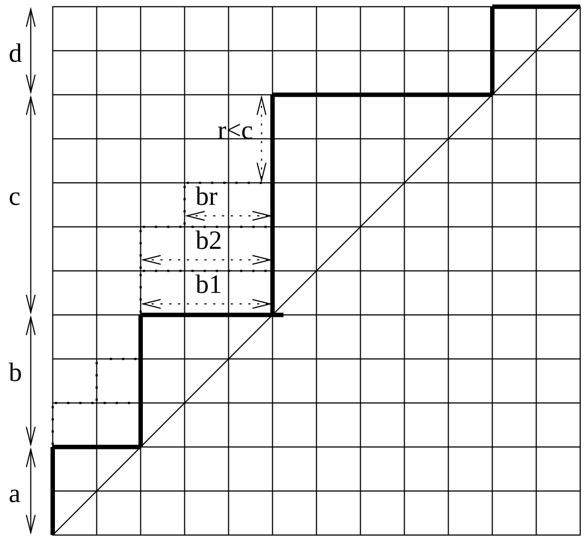


Figure 2.2: Add strips to $\Pi^{(1)}$ to obtain $\Pi^{(2)}$

Now τ is ready.

$$\begin{aligned}\tau =& \rho_1 \cdots \rho_{a+b-b_1}(a+b+c)\rho_{a+b-b_1+1} \cdots \rho_{a+b-b_2}(a+b+c-1) \\ & \rho_{a+b-b_2+1} \cdots\cdots \rho_{a+b-b_r}(a+b+c-(r-1))\rho_{a+b-b_r+1} \cdots \rho_{a+b} \\ & (a+b+c-r) \cdots (a+b+1)\; n(n-1) \cdots (a+b+c+1).\end{aligned}$$

Intuitively, in the process to find τ, we just move $(a+b+c)$ b_1 positions left to correspond to the b_1 squares in the newly added first row, and $(a+b+c-1)$ b_2 positions left, etc, while leave the remaining untouched.

Obviously, $area(\Pi^{(2)}) - area(\Pi^{(2)}) = b_1 + \cdots + b_r$. Note that for any permutation ς, we have $inv(\varsigma) = inv(\varsigma^{-1})$. Therefore,

$$\begin{aligned}inv(\sigma^{(2)}) - inv(\sigma^{(1)}) &= inv(\tau) - inv(\rho) \\ &= b_1 + \cdots + b_r,\end{aligned}$$

and hence we have the first statistical identity for $\sigma^{(2)}$ and $\Pi^{(2)}$ by the inductive hypothesis.

Since B is the balanced path of all the $\Pi^{(j)}$s, $bounce(\Pi^{(2)}) = bounce(\Pi^{(1)})$ by the definition of the bounce statistic. Next we prove that $maj(\sigma^{(2)}) = maj(\sigma^{(1)})$ by showing that $\sigma^{(1)} = \tau^{-1}$ and $\sigma^{(2)} = \rho^{-1}$ have the same descent set, so that we have the desired second statistical identity for $\sigma^{(2)}$ and $\Pi^{(2)}$.

Lemma 2.1.1. *For any integer i, with $1 \leq i \leq n-1$,*

$$i \in Des(\rho^{-1}) \;\Leftrightarrow\; i \in Des(\tau^{-1}).$$

Proof. There are three cases for i.

- $a+b+c+1 \leq i \leq n$. This case is trivial.
- $1 \leq i \leq a+b$. Observe that by the construction of τ,

$$\begin{aligned} i \in Des(\rho^{-1}) \;&\Leftrightarrow\; \exists x < y, s.t. \begin{cases} 1 \leq x < y \leq a+b, \\ \rho(x) = i+1, \\ \text{and } \rho(y) = i. \end{cases} \\ &\Leftrightarrow\; \exists x' < y', s.t. \begin{cases} 1 \leq x' < y' \leq a+b+r, \\ \tau(x') = i+1, \\ \text{and } \tau(y') = i. \end{cases} \\ &\Leftrightarrow\; i \in Des(\tau^{-1}). \end{aligned}$$

- $a+b+1 \leq i \leq a+b+c$. Then $\rho(i) = a+b+1+(a+b+c-r)$. Because $\rho^{-1}(a+b+c+1) = n$, $i \in Des(\rho^{-1})$ iff $i \neq a+b+c$. On the other hand, this is also the case for τ since it is also true that

$$\tau^{-1}(a+b+c) < \tau^{-1}(a+b+c-1) < \cdots < \tau^{-1}(a+b),$$
$$\text{and } \tau^{-1}(a+b+c+1) = n.$$

$\square$

So we have found a map g from Dyck paths to permutations preserving the statistical identities. The last thing we need to prove is that g is an injection so that it is reversible.

Theorem 2.1.2. *The map g described in the above algorithm is an injection.*

Proof. Given two Dyck paths Π_1 and Π_2, we prove that they result in different images.

Case 1: suppose Π_1 and Π_2 have the same bounce path B and they are obtained by the following procedures, respectively,

$$B = \Pi_1^{(0)} \to \Pi_1^{(1)} \to \cdots \to \Pi_1^{(m)} = \Pi_1,$$
$$B = \Pi_2^{(0)} \to \Pi_2^{(1)} \to \cdots \to \Pi_2^{(m)} = \Pi_2.$$

where $\Pi^{(j)}$ is *j-right balanced*, $0 \leq j \leq m$. For each j, let $\sigma_1^{(j)} = g(\Pi_1^{(j)})$ and $\sigma_2^{(j)} = g(\Pi_2^{(j)})$. So, $g(\Pi_1) = g(\Pi_1^{(m)}) = \sigma_1^{(m)}$ and $g(\Pi_2) = g(\Pi_2^{(m)}) = \sigma_2^{(m)}$.

Assume the two procedures do not agree with each other for the first time at the i^{th} step, i.e., $\Pi_1^{(j)} = \Pi_2^{(j)}$, and therefore $\sigma_1^{(j)} = \sigma_2^{(j)}, 0 \leq j \leq i-1$, but $\Pi_1^{(i)} \neq \Pi_2^{(i)}$. For convenience, let $\rho = (\sigma_1^{(i)})^{-1}$ and $\tau = (\sigma_2^{(i)})^{-1}$. W.O.L.G, assume the first disagreement of Π_1 and Π_2 is that Π_1 adds a strip of more squares than Π_2 does to the same row of the j^{th} block. As a result, while constructing ρ there will be an integer x moving r places left, passing r numbers $y_1, \cdots, y_r$; but in the process of constructing τ the same integer x only moves s places left, with $s < r$, passing the very s numbers $y_1, \cdots, y_s$. Notice that the relevant order of x and $y_1, \cdots, y_r$ will never change again after that in the later process of constructing $\sigma_1^{(m)}$ and $\sigma_2^{(m)}$. So, x is to the right of y_r in $(\sigma_1^{(m)})^{-1}$, but to the left of y_r in $(\sigma_2^{(m)})^{-1}$, and thus $\sigma_1^{(m)} \neq \sigma_2^{(m)}$.

Case 2: suppose Π_1 and Π_2 have different bounce paths, respectively B_1 and B_2, and Π_1 and Π_2 are obtained by the following procedures,

$$B_1 = \Pi_1^{(0)} \to \Pi_1^{(1)} \to \cdots \to \Pi_1^{(m_1)} = \Pi_1,$$
$$B_2 = \Pi_2^{(0)} \to \Pi_2^{(1)} \to \cdots \to \Pi_2^{(m_2)} = \Pi_2.$$

Also let $\sigma_1^{(j)} = g(\Pi_1^{(j)})$ and $\sigma_2^{(j)} = g(\Pi_2^{(j)})$. So $g(\Pi_1) = \sigma_1^{(m_1)}$ and $g(\Pi_2) = \sigma_2^{(m_2)}$.

W.O.L.G., assume B_1 and B_2 do not agree for the first time at the i^{th} block (from bottom to top), and that the i^{th} block of B_1 is larger in size than the i^{th} block of B_2. Then, the i^{th} descending block of $(\sigma_1^{(0)})^{-1}$ will have the form $(x+r)\cdots(x+1)x$ and accordingly, the i^{th} descending block of $(\sigma_2^{(0)})^{-1}$ has the form $(x+s)\cdots(x+1)x$ with $r \geq s+1$. Be aware that $(x+s+1)$ is always to the left of $(x+s)$ in every $(\sigma_1^{(j)})^{-1}$ during our procedure of reaching Π_1 because we maintain the relative order of each block. Hence $(x+s+1)$ is to the left of $(x+s)$ in $(\sigma_1^{(m_1)})^{-1}$. On the other hand, observe that the $(i+1)^{st}$ descending block of $(\sigma_2^{(0)})^{-1}$ is of the form $y(y-1)\cdots(x+s+1)$. That is, $(x+s+1)$ is at the the end of the next block of $(x+s)$ in $(\sigma_2^{(0)})^{-1}$. By the rule of our algorithm, $(x+s+1)$ stays unmoved during the modification from $(\sigma_2^{(i)})^{-1}$ to $(\sigma_2^{(i+1)})^{-1}$ (again, because otherwise Π_2 should have a different bounce path from B_2). Therefore $(x+s+1)$ is to the right of $(x+s)$ in $(\sigma_2^{(m_2)})^{-1}$, and so $\sigma_1^{(m_1)} \neq \sigma_2^{(m_2)}$. $\square$

Hence g is a bijection between $\mathcal{D}_n$ and a subset of S_n, which we call T_n, and accordingly we have the following corollary.

Corollary 2.1.3. $C_n(q,t)$*, the* (q,t)*-Catalan sequence is equal to the following distribution* *function defined on T_n.*

$$C_n(q,t) = F_n(q,t) = \sum_{\sigma \in T_n} q^{inv(\sigma)} t^{\binom{n}{2} - maj(\sigma)}.$$

Example 2.1.1. When $n = 3$, $T_3 = \{321, 231, 213, 132, 123\}$.

When $n = 4$, $T_4 = \{4321, 3214, 1324, 2134, 1234, 2143, 1243, 1432, 2314, 3421, 3241, 2431, 3142, 1342\}$.

2.1.2 The Foata-Schützenberger Involution

An involution ψ is described in [FS78] that interchanges inv and maj and preserves the descent set. More specifically, we have the following theorem.

Theorem 2.1.4. [FS78] *There exists an involution $\psi : S_n \to S_n$ with the property that $inv(\psi(\sigma)) = maj(\sigma)$ and $inv(\sigma) = maj(\psi(\sigma))$ hold simultaneously.*

Lemma 2.1.5. *The following polynomial is symmetric in q and t:*

$$S_n(q,t) = \sum_{\sigma \in S_n} q^{inv(\sigma)} t^{\binom{n}{2} - maj(\sigma)}.$$

Proof. Let c be the "complement" map such that $c(\sigma) = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$ for $\sigma \in S_n$. Note that $\kappa = c\psi$, where ψ is the Foata-Schützenberger involution in Theorem 2.1.4, is a bijection from S_n to S_n. Furthermore,

$$\begin{aligned} inv(\kappa(\sigma)) &= \binom{n}{2} - inv(\psi(\sigma)) \\ &= \binom{n}{2} - maj(\sigma), \end{aligned}$$

and

$$\begin{aligned}\binom{n}{2} - maj(\kappa(\sigma)) &= \binom{n}{2} - (\binom{n}{2} - maj(\psi(\sigma))) \\ &= maj(\psi(\sigma)) \\ &= inv(\sigma).\end{aligned}$$

Therefore,

$$\begin{aligned}S_n(q,t) &= \sum_{\sigma \in S_n} q^{inv(\sigma)} t^{\binom{n}{2} - maj(\sigma)} \\ &= \sum_{\sigma \in S_n} q^{inv(\kappa(\sigma))} t^{\binom{n}{2} - maj(\kappa(\sigma))} \\ &= \sum_{\sigma \in S_n} q^{\binom{n}{2} - maj(\sigma)} t^{inv(\sigma)} \\ &= S_n(t,q).\end{aligned}$$

□

Notice that if we could replace S_n by T_n in the above theorem, then the symmetry of the (q,t)-Catalan polynomial would be proved due to the bijection g between $\mathcal{D}_n$ and T_n. That however, would require a proof for $\kappa = c\psi$ to be closed on T_n and would require a better understanding of the set T_n.

2.2 The Theory of Permutation Paths

2.2.1 The $n!$ Permutation Paths and the Bijection f between S_n and $\mathcal{P}_n$

As an attempt to extend the idea of Dyck paths, we introduce the notion of Permutation paths and develop the related theory.

Definition 2.2.1. A Permutation path of order n is a lattice path from $(0, 0)$ to (n, n) which never goes below the main diagonal (i, i), $0 \leq i \leq n$, or above the line $y = n$, and consists of NORTH $(0, 1)$, EAST $(1, 0)$ and SOUTH $(0, -1)$ steps but never repeats (i.e. no NORTH step followed or preceded by a SOUTH step).

Let $\mathcal{P}_n$ denote the collection of Permutation paths of order n. Figure 2.3 is an illustration of the 6 members in $\mathcal{P}_3$.

Theorem 2.2.1.

$$|\mathcal{P}_n| = n! \ .$$

Proof. Note that any Permutation path $\Pi \in \mathcal{P}_n$ consists of some NORTH steps, some SOUTH steps and exactly n EAST steps made at different columns. In more detail, for j from 1 to $n-1$, these n EAST steps are in the form of $(j-1, h_j) \to (j, h_j)$ where h_j could be any integer satisfying $j \leq h_j \leq n$ because Π never goes below the main diagonal or above the line $y = n$. In fact Π is uniquely decided by these EAST steps or equivalently the sequence of their heights (y-values) $(h_1, \cdots, h_n)$. Once these EAST steps are fixed, we just connect them up by continuous NORTH or SOUTH steps, or possibly an empty vertical move if $h_j = h_{j+1}$. Since repeats are not allowed, the connection is unique. Therefore the number of Permutation paths of order n is equal to the number of sequences $(h_1, \cdots, h_n)$. For each j, since $j \leq h_j \leq n$, there are $n + 1 - j$ ways to choose h_j. Furthermore, the choices of different h_j's are independent. So we are done. $\square$

The cardinality of $n!$ naturally motivates us to give a bijection between $\mathcal{P}_n$ and the symmetric group S_n. Actually the previous proof already provides hints of this bijection.

Lemma 2.2.2. *There exists a bijection f between the symmetric group S_n and the Permutation paths $\mathcal{P}_n$.*

Proof. For any $\Pi \in \mathcal{P}_n$, define the *height sequence* of Π to be the sequence of the heights (y-values) of Π's EAST steps, from left to right, as in the previous proof. Denote it by $h(\Pi) = (h_1^\Pi, \ldots, h_n^\Pi)$. Clearly $j \leq h_j^\Pi \leq n$ for each j and any integer vector satisfying this requirement is a height sequence for some uniquely decided Permutation path.

Given $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, find its "lifted word" $l(\sigma) = l_1^\sigma \cdots l_n^\sigma$ where for $1 \leq j \leq n$, l_j^σ is what σ_j would become if we map $\{\sigma_j, \cdots, \sigma_n\}$ to the set $\{j, \cdots, n\}$ keeping the relative order of each element.

For example, if $n = 6$ and $\sigma = \ 6\ 2\ 4\ 3\ 5\ 1$, then $l(\sigma) = 6\ 3\ 5\ 5\ 6\ 6$: $l_1^\sigma = 6$ because $\sigma_1 = 6$ is the biggest among $\{6, 2, 4, 3, 5, 1\}$ when the set $\{6, 2, 4, 3, 5, 1\}$ is mapped to $\{1, 2, 3, 4, 5, 6\}$ where 6 is also the biggest; $l_2^\sigma = 3$ because $\sigma_2 = 2$ is the second smallest in $\{2, 4, 3, 5, 1\}$ when the set $\{2, 4, 3, 5, 1\}$ is mapped to $\{2, 3, 4, 5, 6\}$ where the second smallest element is 3, etc. Notice that l_1^σ is always equal to σ_1, l_n^σ is always n and that $i \leq l_i^\sigma \leq n$ for every i.

So, $l(\sigma)$ is a height sequence. Find its corresponding Permutation path Π and let $f(\sigma) = \Pi$.

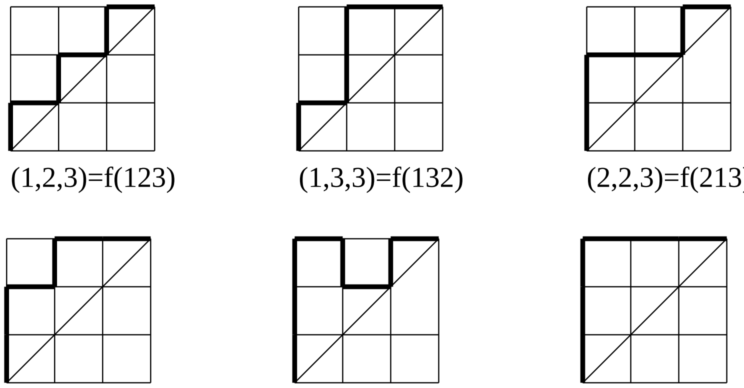
Conversely, given any Permutation path Π, locate its height sequence $h(\Pi)$. Actually we will use $h(\Pi)$ as the "lifted word" to find $\sigma = f^{-1}(\Pi)$. Let $\sigma_1 = h_1^\Pi$. For i from 2 to n, let σ_i be the $(h_i^\Pi + 1 - i)^{th}$ smallest number in the set $\{1, \cdots, n\} - \{\sigma_1, \cdots, \sigma_{i-1}\}$. Clearly $l(\sigma) = h(\Pi)$ and hence the permutation σ so obtained is $f^{-1}(\Pi)$.

So we have established the bijection f as desired. $\square$

Sometimes we use the height sequence to represent a Permutation path for convenience.

That is, we may write Π directly as $\Pi = (h_1^\Pi, \ldots, h_n^\Pi)$, where $h(\Pi)$ is the height sequence of Π.

Example 2.2.1. When $n = 3$, there are $3! = 6$ Permutation paths, and their correspondence with the permutations in S_3 through f is indicated in Figure 2.3.



(1,2,3)=f(123) (1,3,3)=f(132) (2,2,3)=f(213)

(2,3,3)=f(231) (3,2,3)=f(312) (3,3,3)=f(321)

Figure 2.3: Correspondence between $\mathcal{P}_3$ and S_3 under f

The $area$ statistic, previously defined on Dyck paths $\mathcal{D}_n$, may be extended to $\mathcal{P}_n$ naturally since a Permutation path never goes below the main diagonal. Simply, we let $area(\Pi)$ be the number of complete squares between Π and the main diagonal line $y = x$. This agrees with the old definition of $area$ on $\mathcal{D}_n$, which is a subset of $\mathcal{P}_n$, if Π is also a Dyck path.

The bijection f has the nice property of mapping inv to $area$.

Theorem 2.2.3. *f is a weight-preserving bijection between S_n and $\mathcal{P}_n$ that maps the inversion statistic to the area statistic. Namely, for any $\sigma \in S_n$, we have*

$$inv(\sigma) = area(f(\sigma)).$$

Proof. Let $f(\sigma) = \Pi$ and $h(\Pi) = (h_1^\Pi, \cdots, h_n^\Pi)$. Notice that

$$area(\Pi) = \sum_{i=1}^{n} h_i^\Pi - i$$

and for $1 \leq i \leq n-1$ ($h_n^\Pi - n = 0$) we have

$$h_i^\Pi - i = |\{j : \sigma_i > \sigma_j \text{ and } i < j \leq n\}|.$$

So it is clear. $\square$

Corollary 2.2.4.

$$\sum_{\Pi \in \mathcal{P}_n} q^{area(\Pi)} = [n]!.$$

Proof. Recall that the inv statistic is Mahonian on S_n [Mac60],

$$\sum_{\sigma \in S_n} q^{inv(\sigma)} = [n]!.$$

So the conclusion follows from Theorem 2.2.3. Alternatively, it could also be obtained easily by induction. $\square$

As we did for Dyck paths, we may associate each Permutation path with an appropriate word. Still encode each N step by a 0, each E step by a 1, and in addition encode each S step by a special character 0^*.

Definition 2.2.2. A Permutation word of order n is a permutation of the multiset $\{0^{n+s}1^n0^{*s}\}$, where $1 \leq s \leq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, with the property that for $1 \leq i \leq 2n+2s$, in the initial subword $w = w_1 w_2 \cdots w_i$,

- the number of 0's is at least as many as the sum of the numbers of 1's and the number of 0^*'s;

- the number of 0's minus the number of 0^*'s is at most n.

We let AW_n denote the set of Permutation words of order n.

Definition 2.2.3. The inversion statistic of a Permutation word $w = w_1 w_2 \cdots w_{2n+2s}$ is defined to be

$$inv(w) = \sum_{i:\ w_i=1} n_1(i) - n_2(i),$$

where $n_1(i)$ is the number of 0's after w_i and $n_2(i)$ is the number of 0^*'s after w_i.

The inversion statistic so defined on AW_n is apparently an extension of the inversion statistic on the Catalan words CW_n, where $s = 0$, i.e. no 0^*'s exist.

Corollary 2.2.5.

$$\sum_{w \in AW_n} q^{\binom{n}{2} - inv(w)} = [n]!.$$

Proof. For any $w \in AW_n$, it is easy to see that

$$\binom{n}{2} - inv(w) = area(\Pi(w)),$$

where $\Pi(w)$ is the Permutation path that w corresponds to. □

2.2.2 Restricting f to Some Pattern Forbidding Permutations

Since Dyck paths $\mathcal{D}_n$ is a subset of $\mathcal{P}_n$ and at least the $area$ statistic is extended to $\mathcal{P}_n$ in a nice way, we study $f^{-1}(\mathcal{D}_n)$ and some other related objects in order to understand the Catalan phenomena as well as derive more general theories.

First we need some preliminary background on the theory of patterns.

Given permutations $\tau \in S_k$ and $\sigma \in S_n$, we define an *occurrence* of the pattern τ in σ to be a choice of k slots

$$1 \le i_1 < \cdots < i_k \le n$$

such that the sequence $\sigma_{i_1}, \ldots, \sigma_{i_k}$ is in the same order of relative size as the sequence $\tau_1, \ldots, \tau_k$. In other words, for $1 \leq j_1 < j_2 \leq k$,

$$\sigma_{i_{j_1}} < \sigma_{i_{j_2}} \text{ iff } \tau_{j_1} < \tau_{j_2}.$$

Sometimes we also say that $\{\sigma_{i_1}, \cdots, \sigma_{i_k}\}$ is a τ-$occurrence$ in σ.

Accordingly, if σ does not contain any τ-occurrences of the pattern , we say that σ is τ-$avoiding$. Denote the set of all τ-avoiding permutations in S_n by $S_n(\tau)$ [Pri97].

Example 2.2.2. Consider $\sigma = 51324 \in S_5$ and $\tau = 123 \in S_3$. σ is NOT τ-avoiding because

$$\{\sigma_2, \sigma_3, \sigma_5\} = \{1, 3, 4\}$$

is a τ-occurence in σ. Notice that

$$\{\sigma_2, \sigma_4, \sigma_5\} = \{1, 2, 4\}$$

is also a τ-occurence in σ, but finding one occurence is sufficient for our purpose here.

Alternatively, let's consider $\sigma' = 32541 \in S_5$ and $\tau' = 312 \in S_3$. Then σ' is τ'-avoiding because we can not find any 312-occurence in $\sigma' = 32541$. So we can say that

$$32541 \in S_5(312).$$

The theory of pattern avoidance has been studied extensively. It is now well known (see, for example, [Knu73]) that for any $\tau \in S_3$, $|S_n(\tau)| = C_n$. Partly because of this, we are motivated to consider $f^{-1}(\mathcal{D}_n)$, and we find that this pre-image is indeed $S_n(312)$. In fact, there have been two direct bijections between $S_n(312)$ and $\mathcal{D}_n$ which have occurred recently in the literature: [Kra01] and [BK01]. The latter also gives a weight-preserving bijection exchanging the $inversion$ and $area$ statistics but it is different from our f.

Let's consider the restriction of f on $S_n(312)$, the 312-*avoiding* permutations, and call it f^*. We prove that f^* is a bijective map between $S_n(312)$ and Dyck paths $\mathcal{D}_n$, a subset of the image set of Permutation paths.

Theorem 2.2.6. *f^* is a weight-preserving bijection between $S_n(312)$ and Dyck paths $\mathcal{D}_n$ that maps the inversion statistic to the area statistic, and therefore*

$$\sum_{\sigma \in S_n(312)} q^{inv(\sigma)} = \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)},$$

where the right hand side is Carlitz-Riordan's q-Catalan polynomial $C_n(q)$ [CR64] *which satisfies the recurrence*

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q).$$

Proof. Any Permutation path $\Pi \in \mathcal{P}_n$ can be uniquely represented by its height vector $h(\Pi) = (h_1^\Pi, \ldots, h_n^\Pi)$. Notice that $\Pi \in \mathcal{D}_n$ if and only if for $1 \le i \le n-1$, so we have

$$h_i^\Pi \le h_{i+1}^\Pi.$$

Given $\sigma \in S_n(312)$, we prove $f(\sigma) = \Pi \in \mathcal{D}_n$. Recall that $h(\Pi) = l(\sigma)$. This means for $1 \le i \le n$, $h_i^\Pi = l_i^\sigma$ is what σ_i would become if we map $\{\sigma_i, \cdots, \sigma_n\}$ to the set $\{i, \cdots, n\}$ keeping the relative order of each element. Revising this a little bit, let $(l_{i+1}^\sigma)^-$ denote what σ_{i+1} would be at the "previous stage", i.e, when we map $\{\sigma_i, \cdots, \sigma_n\}$ to $\{i, \cdots, n\}$ rather than replacing i by $i+1$ (for which we would get l_{i+1}^σ). Now note that

- If $\sigma_{i+1} > \sigma_i$, then $l_{i+1}^\sigma = (l_{i+1}^\sigma)^- > l_i^\sigma$.

- If $\sigma_{i+1} < \sigma_i$, then $l^\sigma_{i+1} = (l^\sigma_{i+1})^- + 1$. So

$$\begin{aligned}
& l^\sigma_{i+1} \geq l^\sigma_i \\
& \Leftrightarrow (l^\sigma_{i+1})^- \geq l^\sigma_i - 1 \\
& \Leftrightarrow \nexists k, \text{s.t. } i < i+1 < k \text{ and } \sigma_{i+1} < \sigma_k < \sigma_i.
\end{aligned}$$

From the above conditions, it is clear that $\sigma \in S_n(312)$ implies $\Pi \in \mathcal{D}_n$.

Conversely, given $\Pi \in \mathcal{D}_n$, consider its pre-image $f^{-1}(\Pi) = \sigma$. Suppose $\sigma \notin S_n(312)$. Take a *minimal* 312-occurence $\{\sigma_i, \sigma_j, \sigma_k\}$ in the sense

$$\begin{aligned}
& i < j < k, \\
& \sigma_j < \sigma_k < \sigma_i, \\
& \text{and } |j-i| + |k-i| \text{ is } \textit{minimal}.
\end{aligned}$$

$\Pi \in \mathcal{D}_n$ implies that $h^\Pi_i \leq h^\Pi_{i+1}$, and hence $l^\sigma_i \leq l^\sigma_{i+1}$. This requires $\sigma_i - \sigma_{i+1} \leq 1$, and hence $j \neq i+1$. Then what about σ_{i+1}? If $\sigma_{i+1} > \sigma_k$, then $\{\sigma_{i+1}, \sigma_j, \sigma_k\}$ would be another 312-occurence, violating the minimality. So $\sigma_{i+1} < \sigma_k$. But then $\{\sigma_i, \sigma_{i+1}, \sigma_k\}$ would be a "less" 312-occurence. This shows that $\sigma \notin S_n(312)$ is impossible.

Therefore $f(S_n(312)) = \mathcal{D}_n$, or we may say that there exists a bijection f^* from $S_n(312)$ to $\mathcal{D}_n$.

Since f is weight-preserving, its restriction f^* is also weight-preserving. $\square$

The above result gives rise to more general questions. Since we now know $f^{-1}(S_n(312)) = \mathcal{D}_n$, what is $f^{-1}(S_n(k12\ldots k-1))$ for general k? We answer this question partially by giving a lower bound.

For convenience, let the restriction of f on $S_n(k12\cdots k-1)$, the set of $k12\cdots(k-1)$-*avoiding* permutations, be $f^{(k)}$. To explain our combinatorial interpretation, we need the following definition.

Definition 2.2.4. An m-$cave$ of a Permutation path Π, with height sequence $h(\Pi) = h_1^{\Pi} h_2^{\Pi} \cdots h_n^{\Pi}$, is a step i satisfying that

$$max\{h_1^{\Pi}, \ldots, h_{i-1}^{\Pi}\} - h_i^{\Pi} = m,$$

where $m \geq 1$. For an m-cave c, sometimes we say that c is of depth m. An m-$triangle$ is a sequence of caves, $c_1, c_2, \cdots, c_m$, not necessarily continuous but in order from left to right, where c_j is of depth *at least* $m + 1 - j$ for each $1 \leq j \leq m$. If a Permutation path Π does not contain any m-triangle, we say that P is m-$triangle$-$forbidding$. Denote the set of m-triangle-forbidding Permutation paths of order n by $\mathcal{F}_{n,m}$.

Geometrically, an m-cave of $\Pi \in \mathcal{P}_n$ is a step which is m squares down compared with the highest level that Π has reached earlier (observe that the highest level that Π will reach later is always $y = n$). The juxtaposition of the not-necessarily continuous sequence of caves in an m-triangle contains an isosceles right triangle with leg length m. Since we require $m \geq 1$, a Dyck path has no m-cave or m-triangle. In addition, $\mathcal{F}_{n,1} = \mathcal{D}_n$. The following figure illustrates a path $\Pi \in \mathcal{P}_8$ with four caves: c_1 at step 2 is of depth 1, c_2 at step 4 is of depth 1, c_3 at step 5 is of depth 3 and c_4 at step 6 is of depth 2. The cave c_1 itself is a 1-triangle, and so is every other cave. The sequence of caves $\{c_3, c_4\}$ forms the only 2-triangle of Π and there is no m-triangle for $m \geq 3$.

The next theorem provides a combinatorial interpretation of the $(m1 \ldots m{-}1)$-avoiding permutations in terms of the $(m-2)$-triangle-forbidding paths.

Theorem 2.2.7. *For $m \geq 3$, if $\sigma \in S_n$ contains an $m1 \cdots m-1$ pattern, then the Permutation path $\Pi = f(\sigma)$ must have an $(m-2)$-triangle. That is,*

$$\mathcal{F}_{n,m-2} \subseteq f(S_n(m1 \cdots m-1)).$$

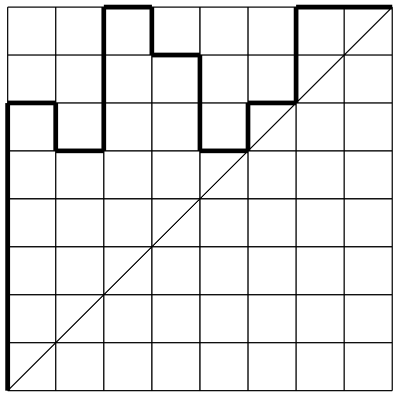


Figure 2.4: A Permutation path $\Pi \in \mathcal{P}_8$ with 4 caves and a 2-triangle

When $m = 3$*,* $\subseteq$ *is replaced by equality.*

Proof. For any $\sigma \notin S_n(m12\ldots m{-}1)$, we prove $\Pi = f(\sigma)$ contains some $(m{-}2)$-triangle. Assume $\{\sigma_{i_1}, \cdots, \sigma_{i_m}\}$ is a $(m12\ldots m-1)$-occurence in σ, namely

$$i_1 < i_2 < \cdots < i_m,$$

$$\sigma_{i_2} < \cdots \sigma_{i_m} < \sigma_{i_1}.$$

In fact, for j from 2 to $m-1$, we show that there is a d_j-cave at step i_j, where $d_j \geq m-j$. Note that

$$h^{\Pi}_{i_2} < \cdots < h^{\Pi}_{i_{m-1}}.$$

Furthermore because $\sigma_{i_{m-1}} < \sigma_{i_m} < \sigma_{i_1}$, we have

$$h^{\Pi}_{i_{m-1}} < h^{\Pi}_{i_1}.$$

So,

$$max\{h^{\Pi}_1, \ldots, h^{\Pi}_{i_j-1}\} - h^{\Pi}_{i_j} \geq h^{\Pi}_{i_1} - h^{\Pi}_{i_j} \geq m-j.$$

Hence at step $i_j, 2 \leq i_j \leq m-1$, we have a cave of depth at least $m-j$ and therefore

$\Pi = f(\sigma)$ contains an $(m-2)$-triangle.

$m = 3$ is the case of Dyck paths discussed in Theorem 2.2.6. $\square$

Remark 2.2.1. Conversely, given $\sigma \in S_n(m12\ldots m-1)$, $\Pi = f(\sigma)$ does not necessarily forbid $(m-2)$-triangles. For example, let $n = 8$ and $m = 4$, $\sigma = 57836241 \in S_8(4123)$, but the corresponding Permutation path Π is not 2-triangle-forbidding. See Figure 2.5.

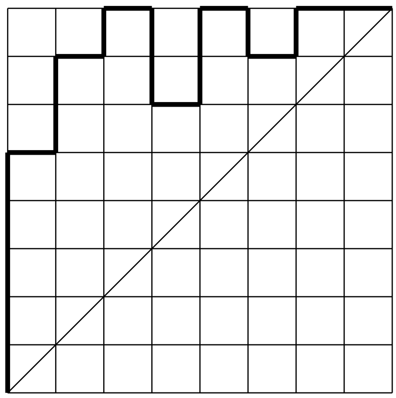


Figure 2.5: $\sigma = 57836241 \in S_8(4123)$ but Π is not 2-triangle-forbidding.

Nonetheless, we now know that in terms of cardinality,

$$|\mathcal{F}_{n,m-2}| \leq |S_n(m12\ldots m-1)|.$$

A recent result of Backelin, West and Xin ([BWG], see [SW02]) implies that for any $k \in \mathbb{N}$,

$$|S_n(m12\ldots m-1) = S_n(12\ldots m)|.$$

So Theorem 2.2.7 provides a lower bound estimate of the number of permutations in S_n whose longest increasing subsequence has length m.

2.2.3 The Signed Permutation Paths $\mathcal{B}_n$

In this section we introduce the notion of Signed Permutation paths. This may be viewed as a generalization of both Permutation paths and Schröder paths.

Definition 2.2.5. A Signed Permutation path of order n is a lattice path from $(0,0)$ to (n,n) which never goes below the main diagonal (i,i), $0 \leq i \leq n$, or above the line $y = n$, and consists of NORTH $(0,1)$, EAST $(1,0)$, SOUTH $(0,-1)$ and Diagonal $(1,1)$ steps but never repeats (i.e. no NORTH step followed or preceded by a SOUTH step).

Let $\mathcal{B}_n$ denote the collection of Signed Permutation paths of order n. Figure 2.6 is an illustration of a Signed Permutation path in $\mathcal{B}_8$.

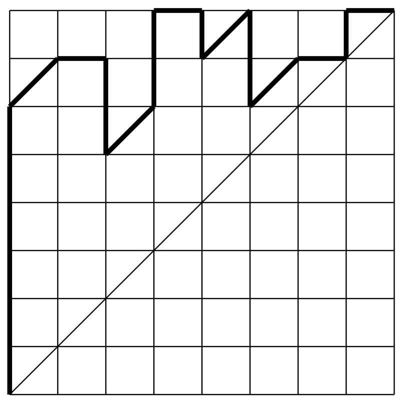


Figure 2.6: A Signed Permutation path.

Similar to the situation before, $\mathcal{B}_n$ is closely related to the set of signed permutations. A *signed permutation* is a permutation $\sigma \in S_n$ where each σ_i has a plus or minus sign attached to it [HLR]. The set of signed permutations is also called the hyperoctahedral group, which is denoted by B_n and studied by Reiner [Rei93].

Definition 2.2.6. For a signed permutation $\sigma \in B_n$, the absolute-value inversion statistic $\overline{inv}$ is defined to be

$$\overline{inv}(\sigma) = inv(|\sigma|),$$

where $|\sigma|$ denotes the ordinary permutation obtained from σ by removing the plus or minus sign attached to each σ_i.

Theorem 2.2.8. *There exists a weight-preserving bijection φ between B_n and $\mathcal{B}_n$ that maps the absolute-value inversion statistic to the area statistic. Namely, for any $\sigma \in B_n$, we have*

$$\overline{inv}(\sigma) = area(\varphi(\sigma)).$$

Proof. Given $\sigma \in B_n$, find $f(|\sigma|) = \Pi$, where f is our old weight-preserving bijection between S_n and $\mathcal{P}_n$. Suppose the height sequence of Π is $h(\Pi) = (h_1^\Pi, \ldots, h_n^\Pi)$. Note h_i^Π means that at the i^{th} column, the E step of Π goes from $(i-1, h_i^\Pi)$ to (i, h_i^Π). Now for each $1 \leq i \leq n$, if σ_i is positive, leave it untouched; if σ_i negative, then change the original E step to a D step which goes $(i-1, h_i^\Pi - 1)$ to (i, h_i^Π). The path so modified from Π is a Signed Permutation path and we let it be $\varphi(\sigma)$.

Conversely, given $\Pi \in \mathcal{B}_n$, first raise each of its D steps to an E step with the same height of the ending height of the D step. Connect where appropriate. Find the preimage under f of the Permutation path thus obtained and call it τ. For $1 \leq i \leq n$, if Π has an E step at column i, let $\sigma_i = \tau_i$; if Π has a D step at column i, let $\sigma_i = -\tau_i$. The signed permutation σ so obtained is $\varphi^{-1}(\Pi)$.

In our construction of the map φ, the Signed Permutation path has the same area as the Permutation path modified by changing D steps to E steps, and $\overline{inv}(\sigma) = inv(|\sigma|)$. Hence by Theorem 2.2.3, our conclusion follows. □

In Figure 2.6, $\varphi^{-1}(\Pi) = (-7)6(-4)8(-5)(-2)13$. Clearly, it is true that $area(\Pi) = 22 = inv(76485213)$.

Corollary 2.2.9.

$$\sum_{\Pi \in \mathcal{B}_n} q^{area(\Pi)} = 2^n [n]!.$$

Proof. For each $\sigma \in S_n$, there are 2^n ways to attach plus or minus signs to its entries to make it a signed permutation. Each of the 2^n signed permutation has the same absolute-value inversion statistic as $inv(\sigma)$. So it is clear from Theorem 2.2.8. $\square$

Chapter 3

Tree Enumeration Theorems and the (q, t)-Parking Function Polynomial

3.1 Haglund and Loehr's (q, t)-Parking Function Polynomial

The standard definition of *parking function* is as follows [Sta99, Ex.5.49, pages 94-95]: For fixed n, there are n parking places $1, 2, \ldots, n$ (in that order) on a one-way street. Cars $C_1, \ldots, C_n$ enter that street in that order and try to park. Each car C_i has a preferred space a_i. A car will drive to its preferred space and try to park there. If the space is already occupied, the car will park in the next available space. If the car must leave the street without parking, then the process fails. If $P = (a_1, \ldots, a_n)$ is a sequence of preferences that allows every car to park, then we call P a *parking function*. It is easy to see that a sequence $(a_1, \ldots, a_n)$ is a parking function if and only if the increasing rearrangement $(b_1, \ldots, b_n)$ of $(a_1, \ldots, a_n)$ satisfies $b_i \leq i$. It is also known that the number of parking functions of length n is given by

$$Park(n) = (n+1)^{n-1},$$

which is equal to the number of labelled trees on the labeling set $\{0, 1, 2, \ldots, n\}$.

As introduced in [HL], a parking function P can also be obtained by starting with a Dyck path D and placing n "cars", denoted by the integers 1 through n, in the squares immediately to the right of the vertical segments of D, with the restriction that if car i is placed immediately on top of car j, then $i > j$. It is easy to see this definition is in bijection with the one defined earlier: having cars $i_1, \ldots, i_j$ at column i is equivalent to say that exactly those cars $C_{i_1}, \ldots, C_{i_j}$ have i as their preferred space. For any parking function P, let $D(P)$ be the Dyck path that P corresponds to, i.e., D is obtained by removing the cars from P. Let $\mathbb{P}_n = \{P : D(P) \in \mathcal{D}_n\}$ be the parking functions on n cars. An example of a parking function is given in Figure 3.1.

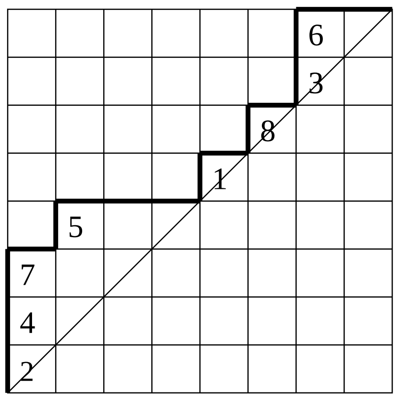


Figure 3.1: A parking function $P \in \mathcal{P}_8$ with $area(D(P)) = 6$.

Haglund and Loehr introduced a distribution function over the set of parking functions defined in this manner and made the following conjecture.

Conjecture 3.1.1. *Define*

$$R_n(q,t) = \sum_{P:D(P)\in\mathcal{D}_n} q^{area(D(P))} t^{dinv(P)},$$

where the sum is over all parking functions on n cars, and $dinv$ is another statistic on Dyck paths (which we shall not discuss here, see [HL]*). Then $R_n(q,t) = \mathcal{H}_n(q,t)$, where $\mathcal{H}_n(q,t)$ denotes the Hilbert Series of the space of diagonal harmonics (see* [Hai94]*).*

Although Conjecture 3.1.1 has been verified for $n \leq 11$, by Garsia, Haglund, Loehr and Ulyanov using Maple, it remains a conjecture. Even the symmetry of $R_n(q,t)$ in q and t is not proved. The main obstacle is the lack of a recurrence for $R_n(q,t)$. As a partial result, Loehr proved that $R_n(q,1) = R_n(1,q)$ [Loe03].

In the second section of this chapter, we prove some labelled tree counting theorems, which are meaningful by themselves, and use the "*Least-Child-Being-Monk*" theorem to get a recurrence relation on $R_n(1,1)$. In the last section, we prove that $R_n(q,1) = M_n(q)$, where $M_n(q)$ is the $area$ statistic for the major sequence [Kre80], thus establishing the equivalence between the $q = 1$ or $t = 1$ specialization of (q,t)-parking functions and a group of other combinatorial statistics and thereby getting another recurrence on $R_n(q,1) = R_n(1,q)$ by the known facts.

3.2 Counting Special Families of Labelled Trees

In the process of looking for a recurrence relation on $R_n(q,1)$, we found some interesting theorems about enumerating special families of labelled trees. Before proceeding to the main theorems, we first cite a lemma of L. E. Clarke to be used in our proof.

Lemma 3.2.1. ([Cla58], see also [Moo70] or [Ber76]) *The number of forests consisting* *of k rooted trees on $n-j$ nodes is*

$$\binom{n-j}{k} k(n-j)^{n-j-k-1}.$$

Throughout this section we consider rooted trees and rooted forests, where the notions "child" and "descendant" are defined in the standard way: node i is a *child* of node j if i is exactly one edge further away from the root; node i is a *descendant* of node j if i is one or more edges further away from the root. But sometimes we may drop the word "rooted" if there is no confusion. The convention we use is that every free tree corresponds to a rooted tree naturally by designating the least labelled vertex to be the root. Our main concern is to count families of labelled trees with some special structures, and the results will have no difference if we designate a different root, which we will do occasionally.

Definition 3.2.1. A labelled tree rooted at its least labelled vertex is *Least-Child-Being-Monk* if it has the property that the least labelled child of 0 has no children (or equivalently, is a leaf).

Theorem 3.2.2. *("Least-Child-Being-Monk") Define $\mathcal{T}_{n+1,0}$ to be the set of trees labelled on $\{0, 1, 2, ..., n+1\}$ with the Least-Child-Being-Monk property. Then the cardinality of $\mathcal{T}_{n+1,0}$, which we denote by $t_{n+1,0}$, is equal to n^n.*

Example 3.2.1. When $n = 2$, there are altogether 4 trees labelled on $\{0, 1, 2, 3\}$ having the Least-Child-Being-Monk property, as illustrated by Figure 3.2.

Proof. Nontrivially, assume $n \geq 2$ so that 0 has more than one descendants. If the least child of 0 is i and 0 has j other children, then these j children are selected randomly from $i+1, \cdots, n, n+1$. Now that we have 0 and its children i, c_1, c_2,$\cdots$,c_j, we only need to build the other $n-j$ nodes into a forest of k rooted trees and attach these roots of the forest to some or all of the "free children" $c_1, c_2, \cdots, c_j$ in j^k ways, but that is exactly counted by

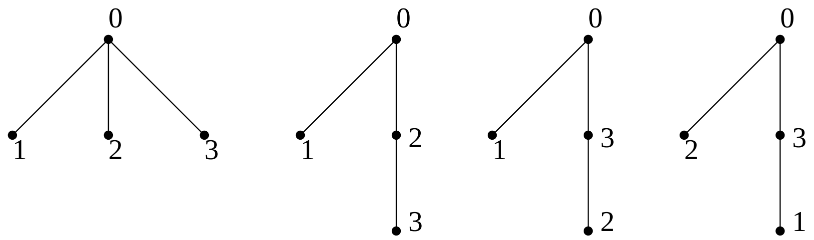


Figure 3.2: The number of "Least-Child-Being-Monk" trees on $\{0,1,2,3\}$ is $2^2 = 4$.

Lemma 3.2.1. Therefore,

$$
\begin{aligned}
|\mathcal{T}_{n+1,0}| &= \sum_{i=1}^{n+1} \sum_{j=0}^{n+1-i} \binom{n+1-i}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} k(n-j)^{n-j-k-1}\, j^k \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n-j}{k} k(n-j)^{n-j-k-1} j^k \sum_{i=1}^{n+1} \binom{n+1-i}{j} \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n-j}{k} k(n-j)^{n-j-k-1} j^k \binom{n+1}{j+1} \\
&= \sum_{j=0}^{n} (n-j)^{n-j-1} \binom{n+1}{j+1} \frac{j}{n-j} \sum_{k=0}^{n-j} \binom{n-j}{k} k(\frac{j}{n-j})^{k-1} \\
&= \sum_{j=0}^{n} (n-j)^{n-j-1} \binom{n+1}{j+1} \frac{j}{n-j}\ \frac{d[(1+x)^{n-j}]}{dx}\Big|_{x=\frac{j}{n-j}} \\
&= \sum_{j=0}^{n} (n-j)^{n-j-1} \binom{n+1}{j+1} \frac{j}{n-j}\ (n-j)(1+\frac{j}{n-j})^{n-j-1} \\
&= \sum_{j=0}^{n} \binom{n+1}{j+1} j n^{n-j-1} \\
&= n^{n-1}(\sum_{j=0}^{n} \binom{n+1}{j+1} (j+1) n^{-j} - \sum_{j=0}^{n} \binom{n+1}{j+1} n^{-j}).
\end{aligned}
$$

Since

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n+1}{j+1} (j+1) n^{-j} \\
&\quad = \sum_{k=1}^{n+1} \binom{n+1}{k} (k) (\frac{1}{n})^{k-1} \\
&\quad = \frac{d[(1+x)^{n+1}]}{dx} |_{x=\frac{1}{n}} \\
&\quad = \frac{(n+1)^{n+1}}{n^n}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n+1}{j+1} n^{-j} \\
&\quad = n \sum_{k=1}^{n+1} \binom{n+1}{k} (\frac{1}{n})^{k} \\
&\quad = \frac{(n+1)^{n+1}}{n^n} - n,
\end{aligned}$$

we have

$$\sum_{j=0}^{n} \binom{n+1}{j+1} (j+1) n^{-j} - \sum_{j=0}^{n} \binom{n+1}{j+1} n^{-j} = n.$$

Therefore,

$$|\mathcal{T}_{n+1,0}| = n^{n-1} \cdot n = n^n.$$

□

Corollary 3.2.3.

$$R_n(1,1) = \sum_{i=1}^{n} (i-1)^{i-1} \binom{n}{i} R_{n-i}(1,1)$$

Proof. Define $\mathbb{P}_n^*$, *primary parking functions* of order n, to be the subset of $\mathbb{P}_n$ which touches the main diagonal $y = x$ only at $(0, 0)$ and (n, n), and let

$$R_n^*(q, t) := \sum_{P \in \mathbb{P}_n^*} q^{area(P)} t^{dinv(P)}.$$

Figure 3.3 shows an example of a primary parking function, while the parking function illustrated by Figure 3.1 is *not* primary.

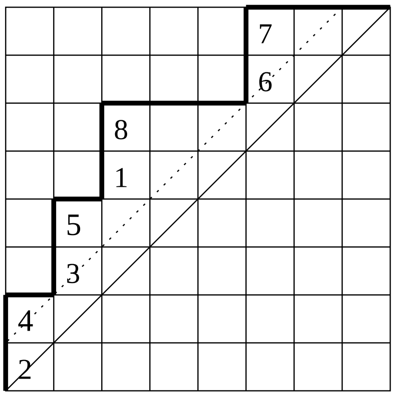


Figure 3.3: A primary parking function $P \in \mathcal{P}_8^*$, with $area(D(P)) = 12$.

Then, by decomposing the parking functions in $\mathbb{P}_n$,

$$R_n(1, 1) = \sum_{i=1}^{n} R_i^*(1, 1) \binom{n}{i} R_{n-i}(1, 1).$$

So it suffices to show

$$R_i^*(1, 1) = (i - 1)^{i-1}.$$

$R_i^*(1, 1)$ counts the number of primary parking functions of order i. [HL] provides a bijection between $\mathcal{P}_i^*$ and $\mathcal{T}_{i,0}$. Therefore it follows from Theorem 3.2.2. $\square$

Remark 3.2.1. The primary parking functions are also counted by a different way in [Sta99], where a different definition of primary parking functions is used.

Corollary 3.2.4. *When n goes to infinity, the probability for a labelled tree to be "Least-Child-Being-Monk" is e^{-2}.*

Proof. Using Cayley's formula and Theorem 3.2.2, the desired probability is

$$\lim_{n\to\infty} \frac{n^n}{(n+2)^n} = \frac{1}{e^2}.$$

$\square$

Corollary 3.2.5. *Define $f_{n,0}$ to be the number of rooted forests on n nodes such that each tree in the forest is rooted at its least labelled vertex and has the "Least-Child-Being-Monk" property. Then*

$$f_{n,0} = \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0}} \frac{\binom{n}{n_1,\cdots,n_k}|(n_1-2)^{n_1-2}\cdots(n_k-2)^{n_k-2}|}{k!},$$

where the $|\ |$ is to ensure its validity when some n_i takes the value of 1.

Proof. Note that $t_{n+1,0} = n^n$ actually counts labelled trees with the "Least-Child-Being-Monk" property on $n+2$ vertices (and it does not make any difference which vertex is the root). Given n vertices, our task is to partition them into k groups and build the nodes in each group into a tree with the "Least-Child-Being-Monk" property so that the least labelled vertex is the root in each tree. The conclusion readily follows. $\square$

The initial terms of $\{f_{n,0}\}_{n\geq 1}$ are 1, 2, 5, 18, 93, 104, ...

Remark 3.2.2. Let $t_n^* = t_{n-1,0}$ so that t_n^* denotes the number of rooted trees on n nodes with the "Least-Child-Being-Monk" property, and define exponential generating functions of t_n^* and $f_{n,0}$ by $T(x)$ and $F(x)$, respectively. Then by the Exponential Formula [Wil94], we have

$$F(x) = e^{T(x)},$$

which is equivalent to our formula for $f_{n,0}$ as in the Corollary. Furthermore, let $f_{n,0}^{(k)}$ be the number of rooted forests on n nodes that consist of k rooted trees with "Least-Child-Being-Monk" property, and introduce the 2-variable generating function

$$F(x,y) = \sum_{n,k\geq 0} f_{n,0}^{(k)} \frac{x^n}{n!} y^k.$$

Again by the the Exponential Formula [Wil94], we have

$$F(x,y) = e^{yT(x)}.$$

It is not hard to derive from here that

$$\begin{aligned} f_{n,0}^{(k)} &= [\frac{x^n}{n!} y^k] F(x,y) \\ &= [\frac{x^n}{n!} y^k] e^{yT(x)} \\ &= \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0}} \frac{\binom{n}{n_1,\cdots,n_k}(n_1-2)^{n_1-2}\cdots(n_k-2)^{n_k-2}}{k!}. \end{aligned}$$

This is a refinement of Corollary 3.2.5, or as we may say, another proof.

The "Least-Child-Being-Monk" theorem has some nice generalizations. Instead of requiring the "Least-Child-Being-Monk", we may let the least labelled child of 0 have p descendants.

Theorem 3.2.6. *Define $\mathcal{T}_{n+1,p}$ to be the set of labelled trees on $\{0,1,2,...,n+1\}$, such that the total number of descendants of the least labelled child of 0 is p. Then, the cardinality of $\mathcal{T}_{n+1,p}$, denoted by $t_{n+1,p}$, is equal to*

$$(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p}.$$

Example 3.2.2. The case $p = 0$ is dealt with in Theorem 3.2.2. Figure 3.4 is an illustration of $\mathcal{T}_{3,1}$ for the case $n = 2$ and $p = 1$. There are a total of $t_{3,1} = 3$ trees labelled on $\{0, 1, 2, 3\}$ so that the least child of 0 has exactly 1 descendant.

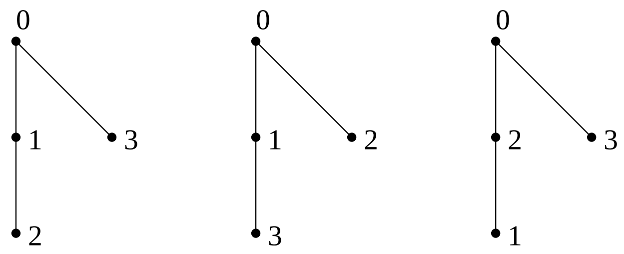


Figure 3.4: The 3 members of $\mathcal{T}_{3,1}$.

First Proof. The proof is similar to the $p = 0$ case, though a little more complicated.

$$t_{n+1,p} = \sum_{i=1}^{n+1} \sum_{j=0}^{n+1-i-p} \binom{n+1-i}{j} \binom{n-j}{p} (p+1)^{p-1}$$

$$\sum_{k=0}^{n-j-p} \binom{n-j-p}{k} k(n-j-p)^{n-j-k-1-p}\, j^k$$

$$= (p+1)^{p-1} \sum_{j=0}^{n-p} \sum_{k=0}^{n-j-p} \binom{n-j-p}{k} k(n-j-p)^{n-j-p-k-1} j^k \binom{n-j}{p}$$

$$\sum_{i=1}^{n+1} \binom{n+1-i}{j}$$

$$= (p+1)^{p-1} \sum_{j=0}^{n-p} \sum_{k=0}^{n-j-p} \binom{n-j-p}{k} k(n-j-p)^{n-j-p-k-1} j^k \binom{n-j}{p} \binom{n+1}{j+1}$$

$$= (p+1)^{p-1} \sum_{j=0}^{n-p} \binom{n-j}{p} \binom{n+1}{j+1} (n-j-p)^{n-j-p-1} \frac{j}{n-j-p}$$

$$\sum_{k=0}^{n-j-p} \binom{n-j-p}{k} k(\frac{j}{n-j-p})^{k-1}$$

$$= (p+1)^{p-1} \sum_{j=0}^{n-p} \binom{n-j}{p} \binom{n+1}{j+1} (n-j-p)^{n-j-p-1}$$

$$\frac{j}{n-j-p} \frac{d[(1+x)^{n-j-p}]}{dx} |_{x=\frac{j}{n-j-p}}$$

$$= (p+1)^{p-1} \sum_{j=0}^{n-p} \binom{n-j}{p} \binom{n+1}{j+1} (n-j-p)^{n-j-p-1}$$

$$\frac{j}{n-j-p} (n-j-p)(1+\frac{j}{n-j-p})^{n-j-p-1}$$

$$= (p+1)^{p-1} \sum_{j=0}^{n-p} \binom{n-j}{p} \binom{n+1}{j+1} j(n-p)^{n-j-p-1}$$

$$= (p+1)^{p-1}(n-p)^{n-p-1} \frac{(n+1)!}{p!\,(n-p+1)!} \sum_{j=0}^{n-p} \binom{n-p+1}{j+1} j(n-p)^{-j}.$$

Analogous to the previous proof, we have

$$\begin{aligned}
&\sum_{j=0}^{n-p} \binom{n-p+1}{j+1} (j+1)(n-p)^{-j} \\
&\quad = \sum_{k=1}^{n-p+1} \binom{n-p+1}{k} (k)(\frac{1}{n-p})^{k-1} \\
&\quad = \frac{d[(1+x)^{n-p+1}]}{dx}|_{x=\frac{1}{n-p}} \\
&\quad = \frac{(n-p+1)^{n-p+1}}{(n-p)^{(n-p)}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{n-p}\binom{n-p+1}{j+1}(n-p)^{-j}\\
&\quad=(n-p)\sum_{k=1}^{n-p+1}\binom{n-p+1}{k}(\frac{1}{n-p})^{k}\\
&\quad=\frac{(n-p+1)^{n-p+1}}{(n-p)^{(n-p)}}-(n-p).
\end{aligned}$$

So,

$$\sum_{j=0}^{n-p}\binom{n-p+1}{j+1}j(n-p)^{-j}=n-p,$$

and hence,

$$\begin{aligned}
t_{n+1,p}&=(p+1)^{p-1}(n-p)^{n-p-1}\binom{n+1}{p}(n-p)\\
&=(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p}.
\end{aligned}$$

□

Second Proof. Notice that we just need to (1) choose p nodes from $\{1,\cdots,n+1\}$ as the descendants of the least child of 0, (2) arrange the p nodes into a rooted forest, (3) build the remaining $n-j+1$ nodes as well as 0 into a tree with the Least-Child-Being-Monk property, and (4) attach all the roots of the forest obtained in the second step to the least child of 0 obtained in the third step. Clearly, the numbers of ways to realize the first two steps are $\binom{n+1}{p}$ and $(p+1)^{p-1}$, respectively. By the “Least-Child-Being-Monk” theorem, there are in total $(n-p)^{n-p}$ ways in the third step. The fourth step is done in a unique way. So it is clear.

□

Remark 3.2.3. If we add up all the p's, i.e, the total number of descendants of the least labelled child of 0, then we get the following identity by Cayley's formula:

$$\sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p} = (n+2)^n,$$

which is equivalent to

$$\sum_{p=0}^{n+1}(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p} = 0$$

and further becomes

$$\sum_{p=0}^{n}(n-1-p)^{n-1-p}(p+1)^{p-1}\binom{n}{p} = 0 \tag{3.2.1}$$

when we drop the scale from $n+1$ to n.

Eq. (3.2.1) reminds us of *Abel's indentity*, a striking generalization of the binomial theorem.

Theorem 3.2.7. (Abel's identity, see [Abe26], [Com74] or [Str92]) *For all* x, y, z, *we have:*

$$\sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k} = (x+y)^n.$$

If we let $k = p, x = 1, z = -1$ and $y = n-1$, then we have the specialization

$$\sum_{p=0}^{n}(n-1-p)^{n-p}(p+1)^{p-1}\binom{n}{p} = n^n.$$

This is very similar to (3.2.1); however (3.2.1) can not be derived from any direct specialization of Abel's identity. Nevertheless, our identity is indeed obtainable if we apply

Theorem 3.2.7 twice. Since $\binom{n+1}{p} = \binom{n}{p} + \binom{n}{p-1}$,

$$\begin{aligned}&\sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p}\\ &= \sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n}{p} + \sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n}{p-1}.\end{aligned}$$

In Abel's identity, let $k = p, x = 1, z = -1$ and $y = n$, then

$$\sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n}{p} = (n+1)^n. \tag{3.2.2}$$

On the other hand,

$$\begin{aligned}&\sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n}{p-1}\\ &= \sum_{p=1}^{n+1}(n-p)^{n-p}(p+1)^{p-1}\binom{n}{p-1} - (-1)^{-1}(n+2)^n\binom{n}{n}\\ &= \sum_{j=0}^{n}(n-j-1)^{n-j-1}(j+2)^j\binom{n}{j} + (n+2)^n.\end{aligned}$$

In Abel's identity, let $k = n-j, x = -1, z = -1$ and $y = n+2$, then

$$\sum_{j=0}^{n}\binom{n}{j}(-1)(n-j-1)^{n-j-1}(j+2)^j = (n+1)^n.$$

So,

$$\sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n}{p-1} = -(n+1)^n + (n+2)^n. \tag{3.2.3}$$

Sum up (3.2.2) and (3.2.3), again we have

$$\sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p} = (n+2)^n.$$

Corollary 3.2.8. *When n goes to infinity, the probability for a labelled tree on $\{0, 1, 2, \cdots, n\}$ to have the property that the least labelled child of 0 has exactly p descendants is*

$$\frac{(p+1)^{p-1}}{p\,!}\, e^{-2-p}.$$

Proof.

$$\lim_{n\to\infty} \frac{(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p}}{(n+2)^n} = \frac{(p+1)^{p-1}}{p\,!}\, e^{-2-p}.$$

$\square$

Let $\mathcal{T}_{n+1}$ be the collection of all labelled trees on $\{0, 1, \cdots, n+1\}$. For any $T \in \mathcal{T}_{n+1}$, let $y(T)$ denote the size of the "youngest descendant branch", i.e., the number of descendants of the least labelled child of 0. Furthermore, let

$$Y_n(q) = \sum_{T\in\mathcal{T}_n} q^{y(T)}.$$

A different way to state Theorem 3.2.6 is

$$Y_{n+1}(q) = \sum_{p=0}^{n}(n-p)^{n-p}(p+1)^{p-1}\binom{n+1}{p} q^p.$$

Another generating function worth studying is

$$Z_n(q) = \sum_{T\in\mathcal{T}_{n+1}} q^{z(T)},$$

where $z(T) = deg_T(0) - 1$, i.e., $z(T)$ is the number of children of 0 excluding the least labelled one.

Theorem 3.2.9.

$$Z_{n+1}(q) = \sum_{j=0}^{n} \sum_{p=0}^{n-j} (p+1)^{p-1} j (n-p)^{n-p-j-1} \binom{n+1}{p, j+1, n-p-j} q^j.$$

Proof. Let

$$Z_{n+1,p}(q) := \sum_{T \in \mathcal{T}_{n+1,p}} q^{z(T)}.$$

It suffices to show that for $1 \le p \le n$,

$$Z_{n+1,p}(q) = \sum_{j=0}^{n-p} (p+1)^{p-1} j (n-p)^{n-p-j-1} \binom{n+1}{p, j+1, n-p-j} q^j.$$

Similar to the proof of Theorem 3.2.6 and following exactly the same process as we went through, we obtain

$$\begin{aligned} Z_{n+1,p}(q) = & \sum_{i=1}^{n+1} \sum_{j=0}^{n+1-i-p} \binom{n+1-i}{j} \binom{n-j}{p} (p+1)^{p-1} q^j \\ & \sum_{k=0}^{n-j-p} \binom{n-j-p}{k} k (n-j-p)^{n-j-k-1-p} j^k \\ = & \cdots \\ = & (p+1)^{p-1} (n-p)^{n-p-1} \frac{(n+1)!}{p!\,(n-p+1)!} \sum_{j=0}^{n-p} \binom{n-p+1}{j+1} j (\frac{q}{n-p})^j. \end{aligned}$$

Now it is not hard to see the coefficient of q^j of the above formula is

$$(p+1)^{p-1} j (n-p)^{n-p-j-1} \binom{n+1}{p, j+1, n-p-j}.$$

$\square$

Remark 3.2.4. Theorem 3.2.9 covers and refines Theorem 3.2.2 and Theorem 3.2.6. For instance,

$$[q^j]Z_{n+1,p}(q) = (p+1)^{p-1} j(n-p)^{n-p-j-1} \binom{n+1}{p, j+1, n-p-j}$$

gives the number of labelled trees on $\{0, 1, \cdots, n+1\}$ such that the "least child" of 0 has exactly p descendants and $deg_T(0) = j+1$. In addition, if we specialize $q = 1$ and $p = 0$ in Theorem 3.2.9, then we get

$$\begin{aligned} Z_{n+1,0}(1) &= \sum_{j=0}^{n} j n^{n-j-1} \binom{n+1}{j+1} \\ &= n^n, \end{aligned}$$

which is the value of $|\mathcal{T}_{n+1,0}|$ as in Theorem 3.2.2 or $t_{n+1,0}$ as in Theorem 3.2.6.

From a different point of view, we can also extend the behavior of the "least child of 0 being single" to a "hereditary" property. That is, we want every "least child" in a rooted tree, instead of just the least child of 0, to be single. According to some old European Church tradition, the youngest child of each family will serve as a priest, so as to have no children. Translating this situation into graph theory, we are motivated to consider the following counting question: what is the number of labelled trees such that the least labelled child of *any vertex* has no children?

Definition 3.2.2. A labelled tree rooted at its least labelled vertex is *Hereditary-Least-Single* if it has the property that *every least child* in this tree has no children (or equivalently, is a leaf).

Lemma 3.2.10. *(Hereditary-Least-Single Trees Pre-recurrence) Let the number of Hereditary-Least-Single trees with n vertices be h_n. Then h_n satisfies the following recurrence:*

$$h_n = \sum_{k=0}^{n-2} \binom{n-1}{k+1} \sum_{\substack{n_1+\cdots+n_k=n-k-2 \\ n_i \geq 0}} \binom{n-k-2}{n_1, \cdots, n_k} h_{n_1+1} \cdots h_{n_k+1}, n \geq 2,$$

$$h_1 = 1.$$

Proof. Firstly, we choose $k+1$ children of the root: $c_0 < c_1 < \cdots < c_k$. Secondly, for each of the k children except the least, let the number of descendants that c_i will have be $n_i \geq 0$, $i = 1 \ldots k$. Lastly, for each i, build up a subtree rooted at c_i together with its n_i descendants. Note that here is a trick: although c_i may not be the least labelled vertex in this subtree, the number of such subtrees is exactly h_{n_i+1} since the Hereditary-Least-Single property is essentially only concerned with every vertex except the root. $\square$

Theorem 3.2.11. *Consider the exponential generating function $H(x) = \sum_{n\geq 0} \frac{h_{n+1}}{n!} x^n$. Then $H(x)$ satisfies the simple functional equation*

$$H^2(x) - H(x) + 1 = e^{xH(x)}.$$

Proof. By Lemma 3.2.10, for $n \geq 2$,

$$\begin{aligned} h_n &= \sum_{k=0}^{n-2} \binom{n-1}{k+1} (n-k-2)! \sum_{n_1+\cdots+n_k=n-k-2} \frac{h_{n_1+1}}{n_1!} \cdots \frac{h_{n_k+1}}{n_k!} \\ &= \sum_{k=0}^{n-2} \binom{n-1}{k+1} (n-k-2)! [x^{n-k-2}] H^k(x) \\ &= (n-1)! \sum_{k=0}^{n-2} [x^{n-k-2}] \frac{H^k(x)}{(k+1)!}. \end{aligned}$$

Here by the symbol $[x^n]f(x)$ we mean the coefficients of x^n in the series $f(x)$. So on one hand,

$$[x^{n-1}]H(x) = \frac{h_n}{(n-1)!} = \sum_{k=0}^{n-2}[x^{n-k-2}]\frac{H^k(x)}{(k+1)!}.$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{n-2}[x^{n-k-2}]\frac{H^k(x)}{(k+1)!} &= \sum_{k=0}^{n-2}[x^{n-1}]\frac{x^{k+1}H^k(x)}{(k+1)!} \\ &= [x^{n-1}]\sum_{k=0}^{\infty}\frac{x^{k+1}H^k(x)}{(k+1)!} \\ &= [x^{n-1}]\frac{1}{H(x)}\sum_{k=0}^{\infty}\frac{(xH(x))^{k+1}}{(k+1)!} \\ &= [x^{n-1}]\frac{e^{xH(x)}-1}{H(x)},\end{aligned}$$

for any $n \geq 2$. Therefore,

$$H(x) = \frac{e^{xH(x)}-1}{H(x)} + C,$$

where C is a constant to be determined. By checking the constant term on both sides, we see $C = 1$ and the equation follows.

□

Theorem 3.2.12. *(Hereditary-Least-Single Trees Recurrence) h_n satisfies the following recurrence:*

$$\begin{aligned}h_n =& (n-1)h_{n-1} - 2\sum_{1\leq i\leq n-2} h_{n-i}h_{i+1}\binom{n-2}{i-1, n-i-1} \\ &+ \sum_{1\leq i\leq n-2}\ \sum_{1\leq j\leq n-i-1} ih_ih_jh_{n+1-i-j}\binom{n-2}{i-1, j-1, n-i-j}.\end{aligned}$$

Proof. For simplicity, let

$$H(x) = \sum_{n\leq 0} a_n x^n$$

and

$$a_0 = 1,$$

s.t.

$$a_n = \frac{h_{n+1}}{n!}.$$

Take the derivative with respect to x on both sides of

$$xH = \ln(H^2 - H + 1)$$

to obtain

$$xH' + H = \frac{2HH' - H'}{H^2 - H + 1}.$$

Therefore

$$(xH' + H)(H^2 - H + 1) = 2HH' - H'.$$

For the above equation, since

$$\begin{aligned}
[x^n]\text{ LHS } &= \sum_{0 \le m \le n} [x^m](xH' + H)[x^{n-m}](H^2 - H + 1) \\
&= \sum_{0 \le m \le n} (ma_m + a_m)\Big(\sum_{0 \le r \le n-m} a_r a_{n-m-r} - a_{n-m} + \delta_{n-m,0}\Big) \\
&= \sum_{0 \le m \le n} (m+1)\Big(\sum_{0 \le r \le n-m} a_r a_{n-m-r} a_m - a_m a_{n-m} + \delta_{n-m,0} a_m\Big),
\end{aligned}$$

and

$$\begin{aligned}
[x^n]\text{ RHS} &= \sum_{0\le m\le n}[x^m](2H-1)[x^{n-m}](H') \\
&= \sum_{0\le m\le n}(2a_m-\delta_{m,0})(n-m+1)a_{n-m+1} \\
&= \sum_{0\le m\le n}(2a_{n-m}-\delta_{n-m,0})(m+1)a_{m+1} \\
&= (n+1)a_{n+1}+\sum_{0\le m\le n-1}2a_{n-m}(m+1)a_{m+1},
\end{aligned}$$

we have

$$\begin{aligned}
&a_{n+1} \\
&= \frac{[x^n]\text{ LHS } - \sum_{0\le m\le n-1}2a_{n-m}(m+1)a_{m+1}}{n+1} \\
&= \frac{(n+1)a_n+\sum_{0\le m\le n-1}(m+1)(\sum_{0\le r\le n-m}a_ra_{n-m-r}a_m-a_ma_{n-m}-2a_{n-m}a_{m+1})}{n+1} \\
&= a_n+\frac{\sum_{0\le m\le n-1}(m+1)(\sum_{0\le r\le n-m-1}a_ra_{n-m-r}a_m-2a_{n-m}a_{m+1})}{n+1}.
\end{aligned}$$

Hence,

$$a_{n-1}=a_{n-2}+\frac{\sum_{0\le m\le n-3}(m+1)(\sum_{0\le r\le n-m-3}a_ra_{n-2-m-r}a_m-2a_{n-2-m}a_{m+1})}{n-1},$$

which is equivalent to

$$\frac{h_n}{(n-1)!} = \frac{h_{n-1}}{(n-2)!} + \frac{\sum_{0\le m\le n-3}(m+1)(\sum_{0\le r\le n-m-3}\frac{h_{r+1}}{r!}\frac{h_{n-1-m-r}}{(n-2-m-r)!}\frac{h_{m+1}}{m!} - 2\frac{h_{n-1-m}}{(n-2-m)!}\frac{h_{m+2}}{(m+1)!})}{n-1}.$$

Therefore,

$$\begin{aligned}
h_n &= (n-1)h_{n-1} - \sum_{0\le m\le n-3} 2h_{n-1-m}h_{m+2}\binom{n-2}{n-2-m, m} \\
&\quad + \sum_{0\le m\le n-3}\ \sum_{0\le r\le n-m-3}(m+1)h_{r+1}h_{n-m-r-2+1}h_{m+1}\binom{n-2}{r, m, n-2-m-r} \\
&= (n-1)h_{n-1} - \sum_{1\le i\le n-2} 2h_{n-i}h_{i+1}\binom{n-2}{n-i-1, i-1} \\
&\quad + \sum_{1\le i\le n-2}\ \sum_{1\le j\le n-1-i} ih_ih_jh_{n-i-j+1}\binom{n-2}{i-1, j-1, n-i-j}.
\end{aligned}$$

So we are done. □

The following list computed by Maple using our recurrence contains $\{h_n\}_{n=1}^{10}$: 1, 1, 1, 4, 15, 96, 665, 6028, 60907, 725560 ...

3.3 Major Sequences, Tree Inversion Enumerator and the Tutte Polynomial

In this section we investigate several interesting combinatorial objects related to $R_n(q,t)$, and establish the equivalence between $R_n(q,1)$ and a group of combinatorial formulae: the area enumerator for major sequences, the inversion enumerator for labelled trees and the

Tutte polynomial specialized at $(1, q)$. Loehr [Loe03] recently proved that $R_n(q, 1) = R_n(1, q)$, so we can also include $R_n(1, q)$ in this equivalence family.

An integer sequence $\hat{s} = (s_1, \cdots, s_n)$ is called a *major sequence* of length n [Kre80] if its non-decreasing rearrangement $(z_1, \cdots, z_n)$ satisfies

$$i \leq z_i \leq n, \text{ for all } 1 \leq i \leq n.$$

The area of a major sequence $\hat{s} = (s_1, \cdots, s_n)$ is defined as

$$area(\hat{s}) = \sum_{i=1}^{n} s_i - \binom{n+1}{2}.$$

If we denote by $\mathbb{M}_n$ the set of major sequences of length n, then we can further define the area enumerator for major sequences of length n to be

$$M_n(q) = \sum_{\hat{s} \in \mathbb{M}_n} q^{area(\hat{s})}.$$

Theorem 3.3.1.

$$R_n(q, 1) = M_n(q)$$

Proof. Notice that there is a simple bijection between $\mathbb{M}_n$ and $\mathbb{P}_n$. Given a major sequence $\hat{s} = (s_1, \cdots, s_n)$, it corresponds to its "complement" $(n + 1 - s_1, \cdots, n + 1 - s_n) = (f_1, \cdots, f_n)$, which corresponds to a standard parking function P as defined in [Sta99], which corresponds to a parking function P' of ours as defined in [HL], and which further corresponds to its reflection about the NW-SE diagonal.

Next it suffices to show that if $\hat{z} = (z_1, \cdots, z_n)$ is the non-decreasing rearrangement of $\hat{s}$, then we have $area(\hat{z}) = area(P')$. In fact,

$$
\begin{aligned}
&\text{\# of squares counted in the } i^{th} \text{ column of } P' \\
&= \text{\# of squares counted in the } i^{th} \text{ row of } P \\
&= (n-i) - (\text{the } i^{th} \text{ biggest } f_j - 1) \\
&= (n-i) - (n+1-z_i-1) \\
&= z_i - i.
\end{aligned}
$$

So we are done. $\square$

To define the Tutte polynomial, denoted by $T_G(x, y)$, we make the convention that G is a connected graph (actually $T_G(x, y)$ is just the product of the Tutte polynomials of components of G, if G is disconnected). Recursively, the Tutte polynomial could be defined by the following set of rules [Ste02]:

1. If G has no edges, then $T_G(x, y) = 1$.

2. If e is an edge of G that is neither a loop nor an isthmus, then

$$T_G(x, y) = T_{G'_e}(x, y) + T_{G''_e}(x, y),$$

   where G'_e is the graph G with the edge e *deleted* and G''_e is the graph G with the edge e *contracted*.

3. If e is an isthmus, then $T_G(x, y) = xT_{G'_e}(x, y)$.

4. If e is a loop, then $T_G(x, y) = yT_{G''_e}(x, y)$.

From a different prospective, which is "graph theoretical" as well but apparently more "combinatorial statistical"), $T_G(x, y)$ is also defined in terms of two graph theory statistics: the external activity and internal activity [GS96]. Now suppose we are given G and the lexicographic ordering of its edges, i.e., edge ij with $i < j$ is smaller than edge kl with $k < l$ iff $(i < k)$ or $(i = k, j < l)$. Consider a spanning tree T of G. An edge $e \in G - T$ is *externally active* if it is the largest edge in the unique *cycle* contained in $T \cup e$. We let

$$\mathcal{EA}_G(T) = \text{set of external active edges of } T$$

and

$$ea_G(T) = |\mathcal{EA}_G(T)|.$$

Dually, an edge $e \in T$ is *internally active* if it is the largest edge in the unique *cocycle* contained in $(G - T) \cup e$. We let

$$\mathcal{IA}_G(T) = \text{set of internal active edges of } T$$

and

$$ia_G(T) = |\mathcal{IA}_G(T)|.$$

Finally, the Tutte polynomial is defined as [Tut54]

$$T_G(x, y) = \sum_{T \subseteq G} x^{ia_G(T)} y^{ea_G(T)}.$$

Beissinger and Peled [BP97] proved bijectively that

$$M_n(q) = T_{K_{n+1}}(1, q),$$

where K_{n+1} is the completed graph on $\{0, 1, \cdots, n\}$. Henceforth, together with Theorem 3.3.1, we get the following corollary.

Corollary 3.3.2.

$$R_n(q,1) = T_{K_{n+1}}(1,q).$$

Remark 3.3.1. We remark that $T_{K_{n+1}}(q,t) = R_n(q,t)$ is false in general. In fact, the Tutte polynomial is not even symmetric. For example,

$$T_{K_5}(q,1) = q^4 + 6q^3 + 21q^2 + 46q + 51$$

while

$$T_{K_5}(1,q) = q^6 + 4q^5 + 10q^4 + 20q^3 + 30q^2 + 36q + 24.$$

Yet another statistic is the *tree inversion enumerator*. If T is a tree on $\{0,1,\cdots,n\}$, an *inversion* of T is a pair i,j such that $1 \le i < j \le n$ and j lies on the path from 0 to i in T. We denote by $inv(T)$ the number of inversions of T. The *inversion enumerator for trees on* $\{0,1,\cdots,n\}$ is defined as

$$I_n(q) = \sum_T q^{inv(T)},$$

where the sum is over all trees in $\{0,1,\cdots,n\}$.

Björner [Bjö92] discovered that

$$I_n(q) = T_{K_{n+1}}(1,q),$$

by considering the generating function of the number of connected spanning subgraphs with k edges of a complete graph and using his results in matroids as well as a result of Gessel and Wang [GW79]. A bijective proof was given by Beissinger [Bei82].

Kreweras [Kre80] proved that

$$I_n(q) = M_n(q),$$

by a bijective method and also by proving that they both satisfy the same recurrence relation:

$$Stat_n(q) = \sum_{i=1}^{n} \binom{n-1}{i-1} [i] \; Stat_{i-1}(q) \, Stat_{n-i}(q).$$

Recently, Loehr proved in his dissertation [Loe03] that

$$R_n(q,1) = R_n(1,q),$$

by a nontrivial combinatorial argument.

Therefore we have the following conclusion.

Corollary 3.3.3. *The 5 combinatorial statistics are all equal, i.e.*

$$R_n(q,1) = R_n(1,q) = M_n(q) = T_{K_{n+1}}(1,q) = I_n(q),$$

and they all satisfy the following recurrence:

$$Stat_1(q) = 1,$$

$$Stat_n(q) = \sum_{i=1}^{n} \binom{n-1}{i-1} [i] \; Stat_{i-1}(q) \, Stat_{n-i}(q).$$

Chapter 4

The Limit Case of the (q,t)-Schröder Theorem

4.1 Background and Basic Properties

Throughout this chapter we use the standard notation:

$$(a;q)_k := (1-a)(1-qa)\cdots(1-q^{k-1}a)$$

is the "q-rising factorial" and $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient.

A closed-form expression of the (q,t)-Schröder polynomial, which doesn't reference the bounce or area statistic, is obtained in [EHKK03]:

Theorem 4.1.1. *For all* $n, d \in \mathbb{N}$*,*

$$S_{n+d,d}(q,t) = \sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n,\ a_i>0 \\ b_0+\cdots+b_k=d,\ b_i\geq 0}} q^{\sum_{i=1}^{k}\binom{a_i}{2}} t^{\sum_{i=0}^{k-1}(k-i)b_i + \sum_{i=1}^{k-1}(k-i)a_i}$$
$$\begin{bmatrix} b_0+a_1-1 \\ b_0 \end{bmatrix} \begin{bmatrix} b_k+a_k \\ b_k \end{bmatrix} \times \prod_{i=1}^{k-1} \begin{bmatrix} b_i+a_i+a_{i+1}-1 \\ b_i, a_i, a_{i+1}-1 \end{bmatrix}.$$

Recall that in [Hag04], Haglund proved the (q,t)-*Schröder Theorem*. A corollary is the limit case:

Theorem 4.1.2. [Hag04] *For* $n \in \mathbb{N}$,

$$\lim_{d\to\infty} S_{n+d,d}(q,t) = [z^n] \prod_{i,j\geq 0} (1+q^i t^j z).$$

On the other hand, if we let d go to infinity, then the upper bound restriction of $b_0, \ldots, b_k$ on the right hand side of Theorem 4.1.1 is removed. So,

$$\begin{aligned}
\lim_{d\to\infty} S_{n+d,d}(q,t) &= \lim_{d\to\infty} \sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n,\ a_i>0 \\ b_0+\cdots+b_k=d,\ b_i\geq 0}} q^{\sum_{i=1}^{k}\binom{a_i}{2}} t^{\sum_{i=0}^{k-1}(k-i)b_i+\sum_{i=1}^{k-1}(k-i)a_i} \\
&\quad \begin{bmatrix} b_0+a_1-1 \\ b_0 \end{bmatrix} \begin{bmatrix} b_k+a_k \\ b_k \end{bmatrix} \times \prod_{i=1}^{k-1} \begin{bmatrix} b_i+a_i+a_{i+1}-1 \\ b_i, a_i, a_{i+1}-1 \end{bmatrix}. \\
&= \sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n \\ a_i>0}} q^{\sum_{i=1}^{k}\binom{a_i}{2}}\ t^{\sum_{i=1}^{k-1}(k-i)a_i} \sum_{b_0=0}^{\infty} t^{kb_0} \begin{bmatrix} b_0+a_1-1 \\ b_0 \end{bmatrix} \\
&\quad \lim_{b_k\to\infty} \begin{bmatrix} b_k+a_k \\ b_k \end{bmatrix} \times \prod_{i=1}^{k-1} \begin{bmatrix} a_i+a_{i+1}-1 \\ a_i \end{bmatrix} \sum_{b_i=0}^{\infty} t^{(k-i)b_i} \begin{bmatrix} b_i+a_i+a_{i+1}-1 \\ a_i+a_{i+1}-1 \end{bmatrix}.
\end{aligned}$$

For fixed $a_1, \ldots, a_k$, using the q-binomial theorem, we have

$$\begin{aligned}
\sum_{b_0=0}^{\infty} t^{kb_0} \begin{bmatrix} b_0+a_1-1 \\ b_0 \end{bmatrix} &= \frac{1}{(t^k;q)_{a_1}} \\
\sum_{b_i=0}^{\infty} t^{(k-i)b_i} \begin{bmatrix} b_i+a_i+a_{i+1}-1 \\ a_i+a_{i+1}-1 \end{bmatrix} &= \frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}}.
\end{aligned}$$

Furthermore, for $q < 1$,

$$\lim_{b_k\to\infty}\begin{bmatrix} b_k + a_k \\ b_k \end{bmatrix} = \lim_{b_k\to\infty}\frac{(1-q^{b_k+1})\cdots(1-q^{b_k+a_k})}{(1-q)\cdots(1-q^{a_k})}$$
$$= \frac{1}{(q;q)_{a_k}}.$$

Hence, we have derived an identity from the limit case of (q,t)-Schröder theorem.

Theorem 4.1.3. *For $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n}\sum_{\substack{a_1+\cdots+a_k=n\\ a_i>0}} q^{\sum_{i=1}^{k}\binom{a_i}{2}}\, t^{\sum_{i=1}^{k-1}(k-i)a_i}\,\frac{1}{(t^k;q)_{a_1}(q;q)_{a_k}}$$
$$\times\prod_{i=1}^{k-1}\begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix}\frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}}$$
$$= [z^n]\prod_{i,j\geq 0}\left(1+q^i t^j z\right).$$

However, because Haglund's proof makes heavy use of symmetric function identities and plethystic machinery, the combinatorics behind it is not understood. Therefore it is worthwhile to study it combinatorially. This chapter will be devoted to obtaining some partial results and develop related theorems by looking at some special cases.

The right hand side of Theorem 4.1.3 has been studied in connection with bipartite partitions. Wright [Wri61] conjectured and Gordon [Gor63] proved that the right hand side times $(q;q)_n(t;t)_n$ is a polynomial of degree $\frac{1}{2}n(n-1)$ in each variable. In fact, it is convenient to multiply both sides of Theorem 4.1.3 by $(q;q)_n(t;t)_n$, so we will do that and thereby get the following modified version of Theorem 4.1.3.

Theorem 4.1.4. *For* $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n \\ a_i>0}} q^{\sum_{i=1}^{k}\binom{a_i}{2}} \ t^{\sum_{i=1}^{k-1}(k-i)a_i} \ \frac{1}{(t^k;q)_{a_1}(q;q)_{a_k}}$$

$$\times \prod_{i=1}^{k-1} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix} \frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}} \ \times \ (q;q)_n(t;t)_n$$
$$= [z^n] \prod_{i,j\geq 0} (1+q^i t^j z) \ \times \ (q;q)_n(t;t)_n$$

Furthermore, it could be shown that the right hand side of Theorem 4.1.4 is equal to $\sum_{\sigma\in S_n} q^{maj(\sigma)} \ t^{\binom{n}{2}-maj(\sigma^{-1})}$. In fact, a result in [Sta99, page 385] establishes

$$\sum_{\sigma\in S_n} q^{maj(\sigma)} \ t^{maj(\sigma^{-1})} = [z^n] \prod_{i,j\geq 0} \frac{1}{1-q^i t^j z} \ \times \ (q;q)_n(t;t)_n. \tag{4.1.1}$$

Wright [Wri61, page 884] showed that

$$[z^n] \prod_{i,j\geq 0} \frac{1}{1-q^i(\frac{1}{t})^j z} = (-1)^n t^n [z^n] \prod_{i,j\geq 0} (1+q^i t^j z). \tag{4.1.2}$$

Combing (4.1.1) and (4.1.2) together, we have

$$\begin{aligned}
&\sum_{\sigma\in S_n} q^{maj(\sigma)} \ t^{\binom{n}{2}-maj(\sigma^{-1})} \\
&= t^{\binom{n}{2}} (q;q)_n (\frac{1}{t};\frac{1}{t})_n [z^n] \prod_{i,j\geq 0} \frac{1}{1-q^i(\frac{1}{t})^j z} \\
&= (q;q)_n (t;t)_n \frac{(-1)^n}{t^n} [z^n] \prod_{i,j\geq 0} \frac{1}{1-q^i(\frac{1}{t})^j z} \\
&= (q;q)_n (t;t)_n [z^n] \prod_{i,j\geq 0} (1+q^i t^j z).
\end{aligned}$$

Therefore we have another modified version of Theorem 4.1.3.

Theorem 4.1.5. *For* $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n \\ a_i>0}} q^{\sum_{i=1}^{k} \binom{a_i}{2}} \; t^{\sum_{i=1}^{k-1} (k-i)a_i} \; \frac{1}{(t^k;q)_{a_1}(q;q)_{a_k}}$$

$$\times \prod_{i=1}^{k-1} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix} \frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}} \; \times \; (q;q)_n(t;t)_n$$

$$= \sum_{\sigma \in S_n} q^{maj(\sigma)} \; t^{\binom{n}{2} - maj(\sigma^{-1})}.$$

In the rest of this section we analyze some basic properties of Theorem 4.1.4. Let $F_n(q,t)$ denote the left hand side of Theorem 4.1.4 and $G_n(q,t)$ denote the right hand side. First let's review some nice facts. Carlitz [Car56] proved the following properties of G_n:

$$G_n(q,t) = (qt)^{\binom{n}{2}} G_n(\frac{1}{q}, \frac{1}{t}), \tag{4.1.3}$$

$$G_n(q,0) = q^{\binom{n}{2}}, \tag{4.1.4}$$

$$G_n(q,1) = \frac{1-q^n}{1-q} \, G_{n-1}(q,1), \tag{4.1.5}$$

$$G_n(q,q) = q^{n(n-1)} G_n(\frac{1}{q}, \frac{1}{q}), \tag{4.1.6}$$

$$G_n(1,1) = n. \tag{4.1.7}$$

We prove that all of the above hold in exactly the same way for $F_n(q,t)$. Namely,

$$F_n(q,t) = (qt)^{\binom{n}{2}} F_n(\frac{1}{q}, \frac{1}{t}), \tag{4.1.8}$$

$$F_n(q,0) = q^{\binom{n}{2}}, \tag{4.1.9}$$

$$F_n(q,1) = \frac{1-q^n}{1-q}\, F_{n-1}(q,1), \tag{4.1.10}$$

$$F_n(q,q) = q^{n(n-1)} F_n(\frac{1}{q}, \frac{1}{q}), \tag{4.1.11}$$

$$F_n(1,1) = n. \tag{4.1.12}$$

Note that (4.1.8) implies (4.1.11) and (4.1.10) implies (4.1.12). So we only need to prove (4.1.8), (4.1.9) and (4.1.10).

Lemma 4.1.6.

$$F_n(q,t) = (qt)^{\binom{n}{2}} F_n(\frac{1}{q}, \frac{1}{t}).$$

Proof. Let $f_n(q,t) = F_n(q,t)/(q;q)_n(t;t)_n$. Since

$$\frac{(q;q)_n(t;t)_n}{(\frac{1}{q};\frac{1}{q})_n(\frac{1}{t};\frac{1}{t})_n} = q^n t^n (qt)^{\binom{n}{2}},$$

it suffices to show

$$\frac{f_n(\frac{1}{q},\frac{1}{t})}{f_n(q,t)} = q^n t^n.$$

For any given $k, 1 \le k \le n$, and $a_1 + \cdots + a_k = n$ with $a_i > 0$, define

$$b(q,t) = q^{\sum_{i=1}^{k} \binom{a_i}{2}}\, t^{\sum_{i=1}^{k-1} (k-i)a_i},$$

$$r_0(q,t) = \frac{1}{(t^k;q)_{a_1}(q;q)_{a_k}},$$

$$r_i(q,t) = \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix} \frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}}, i = 1 \ldots k-1.$$

Now it will be enough to prove

$$\prod_{i=0}^{k-1} \frac{r_i(\frac{1}{q}, \frac{1}{t})}{r_i(q,t)} = \frac{b(q,t)}{b(\frac{1}{q}, \frac{1}{t})} q^n t^n.$$

In fact, for $1 \leq i \leq k-1$,

$$\frac{\begin{bmatrix} a_i+a_{i+1}-1 \\ a_i \end{bmatrix}_{\frac{1}{q}}}{\begin{bmatrix} a_i+a_{i+1}-1 \\ a_i \end{bmatrix}} = \frac{q^{\binom{a_i}{2}+\binom{a_{i+1}}{2}+a_i}}{q^{\binom{a_i+a_{i+1}-1}{2}}},$$

and

$$\frac{\frac{1}{((\frac{1}{t})^{k-i};\frac{1}{q})_{a_i+a_{i+1}}}}{\frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}}} = t^{(k-i)(a_i+a_{i+1})} q^{\binom{a_i+a_{i+1}-1}{2}}.$$

Hence,

$$\frac{r_i(\frac{1}{q}, \frac{1}{t})}{r_i(q,t)} = q^{\binom{a_i}{2}} q^{\binom{a_{i+1}}{2}} q^{a_i} t^{(k-i)a_i} t^{(k-(i+1))a_{i+1}} t^{a_{i+1}}.$$

In addition,

$$\frac{r_0(\frac{1}{q}, \frac{1}{t})}{r_0(q,t)} = \frac{(t^k;q)_{a_1}(q;q)_{a_k}}{\left((\frac{1}{t})^k;\frac{1}{q}\right)_{a_1}(\frac{1}{q};\frac{1}{q})_{a_k}} = q^{\binom{a_1}{2}} q^{\binom{a_k}{2}} q^{a_k} t^{(k-1)a_1} t^{a_1}.$$

Therefore finally

$$\prod_{i=0}^{k-1} \frac{r_i(\frac{1}{q}, \frac{1}{t})}{r_i(q,t)} = q^{2\sum_{i=1}^{k}\binom{a_i}{2}} t^{2\sum_{i=1}^{k}(k-i)a_i} q^n t^n = \frac{b(q,t)}{b(\frac{1}{q}, \frac{1}{t})} q^n t^n.$$

$\square$

Lemma 4.1.7.

$$F_n(q,0) = q^{\binom{n}{2}}.$$

Proof. Consider $F_n(q,0)$. The only non-zero term occurs when $\sum_{i=1}^{k-1}(k-i)a_i = 0$, i.e., when $k=1$. The rest is clear. $\square$

Corollary 4.1.8.

$$\lim_{t\to\infty} \frac{F_n(q,t)}{G_n(q,t)} = 1.$$

Proof. By Lemma 4.1.6 and Lemma 4.1.7,

$$\lim_{t\to\infty} \frac{F_n(q,t)}{t^{\binom{n}{2}}} = q^{\binom{n}{2}} \lim_{t\to\infty} F_n(\frac{1}{q}, \frac{1}{t}) = q^{\binom{n}{2}} F_n(\frac{1}{q}, 0) = 1.$$

Similarly, from (4.1.3) and (4.1.4), we also have

$$\lim_{t\to\infty} \frac{G_n(q,t)}{t^{\binom{n}{2}}} = 1.$$

□

Lemma 4.1.9.

$$F_n(q,1) = \frac{1-q^n}{1-q} F_{n-1}(q,1).$$

Proof. Consider $F_n(q,1)$. The multiplicity of zero's on top is n, generated by $(t,t)_n$. In order for the bottom to cancel these many zero's, k must be n. It is easy to see that immediately

$$F_n(q,1) = \frac{(q)_n}{(1-q)^n}.$$

□

From Lemma 4.1.7 and Lemma 4.1.9, $F_n(q,0) = G_n(q,0)$ and $F_n(q,1) = G_n(q,1)$ for any given q.

4.2 Investigation of Zeros

The following is a theorem by Carlitz.

Theorem 4.2.1. [Car56] *If ξ is a primitive g^{th} root of unity, where $g|n$, then*

$$G_n(q,\xi) = \frac{(-1)^{n-\frac{n}{g}}(q)_n}{(1-q^g)^{\frac{n}{g}}}.$$

We are happy to see exactly the same hold for F_n.

Theorem 4.2.2. *If ξ is a primitive g^{th} root of unity, where $g|n$, then*

$$F_n(q,\xi) = \frac{(-1)^{n-\frac{n}{g}}(q)_n}{(1-q^g)^{\frac{n}{g}}}.$$

Proof. Since $g|n$, $1-\xi^n = 0$. Hence,

$$\frac{(\xi)_n}{(\xi)_k} \neq 0 \Leftrightarrow k = n.$$

So,

$$\begin{aligned} F_n(q,\xi) &= \frac{\xi^{\sum_{i=1}^{k-1}(n-i)}(q)_n(\xi)_n}{(\xi^n;q)_1\,(q;q)_1\,(\xi^{n-1};q)_2\cdots(\xi;q)_2} \\ &= \frac{\xi^{\frac{n(n-1)}{2}}(q)_n}{(1-\xi^{n-1}q)(1-\xi^{n-2}q)\cdots(1-\xi q)(1-q)} \\ &= \frac{\xi^{\frac{n(n-1)}{2}}(q)_n}{((1-\xi^{g-1}q)(1-\xi^{g-2}q)\cdots(1-q))^{\frac{n}{g}}}. \end{aligned}$$

Now on the bottom,

$$\begin{aligned} &(1-\xi^{g-1}q)(1-\xi^{g-2}q)\cdots(1-q) \\ &= q^g(x-1)\cdots(x-\xi^{g-1})|_{x=\frac{1}{q}} \\ &= q^g(x^g-1)|_{x=\frac{1}{q}} \\ &= 1-q^g. \end{aligned}$$

On the top, let $\xi = \frac{2\pi i j}{g}$, where $\gcd(j, g) = 1$ and suppose $n = yg$, where $y \in \mathbb{N}$. Then,

$$\xi^{\frac{n(n-1)}{2}} = e^{\frac{2\pi i j}{g}\frac{n(n-1)}{2}} = e^{\pi i j y(n-1)} = (-1)^{jy(n-1)}.$$

Notice that $jy(n-1) - (n-y) = jy(yg-1) - (yg-y) = y(jyg - j - g + 1)$ must be even because j and g should have different parities. Therefore,

$$\xi^{\frac{n(n-1)}{2}} = (-1)^{n-y} = (-1)^{n-\frac{n}{g}}.$$

So we are done. $\square$

Remark 4.2.1. G_n is obviously symmetric from its definition. For the case discussed above we also have the symmetry for F_n: $F_n(q,\xi) = F_n(\xi, q)$. To see that, notice $F_n(\xi, q)$ only has one non-zero term, i.e. the one where $a_k = n$ in order to cancel $1 - \xi^n$ which is generated by $(\xi;\xi)_n$ on the top, and thus $k = 1$. The rest is similar to the proof of Theorem 4.2.2. In the end we have the same expression

$$F_n(\xi, q) = \frac{(-1)^{n-\frac{n}{g}}(q)_n}{(1-q^g)^{\frac{n}{g}}},$$

so that $F_n(q,\xi) = F_n(\xi, q)$.

For any fixed q, let $d_q(t) := F_n(q,t) - G_n(q,t)$. Theorem 4.2.2 shows that $d_q(t)$ has n zero's, since every n^{th} root of unity is a primitive g^{th} root, for some $g|n$. By (4.1.4) and Lemma 4.1.7, $t = 0$ is another zero. So altogether we have found $n+1$ zeros of $d_q(t)$, where q is any given real number, which is a partial support for $F_n(q,t)$ and $G_n(q,t)$ to be equal.

Next we present a theorem about F_n, parallel to a result of Gordon about G_n.

Theorem 4.2.3. [Gor63] *Assume* $\omega = e^{\frac{2\pi i u}{h}}$ *and* $\xi = e^{\frac{2\pi i v}{g}}$*, where* $\gcd(u, h) = \gcd(v, g) = 1, g < h \leq n$ *and* $n = yh + s = xg + r$ *(so that* $1 \leq y \leq x$*), then*

$$G_n(\omega, \xi) = 0.$$

Theorem 4.2.4. *Assume* $\omega = e^{\frac{2\pi i u}{h}}$ *and* $\xi = e^{\frac{2\pi i v}{g}}$*, where* $\gcd(u, h) = \gcd(v, g) = 1, g < h \leq n$ *and* $n = yh + s = xg + r$ *(so that* $1 \leq y \leq x$*), then*

$$F_n(\omega, \xi) = 0.$$

Proof. For each selection of k and $a_1, a_2, \cdots, a_k$ with $a_1 + a_2 + \cdots + a_k = n$, clearly the number of zero's on the numerator of $F_n(\omega, \xi)$ is $x + y$. We only need to make sure that the number of zero's generated by the bottom is something smaller. Let $a_0 = 0$, then

$$\text{The bottom} = (\omega, \omega)_{a_k} \prod_{j=0}^{k-1} (1 - \xi^{k-j}) \cdots (1 - \xi^{k-j} \omega^{a_j + a_{j+1} - 1}).$$

For any fixed j and $0 \leq p \leq a_j + a_{j+1} - 1$,

$$1 - \xi^{k-j} \omega^p = 0 \Leftrightarrow \frac{v}{g}(k - j) + \frac{u}{h} p \in \mathbb{N}.$$

Note that if p_i, $i = 1, 2$, makes

$$\frac{v}{g}(k - j) + \frac{u}{h} p_i \in \mathbb{N},$$

then

$$\frac{u}{h}(p_1 - p_2) \in \mathbb{N} \Leftrightarrow h | (p_1 - p_2),$$

which says that the difference between any two "good" p's is at least h.

Next, let $f = \gcd(g, h), g = g_1 f, h = h_1 f$ and $\frac{v}{g}(k-j) + \frac{u}{h}p = m \in \mathbb{N}$. Then,

$$hv(k-j) + gup = mgh \Rightarrow h_1 v(k-j) + g_1 up = mg_1 h_1 f$$

and therefore,

$$\gcd(g_1 u, h_1) = 1 \Rightarrow h_1 | p \tag{4.2.1}$$

$$\gcd(h_1 v, g_1) = 1 \Rightarrow g_1 | (k-j). \tag{4.2.2}$$

So, only when $j = k - g_1, j = k - 2g_1, \cdots$, there could possibly be zero's on the bottom.

- Case 1: $g_1 = 1$. Then $g = f \mid h \Rightarrow h \geq 2g$. Therefore,

$$\begin{aligned}
&\text{The number of zero's on the bottom} \\
&\leq \lfloor \frac{a_k}{h} \rfloor + \sum_{j=0}^{k-1} \lfloor \frac{a_j + a_{j+1}}{h} \rfloor \\
&\leq \lfloor \frac{(a_0 + a_1) + \cdots + (a_{k-1} + a_k) + a_k}{h} \rfloor \\
&= \lfloor \frac{2n}{h} \rfloor \\
&\leq \lfloor \frac{n}{g} \rfloor \\
&= x \\
&< x + y.
\end{aligned}$$

- Case 2: $g_1 \geq 2$. Then,

$$\begin{aligned}
&\text{The number of zero's on the bottom} \\
&\leq \lfloor \frac{a_k}{h} \rfloor + \lfloor \frac{a_{k-g_1} + a_{k-g_1+1}}{h} \rfloor + \cdots + \lfloor \frac{a_{k-\lfloor \frac{k}{g_1} \rfloor g_1} + a_{k-\lfloor \frac{k}{g_1} \rfloor g_1+1}}{h} \rfloor \\
&\leq \lfloor \frac{a_k + a_{k-1} + \cdots + a_1 + a_0}{h} \rfloor \\
&= \lfloor \frac{n}{h} \rfloor \\
&= y \\
&< x + y.
\end{aligned}$$

So, in either case, $F_n(\omega, \xi) = 0$. $\square$

By Theorem 4.2.4 and the following result, we also have symmetry of $F_n(q,t)$ for such values of q and t.

Theorem 4.2.5. *Assume* $\omega = e^{\frac{2\pi i u}{h}}$ *and* $\xi = e^{\frac{2\pi i v}{g}}$, *where* $\gcd(u,h) = \gcd(v,g) = 1, g < h \leq n$ *and* $n = yh + s = xg + r$ $(so\ that\ 1 \leq y \leq x)$, *then*

$$F_n(\xi, \omega) = 0$$

Proof. The proof is similar to but simpler than the proof of Theorem 4.2.4. If we use the same notation, then (4.2.1) and (4.2.2) in the proof of Theorem 4.2.4 will become $g_1|(p_1-p_2)$ and $h_1|(k-j)$, respectively. Moreover, only when $j = k-g_1, j = k-2g_1, \cdots,$

could there be zero's on the bottom. Note $h > g \Rightarrow h_1 > g_1 \geq 1 \Rightarrow h_1 \geq 2$. So nicely,

The number of zero's on the bottom

$$\leq \lfloor \frac{a_k}{g} \rfloor + \lfloor \frac{a_{k-h_1} + a_{k-h_1+1}}{g} \rfloor + \lfloor \frac{a_{k-2h_1} + a_{k-2h_1+1}}{g} \rfloor + \cdots + \lfloor \frac{a_{k-\lfloor \frac{k}{h_1} \rfloor h_1} + a_{k-\lfloor \frac{k}{h_1} \rfloor h_1+1}}{g} \rfloor$$
$$\leq \lfloor \frac{a_k + (a_{k-h_1+a_{k-h_1+1}}) + (a_{k-2h_1} + a_{k-2h_1+1}) + \cdots + (a_{k-\lfloor \frac{k}{h_1} \rfloor h_1} + a_{k-\lfloor \frac{k}{h_1} \rfloor h_1+1})}{g} \rfloor$$
$$\leq \lfloor \frac{a_k + a_{k-1} + \cdots + a_1 + a_0}{g} \rfloor$$
$$= \lfloor \frac{n}{g} \rfloor$$
$$= x$$
$$< x + y.$$

□

4.3 When $q = t$

In this section we refer to the version of Theorem 4.1.5. That is, we use the same $F_n(q,t)$ as previously but let

$$G_n(q,t) := \sum_{\sigma \in S_n} q^{maj(\sigma)}\, t^{\binom{n}{2} - maj(\sigma^{-1})}.$$

Furthermore, for $1 \leq k \leq n$, define

$$F_{n,k}(q,t) := \sum_{\substack{a_1+\cdots+a_k=n \\ a_i>0}} q^{\sum_{i=1}^{k} \binom{a_i}{2}}\, t^{\sum_{i=1}^{k-1}(k-i)a_i}\, \frac{1}{(t^k;q)_{a_1}(q;q)_{a_k}}$$
$$\times \prod_{i=1}^{k-1} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix} \frac{1}{(t^{k-i};q)_{a_i+a_{i+1}}} \times (q;q)_n (t;t)_n,$$

and

$$G_{n,k}(q,t) := \sum_{\sigma \in S_{n,k}} q^{maj(\sigma)}\, t^{\binom{n}{2}-maj(\sigma^{-1})},$$

where $S_{n,k} = \{\sigma \in S_n : is_n(\sigma) = k\}$ and $is_n(\sigma)$ denotes the length of the longest increasing subsequence of $\sigma \in S_n$. By definition, $F_n(q,t) = \sum_{i=1}^{n} F_{n,i}(q,t)$ and $G_n(q,t) = \sum_{i=1}^{n} G_{n,i}(q,t)$. For brevity, we also let

$$f_{n,k}(q,t) = \sum_{i=1}^{k} F_{n,i}(q,t)$$

and

$$\begin{aligned} g_{n,k}(q,t) &= \sum_{i=1}^{k} G_{n,i}(q,t) \\ &= \sum_{i=1}^{k} \sum_{\sigma \in S_{n,i}} q^{maj(\sigma)}\, t^{\binom{n}{2}-maj(\sigma^{-1})}. \end{aligned}$$

Here we investigate the interesting case where $q = t$. One property of the (q,q)-case comes from Lemma 4.1.9 and (4.1.6):

$$F_n(q,q) = q^{n(n-1)} F_n(\frac{1}{q}, \frac{1}{q})$$

and

$$G_n(q,q) = q^{n(n-1)} G_n(\frac{1}{q}, \frac{1}{q}).$$

This property can be refined to the following.

Lemma 4.3.1.

$$F_{n,k}(q,q) = q^{n(n-1)} F_{n,k}(\frac{1}{q}, \frac{1}{q}),$$

and

$$G_{n,k}(q,q) = q^{n(n-1)} G_{n,k}(\frac{1}{q},\frac{1}{q})$$

Proof. The first equality is immediate from the proof of Lemma 4.1.6. Observe that the second equality will be true as soon as we prove that $S_{n,k}$ is closed under the inverse operation of symmetric groups. In fact, it is easy to see that if $is_n(\sigma) = k$ then $is_n(\sigma^{-1}) \geq k$, and vice versa, so it is clear. □

The main conjecture regarding the (q,q) case is the following.

Conjecture 4.3.1. *For* $1 \leq k \leq n$,

$$F_{n,k}(q,q) = G_{n,k}(q,q).$$

Equivalently, for $1 \leq k \leq n$,

$$f_{n,k}(q,q) = g_{n,k}(q,q).$$

Conjecture 4.3.1 has been verified by using Maple for small values of n and k, but the proof appears to be quite hard even for $k = 2$. It is not even clear that $F_{n,k}(q,q)$ or $f_{n,k}(q,q)$ has to be polynomial for general n and k.

Example 4.3.1. When $k = 1$,

$$f_{n,1}(q,q) = F_{n,1}(q,q) = q^{\binom{n}{2}} \frac{(q;q)_n(q;q)_n}{(q;q)_n(q;q)_n} = q^{\binom{n}{2}},$$

and

$$g_{n,1}(q,q) = G_{n,1}(q,q) = q^{maj(n\cdots 21)+\binom{n}{2}-maj((n\cdots 21)^{-1})} = q^{\binom{n}{2}}.$$

In [FH85], $c_n(\lambda;q)$ is introduced as a kind of q-Catalan number with parameter λ by means of the expansion formula

$$z = \sum_{n=1}^{\infty} \frac{c_n(\lambda;q) z^n}{q^{\binom{n}{2}}(-q^{-n}z)_n(-q^{\lambda}z)_n}.$$

Fürlinger and Hofbauer showed that

$$c_n(\lambda;q) = \frac{1}{[n]} \sum_{k=1}^{n} q^{k^2+\lambda k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix},$$

and in particular

$$c_n(1;q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Lemma 4.3.2. *When $k=2$,*

$$f_{n,2}(q,q) = q^{\binom{n}{2}} c_n(1-n,q).$$

Proof.

$$\begin{aligned}
F_{n,2}(q,q) &= \sum_{a_1=1}^{n-1} q^{\binom{a_1}{2}+\binom{n-a_1}{2}+a_1} \frac{(q;q)_n^2}{(q^2;q)_{a_1}(q;q)_{n-a_1}} \begin{bmatrix} n-1 \\ a_1 \end{bmatrix} \frac{1}{(q;q)_n} \\
&= \sum_{a_1=1}^{n-1} q^{\binom{a_1}{2}+\binom{n-a_1}{2}+a_1} \frac{(q;q)_n}{(q;q)_{a_1}(q;q)_{n-a_1}} \frac{1}{[a_1+1]} \begin{bmatrix} n-1 \\ a_1 \end{bmatrix} \\
&= \sum_{a_1=1}^{n-1} q^{a_1^2+a_1(1-n)+\binom{n}{2}} \begin{bmatrix} n \\ a_1 \end{bmatrix} \begin{bmatrix} n \\ a_1+1 \end{bmatrix} \frac{1}{[n]} \\
&= q^{\binom{n}{2}} \frac{1}{[n]} \left(\sum_{k=1}^{n} q^{k^2+(1-n)k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix} - [n] - 0 \right) \\
&= q^{\binom{n}{2}} (c_n(1-n;q) - 1).
\end{aligned}$$

So,

$$f_{n,2}(q,q) = F_{n,1}(q,q) + F_{n,2}(q,q) = q^{\binom{n}{2}} c_n(1-n;q).$$

$\square$

Remark 4.3.1. In the summation above,

$$\frac{1}{[n]}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ k-1 \end{bmatrix}$$

is the q-analog of the $Runyon$ numbers $r_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ [Rio68, page 17], which counts the number of Dyck paths $\Pi \in \mathcal{D}_n$ with $k-1$ "valleys" (consecutive EN pair steps) or equivalently k "peaks" (consecutive NE pair steps), i.e. $des(w(\Pi)) = k-1$. Fürlinger and Hofbauer also obtained the following combinatorial interpretation of $c_n(\lambda, q)$:

$$c_n(\lambda, q) = \sum_{w \in CW_n} q^{maj(w)+(\lambda-1)des(w)}.$$

Corollary 4.3.3. *$f_{n,2}(q,q)$ is a polynomial with non negative coefficients.*

Proof. For brevity let $des(w) = d$. Then,

$$\begin{aligned}
&\binom{n}{2} + maj(w) + ((1-n)-1)des(w) \\
&\geq \binom{n}{2} + (2+4+\cdots+2d) - nd \\
&= \binom{n}{2} + d^2 + d - nd \\
&= (d - \frac{n-1}{2})^2 + \frac{n^2-1}{4} \\
&\geq \begin{cases} \frac{n^2-1}{4}, & \text{if n is odd;} \\ \frac{n^2}{4}, & \text{if n is even.} \end{cases}.
\end{aligned}$$

So $f_{n,2}(q,q)$ is a polynomial of positive coefficients, following Lemma 4.3.2 and Remark 4.3.1.

$\square$

Computationally, we found that

$$f_{n,2}(q,q) = g_{n,2}(q,q)$$

for n up to 8. Since $S_{n,1} \cup S_{n,2} = S_n(123)$, where $S_n(123)$ stands for the set of 123-avoiding permutations in S_n, we can rewrite the conjecture $f_{n,2}(q,q) = g_{n,2}(q,q)$ in the following way.

Conjecture 4.3.2.

$$\sum_{w \in CW_n} q^{maj(w)-ndes(w)} = \sum_{\sigma \in S_n(123)} q^{maj(\sigma)-maj(\sigma^{-1})}.$$

Although the conjecture appears to be formidable, it is still interesting to consider some related situation. If restricted to the set of "involutions" on both sides, then as a partial result, we prove a similar identity.

Definition 4.3.1. The *inverse* of a Catalan word $w \in CW_n$ is defined to be

$$w^{-1} = \overline{r(w)},$$

where r denotes the reverse operation and $^{-}$ denotes the complement operation that exchanges 0 and 1. We say w is an $involution$ if and only if $w = w^{-1}$.

Example 4.3.2. When n=3,

$$(000111)^{-1} = 000111,$$

$$(001011)^{-1} = 001011,$$

$$(001101)^{-1} = 010011,$$

$$(010011)^{-1} = 001101,$$

$$(010101)^{-1} = 010101.$$

So the involution subset of CW_3 consists of 000111, 001011 and 010101.

It is easy to see that $w^{-1} \in CW_n$ if and only if $w \in CW_n$, so the inverse operation is closed on CW_n. Geometrically, given w, we may obtain w^{-1} by finding the Dyck path Π that w corresponds to under the natural map, reflecting Π over the NW-SE main diagonal to obtain a new Dyck path P, and then taking the Catalan word that corresponds to P. For this reason, we also define the *inverse* of a Dyck path Π by $\Pi^{-1} = P$ through this reflection, and say Π is an $involution$ if and only if $\Pi = \Pi^{-1}$. Figure 4.1 illustrates a Dyck path together with its inverse. Note the reflection over the dashed NW-SE diagonal.

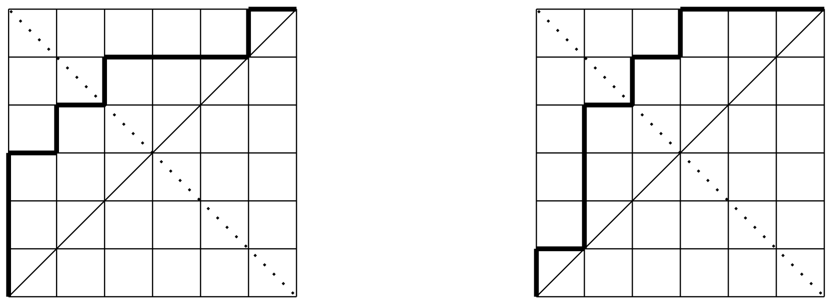


Figure 4.1: A Dyck path (on the left) and its inverse (on the right).

Theorem 4.3.4.

$$\sum_{\substack{w \in CW_n: \\ w \text{ is an involution}}} q^{maj(w)-ndes(w)} = \sum_{\substack{\sigma \in S_n(123): \\ \sigma \text{ is an involution}}} q^{maj(\sigma)-maj(\sigma^{-1})}.$$

Proof. For $w \in CW_n$,

$$\begin{aligned} maj(w^{-1}) &= \sum_{i \in des(w^{-1})} i \\ &= \sum_{2n-i \in des(w)} i \\ &= \sum_{j \in des(w)} 2n - j \\ &= 2ndes(w) - maj(w). \end{aligned}$$

Clearly $des(w^{-1}) = des(w)$. Thus

$$\begin{aligned} maj(w^{-1}) - ndes(w^{-1}) &= (2ndes(w) - maj(w)) - ndes(w) \\ &= ndes(w) - maj(w) \\ &= -(maj(w) - ndes(w)). \end{aligned}$$

If w is an involution, this implies $maj(w) - ndes(w) = 0$. So it suffices to prove that the number of involutions in CW_n, which is also the number of Dyck path involutions in $\mathcal{D}_n$, is equal to the number of involutions in $S_n(123)$.

On one hand, Simion and Schmidt [SS85] showed that the number of involutions in $S_n(123)$ is equal to

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

On the other hand, we can decompose the set of Dyck path involutions according to the place where they reach the NW-SE diagonal about which they are symmetric. For $k \geq m$, let $S(k,m)$ be the set of all 01 words $w = w_1 w_2 \cdots w_{k+m}$ consisting of k 0s and m 1s; let $S_+(k,m)$ denote the subset of $S(k,m)$ consisting of those words, such that no initial segment contains more 1s than 0s; and let $S_-(k,m) = S(k,m) \setminus S_+(k,m)$. Obviously $|S(k,m)| = \binom{k+m}{k}$. Fürlinger and Hofbauer [FH85] constructed a bijection between $S_-(k,m)$ and $S(k+1,m-1)$. Furthermore, since every Dyck path involution is uniquely decided by its beginning half (the part from (0, 0) to where it reaches the NW-SE diagonal), there is an apparent bijection between $S_+(k,n-k)$ and the subset of Dyck path involutions in $\mathcal{D}_n$ that reach the NW-SE diagonal at $(n-k,k)$. Therefore, the number of Dyck path involutions in $\mathcal{D}_n$ is equal to

$$\begin{aligned}\sum_{k=\lceil\frac{n}{2}\rceil}^{n} |S_+(k,n-k)| &= \sum_{k=\lceil\frac{n}{2}\rceil}^{n} |S(k,n-k)| - |S_-(k,n-k)| \\ &= \sum_{k=\lceil\frac{n}{2}\rceil}^{n} \binom{n}{k} - |S(k+1,n-k-1)| \\ &= \sum_{k=\lceil\frac{n}{2}\rceil}^{n} \binom{n}{k} - \binom{n}{k+1} \\ &= \binom{n}{\lceil\frac{n}{2}\rceil}.\end{aligned}$$

So we are done. $\square$

4.4 Agreement of t^k Coefficients

Theorem 4.1.5 would be proved if the coefficients of t^k agreed on both sides. It's trivial to see that the constant terms, i.e. the coefficients of t^0 on the two sides, are both equal to $q^{\binom{n}{2}}$.

We will show that this is also true for $k = 1, 2, 3$ and look for a pattern to generalize.

Lemma 4.4.1. *The coefficients of t^1 on the two sides of Theorem 4.1.5 are both equal to*

$$q^{\binom{n-1}{2}}\left(\begin{bmatrix} n \\ 1 \end{bmatrix} - q^{(n-1)}\right).$$

Proof. There are two cases on the *left hand side*:

- $k = 1$ and so $a_1 = n$.
$$\text{LHS} = q^{\binom{n}{2}} \frac{(t;t)_n (q;q)_n}{(t;q)_n (q;q)_n}.$$
So we only need to find the coefficient of t in
$$q^{\binom{n}{2}} \left((1+tq)(1+tq^2)\cdots(1+tq^{n-1})\right)$$
which is $q^{\binom{n}{2}} q[n-1]$.

- $k = 2$, so a_1 must be 1 and $a_2 = n-1$.
$$\text{LHS} = q^{\binom{n-1}{2}} t \, \frac{(t;t)_n (q;q)_n}{(t^2;q)_1 (q;q)_n} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \frac{1}{(t;q)_n}.$$
So the coefficient of t is
$$q^{\binom{n-1}{2}} (1-q^n) \begin{bmatrix} n-1 \\ 1 \end{bmatrix}.$$

Summing up the two cases we get $q^{\binom{n-1}{2}} \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - q^{(n-1)} \right)$.

Right hand side:

$$\begin{aligned} [t^1] \text{ RHS} &= \sum_{\binom{n}{2} - maj(\sigma^{-1}) = 1} q^{maj(\sigma)} \\ &= \sum_{\binom{n}{2} - maj(\sigma) = 1} q^{maj(\sigma^{-1})}. \end{aligned}$$

Note that
$$\binom{n}{2} - maj(\sigma) = 1 \Leftrightarrow \sigma_1 < \cdots < \sigma_{n-1} > \sigma_n.$$

So $\sigma_2 = n$. Let $\sigma_1 = i$, $1 \le i \le n-1$, then i will be the only non-descent of σ^{-1}. Hence

$$\begin{aligned}\sum_{\binom{n}{2}-maj(\sigma)=1} q^{maj(\sigma^{-1})} &= \sum_{i=1}^{n-1} q^{\binom{n}{2}-i} \\ &= q^{\binom{n-1}{2}}[n-1] \\ &= q^{\binom{n-1}{2}} \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - q^{(n-1)} \right).\end{aligned}$$

□

Theorem 4.4.2. *The coefficients of t^2 on the two sides of Theorem 4.1.5 are both equal to*

$$q^{\binom{n-2}{2}+1} \left(\begin{bmatrix} n \\ 2 \end{bmatrix} - q^{2(n-2)} \right).$$

Proof. First consider the LHS. Notice that $\sum_{i=1}^{k-1}(k-i)a_i$ will be bigger than 2 if k is bigger than 2. Thus in order to generate t^2 we only have the following three situations:

- $k = 1$ and $a_1 = n$

  LHS is easily reduced to $q^{\binom{n}{2}}(t;t)_n/(t;q)_n$. So we only need to get the coefficient of t^2 in

  $$q^{\binom{n}{2}}(1-t^2)(1+tq+(tq)^2)\cdots(1+tq^{n-1}+(tq^{n-1})^2),$$

  which is clearly $q^{\binom{n}{2}}(-1+q^2[n-1]_{q^2}+q^3\begin{bmatrix} n-1 \\ 2 \end{bmatrix})$.

- $k = 2,\ a_1 = 1$ and $a_2 = n-1$

  In this case,

$$\text{LHS} = q^{\binom{n-1}{2}}\, t\, \frac{(t;t)_n\,(q;q)_n}{(t^2;q)_1\,(q;q)_{n-1}} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \frac{1}{(t;q)_n}\,(t;q)_n\,.$$

So it suffices to find the coefficient of t in

$$q^{\binom{n-1}{2}+1}\,(1-q^n)\,[n-1]\,(1+tq)\cdots(1+tq^{n-1}).$$

That is clearly $q^{\binom{n-1}{2}+1}(1-q^n)\,[n-1]^2$.

- $k=2,\ a_1=2$ and $a_2=n-2$

  Similarly, the coefficient of t^2 in

$$q^{\binom{n-2}{2}+1}\,t^2\,\frac{(t;t)_n\,(q;q)_n}{(t^2;q)_2\,(q;q)_{n-2}} \begin{bmatrix} n-1 \\ 2 \end{bmatrix} \frac{1}{(t;q)_n}\,(t;q)_n$$

  is $q^{\binom{n-2}{2}+1}(1-q^n)(1-q^{n-1})\left[\begin{smallmatrix} n-1 \\ 2 \end{smallmatrix}\right])$.

Summing up the contributions from the three cases, we find the coefficient of t^2 on the LHS is equal to

$$\begin{aligned} &\frac{q^{\binom{n-2}{2}}}{(1-q)\,(1-q^2)}\,(q-q^n-q^{n+1}-q^{2n-3}+q^{2n-2}+q^{2n-1}) \\ &= q^{\binom{n-2}{2}+1}\,\left(\begin{bmatrix} n \\ 2 \end{bmatrix} - q^{2(n-2)}\right). \end{aligned}$$

Next consider the RHS. The coefficient of t^2 on the RHS is equal to

$$\sum_{\binom{n}{2}-maj(\sigma^{-1})\ =2} q^{maj(\sigma)} = \sum_{\binom{n}{2}-maj(\sigma)=2} q^{maj(\sigma^{-1})}$$

Observe that

$$\begin{aligned}
&\binom{n}{2} - maj(\sigma) = 2 \\
&\Leftrightarrow Des(\sigma) = \{1, 2, \cdots, n-1\} \backslash \{2\} \\
&\Leftrightarrow \sigma_1 < \sigma_2 < \cdots < \sigma_{n-2} > \sigma_{n-1} > \sigma_n.
\end{aligned}$$

There are three possible cases according to the value of σ_3. We call the subsets of S_n in these cases $\Pi_1,\ \Pi_2$ and Π_3 respectively.

- $\sigma_3 = n-1$. Then we must have $\sigma_1 = n$. Let $\sigma_2 = k$, where $1 \leq k \leq n-2$. It is easy to see that the only non-descent of σ^{-1} is k where $\sigma^{-1}(k) = 2$ and $\sigma^{-1}(k+1) = n-k+1$. Therefore, $maj(\sigma^{-1})\ = \binom{n}{2} - k$ and so we have,

$$\begin{aligned}
\sum_{\sigma \in \Pi_1} q^{maj(\sigma^{-1})} &= \sum_{k=1}^{n-2} q^{\binom{n}{2}-k} \\
&= q^{\binom{n-1}{2}}\ (q + \ldots + q^{n-2}) \\
&= q^{\binom{n-1}{2}+1}\ [n-2].
\end{aligned}$$

- $\sigma_3 = n$ and $\sigma_2 = k,\ \sigma_1 = j,$ with $1 \leq k < j-1 \leq n$. Note that the case of $k = j-1$ will be dealt with differently next. Now,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-j+2 & n-j+3 & \cdots & n-k+1 & n-k+2 & \cdots \\ j & k & n & \cdots & j+1 & j-1 & \cdots & k+1 & k-1 & \cdots \end{pmatrix}$$

implies that

$$\sigma^{-1} = \begin{pmatrix} \cdots & k-1 & k & k+1 & \cdots & j-1 & j & j+1\cdots & n \\ \cdots & n-k+2 & 2 & n-k+1 & \cdots & n-j+3 & 1 & n-j+2\cdots & 3 \end{pmatrix}.$$

So,

$$\begin{aligned} \sum_{\sigma\in\Pi_2} q^{maj(\sigma^{-1})} &= \sum_{1\le k<j-1\le n-2} q^{\binom{n}{2}-k-j} \\ &= q^{\binom{n-2}{2}} \sum_{1\le k<j-1\le n-2} q^{(n-2-k)+(n-1-j)} \\ &= q^{\binom{n-2}{2}} \sum_{0\le r<s\le n+s} q^{r+s} \\ &= q^{\binom{n-2}{2}+1} \begin{bmatrix} n-2 \\ 2 \end{bmatrix}. \end{aligned}$$

- $\sigma_3 = n,\ \sigma_2 = k,$ and $\sigma_1 = k+1,$ with $1 \le k \le n-2$. In this case,

$$\begin{aligned} \sigma^{-1} &= \begin{pmatrix} k+1 & k & n & \cdots & k+2 & k-1 & \cdots \\ 1 & 2 & 3 & \cdots & n-k+1 & n-k+2 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} \cdots & k-1 & k & k+1 & k+2 & \cdots & n \\ \cdots & n-k+2 & 2 & 1 & n-k+1 & \cdots & 3 \end{pmatrix}. \end{aligned}$$

So,

$$\begin{aligned}
\sum_{\sigma\in\Pi_3} q^{maj(\sigma^{-1})} &= \sum_{1\le k\le n-2} q^{\binom{n}{2}-(k+1)} \\
&= q^{\binom{n-1}{2}} \sum_{1\le k\le n-2} q^{n-1-(k+1)} \\
&= q^{\binom{n-1}{2}} \sum_{0\le r\le n-3} q^{r} \\
&= q^{\binom{n-1}{2}}[n-2].
\end{aligned}$$

Adding together, the coefficient of t^2 on the RHS is also

$$q^{\binom{n-2}{2}+1} \left(\begin{bmatrix} n \\ 2 \end{bmatrix} - q^{2(n-2)} \right).$$

$\square$

Theorem 4.4.3. *The coefficients of t^3 on the two sides of Theorem 4.1.5 are both equal to*

$$q^{\binom{n-3}{2}+3} \begin{bmatrix} n-3 \\ 3 \end{bmatrix} + q^{\binom{n-2}{2}+1} [3] \begin{bmatrix} n-3 \\ 2 \end{bmatrix} + q^{\binom{n-1}{2}} [3] \begin{bmatrix} n-3 \\ 1 \end{bmatrix} + q^{\binom{n-2}{2}} \begin{bmatrix} n-1 \\ 2 \end{bmatrix}.$$

Proof. There are 5 possible cases on the *left hand side*.

- $k=1$ and $a_1=n$

$$\begin{aligned}
&[t^3]\{q^{\binom{n}{2}}(t;t)_n/(t;q)_n\} \\
&= [t^3]\{q^{\binom{n}{2}}(1-t^2)(1-t^3)(1+tq+(tq)^2+(tq)^3) \\
&\cdots(1+tq^{n-1}+(tq^{n-1})^2+(tq^{n-1})^3\} \\
&= q^{\binom{n}{2}}(q^6\begin{bmatrix}n-1\\3\end{bmatrix}-q[n-1]+\sum_{i=1}^{n-1}q^{2i}(q[n-1]-q^i) \\
&-1+q^3+\cdots+q^{3(n-1)}) \\
&= q^{\binom{n}{2}}(q^6\begin{bmatrix}n-1\\3\end{bmatrix}+q[n-1](q^2[n-1]_{q^2}-1)-1).
\end{aligned}$$

- $k=2$, $a_1=1$ and $a_2=n-1$

$$\begin{aligned}
&[t^3]\{q^{\binom{n-1}{2}}t\frac{(t;t)_n(q;q)_n}{(t^2;q)_1(q;q)_{n-1}}\begin{bmatrix}n-1\\1\end{bmatrix}\frac{1}{(t;q)_n}\} \\
&= [t^2]\{q^{\binom{n-1}{2}}[n-1](1-q^n)(1+tq+t^2q^2)\cdots(1+tq^{n-1}+(tq^{n-1})^2)\} \\
&= q^{\binom{n-1}{2}}[n-1](1-q^n)(q^2[n-1]_{q^2}+q^3\begin{bmatrix}n-1\\2\end{bmatrix}).
\end{aligned}$$

- $k=2$, $a_1=2$ and $a_2=n-2$

$$\begin{aligned}
&[t^3]\{q^{\binom{n-2}{2}+1}t^2\frac{(t;t)_n(q;q)_n}{(t^2;q)_2(q;q)_{n-2}}\begin{bmatrix}n-1\\2\end{bmatrix}\frac{1}{(t;q)_n}\} \\
&= [t^1]\{q^{\binom{n-2}{2}+1}(1-q^n)(1-q^{n-1})\begin{bmatrix}n-1\\2\end{bmatrix}\frac{(t;t)_n}{(t;q)_n}\} \\
&= [t^1]\{q^{\binom{n-2}{2}+1}(1-q^n)(1-q^{n-1})\begin{bmatrix}n-1\\2\end{bmatrix}(1+tq)\cdots(1+tq^{n-1})\} \\
&= q^{\binom{n-2}{2}+1}(1-q^n)(1-q^{n-1})\begin{bmatrix}n-1\\2\end{bmatrix}q[n-1].
\end{aligned}$$

- $k=2$, $a_1=3$ and $a_2=n-3$

$$[t^3]\{q^{\binom{n-3}{2}+3}t^3\frac{(t;t)_n(q;q)_n}{(t^2;q)_3(q;q)_{n-3}}\begin{bmatrix}n-1\\3\end{bmatrix}\frac{1}{(t;q)_n}\}$$
$$=q^{\binom{n-3}{2}+3}(1-q^n)(1-q^{n-1})(1-q^{n-2})\begin{bmatrix}n-1\\3\end{bmatrix}.$$

- $k=3$, $a_1=1$, $a_2=1$ and $a_3=n-2$

$$[t^3]\{q^{\binom{n-2}{2}}t^3\frac{(t;t)_n(q;q)_n}{(t^3;q)_1(q;q)_{n-2}}\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{(t^2;q)_2}\begin{bmatrix}n-2\\1\end{bmatrix}\frac{1}{(t;q)_{n-1}}\}$$
$$=q^{\binom{n-2}{2}}(1-q^n)(1-q^{n-1})[n-2].$$

It is not hard to verify that the sum of the results in the 5 cases above is equal to the desired formula.

Right hand side: there are altogether 8 possible cases.

$$[t^3]\ \text{RHS}=\sum_{\binom{n}{2}-maj(\sigma^{-1})=3}q^{maj(\sigma)}$$
$$=\sum_{\binom{n}{2}-maj(\sigma)=3}q^{maj(\sigma^{-1})}.$$

$\binom{n}{2}-maj(\sigma)=3$ implies either

$$Des(\sigma)=\{1,2,\cdots,n-1\}\backslash\{1,2\},$$

or

$$Des(\sigma)=\{1,2,\cdots,n-1\}\backslash\{3\}.$$

Case 1 $Des(\sigma) = \{1, 2, \cdots, n-1\} \backslash \{1, 2\}$. This means

$$\sigma_1 < \sigma_2 < \sigma_3 > \cdots > \sigma_n,$$

which implies that $\sigma_3 = n$. Let $\sigma_1 = k$ and $\sigma_2 = j$, where $1 \leq k < j \leq n-1$. Then we have

$$\begin{aligned}\sigma^{-1} &= \begin{pmatrix} 1 & 2 & 3 & \cdots & n-j+2 & \cdots & n-k+1-1 & \cdots \\ k & j & n & \cdots & j+1 & \cdots & k+1 & \cdots \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \cdots & k-1 & k & \cdots & j & j+1 & \cdots & n \\ \cdots & n-k+2 & 1 & \cdots & 2 & n-j+2 & \cdots & 3 \end{pmatrix}.\end{aligned}$$

Therefore, $Des(\sigma^{-1}) = \{1, 2, \cdots, n-1\} \backslash \{k, j\}$. Hence,

$$\begin{aligned}\sum_{\binom{n}{2} - maj(\sigma) = 3} q^{maj(\sigma^{-1})} &= \sum_{1 \leq k < j \leq n-1} q^{\binom{n}{2} - k - j} \\ &= q^{\binom{n-2}{2} - 1} \sum_{1 \leq k < j \leq n-1} q^{(n-1-k)+(n-1-j)} \\ &= q^{\binom{n-2}{2} - 1} \sum_{0 \leq r < s \leq n-2} q^{r+s} \\ &= q^{\binom{n-2}{2}} \begin{bmatrix} n-1 \\ 2 \end{bmatrix}.\end{aligned}$$

Case 2 $Des(\sigma) = \{1, 2, \cdots, n-1\} \backslash \{3\}$. This means

$$\sigma_1 > \cdots > \sigma_{n-3} < \sigma_{n-2} > \sigma_{n-1} > \sigma_n.$$

Apparently $\sigma_4 \geq n-2$. We explore the three situations according to $\sigma_4 = n-2$, $\sigma_4 = n-1$, or $\sigma_4 = n$.

Subcase 2.1 $\sigma_4 = n-2$. It must be true that $\sigma_2 = n-1$ and $\sigma_1 = n$. Let $\sigma_3 = k$, where $1 \leq k \leq n-3$, then

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-k+1 & n-k+2 & \cdots \\ n & n-1 & k & n-2 & \cdots & k+1 & k-1 & \cdots \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \cdots & k-1 & k & k+1 & \cdots & n-2 & n-1 & n \\ \cdots & n-k+2 & 3 & n-k+1 & \cdots & 4 & 2 & 1 \end{pmatrix}.$$

Hence $maj(\sigma^{-1}) = \binom{n}{2} - k$ and accordingly

$$\begin{aligned}\sum_{\binom{n}{2}-maj(\sigma)=3} q^{maj(\sigma^{-1})} &= \sum_{1\leq k\leq n-3} q^{\binom{n}{2}-k} \\ &= q^{\binom{n-1}{2}} \sum_{1\leq k\leq n-3} q^{n-1-k} \\ &= q^{\binom{n-1}{2}} \sum_{0\leq r\leq n-4} q^{r+2} \\ &= q^{\binom{n-1}{2}+2}[n-3].\end{aligned}$$

Subcase 2.2 $\sigma_4 = n-1$, then $\sigma_3 = k$. Let $\sigma_3 = k$ and $\sigma_2 = j$, where $1 \leq k < j \leq n-2$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-j+2 & n-j+3 & \cdots \\ n & j & k & n-1 & \cdots & j+1 & j-1 & \cdots \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \cdots & k & j-1 & j & j+1 & \cdots & n-1 & n \\ \cdots & 3 & n-j+3 & 2 & n-j+2 & \cdots & 4 & 1 \end{pmatrix}.$$

It is a little tricky here because it makes a difference if $k = j - 1$ or not. In fact,

$$k = j - 1 \Rightarrow maj(\sigma^{-1}) = \binom{n}{2} - j,$$
$$k < j - 1 \Rightarrow maj(\sigma^{-1}) = \binom{n}{2} - k - j.$$

Therefore,

$$\begin{aligned}
\sum_{\binom{n}{2} - maj(\sigma) = 3} q^{maj(\sigma^{-1})} &= \sum_{1 \le k < j-1 \le n-3} q^{\binom{n}{2} - k - j} + \sum_{2 \le j \le n-2} q^{\binom{n}{2} - j} \\
&= q^{\binom{n-2}{2}+2} \sum_{1 \le k < j-1 \le n-3} q^{(n-2-j)+(n-3-k)} + q^{\binom{n-1}{2}} \sum_{2 \le j \le n-2} q^{n-1-j} \\
&= q^{\binom{n-2}{2}+2} \sum_{0 \le r < s \le n-4} q^{r+s} + q^{\binom{n-1}{2}} \sum_{0 \le r \le n-4} q^{r+1} \\
&= q^{\binom{n-2}{2}+3} \begin{bmatrix} n-3 \\ 2 \end{bmatrix} + q^{\binom{n-1}{2}+1} [n-3].
\end{aligned}$$

Subcase 2.3 $\sigma_4 = n$. Let $\sigma_3 = k$, $\sigma_2 = j$, and $\sigma_1 = i$, where $1 \le k < j < i \le n-1$. This is a complicated case and we will only illustrate the idea.

$$\begin{aligned}
\sigma^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-i+3 & \cdots \\ i & j & k & n & \cdots & i+1 & \cdots \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \cdots & k & \cdots & j & \cdots & i & i+1 & \cdots & n \\ \cdots & 4 & \cdots & 2 & \cdots & 1 & n-i+3 & \cdots & 4 \end{pmatrix}.
\end{aligned}$$

The only possible non-descents of σ^{-1} are thus spots k, j and i. There are different situations dependent on the values of $j - k$ and $i - j$.

- **Sub-subcase 2.3.1**: $j-k>1$ and $i-j>1$. This implies

$$maj(\sigma^{-1}) = \binom{n}{2} - i - j - k.$$

$$\begin{aligned}
\text{So,} \sum_{\binom{n}{2}-maj(\sigma)=3} & q^{maj(\sigma^{-1})} \\
&= \sum_{1\le k<j-1<i-2\le n-3} q^{\binom{n}{2}-i-j-k} \\
&= q^{\binom{n-3}{2}} \sum_{1\le k<j-1<i-2\le n-3} q^{((n-3)-(i-2))+((n-3)-(j-1))+((n-3)-k)} \\
&= q^{\binom{n-3}{2}} \sum_{0\le r<s<t\le n-4} q^{r+s+t} \\
&= q^{\binom{n-3}{2}+3} \begin{bmatrix} n-3 \\ 3 \end{bmatrix}.
\end{aligned}$$

- **Sub-subcase 2.3.2**: $j-k>1$ while $i-j=1$. This implies

$$maj(\sigma^{-1}) = \binom{n}{2} - i - k.$$

$$\begin{aligned}
\text{So,} \sum_{\binom{n}{2}-maj(\sigma)=3} q^{maj(\sigma^{-1})} &= \sum_{1\le k<i-2\le n-3} q^{\binom{n}{2}-i-k} \\
&= q^{\binom{n-2}{2}+1} \sum_{1\le k<i-2\le n-3} q^{((n-3)-(i-2))+((n-3)-k)} \\
&= q^{\binom{n-2}{2}+1} \sum_{0\le r<s\le n-4} q^{r+s} \\
&= q^{\binom{n-2}{2}+2} \begin{bmatrix} n-3 \\ 2 \end{bmatrix}.
\end{aligned}$$

- **Sub-subcase 2.3.3**: $i-j>1$ while $j-k=1$. This implies

$$maj(\sigma^{-1})=\binom{n}{2}-i-j.$$

$$\begin{aligned}
\text{So,} \sum_{\binom{n}{2}-maj(\sigma)=3} q^{maj(\sigma^{-1})} &= \sum_{2\le j<i-1\le n-2} q^{\binom{n}{2}-i-j}\\
&= q^{\binom{n-2}{2}} \sum_{2\le j<i-1\le n-2} q^{(n-2-(i-1))+(n-2-j)}\\
&= q^{\binom{n-2}{2}} \sum_{0\le r<s\le n-4} q^{r+s}\\
&= q^{\binom{n-2}{2}+1} \begin{bmatrix} n-3\\ 2\end{bmatrix}.
\end{aligned}$$

- **Sub-subcase 2.3.4**: $i-j=1$ and $j-k=1$. This implies

$$maj(\sigma^{-1})=\binom{n}{2}-i.$$

$$\begin{aligned}
\text{So,} \sum_{\binom{n}{2}-maj(\sigma)=3} q^{maj(\sigma^{-1})} &= \sum_{3\le i\le n-1} q^{\binom{n}{2}-i}\\
&= q^{\binom{n-1}{2}} \sum_{3\le i\le n-1} q^{n-1-i}\\
&= q^{\binom{n-1}{2}} \sum_{0\le r\le n-4} q^{r}\\
&= q^{\binom{n-1}{2}}[n-3].
\end{aligned}$$

Adding up the results of Case 1, Subcases 2.1, 2.2, Sub-subcases 2.3.1, 2.3.2, 2.3.3 and 2.3.4, we have the desired sum.

Chapter 5

Higher Dimensional Schröder Theory

5.1 m-Schröder Paths and m-Schröder Number

While the standard Catalan and Schröder theories both have been extensively studied, people have only begun to investigate higher dimensional versions of the Catalan number (see [HPW99] and [GH96]). In this chapter, we study a yet more general case, namely the higher dimensional Schröder theory. We introduce and derive results about the m-Schröder paths and words. First let's introduce the notions of generalized Dyck and Schröder paths.

Definition 5.1.1. An m-*Dyck path* of order n is a lattice path from $(0,0)$ to (mn,n) which never goes below the main diagonal $\{(mi,i) : 0 \leq i \leq n\}$, with steps $(0,1)$ (or NORTH, for brevity *N*) and $(1,0)$ (or EAST, for brevity *E*). Let $\mathcal{D}_n^m$ denote the set of all m-Dyck paths of order n.

A 2-Dyck path of order 6 is illustrated in Figure 5.1.

As in the $m=1$ case, given $\Pi \in \mathcal{D}_n^m$, we encode each N step by a 0 and each E step by a 1 so as to obtain a word $w(\Pi)$ of n 0's and mn 1's. This clearly provides a bijection

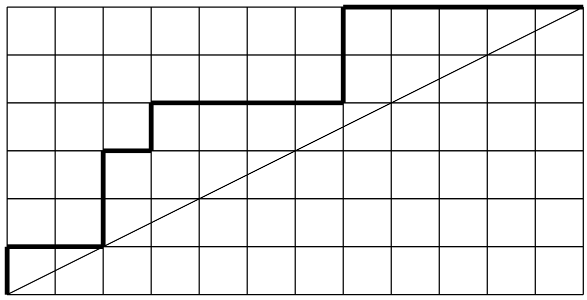


Figure 5.1: A 2-Dyck path in $\mathcal{D}_6^2$.

between $\mathcal{D}_n^m$ and CW_n^m, where

$$CW_n^m = \{w \in M_{n,mn} | \begin{array}{c}\text{at any initial segment of } w\text{, the number of 0's times}\\ m \text{ is at least as many as the number of 1's.}\end{array}\}$$

We call this special subset of 01 words, CW_n^m, *Catalan words of order* n *and dimension* m.

It is shown in [HP91] (see also [HPW99]) that the number of m-Dyck paths, denoted by C_n^m, is equal to

$$\frac{1}{mn+1}\binom{mn+n}{n},$$

which we call the m-Catalan number. In fact, Cigler [Cig87] proved that the number of m-Dyck paths with k peaks, i.e., those with exactly k consecutive NE pairs, is the generalized Runyon number,

$$R_{n,k}^m = \frac{1}{n}\binom{n}{k}\binom{mn}{k-1}.$$

Now we turn to the more general m-Schröder theory.

Definition 5.1.2. An *m-Schröder path* of order n is a lattice path from $(0,0)$ to (mn,n) which never goes below the main diagonal $\{(mi,i) : 0 \leq i \leq n\}$, with steps $(0,1)$ (or NORTH, for brevity *N*), $(1,0)$ (or EAST, for brevity *E*) and (1,1) (or Diagonal, for brevity *D*). Let $\mathcal{S}_n^m$ denote the set of all m-Schröder paths of order n.

Definition 5.1.3. An m-*Schröder path* of order n and with d diagonal steps is a lattice path from $(0, 0)$ to (mn, n) which never goes below the main diagonal $\{(mi, i) : 0 \leq i \leq n\}$, with $(0, 1)$ (or NORTH, for brevity *N*), $(1, 0)$ (or EAST, for brevity *E*) and exactly ***d*** (1,1) (or Diagonal, for brevity *D*) steps. Let $\mathcal{S}^m_{n,d}$ denote the set of all m-Schröder paths of order n and with ***d*** diagonal steps.

A 2-Schröder path of order 6 and with 4 diagonal steps is illustrated in Figure 5.2.

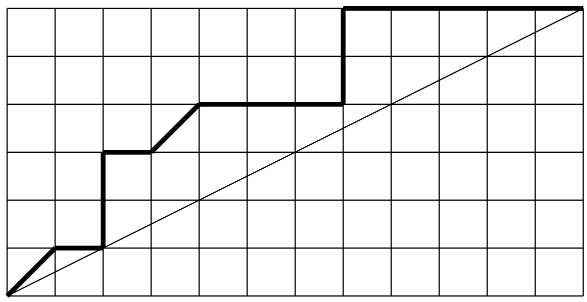


Figure 5.2: A 2-Schröder path in $\mathcal{S}^2_{6,4}$.

Theorem 5.1.1. *The number of m-Schröder paths of order n and with d diagonal steps, denoted by $S^m_{n,d}$, is equal to*

$$\frac{1}{mn - d + 1}\binom{mn + n - d}{mn - d, n - d, d}.$$

Proof. For an m-Dyck path Π, let its number of peaks, or consecutive NE pairs, be denoted by $peak(\Pi)$. Notice that any m-Schröder path with d diagonal steps can be obtained uniquely by choosing d of the peaks of a uniquely decided m-Dyck path Π of the same order, and changing each of the chosen consecutive NE pair steps to a Diagonal step. Conversely, given an m-Dyck path Π of order n, choosing d of its peaks (if there are d to choose) and changing them to D steps will give a path in $\mathcal{S}^m_{n,d}$. For example, the 2-Schröder path as illustrated in Figure 5.2 is one of $\binom{4}{2} = 6$ paths in $\mathcal{S}^2_{6,4}$ that can be obtained from the 2-Dyck path shown in Figure 5.1. Hence,

$$\begin{aligned}
S^m_{n,d} &= \sum_{\Pi \in \mathcal{D}^m_n} \binom{peak(\Pi)}{d} \\
&= \sum_{k \geq d} \binom{k}{d} R^m_{n,k} \\
&= \sum_{k \geq d} \binom{k}{d} \frac{1}{n} \binom{n}{k} \binom{mn}{k-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\binom{n}{d}}{n} \sum_{k \geq d} \binom{n-d}{n-k} \binom{mn}{k-1} \\
&= \frac{\binom{n}{d}}{n} \binom{mn+n-d}{n-1} \\
&= \frac{1}{mn-d+1} \binom{mn+n-d}{d, n-d, mn-d}
\end{aligned}$$

Above we used the Vandermonde Convolution (see, say, [Com74, page 44]). □

As a generalization of the $m = 1$ case, we name

$$S^m_n = \sum_{d=0}^{n} \frac{1}{mn-d+1} \binom{mn+n-d}{mn-d, n-d, d}$$

the m-*Schröder number*.

5.2 q-m-Schröder Polynomials

When Bonin, Shapiro and Simion [BSS93] studied q-analogues of the Schröder numbers, they obtained several classical results for several single variable analogue cases. Here we generalize some of them to the m case.

Definition 5.2.1. Define the m-*Narayana* polynomial $d^m_n(q)$ over the m-Schröder paths of order n to be

$$d_n^m(q) = \sum_{\Pi \in \mathcal{S}_n^m} q^{\mathrm{diag}(\Pi)},$$

where $\mathrm{diag}(\Pi)$ is the number of D steps in the path Π.

Theorem 5.2.1. *$d_n^m(q)$ has $q = -1$ as a root.*

Proof. We use the idea of [BSS93]. The statement is equivalent to say that there are as many m-Schröder paths of order n with an even number of D steps as there are with an odd number of D steps. For any $\Pi \in \mathcal{S}_n^m$, there must be some first occurrence of either a consecutive NE pair of steps, or a D step. According to which occurs first, either replace the consecutive NE pair by a D, or replace the D with a consecutive NE pair. Notice that this presents a bijection between the two sets of objects we wish to show have the same cardinality. $\square$

In [FH85], there is a refined q-analogue identity,

$$\sum_{k\geq 1} \sum_{w \in CW_{n,k}} q^{majw} = \sum_{k \geq 1} \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{k(k-1)} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}, \tag{5.2.1}$$

where $CW_{n,k}$ is the set of Catalan words consisting of n 0's, n 1's, with k ascents (i.e. $k-1$ descents or the corresponding Dyck path has k peaks). As for the m-version, Cigler proved there are exactly

$$\frac{1}{n}\binom{n}{k}\binom{mn}{k-1}$$

m-Dyck paths with k peaks [Cig87]. In order to generalize the results of [FH85], we prove the following q-identity.

Theorem 5.2.2.

$$\sum_{k \geq d} \begin{bmatrix} k \\ d \end{bmatrix} \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} mn \\ k-1 \end{bmatrix} q^{(k-d)(k-1)} = \frac{1}{[mn-d+1]} \begin{bmatrix} mn+n-d \\ d, n-d, mn-d \end{bmatrix}.$$

Before we proceed to the proof of Theorem 5.2.2, we cite the q-Vandermonde Convolution, which may be obtained as a corollary of the q-binomial theorem.

Lemma 5.2.3. [Hagon] *The q-Vandermonde Convolution.*

$$\sum_{j=0}^{h} q^{(n-j)(h-j)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ h-j \end{bmatrix} = \begin{bmatrix} m+n \\ h \end{bmatrix}.$$

Proof. Proof of Theorem 5.2.2.

$$\begin{aligned}
&\sum_{k \geq d} \begin{bmatrix} k \\ d \end{bmatrix} \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} mn \\ k-1 \end{bmatrix} q^{(k-d)(k-1)} \\
&= \frac{\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]}{[n]} \sum_{k=d}^{n} \begin{bmatrix} n-d \\ n-k \end{bmatrix} \begin{bmatrix} mn \\ k-1 \end{bmatrix} q^{(k-d)(k-1)} \\
&= \frac{\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]}{[n]} \sum_{j=0}^{n-d} \begin{bmatrix} n-d \\ j \end{bmatrix} \begin{bmatrix} mn \\ n-1-j \end{bmatrix} q^{(n-d-j)(n-1-j)} (q\text{-Vandermonde Convolution}) \\
&= \frac{\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]}{[n]} \begin{bmatrix} mn+n-d \\ n-1 \end{bmatrix} \\
&= \frac{1}{[mn-d+1]} \begin{bmatrix} mn+n-d \\ d, n-d, mn-d \end{bmatrix}.
\end{aligned}$$

□

Remark 5.2.1. It is difficult to find a combinatorial interpretation for the left hand side of Theorem 5.2.2. As a matter of fact, the most straightforward generalization of (5.2.1) even fails for the 2-Dyck paths:

$$\sum_{w \in CW_2^2} q^{majw} = 1 + q^2 + q^3 \neq \frac{[1]}{[5]} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 1 + q^2 + q^4.$$

5.3 (q, t)-m-Schröder Statistics and the Shuffle Conjecture

Similar to the manner of [HL], for an m-Dyck path of order n, we may associate it with *m-parking functions* by placing one of the n "cars", denoted by the integers 1 through n, in the square immediately to the right of each N step of D, with the restriction that if car i is placed immediately on top of car j, then $i > j$. Let $\mathbb{P}_n^m$ denote the collection of m-parking functions on n cars.

Definition 5.3.1. Given an m-parking function, its *m-reading word* is obtained by reading from NE to SW line by line, starting from the lines farther from the m-diagonal $x = my$.

Figure 5.3 illustrates an m-parking function with 231 as its m-reading word. The first line we look at is the line connecting cars 2 and 3. We read it from NE to SW so that 2 is before 3. Then the next line is the m-diagonal $x = my$ which contains car 1.

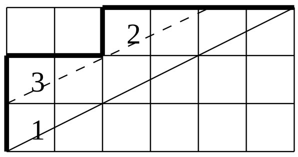


Figure 5.3: An m-parking function whose m-reading word is 231.

Definition 5.3.2. Given an m-parking function, its *natural expansion* is defined as follows: starting from (0, 0), each N step, together with the car to its right, is duplicated m times, the car within the N step is duplicated m times and put one to each of the m N steps duplicated; leave each E step untouched.

Figure 5.4 illustrates the natural expansion of the m-parking function shown in Figure 5.3. Note that the natural expansion of an m-parking function is kind of a "non-strict" standard parking function in the sense that if car placing i immediately on top of car j implies that $i \geq j$ instead of $i > j$.

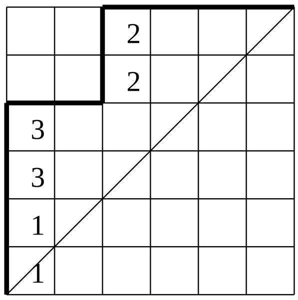


Figure 5.4: The natural expansion of an m-parking function.

Definition 5.3.3. [Sta99, page 482, Ex. 7.93] For two words $u = (u_1, \dots, u_k) \in S_k$ and $v = (v_1, \dots, v_l) \in S(k+1, k+l)$, where $S(m+1, m+l)$ denotes all the permuted words of $\{k+1, \cdots, k+l\}$, $sh(u, v)$ or $sh((u_1, \dots, u_k), (v_1, \dots, v_l))$ is the set of *shuffles* of u and v, i.e., $sh(u, v)$ consists of all permutations $w = (w_1, \dots, w_{k+l}) \in S_{k+l}$ such that both u and v are subsequences of w.

If the m-reading word of an m-parking function P is a shuffle of the two words $(n-d+1, \cdots, n)$ and $(n-d, \cdots, 2, 1)$, the increasing order of $(n-d+1, \cdots, n)$ will imply that any single N segment of P contains at most 1 of $\{n-d+1, \cdots, n\}$. Furthermore, each of $\{n-d+1, \cdots, n\}$ should occupy the top spot of some N segment. Hence if we change these d top N steps all to D steps and remove the cars in the m-parking function, we will get an m-Schröder path with d diagonal steps. Conversely, given a path $\Pi \in \mathcal{S}^m_{n,d}$, we may change its d diagonal steps to d NE pairs; after that place cars $\{n-d+1, \cdots, n\}$ to the right of the d new N steps, and place cars $\{n-d, \cdots, 2, 1\}$ to the right of the other $n-d$ D steps in the uniquely right order so that the m-reading word of the m-parking function formed is a shuffle of the two words $(n-d+1, \cdots, n)$ and $(n-d, \cdots, 2, 1)$. In this way every m- Schröder corresponds to an m-parking function of the particular type. Because it is easier to manipulate when there are no D steps, we define the m-Schröder polynomial in the following way.

Definition 5.3.4. The m-Schröder polynomial is defined as

$$S_{n,d}^m(q,t) = \sum_{\substack{\Pi:\ \Pi \in \mathbb{P}_n^m \text{ and the } m\text{-reading word of } \Pi \\ \in sh((n-d+1,\cdots,n),(n-d,\cdots,1))}} q^{dinv_m(\Pi)} t^{area(\Pi)},$$

where $dinv_m(\Pi) = dinv(\hat{\Pi})$, $\hat{\Pi}$ is the natural expansion of Π, and $dinv$ is the obvious generalization of the statistic on parking functions introduced in [HL].

The m-Shuffle Conjecture is due to Haglund, Haiman, Loehr, Remmel and Ulyanov.

Conjecture 5.3.1. [HHL$^+$]

$$S_{n,d}^m(q,t) =< \nabla^m e_n, e_{n-d} h_d >,$$

where ∇ *is a linear operator defined in terms of the modified Macdonald polynomials (for details see* [HHL$^+$]*).*

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