ZEROS, CRITICAL POINTS, AND COEFFICIENTS OF RANDOM FUNCTIONS

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A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2014

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Acknowledgments

It has been a long journey - almost half a decade - and this journey would not have been possible without some truly exceptional and amazing people!

I would first like to thank my advisor, Robin Pemantle, for guiding me through my years at Penn. He's been incredibly helpful and encouraging and I've learnt a lot from him. Apart from being a prolific mathematician, he's also an amazing human being, and it has been a great privilege to work with him.

I was lucky to have Andreea Nicoara, Tony Pantev, Stephen Shatz, J. Michael Steele (Statistics) instruct me in courses that have formed strong foundations. I want to thank Philip Gressman and Andreaa Nicoara for stimulating discussions and helpful suggestions that changed the way I was thinking about my very first research problem and led to my first ever Eureka! moment. Also, I'm grateful to Andreea Nicoara and J. Michael Steele for serving on my orals' committee, and Jerry Kazdan and J. Michael Steele for being a part of my defense committee. In my second year at Penn, I was a teaching assistant to Jerry Kazdan and Nakia Rimmer and learnt a lot from them about teaching mathematics.

An "Acknowledgements" section in a dissertation that comes out of the Penn mathematics department would be incomplete (and dishonest!) without the mention of four women in the department office - Janet Burns, Monica Pallanti, Paula Scarborough and Robin Toney. We are all extremely lucky to have them! A special mention to Janet Burns, who's retiring at the end of this year - I find it very hard to imagine our mathematics department without her, and am glad to have had her throughout my time here.

Starting from my very first year here, I've had many wonderful colleagues. My time at "the zoo" (i.e. first years' office) with Deborah Crook, Tyler Kelly, David Lonoff, Haggai Nuchi, Pooya Ronagh, and others, was very enriching, both mathematically and personally, and I am happy those friendships have continued. Special thanks to Deborah Crook for being my office mate for the subsequent years and tolerating the sight of my crazy desk. I've found many friends here, including Brett and Fatema Frankel, Torin Greenwood, Ricardo Mendes, Julius Poh, Radmila Sazdanovic, Elaine So, and so many more! Finally, I don't know where I would have been had I not received the friendship and support of two amazing people - Jonathan Kariv and Sashka Kjuchukova.

One of the most defining times of my life has been the time I spent as an undergraduate student in St. Xavier's College, Kolkata. The friendships forged then remain just as strong today and I'm lucky to have so many people who, despite me having been half way across the earth for the past five years, still share my tears and joy. Like any Indian, I have a huge extended family comprising numerous cousins, aunts, uncles and recently, nieces and nephews - they have always showered enormous love on me. I would especially like to thank all of my grandparents, none of whom are alive today - I know this would have meant a lot to them. My parents, Bimala and Shiva Subramanian, have worked hard and made many sacrifices to provide me a comfortable childhood and make sure I got the best education I needed, and I'm extremely grateful to them for being with me at every step. Finally, I can't thank my husband Doby Rahnev enough for being such an amazingly loving and supportive life partner!

ABSTRACT

ZEROS, CRITICAL POINTS, AND COEFFICIENTS OF RANDOM FUNCTIONS

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Traditional approaches to the study of random polynomials and random analytic functions have focussed on answering questions regarding the behavior and/or location of zeros of these functions, where the randomness in these functions arises from the choice of coefficients. In this thesis, we shall flip this model - we consider random polynomials and random analytic functions where the source of randomness is in the choice of zeros. While first chapter is devoted to an introduction into the field, in the next two chapters, we consider random polynomials whose zeros are chosen IID using some distribution. The second chapter answers questions regarding the asymptotic distribution of the critical points of a random polynomial whose zeros are IID on a circle on the complex plane. The fourth chapter describes the asymptotic behavior of the coefficients of a random polynomial whose zeros are IID Rademacher random variables. In the third chapter, we consider a random entire function that vanishes at a Poisson point process of intensity 1 on $\mathbb{R}$. We give results on the asymptotic behavior of the coefficients as well as the resulting zero set on repeatedly differentiating this function.

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Chapter 1

Introduction

Interest in the zeros of a function can be traced back almost to the time in mathematical history when the concept of "function" came into being. Be it the Riemann Zeta function, the study of stable polynomials or the entire field of algebraic geometry, the zeros of a function form the center of the universe of many areas in mathematics.

Numerous questions have been asked and answered about the relation between zeros and critical points of a function. One of the oldest among these is the Rolle's theorem - a theorem that describes location of critical points, provided the function has all real zeros. For polynomials with complex coefficients, the analogous result, called Gauss-Lucas theorem, tells us that the critical points are all contained in the complex hull of the zeros. Despite a very natural interest in the idea of deriving information about critical points from the zeros of a function, generalizations of Rolle's and Gauss-Lucas theorems have been few. This fact is demonstrated by two long standing conjectures by Blagovest Sendov and Steve Smale, both of which have been proved and studied extensively for special cases, but in their full generality, still remain open.

Sendov formed his famous conjecture in the 1950's, which states that if the roots $z_1, z_2, ..., z_n$ of a convex polynomial all lie inside the closed unit disc, then for each root of the polynomial, the closed unit disc centered at the root must contain at least one critical point. In some ways, this problem seems only a small step away from Gauss-Lucas theorem, but given that it has been open for six decades, it clearly is not so. Smale's conjecture states that, if f is a polynomial of degree n with at least one root 0 and $f'(0) \neq 0$, then,

$$\min\left\{\left|\frac{f(\xi)}{\xi f'(0)}\right| : f'(\xi) = 0\right\} \leq K,$$

where $K = 1$ or $\frac{n-1}{n}$.

Mentioned above are only the deterministic results about the function characteristics. An interesting route in thinking about zeros and critical points is by bringing randomness and probabilistic results into the picture. Recently, the area of random polynomials and random analytic functions has been a very active one. In the most classical problems, a "random polynomial" is formed by fixing its degree and having its coefficients be identically and independently distributed (IID) random variables with some desired law. Questions regarding the properties of the zeros of these functions are then explored. For example, Mark Kac [8] gave an explicit formula for the expected number of zeros of a random polynomial in any measurable subset of the real numbers, where coefficients of the polynomial are IID standard normal variates. Jon Ben Hough, Manjunath Krishnapur, Yuval Peres and Bálint Virág, in their book [7], study the probabilistic properties of zeros of a complex analytic function whose coefficients are IID standard complex Gaussian.

A natural variation from the classical random function questions is to flip the model. That is, if we start with the zeros of a polynomial or an analytic function to be IID random variables, then form these functions in a canonical way, how much can be said about properties of this random function, including its coefficients and critical points?

Very recently, this approach was explored by Robin Pemantle and Igor Rivin in [14], where it was conjectured that if f is a polynomial of degree n, whose n roots are chosen IID using a law μ on the complex plane, then the empirical distribution of the roots of f' converge weakly to μ as $n \to \infty$. In Chapter 2, we shall prove that this is indeed true in the case where the zeros are chosen IID from a distribution that is supported on a circle in the complex plane.

In Chapter 3, we shall consider a random polynomial of degree N whose zeros are IID ± 1 with probability $\frac{1}{2}$ each. We shall show that the coefficients of this random polynomial exhibit a sinusoidal behavior asymptotically.

In the last chapter of this thesis, Chapter 4, we shall consider a random entire function, f, that vanishes exactly at the points of a Poisson process of intensity 1 on the real line. We shall prove that ratios of alternating coefficients in the power series expansion of this function display a curious convergence behavior. This fact, along with the fact that the critical points of the function is translation invariant over the choice or origin, will lead us to show that the resulting zero set on repeatedly differentiating the function f converges to a uniform random translate of the set of integers.

Chapter 2

Critical points of random polynomials

2.1 Notations and Background

Say, $Z_1, Z_2, ...$ is a sequence of points chosen i.i.d. with respect to some distribution μ on the unit circle. Write, $Z_k = \exp(2\pi i\theta_k)$, so that $\{\theta_k\}$ is a collection of IID random variables whose common law is supported on $[0, 1]$, which we denote by ν.

Let

$$p_n(z) = (z - Z_1)(z - Z_2)...(z - Z_n),$$

and $y_1^{(n)}, y_2^{(n)}, ..., y_{n-1}^{(n)}$ be the roots of $p_n'(z)$.

For $k \geq 1$, let $c_k = \mathbb{E}(Z^k)$, where $Z \sim \mu$. Denote by $\mathbb{Z}(f)$ the empirical distribution of the roots of a random polynomial f. That is, if f has roots $X_1, X_2, ..., X_m$, then $\mathbb{Z}(f) = \frac{1}{m}\sum_{j=1}^{m} \delta_{X_j}$. We shall write $\mathbb{D}$ for the open unit disc, and $\mathcal{C}$ for the unit circle.

In their paper, [14], the authors conjectured that, for any distribution μ on the closed unit disc, $\mathbb{Z}(p_n')$ converges weakly to μ. That paper also proves the following proposition.

Proposition 2.1.1. *Let μ be the uniform measure on $\mathcal{C}$. Then $\mathbb{Z}(p_n')$ converges to $\mathcal{C}$ in probability, that is, $P(\mathbb{Z}(S) \geq \epsilon) \to 0)$ for any $\epsilon > 0$ and any closed set $S \subset \mathbb{D}$, disjoint from $\mathcal{C}$.* □

In this note, we shall generalize this to prove that

Lemma 2.1.2. *For any distribution μ on $\mathcal{C}$, $\mathbb{Z}(p_n')$ converges to $\mathcal{C}$ in probability. In fact, if μ is not uniform on $\mathcal{C}$, the convergence is almost everywhere.*

The above leads us to prove our main result, which is a special case of the aforementioned conjecture in [14]:

Theorem 2.1.3. *For any distribution μ on $\mathcal{C}$, $\mathbb{Z}(p_n')$ converges weakly to μ on $\mathcal{C}$.*

The proof, as shall be seen in forthcoming sections, can be divided in to two parts, the latter following a pattern similar to the proof of Weyl's equidistribution criterion (see, for example [1]). The former requires the following theorem (proved both in [11] and in [2]) regarding a companion matrix of the critical points.

Proposition 2.1.4. *If $z_1, z_2, ..., z_n \in \mathbb{C}$, and $y_1, y_2, ..., y_{n-1}$ are the critical points of the polynomial $p_n(z) = (z - z_1)(z - z_2)...(z - z_n)$, then, the matrix*

$$D\left(I - \frac{J}{n}\right) + \frac{z_n}{n}J \tag{2.1.1}$$

has $y_1, y_2, ..., y_{n-1}$ as its eigenvalues, where $D = diag(z_1, z_2, ..., z_{n-1})$, I is the identity matrix of order $n-1$ and J is the $(n-1) \times (n-1)$ matrix of all entries 1. □

2.2 Proofs of Lemma 2.1.2 and Theorem 2.1.3

We first begin by proving a small lemma.

Lemma 2.2.1. *Let μ be a distribution on the unit circle $\mathcal{C}$ with $c_k = \mathbb{E}(Z^k)$, where $Z \sim \mu$. Then $c_k = 0$ for all $k \geq 1$ if and only if μ is uniform on $\mathcal{C}$.*

Proof. Clearly if μ is uniform on $\mathcal{C}$ then $c_k = 0$ for all $k \geq 1$. Now say μ is not uniform on the circle but we still have $c_k = 0$ for all $k \geq 1$. Then the law ν is not uniform on $[0, 1]$. Now, if $Z_1, Z_2, ...$ are points on $\mathcal{C}$, chosen i.i.d. using μ, and if we write $Z_j = \exp(2\pi i\theta_j), j = 1, 2, ...,$ then $\theta_1, \theta_2, ...$ are points in $[0, 1]$ that are i.i.d. ν.

By the Strong Law of Large Numbers, for all $k \geq 1$,

$$\frac{Z_1^k + Z_2^k + ...Z_n^k}{n} \xrightarrow{a.s.} 0,$$

and so by Weyl's criterion, for any $0 \leq a < b \leq 1$,

$$\frac{\sum_{j=1}^{n} \mathbb{1}_{\{\theta_j \in [a,b]\}}}{n} \xrightarrow{a.s.} b - a.$$

But $\mathbb{1}_{\{\theta_j \in [a,b]\}}, j = 1, 2, ...$ are i.i.d. random variables taking values 0 or 1 with expectation $\nu([a, b])$. Therefore,

$$\frac{\sum_{j=1}^{n} \mathbb{1}_{\{\theta_j \in [a,b]\}}}{n} \xrightarrow{a.s.} \nu([a, b]).$$

Since ν is not uniform on $[0, 1]$, we have arrived at a contradiction. So, there must exist at least one non-zero c_k. □

We proceed to use this fact for the proof of Lemma 2.1.2.

Proof of Lemma 2.1.2. Assume μ is not the uniform distribution on the circle (as the uniform case has been taken care of in [14]). Then, as mentioned above, there is at least one non-zero c_k. Thus the power series function $f(z) = \sum_{k=0}^{\infty} \bar{c}_{k+1} z^k$ exists at every point $z \in \mathbb{D}$, is analytic there (since $|c_k| < 1, \forall k$), and so has only finitely many zeros inside any r-ball, where $r < 1$.

Define

$$V_n(z) = \frac{p_n'(z)}{np_n(z)} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - Z_j}.$$

V_n has $n-1$ zeros, which are exactly the zeros of $p_n'(z)$, and n poles, which are exactly the zeros of $p_n(z)$. Thus $V_n(z)$ is analytic inside $\mathbb{D}$. We shall show that as $n \to \infty$, V_n converges inside the disc to $-f$, uniformly over compact sets. To see this, note that for $z \in \mathbb{D}$,

$$V_n(z) = \frac{1}{n}\sum_{j=1}^{n} \frac{-1/Z_j}{1 - z/Z_j} = -\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{\infty} \bar{Z}_j^{k+1} z^k = -\sum_{k=0}^{\infty} \bar{a}_n^{k+1} z^k,$$

where, we write a_n^{k+1} for the kth power sum average $\frac{Z_1^k + Z_2^k + \ldots + Z_n^k}{n}$. By Strong Law of Large Numbers, $a_n^k \xrightarrow{a.s.} c_k$ for all $k \geq 1$.

Let $0 < r < 1$. Given any $\delta > 0, \exists K \geq 1$ such that

$$\sum_{k=K}^{\infty} r^k = \frac{r^k}{1-r} < \frac{\delta}{4}.$$

Corresponding to the chosen K, there exists an $N \geq 1$ such that,

$$|a_n^k - c_k| < \frac{\delta(1-r)}{2},$$

$\forall n \geq N$ and $\forall k = 1, 2, ..., K-1$. Therefore, $\forall n \geq N$ and all $z \in B_r(0)$,

$$\begin{aligned} |V_n(z) + f(z)| &\leq \sum_{k=0}^{K-1} |a_n^k - c_k| r^k + \sum_{k=K+1}^{\infty} |a_n^k - c_k| r^k \\ &\leq \frac{\delta(1-r)}{2} \cdot (1 + r + r^2 + ... + r^{K-1}) + 2 \cdot \frac{\delta}{4} < \delta, \end{aligned}$$

which proves uniform convergence of V_n to $-f$ over compact sets.

Using Hurwitz's theorem (see [3]), given any $0 < r < 1$, there exists an $M \geq 1$ for which V_n and f have the same number of zeros inside $B_r(0)$ for all $n \geq M$. That is, p_n' and f shall have the same number of zeros inside $B_r(0)$ for all $n \geq M$. But, as discussed above, f has only finitely many zeros inside $B_r(0)$. Thus $\mathbb{Z}(p_n')$ converges to the unit circle almost surely. $\square$

Our main result, Theorem 2.1.3, will be a consequence of the following proposition.

Proposition 2.2.2. *Given any sequence of points $z_1, z_2, ...$ with $|z_n| \leq M$ for all n, and $\frac{z_1^k+z_2^k+...z_n^k}{n} \to c_k$ as $n \to \infty$, $\forall k \geq 1$, the critical points $y_1^{(n)}, y_2^{(n)}, ..., y_{n-1}^{(n)}$ of $p_n(z) = (z-z_1)(z-z_2)...(z-z_n)$ also satisfy*

$$\frac{(y_1^{(n)})^k + (y_2^{(n)})^k + ... + (y_{n-1}^{(n)})^k}{n-1} \longrightarrow c_k \text{ as } n \to \infty,$$

$\forall k \geq 1$.

Proof. Note that, it is easy to see that this theorem holds true for $k = 1$, because the average of the critical points is exactly equal to the average of the roots (by comparing the coefficients of z^{n-1} in $p_n(z)$ with z^{n-2} of $p_n'(z)$). To prove the result for general k, we use Proposition 2.1.4 to see that for $k \geq 2$, $(y_1^{(n)})^k, (y_2^{(n)})^k, ..., (y_{n-1}^{(n)})^k$ are the eigenvalues of $[D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J]^k$, and so,

$$(y_1^{(n)})^k + (y_2^{(n)})^k + ... + (y_{n-1}^{(n)})^k = Tr\left[D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J\right]^k.$$

Note that the expansion of $[D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J]^k$ is the sum of all terms such as

$$D^{l_1}\left(-\frac{DJ}{n}\right)^{l_2}\left(\frac{z_n}{n}J\right)^{l_3} D^{l_4}\left(-\frac{DJ}{n}\right)^{l_5}\left(\frac{z_n}{n}J\right)^{l_6} ...D^{l_{3k-2}}\left(-\frac{DJ}{n}\right)^{l_{3k-1}}\left(\frac{z_n}{n}J\right)^{l_{3k}} \tag{2.2.1}$$

where the exponents $l_1, l_2, ..., l_{3k}$ are non-zero integers, with $l_{3j-2}+l_{3j-1}+l_{3j} = 1$ for all $j = 1, 2, .., k$. Clearly the number of such terms is 3^k, which does not depend on n, and so, if we find that the trace of the matrix in the expression (2.2.1) converges as $n \to \infty$ to $a_{l_1,l_2,...l_{3k}}$, then the trace of $[D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J]^k$ converges to $\sum a_{l_1,l_2,...,l_{3k}}$.

Henceforth, we fix $l_1, l_2, ...l_{3k}$. Now, note that $J^m = (n-1)^{m-1}J^{m-1}$ for any $m \geq 1$, and

$$(D^pJ)(D^qJ) = \left(\sum_{i=1}^{n-1} z_i^q\right)(D^pJ),$$

for any $p, q \geq 0$.

The above tells us that there exists $p, q, s_0, s_1, s_2, ..., s_{k-1} \geq 0$ such that, term (2.2.1) is of the form

$$(-1)^p \cdot z_n^q \cdot \left(\frac{n-1}{n}\right)^{s_0} \cdot \left(\frac{\sum_{i=1}^{n-1} z_i}{n}\right)^{s_1} \cdot \left(\frac{\sum_{i=1}^{n-1} z_i^2}{n}\right)^{s_2} \cdot ... \cdot \left(\frac{\sum_{i=1}^{n-1} z_i^{k-1}}{n}\right)^{s_{k-1}} \cdot M, \tag{2.2.2}$$

where the numbers $p, q, s_0, s_1, ..., s_{k-1}$ are determined solely by the l_i's (and so, are independent of n).

Also, M can only be one of the following terms: D^k or $\frac{D^mJ}{n}$ or $\frac{D^{m_1}J}{n}D^{m_2}$ for some $m, m_1, m_2 \geq 0$, which are fixed, $\leq k$, and dependent only on the l_i's. Furthermore, the scalar coefficient in (2.2.2) is always $O(1)$.

Observe that, if $M = D^k$, then the scalar coefficient in (2.2.2) is equal to 1 and $\frac{Tr(M)}{n} \to c_k$. On the other hand, if $M = \frac{D^m J}{n}$, then

$$Tr(M) = \frac{z_1^m + z_2^m + ... + z_{n-1}^m}{n} = o(n),$$

and if $M = \frac{D^{m_1} J}{n} D^{m_2}$,

$$\begin{aligned} Tr(M) &= Tr\left(D^{m_1+m_2}\frac{J}{n}\right) \\ &= \frac{z_1^{m_1+m_2} + z_2^{m_1+m_2} + ... + z_{n-1}^{m_1+m_2}}{n} = o(n). \end{aligned}$$

Thus,

$$\frac{Tr\left[D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J\right]^k}{n} \longrightarrow c_k \text{ as } n \to \infty.$$

□

We now have all the tools required to prove our main result, namely Theorem 2.1.3.

Proof of Theorem 2.1.3. Say we write,

$$y_j^{(n)} = r_j^{(n)} \exp(2\pi i \phi_j^{(n)}), j = 1, 2, ..., n-1.$$

The proof will consist of three major segments. Our first task is to prove that

$$\frac{1}{n-1}\sum_{j=1}^{n-1}(r_j^{(n)})^k \xrightarrow{P} 1.$$

In fact, unless μ is uniform on the circle, we will show that

$$\frac{1}{n-1}\sum_{j=1}^{n-1}(r_j^{(n)})^k \xrightarrow{a.s.} 1.$$

Next, we shall use the above information to show that

$$\frac{\exp(2k\pi i\phi_1^{(n)}) + \exp(2k\pi i\phi_2^{(n)}) + ... + \exp(2k\pi i\phi_{n-1}^{(n)})}{n-1} \xrightarrow{P} c_k.$$

(Again, the convergence is almost sure, unless μ is uniform on $\mathcal{C}$.)

Finally, using arguments analogous to those in the proof of Weyl's equidistribution criterion, we shall arrive at our final result.

Assume, initially, that μ is not the uniform law on $\mathcal{C}$. For the first task as noted above, observe that, by Lemma 2.1.2, given any $\epsilon > 0$,

$$\frac{1}{n-1}\sum_{j=1}^{n-1}\mathbb{1}_{\{r_j^{(n)}\in[1-\epsilon,1]\}} \xrightarrow{a.s.} 1.$$

Now, for any fixed positive integer k, $(1-\epsilon)^k\mathbb{1}_{\{r_j^{(n)}\in[1-\epsilon,1]\}} \leq (r_j^{(n)})^k \leq 1$, and so

$$(1-\epsilon)^k \cdot \frac{1}{n-1}\sum_{j=1}^{n-1}\mathbb{1}_{\{r_j^{(n)}\in[1-\epsilon,1]\}} \leq \frac{1}{n-1}\sum_{j=1}^{n-1}(r_j^{(n)})^k \leq 1. \tag{2.2.3}$$

Clearly then, a simple squeeze theorem argument gives us

$$\frac{1}{n-1}\sum_{j=1}^{n-1}(r_j^{(n)})^k \xrightarrow{a.s.} 1. \tag{2.2.4}$$

Now, from Proposition 2.2.2, for any positive integer k,

$$\frac{(y_1^{(n)})^k + (y_2^{(n)})^k + ... + (y_{n-1}^{(n)})^k}{n-1} \xrightarrow{a.s.} c_k,$$

$$\implies \frac{(r_1^{(n)})^k\exp(2k\pi i\phi_1^{(n)}) + (r_2^{(n)})^k\exp(2k\pi i\phi_2^{(n)}) + ... + (r_{n-1}^{(n)})^k\exp(2k\pi i\phi_{n-1}^{(n)})}{n-1} \xrightarrow{a.s.} c_k.$$

Note that (2.2.4) gives us that

$$\left|\frac{1}{n-1}\sum_{j=1}^{n-1}(1-(r_j^{(n)})^k)\exp(2k\pi i\phi_j^{(n)})\right| \le \frac{1}{n-1}\sum_{j=1}^{n-1}(1-(r_j^{(n)})^k) \xrightarrow{a.s.} 0,$$

and so,

$$\frac{\exp(2k\pi i\phi_1^{(n)})+\exp(2k\pi i\phi_2^{(n)})+...+\exp(2k\pi i\phi_{n-1}^{(n)})}{n-1} \xrightarrow{a.s.} c_k. \tag{2.2.5}$$

Now, for the final stage of our proof,

$$c_k = \mathbb{E}(Z^k), \text{ where, } Z \sim \mu.$$
$$\implies c_k = \mathbb{E}(\exp(2k\pi i\Theta)) = \mathbb{E}(\cos(2k\pi\Theta)) + i\mathbb{E}(\sin(2k\pi\Theta)), \text{ where, } \Theta \sim \nu.$$

So, (2.2.5) gives,

$$\frac{\cos(2k\pi\phi_1^{(n)})+\cos(2k\pi\phi_2^{(n)})+...+\cos(2k\pi\phi_{n-1}^{(n)})}{n} \xrightarrow{a.s.} \mathbb{E}(\cos(2k\pi\Theta)),$$
$$\frac{\sin(2k\pi\phi_1^{(n)})+\sin(2k\pi\phi_2^{(n)})+...+\sin(2k\pi\phi_{n-1}^{(n)})}{n} \xrightarrow{a.s.} \mathbb{E}(\sin(2k\pi\Theta)).$$

Then, for any trigonometric polynomial $q(x)$,

$$\frac{\sum_{j=1}^{n-1} q(\phi_j^{(n)})}{n} \xrightarrow{a.s.} \mathbb{E}(q(\Theta)). \tag{2.2.6}$$

Let f be a continuous real-valued function on [0, 1] and fix $\epsilon > 0$. By Stone-Weierstrass theorem ([16]), there exists a trigonometric polynomial q such that $|f - q| < \epsilon$. So,

$$\left|\frac{\sum_{j=1}^{n-1} f(\phi_j^{(n)})}{n} - \mathbb{E}(f(\Theta))\right| \le \left|\frac{\sum_{j=1}^{n-1} f(\phi_j^{(n)})}{n} - \frac{\sum_{j=1}^{n-1} q(\phi_j^{(n)})}{n}\right|$$
$$+ \left|\frac{\sum_{j=1}^{n-1} q(\phi_j^{(n)})}{n} - \mathbb{E}(q(\Theta))\right| + \mathbb{E}|q(\Theta) - f(\Theta)|.$$

The first and third terms on the right hand side are each $< \epsilon$ while the second term goes to 0 almost surely, by (2.2.6). Hence for any f continuous on [0, 1],

$$\frac{\sum_{j=1}^{n-1} f(\phi_j^{(n)})}{n} \xrightarrow{a.s.} \mathbb{E}(f(\Theta)), \tag{2.2.7}$$

and this holds for complex-valued continuous functions as well (which is easily seen by comparing the real and imaginary parts). Thus, the joint empirical distribution of $\phi_j^{(n)}, j = 1, 2, ..., n-1$, converges weakly to ν, which means that the joint empirical distribution of $\exp(2\pi i \phi_j^{(n)}), j = 1, 2, ..., n-1$, converges weakly to μ. This, along with Lemma 2.1.2, gives us the desired result for μ not uniform on $\mathcal{C}$.

Now suppose μ is the uniform law on the unit circle. Then,

$$\frac{1}{n-1} \sum_{j=1}^{n-1} \mathbb{1}_{\{r_j^{(n)} \in [1-\epsilon, 1]\}} \xrightarrow{P} 1,$$

and as before, using (2.2.3) we get,

$$\frac{1}{n-1} \sum_{j=1}^{n-1} (r_j^{(n)})^k \xrightarrow{P} 1,$$

for any positive integer k.

Note that the above is a slightly weaker version of (2.2.4), since the convergence is now in probability, and not almost sure.

For the rest of the proof, we can follow the same arguments as in the non-uniform case, except that the almost sure convergence in each of the statements will be replaced by convergence in probability. Thus we shall arrive at

$$\frac{\sum_{j=1}^{n-1} f(\phi_j^{(n)})}{n} \xrightarrow{P} \mathbb{E}(f(\Theta)),$$

for any continuous function $f : [0,1] \to \mathbb{C}$. Then, as before, the joint empirical distribution of $\phi_j^{(n)}, j = 1, 2, ..., n-1$, converges weakly to ν (which is the uniform law on $[0,1]$), and so, the joint empirical distribution of $\exp(2\pi i \phi_j^{(n)}), j = 1, 2, ..., n-1$, converges weakly to uniform on $\mathcal{C}$. Lemma 2.1.2 then gives us the desired result. $\square$

Chapter 3

Rademacher zeros

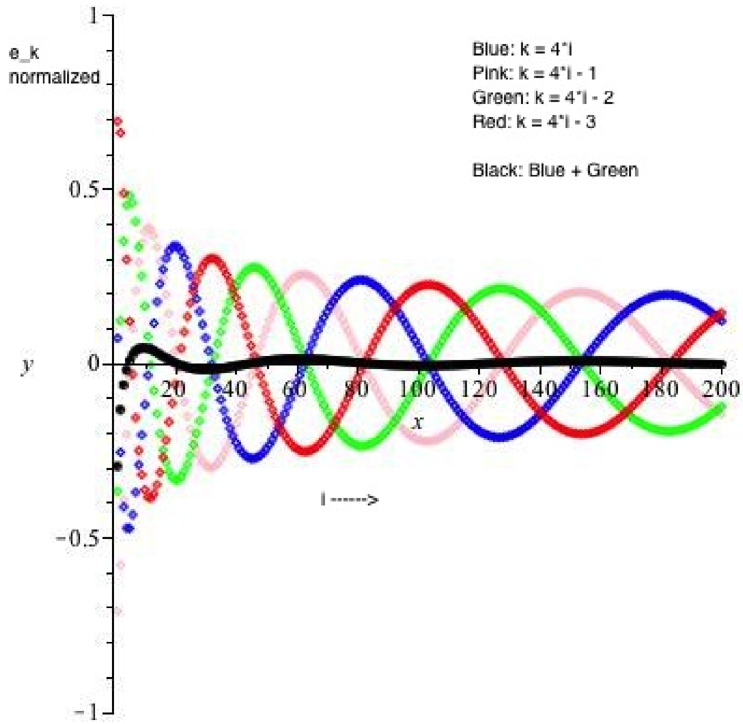
3.1 Introduction and statement of the main result

Let $X_1, X_2, X_3, \ldots$ be identically and independently distributed Rademacher random variables (that is, $\mathbb{P}(X_1 = -1) = \frac{1}{2} = \mathbb{P}(X_1 = +1)$). A random polynomial that takes $X_1, X_2, \ldots, X_N$ to be its zeros is

$$f_N(z) := \prod_{j=1}^{N} \left(1 - \frac{z}{X_j}\right), z \in \mathbb{C}.$$

Note that, the coefficient of z^k in f_N is just $(-1)^k e_{k,N}$, where, $e_{k,N}$ is the kth elementary symmetric function of $X_1, X_2, \ldots, X_N$.

The main theorem in this chapter explores the behavior of $e_{k,N}$ asymptotically as we let $N \to \infty, k \to \infty$, and $k^2/N \to 0$. As demonstrated in the picture below,



this theorem explains a sinusoidal behavior displayed by the elementary symmetric functions of these Rademacher random variables, when suitably normalized.

Theorem 3.1.1. *Let* $X_1, X_2, ...$ *be i.i.d. random variables with each* X_i *taking value* -1 *with probability* $1/2$ *and* 1 *with probability* $1/2$. *Let* $e_{k,N}$ *be the* k*th elementary symmetric function of* $X_1, X_2, ...X_N$, $\alpha := \frac{X_1+X_2+...+X_N}{\sqrt{N}}$ *and* $\Theta := (2\pi)^{-1}\arctan\left(\frac{e_{k,N}}{\sqrt{k/N}e_{k+1}}\right)$. *Then, if* d_H *is the Hausdorff distance between*

$$\mathcal{Y} := \left\{\left(t, -\sin\left(\frac{t\alpha}{4} - 2\pi\Theta\right)\right) : 0 \leq t \leq M\right\}$$

and

$$\Xi := \left\{ \left(t, \frac{e_{k+t\sqrt{k}}}{\sqrt{e_{k,N}^2 + \frac{k}{N} e_{k+1}^2}} \cdot \left(\frac{k}{N}\right)^{t\sqrt{k}/2} \right) : t = 0, \frac{4}{\sqrt{k}}, \frac{8}{\sqrt{k}}, ..., \lfloor M \rfloor_0 \right\}$$

where $\lfloor M \rfloor_0$ *is the highest value that is* $\leq M$ *and equals a multiple of* $4/\sqrt{k}$*, and* M *is any positive integer, then*

$$d_H \xrightarrow{P} 0$$

as $k \to \infty, N \to \infty$ *under the constraint that* $N/k^2 \to \infty$*.*

This theorem is proven by using a method of "steepest descent". The idea is to first express the elementary symmetric polynomials as an integral, as described here. Using Cauchy's integral formula, we have

$$f_N^{(k)}(0) = \frac{k!}{2\pi i} \int_\Gamma \frac{f_N(z)}{z^{k+1}} dz,$$

where Γ is a simple continuous loop around the origin. Note that, $f_N^{(k)}(0) = (-1)^k k! e_{k,N}$. So,

$$e_{k,N} = \frac{(-1)^k}{2\pi i} \int_\Gamma \frac{f_N(z)}{z^{k+1}} dz.$$

We then try to choose the loop Γ such that it is has only two points where the integrand is not negligible, thus allowing us to approximate the above integral with respect to the values the integrand takes at the said points. This gives an expression for the $e_{k,N}$'s that is easy to analyze.

In the next section, we define a function $\phi_{k,N}$ that is analytic away from the real line, and study its higher derivatives as well as its critical points. As demonstrated in the subsequent sections, this exercise is crucial to obtaining a loop of steepest descent, and hence the formula for $e_{k,N}$'s discussed above.

3.2 The function $\phi_{k,N}$ and its derivatives

Define

$$\phi_{k,N}(z) = \log\left(\frac{f_N(z)}{z^k}\right).$$

Clearly, $\phi_{k,N}$ will be holomorphic in regions that are away from the real axis. Let $\sigma_{k,N}$ be a critical point of $\phi_{k,N}$ (i.e. $\sigma_{k,N}$ is a zero of $\phi'_{k,N}$). We shall show below that its conjugate, $\overline{\sigma_{k,N}}$, is also a critical point, and shall use a convention by which, $\sigma_{k,N}$ refers to the critical point on the positive side of the imaginary axis.

Let us write, $b := X_1 + X_2 + ... + X_N$. Then, we can write f_N as

$$f_N(z) = (1-z)^n(1+z)^{n+b},$$

where $2n + b = N$. Note that, $b/N \xrightarrow{a.s.} 0$, and $b/\sqrt{N} \xrightarrow{d} N(0,1)$.

Thus, we can write,

$$\phi_{k,N}(z) = (n+b)\log(1-z) + n\log(1+z) - k\log z,$$

$$\phi'_{k,N}(z) = -\frac{n+b}{1-z} + \frac{n}{1+z} - \frac{k}{z}, \text{ and,}$$

$$\phi^{(r)}_{k,N}(z) = (r-1)!\left[-\frac{n+b}{(1-z)^r} - (-1)^r\frac{n}{(1+z)^r} + (-1)^r\frac{k}{z^r}\right], r \geq 1.$$

3.2.1 The critical points of $\phi_{k,N}$

Lemma 3.2.1. *For N sufficiently large, $\phi_{k,N}(z)$ has two roots, that are complex conjugates. Moreover, if we name the roots $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$, then as $N \to \infty, k \to \infty, N/k \to \infty$, we get,*

$$\frac{\sigma_{k,N}}{i\sqrt{\frac{k}{N}}} \xrightarrow{P} 1, \text{ and,}$$

$$\frac{\overline{\sigma_{k,N}}}{-i\sqrt{\frac{k}{N}}} \xrightarrow{P} 1.$$

Proof. The zeros of $\phi_{k,N}(z)$ are given by

$$\begin{aligned}
\phi'_{k,N}(z) &= 0,\\
-\frac{n+b}{1-z} + \frac{n}{1+z} - \frac{k}{z} &= 0,\\
-(n+b)z(1+z) + n(1-z)z - k(1-z)(1+z) &= 0,\\
(2n+b-k)z^2 + bz + k &= 0,\\
(N-k)z^2 + bz + k &= 0.
\end{aligned}$$

Therefore, there are two zeros, namely,

$$\frac{-b \pm \sqrt{b^2 - 4k(N-k)}}{2(N-k)} = \frac{-b \pm \sqrt{k(N-k)}\sqrt{\frac{b^2}{k(N-k)} - 4}}{2(N-k)}.$$

Since $b^2/(N-k) \xrightarrow{d} N(0,1)$, we have that

$$\frac{b^2}{k(N-k)} \xrightarrow{P} 0.$$

Thus, with probability $\longrightarrow 1$,

$$\frac{b^2}{k(N-k)} - 4 \leq 0,$$

and so, the roots of $\phi'_{k,N}(z)$ are complex conjugates.

Next, write,

$$\begin{aligned}
\sigma_{k,N} &= \frac{-b + i\sqrt{4k(N-k) - b^2}}{2(N-k)}\\
&= \sqrt{\frac{k}{N-k}}\left[\frac{-b}{2\sqrt{k(N-k)}} + i\sqrt{1 - \frac{b^2}{4k(N-k)}}\right]
\end{aligned}$$

Thus as $N \to \infty, k \to \infty, N/k \to \infty$, we have,

$$\frac{\sigma_{k,N}}{i\sqrt{\frac{k}{N}}} \xrightarrow{P} 1.$$

Similarly,

$$\frac{\overline{\sigma_{k,N}}}{-i\sqrt{\frac{k}{N}}} \xrightarrow{P} 1.$$

□

3.2.2 Higher derivatives of $\phi_{k,N}$ at $\sigma_{k,N}$

Lemma 3.2.2. *For $r \geq 3$,*

$$\frac{\phi^{r}_{k,N}(\sigma_{k,N})}{(-1)^r(r-1)!N^{\frac{r}{2}}k^{1-\frac{r}{2}}} \xrightarrow{P} 1,$$

and consequently,

$$\frac{\sigma^3_{k,N}\phi^{(3)}_{k,N}(\sigma_{k,N})}{2ik} \xrightarrow{P} 1,$$

as $N \to \infty, k \to \infty, N/k \to \infty$.

In the case of $r = 2$, $\frac{\phi^{(2)}_{k,N}(\sigma_{k,N})}{-2N} \xrightarrow{P} 1$, which gives, for any $\eta < \frac{1}{2}$,

$$k^{\eta} \cdot \left[\frac{\sigma^2_{k,N}\phi^{(2)}_{k,N}(\sigma_{k,N})}{2k} - 1\right] \xrightarrow{P} 0,$$

as $N \to \infty, k \to \infty, N/k \to \infty$.

Proof. We have, the formula,

$$\phi^{(r)}_{k,N}(\sigma_{k,N}) = (r-1)!\left(-\frac{n+b}{(1-\sigma_{k,N})^r} + (-1)^{r-1}\frac{n}{(1+\sigma_{k,N})^r} + (-1)^r\frac{k}{\sigma^r_{k,N}}\right), r \geq 1.$$

Now, for $r \geq 3$, the dominant term is $\frac{k}{\sigma_{k,N}^r}$, since the first two terms will be of order N and the last term is of order $k(N/k)^{r/2}$. Hence, for $r \geq 3$,

$$\frac{\phi_{k,N}^r(\sigma_{k,N})}{(-1)^r(r-1)!N^{\frac{r}{2}}k^{1-\frac{r}{2}}} \xrightarrow{P} 1.$$

Then,

$$\sigma_{k,N}^3\phi_{k,N}^{(3)}(\sigma_{k,N}) \sim -i\frac{k^{3/2}}{N^{3/2}} \cdot -2N^{3/2}k^{1-3/2} = 2ik.$$

Next,

$$\begin{aligned}
\phi_{k,N}^{(2)}(\sigma_{k,N}) &= -\frac{n+b}{(1-\sigma_{k,N})^2} - \frac{n}{(1+\sigma_{k,N})^2} + \frac{k}{\sigma_{k,N}^2} \\
&= \frac{-(n+b)(1+\sigma_{k,N})^2\sigma_{k,N}^2 - n(1-\sigma_{k,N})^2\sigma_{k,N}^2 + k(1-\sigma_{k,N})^2(1+\sigma_{k,N})^2}{(1-\sigma_{k,N})^2(1+\sigma_{k,N})^2\sigma_{k,N}^2} \\
&= \frac{\sigma_{k,N}^4(-N+k) - 2b\sigma_{k,N}^3 - \sigma_{k,N}^2(N+2k) + k}{(1-\sigma_{k,N})^2(1+\sigma_{k,N})^2\sigma_{k,N}^2} \\
&\sim \frac{-k^2/N + 2ik^{3/2}/N + 2k + 3k/N}{-k/N} = k - 2N - 3 - 2i\sqrt{k} \sim -2N.
\end{aligned}$$

Lastly, note that,

$$\sigma_{k,N}^2\phi_{k,N}^{(2)}(\sigma_{k,N}) = \frac{\sigma_{k,N}^4(-N+k) - 2b\sigma_{k,N}^3 - \sigma_{k,N}^2(N+2k) + k}{(1-\sigma_{k,N})^2(1+\sigma_{k,N})^2}.$$

While the denominator converges to 1, as $N \to \infty$, and the first two terms of the numerator converges to 0 as $N \to \infty$ and $k/N \to 0$, the last two terms of the numerator equals

$$\begin{aligned}
-\sigma_{k,N}^2(N+2k) + k = k + k\frac{N+2k}{N-k} - \frac{b^2(N+2k)}{2(N-k)^2} \\
+ ib\sqrt{k}\frac{N+2k}{(N-k)^{3/2}}\sqrt{1 - \frac{b^2}{4k(N-k)}}.
\end{aligned}$$

So,

$$\frac{-\sigma_{k,N}^2(N+2k)+k}{2k} - 1 = \frac{3k}{2(N-k)} - \frac{b^2(N+2k)}{4k(N-k)^2} + i\frac{b(N+2k)}{\sqrt{k}(N-k)^{3/2}}\sqrt{1-\frac{b^2}{4k(N-k)}}.$$

Then, for any $\eta < 1/2$,

$$k^\eta \cdot \left[\frac{-\sigma_{k,N}^2(N+2k)+k}{2k} - 1\right] \xrightarrow{P} 0.$$

Thus,

$$k^\eta \cdot \left[\frac{\sigma_{k,N}^2\phi_{k,N}^{(2)}(\sigma_{k,N})}{2k} - 1\right] \xrightarrow{P} 0.$$

□

We define, for $t \in [-\pi, \pi]$,

$$g_{k,N}(t) = \phi_{k,N}(\sigma_{k,N}e^{it}).$$

Note that, in any ball that does not contain the origin, $\phi_{k,N}(z)$ is analytic. Therefore, taking $\epsilon > 0$ to be smaller than $\pi/2$, we can use Taylor's expansion for the real and imaginary parts of $g_{k,N}(t)$ over $t \in [-\epsilon, \epsilon]$, to get that, there exist t_1, t_2 in $(-\epsilon, \epsilon)$ such that,

$$g_{k,N}(t) = g_{k,N}(0) + tg'_{k,N}(0) + \frac{t^2}{2!}g_{k,N}^{(2)}(0) + \frac{t^3}{3!}\left(Reg_{k,N}^{(3)}(t_1) + iImg_{k,N}^{(3)}(t_2)\right).$$

Lemma 3.2.3.

$$g'_{k,N}(0) = 0,$$
$$\frac{g^{(2)}_{k,N}(0)}{-2k} \xrightarrow{P} 1, \textit{ and,}$$
$$g^{(3)}_{k,N}(t) = O(k), \forall t \in (-\pi, \pi).$$

Proof. We have,

$$g'_{k,N}(t) = \sigma_{k,N} i e^{it} \phi'_{k,N}(\sigma_{k,N} e^{it}).$$
$$g^{(2)}_{k,N}(t) = -\sigma_{k,N} e^{it} \phi'_{k,N}(\sigma_{k,N} e^{it}) - \sigma^2_{k,N} e^{2it} \phi^{(2)}_{k,N}(\sigma_{k,N} e^{it}).$$
$$g^{(3)}_{k,N}(t) = -\sigma_{k,N} i e^{it} \phi'_{k,N}(\sigma_{k,N} e^{it}) - 3i\sigma^2_{k,N} e^{2it} \phi^{(2)}_{k,N}(\sigma_{k,N} e^{it}) - i\sigma^3_{k,N} e^{3it} \phi^{(3)}_{k,N}(\sigma_{k,N} e^{it}).$$

Then, at $t = 0$ we get,

$$g'_{k,N}(0) = \sigma_{k,N} i \phi'_{k,N}(\sigma_{k,N}) = 0.$$
$$g^{(2)}_{k,N}(0) = -\sigma_{k,N} \phi'_{k,N}(\sigma_{k,N}) - \sigma^2_{k,N} \phi^{(2)}_{k,N}(\sigma_{k,N}),$$
$$\Longrightarrow \frac{g^{(2)}_{k,N}(0)}{-2k} \xrightarrow{P} 1, \text{ by Lemma 3.2.2.}$$

Also, by Lemma 3.2.2,

$$g^{(3)}_{k,N}(0) = -\sigma_{k,N} i \phi'_{k,N}(\sigma_{k,N}) - 3i\sigma^2_{k,N} \phi^{(2)}_{k,N}(\sigma_{k,N}) - i\sigma^3_{k,N} \phi^{(3)}_{k,N}(\sigma_{k,N}) = O(k).$$

Now, for any $t, s \in [0, 2\pi]$, and writing $w = \sigma_{k,N}e^{it}, y = \sigma_{k,N}e^{is}$, we have,

$$
\begin{aligned}
&|3w^2\phi^{(2)}_{k,N}(w) + w^3\phi^{(3)}_{k,N}(w) - 3y^2\phi^{(2)}_{k,N}(y) - y^3\phi^{(3)}_{k,N}(y)| \\
&\qquad \le (n+b)\left|\frac{3w^2}{(1-w)^2} - \frac{3y^2}{(1-y)^2}\right| + n\left|\frac{3w^2}{(1+w)^2} - \frac{3y^2}{(1+y)^2}\right| \\
&\qquad + (n+b)\left|\frac{2w^3}{(1-w)^3} - \frac{2y^3}{(1-y)^3}\right| + n\left|\frac{2w^3}{(1+w)^3} - \frac{2y^3}{(1+y)^3}\right| \\
&\qquad = 3(n+b)\frac{|w-y||w+y-2wy|}{(1-w)^2(1-y)^2} \\
&\qquad + 2(n+b)\frac{|w-y||w^2+wy+y^2-3wy(w+y)+3w^2y^2|}{(1-w)^3(1-y)^3} \\
&\qquad + 3n\frac{|w-y||w+y+2wy|}{(1+w)^2(1+y)^2} \\
&\qquad + 2n\frac{|w-y||w^2+wy+y^2+3wy(w+y)+3w^2y^2|}{(1+w)^3(1+y)^3} \\
&\qquad = O(k) + O(k^{3/2}/\sqrt{N}) = O(k).
\end{aligned}
$$

Next,

$$
\begin{aligned}
|w\phi'_{k,N}(w) - y\phi'_{k,N}(y)| &= \left|-(n+b)\frac{w}{1-w} + (n+b)\frac{y}{1-y} + n\frac{w}{1+w} - n\frac{y}{1+y}\right| \\
&= \left|-(n+b)\frac{w-y}{(1-w)(1-y)} + n\frac{w-y}{(1+w)(1+y)}\right| \\
&= |w-y|\left|\frac{-b(1+wy) - N(w+y)}{(1-w)(1-y)(1+w)(1+y)}\right| = O(k).
\end{aligned}
$$

The above show that, if we substitute $s = 0$,

$$g^{(3)}_{k,N}(t) = O(k).$$

□

3.2.3 The ratio $\frac{f_N(\sigma_{k+r,N})}{f_N(\sigma_{k,N})}$, where $r \leq M \cdot \sqrt{k}$

In this section, we shall give approximations for

$$\frac{f_N(\sigma_{k+r,N})}{f_N(\sigma_{k,N})}, r \leq M \cdot \sqrt{k},$$

(M being a constant) as $N \to \infty, k \to \infty, N/k^2 \to \infty$.

We may write $\sigma_{k,N}$ as

$$\sigma_{k,N} := i\sqrt{\frac{k}{N-k}}\left[\sqrt{1-\frac{b^2}{4k(N-k)}} + i\frac{b}{2\sqrt{k(N-k)}}\right] = i\sqrt{\frac{k}{N-k}}e^{i\theta_{k,N}}. \quad (3.2.1)$$

Lemma 3.2.4.

$$f_N(\sigma_{k,N}) \cdot \exp(n\sigma_{k,N}^2 + b\sigma_{k,N}) \xrightarrow{P} 1,$$

as, $N \to \infty, k \to \infty,$ *and* $N/k^2 \to \infty.$

Proof. We have,

$$\begin{aligned} f_N(\sigma_{k,N}) &= (1-\sigma_{k,N}^2)^n(1-\sigma_{k,N})^b, \\ \implies \log[f_N(\sigma_{k,N})] &= n\log(1-\sigma_{k,N}^2) + b\log(1-\sigma_{k,N}), \\ &= n\left[-\sigma_{k,N}^2 - \frac{\sigma_{k,N}^4}{2} - \frac{\sigma_{k,N}^6}{3} - \cdots\right] \\ &+ b\left[-\sigma_{k,N} - \frac{\sigma_{k,N}^2}{2} - \frac{\sigma_{k,N}^3}{3} - \cdots\right] \end{aligned}$$

Then,

$$
\begin{aligned}
|\log[f_N(\sigma_{k,N})] + n\sigma_{k,N}^2 + b\sigma_{k,N}| &\le n\left|\frac{|\sigma_{k,N}|^4}{2} + \frac{|\sigma_{k,N}|^6}{3} + \cdots\right| \\
&+ |b|\left|\frac{|\sigma_{k,N}|^2}{2} + \frac{|\sigma_{k,N}|^3}{3} + \cdots\right| \\
&\le n\left||\sigma_{k,N}|^4 + |\sigma_{k,N}|^6 + ...\right| \\
&+ |b|\left||\sigma_{k,N}|^2 + |\sigma_{k,N}|^3 + ...\right| \\
&= n\frac{|\sigma_{k,N}|^4}{1-|\sigma_{k,N}|^2} + |b|\frac{|\sigma_{k,N}|^2}{1-|\sigma_{k,N}|} \\
&= n\frac{\left(\frac{k}{N-k}\right)^2}{1-\frac{k}{N-k}} + |b|\frac{\frac{k}{N-k}}{1-\sqrt{\frac{k}{N-k}}} \xrightarrow{P} 0,
\end{aligned}
$$

as, $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$.

Thus,

$$f_N(\sigma_{k,N}) \cdot \exp(n\sigma_{k,N}^2 + b\sigma_{k,N}) \xrightarrow{P} 1, \tag{3.2.2}$$

as, $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$.

□

Lemma 3.2.5. *Write $\alpha := b/\sqrt{N}$. Let $r \le M \cdot \sqrt{k}$, $M > 0$ being a constant. Then,*

$$\frac{f_N(\sigma_{k,N})}{f_N(\sigma_{k+r,N})} \cdot \exp\left(\frac{r}{2} + i\frac{\alpha r}{2\sqrt{k}}\right) \xrightarrow{P} 1,$$

as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$.

Proof. From (3.2.2),

$$\frac{f_N(\sigma_{k,N})}{f_N(\sigma_{k+r})} \cdot \exp(n(\sigma_{k,N}^2 - \sigma_{k+r,N}^2) + b(\sigma_{k,N} - \sigma_{k+r,N})) \xrightarrow{P} 1, \tag{3.2.3}$$

for $r \leq M \cdot \sqrt{k}$, for a constant $M > 0$, as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$. Thus, we need approximations for

$$\exp(n(\sigma_{k,N}^2 - \sigma_{k+r,N}^2) + b(\sigma_{k,N} - \sigma_{k+r}, N)).$$

Towards this, we have,

$$n(\sigma_{k,N}^2 - \sigma_{k+r,N}^2) = n\left(-\frac{k}{N-k}e^{2i\theta_{k,N}} + \frac{k+r}{N-k-r}e^{2i\phi_{k+r}}\right).$$

So,

$$\begin{aligned}
\left|\exp(n(\sigma_{k,N}^2 - \sigma_{k+r,N}^2))\right| &= \exp\left[n\left(-\frac{k}{N-k}\cos(2\theta_{k,N}) + \frac{k+r}{N-k-r}\cos(2\theta_{k+r,N})\right)\right] \\
&= \exp\left[n\left(-\frac{k}{N-k}\left(1 - \frac{b^2}{2k(N-k)}\right)\right.\right. \\
&\left.\left. + \frac{k+r}{N-k-r}\left(1 - \frac{b^2}{2(k+r)(N-k-r)}\right)\right)\right] \\
&= \exp\left[n\left(\frac{rN}{(N-k)(N-k-r)}\right.\right. \\
&\left.\left. - \frac{b^2}{2}\left(\frac{1}{(N-k-r)^2} - \frac{1}{(N-k)^2}\right)\right)\right] \\
&= \exp\left[\frac{N-b}{2}\frac{rN}{(N-k)(N-k-r)}\right. \\
&\left. + \frac{nb^2}{2}\frac{r^2 - 2rk + 2rN}{(N-k)^2(N-k-r)^2}\right].
\end{aligned}$$

Note that, for $r \leq M \cdot \sqrt{k}$, $\frac{nb^2}{2}\frac{r^2-2rk+2rN}{(N-k)^2(N-k-r)^2} \to 0$ as $N \to \infty, k \to \infty, N/k^2 \to \infty$. Also,

$$\frac{(N-b)Nr}{(N-k)(N-k-r)} = \frac{N^2r}{(N-k)(N-k-r)} + \frac{brN}{(N-k)(N-k-r)},$$

where, for $r \leq M \cdot \sqrt{k}$, the second term again goes to 0 in probability, as $N \to$

$\infty, k \to \infty, N/k^2 \to \infty$. As for the first term, we see that,

$$r\left[1 - \frac{N^2}{(N-k)(N-k-r)}\right] = r\left[\frac{-2Nk + k^2 - rN + rk}{(N-k)(N-k-r)}\right],$$

which, again, for $r \leq \sqrt{k}$ clearly goes to 0, as $N \to \infty, k \to \infty, N/k^2 \to \infty$. Therefore,

$$\left|\exp(n(\sigma^2_{k,N} - \sigma^2_{k+r,N}))\right| e^{-r/2} \xrightarrow{P} 1, \tag{3.2.4}$$

as $N \to \infty, k \to \infty, N/k^2 \to \infty$.

Next,

$$\begin{aligned}\arg\left[\exp(n(\sigma^2_{k,N} - \sigma^2_{k+r,N}))\right] &= n\left(-\frac{k}{N-k}\sin(2\theta_{k,N}) + \frac{k+r}{N-k-r}\sin(2\theta_{k+r,N})\right) \\ &= n\left[-\frac{k}{N-k}(\sin(2\theta_{k,N}) - \sin(2\theta_{k+r,N}))\right. \\ &\quad \left. + \sin(2\theta_{k+r,N})\left(-\frac{k}{N-k} + \frac{k+r}{N-k-r}\right)\right]. \end{aligned} \tag{3.2.5}$$

From (3.2.1), we get

$$\sin(2\theta_{k+r,N}) = \frac{b}{\sqrt{(k+r)(N-k-r)}}\sqrt{1 - \frac{b^2}{4(k+r)(N-k-r)}},$$

which makes the second term in (3.2.5) into

$$\sin(2\theta_{k+r,N})\frac{rnN}{(N-k)(N-k-r)} = \frac{rnNb\sqrt{4(k+r)(N-k-r) - b^2}}{2(k+r)(N-k)(N-k-r)^2}.$$

Thus,

$$\begin{aligned}
\sin(2\theta_{k+r,N})&\frac{rnN}{(N-k)(N-k-r)} - \frac{br}{2\sqrt{Nk}} \\
&= \frac{rnNb\sqrt{4(k+r)(N-k-r)-b^2}}{2(k+r)(N-k)(N-k-r)^2} - \frac{br}{2\sqrt{Nk}} \\
&= \frac{br}{2\sqrt{Nk}}\left\{\frac{nN\sqrt{Nk}\sqrt{4(k+r)(N-k-r)-b^2}}{(k+r)(N-k)(N-k-r)^2} - 1\right\}.
\end{aligned}$$

While the term inside the braces goes to 0 in probability as $N \to \infty, k \to \infty, N/k^2 \to \infty$, we know that $b/\sqrt{N} \xrightarrow{d} N(0,1)$. Therefore,

$$\sin(2\theta_{k+r,N})\frac{rnN}{(N-k)(N-k-r)} - \frac{br}{2\sqrt{Nk}} \xrightarrow{P} 0,$$

which as $N \to \infty, k \to \infty, N/k^2 \to \infty$.

The first term in (3.2.5) is

$$\begin{aligned}
\frac{-nk}{N-k}&(\sin(2\theta_{k,N}) - \sin(2\theta_{k+r,N})) \\
&= \frac{-nk}{N-k}\left(\frac{b}{\sqrt{k(N-k)}} - \frac{b}{\sqrt{(k+r)(N-k-r)}}\right) \\
&= \frac{-bnk}{N-k} \cdot \frac{\frac{1}{k(N-k)} - \frac{1}{(k+r)(N-k-r)}}{\frac{1}{\sqrt{k(N-k)}} + \frac{1}{\sqrt{(k+r)(N-k-m)}}} \\
&= \frac{-bn}{(N-k)^2} \cdot \frac{\frac{-2rk+rN-r^2}{(k+r)(N-k-r)}}{\frac{1}{\sqrt{k(N-k)}} + \frac{1}{\sqrt{(k+r)(N-k-r)}}},
\end{aligned}$$

which tends to 0 in probability as $N \to \infty, k \to \infty, N/k^2 \to \infty$, for $r \le M \cdot \sqrt{k}$.

Using this, along with (3.2.4), we get, for $r \le M \cdot \sqrt{k}$,

$$\exp(n(\sigma_{k,N}^2 - \sigma_{k+r,N}^2)) \cdot \exp\left\{-\left(\frac{r}{2} + i\frac{br}{2\sqrt{Nk}}\right)\right\} \xrightarrow{P} 1, \qquad (3.2.6)$$

as $N \to \infty, k \to \infty, N/k^2 \to \infty$.

Finally,

$$b(\sigma_k - \sigma_{k+r,N}) = b\left\{ i\sqrt{\frac{k}{N-k}}\left(-\frac{b}{2\sqrt{k(N-k)}} + i\sqrt{1-\frac{b^2}{4k(N-k)}}\right) \right.$$
$$-i\sqrt{\frac{k+r}{N-k-r}}\left(-\frac{b}{2\sqrt{(k+r)(N-k-r)}}\right.$$
$$\left.\left. + \ i\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}\right)\right\}$$

The imaginary part of the above expression is

$$i\frac{b^2}{2}\cdot\frac{-r}{(N-k)(N-k-r)},$$

which, for $r \leq M\cdot\sqrt{k}$, converges in probability to 0 as $N \to \infty, k \to \infty, N/k^2 \to \infty$.

As for the real part,

$$b\left(\sqrt{\frac{k+r}{N-k-r}}\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}} - \sqrt{\frac{k}{N-k}}\sqrt{1-\frac{b^2}{4k(N-k)}}\right)$$

is equal to

$$b\cdot\frac{\frac{k+r}{N-k-r}\left(1-\frac{b^2}{4(k+r)(N-k-r)}\right) - \frac{k}{N-k}\left(1-\frac{b^2}{4k(N-k)}\right)}{\sqrt{\frac{k+r}{N-k-r}}\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}} + \sqrt{\frac{k}{N-k}}\sqrt{1-\frac{b^2}{4k(N-k)}}},$$

which evaluates out as

$$b \cdot \frac{\frac{r}{(N-k)(N-k-r)} - \frac{b^2}{4} \cdot \left(\frac{1}{(N-k-r)^2} - \frac{1}{(N-k)^2}\right)}{\sqrt{\frac{k+r}{N-k-r}}\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}} + \sqrt{\frac{k}{N-k}}\sqrt{1-\frac{b^2}{4k(N-k)}}}$$

$$\sim b \cdot \frac{\frac{r}{N^2} - \frac{b^2}{4} \cdot \frac{r^2-2Nr+2kr}{N^4}}{2\sqrt{\frac{k}{N}}} \xrightarrow{P} 0.$$

These give us

$$b(\sigma_k - \sigma_{k+r,N}) \xrightarrow{P} 0, \tag{3.2.7}$$

for $r \leq \sqrt{k}$ and $N \to \infty, k \to \infty, N/k^2 \to \infty$.

Thus, we have from (3.2.3), (3.2.6) and (3.2.7),

$$\frac{f_N(\sigma_{k,N})}{f_N(\sigma_{k+r,N})} \cdot \exp\left(\frac{r}{2} + i\frac{br}{2\sqrt{Nk}}\right) \xrightarrow{P} 1,$$

for $r \leq M \cdot \sqrt{k}$ and $N \to \infty, k \to \infty, N/k^2 \to \infty$, which proves the theorem. □

3.3 The asymptotic sinusoidal behavior of $e_{k,N}$

Going back to the Cauchy integral expression we had for $e_{k,N}$,

$$e_{k,N} = \frac{(-1)^k}{2\pi i}\int_\Gamma \frac{f_N(z)}{z^{k+1}}dz.$$

We now choose Γ to be the circle centered at the origin of radius $|\sigma_{k,N}|$. Clearly then, Γ passes through both $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$. We divide Γ into four regions - $\Gamma_1, \overline{\Gamma_1}, \Gamma_2$, and

Γ_2' as follows. Any point $z \in \Gamma$ can be written as $z = \sigma_{k,N}e^{it}, t \in [-\pi, \pi]$. We let Γ_1 be the set of all points $z = \sigma_{k,N}e^{it} \in \Gamma$, for which $|t| \le k^{-\delta}$, where δ is fixed and $\frac{1}{3} < \delta < \frac{1}{2}$. We let $\overline{\Gamma_1}$ be the collection of points in Γ that are complex conjugates of the points in Γ_1. Γ_2 and Γ_2' are the regions on the remaining left and right arcs respectively.

3.3.1 Evaluating the Cauchy integral over the "nice arcs"

Lemma 3.3.1.

$$\sqrt{k}\frac{\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz}{\exp(g_{k,N}(0))} - i\gamma\left(\frac{1}{2}, k^{1-\delta}\right) \longrightarrow 0 \ \textit{as, } N \to \infty,$$

where $\gamma(x, y)$ represents the lower incomplete gamma function.

Proof. We may use the Taylor's expansion of $g_{k,N}(t)$ in the previous section to get

$$\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz = i\int_{-k^{-\delta}}^{k^{-\delta}}\exp\left[g_{k,N}(0) - C_{N,k}t^2 + \frac{t^3}{6}\left(Reg_{k,N}^{(3)}(t_1) + iImg_{k,N}^{(3)}(t_2)\right)\right]dt,$$

where $C_{N,k} = \sigma_{k,N}^2\phi_{k,N}^{(2)}(\sigma_{k,N})/2$ (which is $\sim k$) and $t_1, t_2 \in (-k^{-\delta}, k^{-\delta})$.

Because $g_{k,N}^{(3)}(t) = O(k)$ (Lemma 3.2.3), we have that, for $t \in (-k^{-\delta}, k^{-\delta})$,

$$\frac{t^3}{6}\left(Reg_{k,N}^{(3)}(t_1) + iImg_{k,N}^{(3)}(t_2)\right) \longrightarrow 0, \text{ and so,}$$
$$\exp\left(\frac{t^3}{6}\left(Reg_{k,N}^{(3)}(t_1) + iImg_{k,N}^{(3)}(t_2)\right)\right) \longrightarrow 1,$$

the convergence being uniform over $(-k^{-\delta}, k^{-\delta})$.

Next,

$$\sqrt{k}\left[\exp\left(-\frac{C_{N,k}}{k}\cdot kt^2\right) - \exp(-kt^2)\right]$$

$$= \sqrt{k}\exp(-kt^2)\left[\exp\left(-\frac{C_{N,k}}{k}\cdot kt^2 + kt^2\right) - 1\right]$$

$$= \sqrt{k}\exp(-kt^2)\left[\exp\left(\left(1-\frac{C_{N,k}}{k}\right)kt^2\right) - 1\right]$$

As $|t| \leq k^{-\delta}, kt^2 \leq k^{1-2\delta}$. Since $\delta \in (\frac{1}{3}, \frac{1}{2})$, Lemma 3.2.2 gives us

$$\left(1-\frac{C_{N,k}}{k}\right)kt^2 \longrightarrow 0, \text{which means,}$$

$$\sqrt{k}\left[\exp\left(-\frac{C_{N,k}}{k}\cdot kt^2\right) - \exp(-kt^2)\right] \longrightarrow 0,$$

where, as before, the convergence is uniform over $(-k^{-\delta}, k^{-\delta})$.

Finally,

$$\int_{-k^{-\delta}}^{k^{-\delta}} \exp(-kt^2)dt = 2\int_0^{k^{-\delta}} \exp(-kt^2)dt$$

$$= \frac{1}{\sqrt{k}}\int_0^{k^{1-2\delta}} \frac{e^{-x}}{\sqrt{x}}dx, \text{ where, } x = kt^2,$$

$$= \frac{1}{\sqrt{k}}\gamma\left(\frac{1}{2}, k^{1-2\delta}\right).$$

Stitching all of this together, we have,

$$\sqrt{k}\frac{\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz}{\exp(g_{k,N}(0))} - i\gamma\left(\frac{1}{2}, k^{1-2\delta}\right) \longrightarrow 0.$$

□

A similar result holds for $\overline{\Gamma_1}$, which can be proved by simply taking conjugates in Lemma 3.3.1.

Corollary 3.3.2.

$$\sqrt{k}\frac{\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz}{\exp(g_{k,N}(0))} - i\gamma\left(\frac{1}{2}, k^{1-\delta}\right) \longrightarrow 0 \ \textit{as, } N \to \infty.$$

3.3.2 Evaluating the Cauchy integral over the "bad arcs"

In this section, we see that when the Cauchy integral is evaluated over the arcs Γ_2 and Γ_2', it turns out to be negligible with respect to $\exp(g_{k,N}(0))$.

Lemma 3.3.3.

$$\sqrt{k}\frac{\sigma_{k,N}^k}{f_N(\sigma_{k,N})}\int_{\Gamma_2}\frac{f_N(z)}{z^{k+1}}dz \longrightarrow 0, \ \textit{as, } N \to \infty, \ \textit{and,}$$
$$\sqrt{k}\frac{\sigma_{k,N}^k}{f_N(\sigma_{k,N})}\int_{\Gamma_2'}\frac{f_N(z)}{z^{k+1}}dz \longrightarrow 0, \ \textit{as, } N \to \infty.$$

Proof. We have, for $z = \sigma_{k,N}e^{it} \in \Gamma_2 \cup \Gamma_2'$,

$$\begin{aligned}\frac{f_N(z)}{f_N(\sigma_{k,N})} &= \frac{(1-\sigma_{k,N}e^{it})^{n+b}(1+\sigma_{k,N}e^{it})^n}{(1-\sigma_{k,N})^{n+b}(1+\sigma_{k,N})^n} \\ &= \frac{(1-\sigma_{k,N}^2e^{2it})^n}{(1-\sigma_{k,N}^2)^n}\cdot\frac{(1-\sigma_{k,N}e^{it})^b}{(1-\sigma_{k,N})^b} \\ &= \left\{1+(1-e^{2it})\frac{\sigma_{k,N}^2}{1-\sigma_{k,N}^2}\right\}^n\cdot\left\{1+(1-e^{it})\frac{\sigma_{k,N}}{1-\sigma_{k,N}}\right\}^b.\end{aligned}$$

For the first term above, note that,

$$\begin{aligned}\frac{\sigma_{k,N}^2}{1-\sigma_{k,N}^2} &= -\frac{k}{N}\cdot\frac{1-\frac{b^2}{2k(N-k)}+i\frac{b}{\sqrt{k(N-k)}}\sqrt{1-\frac{b^2}{4k(N-k)}}}{1-\frac{b^2}{2N(N-k)}-i\frac{b}{N\sqrt{N-k}}\sqrt{k-\frac{b^2}{4(N-k)}}}, \\ &= -\frac{k}{N}\cdot\alpha_N,\end{aligned}$$

where $\alpha_N \to 1$. Therefore,

$$\frac{\left\{1 + (1 - e^{2it}) \frac{\sigma_{k,N}^2}{1-\sigma_{k,N}^2}\right\}^n}{\exp(-\frac{1}{2}k\alpha_N(1 - e^{2it}))} \longrightarrow 1.$$

Now, $1 - e^{2it} = 1 - \cos 2t - i \sin 2t = 2\sin^2 t - i \sin 2t$.

$$\frac{1}{2}k \cdot 2\sin^2 t = k \sin^2 t \geq k\frac{t^2}{2}, \text{ for, } |t| \leq 1.39.$$

Then, as $|t| \geq k^{-\delta}$,

$$\frac{1}{2}k \cdot 2\sin^2 t \geq \frac{k^{1-2\delta}}{2} \longrightarrow \infty, \text{ since, } \delta \in \left(\frac{1}{3}, \frac{1}{2}\right).$$

Likewise, for $|\pi - t| \leq 1.39$, since $|\pi - t| \geq k^{-\delta}$ as well in Γ_2 and Γ_2', we have,

$$\frac{1}{2}k \cdot 2\sin^2 t = \frac{1}{2}k \cdot 2\sin^2(\pi - t) \geq \frac{k^{1-2\delta}}{2}.$$

Next,

$$\frac{\sigma_{k,N}}{1 - \sigma_{k,N}} = i\sqrt{\frac{k}{N-k}} \cdot \left[\frac{\sqrt{1 - \frac{b^2}{4k(N-k)}} + i\frac{b}{\sqrt{k(N-k)}}}{1 + \frac{b}{2(N-k)} - i\sqrt{\frac{k}{N-k} - \frac{b^2}{4(N-k)^2}}}\right] = i\sqrt{\frac{k}{N-k}} \cdot \beta_N,$$

where, $\beta_N \to 1$. So,

$$\begin{aligned}\left\{1 + (1 - e^{it})\frac{\sigma_{k,N}}{1 - \sigma_{k,N}}\right\}^b &= \left\{1 + (1 - e^{it})i\sqrt{\frac{k}{N-k}} \cdot \beta_N\right\}^b \\ &= \left\{1 + \beta_N \sin t\sqrt{\frac{k}{N-k}} + i\beta_N(1 - \cos t)\sqrt{\frac{k}{N-k}}\right\}^b\end{aligned}$$

Because $k/\sqrt{N-k} \to 0$, and $|b| \leq A\sqrt{N}$, we get that there exists a constant C for which,

$$\frac{\left\{1 + \beta_N \sin t\sqrt{\frac{k}{N-k}} + i\beta_N(1 - \cos t)\sqrt{\frac{k}{N-k}}\right\}^b}{\exp(C\sqrt{k})}$$

is bounded.

Therefore,

$$\sqrt{k}\frac{f_N(z)}{f_N(\sigma_{k,N})} \sim \sqrt{k}\exp\left(-\frac{1}{2}k\alpha_N(1-e^{2it})\right)\exp(C\sqrt{k})\cdot\frac{\left\{1+(1-e^{it})\frac{\sigma_{k,N}}{1-e_{k,N}}\right\}^b}{\exp(C\sqrt{\sigma_{k,N}k})},$$

and so, $\forall z \in \Gamma_2$,

$$\sqrt{k}\frac{f_N(z)}{f_N(\sigma_{k,N})} \longrightarrow 0,$$

with the convergence being uniform over Γ_2.

Thus,

$$\left|\sqrt{k}\frac{\sigma_{k,N}^k}{f_N(\sigma_{k,N})}\int_{\Gamma_2}\frac{f_N(z)}{z^{k+1}}dz\right| \leq \frac{1}{|\sigma_{k,N}|}\int_{\Gamma_2}\sqrt{k}\left|\frac{f_N(z)}{f_N(\sigma_{k,N})}\right||dz| \longrightarrow 0.$$

Similarly,

$$\left|\sqrt{k}\frac{\sigma_{k,N}^k}{f_N(\sigma_{k,N})}\int_{\Gamma_2'}\frac{f_N(z)}{z^{k+1}}dz\right| \leq \frac{1}{|\sigma_{k,N}|}\int_{\Gamma_2'}\sqrt{k}\left|\frac{f_N(z)}{f_N(\sigma_{k,N})}\right||dz| \longrightarrow 0.$$

□

3.3.3 An expression for $\sqrt{k}\cdot e_{k,N}$

Using Lemmas 3.3.1, 3.3.2 and 3.3.3, we get the following expression for $e_{k,N}$.

Proposition 3.3.4.

$$\sqrt{k}\cdot e_{k,N} = \left(\frac{N-k}{k}\right)^{k/2} Re\left\{i^k\left(\frac{1}{\sqrt{\pi}}+\mathcal{Y}_{k,N}\right)f_N(\sigma_{k,N})e^{-ik\theta_{k,N}}\right\},$$

where $\mathcal{Y}_{k,N}\to 0$ *as* $N\to\infty, k\to\infty, N/k\to\infty,$ *where* $\theta_{k,N} = \arg\left(\sigma_{k,N}-\frac{\pi}{2}\right)$.

Proof. We have,

$$\sqrt{k}(-1)^k e_{k,N} = \frac{\sqrt{k}}{2\pi i}\left(\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz + \int_{\overline{\Gamma_1}}\frac{f_N(z)}{z^{k+1}}dz + \int_{\Gamma_2}\frac{f_N(z)}{z^{k+1}}dz + \int_{\Gamma_2'}\frac{f_N(z)}{z^{k+1}}dz\right).$$

Now, using Lemmas 3.3.1 and 3.3.2, we have, $\sqrt{k}\cdot\frac{\sigma_{k,N}^k}{f_N(\sigma_{k,N})}\cdot\frac{1}{i}\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz$ equals,

$$\left[\sqrt{k}\frac{\int_{\Gamma_1}\frac{f_N(z)}{z^{k+1}}dz}{i\exp(g_{k,N}(0))} - \gamma\left(\frac{1}{2},k^{1-\delta}\right)\right] + \gamma\left(\frac{1}{2},k^{1-\delta}\right),$$

and, $\sqrt{k}\cdot\frac{\overline{\sigma_{k,N}}^k}{f_N(\overline{\sigma_{k,N}})}\cdot\frac{1}{i}\int_{\overline{\Gamma_1}}\frac{f_N(z)}{z^{k+1}}dz$ equals

$$\left[\sqrt{k}\frac{\int_{\overline{\Gamma_1}}\frac{f_N(z)}{z^{k+1}}dz}{i\exp(\overline{g_{k,N}(0)})} - \gamma\left(\frac{1}{2},k^{1-\delta}\right)\right] + \gamma\left(\frac{1}{2},k^{1-\delta}\right),$$

where the first terms in the [−] brackets in the above equations are complex conjugates of each other and converge to 0 as $N\to\infty, k\to\infty, N/k\to\infty$. Also, note that, as $\gamma(x,y)$ is the lower incomplete gamma function, and as $\delta<1$, $\gamma\left(\frac{1}{2},k^{1-\delta}\right)\to\sqrt{\pi}$ as $k\to\infty$.

Therefore, applying Lemma 3.3.3 to $\int_{\Gamma_2}\frac{f_N(z)}{z^{k+1}}dz$ and $\int_{\Gamma_2'}\frac{f_N(z)}{z^{k+1}}dz$, we get,

$$\begin{aligned}\sqrt{k}(-1)^k e_{k,N} &= \frac{1}{2\pi}\left\{\frac{f_N(\sigma_{k,N})}{\sigma_{k,N}^k}\left(\sqrt{\pi}+\mathcal{Y}'_{k,N}\right) + \frac{f_N(\overline{\sigma_{k,N}})}{\overline{\sigma_{k,N}}^k}\left(\sqrt{\pi}+\overline{\mathcal{Y}'_{k,N}}\right)\right\}\\ &= Re\left\{\left(\frac{1}{\sqrt{\pi}}+\mathcal{Y}_{k,N}\right)\frac{f_N(\sigma_{k,N})}{\sigma_{k,N}^k}\right\},\end{aligned}$$

where $\mathcal{Y}_{k,N} = \mathcal{Y}'_{k,N}/\pi \to 0$, as $N \to \infty, k \to \infty, N/k \to \infty$.

Furthermore, writing $\sigma_{k,N} = |\sigma_{k,N}|e^{i\theta_{k,N}}$, we have,

$$\begin{aligned}
\sqrt{k}(-1)^k e_{k,N} &= Re\left\{\left(\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}\right)\frac{f_N(\sigma_{k,N})\overline{\sigma_{k,N}}^k}{|\sigma_{k,N}|^{2k}}\right\} \\
&= Re\left\{\left(\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}\right)\left(\frac{N-k}{k}\right)^k f_N(\sigma_{k,N})\left(-i\sqrt{\frac{k}{N-k}}\right)^k e^{-ik\theta_{k,N}}\right\} \\
&= Re\left\{(-i)^k\left(\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}\right)\left(\frac{N-k}{k}\right)^{k/2} f_N(\sigma_{k,N})e^{-ik\theta_{k,N}}\right\} \\
\implies \sqrt{k}e_{k,N} &= \left(\frac{N-k}{k}\right)^{k/2} Re\left\{i^k\left(\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}\right) f_N(\sigma_{k,N})e^{-ik\theta_{k,N}}\right\}.
\end{aligned}$$

□

We write

$$\mathcal{G}_{k,N} := i^k\left(\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}\right) f_N(\sigma_{k,N})e^{-ik\theta_{k,N}}$$

so that,

$$\sqrt{k} \cdot e_{k,N} = \left(\frac{N-k}{k}\right)^{k/2} Re(\mathcal{G}_{k,N}).$$

As a final step to proving Theorem 3.1.1, we want to evaluate the ratio $\frac{\mathcal{G}_{k+r,N}}{\mathcal{G}_{k,N}}$ asymptotically for $r \leq M \cdot \sqrt{k}$, $M > 0$ being a constant, as $N \to \infty, k \to \infty, N/k^2 \to \infty$.

3.3.4 The ratio $\frac{\mathcal{G}_{k+r,N}}{\mathcal{G}_{k,N}}$, for $r \leq M \cdot \sqrt{k}$

Lemma 3.3.5. *Let $\alpha = b/\sqrt{N}$ and $r \leq M \cdot \sqrt{k}$, $M > 0$ being a constant. Then,*

$$(-i)^r \cdot \frac{\mathcal{G}_{k+r,N}}{\mathcal{G}_{k,N}} \cdot \exp\left(-\frac{r}{2} - i\frac{\alpha r}{4\sqrt{k}}\right) \xrightarrow{P} 1,$$

as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$.

Proof. We have,

$$\frac{\mathcal{G}_{k+r,N}}{\mathcal{G}_{k,N}} = i^r \cdot \frac{\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k+r,N}}{\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}} \cdot \frac{f_N(\sigma_{k+r,N})}{f_N(\sigma_{k,N})} \cdot e^{i\left(k\theta_{k,N} - (k+r)\theta_{k+r,N}\right)}.$$

From Lemma 3.2.5,

$$\frac{f_N(\sigma_{k,N})}{f_N(\sigma_{k+r,N})} \cdot \exp\left(\frac{r}{2} + i\frac{\alpha r}{2\sqrt{k}}\right) \xrightarrow{P} 1,$$

where $\alpha = b/\sqrt{N}$, as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$.

So we now need to look at $e^{i\left(k\theta_{k,N} - (k+r)\theta_{k+r,N}\right)}$. We have,

$$\begin{aligned}
k\theta_{k,N} &- (k+r)\theta_{k+r,N} \qquad (3.3.1)\\
&= k \arctan\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1 - \frac{b^2}{4k(N-k)}}}\right) - (k+r)\arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1 - \frac{b^2}{4(k+r)(N-k-r)}}}\right)\\
&= k\left[\arctan\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1 - \frac{b^2}{4k(N-k)}}}\right) - \arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1 - \frac{b^2}{4(k+r)(N-k-r)}}}\right)\right]\\
&- r\arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1 - \frac{b^2}{4(k+r)(N-k-r)}}}\right). \qquad (3.3.2)
\end{aligned}$$

Now, note that, as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$,

$$\frac{r \arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}\right)}{\frac{br}{2\sqrt{kN}}} \xrightarrow{P} 1.$$

Since $b/\sqrt{N} \xrightarrow{d} N(0,1)$, this gives,

$$r \arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}\right) - \frac{br}{2\sqrt{kN}} \xrightarrow{P} 0. \tag{3.3.3}$$

$$\tag{3.3.4}$$

Next,

$$\begin{aligned}
&\arctan\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\right) - \arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}\right)\\
&= \arctan\left(\frac{\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}} - \frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}}{1+\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}}\right)\\
&= \arctan\left(\frac{\frac{b}{2}\left(\frac{1}{\sqrt{k(N-k)}}\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}} - \frac{1}{\sqrt{(k+r)(N-k-r)}}\sqrt{1-\frac{b^2}{4k(N-k)}}\right)}{\sqrt{1-\frac{b^2}{4k(N-k)}}\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}+\frac{b^2}{4}\frac{1}{\sqrt{k(k+r)(N-k)(N-k-r)}}}\right).
\end{aligned}$$

Since the numerator of the expression inside arctan is $\sim \frac{br}{4N^{1/2}k^{3/2}}$, we get

$$\frac{k\left[\arctan\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\right) - \arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}\right)\right]}{\frac{br}{4\sqrt{Nk}}} \xrightarrow{P} 1$$

as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$. Again, since $b/\sqrt{N} \xrightarrow{d} N(0,1)$, this gives,

$$k\left[\arctan\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\right) - \arctan\left(\frac{\frac{b}{2\sqrt{(k+r)(N-k-r)}}}{\sqrt{1-\frac{b^2}{4(k+r)(N-k-r)}}}\right)\right] - \frac{br}{4\sqrt{Nk}} \xrightarrow{P} 0,$$

as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$. This, along with (3.3.3) and (3.3.2), gives us,

$$k\theta_{k,N} - (k+r)\theta_{k+r,N} + \frac{\alpha r}{4\sqrt{k}} \xrightarrow{P} 0,$$

as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$. Hence,

$$\begin{aligned}
&(-i)^r \frac{\mathcal{G}_{k+r,N}}{\mathcal{G}_{k,N}} \cdot \exp\left(-\frac{r}{2} - i\frac{\alpha r}{4\sqrt{k}}\right) \\
&\quad = \frac{\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k+r,N}}{\frac{1}{\sqrt{\pi}} + \mathcal{Y}_{k,N}} \cdot \frac{f_N(\sigma_{k+r,N})}{f_N(\sigma_{k,N})} \cdot \exp\left(-\frac{r}{2} - i\frac{\alpha r}{2\sqrt{k}}\right) \cdot e^{i\left(k\theta_{k,N} - (k+r)\theta_{k+r,N}\right)} \cdot e^{i\frac{\alpha r}{4\sqrt{k}}} \\
&\quad \xrightarrow{P} 1,
\end{aligned}$$

as $N \to \infty, k \to \infty$, and $N/k^2 \to \infty$.

□

3.3.5 Completing the proof of Theorem 3.1.1

We now finish the proof of the main theorem in this chapter, namely Theorem 3.1.1.

Proof. From Lemma 3.3.5, if r is a multiple of 4, we get,

$$\begin{aligned}
&\frac{\mathcal{G}_{k,N} \cdot e^{i\frac{r\alpha}{4\sqrt{k}}} - \mathcal{G}_{k+r,N} \cdot e^{-r/2}}{\mathcal{G}_{k,N}} \xrightarrow{P} 0 \\
\implies &\frac{\mathcal{G}_{k,N} \cdot e^{i\frac{r\alpha}{4\sqrt{k}}} - \mathcal{G}_{k+r,N} \cdot e^{-r/2}}{|\mathcal{G}_{k,N}|} \xrightarrow{P} 0 \\
\implies &\frac{Re(\mathcal{G}_{k,N} \cdot e^{i\frac{r\alpha}{4\sqrt{k}}} - \mathcal{G}_{k+r,N} \cdot e^{-r/2})}{|\mathcal{G}_{k,N}|} \xrightarrow{P} 0 \\
\implies &\frac{Re(\mathcal{G}_{k,N}) \cos\left(\frac{r\alpha}{4\sqrt{k}}\right) + Im(\mathcal{G}_{k,N}) \sin\left(\frac{r\alpha}{4\sqrt{k}}\right) - Re(\mathcal{G}_{k+r,N}) \cdot e^{-r/2}}{\sqrt{Re(\mathcal{G}_{k,N})^2 + Im(\mathcal{G}_{k,N})^2}} \xrightarrow{P} 0.
\end{aligned}$$

Multiplying numerator and denominator by $\frac{1}{\sqrt{k}}\left(\frac{N-k}{k}\right)^{k/2}$, and noting that,

$$\frac{\frac{1}{\sqrt{k}}\left(\frac{N-k}{k}\right)^{k/2}}{\frac{1}{\sqrt{k+r}}\left(\frac{N-k-r}{k+r}\right)^{k/2}} \sim e^{r/2},$$

we get,

$$\frac{e_{k,N}\cos\left(\frac{r\alpha}{4\sqrt{k}}\right)+\frac{1}{\sqrt{k}}\left(\frac{N-k}{k}\right)^{k/2}Im(\mathcal{G}_{k,N})\sin\left(\frac{r\alpha}{4\sqrt{k}}\right)-e_{k+r,N}\cdot\left(\frac{k}{N-k}\right)^{r/2}}{\sqrt{e_{k,N}^2+\frac{1}{k}\left(\frac{N-k}{k}\right)^{k}Im(\mathcal{G}_{k,N})^2}}\xrightarrow{P}0. \tag{3.3.5}$$

Next, let $\beta_{k,N}$ denote the argument of $\frac{f(\sigma_{k,N})}{\sigma_{k,N}^k}$. Then, from (3.2.2), we get,

$$e^{i\beta_{k,N}}\cdot\exp\{nIm(\sigma_{k,N}^2)+bIm(\sigma_{k,N})+k\frac{\pi}{2}+k\theta_{k,N}\}\longrightarrow 1,$$

as $N\to\infty, k\to\infty, N/k^2\to\infty$. We wish to show that $\beta_{k,N}$ does now come too close to $\frac{\pi}{2}$ too often. Observe that,

$$\begin{aligned} nIm(\sigma_{k,N}^2)+bIm(\sigma_{k,N}) &= n\frac{k}{N-k}\left(-\frac{b}{\sqrt{k(N-k)}}\sqrt{1-\frac{b^2}{4k(N-k)}}\right)\\ &\quad+b\sqrt{\frac{k}{N-k}}\sqrt{1-\frac{b^2}{4k(N-k)}}\\ &=\left(1-\frac{n}{N-k}\right)b\sqrt{\frac{k}{N-k}}\sqrt{1-\frac{b^2}{4k(N-k)}}. \end{aligned}$$

Next,

$$\begin{aligned} \theta_{k,N} &= \arctan\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\right)+\frac{\pi}{2}\\ &=\frac{\pi}{2}+\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}-\frac{1}{3}\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\right)^3\\ &\quad+\frac{1}{5}\left(\frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\right)^5+\cdots.\\ \implies k\theta_k &= k\frac{\pi}{2}+\frac{1}{2}\cdot\frac{b\sqrt{\frac{k}{N-k}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}+O\left(\frac{1}{\sqrt{k}}\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&nIm(\sigma_k^2) + bIm(\sigma_k) + k\theta_k \\
&\qquad = \left(1 - \frac{n}{N-k}\right) b\sqrt{\frac{k}{N-k}}\sqrt{1 - \frac{b^2}{4k(N-k)}} + k\frac{\pi}{2} \\
&\qquad + k \cdot \frac{\frac{b}{2\sqrt{k(N-k)}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}} + O\left(\frac{1}{\sqrt{k}}\right) \\
&\qquad = k\frac{\pi}{2} + O\left(\frac{1}{\sqrt{k}}\right) \\
&\qquad + \frac{b\sqrt{\frac{k}{N-k}}}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\left[\left(1 - \frac{n}{N-k}\right)\left(1 - \frac{b^2}{4k(N-k)}\right) + \frac{1}{2}\right].
\end{aligned}
$$

Let

$$
a_{k,N} := \frac{b}{\sqrt{1-\frac{b^2}{4k(N-k)}}}\left[\left(1 - \frac{n}{N-k}\right)\left(1 - \frac{b^2}{4k(N-k)}\right) + \frac{1}{2}\right].
$$

We shall be done if we can show that $a_{k,N}\sqrt{\frac{k}{N-k}}$ does not hover too close to multiples of $\frac{\pi}{2}$. Notice that $a_{k,N}/\sqrt{k} \xrightarrow{d} N(0,1)$. Let $\Phi(x)$ denote the cumulative distribution function of the standard normal distribution and $\mathcal{N}$ denote a standard normal variate. Choose an $\epsilon > 0$ that is < 1. Then, the intervals $[m\pi - \epsilon, m\pi + \epsilon]$ are going to be disjoint over $m \in \mathbb{Z}$. Then, fixing $M > 0$ (to be chosen suitably later),

$$
\begin{aligned}
&\mathbb{P}\left(\inf_{0\le m\le M\sqrt{k}}\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right| \le \epsilon\right) - \mathbb{P}\left(\inf_{0\le m\le M\sqrt{k}}\left|m\pi - \mathcal{N}\sqrt{k}\right| \le \epsilon\right) \\
&\quad = \sum_{0\le m\le M\sqrt{k}}\left[\mathbb{P}\left(\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right| \le \epsilon\right) - \mathbb{P}\left(\left|m\pi - \mathcal{N}\sqrt{k}\right| \le \epsilon\right)\right]
\end{aligned}
$$

By Berry-Esseen's theorem, there exists a constant C such that each term inside the sum at the right hand side above is $\leq \frac{C}{\sqrt{N-k}}$. Thus

$$\mathbb{P}\left(\inf_{0\leq m\leq M\sqrt{k}}\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right|\leq\epsilon\right) - \mathbb{P}\left(\inf_{0\leq m\leq M\sqrt{k}}\left|m\pi - \mathcal{N}\sqrt{k}\right|\leq\epsilon\right) \leq \frac{CM\sqrt{k}}{\sqrt{N-k}}, \tag{3.3.6}$$

which goes to 0 as $N\to\infty, k\to\infty, N/k^2\to\infty$. Also, using Berry-Esseen again,

$$\begin{aligned}\mathbb{P}\left(\inf_{m>M\sqrt{k}}\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right|\leq\epsilon\right) &\leq \mathbb{P}\left(a_{k,N}\sqrt{\frac{k}{N-k}}\geq -\epsilon + M\pi\sqrt{k}\right)\\ &\leq \frac{C}{\sqrt{N-k}} + 1 - \Phi\left(\frac{-\epsilon}{\sqrt{k}} + M\pi\right).\end{aligned}$$

We now specify our choice of M to be large enough so that $\Phi\left(-1+M\pi\right) > 1-\delta/6$, so that,

$$\mathbb{P}\left(\inf_{m>M\sqrt{k}}\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right|\leq\epsilon\right) \leq \frac{C}{\sqrt{N-k}} + \delta. \tag{3.3.7}$$

Finally,

$$\begin{aligned}\mathbb{P}\left(\inf_{0\leq m\leq M\sqrt{k}}\left|m\pi - \mathcal{N}\sqrt{k}\right|\leq\epsilon\right) &= \sum_{0\leq m\leq M\sqrt{k}}\int_{\frac{m\pi-\epsilon}{\sqrt{k}}}^{\frac{m\pi+\epsilon}{\sqrt{k}}}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\\ &\leq \frac{1}{\sqrt{2\pi}}\sum_{0\leq m\leq M\sqrt{k}}\exp\left(-\left(\frac{m\pi-\epsilon}{\sqrt{k}}\right)^2\right)\cdot\frac{2\epsilon}{\sqrt{k}}\\ &\leq \frac{1}{\sqrt{2\pi}}\frac{\frac{2\epsilon}{\sqrt{k}}}{1-\exp(-\pi/\sqrt{k})} \longrightarrow 0.\end{aligned} \tag{3.3.8}$$

From (3.3.6), (3.3.7) and (3.3.8), we get that, given any $\delta > 0$, we can find $N, k, N/k^2$ sufficiently large, so that

$$\mathbb{P}\left(\inf_{m\in\mathbb{Z}^+}\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right|\leq\epsilon\right) < \frac{\delta}{2}.$$

By symmetry, we get,

$$\mathbb{P}\left(\inf_{m\in\mathbb{Z}}\left|m\pi - a_{k,N}\sqrt{\frac{k}{N-k}}\right| \leq \epsilon\right) < \delta. \tag{3.3.9}$$

By Lemma 3.3.5,

$$\frac{\mathcal{G}_{k+1,N}}{\mathcal{G}_{k,N}} \xrightarrow{P} i\sqrt{e}$$

$$\implies \frac{\frac{Re(\mathcal{G}_{k+1,N})}{Re(\mathcal{G}_{k,N})} + \sqrt{e}\frac{Im(\mathcal{G}_{k,N})}{Re(\mathcal{G}_{k,N})} + i\left(\frac{Im(\mathcal{G}_{k+1,N})}{Re(\mathcal{G}_{k,N})} - \sqrt{e}\right)}{1 + i\frac{Im(\mathcal{G}_{k,N})}{Re(\mathcal{G}_{k,N})}} \xrightarrow{P} 0.$$

Using (3.3.9) we get,

$$\frac{Re(\mathcal{G}_{k+1,N})}{Re(\mathcal{G}_{k,N})} + \sqrt{e}\frac{Im(\mathcal{G}_{k,N})}{Re(\mathcal{G}_{k,N})} \xrightarrow{P} 0.$$

Again, multiplying numerator and denominator by $\frac{1}{\sqrt{k}}\left(\frac{N-k}{k}\right)^{k/2}$, and noting that,

$$\frac{\frac{1}{\sqrt{k}}\left(\frac{N-k}{k}\right)^{k/2}}{\frac{1}{\sqrt{k+1}}\left(\frac{N-k-1}{k+1}\right)^{k/2}} \sim \sqrt{e},$$

we get,

$$\frac{\sqrt{\frac{k}{N-k}}e_{k+1,N}}{e_{k,N}} + \frac{\frac{1}{\sqrt{k}}\left(\frac{N-k}{k}\right)^{k/2} Im(\mathcal{G}_{k,N})}{e_{k,N}} \xrightarrow{P} 0 \tag{3.3.10}$$

Therefore, from (3.3.10) and (3.3.5),

$$\frac{e_{k,N}\cos\left(\frac{r\alpha}{4\sqrt{k}}\right) + \sqrt{\frac{k}{N-k}}e_{k+1,N}\sin\left(\frac{r\alpha}{4\sqrt{k}}\right) - e_{k+r,N}\cdot\left(\frac{k}{N-k}\right)^{r/2}}{\sqrt{e_{k,N}^2 + \frac{k}{N-k}(e_{k+1,N})^2}} \xrightarrow{P} 0,$$

$$\implies -\sin\left(\frac{r\alpha}{4\sqrt{k}} - 2\pi\Theta\right) - \frac{e_{k+r,N}\cdot\left(\frac{k}{N-k}\right)^{r/2}}{\sqrt{e_{k,N}^2 + \frac{k}{N-k}(e_{k+1,N})^2}} \xrightarrow{P} 0.$$

This clearly gives us

$$\sup_{t=0,4/\sqrt{k},\ldots,\lfloor M\rfloor_0}\left(\inf_{0\le s\le M}\left|-\sin\left(\frac{s\alpha}{4}-2\pi\Theta\right)-\frac{e_{k+r,N}\cdot\left(\frac{k}{N-k}\right)^{t\sqrt{k}/2}}{\sqrt{e_{k,N}^2+\frac{k}{N-k}(e_{k+1,N})^2}}\right|\right)\xrightarrow{P}0. \tag{3.3.11}$$

Also,

$$\inf_{t=0,4/\sqrt{k},\ldots,\lfloor M\rfloor_0}\left|-\sin\left(\frac{s\alpha}{4}-2\pi\Theta\right)-\frac{e_{k+r,N}\cdot\left(\frac{k}{N-k}\right)^{t\sqrt{k}/2}}{\sqrt{e_{k,N}^2+\frac{k}{N-k}(e_{k+1,N})^2}}\right|\xrightarrow{P}0,$$

and owing to the uniform continuity of $\sin\left(\frac{s\alpha}{4}-2\pi\Theta\right)$ in $[0,M]$, we get,

$$\sup_{0\le s\le M}\inf_{t=0,4/\sqrt{k},\ldots,\lfloor M\rfloor_0}\left|-\sin\left(\frac{s\alpha}{4}-2\pi\Theta\right)-\frac{e_{k+r,N}\cdot\left(\frac{k}{N-k}\right)^{t\sqrt{k}/2}}{\sqrt{e_{k,N}^2+\frac{k}{N-k}(e_{k+1,N})^2}}\right|\xrightarrow{P}0. \tag{3.3.12}$$

The equations (3.3.11) and (3.3.12) together imply that,

$$d_H(\mathcal{Y},\Xi)\xrightarrow{P}0.$$

□

Chapter 4

Poisson point process

4.1 Introduction

4.1.1 Overview and Notations

Let $\{X_j : j \in \mathbb{N}\}$ represent points of a Poisson process of intensity 1 on the real line. In this chapter, we wish to define an entire function, f, that vanishes at exactly the points $X_j, \forall j$, and explore the resulting zero set on taking repeated derivatives of the function f. As we shall see, answers to questions regarding the resulting zero set are locked in with the behavior of the coefficients of powers of z in the power series expansion of the function f.

Although Weierstrass's Product rule does ensure the existence of (infinitely many!) entire functions that vanish at exactly the points $X_j, \forall j$, we are interested in a class of functions $f_x, x \in \mathbb{R}$, that are defined as

$$f_x(z) := \lim_{N\to\infty} \prod_{j:|X_j - x|\le N} \left(1 - \frac{z}{X_j}\right).$$

As we shall see in the next section, $\forall x \in \mathbb{R}$, $f_x(z)$ is entire and the critical points of f_x are translationally invariant with respect to x.

We shall write

$$f(z) := f_0(z) = \lim_{N\to\infty} \prod_{j:|X_j|\le N} \left(1 - \frac{z}{X_j}\right).$$

The coefficient of z^k in the power series expansion for f will be $(-1)^k$ times the elementary symmetric function of $\frac{1}{X_j}$'s, given by

$$e_k := \lim_{N\to\infty} \sum_{1\le j_1<j_2<\ldots<j_k:|X_{j_l}|\le N} \frac{1}{X_{j_1}X_{j_2}\ldots X_{j_k}}.$$

4.1.2 Main Results

The two-step ratio of the elementary symmetric functions, e_{k+2}/e_k display an interesting convergence property as $k \to \infty$, as we shall show with the following theorem.

Theorem 4.1.1. *Let $\{X_j : j \in \mathbb{N}\}$ represent points of a Poisson process of intensity 1 on the real line, and let,*

$$e_k := \lim_{N\to\infty} \sum_{1\le j_1<j_2<\ldots<j_k:|X_{j_l}|\le N} \frac{1}{X_{j_1}X_{j_2}\ldots X_{j_k}}.$$

Then,

$$\frac{k^2 e_{k+2}}{e_k} \xrightarrow{P} -\pi^2,$$

as $k \to \infty$.

Note that the behavior described in the above theorem holds true for the function $\cos(\pi z) + Y \sin(\pi z)$, where Y has a standard Cauchy distribution. The zeros of this function are $\mathbb{Z} + U$, where $U \sim \text{Uniform}(0, 1)$. Following this link, we arrive at the following result.

Theorem 4.1.2. *Let* $\{X_j : j \in \mathbb{N}\}$ *represent points of a Poisson process of intensity 1 on the real line, and let,*

$$f(z) := \lim_{N\to\infty} \prod_{j:|X_j|\le N} \left(1 - \frac{z}{X_j}\right).$$

The zero set of the nth derivative of f, $f^{(n)}$, *converges in distribution, as* $n \to \infty$, *to* $\mathbb{Z} + U$, *where* $U \sim$ *Uniform*$(0, 1)$.

4.1.3 The Cauchy Integral expression for e_k

As in the previous chapter, we shall express the elementary symmetric functions in terms of a Cauchy integral about a loop of steepest descent, and evaluate that integral over different arcs of the main loop.

By Cauchy's Integral formula,

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(z)}{z^{k+1}} dz,$$

where Γ is a simple continuous loop around the origin. Note that, $f^{(k)}(0) = (-1)^k k! e_k$. So,

$$e_k = \frac{(-1)^k}{2\pi i} \int_\Gamma \frac{f(z)}{z^{k+1}} dz. \tag{4.1.1}$$

Define

$$\phi_k(z) = \log\left(\frac{f(z)}{z^k}\right).$$

Away from the real line, ϕ_k is analytic. As in the previous chapter, the critical points of ϕ_k determine the loop of steepest descent, and the higher derivatives of ϕ_k help in solving the Cauchy integral in (4.4.1).

4.2 Existence of the function f and its properties

Despite not following the Weierstrass's Product rule, we shall show that the function f still exists and is entire. Towards this, we first need the following result about sums of $\frac{1}{X_j}$'s.

Lemma 4.2.1. *For every $r \geq 1$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the sum*

$$\sum_{j:|X_j - x| \leq N} \frac{1}{(z - X_j)^r}$$

converges (conditionally) almost surely to a finite complex number, as $N \to \infty$.

Proof. This result may be proved using a slight variation of Kolmogorov's three-series theorem.

We have, given any $A > 0$,

$$\sum_j \mathbb{P}\left(\frac{1}{|z - X_j|^r} \geq A\right) = \sum_j \mathbb{P}\left(X_j \in \overline{\mathbb{D}(z, A^{-1/r})}\right),$$

where $\mathbb{D}(\omega, \rho)$ denotes the ball of radius ρ centered at ω. Thus, fixing $M > A^{-1/r}$,

$$\begin{aligned}\sum_j \mathbb{P}\left(\frac{1}{|z - X_j|^r} \geq A\right) &= \sum_j \mathbb{P}\left(\left\{X_j \in \overline{\mathbb{D}(z, A^{-1/r})}\right\} \cap \{|X_j| \leq M\}\right) \\ &\leq \sum_j \mathbb{P}\left(|X_j| \leq M\right) = \mathbb{E}\left(\sum_j \mathbb{1}_{|X_j| \leq M}\right) = \mathbb{E}(\mathcal{R}_M),\end{aligned}$$

where $\mathcal{R}_M$ denotes the number of points of the Poisson process within $[-M, M]$. Note that $\mathcal{R}_M$ has a Poisson distribution with mean $2M$. Thus,

$$\sum_j \mathbb{P}\left(\frac{1}{|z - X_j|^r} \geq A\right) \leq 2M < \infty.$$

By Borel-Cantelli Lemma, this implies that for j sufficiently large, $\frac{1}{|z - X_j|^r} \leq A$ almost surely. Write,

$$Y_j := \frac{1}{(z - X_j)^r} \mathbb{1}_{\left\{\frac{1}{|z - X_j|^r} \leq A\right\}}.$$

Thus,

$$\sum_{j : |X_j - x| \leq N} \frac{1}{(z - X_j)^r}$$

converges as $N \to \infty$ if and only if

$$\sum_{j : |X_j - x| \leq N} Y_j$$

converges as $N \to \infty$.

Next,

$$\mathbb{E}\left(\sum_{j:|X_j-x|\le N} Y_j\right) = \mathbb{E}\mathbb{E}\left[\sum_{j:|X_j-x|\le N} Y_j \middle| \mathcal{R}_N\right],$$

where $\mathcal{R}_N$ denotes the number of points of the Poisson process within $[-N+x, N+x]$. As before, $\mathcal{R}_N$ has a Poisson distribution with mean $2N$. Moreover, conditional on $\mathcal{R}_N$, the poisson points inside $[-N+x, N+x]$ is i.i.d uniformly distributed inside that interval. Thus,

$$\mathbb{E}\left(\sum_{j:|X_j-x|\le N} Y_j\right) = \mathbb{E}\left[\mathcal{R}_N \cdot Q_r\right],$$

where

$$\begin{aligned} Q_r &= \frac{1}{2N}\int_{-N+x}^{N+x} \frac{1}{(z-u)^r} \mathbb{1}_{\left\{\frac{1}{|z-u|^r} \le A\right\}} du \\ &= \frac{1}{2N}\int_{-N+x}^{N+x} \frac{1}{(z-u)^r} \mathbb{1}_{|z-u|\ge A^{-1/r}} du. \end{aligned}$$

For $r > 1$, if z is not real, we can choose A large enough, so that

$$\begin{aligned} Q_r &= \frac{1}{2N}\int_{-N+x}^{N+x} \frac{1}{(z-u)^r} du = \frac{1}{2N}\left[\frac{1}{r-1}\left\{\frac{1}{(z-N-x)^{r-1}} - \frac{1}{(z+N-x)^{r-1}}\right\}\right], \\ \implies & \lim_{N\to\infty} \mathbb{E}\left(\sum_{j:|X_j-x|\le N} Y_j\right) = 0, \end{aligned}$$

and if z is a real number, then

$$\begin{aligned} Q_r &= \frac{1}{2N}\left[\frac{1}{r-1}\left\{\frac{1}{(z-N-x)^{r-1}} - \frac{2}{A^{r-1}} - \frac{1}{(z+N-x)^{r-1}}\right\}\right] \\ &\implies \lim_{N\to\infty} \mathbb{E}\left(\sum_{j:|X_j-x|\le N} Y_j\right) = -\frac{2}{A^{r-1}}, \end{aligned}$$

a finite limit in both the cases.

For $r = 1$, writing $z = rei\theta$, if z is not real, we get,

$$
\begin{aligned}
Q_1 &= \frac{1}{2N} \int_{-N+x}^{N+x} \frac{1}{z-u} \mathbb{1}_{\left\{\frac{1}{|z-u|} \leq A\right\}} du \\
&= \frac{1}{2N} \int_{-N+x}^{N+x} \frac{1}{r\cos\theta + ir\sin\theta - u} \mathbb{1}_{\left\{\frac{1}{|z-u|} \leq A\right\}} du \\
&= \frac{1}{2N} \int_{-N+x}^{N+x} \frac{r\cos\theta - ir\sin\theta - u}{(r\cos\theta - u)^2 + r^2\sin^2\theta} \mathbb{1}_{\left\{\frac{1}{|z-u|} \leq A\right\}} du \\
&= \frac{1}{2N} \int_{-N+x}^{N+x} \frac{r\cos\theta - u}{(r\cos\theta - u)^2 + r^2\sin^2\theta} \mathbb{1}_{\left\{\frac{1}{|z-u|} \leq A\right\}} du \\
&- \frac{ir\sin\theta}{2N} \int_{-N+x}^{N+x} \frac{1}{(r\cos\theta - u)^2 + r^2\sin^2\theta} \mathbb{1}_{\left\{\frac{1}{|z-u|} \leq A\right\}} du \\
&= \frac{-1}{2N} \log\left|\frac{N+x-z}{N-x+z}\right| - \frac{i}{2N} \arctan\left(\frac{N+x-r\cos\theta}{r\sin\theta}\right) \\
&+ \frac{i}{2N} \arctan\left(\frac{-N+x-r\cos\theta}{r\sin\theta}\right) + \frac{1}{2N}\log A - \frac{i}{N}\arctan\left(\frac{A^{-1} - r\sin\theta}{r\sin\theta}\right).
\end{aligned}
$$

Therefore, $\lim_{N\to\infty} \mathbb{E}\left(\sum_{j:|X_j - x| \leq N} Y_j\right)$ exists finitely. If z is real, then,

$$
\begin{aligned}
Q_1 &= \frac{1}{2N} \int_{-N+x}^{N+x} \frac{1}{z-u} \mathbb{1}_{\left\{\frac{1}{|z-u|} \leq A\right\}} du \\
&= \frac{-1}{2N} \log\left|\frac{N+x-z}{N-x+z}\right| + \frac{1}{2N}\log A,
\end{aligned}
$$

and again, $\lim_{N\to\infty} \mathbb{E}\left(\sum_{j:|X_j - x| \leq N} Y_j\right)$ exists finitely. Thus, for all $r > 1$, $\mathbb{E}\left(\sum_{j:|X_j - x| \leq N} Y_j\right)$ has a finite limit as N goes to infinity.

Next,

$$\mathbb{E}\left(\sum_{j:|X_j-x|\le N}|Y_j|^2\right) = \mathbb{E}\left[\mathcal{R}_N\cdot\frac{1}{2N}\int_{-N+x}^{N+x}\frac{1}{|z-u|^{2r}}\mathbb{1}_{|z-u|\ge A^{-1/r}}du\right],$$
$$= \int_{-N+x}^{N+x}\frac{1}{|z-u|^{2r}}\mathbb{1}_{|z-u|\ge A^{-1/r}}du$$
$$= \int_{-N+x}^{N+x}\left\{\frac{1}{\overline{z}-z}\left(\frac{1}{z-u}-\frac{1}{\overline{z}-u}\right)\right\}^r\mathbb{1}_{|z-u|\ge A^{-1/r}}du,$$

which would again have a finite limit as N goes to infinity. Now, using Kolmogorov's inequality, we get,

$$\mathbb{P}\left(\max_{1\le n\le N}\left|\sum_{j:m\le|X_j-x|\le n}Y_j - \mathbb{E}\left(\sum_{j:m\le|X_j-x|\le n}Y_j\right)\right|\ge\lambda\,\middle|\,\mathcal{R}_N\right)$$
$$\le\frac{1}{\lambda^2}\mathbb{E}\left(\sum_{j:m\le|X_j-x|\le N}|Y_j|^2\,\middle|\,\mathcal{R}_N\right)$$
$$\implies \mathbb{P}\left(\max_{1\le n\le N}\left|\sum_{j:m\le|X_j-x|\le n}Y_j - \mathbb{E}\left(\sum_{j:m\le|X_j-x|\le N}Y_j\right)\right|\ge\lambda\right)$$
$$\le\frac{1}{\lambda^2}\mathbb{E}\left(\sum_{j:m\le|X_j-x|\le N}|Y_j|^2\right),$$

where $\mathcal{R}_N$ denotes the number of poisson points inside $[-N, N]$. Thus, as

$$\lim_{N\to\infty}\mathbb{E}\left(\sum_{j:|X_j-x|\le N}Y_j\right)$$

exists finitely, given any $\epsilon > 0$,

$$\mathbb{P}\left(\limsup_{N\to\infty}\sum_{j:|X_j-x|\le N}Y_j - \limsup_{N\to\infty}\sum_{j:|X_j-x|\le N}Y_j\ge\epsilon\right)$$
$$\le\mathbb{P}\left(2\max_{1\le n\le N}\left|\sum_{j:m\le|X_j-x|\le n}Y_j - \mathbb{E}\left(\sum_{j:m\le|X_j-x|\le N}Y_j\right)\right|\ge\epsilon\right)$$
$$\le\frac{4}{\epsilon^2}\mathbb{E}\left(\sum_{j:m\le|X_j-x|\le N}|Y_j|^2\right),$$

which converges to 0 as $m \to \infty$. Hence $\lim_{N\to\infty} \sum_{j:|X_j - x| \le N} Y_j$ exists, almost surely, which in turn implies that $\lim_{N\to\infty} \sum_{j:|X_j - x| \le N} \frac{1}{(z-X_j)^r}$ exists almost surely.

□

The above lemma sets the stage to prove our result about the existence of f.

Lemma 4.2.2. *Define, for $x \in \mathbb{R}$,*

$$f_{x,N}(z) := \prod_{j:|X_j - x| \le N} \left(1 - \frac{z}{X_j}\right).$$

Then, as $N \to \infty$, $f_{x,N}(z)$ converges almost surely and uniformly on compact subsets of $\mathbb{C}$.

Proof. Let K be a compact subset of $\mathbb{C}$. Let $M = \max\{\lfloor \max\{|z| : z \in K\} \rfloor + 1, x\}$. Then, if $|X_j - x| \ge 3M$, $|z/X_j| \le 1/2, \forall z \in K$, and so

$$\left|1 - \frac{z}{X_j}\right| \ge \frac{1}{2}.$$

So, for $|X_j - x| \ge 3M$, we can take the principal logarithm of $1 - z/X_j$ as an analytic function on K, and we have,

$$\log\left(1 - \frac{z}{X_j}\right) = -\left(\frac{z}{X_j} + \frac{z^2}{2X_j^2} + \frac{z^3}{3X_j^3} + ...\right).$$

$$\implies \left|-\log\left(1-\frac{z}{X_j}\right)-\frac{z}{X_j}\right| = \left|\frac{z^2}{2X_j^2}+\frac{z^3}{3X_j^3}+\frac{z^4}{4X_j^4}+...\right|,$$

$$\leq \frac{|z|}{|X_j|^2}\left|\frac{1}{2}+\frac{1}{3}\cdot\frac{z}{X_j}+\frac{1}{4}\cdot\frac{z^2}{X_j^2}+\cdots\right|,$$

$$\leq \frac{|z|^2}{|X_j|^2}\left[\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2^2}+...\right],$$

$$\leq \frac{|z|^2}{|X_j|^2}.$$

$$\implies \sum_{3M\leq|X_j-x|\leq N}\left|-\log\left(1-\frac{z}{X_j}\right)-\frac{z}{X_j}\right| \leq |z|^2\sum_{3M\leq|X_j-x|\leq N}\frac{1}{|X_j|^2}$$

$$\leq M^2\sum_{3M\leq|X_j-x|\leq N}\frac{1}{|X_j|^2}.$$

Since $X_1, X_2, ...$ are Poisson points, $\sum_{3M\leq|X_j-x|\leq N}\frac{1}{|X_j|^2}$ converges as $N\to\infty$. Thus, by the above inequalities,

$$\sum_{3M\leq|X_j-x|\leq N}\left|-\log\left(1-\frac{z}{X_j}\right)-\frac{z}{X_j}\right|$$

converges as $N\to\infty$, and moreover, this convergence is uniform over K, as it is dominated by the convergence of the sums of inverse squares of $|X_j|$. So, we may write

$$\sum_{3M\leq|X_j-x|\leq N}\log\left(1-\frac{z}{X_j}\right) = -z\sum_{3M\leq|X_j-x|\leq N}\frac{1}{X_j}+G_N(z),$$

where G_N converges uniformly over K. Also, from Lemma 4.2.1 the sum

$$\sum_{3M\leq|X_j-x|\leq N}\frac{1}{X_j}$$

converges (conditionally) as $N\to\infty$. Hence,

$$-z\sum_{3M\leq|X_j-x|\leq N}\frac{1}{X_j}$$

converges (conditionally) uniformly over K as $N \to \infty$. Therefore,

$$\sum_{3M \leq |X_j - x| \leq N} \log\left(1 - \frac{z}{X_j}\right)$$

converges uniformly over K as $N \to \infty$, which implies that,

$$\prod_{3M \leq |X_j - x| \leq N} \left(1 - \frac{z}{X_j}\right)$$

converges uniformly over K as $N \to \infty$. Multiplying this by

$$\prod_{|X_j - x| < 3M} \left(1 - \frac{z}{X_j}\right),$$

which is, a.s., a finite product, we have that,

$$\prod_{|X_j - x| \leq N} \left(1 - \frac{z}{X_j}\right)$$

converges uniformly over K. Thus we have the a.s. uniform convergence in compact subsets of $\mathbb{C}$, which also implies that

$$f_x(z) := \lim_{N\to\infty} f_{x,N}(z) = \lim_{N\to\infty} \prod_{|X_j - x| \leq N} \left(1 - \frac{z}{X_j}\right)$$

is an analytic function on $\mathbb{C}$ that vanishes at exactly the Poisson points $X_1, X_2, \cdots$.

□

Now that we have established that for each $x \in \mathbb{R}$, f_x exists and is entire, we are going to state and prove the following result on the translation invariance of the critical points of f_x.

Lemma 4.2.3. *The logarithmic derivative of $f_{x,N}$, $\frac{f'_{x,N}}{f_{x,N}}$, converges almost surely and uniformly in compact subsets of $\mathbb{C}$ to a function independent of x.*

Proof. Assume, without any loss of generality that, $x > 0$. We shall show that

$$\lim_{N\to\infty} \frac{f'_{x,N}}{f_{x,N}} = \lim_{N\to\infty} \frac{f'_{0,N}}{f_{0,N}}, \text{ a.s., } \forall x \in \mathbb{R},$$

so that,

$$\frac{f'_x}{f_x} = \frac{f'}{f}, \text{ a.s., } \forall x \in \mathbb{R}.$$

We have,

$$\frac{f'_{x,N}(z)}{f_{x,N}(z)} = \sum_{j:|X_j - x| \le N} \frac{1}{z - X_j}$$

and so,

$$\begin{aligned}\frac{f'_{x,N}(z)}{f_{x,N}(z)} - \frac{f'_{0,N}(z)}{f_{0,N}(z)} &= \sum_{j:|X_j - x| \le N} \frac{1}{z - X_j} - \sum_{j:|X_j| \le N} \frac{1}{z - X_j} \\ &= \sum_{j:X_j \in (N,N+x]} \frac{1}{z - X_j} - \sum_{j:X_j \in [-N,-N+x)} \frac{1}{z - X_j}.\end{aligned}$$

Now, let $\mathcal{R}_N(x)$ be the random variable denoting the number of Poisson points in $(N, N + x]$. Then,

$$\mathbb{E}\left|\sum_{j:X_j \in (N,N+x]} \frac{1}{z - X_j}\right|^2 = \mathbb{E}\mathbb{E}\left[\left|\sum_{j:X_j \in (N,N+x]} \frac{1}{z - X_j}\right|^2 \middle| \mathcal{R}_N(x)\right].$$

Taking N large enough so that $|z| \le \frac{N}{2}$, we get,

$$\begin{aligned}\mathbb{E}\left|\sum_{j:X_j \in (N,N+x]} \frac{1}{z - X_j}\right|^2 &\le \mathbb{E}\mathbb{E}\left[\sum_{j:X_j \in (N,N+x]} \frac{4}{N^2} \middle| \mathcal{R}_N(x)\right] \\ &= \mathbb{E}\left(\frac{4\mathcal{R}_N(x)}{N^2}\right).\end{aligned}$$

Now, note that, since the X_j's are a Poisson point process of intensity 1 on $\mathbb{R}$, $\mathcal{R}_N(x)$ has Poisson distribution with mean $N + x - N = x$. Therefore,

$$\mathbb{E}\left|\sum_{j:X_j\in(N,N+x]} \frac{1}{z - X_j}\right|^2 \le \frac{4x}{N^2} \longrightarrow 0,$$

as $N \to \infty$. Thus, by Fatou's Lemma,

$$\mathbb{E}\left|\lim_{N\to\infty} \sum_{j:X_j\in(N,N+x]} \frac{1}{z - X_j}\right|^2 \le \lim_{N\to\infty} \mathbb{E}\left|\sum_{j:X_j\in(N,N+x]} \frac{1}{z - X_j}\right|^2 = 0.$$

Therefore,

$$\sum_{j:X_j\in(N,N+x]} \frac{1}{z - X_j} \xrightarrow{a.s.} 0.$$

By symmetry,

$$\sum_{j:X_j\in[-N,-N+x)} \frac{1}{z - X_j} \xrightarrow{a.s.} 0.$$

Thus,

$$\frac{f'_{x,N}(z)}{f_{x,N}(z)} - \frac{f'_{0,N}(z)}{f_{0,N}(z)} \xrightarrow{a.s.} 0,$$

which proves the lemma.

□

4.3 The logarithmic derivative of f

4.3.1 Expectations of various power sums of $\frac{1}{X_j}$'s

In this segment, we shall compute the expectations of some crucial quantities that tend to occur throughout the proofs of the main theorems of this chapter. We first introduce some notations, for ease of writing:

$$\sum_{*} := \lim_{N\to\infty} \sum_{j:|X_j|\le N},$$

$$\prod_{*} := \lim_{N\to\infty} \prod_{j:|X_j|\le N},$$

$$\mathcal{R}_N := \sum_{j} \mathbb{1}_{X_j\in[-N,N]}, \text{ and,}$$

$$\mathbb{E}(\cdot|N,\mathcal{R}_N) := \mathbb{E}(\cdot|X_j \in [-N,N], \mathcal{R}_N \text{ poisson points in } [-N,N]).$$

Lemma 4.3.1.

$$\mathbb{E}\left[\sum_{*} \frac{1}{z-X_j}\right] = -\pi i,$$

if z is in the upper half plane, and,

$$\mathbb{E}\left[\sum_{*} \frac{1}{z-X_j}\right] = \pi i,$$

if z is in the lower half plane.

Proof. Note that, conditioning on $\mathcal{R}_N$, the poisson points X_j that are contained in $[-N,N]$ are identically and independently distributed as Uniform$[-N,N]$. So, writing $z = re^{i\theta}$, we get,

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{z-X_j}\middle| N, \mathcal{R}_N\right] &= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{z-x}dx \\
&= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{r\cos\theta + ir\sin\theta - x}dx \\
&= \frac{1}{2N}\int_{x\in[-N,N]}\frac{r\cos\theta - ir\sin\theta - x}{(r\cos\theta - x)^2 + r^2\sin^2\theta}dx \\
&= \frac{1}{2N}\int_{x\in[-N,N]}\frac{r\cos\theta - x}{(r\cos\theta - x)^2 + r^2\sin^2\theta}dx \\
&- \frac{ir\sin\theta}{2N}\int_{x\in[-N,N]}\frac{1}{(r\cos\theta - x)^2 + r^2\sin^2\theta}dx \\
&= \frac{-1}{4N}\log[(r\cos\theta - x)^2 + r^2\sin^2\theta]\Big|_{-N}^{N} \\
&- \frac{i}{2N}\arctan\left(\frac{x - r\cos\theta}{r\sin\theta}\right)\Bigg|_{-N}^{N} \\
&= \frac{-1}{2N}\log\left|\frac{N-z}{N+z}\right| - \frac{i}{2N}\arctan\left(\frac{N - r\cos\theta}{r\sin\theta}\right) \\
&+ \frac{i}{2N}\arctan\left(\frac{-N - r\cos\theta}{r\sin\theta}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\sum_{j:|X_j|\le N}\frac{1}{z-X_j}\middle| \mathcal{R}_N\right] =& \mathcal{R}_N\left[\frac{-1}{2N}\log\left|\frac{N-z}{N+z}\right| - \frac{i}{2N}\arctan\left(\frac{N - r\cos\theta}{r\sin\theta}\right)\right. \\
&\left. + \frac{i}{2N}\arctan\left(\frac{-N - r\cos\theta}{r\sin\theta}\right)\right] \\
\implies \mathbb{E}\left[\sum_{j:|X_j|\le N}\frac{1}{z-X_j}\right] =& -\log\left|\frac{N-z}{N+z}\right| - i\arctan\left(\frac{N - r\cos\theta}{r\sin\theta}\right) \\
&+ i\arctan\left(\frac{-N - r\cos\theta}{r\sin\theta}\right)
\end{aligned}$$

since, $\mathcal{R}_N = \text{Poisson}(2N)$. Taking $N \to \infty$, we get,

$$\mathbb{E}\left[\sum_{*}\frac{1}{z-X_j}\right] = -\pi i,$$

for z in the upper half plane, and,

$$\mathbb{E}\left[\sum_{*} \frac{1}{z - X_j}\right] = \pi i,$$

for z in the lower half plane. $\square$

Lemma 4.3.2. *For any $m \in \mathbb{N}$ with $m \geq 2$,*

$$\mathbb{E}\left[\sum_{*} \frac{1}{(z - X_j)^m}\right] = 0, \forall z \in \mathbb{C}.$$

Proof.

$$\begin{aligned}\mathbb{E}\left[\frac{1}{(z - X_j)^m}\middle| N, \mathcal{R}_N\right] &= \frac{1}{2N}\int_{x\in[-N,N]} \frac{1}{(z-x)^m}dx \\ &= \frac{1}{2N}\cdot\frac{1}{m-1}\left\{\frac{1}{(z-N)^{m-1}} - \frac{1}{(N+z)^{m-1}}\right\}.\end{aligned}$$

Then, noting that $\mathcal{R}_N$ has Poisson distribution with mean $2N$,

$$\begin{aligned}&\mathbb{E}\left[\sum_{j:|X_j|\leq N} \frac{1}{(z - X_j)^m}\middle| \mathcal{R}_N\right] = \frac{\mathcal{R}_N}{2N(m-1)}\left\{\frac{1}{(z-N)^{m-1}} - \frac{1}{(N+z)^{m-1}}\right\}, \\ \implies &\mathbb{E}\left[\sum_{j:|X_j|\leq N} \frac{1}{(z - X_j)^m}\right] = \frac{1}{m-1}\left\{\frac{1}{(z-N)^{m-1}} - \frac{1}{(N+z)^{m-1}}\right\}.\end{aligned}$$

Thus,

$$\mathbb{E}\left[\frac{1}{(z - X_j)^m}\right] = \lim_{N\to\infty}\frac{1}{m-1}\left\{\frac{1}{(z-N)^{m-1}} - \frac{1}{(N+z)^{m-1}}\right\} = 0.$$

$\square$

Lemma 4.3.3.

$$\mathbb{E}\left[\sum_{*} \frac{1}{|z - X_j|^2}\right] = \frac{\pi}{|Im(z)|}, \forall z \in \mathbb{C}.$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\left.\frac{1}{|z-X_j|^2}\right| N, \mathcal{R}_N\right] &= \frac{1}{2N}\int_{x\in[-N,N]} \frac{1}{(z-x)\cdot(\bar{z}-x)}dx \\
&= \frac{1}{2N(\bar{z}-z)}\left[\int_{x\in[-N,N]}\frac{1}{z-x}dx - \int_{x\in[-N,N]}\frac{1}{\bar{z}-x}dx\right] \\
&= \frac{1}{\bar{z}-z}\left\{\mathbb{E}\left[\left.\frac{1}{z-X_j}\right| N, \mathcal{R}_N\right] - \mathbb{E}\left[\left.\frac{1}{\bar{z}-X_j}\right| N, \mathcal{R}_N\right]\right\}.
\end{aligned}$$

Thus,

$$\mathbb{E}\left[\sum_* \frac{1}{|z-X_j|^2}\right] = \frac{1}{\bar{z}-z}\left\{\mathbb{E}\left[\sum_* \frac{1}{z-X_j}\right] - \mathbb{E}\left[\sum_* \frac{1}{\bar{z}-X_j}\right]\right\}.$$

So, using Lemma 4.3.1, if z is in the upper half plane,

$$\mathbb{E}\left[\sum_* \frac{1}{|z-X_j|^2}\right] = \frac{\pi}{Im(z)},$$

and if z is in the lower half plane,

$$\mathbb{E}\left[\sum_* \frac{1}{|z-X_j|^2}\right] = -\frac{\pi}{Im(z)},$$

which proves the result.

□

Lemma 4.3.4.

$$\mathbb{E}\left[\sum_* \frac{1}{|z-X_j|^4}\right] = \frac{\pi}{|Im(z)|^3}, \quad \textit{and,}$$

$$\mathbb{E}\left[\frac{1}{|z-X_j|^6}\right] = \frac{3\pi}{4|Im(z)|^5}.$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|z-X_j|^4}\middle| N, \mathcal{R}_N\right] &= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{|z-x|^4}dx \\
&= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{((z-x)(\bar{z}-x))^2}dx \\
&= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{(z-\bar{z})^2}\left(\frac{1}{z-x}-\frac{1}{\bar{z}-x}\right)^2 dx \\
&= \frac{1}{2N(z-\bar{z})^2}\int_{-N}^{N}\left(\frac{1}{(z-x)^2}+\frac{1}{(\bar{z}-x)^2}-\frac{2}{|z-x|^2}\right)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|z-X_j|^4}\middle| N, \mathcal{R}_N\right] =& \frac{1}{(z-\bar{z})^2}\left[\mathbb{E}\left(\frac{1}{(z-X_j)^2}\middle| N, \mathcal{R}_N\right)+\mathbb{E}\left(\frac{1}{(\bar{z}-X_j)^2}\middle| N, \mathcal{R}_N\right)\right. \\
&\left. -2\mathbb{E}\left(\frac{1}{|z-X_j|^2}\middle| N, \mathcal{R}_N\right)\right] \\
\implies \mathbb{E}\left[\frac{1}{|z-X_j|^4}\right] =& \frac{1}{(z-\bar{z})^2}\left[\mathbb{E}\left(\frac{1}{(z-X_j)^2}\right)+\mathbb{E}\left(\frac{1}{(\bar{z}-X_j)^2}\right)\right. \\
&\left. -2\mathbb{E}\left(\frac{1}{|z-X_j|^2}\right)\right].
\end{aligned}$$

From Lemmas 4.3.2 and 4.3.3,

$$\mathbb{E}\left[\frac{1}{|z-X_j|^4}\right] = \frac{\pi}{|Im(z)|^3}.$$

Next,

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{|z-X_j|^6}\middle| N,\mathcal{R}_N\right] &= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{|z-x|^6}dx \\
&= \frac{1}{2N}\int_{x\in[-N,N]}\frac{1}{((z-x)(\bar{z}-x))^3}dx \\
&= \frac{1}{2N(\bar{z}-z)^3}\int_{x\in[-N,N]}\left[\frac{1}{(z-x)^3}-\frac{1}{(\bar{z}-x)^3}\right. \\
&\left. -\frac{3}{(z-x)^2(\bar{z}-x)}+\frac{3}{(z-x)(\bar{z}-x)^2}\right]dx \\
&= \frac{1}{2N(\bar{z}-z)^3}\int_{-N}^{N}\left[\frac{1}{(z-x)^3}-\frac{1}{(\bar{z}-x)^3}-\frac{3(\bar{z}-z)}{|z-x|^4}\right]dx.
\end{aligned}
$$

Therefore, using Lemmas 4.3.2 and 4.3.3,

$$
\mathbb{E}\left[\frac{1}{|z-x_j|^6}\right] = \frac{3\pi}{4|Im(z)|^5}.
$$

□

4.3.2 The function ϕ_k and its derivatives

We have,

$$
\begin{aligned}
\phi_k(z) &= \log\left(\frac{f(z)}{z^k}\right) \\
&= \sum_{*}\log\left(1-\frac{z}{X_j}\right) - k\log z. \\
\implies \phi_k'(z) &= \sum_{*}\frac{1}{z-X_j}-\frac{k}{z}.
\end{aligned}
$$

Let us make a change of variables by assigning $y = \frac{z}{k}$, so that the expression for $\phi_k(z)$ becomes

$$\sum_{*} \frac{1}{ky - X_j} - \frac{1}{y}.$$

Write,

$$h_k(y) := \sum_{*} \frac{1}{ky - X_j}.$$

Lemma 4.3.5. *Let K be a compact subset of $\mathbb{C}$ that is either contained entirely in the upper-half plane or entirely in th lower-half plane. Then, there exists an $M > 0$ corresponding to K such that, for every $\delta > 0$, there exists $k_0 \in \mathbb{N}$ for which, $\mathbb{P}(\sup_{y \in K} |h_k'(y)| \leq M) > 1 - \delta, \forall k \geq k_0$. Thus, with high probability, for k sufficiently large, h_k' is uniformly bounded on compact subsets of the upper half plane and compact subsets of the lower half plane.*

Consequently, h_k and its higher derivatives are Lipschitz on compact subsets of the upper half plane and compact subsets of the lower half plane.

Proof. Let $y \in K$ and $l = \min_K Im(y)$. We have,

$$\begin{aligned} h_k'(y) &= \sum_{*} \frac{-k}{(ky - X_j)^2}, \\ \implies |h_k'(y)| &\leq k \sum_{*} \frac{1}{|ky - X_j|^2} \\ &\leq k \sum_{*} \frac{1}{(kRe(y) - X_j)^2 + k^2 Im(y)^2} \\ &\leq k \sum_{*} \frac{1}{(kRe(y) - X_j)^2 + k^2 l^2} = k(S_1 + S_2), \end{aligned}$$

where

$$S_1 = \sum_{j:|X_j| \leq k|Re(y)|} \frac{1}{(kRe(y) - X_j)^2 + k^2l^2}, \text{ and,}$$

$$S_2 = \sum_{j:|X_j| > k|Re(y)|} \frac{1}{(kRe(y) - X_j)^2 + k^2l^2}.$$

Let, $\mathcal{R}_{kL}$ denote the number of poisson points between $-kL$ and kL, where $L = \max_K Re(y)$. Then,

$$S_1 \leq \frac{\mathcal{R}_{kL}}{k^2l^2}.$$

Now, since $\mathcal{R}_k L$ has Poisson distribution with mean $2kL$, there exists k_1 such that, $\forall k \geq k_1$,

$$\begin{aligned}
\mathbb{P}(\mathcal{R}_{kL} > 8kL) &= \sum_{m=\lfloor 8kL \rfloor + 1}^{\infty} e^{-2kL} \frac{(2kL)^m}{m!}, \\
&\leq 2 \sum_{m=\lfloor 8kL \rfloor + 1}^{\infty} e^{-2kL} \frac{(2kL)^m}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} \\
&\leq 2e^{-2kL} \sum_{m=\lfloor 8kL \rfloor + 1}^{\infty} e^m \frac{1}{4^m} \\
&\leq 2e^{-2kL} \left(\frac{e}{4}\right)^{\lfloor 8kL \rfloor} \leq 2e^{-2kL} < \frac{\delta}{2},
\end{aligned}$$

where the second inequality above follows from Stirling's formula. If we write A_{kL} to be the event $\{\mathcal{R}_{kL} \leq 8kL\}$, we get that on $A_k L$,

$$S_1 \leq \frac{8kL}{k^2l^2} = \frac{8}{kL}.$$

Next, let I_m denote the interval $\left(k|Re(y)| + k^{\frac{m(m-1)}{2}}, k|Re(y)| + k^{\frac{m(m+1)}{2}}\right]$. Then, $|X_j| > k|Re(y)| \Leftrightarrow |X_j| \in \cup_{m \geq 1} I_m$. Write $\mathcal{R}_{km}$ to be the number of j such that

$|X_j| \in I_m$. Then, $\mathcal{R}_{km}$ has Poisson distribution with mean $2km$. Following similar computations as before, there exists k_2 such that, $\forall k \geq k_2$ and $\forall m \geq 1$,

$$\mathbb{P}(\mathcal{R}_{km} > 8km) \leq 2e^{-2km}.$$

Thus,

$$\mathbb{P}(\mathcal{R}_{km} > 8km \text{ for at least one } m \geq 1) \leq 2\sum_{m=1}^{\infty} e^{-2km} \leq 2e^{-2k}.$$

Hence, there exists k_3, such that, $\forall k \geq k_3$,

$$\mathbb{P}(\mathcal{R}_{km} > 8km \text{ for at least one } m \geq 1) < \frac{\delta}{2}.$$

If we write B_k to be the event, $\{\mathcal{R}_{km} \leq 8km, \forall m \geq 1\}$, then, on B_k,

$$\begin{aligned} S_2 &= \sum_{m=1}^{\infty} \sum_{j:|X_j| \in I_m} \frac{1}{(kRe(y) - X_j)^2 + k^2 l^2}, \\ &\leq \sum_{m=1}^{\infty} \frac{\mathcal{R}_{km}}{k^2 \frac{m^2(m-1)^2}{4} + k^2 l^2}, \\ &\leq \frac{8}{k} \sum_{m=1}^{\infty} \frac{1}{\frac{m^2(m-1)^2}{4} + l^2} = \frac{8s}{l}, \end{aligned}$$

where s is an infinite sum that converges and is independent of k.

Therefore, writing $k_0 = \max k_1, k_3$, we have, $\forall k \geq k_0$, on $A_{kL} \cap B_k$,

$$\begin{aligned} |h_k'(y)| &\leq k \left(\frac{8}{kL} + \frac{8s}{k} \right) = \frac{8(1 + Ls)}{L}, \\ \implies \sup_{y \in K} &\leq \frac{8(1 + Ls)}{L}, \end{aligned}$$

where, we note that $\mathbb{P}(A_{kL} \cap B_k) \geq 1 - \frac{\delta}{2} + 1 - \frac{\delta}{2} - 1 = 1 - \delta$, which proves the first part of this lemma.

Now, write $c(K)$ to be the convex hull of K. Then $c(K)$ is also a compact set that is contained entirely either in the upper half plane or the lower half plane. Now, using Mean value theorem, we know that, given any two points $y_1, y_2 \in K$, there exists a point w in the line segment joining y_1 and y_2 such that,

$$|h_k(y_1) - h_k(y_2)| \leq |y_1 - y_2||h_k'(w)|.$$

Since $y_1, y_2, w \in c(K)$, then applying the first part of the lemma to $c(K)$, we have that, given any $\delta > 0$, there exists $k_0 \geq 1$ such that $\forall k \geq k_0$,

$$\mathbb{P}\left(|h_k(y_1) - h_k(y_2)| \leq \frac{8(1+Ls)}{L}|y_1 - y_2|\right) > 1 - \delta.$$

Finally, to show that higher derivatives of h_k are also uniformly Lipschitz on compact subsets of the upper half plane or lower half plane, with high probability, for sufficiently large k, we simply use the Cauchy's integral formula. Given K, there exists $r > 0$, such that $K_r = \{z : \exists y \in K$ such that $|z - y| \leq r$ is also a compact set that is contained entirely either in the upper half plane or in the lower half plane. So, we can apply the uniform Lipschitz condition to K_r. For $p \geq 1$ and any $y_1, y_2 \in K$,

$$h_k^{(p)}(y_1) - h_k^{(p)}(y_2) = \frac{p!}{2\pi i}\int_{|w-y_1|=r} \frac{h_k(w)}{(w-y_1)^{p+1}}dw - \frac{p!}{2\pi i}\int_{|w-y_2|=r} \frac{h_k(w)}{(w-y_1)^{p+1}}dw.$$

If we make the substitution, $w = y_1 + re^{it}$ in the first integral, and $w = y_2 + re^{it}$ in the second, we get,

$$h_k^{(p)}(y_1) - h_k^{(p)}(y_2) = \frac{p!}{2\pi}\int_0^{2\pi} \frac{h_k(y_1 + re^{it})}{r^p e^{ipt}} dt - \frac{p!}{2\pi}\int_0^{2\pi} \frac{h_k(y_2 + re^{it})}{r^p e^{ipt}} dt$$
$$= \frac{p!}{2\pi r^p}\int_0^{2\pi} \frac{h_k(y_1 + re^{it}) - h_k(y_2 + re^{it})}{e^{ipt}} dt.$$

Thus,

$$|h_k^{(p)}(y_1) - h_k^{(p)}(y_2)| \leq \frac{p!}{2\pi r^p}\int_0^{2\pi} |h_k(y_1 + re^{it}) - h_k(y_2 + re^{it})| dt$$
$$\leq \frac{p!}{r^p} M|y_1 - y_2|,$$

with probability $> 1 - \delta$ for $k \geq k_0$.

□

Lemma 4.3.6. *Let B_k be a ball in $\mathbb{C}$ such that its closure, $\overline{B_k}$ lies entirely in the upper half plane and the radius of B_k is $k^{-1/2+\delta}$, where $0 < \delta < \frac{1}{2}$. Then, there exists $\eta_0 > 0$ such that, for all $\eta > \eta_0$, $\mathbb{P}(\sup_{z \in \overline{B_k}} |h_k(z) + i\pi| > k^{-1/2+\eta}) \longrightarrow 0$ as $k \to \infty$.*

In particular, if B_k has radius $k^{-1/3}$, then the above condition is satisfied for $\eta_0 = \frac{1}{12}$.

Proof. By Lemma 4.3.1,

$$\mathbb{E}h_k(y) = -\pi i,$$

if y is in the upper half plane, and,

$$\mathbb{E}h_k(y) = \pi i,$$

if y is in the lower half plane. Also, by Lemma 4.3.3,

$$Var|h_k(y)| \leq \mathbb{E}|h_k(y)|^2 = \frac{\pi}{|Im(ky)|} = \frac{1}{k} \cdot \frac{\pi}{|Im(y)|}.$$

$$\implies \sup_{y \in B_k} Var|h_k(y)| \leq \frac{2\pi^2}{k}.$$

By Cauchy Schwartz's inequality, if $y \in B_k$,

$$\mathbb{P}\left(|h_k(y) + i\pi| \geq \frac{1}{2} k^{-\frac{1}{2}+\eta} \right) \leq \frac{8\pi^2}{k} \cdot k^{1-2\eta} = 8\pi^2 k^{-2\eta}.$$

Now, let E_k be the event $\{\omega : \sup_{y \in \overline{B_k}} |h_k(y)(\omega) + i\pi| > k^{-1/2+\eta}\}$. Since $\overline{B_k}$ is compact, the supremum of $|h_k(y)(\omega) + i\pi|$ is attained at, say, $y_k \in B_k$. Thus, using the fact that h_k is uniformly Lipschitz over compact sets, we get that, for k sufficiently large, for all $\omega \in E_k$,

$$|h_k(y)(\omega) + i\pi| > \frac{k^{-1/2+\eta}}{2},$$

for all $y \in B_k$ that are at a distance $k^{-1/2+\eta-\epsilon}$ from the center of the ball, where $\eta > 0$ is chosen so that $\eta - \epsilon < \delta$. Thus,

$$\mathbf{m}\left\{ z \in \overline{B_k} : |h_k(y) + i\pi| > \frac{k^{-1/2+\eta}}{2} \right\} \geq \pi k^{-1+2\eta-2\epsilon}, \forall \omega \in E_k$$

$$\implies \mathbb{E}\mathbf{m}\left\{ z \in \overline{B_k} : |h_k(y) + i\pi| > \frac{k^{-1/2+\eta}}{2} \right\} \geq \mathbb{P}(E_k) \cdot \pi k^{-1+2\eta-2\epsilon}.$$

Also, using Fubini's theorem,

$$\mathbb{E}\mathbf{m}\left\{ z \in \overline{B_k} : |h_k(y) + i\pi| > \frac{k^{-1/2+\eta}}{2} \right\} = \int_{\overline{B_k}} \mathbb{P}\left(|h_k(y) + i\pi| > \frac{k^{-1/2+\eta}}{2} \right) d\mathbf{m},$$

$$\leq \pi k^{-1+2\delta} \cdot 8\pi^2 k^{-2\eta} = 8\pi^3 k^{-1+2\delta-2\eta}.$$

Therefore,

$$\mathbb{P}(E_k) \leq 8\pi^2 k^{2(\delta-2\eta+\epsilon)}.$$

Thus, we shall be done if we choose ϵ small enough and η_0 large enough so that $\delta - 2\eta + \epsilon > 0, \forall \eta > \eta_0$.

In particular, when the ball has radius $k^{-1/3}$, we find that all of the above holds true for $\eta > \frac{1}{12}$ and $\epsilon < \frac{1}{66}$. Thus, in this case,

$$\mathbb{P}\left(\sup_{y\in\overline{B_k}} |h_k(y) + i\pi| > k^{-1/2+1/11}\right) \longrightarrow 0, \text{ as, } k \to \infty.$$

□

Lemma 4.3.7. *With probability* $\longrightarrow 1$*, the equation*

$$h_k(y) - \frac{1}{y} = 0$$

has a solution in the upper half plane of the form $\frac{i}{\pi} + o(k^{-1/2+1/11})$ *and a solution in the lower half plane of the form* $\frac{-i}{\pi} + o(k^{-1/2+1/11})$.

Proof. We shall first show that a solution to the equation $h_k(y) - \frac{1}{y} = 0$ exists inside a ball centered at i/π of radius $k^{-1/3}$, and then demonstrate that this solution must be of the form $\frac{i}{\pi} + o(k^{-1/2+1/11})$. The proof for a solution of the form $\frac{-i}{\pi} + o(k^{-1/2+1/11})$ is similar.

From Lemma 4.3.6, we know that

$$\mathbb{P}\left(\sup_{y:\left|y-\frac{i}{\pi}\right|\leq k^{-1/3}} |h_k(y) + i\pi| \leq k^{-1/2+1/11}\right) \longrightarrow 1, \text{ as, } k \to \infty.$$

Writing A_k as the event $\{\omega : \sup_{y:\left|y-\frac{i}{\pi}\right|\leq k^{-1/3}} |h_k(y)(\omega) + i\pi| \leq k^{-1/2+1/11}\}$, we have that, $\forall \omega \in A_k$, and all y such that $|y - \frac{i}{\pi}| = k^{-1/3}$,

$$\begin{aligned}\left|\left(h_k(y)(\omega) - \frac{1}{y}\right) - \left(-i\pi - \frac{1}{y}\right)\right| &= |h_k(y)(\omega) + i\pi| \\ &\leq k^{-1/2+1/11} < \left|-i\pi - \frac{1}{y}\right|,\end{aligned}$$

for k sufficiently large. Thus, by Rouche's theorem, $h_k(y)(\omega) - \frac{1}{y}$ and $-i\pi - \frac{1}{y}$ have the same number of zeros inside the disc centered at i/π of radius $k^{-1/3}$. Since $-i\pi - \frac{1}{y}$ has exactly one zero, namely i/π, this implies that $h_k(y)(\omega) - \frac{1}{y}$ has exactly one zero as well.

Now, let y_k denote this solution in the upper half plane. Then, by Lemma 4.3.6,

$$h_k(y_k) = -\pi i \left(1 + c_k k^{-1/2+1/11}\right),$$

where $c_k \xrightarrow{P} 0$. Since $y_k = \frac{1}{h_k(y_k)}$,

$$y_k = \frac{i}{\pi}\left(1 + c_k k^{-1/2+1/11}\right)^{-1}.$$

On choosing k large enough so that $M \leq \sqrt{k}$,

$$\begin{aligned}y_k &= \frac{i}{\pi}\left(1 - c_k k^{-1/2+1/11} + c_k^2 k^{-1+2/11} - c_k^3 k^{-3/2+3/11} + ...\right) \\ &= \frac{i}{\pi} + o\left(k^{-1/2+1/11}\right).\end{aligned}$$

The solution in the lower half plane is obtained by simply taking conjugates in the above equations. Thus, as $P(A_k) \longrightarrow 1$, with probability $\longrightarrow 1$, the two solutions to the equation

$$h_k(y) - \frac{1}{y} = 0$$

are y_k and $\overline{y_k}$, where

$$y_k = \frac{i}{\pi} + o\left(k^{-1/2+1/11}\right).$$

$\square$

We shall write $\sigma_k = ky_k = k\left(\frac{i}{\pi} + o\left(k^{-1/2+1/11}\right)\right)$. Thus, from the above lemma, with probability $\longrightarrow 1$, $\phi_k'(\sigma_k) = 0 = \phi_k'(\overline{\sigma_k})$. We now try to get estimates on the higher derivatives of ϕ_k.

Lemma 4.3.8. *For any* $r \leq \frac{1}{2} - \frac{1}{11}$,

$$k^r\left(\frac{\sigma_k^2\phi_k^{(2)}(\sigma_k)}{k} - 1\right) \xrightarrow{P} 0,$$

and,

$$k^r\left(\frac{\overline{\sigma_k}^2\phi_k^{(2)}(\overline{\sigma_k})}{k} - 1\right) \xrightarrow{P} 0.$$

Proof. We have,

$$\begin{aligned}
\phi_k^{(2)}(ky) &= \sum_{*} \frac{-1}{(ky - X_j)^2} + \frac{k}{k^2y^2}, \\
\implies \frac{k^2y^2\phi_k^{(2)}(ky)}{k} - 1 &= y^2h_k'(y) \\
\implies \frac{\sigma_k^2\phi_k^{(2)}(\sigma_k)}{k} - 1 &= y_k^2h_k'(y_k).
\end{aligned}$$

By Lemma 4.3.5, h_k' is Lipschitz in compact subsets of the upper half plane, and is hence Lipschitz in a ball around $\frac{i}{\pi}$ that lies entirely in the upper half plane. Now, from Lemmas 4.3.2 and 4.3.4 we have, for any fixed y in the upper half plane,

$$\mathbb{E}h_k'(y) = 0, \text{ and,}$$
$$Var(h_k'(y)) = \frac{\pi}{k|Im(y)|^3}.$$

Thus, by Cauchy-Schwartz Inequality,

$$\begin{aligned}\mathbb{P}\left(|h_k'(y)| > \frac{k^{-1/2+\delta}}{2}\right) &\leq \frac{\pi}{k|Im(y)|^3} \cdot 4k^{1-2\delta} \\ &\leq 4\pi^2 k^{-2\delta},\end{aligned}$$

when $y = i/\pi$. So, using the fact that h_k' is Lipschitz, we get,

$$\begin{aligned}\mathbb{P}\left(|h_k'(y_k)| > k^{-1/2+\delta}\right) &\leq \mathbb{P}\left(|h_k'(y)| > \frac{k^{-1/2+\delta}}{2}\right) \\ &\quad + \mathbb{P}\left(\left|h_k'(y_k) - h_k'\left(\frac{i}{\pi}\right)\right| > \frac{k^{-1/2+\delta}}{2}\right) \\ &\leq 4\pi^2 k^{-2\delta} + \mathbb{P}\left(\left|y_k - \frac{i}{\pi}\right| > M\frac{k^{-1/2+\delta}}{2}\right),\end{aligned}$$

M being a constant. Note that $y_k - \frac{i}{\pi} = o(k^{-1/2+1/11})$. Therefore, if we take $\delta = 1/11$, the right hand side of the inequality above goes to 0, which implies that $h_k'(y_k) = o(k^{-1/2+1/11})$. Thus, for any $r \leq \frac{1}{2} - \frac{1}{11}$,

$$k^r \left(\frac{\sigma_k^2 \phi_k^{(2)}(\sigma_k)}{k} - 1\right) \xrightarrow{P} 0.$$

The result for $\overline{\sigma_k}$ is obtained by just taking conjugates.

□

4.4 Evaluating the Cauchy's Integral expression for e_k

As seen in equation (4.4.1), the elementary symmetric polynomials of $\frac{1}{X_j}'s$ can be written as

$$(-1)^k e_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z^{k+1}} dz, \tag{4.4.1}$$

where Γ is a simple continuous loop around the origin. Let us take Γ to be the circle centered at the origin, with radius $|\sigma_k|$. Thus it passes through both σ_k and $\overline{\sigma_k}$.

Let us take Γ_1 to be an arc of Γ that passes through σ_k and extends to an angle of $k^{-\delta}$ on both sides, where δ lies between $\frac{1}{3}$ and $\frac{1}{4}$ (thus, $\delta < \frac{1}{2} - \frac{1}{11}$). Let $\overline{\Gamma_1}$ be the arc that has points that are conjugate to Γ_1. Finally, let Γ_2 are Γ_2' denote the remainder arcs on the left and right respectively, so that Γ_1, $\overline{\Gamma_1}, \Gamma_2$ and Γ_2' together complete the full circle Γ.

4.4.1 Evaluating the Cauchy integral over the "nice arcs"

In this subsection, we will give approximations of the integral in (4.4.1) over the arcs Γ_1 and $\overline{\Gamma_1}$. Define

$$g_k(t) := \phi_k(\sigma_k e^{it}), t \in [-k^{-\delta}, k^{-\delta}].$$

Lemma 4.4.1.

$$\sqrt{k}\frac{\int_{\Gamma_1} \frac{f(z)}{z^{k+1}}dz}{\exp(g_k(0))} - i\sqrt{2}\gamma\left(\frac{1}{2}, \frac{k^{1-2\delta}}{2}\right) dt \xrightarrow{P} 0, \;\; as, \; k \to \infty,$$

where $\gamma(x, y)$ represents the lower incomplete gamma function.

Proof. Note that g_k is continuous over $[-k^{-\delta}, k^{-\delta}]$ and infinitely differentiable over $(-k^{-\delta}, k^{-\delta})$. The Taylor's expansion of $g_k(t)$ gives us,

$$g_k(t) = g_k(0) + tg_k'(0) + \frac{t^2}{2}g_k^{(2)}(0) + \frac{t^3}{6}\left(Reg_k^{(3)}(t_1) + iImg_k^{(3)}(t_2)\right),$$

where t_1 and t_2 are points that lie between 0 and t.

Now, $g_k'(0) = i\sigma_k\phi_k'(\sigma_k) = 0$. Also, $g_k^{(2)}(0) = -\sigma_k^2\phi_k^{(2)}(\sigma_k)$. Thus, be Lemma 4.3.8, for any $r \leq \frac{1}{2} - \frac{1}{11}$,

$$k^r\left(\frac{g_k'(0)}{k} + 1\right) \xrightarrow{P} 0.$$

Thus,

$$\begin{aligned}
&\sup_{|t|\leq k^{-\delta}} \sqrt{k}\left[\exp\left(\frac{t^2}{2}g_k^{(2)}(0)\right) - \exp\left(-\frac{kt^2}{2}\right)\right] \\
&\qquad = \sup_{|t|\leq k^{-\delta}} \sqrt{k}\exp\left(-\frac{kt^2}{2}\right)\exp\left(\frac{kt^2}{2}\left(\frac{g_k^{(2)}(0)}{k} + 1\right)\right) \\
&\qquad \xrightarrow{P} 0.
\end{aligned}$$

Next, note that for any $t \in (-k^{-\delta}, k^{-\delta})$,

$$g_k^{(3)}(t) = -i\sigma_k e^{it}\phi_k'(\sigma_k e^{it}) - 3i\sigma_k^2 e^{2it}\phi_K^{(2)}(\sigma_k e^{it}) - i\sigma_k^3 e^{3it}\phi_k^{(2)}(\sigma_k e^{it})$$
$$= -ik\left[y_k e^{it}h_k(y_k e^{it}) + 3y_k^2 e^{2it}h_k'(y_k e^{it}) + y_k^3 e^{3it}h_k^{(2)}(y_k e^{it})\right].$$

Now, since $h_k(y_k) = 0$, using the Lipschitz condition on h_k we get that $|h_k(y_k e^{it})| \leq M|t| \leq Mk^{-\delta}$, where M is a constant. Similarly, since h_k' is Lipschitz near i/π, and $h_k'(y_k) = o(k^{-1/2+1/11})$, we get that $|h_k'(y_k e^{it})| \leq o(k^{-1/2+1/11}) + O(k^{-\delta}) = O(k^{-\delta})$, uniformly in probability.

Next, from Lemma 4.3.4,

$$\mathbb{E}h_k^{(2)}\left(\frac{i}{\pi}\right) = 0, \text{ and,}$$
$$Var h_k^{(2)}\left(\frac{i}{\pi}\right) = \frac{3\pi^2}{4k}.$$

Thus, using the same methods as in the proof of Lemma 4.3.8 and above, and the fact that $h_k^{(3)}$ is also Lipschitz near $\frac{i}{\pi}$, we get, $h_k^{(2)}(y_k e^{it}) = O(k^{-\delta})$, uniformly in probability. Thus,

$$\sup_{|t|\leq k^{-\delta}} \frac{t^3}{6} g_k^{(3)}(t) \xrightarrow{P} 0.$$

Now,

$$\int_{\Gamma_1} \frac{f(z)}{z^{k+1}}dz = i\int_{-k^{-\delta}}^{k^{-\delta}} \exp\left[g_k(0) + \frac{t^2}{2}g_k^{(2)}(0) + \frac{t^3}{6}\left(Reg_k^{(3)}(t_1) + iImg_{k,N}^{(3)}(t_2)\right)\right] dt.$$

So, stitching everything together, we get,

$$\sqrt{k}\frac{\int_{\Gamma_1}\frac{f(z)}{z^{k+1}}dz}{\exp(g_k(0))} - i\sqrt{k}\int_{-k^{-\delta}}^{k^{-\delta}} \exp\left(-\frac{kt^2}{2}\right) dt \xrightarrow{P} 0.$$

But,

$$\int_{-k^{-\delta}}^{k^{-\delta}} \exp\left(-\frac{kt^2}{2}\right) dt = 2\int_0^{k^{-\delta}} \exp\left(-\frac{kt^2}{2}\right) dt$$
$$= \frac{1}{\sqrt{2k}} \int_0^{k^{1-2\delta}/2} \frac{e^{-x}}{\sqrt{x}} dx, \text{ where, } x = -\frac{kt^2}{2},$$
$$= \frac{1}{\sqrt{2k}} \gamma\left(\frac{1}{2}, \frac{k^{1-2\delta}}{2}\right).$$

Thus,

$$\sqrt{k}\frac{\int_{\Gamma_1} \frac{f(z)}{z^{k+1}} dz}{\exp(g_k(0))} - i\sqrt{2}\gamma\left(\frac{1}{2}, \frac{k^{1-2\delta}}{2}\right) dt \xrightarrow{P} 0.$$

□

Corollary 4.4.2.

$$\sqrt{k}\frac{\overline{\int_{\overline{\Gamma_1}} \frac{f(z)}{z^{k+1}} dz}}{\exp(g_k(0))} - i\sqrt{2}\gamma\left(\frac{1}{2}, \frac{k^{1-2\delta}}{2}\right) dt \xrightarrow{P} 0, \ \ as, \ k \to \infty.$$

Proof. The proof of this is direct - by just taking conjugates in the statement of the above lemma. □

4.4.2 Evaluating the Cauchy integral over the "bad arcs"

In this section, we shall show that

$$\frac{1}{\frac{f(\sigma_k)}{\sigma_k^k}} \int_{\Gamma_2} \frac{f(z)}{z^{k+1}} dz \xrightarrow{P} 0, \text{ and,}$$
$$\frac{1}{\frac{f(\sigma_k)}{\sigma_k^k}} \int_{\Gamma_2'} \frac{f(z)}{z^{k+1}} dz \xrightarrow{P} 0.$$

To do so, we first show that, when K is a subset of the circle Γ lies outside the "nice arcs" Γ_1 and $\overline{\Gamma_1}$, and is far away from the real line,

$$\sup_{z\in K}\left|\frac{f(z)}{f(\sigma_k)}\right| \xrightarrow{P} 0,$$

which automatically implies that integrating $\frac{1}{\frac{f(\sigma_k)}{\sigma_k^k}}\int_K \frac{f(z)}{z^{k+1}}dz \xrightarrow{P} 0$. Next we show that, this condition holds true even when K contains the axis (for this part we have given only a sketch of the proof - details shall be filled in later).

Lemma 4.4.3. *Let β be a fixed angle that is strictly less that $\frac{\pi}{2}$. Then,*

$$\sup_{k^{-\delta}\le t\le\beta}\left|\frac{f(\sigma_k e^{it})}{f(\sigma_k)}\right| \xrightarrow{P} 0, \text{ and,}$$
$$\sup_{k^{-\delta}\le t\le\beta}\left|\frac{f(\sigma_k e^{-it})}{f(\sigma_k)}\right| \xrightarrow{P} 0.$$

Proof. Define

$$\psi_k(t) := \left|\frac{f(\sigma_k e^{it})}{f(\sigma_k)}\right|^2, t\in[-\beta,-k^{-\delta}]\cup[k^\delta,\beta].$$

We can then write,

$$\begin{aligned}
\psi_k(t) &= \frac{f(\sigma_k e^{it})f(\bar{\sigma_k}e^{-it})}{|f(\sigma_k)|^2},\\
\implies \psi_k'(t) &= \frac{\sigma_k i e^{it} f'(\sigma_k e^{it})f(\bar{\sigma_k}e^{-it}) - \bar{\sigma_k} i e^{-it} f(\sigma_k e^{it})f'(\bar{\sigma_k}e^{-it})}{|f(\sigma_k)|^2},\\
&= i\psi_k(t)\left(\sigma_k e^{it}\frac{f'(\sigma_k e^{it})}{f(\sigma_k e^{it})} - \bar{\sigma_k}e^{-it}\frac{f'(\bar{\sigma_k}e^{-it})}{f(\bar{\sigma_k}e^{-it})}\right)\\
&= -2\cdot\psi_k(t)\cdot Im\left(\sigma_k e^{it}\frac{f'(\sigma_k e^{it})}{f(\sigma_k e^{it})}\right).
\end{aligned}$$

Thus, ψ_k' vanishes at $t=0,-2\arg(\sigma_k)$ and at other values of t for which

$$Im\left(\sigma_k e^{it}\frac{f'(\sigma_k e^{it})}{f(\sigma_k e^{it})}\right) = 0. \tag{4.4.2}$$

We shall now try to study the behavior of g and g' over $[-\beta, -k^{-\delta}] \cup [k^{-\delta}, \beta]$.

When $t \in [k^{-\delta}, \beta]$, we know that

$$\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)} = h_k\left(\frac{i}{\pi}e^{it}\right) = -i\pi + o(k^{-\delta}),$$

since $\delta < 1/2$. In fact,

$$\mathbb{P}\left(\left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)} + i\pi\right| \geq \frac{k^{-\delta}}{2}\right) \leq Ck^{-1+2\delta},$$

where C is a constant that depends on β. Then, letting E_k be the event

$$\left\{\omega : \sup_{t\in[k^{-\delta},\beta]} \left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)}(\omega) + i\pi\right| > k^{-\delta}\right\}.$$

Using the fact that h_k is uniformly Lipschitz over compact sets, we get that, for k sufficiently large, for all $\omega \in E_k$,

$$\mathbf{m}\left\{t \in [k^{-\delta}, \beta] : \left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)}(\omega) + i\pi\right| | > \frac{k^{-\delta}}{2}\right\} \geq \pi k^{-\delta},$$

$$\implies \mathbb{E}\mathbf{m}\left\{t \in [k^{-\delta}, \beta] : \left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)}(\omega) + i\pi\right| > \frac{k^{-\delta}}{2}\right\} \geq \mathbb{P}(E_k) \cdot \pi k^{-\delta}.$$

Also, using Fubini's theorem,

$$\begin{aligned}\mathbb{E}\mathbf{m}\left\{t \in [k^{-\delta}, \beta] : \left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)}(\omega) + i\pi\right| > \frac{k^{-\delta}}{2}\right\} \\ = \int_{k^{-\delta}}^{\beta} \mathbb{P}\left(\left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)}(\omega) + i\pi\right| > \frac{k^{-\delta}}{2}\right) d\mathbf{m}, \\ \leq (\beta - k^{-\delta}) \cdot Ck^{-1+2\delta}.\end{aligned}$$

Therefore,

$$\mathbb{P}(E_k) \leq C\beta k^{-1+3\delta} \longrightarrow 0.$$

Thus, with probability $\longrightarrow 1$, $\sup_{t\in[k^{-\delta},\beta]}\left|\frac{f'\left(k\frac{i}{\pi}e^{it}\right)}{f\left(k\frac{i}{\pi}e^{it}\right)}+i\pi\right| \leq k^{-\delta}$. Since $\frac{f'(ky)}{f(ky)} = h_k(y)$, which is uniformly Lipschitz in compact subsets of the upper half plane, this implies that,

$$\mathbb{P}\left(\sup_{t\in[k^{-\delta},\beta]}\left|\frac{f'\left(\sigma_k e^{it}\right)}{f\left(\sigma_k e^{it}\right)}+i\pi\right| \leq k^{-\delta}\right) \longrightarrow 1.$$

Next, we know that,

$$\begin{aligned}\sigma_k e^{it} &= k\left(\frac{i}{\pi}+o(k^{-1/2+1/11})\right)\cdot(\cos t+i\sin t),\ \text{in probability},\\ &= k\left[\left(\frac{-1}{\pi}+o(k^{-1/2+1/11})\right)\sin t+o(k^{-1/2+1/11})\cos t\right.\\ &\left.+i\left\{\left(\frac{1}{\pi}+o(k^{-1/2+1/11})\right)\cos t+o(k^{-1/2+1/11})\sin t\right\}\right].\end{aligned}$$

So,

$$Im\left(\sigma_k e^{it}\cdot\frac{f'(\sigma_k e^{it})}{f(\sigma_k e^{it})}\right) = k\left[\left(1+O(k^{-\delta})\right)\sin t+O(k^{-\delta})\cos t\right].$$

Since $t \geq k^{-\delta}$ in this case, $k\sin t$ is the dominant term here, and is > 0. Therefore, $\psi_k'(t) < 0$ here - that is, ψ_k is decreasing. Moreover, $\sup_{t\in[k^{-\delta},\beta]}\frac{\psi_k'(t)}{\psi(t)} \xrightarrow{P} -\infty$.

Similarly, if $t \in [-\beta, -k^{-\delta}]$,

$$Im\left(\sigma_k e^{it}\cdot\frac{f'(\sigma_k e^{it})}{f(\sigma_k e^{it})}\right) = k\left[\left(1+O(k^{-1/2})\right)\sin t+O(k^{-1/2})\cos t\right].$$

Since $t \leq -k^{-\delta}$ in this case, $k\sin t$ is the dominant term, and is < 0. Therefore, $\psi'(t) > 0$ here - that is, ψ_k is increasing. Also, $\inf_{t\in[-\beta,-k^{-\delta}]}\frac{\psi_k'(t)}{\psi_k(t)} \xrightarrow{P} -\infty$.

The rest of the lemma is now immediate from applying the following claim to $l_k(t) = \log g(t)$ on $[-\beta, -k^{-\delta}]$ and to $l_k(t) = -\log \psi_k(t)$ on $[k^{-\delta}, \beta]$, while noting that ψ_k is maximized at σ_k and $\overline{\sigma_k}$, where it takes the value 1 (meaning $\log \psi_k$ takes value 0).

Claim: Let $l_k(t), k \geq 1$ be a collection of functions defined on an interval (a, b), that are continuous and differentiable there. Suppose that $l_k'(t) > 0, \forall t \in (a, b)$ and that $\lim_{k\to\infty} \sup_{t\in(a,b)} l_k'(t) = \infty$. If there exists $M \in \mathbb{R}$ such that $l_k(t) \leq M, \forall t \in (a, b), \forall k$, then,

$$\lim_{k\to\infty} \inf_{t\in(a,b)} l_k(t) = -\infty, \forall t \in (a, b).$$

Similarly, if there exists $\in \mathbb{R}$ such that $l_k(t) \geq m, \forall t \in (a, b), \forall k$, then,

$$\lim_{k\to\infty} \sup_{t\in(a,b)} l_k(t) = +\infty, \forall t \in (a, b).$$

Proof of Claim: Look at the case with the upper bound first. Given $t \in (a, b)$, choose $t_0 \in (a, b)$, with $t_0 > t$. By the Mean value theorem, there exists $t_1 \in (t, t_0)$ such that,

$$l_k(t) = l_k(t_0) + (t - t_0)l_k'(t_1),$$

$$\leq M + (t - t_0)l_k'(t_1).$$

Taking limits $k \to \infty$ on the right hand side gives $-\infty$. Thus, $\lim_{k\to\infty} \inf_{t\in(a,b)} l_k(t) = -\infty, \forall t \in (a, b)$. The case with the lower bound follows similarly.

□

Lemma 4.4.4. *For any β lying strictly between 0 and $\pi/2$,*

$$\int_{t=\beta}^{\pi-\arg(\sigma_k)} \frac{f(\sigma_k e^{it})}{f(\sigma_k)} dt \longrightarrow 0, \ \textit{and,}$$
$$\int_{t=-\arg(\sigma_k)}^{-\beta} \frac{f(\sigma_k e^{it})}{f(\sigma_k)} dt \longrightarrow 0.$$

Proof. We give here a sketch of a proof:

Let K be a compact subset of the quadrant $\{z \in \mathbb{C} : Re(z) > 0 \text{ and } Im(z) > 0\}$. Let $h_k(z) = \sum_* \frac{1}{kz - X_j}$. We know that $h_k(z) = -\pi i + O(k^{-1/2})$ uniformly over K. Now, let $z \in K$ and $s, t \in [0, \pi/2)$ such that $t > s$ and $|z|e^{it} = z$. Writing $z_s = |z|e^{is}$, we have,

$$\begin{aligned} \frac{f(kz_s)}{f(kz)} &= \lim_{N\to\infty} \prod_{j:|x_j|\le N} \frac{\left(1 - \frac{kz_s}{x_j}\right)}{\left(1 - \frac{kz}{x_j}\right)} \\ &= \lim_{N\to\infty} \prod_{j:|x_j|\le N} \left(1 + \frac{kz - kz_s}{x_j - kz}\right). \end{aligned}$$

So, by Fatou's lemma,

$$\begin{aligned} \mathbb{E}\left|\frac{f(kz_s)}{f(kz)}\right|^2 &\le \lim_{N\to\infty} \mathbb{E} \prod_{j:|x_j|\le N} \left|1 + \frac{kz - kz_s}{x_j - kz}\right|^2 \\ &= \lim_{N\to\infty} \mathbb{E}\mathbb{E}\left[\prod_{j:|x_j|\le N} \left|1 + \frac{kz - kz_s}{x_j - kz}\right|^2 \middle| N, \mathcal{R}_N\right] \\ &= \lim_{N\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left|1 + \frac{kz - kz_s}{u - kz}\right|^2\right)^{\mathcal{R}_N}\right], \end{aligned}$$

where $\mathcal{R}_N \sim \text{Poisson}(2N)$, and $u \sim \text{Uniform}[-N, N]$.

Now,

$$\mathbb{E}\left|1+\frac{kz-kz_s}{u-kz}\right|^2 = 1+(kz-kz_s)\mathbb{E}\left(\frac{1}{u-kz}\right)+(\overline{kz}-\overline{kz_s})\mathbb{E}\left(\frac{1}{u-\overline{kz}}\right)$$
$$+|kz_s-kz|^2\mathbb{E}\frac{1}{|u-kz|^2}.$$

Note that, as z_s, z are in the upper half plane, by Lemmas 4.3.1 and 4.3.3,

$$2N\mathbb{E}\left(\frac{1}{u-kz}\right)\longrightarrow \pi i,$$
$$2N\mathbb{E}\frac{1}{|u-kz|^2}\longrightarrow\frac{\pi}{kIm(z)}.$$

So,

$$\begin{aligned}\mathbb{E}\left|\frac{f(kz_s)}{f(kz)}\right|^2 &\le \lim_{N\to\infty}\exp\Bigg(2N(kz-kz_s)\mathbb{E}\left(\frac{1}{u-kz}\right)+2N(\overline{kz}-\overline{kz_s})\mathbb{E}\left(\frac{1}{u-\overline{kz}}\right)\\ &\quad +2Nk^2|z_s-z|^2\mathbb{E}\frac{1}{|u-kz|^2}\Bigg)\\ &=\exp\left(k(z-z_s)\pi i-k(\overline{z}-\overline{z_s})\pi i+k\frac{\pi|z_s-z|^2}{Im(z)}\right)\\ &=\exp\left(-2k\pi Im(z)+2k\pi Im(z_s)+k\frac{\pi|z_s-z|^2}{Im(z)}\right)\\ &=\exp\left(k\pi\frac{-2Im(z)^2+2Im(z_t)Im(z_s)+|z_s-z|^2}{Im(z)}\right)\\ &=\exp\left(k\pi\frac{-2Im(z)^2+2|z|^2-2Re(z_t)Re(z_s)}{Im(z)}\right)\\ &=\exp\left(k\pi\frac{2Re(z)\{Re(z_t)-Re(z_s)\}}{Im(z)}\right).\end{aligned}$$

Note that, as $0\le s<t<\pi/2$ and $z\in K$, $Re(z_s)>Re(z)>0$ and $Im(z)>0$.

Therefore,

$$\mathbb{E}\left|\frac{f(kz_s)}{f(kz)}\right|^2\longrightarrow 0. \tag{4.4.3}$$

Thus, by Fatou's lemma,

$$\frac{f(kz_s)}{f(kz)} \xrightarrow{P} 0, \forall z \in K.$$

In fact, using Borel-Cantelli, we can even show that

$$\frac{f(kz_s)}{f(kz)} \xrightarrow{a.s.} 0, \forall z \in K. \tag{4.4.4}$$

In order to prove the lemma using this fact, we only need to show that the function $\frac{f(kz_s)}{f(kz)}$ is uniformly continuous, so that (4.4.4) will hold true for $z = y_k e^{it}$, where t is between $\pi/6$ and $\pi/3$. In that case, we can apply Lemma 4.4.3 with β equal to $\pi/3$ to show that

$$\int_{t=-\arg(\sigma_k)}^{-\beta} \frac{f(\sigma_k e^{it})}{f(\sigma_k)} dt \longrightarrow 0.$$

The proof for

$$\int_{t=\beta}^{\pi-\arg(\sigma_k)} \frac{f(\sigma_k e^{it})}{f(\sigma_k)} dt \longrightarrow 0,$$

is exactly the same, with K lying entirely in the quadrant $\{z \in \mathbb{C} : Re(z) < 0$ and $Im(z) > 0\}$.

□

4.4.3 An expression for $\sqrt{k} \cdot e_k$

As a consequence to the results in the previous sections, we obtain the following expression for e_k.

Proposition 4.4.5.

$$\sqrt{k} \cdot e_k = (-1)^k Re \left\{ \frac{f(\sigma_k)}{\sigma_k^k} \left(\sqrt{\frac{1}{2\pi}} + \mathcal{Y}_k \right) \right\},$$

where $\mathcal{Y}_k \xrightarrow{P} 0$.

Proof. We have,

$$\sqrt{k}(-1)^k e_k = \frac{\sqrt{k}}{2\pi i} \left(\int_{\Gamma_1} \frac{f(z)}{z^{k+1}} dz + \int_{\overline{\Gamma_1}} \frac{f(z)}{z^{k+1}} dz + \int_{\Gamma_2} \frac{f(z)}{z^{k+1}} dz + \int_{\Gamma_2'} \frac{f(z)}{z^{k+1}} dz \right).$$

Now, using Lemma 4.4.1 and Corollary 4.4.2, we have, $\sqrt{k} \cdot \frac{\sigma_k^k}{f(\sigma_k)} \cdot \frac{1}{i} \int_{\Gamma_1} \frac{f(z)}{z^{k+1}} dz$ equals,

$$\left[\sqrt{k} \frac{\int_{\Gamma_1} \frac{f(z)}{z^{k+1}} dz}{i \exp(g_k(0))} - \frac{1}{\sqrt{2}} \gamma \left(\frac{1}{2}, \frac{k^{1-2\delta}}{2} \right) \right] + \frac{1}{\sqrt{2}} \gamma \left(\frac{1}{2}, \frac{k^{1-2\delta}}{2} \right),$$

and, $\sqrt{k} \cdot \frac{\overline{\sigma_k}^k}{f_N(\overline{\sigma_k})} \cdot \frac{1}{i} \int_{\overline{\Gamma_1}} \frac{f(z)}{z^{k+1}} dz$ equals

$$\left[\sqrt{k} \frac{\int_{\overline{\Gamma_1}} \frac{f(z)}{z^{k+1}} dz}{i \exp(\overline{g_k(0)})} - \frac{1}{\sqrt{2}} \gamma \left(\frac{1}{2}, \frac{k^{1-2\delta}}{2} \right) \right] + \frac{1}{\sqrt{2}} \gamma \left(\frac{1}{2}, \frac{k^{1-2\delta}}{2} \right),$$

where the first terms in the $[-]$ brackets in the above equations converge to 0 as $k \to \infty$. Also, note that, as $\gamma(x, y)$ is the lower incomplete gamma function, and as $\delta < 1/2 - 1/11$, $\gamma \left(\frac{1}{2}, \frac{k^{1-2\delta}}{2} \right) \to \sqrt{\pi}$ as $k \to \infty$.

Therefore, applying Lemmas 4.4.3 and 4.4.4 to $\int_{\Gamma_2} \frac{f(z)}{z^{k+1}} dz$ and $\int_{\Gamma_2'} \frac{f(z)}{z^{k+1}} dz$, we get,

$$\begin{aligned} \sqrt{k} \cdot (-1)^k \cdot e_k &= \frac{1}{2\pi} \left\{ \frac{f(\sigma_k)}{\sigma_k^k} \left(\sqrt{\frac{\pi}{2}} + \mathcal{Y}_k' \right) + \frac{f(\overline{\sigma_k})}{\overline{\sigma_k}^k} \left(\sqrt{\frac{\pi}{2}} + \overline{\mathcal{Y}_k'} \right) \right\} \\ &= Re \left\{ \left(\frac{1}{\sqrt{2\pi}} + \mathcal{Y}_k \right) \frac{f(\sigma_k)}{\sigma_k^k} \right\}, \end{aligned}$$

where $\mathcal{Y}_k = \mathcal{Y}_k'/\pi \xrightarrow{P} 0$, as $k \to \infty$.

□

We write $\mathcal{G}_k := \left(\frac{1}{\sqrt{2\pi}} + \mathcal{Y}_k\right) \frac{f(\sigma_k)}{\sigma_k^k}$, so that $\sqrt{k} \cdot (-1)^k \cdot e_k = Re(\mathcal{G}_k)$.

4.5 Convergence of the two-step ratio of the elementary polynomials

This section is devoted to proving the two-step ratio convergence in Theorem 4.1.1

4.5.1 The ratio $\mathcal{G}_{k+2}/\mathcal{G}_k$

Lemma 4.5.1.

$$\frac{k^2 \mathcal{G}_{k+2}}{\mathcal{G}_k} \xrightarrow{P} -\pi^2.$$

Proof.

$$\frac{\mathcal{G}_{k+2}}{\mathcal{G}_k} = \frac{\frac{f(\sigma_{k+2})}{\sigma_{k+2}^k}}{\frac{f(\sigma_k)}{\sigma_k^k}} \cdot \frac{1}{\sigma_{k+2}^2} \cdot \frac{c + \mathcal{Y}_{k+2}}{c + \mathcal{Y}_k}. \tag{4.5.1}$$

Note that,

$$\frac{k^2}{\sigma_{k+2}^2} \longrightarrow -\pi^2. \tag{4.5.2}$$

Next,

$$\frac{\frac{f(\sigma_{k+2})}{\sigma_{k+2}^k}}{\frac{f(\sigma_k)}{\sigma_k^k}} = \exp\left(\phi_k(\sigma_{k+2}) - \phi_k(\sigma_k)\right).$$

Using mean value theorem on both the real and imaginary parts of ϕ_k on the line segment between σ_k and σ_{k+2}, we have that, there exists points s_1 and s_2 on this line segment, for which,

$$\phi_k(\sigma_{k+2}) - \phi_k(\sigma_k) = (\sigma_{k+2} - \sigma_k)\{Re\phi_k'(s_1) + iIm\phi_k'(s_2)\}.$$

Next, observe that,

$$\phi_k'(s_1) - \phi_k'(\sigma_k) = h_k\left(\frac{s_1}{k}\right) - h_k\left(\frac{\sigma_k}{k}\right) - \frac{k}{s_1} + \frac{k}{\sigma_k},$$
$$\implies \phi_k'(s_1) - 0 = h_k\left(\frac{s_1}{k}\right) - h_k\left(\frac{\sigma_k}{k}\right) - k\frac{\sigma_k - s_1}{s_1\sigma_k},$$

where

$$h_k(y) = \sum_{*} \frac{1}{ky - X_j}.$$

But, we know that h_k is Lipschitz near i/π and so, we can find a constant M for which

$$|\phi_k'(s_1)| \leq M\left|\frac{\sigma_{k+2} - \sigma_k}{k}\right|,$$

since $|s_1 - \sigma_k| \leq |\sigma_{k+2} - \sigma_k|$. Similarly, we can get

$$|\phi_k'(s_2)| \leq M\left|\frac{\sigma_{k+2} - \sigma_k}{k}\right|.$$

So,

$$|\phi_k(\sigma_{k+2}) - \phi_k(\sigma_k)| \leq M\frac{|\sigma_{k+2} - \sigma_k|^2}{k}$$

Therefore, to show that

$$\phi_k(\sigma_{k+2}) - \phi_k(\sigma_k) \longrightarrow 0,$$

we need,

$$|\sigma_{k+2} - \sigma_k| = o(k^{1/2}).$$

Now, writing $\sigma_{k+2} = (k+2)y_{k+2}$ and $\sigma_k = ky_k$ we see that the above translates to showing that

$$|y_{k+2} - y_k| = o(k^{-1/2}).$$

Since y_k is the solution of the equation $h_k(z) - 1/z = 0$ and y_{k+2} is the solution of the equation $h_{k+2}(z) - 1/z = 0$, we shall be done if we can show that

$$|h_{k+2}(y_{k+2}) - h_k(y_k)| = o(k^{-1/2}),$$

since h_k takes value near $-i\pi$.

Now, note that $h_{k+2}(y_{k+2}) = h_k(y_k')$, where $y_k' = \frac{k+2}{k} y_{k+2}$. So,

$$\begin{aligned} |h_{k+2}(y_{k+2}) - h_k(y_k)| &= |h_k(y_k') - h_k(y_k)| \\ &\leq |y_k' - y_k||h_k'(y_k'')|, \end{aligned}$$

where y_k'' is a point on the line segment joining y_k and y_k'. Now, note that $h_k'(i/\pi) = o(k^{-1/2+1/11})$. Owing to the Lipschitz condition satisfied by h_k' and the fact that y_k'' is at a distance $o(k^{-1/2+1/11})$ of i/π, we get, $h_k'(y_k'') = o(k^{-1/2+1/11})$, which implies,

$$|h_{k+2}(y_{k+2}) - h_k(y_k)| \leq o(k^{-1+2/11}),$$

which proves that $\phi_k(\sigma_{k+2}) - \phi_k(\sigma_k) \xrightarrow{P} 0$. Thus, $\frac{\frac{f(\sigma_{k+2})}{\sigma_{k+2}^k}}{\frac{f(\sigma_k)}{\sigma_k^k}} \xrightarrow{P} 1$. So, equations (4.5.1) and (4.5.2) give us

$$\frac{k^2 \mathcal{G}_{k+2}}{\mathcal{G}_k} \xrightarrow{P} -\pi^2.$$

□

4.5.2 Proof of Theorem 4.1.1

In this section we shall prove the first main theorem of this chapter, Theorem 4.1.1. We first need to show that the one-step ratio of the elementary symmetric functions cannot be too small, and use that to check that the argument of $\mathcal{G}_k$ stays away from odd multiples of $\frac{\pi}{2}$ with high probability. This fact, along with Lemma 4.5.1 gives us the proof.

Lemma 4.5.2.

$$\sup_k \mathbb{P}\left(\left|\frac{ke_{k+1}}{e_k}\right| \leq \delta\right) \longrightarrow 0, \ \textit{as, } \delta \to 0.$$

Proof. (Sketch) Note that the total variation distance between the Poisson point process $\{X_j\}_j$ and the Poisson point process taken together with an independent $X \sim \text{Uniform}(-k, k)$ converges to 0 as $k \to \infty$. Thus, if $\mathbb{Q}$ denotes the probability measure associated with the latter, we shall be done if we can show that

$$\sup_k \mathbb{Q}\left(\left|\frac{k\tilde{e}_{k+1}}{\tilde{e}_k}\right| \leq \delta\right) \longrightarrow 0, \ \text{as, } \delta \to 0,$$

where $\tilde{e}_k = e_k + \frac{e_{k-1}}{X}$. But, we can bound the conditional probability, $\mathbb{Q}\left(\left|\frac{k\tilde{e}_{k+1}}{\tilde{e}_k}\right| \leq \delta \middle| \{X_j\}_j\right)$, by a function of δ that goes to 0 uniformly in k. Taking expectations gives us the result. $\square$

Lemma 4.5.3. *For every $\delta > 0$, there exists $\eta > 0$, and $k_0 \geq 1$, such that*

$$\mathbb{P}\left(\inf_{m\in\mathbb{Z}}\left|\frac{(2m+1)\pi}{2} - Arg(\mathcal{G}_k)\right| < \eta\right) < \delta, \forall k \geq k_0.$$

Proof. Following the same steps as in the proof of Lemma 4.5.1, it is easy to see that,

$$\frac{k\mathcal{G}_{k+1}}{\mathcal{G}_k} \xrightarrow{P} -i\pi.$$

Suppose, the statement of this lemma is untrue. Then there exists $\delta > 0$, an increasing sequence of positive integers $\{k_n\}_n$ and a set A_δ of measure $\geq \delta$, such that, on A_δ,

$$\left|\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}\right| \longrightarrow \infty,$$

and,

$$\frac{\mathcal{G}_{k_n-1}}{k_n\mathcal{G}_{k_n}} \longrightarrow \frac{i}{\pi}.$$

But,

$$\begin{aligned}\frac{\mathcal{G}_{k_n-1}}{k_n\mathcal{G}_{k_n}} - \frac{i}{\pi} &= \frac{\frac{e_{k_n-1}}{k_n e_{k_n}} + i\frac{Im(\mathcal{G}_{k_n-1})}{k_n Re(\mathcal{G}_{k_n})}}{1 + i\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}} - \frac{i}{\pi} \\ &= \frac{\left(\frac{e_{k_n-1}}{k_n e_{k_n}} + \frac{1}{\pi}\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}\right) + i\left(\frac{Im(\mathcal{G}_{k_n-1})}{k_n Re(\mathcal{G}_{k_n})} - \frac{1}{\pi}\right)}{1 + i\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}}.\end{aligned}$$

This implies that, on A_δ,

$$\frac{\frac{e_{k_n-1}}{k_n e_{k_n}} + \frac{1}{\pi}\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}}{1 + i\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}} \longrightarrow 0.$$

But,

$$\begin{aligned} \frac{e_{k_n-1}}{k_n e_{k_n}} + \frac{1}{\pi}\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})} &= \frac{e_{k_n-1}}{k_n e_{k_n}} + \frac{i}{\pi} - \frac{i}{\pi}\left(1 + i\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}\right), \\ \implies \frac{1 - \pi i\frac{e_{k_n-1}}{k_n e_{k_n}}}{1 + i\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}} &\longrightarrow 1. \end{aligned}$$

Since, $\left|\frac{Im(\mathcal{G}_{k_n})}{Re(\mathcal{G}_{k_n})}\right| \longrightarrow \infty$, this gives, on A_δ,

$$\left|\frac{e_{k_n-1}}{k_n e_{k_n}}\right| \longrightarrow \infty.$$

But this contradicts the statement of Lemma 4.5.2, and thus we have arrived at a contradiction, thereby proving the result.

□

We shall now complete the proof of Theorem 4.1.1.

Proof. From Lemma 4.5.1

$$\begin{aligned} \frac{k^2\mathcal{G}_{k+2}}{\mathcal{G}_k} &\xrightarrow{P} -\pi^2 \\ \implies k^2\frac{Re(\mathcal{G}_{k+2}) + iIm(\mathcal{G}_{k+2})}{Re(\mathcal{G}_k) + iIm(\mathcal{G}_k)} &\xrightarrow{P} -\pi^2. \end{aligned}$$

Since $e_k \cdot \sqrt{k} = (-1)^k Re(\mathcal{G}_k)$, we will be done if we can show that $k^2\frac{Re(\mathcal{G}_{k+2})}{Re(\mathcal{G}_k)} \xrightarrow{P} -\pi^2$.

We have,

$$k^2\frac{Re(\mathcal{G}_{k+2}) + iIm(\mathcal{G}_{k+2})}{Re(\mathcal{G}_k) + iIm(\mathcal{G}_k)} + \pi^2 = k^2\frac{\left(\frac{Re(\mathcal{G}_{k+2})}{Re(\mathcal{G}_k)} + \pi^2\right) + i\left(\frac{Im(\mathcal{G}_{k+1})}{Re(\mathcal{G}_k)} + \pi^2\frac{Im(\mathcal{G}_k)}{Re(\mathcal{G}_k)}\right)}{1 + i\frac{Im(\mathcal{G}_k)}{Re(\mathcal{G}_k)}}.$$

By Lemma 4.5.3, except with probability $o(1)$, the denominator $1+i\frac{Im(\mathcal{G}_k)}{Re(\mathcal{G}_k)}$ stays sufficiently far away from 0, which makes $k^2\frac{Re(\mathcal{G}_{k+2})}{Re(\mathcal{G}_k)} \xrightarrow{P} -\pi^2$, and thus $k^2\frac{e_{k+2}}{e_k} \xrightarrow{P} -\pi^2$.

□

4.6 Convergence of the zero set of the nth derivative of f: Theorem 4.1.2

We begin by introducing the following notations:

$$c_k = (k+1)\frac{e_{k+1}}{\pi a_n},$$

$$\beta_k = \sqrt{\{k!e_k\}^2 + \{(k+1)!e_{k+1}\}^2},$$

$$h_k(z) = \frac{k!e_k}{\beta_k}\cos(\pi z) - \frac{(k+1)!e_{k+1}}{\pi\beta_k}\sin(\pi z).$$

Lemma 4.6.1. *For all $m \geq 1$, and any compact subset K of $\mathbb{C}$,*

$$\sup_{z\in K}\left|(-1)^n\frac{f_m^{(n)}(z)}{\beta_n} - h_{n,m}(z)\right| \xrightarrow{P} 0,$$

where $f_m^{(n)}(z)$ and $h_{n,m}$ are polynomials that equal the power series expansion of $f^{(n)}(z)$ and $h_n(z)$, respectively, up to the term z^m.

Proof. Suppose $|z| \leq M, \forall z \in K$. We have,

$$(-1)^n f^{(n)_m}(z) = n!e_n - (n+1)_1 e_{n+1} z + (n+2)_2 e_{n+2} z^2 - \cdots$$
$$+ (-1)^m (n+m)_m e_{n+m} z^m$$

and

$$h_{n,m}(z) = \frac{n!e_n}{\beta_n} - \frac{(n+1)!e_{n+1}}{\beta_n} z - \frac{n!e_n}{\beta_n}\frac{\pi^2 z^2}{2!} + \frac{(n+1)!e_{n+1}}{\beta_n}\frac{\pi^2 z^3}{3!} + ... + \gamma_{n,m} z^m,$$

where, if m is even,

$$\gamma_{n,m} = (-1)^{m/2}\frac{n!e_n}{\beta_n}\frac{\pi^m}{m!},$$

and if m is odd,

$$\gamma_{n,m} = (-1)^{(m+1)/2}\frac{(n+1)!e_{n+1}}{\beta_n}\frac{\pi^{m-1}}{m!}.$$

For ease of notation, I'll only write the proof for m even here. The proof is almost exactly the same of m odd. We have,

$$\begin{aligned}
&(-1)^n \frac{f_m^{(n)}(z)}{\beta_n} - h_{n,m}(z) \\
&\quad = \frac{n!e_n}{\beta_n}\left\{\left((n+1)(n+2)\frac{e_{n+2}}{e_n} + \pi^2\right)\frac{z^2}{2!}\right. \\
&\quad + \left((n+1)(n+2)(n+3)(n+4)\frac{e_{n+4}}{e_n} - \pi^4\right)\frac{z^4}{4!} + \cdots \\
&\quad + \left.\left((n+1)(n+2)\cdots(n+m)\frac{e_{n+m}}{e_n} - (-1)^{m/2}\pi^m\right)\frac{z^m}{m!}\right\} \\
&\quad - \frac{(n+1)!e_{n+1}}{\beta_n}\left\{\left((n+2)(n+3)\frac{e_{n+3}}{e_{n+1}} + \pi^2\right)\frac{z^3}{3!}\right. \\
&\quad + \left((n+2)(n+3)(n+4)(n+5)\frac{e_{n+5}}{e_{n+1}} - \pi^4\right)\frac{z^5}{5!} + \cdots \\
&\quad + \left.\left((n+2)(n+3)\cdots(n+m-1)\frac{e_{n+m-1}}{e_{n+1}} - (-1)^{\frac{m-1}{2}}\pi^{m-1}\right)\frac{z^{m-1}}{(m-1)!}\right\}.
\end{aligned}$$

So, if M is such that $|z| \leq M, \forall z \in K$,

$$\begin{aligned}
\sup_{z\in K}\left|(-1)^n \frac{f_m^{(n)}(z)}{\beta_n} - h_{n,m}(z)\right| \\
&\leq \left|(n+1)(n+2)\frac{a_{n+2}}{a_n} + \pi^2\right| \frac{M^2}{2!} \\
&+ \left|(n+2)(n+3)\frac{a_{n+3}}{a_{n+1}} + \pi^2\right| \frac{M^3}{3!} \\
&+ \left|(n+1)(n+2)(n+3)(n+4)\frac{a_{n+4}}{a_n} - \pi^4\right| \frac{M^4}{4!} \\
&+ \left|(n+2)(n+3)(n+4)(n+5)\frac{a_{n+5}}{a_{n+1}} - \pi^4\right| \frac{M^5}{5!} \\
&+ \cdots + \left|(n+1)(n+2)\cdots(n+m)\frac{a_{n+m}}{a_n} - (-1)^{m/2}\pi^m\right| \frac{M^m}{m!}.
\end{aligned}$$

Thus, by Theorem 4.1.1, since m is fixed here, the above inequality gives,

$$\sup_{z\in K}\left|(-1)^n \frac{f_m^{(n)}(z)}{\beta_n} - h_{n,m}(z)\right| \xrightarrow{P} 0.$$

□

Lemma 4.6.2. *Given $\delta, \epsilon > 0$ and a compact set K, there exists m sufficiently large, such that*

$$\mathbb{P}\left(\sup_{z\in K}\left|\frac{f^{(n)}(z)}{\beta_n} - \frac{f_m^{(n)}(z)}{\beta_n}\right| \geq \epsilon\right) < \delta.$$

Proof. Let $K_1 = \{z \in \mathbb{C} : |z - w| \leq 1$, for some $w \in K\}$. Then, as K_1 is compact, there exists m sufficiently large so that

$$\mathbb{P}\left(\sup_{z\in K_1}|f(z) - f_{n+m}(z)| \geq \epsilon\right) < \delta, \forall n \geq 1. \tag{4.6.1}$$

By Cauchy's integral formula,

$$f^{(n)}(z) - \frac{d^n(f_{n+m})}{dz^n}(z) = \frac{n!}{2\pi i}\int_{w:|w-z|=1} \frac{f(w) - f_{n+m}(w)}{(w-z)^{n+1}}dw,$$

$$\implies f^{(n)}(z) - f_m^{(n)}(z) = \frac{n!}{2\pi i}\int_{w:|w-z|=1} \frac{f(w) - f_{n+m}(w)}{(w-z)^{n+1}}dw,$$

$$\implies \sup_{z\in K}|f^{(n)}(z) - f_m^{(n)}(z)| \leq \frac{n!}{2\pi}\sup_{z\in K_1}|f(z) - f_{n+m}(z)| \cdot 2\pi.$$

Thus,

$$\mathbb{P}\left(\sup_{z\in K}\left|\frac{f^{(n)}(z)}{n!} - \frac{f_m^{(n)}(z)}{n!}\right| \geq \epsilon\right) < \delta, \forall n \geq 1.$$

However, the statement of the lemma needs β_n in the denominator, and not just $n!$, where we note that $\beta_n = n!\sqrt{(e_n)^2 + (n+1)^2 e_{n+1}^2}$. Thus, we wish to be able to modify the expression in (4.6.1) so that we can have $\sqrt{(e_n)^2 + (n+1)^2 e_{n+1}^2}$ in the denominator. This will be true if the higher symmetric functions of $1/X_j$'s are decreasing very quickly. So we wish to show that given any compact set, K, there exists $l \geq 1$ sufficiently large such that $\sum_{j\geq n+l}|e_j z^j| \leq \epsilon|e_n|, \forall z \in K$, with probability $> 1-\delta$. Now, let $M = \sup_{z\in K}|z|$. Then,

$$\begin{aligned}
\sum_{j\geq n+l}|e_j z^j| &\leq \sum_{j\geq n+l}|\mathcal{G}_j|M^j \\
&= \sum_{j\geq n+l}\left|\frac{f(\sigma_j)}{\sigma_j^j}\right|M^j \\
&\leq M^n \max_{j\geq n+l}\left(\left|\frac{f(\sigma_j)}{\sigma_j^n}\right|\right)\sum_{j\geq 1}\frac{M^{l+j-1}}{|\sigma_{n+l+j-1}^{l+j-1}|} \\
&\leq M^n \cdot \left|\frac{f(\sigma_n)}{\sigma_n^n}\right| \cdot \sum_{j\geq 1}\frac{M^{l+j-1}}{|\sigma_{n+l+j-1}^{l+j-1}|} \\
&\leq M^n \cdot |\mathcal{G}_n| \cdot \sum_{j\geq 1}\frac{M^{l+j-1}}{|\sigma_{n+l+j-1}^{l+j-1}|}.
\end{aligned}$$

By Lemma 4.5.3, there exists η such that, with probability $> 1-\delta$, $|\mathcal{G}_n| < \eta|e_n|$. Also, as the infinite sum $\sum_{r=1}^{\infty} \frac{M^r}{r^r} < \infty$, we can find l_0 such that, for all $l \geq l_0$,

$$\frac{M^{n+l+j-1}}{|\sigma_{n+l+j-1}^{l+j-1}|} < \frac{\epsilon}{\eta}.$$

Therefore, with probability $> 1-\delta$,

$$\sum_{j \geq n+l} |e_j z^j| \leq \epsilon |e_n|, \forall z \in K,$$

which proves the desired result.

□

Lemma 4.6.3. *Let w_n be a zero of the function $h_n(z)$. Then, given any $\epsilon > 0$ and any $\rho \in (0,1)$, there exists N (depending on ϵ) such that*

$$\mathbb{P}(f^{(n)} \textit{ has a zero within distance } \rho \textit{ of } w_n) > 1-\epsilon,$$

$\forall n \geq N$.

Proof. We have,

$$\left|(-1)^n \frac{f^{(n)}(z)}{\beta_n} - h_n(z)\right| \leq \left|\frac{f^{(n)}(z)}{\beta_n} - \frac{f_m^{(n)}(z)}{\beta_n}\right| + \left|(-1)^n \frac{f_m^{(n)}(z)}{\beta_n} - h_{n,m}(z)\right| + \left|h_{n,m}(z) - h_n(z)\right|,$$

where $\forall m$. Note that,

$$|h_{n,m}(z) - h_n(z)| \leq \left|\cos(\pi z) - \left(1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \frac{\pi^6 z^6}{6!} + ...\right)\right| + \frac{1}{\pi}\left|\sin(\pi z) - \left(z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \frac{\pi^7 z^7}{7!} + ...\right)\right|.$$

Since the right hand side is completely independent of n and both cos and sin are entire functions, this means that given any compact subset K, and any $\eta > 0$, there exists m, such that,

$$\sup_{z\in K} |h_{n,r}(z) - h_n(z)| < \eta, \forall r \geq m.$$

Thus, using Lemmas 4.6.1 and 4.6.2, given any $\eta > 0$,

$$\mathbb{P}\left(\sup_{z\in K}\left|(-1)^n \frac{f^{(n)}(z)}{\beta_n} - h_n(z)\right| > \eta\right) \longrightarrow 0.$$

Notice that, we can write h_n as

$$h_n(z) = -\sin(\pi(z - d_n)),$$

where $d_n = \frac{1}{\pi}\arctan(e_n/(n+1)e_{n+1})$. So the zeros of h_n are exactly $d_n + \mathbb{Z}$. Let w_n denote one such zero, say, $w_n = d_n + l$, where $l \in \mathbb{Z}$. Then h_n has exactly one zero inside the disc of radius $\rho < 1$ around w_n. Also, the value of h_n at any point $w_n + \rho e^{it}$ on the corresponding circle is $-\sin(\pi(d_n + l + \rho e^{it} - d_n)) = -\sin(\pi(l + \rho e^{it})) = -\sin(\pi\rho e^{it})$, which is independent of n and the choice of the zero of h_n. Therefore, we can write

$$\delta = \inf_{|z-w_n|=\rho} h_n(z),$$

which will be positive and independent of n and the choice of the zero of h_n. For this δ,

$$\mathbb{P}\left(\sup_{|z-w_n|=\rho}\left|(-1)^n \frac{f^{(n)}(z)}{\beta_n} - h_n(z)\right| \leq \frac{\delta}{2}\right) \longrightarrow 1.$$

Writing B_δ^n to be the event $\sup_{|z-w_n|=\rho} \left| (-1)^n \frac{f^{(n)}(z)}{\beta_n} - h_n(z) \right| \le \frac{\delta}{2}$, we have that, on B_δ^n,

$$\sup_{|z-w_n|=\rho} \left| (-1)^n \frac{f^{(n)}(z)}{\beta_n} \right| \ge \frac{\delta}{2}.$$

Thus the logarithmic derivative of $(-1)^n \frac{f^{(n)}(z)}{\beta_n}$ is well defined on the circle $\{z : |z-w_n| = \rho\}$. Moreover,

$$\sup_{|z-w_n|=\rho} \left| \frac{\left((-1)^n \frac{f^{(n)}(z)}{\beta_n} \right)'}{(-1)^n \frac{f^{(n)}(z)}{\beta_n}} - \frac{h_n'(z)}{h_n(z)} \right| \le 2\delta.$$

Thus, on B_δ^n,

$$\left| \int_{|z-w_n|=\rho} \frac{\left((-1)^n \frac{f^{(n)}(z)}{\beta_n} \right)'}{(-1)^n \frac{f^{(n)}(z)}{\beta_n}} dz - \int_{|z-w_n|=\rho} \frac{h_n'(z)}{h_n(z)} dz \right| \le 2\delta.$$

Now, since h_n has exactly one zero in the said disc, by Cauchy's argument principle,

$$\frac{1}{2\pi i} \int_{|z-w_n|=\rho} \frac{h_n'(z)}{h_n(z)} dz = 1.$$

So, if δ is chosen so that $\delta/\pi < 1$, we get, on B_δ^n,

$$\frac{1}{2\pi i} \int_{|z-w_n|=\rho} \frac{\left((-1)^n \frac{f^{(n)}(z)}{\beta_n} \right)'}{(-1)^n \frac{f^{(n)}(z)}{\beta_n}} dz = 1,$$

meaning that $f^{(n)}(z)$ has exactly 1 zero inside the disc of radius ρ centered at w_n. Thus, there exists N (depending on ϵ) such that,

$$\mathbb{P}(f^{(n)} \text{ has a zero within distance } \rho \text{ of } w_n) > 1 - \epsilon,$$

$\forall n \ge N$.

□

We now have all the tools necessary to prove Theorem 4.1.2, a sketch of which we give below.

Proof. From Lemma 4.6.3, we see that the distance between the zeros of $f^{(n)}$ and the zeros of h_n converge to zero in probability. Therefore, we shall be done if we show that the zeros of h_n converge to a uniform translate of $\mathbb{Z}$. We know already that h_n vanishes at $\mathbb{Z} + d_n$, where $d_n = \frac{1}{\pi}\arctan(\frac{e_n}{(n+1)e_{n+1}})$. So, the zeros of $f^{(n)}$ are indeed very close to some translate of the integers.

Now, from Lemma 4.2.3, we have that shifting the origin by a certain amount shifts the zeros of $f^{(n)}$ by the same amount. However, the Poisson point process of intensity 1 on $\mathbb{R}$ is translation invariant. Thus, adding a Uniform$(0, 1)$ variate to each point of the process still gives us a Poisson point process of intensity 1 on $\mathbb{R}$. This implies that the limiting distribution of $(d_n \mod 1)$ has to be a translation invariant distribution as well, which can only be Uniform$(0, 1)$.

□

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