THE SCHOTTKY PROBLEM IN GENUS FIVE

Charles Siegel

A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2012

Ron Donagi, Professor of Mathematics
Supervisor of Dissertation

Jonathan Block, Professor of Mathematics
Graduate Group Chairperson

Dissertation Committee
Tony Pantev, Professor of Mathematics (Chairman)
Ron Donagi, Professor of Mathematics (Advisor)
Jonathan Block, Professor of Mathematics

Acknowledgments

I would like to start by thanking two people without whom this thesis would not have been possible. First, my advisor, Ron Donagi, for all of his advice and the time he spent teaching me mathematics, not least of which included suggesting that I would enjoy working on the Schottky problem in the first place. And second, my wife, Celia Rae, who put up with me through all the inevitable ups and downs of writing a Ph.D. thesis.

Next I would like to the University of Pennsylvania faculty members who particularly helped me along the way, whether by teaching me mathematics or just reminding me that being stuck is alright, so long as I keep working. Most prominent among them are Tony Pantev and Jonathan Block, who in addition to being on my dissertation committee, were also on my oral exam committee and served as graduate group chair while I've been here. Others I'd like to thank include Steve Shatz, Antonella Grassi, Murray Gerstenhaber, Florian Pop, and Eugenio Calabi.

I couldn't have finished without the support of many of my fellow graduate students. I am forever indebted to my former officemates, Alberto Garcia-Raboso and

Tyler Kelly in particular To all of the graduate students at Penn, who form a very supportive community, I am thankful. A special thanks to a graduate student not at Penn, Joseph Walsh at Stony Brook, who has helped me learn and do mathematics since we took undergraduate analysis together at Rutgers.

There are also many professors at other universities who have helped me through mathematical discussions, too many to list in their entirety here, but particularly Sam Grushevsky, Angela Gibney, Elham Izadi, Rob Varley, Eduard Looijenga, Benson Farb, and Chuck Weibel, as well as my undergraduate mentors, Diance Maclagan and Anders Buch.

And finally, I'd like to thank my family, who encouraged my interest in mathematics from a young age, and who never lost faith that I could complete a doctorate in the subject, even when I did.

ABSTRACT

THE SCHOTTKY PROBLEM IN GENUS FIVE

Charles Siegel

Ron Donagi

In this thesis, we present a solution to the Schottky problem in the spirit of Schottky and Jung for genus five curves. To do so, we exploit natural incidence structures on the fibers of several maps to reduce all questions to statements about the Prym map for genus six curves. This allows us to find all components of the big Schottky locus and thus, to show that the small Schottky locus introduced by Donagi is irreducible.

Contents

Chapter 1

Introduction

The Schottky problem has a long history, tied inextricably to curve theory as a whole. It is one of the first questions to ask at the beginning of the theory: how can one characterize which abelian varieties are the Jacobians of curves? The first results were obtained by Schottky himself, and with his collaborator Jung [Sch88, JS09]. The next result in this spirit wasn't until the 1980s, when Igusa proved the Schottky-Jung conjecture in genus 4 [Igu81].

Several other approaches were attempted, many of which successfully characterized Jacobians, such at Shiota's theorem on the KP heirarchy [Shi86], the trisecant conjecture which was recently resolved by Krichever [Kri10], and the general position arguments of Pareschi and Popa [PP08]. Another method, which only gives a weak characterization, that is, a locus where the Jacobians are an irreducible component, is via the singularities of the theta divisor, begun by Andreotti and Mayer

[AM67], pursued in [CM08, CvdG00, GSM07] and many others, has increased our understanding of the moduli of abelian varieties greatly, without solving the Schottky problem.

In [Don87a, Don87b], Donagi showed that the original Schottky-Jung conjecture was incorrect, offered a means of correcting it, and made the bold conjecture that the problems he saw were the only ones that would arise. At that time, he and others were studying the Prym map, which appeared in the original Schottky-Jung conjecture, and obtaining insight into the Andreotti-Mayer loci, but also into the Schottky locus itself [Bea77, Bea82, Deb90, Deb88b, Don92, DS81, DS80, Don81]. This approach gave a second proof of the Schottky-Jung conjecture in genus four, and this paper pushes the approach through to give a proof in genus five.

Chapter 2

Background

We will start by establishing some notation. Denote by $\mathcal{M}_g$ the moduli space of genus g curves and by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g. It is useful in the context of theta functions to recall that $\mathcal{A}_g$ can be written as $\mathbb{H}_g/\operatorname{Sp}(2g,\mathbb{Z})$, where $\mathbb{H}_g$ is the Siegel upper half space of symmetric $g \times g$ matrices over $\mathbb{C}$ whose imaginary part is positive definite.

We will need several level covers of these moduli spaces. All such covers that we use arise naturally over $\mathcal{A}_g$ and are defined for $\mathcal{M}_g$ by pullback. For any abelian variety, we define the Weil pairing as the bilinear form associated to the quadratic form obtained by fixing a symmetric theta divisor Θ and then associating to each point of order two μ the multiplicity (mod 2) at the identity of $\Theta + \mu$. Given that, we define $\mathcal{R}^n\mathcal{A}_g$ to be the moduli space of principally polarized abelian varieties along with a length n flag of isotropic subspaces of the points of order two on A, and similarly for $\mathcal{R}^n\mathcal{M}_g$ and for lifts of maps of moduli spaces.

The corank 1 parts of the boundaries of these moduli spaces will also play a role, so we must understand the components. The boundary of $\mathcal{R}\mathcal{A}_g$ has three irreducible components, $\partial^I\mathcal{R}\mathcal{A}_g$, where the vanishing cycle and the marked semiperiod are equal, $\partial^{II}\mathcal{R}\mathcal{A}_g$ where they are perpendicular adn $\partial^{III}\mathcal{R}\mathcal{A}_g$ where they are not. The situation for $\mathcal{R}\mathcal{M}_g$ is somewhat more complex, with the above components corresponding to double covers of irreducible nodal curves, referred to as Wirtinger, unallowable and Beauville covers, respectively. However, there are two additional classes of components, corresponding to reducible curves that are the union of a genus i curve and a genus $g-i$ curve. One class of components, the ∂_i's, has the point of order two a pullback of one from the genus i curve, whereas the other class, the $\partial_{i,g-i}$'s, have the point of order two a sum of pullbacks from each component.

Though these components exist in all compactifications, we will sometimes want to use the Satake partial compactification of these spaces, sometimes the toroidal partial compactification, which can be described as the Satake partial compactification, blown up along the boundary. We will denote which one we are working on by a superscript s or t.

There is a natural map $\mathcal{J}_g : \mathcal{M}_g \to \mathcal{A}_g$ taking a curve to its Jacobian, and this lifts to a map $\mathcal{R}\mathcal{J}_g : \mathcal{R}\mathcal{M}_g \to \mathcal{R}\mathcal{A}_g$. We will denote the closure of the images of these maps by $\mathcal{J}_g$ and $\mathcal{R}\mathcal{J}_g$ respectively. There is also a natural map $\mathcal{P}_{g+1}$: $\mathcal{R}\mathcal{M}_{g+1} \to \mathcal{A}_g$ with $\mathcal{P}_{g+1}(C,\tilde{C}) = \ker^0(\mathrm{Nm} : \mathcal{J}_g(\tilde{C}) \to \mathcal{J}_g(C))$, and it lifts to

$\mathcal{RP}_{g+1} : \mathcal{R}^2\mathcal{M}_{g+1} \to \mathcal{RA}_g$.

Lemma 2.1 ([Mum74]). *Let $(C, \mu) \in \mathcal{RM}_g$, $(\mu) = \{0, \mu\}$, $\mu^\perp$ the points of order two on $J(C)$ that are Weil orthogonal to μ, and P the Prym variety of (C, μ). Then we have a short exact sequence $0 \to (\mu) \to \mu^\perp \to P_2 \to 0$.*

The Weil pairing defined above is essential, as we will also be using the space $\mathcal{R}^2\mathcal{M}_g$ of triples (C, μ, Λ) where C is a genus g curves, $\mu \in \mathcal{J}_g(C)_2$, and Λ is a rank 2 isotropic subgroup with respect to the Weil pairing containing μ, in short, a partial isotropic flag. The Mumford sequence implies that we actually get a map $\mathcal{RP}_{g+1} : \mathcal{R}^2\mathcal{M}_{g+1} \to \mathcal{RA}_g$ given by $\mathcal{RP}_{g+1}(C, \mu, \Lambda) = (P(C, \mu), \bar{\nu})$, where ν is a nontrivial element of Λ distinct from μ.

Here, we recall some properties of the Prym map that we will need:

Proposition 2.2 (Prym is Proper [DS81, Theorem 1.1], [Bea77, Proposition 6.3]). *The Prym map $\mathcal{P} : \mathcal{RM}_g \to \mathcal{A}_{g-1}$ extends to a proper map $\overline{\mathcal{P}} : \overline{\mathcal{RM}}_g \to \mathcal{A}_{g-1}$.*

Theorem 2.3 ([Bea77]). *For $g \leq 6$, the Prym map is surjective.*

Theorem 2.4 ([DS81, Don81]). *The map $\mathcal{P} : \mathcal{RM}_6 \to \mathcal{A}_5$ has degree 27 and Galois group WE_6.*

The moduli of abelian varieties with level $(2, 4)$ structure is given by the quotient $\mathbb{H}_g/\Gamma_g^{(2,4)}$, where $\Gamma_g^{(2,4)}$ consists of matrices which are the identity modulo 2 and preserve the form $\epsilon^t\delta$ modulo 4, and this structure is extended to other moduli spaces fiberwise. All of the theta maps lift to the moduli spaces with level (2,4) structure, and at that level, we can define certain maps via theta functions. We start with the Riemann theta function, defined by $\theta : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ given by

$$\theta(\Omega, z) = \sum_{n \in \mathbb{Z}^g} \exp[\pi i (n^t \Omega n + 2 n^t z)].$$

From this, we can define theta functions with characteristics, which, for any $\epsilon, \delta \in \mathbb{Q}^g$, gives us

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\Omega, z) = \exp[\pi i(\epsilon^t \Omega \epsilon + 2\epsilon^t (z + \delta))] \theta(\Omega, z + \Omega\epsilon + \delta),$$

and then these can be used to define kth order theta functions, for any $\epsilon \in (\{0, 1, \ldots, k-1\})^g$, have $\theta_k[\epsilon](\Omega, z) = \theta \begin{bmatrix} \epsilon/k \\ 0 \end{bmatrix} (k\Omega, kz)$.

Then, we define on $\mathbb{H}_g$ the maps α_g and β_g where α_g is given by second order theta constants and β_g is given by the functions $\theta \begin{bmatrix} \epsilon/2 & 0 \\ 0 & 1/2 \end{bmatrix} (2\Omega, 0)$ for $\epsilon \in \{0, 1\}^{g-1}$. These maps descend to $\mathcal{A}_{g-1}$ and $\mathcal{RA}_g$, mapping them to the projectivization of $U_g = \{f : \mathbb{F}_2^{2g} \to \mathbb{C}\}$, which, by the Stone-von Neumann theorem comes with a unique irreducible representation of the Heisenberg group $G_g = \Gamma_g / \Gamma_g^{(2,4)}$. These maps, the Prym map and the Jacobian map are related by the following diagram:

Theorem 2.5 (Schottky-Jung Identities [Sch88, JS09, RF74])**.** *The following diagram commutes:*

$$
\begin{array}{ccccc}
 & & \mathcal{RM}_g & & \\
 & \overset{\mathcal{P}}{\swarrow} & & \overset{\mathcal{RJ}}{\searrow} & \\
\mathcal{A}_{g-1} & & & & \mathcal{RA}_g \\
 & \underset{\alpha}{\searrow} & & \underset{\beta}{\swarrow} & \\
 & & \mathbb{P}(U_{g-1})/G_{g-1} & &
\end{array}
$$

And now, we define the Schottky locus to be $\mathcal{RS}_g = \beta^{-1}(\operatorname{im}\alpha)$, the big Schottky $\mathcal{S}_g^{\text{big}}$ locus to be the image of $\mathcal{RS}_g$ in $\mathcal{A}_g$, and the small Schottky locus to be $\mathcal{S}_g^{\text{small}} = \{A \in \mathcal{A}_g | \forall \mu \in A_2 \setminus \{0\}, (A, \mu) \in \mathcal{RS}_g\}$. An immediate consequence of the Schottky-Jung identities is that $\mathcal{RJ}_g$ is an contained in $\mathcal{RS}_g$. In fact, [vGvdG86, Don87a] $\mathcal{RJ}_g$ is a component of $\mathcal{RS}_g$, and so, $\mathcal{J}_g$ is a component of $\mathcal{S}_g$. More surprising was the result [Don87b, Theorem 5.4] that $\mathcal{RC}^0$ is a component of $\mathcal{RS}_5$, where $\mathcal{RC}$ is the locus of intermediate Jacobians of cubic threefolds with a marked point of order two, which splits into $\mathcal{RC}^0$ and $\mathcal{RC}^1$, depending on the parity of the point of order two.

The final result that we will need is theta symmetry. Given genus $g+1$ curve C and Λ an isotropic rank 2 subgroup of $\mathcal{J}_{g+1}(C)_2$, theta symmetry relates the behavior of the three points $(C, \mu, \Lambda) \in \mathcal{R}^2\mathcal{M}_{g+1}$ with respect to $\mathcal{RP}_{g+1}$ and β_g.

Theorem 2.6 (Theta symmetry [Don87b, Theorem 3.1]). *Let $C \in \mathcal{M}_{g+1}$ be a curve of genus $g+1$ and let $\{0, \mu_0, \mu_1, \mu_2\}$ a rank 2 isotropic subgroup of $\mathcal{J}_{g+1}(C)_2$ (thus, $\mu_2 = \mu_0 + \mu_1$). For $i = 0, 1, 2$ we have a Prym variety $P_i = P(C, \mu_i) \in \mathcal{A}_g$ and on it a uniquely determined semiperiod ν_i, the image of μ_j, $j \neq i$ in P_i.*

The point $\beta(P_i, \nu_i)$ is independent of $i = 0, 1, 2$.

Chapter 3

Incidence Relations

For this chapter, we fix a finite field k. Later, we will specialize to the case of $k = \mathbb{F}_2$, which occurs because the natural incidence relations that will arise are trialities. We'll study structures over this field that will give us more information about the fibers of the map β_5.

Definition 3.1 (Line configuration)**.** A line configuration V over a finite field k is a finite set P_V and a set of subsets of P_V, L_V, such that each $\ell \in L_V$ has $|\ell| = |k| + 1$ and if $\ell, \ell' \in L_V$ have $|\ell \cap \ell'| \geq 2$, then $\ell = \ell'$.

Intuitively, P_V is aset of points and L_V is a set of (projective) lines. As such, we will denote by $\mathrm{Lines}_V(p)$ the set $\{\ell \in L_V | p \in \ell\}$.

Example 3.2. A large class of examples arises as follows: let $V \subset \mathbb{P}^n(k)$ be a subvariety. Then we set $P_V = V$ and $L_V = \{\ell \in \mathbb{G}(1, n) | \ell \subset V\}$.

The most fundamental example is $\mathbb{P}^n(k)$, which is of this form.

If $V = (P_V, L_V)$ is a line configuration, then we say that X is a subconfiguration if $P_X \subset P_V$ and $L_X = \{\ell \in L_V | \forall p \in \ell, p \in P_X\}$, and we will identify configurations for which there is a bijection $P_V \to P_{V'}$ which induces a bijection $L_V \to L_{V'}$.

With the above in mind, we will abuse language slightly and say that two lines $\ell, \ell' \in L_V$ are coplanar if there exists a subconfiguration $X \subset V$ isomorphic to $\mathbb{P}^2(k)$ with $\ell, \ell' \subset X$ where k is the field over which V and X are defined.

Definition 3.3 (V-configuration)**.** Let V be a line configuration over k. Then a line configuration W is a V-configuration if for each $p \in P_W$, we have $\phi_p : \mathrm{Lines}_W(p) \to P_V$ a bijection such that $\ell, \ell' \in \mathrm{Lines}_W(p)$ are coplanar if and only if $\phi_p(\ell)$ and $\phi_p(\ell')$ are colinear.

Example 3.4. $\mathbb{P}^n$ is always a $\mathbb{P}^{n-1}$-configuration.

Example 3.5. If V is a collection of n points with $L_V = \emptyset$, then $(\mathbb{P}^1)^n$ is a V-configuration.

Example 3.6. Consider the $n = 5$ case of 3.5, that is, V is the line configuration over $\mathbb{F}_2$ with $|P_V| = 5$ and $L_V = \emptyset$. Fix $S \subset \mathbb{P}^3(\mathbb{C})$ a smooth cubic surface. Then we get a V-configuration W by setting P_W to be the set of lines on S and L_W the set of triples of lines which are coplanar. This works because if we fix a line in S, we have five pairs of intersecting lines which intersect ℓ, and thus are coplanar. This same line configuration can also be relieved as per 3.2. For this, we look at $\mathbb{P}^5(\mathbb{F}_2)$ and the subvariety Q_6^- given by $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_5x_6$. This is a smooth quadric and $O_{\mathbb{F}_2}(6)^-$ acts transitively, so we can look at any point, for instance

$p = [0:0:0:0:0:1]$. Any line through p has two other points, both have the last coordinate 0 and they differ by the 5th coordinate. So the set of lines through p is given by $Q_6^- \cap V(x_5 = x_6 = 0)$, which is $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 \subset \mathbb{P}^3(\mathbb{F}_2)$, which is 5 points, no three of which are colinear. This will be the most useful realization of this configuration and it follows from the isomorphism $WE_6 \cong O_{\mathbb{F}_2}(6)^-$.

Note, that example 3.6 shows that for some line configurations V there exist multiple fundamentally distinct V-configurations. This can be controlled, however, through the imposition of symmetry conditions. To describe them, we first need to define an associated graph:

Definition 3.7 (Graph associated to a configuration)**.** Let V be a line configuration. Then we define the graph associated to V, Γ_V, to have vertex set P_V and an edge between $p, q \in P_V$ if and only if there exists $\ell \in L_V$ such that $p, q \in \ell$. We will also use the natural metric on Γ_V obtained by giving each edge length one. Additionally, we will say that V is connected if Γ_V is.

With the metric, we can define various subsets of V. Fixing $p \in V$, we define $V_i(p) = \{q \in V | d(p,q) = i\}$, and we define $A_{i,j}(p,q) = V_j(p) \cap V_1(q)$ for $q \in V_i(p)$, and the numbers $v_i(p) = |V_i(p)|$ and $a_{i,j}(p,q) = |A_{i,j}(p,q)|$. These numbers are not in general independent of the choices of points. However, in the situation we care about, our configurations occur as the fibers of a map of irreducible varieties, and so for a general fiber, these numbers will be well defined because we can find a path from one starting point to another, and away from special fibers, these numbers will be continuous functions.

Now, we will prove some numerical facts about configurations.

Lemma 3.8. *Let V be a connected configuration where all the relevant numbers are well defined. Then*

1 *For all i,* $v_1 = a_{i,i-1} + a_{i,i} + a_{i,i+1}$

2 *For all i,* $v_1 v_i = a_{i-1,i} v_{i-1} + a_{i,i} v_i + a_{i+1,i} v_{i+1}$

3 $v_0 = 1$, $a_{0,0} = 0$, $a_{0,1} = v_1$, *and* $a_{1,0} = 1$.

For W a connected V-configuration, where these numbers are well defined, we have

4 $a_{2,2} \geq a_{2,1}$

5 $w_1 = |k| \sum_{i=0}^{\infty} v_i$

6 $a_{1,1} = |k|(v_1 + 1) - 1$

Proof. 1 Fix $p \in V$, $i \in \mathbb{N}$, $q \in X_i(p)$. Then

$$\begin{aligned} & A_{i,i-1}(p,q) \cup A_{i,i}(p,q) \cup A_{i,i+1}(p,q) \\ = \;& (V_{i-1}(p) \cap V_1(q)) \cup (V_i(p) \cap V_1(q)) \cup (V_{i+1}(p) \cap V_1(q)) \\ = \;& (V_{i-1}(p) \cup V_i(p) \cup V_{i+1}(p)) \cap V_1(q) \\ = \;& V_1(q). \end{aligned}$$

2 Fix $p \in V$, $i \in \mathbb{N}$. Let $X = \{(a,b) | a \in V_i(p), b \in X_1(a)\}$. Then $|X| = v_i v_1$. But also,

$$\begin{aligned} X &= \{(a,b) | a \in V_i(p) \cap V_1(b), b \in \cup_j V_j(p)\} \\ &= \cup_j \{(a,b) | a \in A_{j,i}(p,b), b \in V_j(p)\} \end{aligned}$$

and so $|X| = \sum_j a_{j,i} v_j$.

3 These all follow directly from the definitions.

4 Fix $q \in W_2(p)$. We prove that no line containing q contains two points of $W_1(p)$. As each line consists of $n+1 \geq 3$ points, this implies that $a_{2,2} \geq a_{2,1}$. Let $a, b \in W_1(p)$ and assume that there is a line $\ell \in L_W$ such that $a, b, q \in \ell$. As $a, b \in W_1(p)$, there exist lines m_1, m_2 through p containing a, b respectively. But then, m_1, m_2 must be coplanar, and so there is a line m containing p, q, so $q \in V_1(p) \cap V_2(p) = \emptyset$, a contradiction.

5 For each line $\ell \in \text{Lines}_W(p)$, fix a bijection $\ell \to \mathbb{P}^1(k)$ such that p is mapped to ∞. Then $\cup_{\ell \in \text{Lines}_W(p)} \ell = \coprod_{\ell \in \text{Lines}_W(p)} \mathbb{A}^1(k)$, and this has cardinality $|k|$ times the number of lines, $|k||P_V|$.

6 Fix $p \in P_W$, ℓ a line through p, $p' \in \ell$ distinct form p. Through p', there are $|P_V|$ lines. One is ℓ, v_1 of them are coplanar with ℓ, and the rest are not. Each coplanar line consists of n points in $V_1(p)$ that are not p', but there are also $n-1$ points of ℓ in $V_1(p)$ other than p', and so $a_{1,1} = |k|(v_1+1) - 1$.

□

Proposition 3.9. *Let V be a connected configuration and W a connected V-configuration where all the relevant numbers are well defined, and with both having diameter 2. Then either $w_2 = |P_V|$ or $w_2 = |k|^2 v_2$.*

Proof. Fix $p \in W$. Let $\tilde{X}$ be the set of triples (ℓ, m, q) in $\text{Lines}_W(p) \times L_W \times W_2$ such that $p \in \ell$, $\ell \cap m \neq \emptyset$ and $q \in m$. There is a natural map $\tilde{X} \to W_2$ which is surjective. Then

$$\begin{aligned}
|\tilde{X}| &= w_2 \cdot |\text{fiber}| \\
&= w_2 |\{\text{paths to } q \in W_2 \text{ from } q\}| \\
&= w_2 |\{\text{points of} W_1(p) \text{connected to } p\}| \\
&= w_2 \cdot |W_1(p) \cap W_1(q)| \text{ for } q \in W_2(p) \\
&= w_2 a_{2,1}
\end{aligned}$$

Now, we also have a map onto $\text{Lines}_W(p) \times W_2(p) = A \coprod B$, where the fiber over A has cardinality 1 and over B has cardinality 0. These are the only possibilities, because if there were two, then we get a plane and $q \in W_1(p)$. So $|A| = |\tilde{X}| = w_2 a_{2,1}$, and $|A| + |B| = |\text{Lines}_W(p) \times W_2(q)| = |P_V| w_2$, so $|B| = w_2(|P_V| - a_{2,1})$.

But, as W is symmetric, we can see that $|V| = |P_V|\alpha$, where α is the number of lines in $\text{Lines}_W(p)$ that do't have a line connecting them to q. Then $\alpha = |P_V| - |\{\text{lines in } \text{Lines}_W(p) \text{ connected to } q\}|$, which is $\alpha = |V| - a_{2,1}$, so $|B| = |P_V|(|P_V| - a_{2,1})$.

So, $w_2(|P_V| - a_{2,1}) = |P_V|(|P_V| - a_{2,1})$, implying that either $w_2 = |P_V|$ or else

$a_{2,1} = |P_V|$. In the latter case, the conditions of lemma 3.8 along with the fact that W is symmetric, and so has diameter at most 2, implies that $w_2 = |k|^2 v_2$. $\square$

Now, we proceed from the general to the specific. We fix our field to be $\mathbb{F}_2$ and define $Q_{2n}^- \subset \mathbb{P}^{2n-1}(\mathbb{F}_2)$ to be the variety defined by $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + \ldots + x_{2n-1}x_{2n} = 0$. For $n \geq 3$, Q_{2n}^- is connected and symmetric.

Lemma 3.10. *Any Q_6^--configuration of diameter 2 must consist of 119 points.*

Proof. It follows immediately from the previous proposition that either $w_2 = 27$ or $w_2 = 64$. But, if $w_2 = 27$, then by 3.8, we have that $a_{2,1} = 64$, which forces $a_{2,2}$ to be negative, which is impossible, so $w_2 = 64$. Thus, the configuration has $w_0 = 1$, $w_1 = 54$ and $w_2 = 64$, thus P_W has 119 points. $\square$

Proposition 3.11. *The general fiber of $\beta_5 : \mathcal{RA}_5 \to \mathbb{P}^{15}/G_4$ is a Q_6^--configuration.*

Proof. Let $(A, \mu) \in \mathcal{RA}_5$ be a general point. Then A is the Prym variety of 27 curves $(X_i, \nu_i) \in \mathcal{RM}_6$, and μ lifts to two distinct points of order two on $\mathcal{J}(X_i)$, μ_i^0, μ_i^1. Then theta symmetry implies that $(\mathcal{P}(X_i, \mu_i^j), \bar{\nu}_i)$ are distinct points in the fiber of β_5. Thu, (A, μ) lies on 27 triples (A, μ), $(\mathcal{P}(X_i, \mu_i^0), \bar{\nu}_i)$, $(\mathcal{P}(X_i, \mu_i^1), \bar{\nu}_i)$, which are lines over $\mathbb{F}_2$, form the configuration Q_6^-, by [Don81], and so the fiber is a Q_6^--configuration. $\square$

Lemma 3.12. *The connected components Q_6^--configuration on the fibers of β_5 has diameter 2.*

Proof. If we blow up the Prym map $\mathcal{RP}_6$ to be finite, and then apply the results of [Don92] section 4, we see that if we start with a generic cubic threefold $(X, \mu) \in \mathcal{RC}^0$, with normal data suppressed for ease of notation, the inverse image under the Prym map consists of 54 points of $\mathcal{R}^2\mathcal{M}_6$, all of which have base curve a quintic plane curve. Exchanging points of order two via theta symmetry, and applying the Prym map, we get 54 Jacobians. Each of those Jacobians will then have preimage of 54 points, 2 of which will be covers of a given quintic, 20 of which will be covers of trigonal curves, and 32 of which will be irreducible singular curves, the Wirtinger double covers. Applying theta symmetry and taking Pryms, we get our original cubic threefold, 21 Jacobians and 32 degenerate abelian varieties.

Now, we must show that nothing new arises when we perform this process on a degenerate abelian variety. By construction, we know that every degeneration we get this way has some irreducible singular curve that it is the Prym variety of. By [Iza91, 6.4], the family of Abel-Prym embeddings of these irreducible curves are all of the Abel-Prym embeddings for generic $A \in \partial\mathcal{A}_5$. So, the only unallowable covers that appear in the general fiber of the Prym map over A are in $\partial^{II}\mathcal{RM}_6$, and thus applying theta symmetry and taking Pryms, we only get other points of $\partial^I\mathcal{RA}_5$ and Jacobians.

Finally, we show that the objects obtained previously don't constitute anything new. First, the Jacobians obtained from the chosen Jacobian in the second step. These Jacobians all must have already been obtained from the cubic threefold, because each Jacobian is the Prym of a unique quintic curve, and will thus determine a cubic threefold. However, no two cubic threefolds can be related in this manner, because the incidence is contained in the fibers of β_5, which induces a birational isomorphism between $\mathcal{RC}^0$ and $\alpha(\mathcal{A}_4)$, and so, generically, must be injective. And last, we have the objects obtained from a given degenerate abelian variety. We have twenty-seven nodal curves, and each occurs as a Wirtinger and as a $\partial^{II}\mathcal{RM}_6$ double cover, which we then take the Pryms of. The Wirtingers must be Jacobians we've already seen, by the above, and so the ∂^{II} double covers would be obtained by applying theta symmetry on those Jacobians first. Thus, the Q_6^--configuration must have diameter 2. $\square$

And so, by Lemma 3.10, the fibers must have degree a multiple of 119.

Chapter 4

Degenerations of abelian varieties

In this chapter, we will show that the Q_6^--configuration on the fibers of β is connected. This amounts to computing the degree of β, which we will do by studying certain degenerations. We will spend this chapter studying the structure of $\partial^{II}\overline{\mathcal{RA}}_5^t$, the degenerations with vanishing cycle orthogonal but not identical to the marked semiperiod.

Lemma 4.1 ([Don87a])**.** *The extension of $\beta_g : \overline{\mathcal{RA}}_g^t \to \mathbb{P}(U_{g-1})/G_{g-1}$ to $\partial^{II}\overline{\mathcal{RA}}_g^t$ is β_{g-1} and the diagram:*

$$\begin{array}{ccc} \mathcal{RA}_{g-1} & \xrightarrow{i_{II}} & \overline{\mathcal{RA}}_g^t \\ \downarrow{\scriptstyle \beta_{g-1}} & & \downarrow{\scriptstyle \beta_g} \\ \mathbb{P}(U_{g-2})/G_{g-2} & \to & \mathbb{P}(U_{g-1})/G_{g-1} \end{array}$$

is Cartesian.

This result tells us that we really can just look inside this boundary component, and we have the following:

Proposition 4.2. *Theta symmetry spans the fibers of β_5 in $\partial^{II}\overline{\mathcal{RA}}_5^t$.*

Proof. By lemma 4.1, we are working with the map $\beta_4 : \mathcal{RA}_4 \to \mathbb{P}^7/G_3$. Now, fix a curve $C \in \mathcal{M}_3$. Then the fiber of β_4 over $\alpha_3(\mathcal{J}(C))$ is $\mathrm{Bl}_0 K(\mathcal{J}(C)) \cup K(\mathcal{J}(C))$[Don88], with the first component consisting of genus 4 Jacobians and the second of degenerate abelian varieties in $\partial^I\overline{\mathcal{RA}}_4$. Over $(B, \tilde{B}) \in \mathcal{RJ}_4$, the fiber of $\mathcal{RP}_5$ is a double cover of $\mathrm{Sym}^2 B \cup \mathrm{Sym}^2 B$, with the first component consisting of trigonal curves and the second consisting of Wirtinger curves[Don92]. Thus, for each $(B, \tilde{B})$, we get a map $\widetilde{\mathrm{Sym}^2 B} \to \mathrm{Bl}_0 K(\mathcal{J}(C))$, because theta symmetry takes trigonal curves to trigonal curves and Wirtinger curves to Wirtinger curves. Each of these maps has two dimensional image, and they're all nonisomorphic, and thus distinct, so the dimension of the union is at least three, so surjectivity follows. □

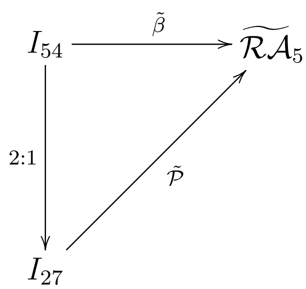
Now that we have a locus where we know that theta symmetry spans the fibers, we need to identfy theta symmetry over that locus:

Theorem 4.3 ([Iza91, 6.5])**.** *Over a general point $X \in \partial\overline{\mathcal{A}}_5^t$, there are 27 distinct objects of $\partial^{II}\overline{\mathcal{RM}}_6$ with Prym variety X, and the Prym map is generically unramified on $\partial^{II}\overline{\mathcal{RM}}_6$.*

This theorem tells us that the fiber in $\partial^{II}\overline{\mathcal{RA}}_5$ has the Q_6^--configuration structure we expect, and more so, the fact that theta symmetry spans the fibers implies connectivity, so the fiber is Q_8^-. It remains, though, to check that β_5 is generically unramified on $\partial^{II}\overline{\mathcal{RA}}_5^t$.

We have a tower of maps $\overline{\mathcal{R}^2\mathcal{M}_6} \overset{\mathcal{RP}}{\to} \overline{\mathcal{RA}}_5^t \overset{\beta}{\to} \mathbb{P}^{15}/G_4$ and we can blow up the tower so that both maps are finite. We denote these spaces by $\widetilde{\mathcal{R}^2\mathcal{M}_6} \to \widetilde{\mathcal{RA}_5} \to \widetilde{\mathbb{P}}$. Set $X = \widetilde{\mathcal{RA}_5} \times_{\mathbb{P}} \widetilde{\mathcal{RA}_5}$. For each point $(x, y) \in X$, we can associate the distance in $\Gamma_{Q_8^-}$, or ∞ if they are not connected. This gives a decomposition $X = I_1 \cup I_{54} \cup I_{64} \cup I$, where I_1 is the diagonal, I_{54} are theta related pairs, I_{64} the pairs theta related to a common point and I the pairs in the same fiber but not theta related. Our goal is to show that $I = \emptyset$.

We will focus on studying I_{54}. It fits into a diagram



where I_{27} is the pullbac of the Prym map to $\widetilde{\mathcal{RA}_5}$ from $\mathcal{A}_5$, and $I_{54} \to I_{27}$ is teh natural double cover. Differentiating these maps, we can see that $\operatorname{Ram}\tilde{\beta} = \operatorname{Ram}\tilde{\mathcal{P}}$.

Lemma 4.4. *Let $f : X \to Y$ be a finite morphism of smooth varieties and $\bar{f} : X \times_Y X \to X$ be the pullback of f along f. Then if p is a ramification point of f and q is a nonramification point in the same fiber, (p, q) is a ramification point of $\bar{f}$.*

Proof. Set $b = f(q) = f(p)$. As p is a ramification point and q is not, we have df_q

an isomorphism and df_p not. Locally, this means that we have an isomorphism of X and Y near q but not near p, and so the map $d\bar{f}_{(p,q)} : T_{p,q}X \times_Y X \to T_qX$ is not an isomorphism. $\square$

Lemma 4.5. *Let $p \in \partial^{II}\widetilde{\mathcal{RA}}_5$. Then if β_5 is ramified at p, I_{54} is ramified over p.*

Proof. Let $p \in \partial^{II}\widetilde{\mathcal{RA}}_5$ be a ramification point of $\tilde{\beta}$. Then either there exists a q such that $(p,q) \in I_{54}$ is unramified, or else not. If there is such a q, then this follows from lemma 4.4. If not, then we must have a fiber of the Prym map which is totally ramified, and this can be seen to be impossible by examining the branch locus of $\mathcal{P}_6$, the quartic double solids[Don92]. $\square$

More useful than the lemma is the contrapositive: that if I_{54} is unramified over p, then β_5 is unramified at p.

Proposition 4.6. *β_5 is generically unramified on $\partial^{II}\overline{\mathcal{RA}}_5^t$.*

Proof. Lemma 4.5 tells us that a point is a ramification point for β_5 only if it is a branch point of $\tilde{\beta}$, which is the same as being a branch point over $\tilde{\mathcal{P}}$. But Izadi's theorem 4.3 tells us that the Prym map is generically unramified over $\partial^{II}\overline{\mathcal{RA}}_5^t$, and so $I_{54} \to \widetilde{\mathcal{RA}}_5$ is generically unramified over $\partial^{II}\widetilde{\mathcal{RA}}_5$, and so, β_5 is as well. $\square$

This so, we have the main result of this chapter:

Theorem 4.7. *the fibers of $\beta_5 : \overline{\mathcal{RA}}_5^t \to \mathbb{P}^{15}/G_4$ are exactly the orbits of theta symmetry.*

Chapter 5

Contracted Loci

Now, we've identified the fibers of β_5 with the orbits of theta symmetry, which gives us the degree of this generically finite map. What remains is to understand the locus where β_5 has infinite fibers. As each abelian variety has finitely many points of order two, and as α_g is finite to one and generically injective for all g [SM94] (in fact, α_g is conjectured to be injective), any infinite fiber must arise from an infinite fiber of the Prym map.

Donagi showed in [Don81] that given $(C, \mu) \in \mathcal{RM}_6$, we obtain the rest of the fiber containing (C, μ) by iterating the tetragonal construction, so long as we allow points to be related via the boundary. Additionally, in [Don92], it is shown that the tetragonal relation on $\overline{\mathcal{RM}}_6 \times \overline{\mathcal{RM}}_6$ is stable after two iterations, and so there cannot be any infinite chains of curves with finitely many g^1_4's. Thus, infinite fibers will correspond to curves with infinitely many g^1_4's and to singular curves.

Theorem 5.1 ([ACGH85, Mum74]). *Let C be a smooth non-hyperelliptic curve of genus $g \geq 4$. Let $d, r \in \mathbb{Z}$ with $0 < 2r \leq d$ and $2 \leq d \leq g-2$. Then, if there exists a $(d-2r-1)$-dimensional family of g^r_d's on C, we must have that C is trigonal, bielliptic or a plane quintic.*

Fixing $g = 6$, $d = 4$ and $r = 1$, the hypotheses on theorem 5.1 are satisfied, and so a curve of genus 6 that has a positive dimensional family of g^1_4's must be hyperelliptic, trigonal, bielliptic, a plane quintic, or singular.

By results in [Don92, Rec74], the image of these loci under $\mathcal{P}_6$ is contained in the union of the locus of products, $\mathcal{J}_5$, $\mathcal{C}$, $\partial\mathcal{A}_5$, the locus of bielliptic Pryms and the locus of Beauville Pryms.

But we actually only care about contracted loci over $\alpha_4(\mathcal{A}_4)$, which may be irreducible components of $\mathcal{RS}_5$. As $\dim(\alpha_4(\mathcal{A}_4)) = 10$, any irreducible component of $\mathcal{RS}_5$ must have dimension at least 10. Also, as we will see later, $\overline{\mathcal{RJ}}_5$, $\overline{\mathcal{RC}}^0$, and $\partial^I\overline{\mathcal{RA}}_5$ are the only components of $\mathcal{RS}_5$ with positive local degree, thus, any other component is actually a contracted locus.

Proposition 5.2. *The only component of $\beta_5^{-1}(\alpha_4(\mathcal{A}_4))$ that is blown down by β is $\mathcal{RA}_1 \times \mathcal{A}_4$.*

Proof. The only loci with positive dimensional fibers are $\mathcal{RJ}_5$, $\mathcal{RB}_0$, $\mathcal{RB}_1$, $\mathcal{RB}_2$, $\mathcal{RC}^0$, $\mathcal{RC}^1$, $\partial^I\overline{\mathcal{RA}}_5^t$, $\partial^{II}\overline{\mathcal{RA}}_5^t$, $\partial^{III}\overline{\mathcal{RA}}_5^t$, and $\mathcal{RP}_6(\partial^{III}\overline{\mathcal{R}^2\mathcal{M}}_6)$.

The loci $\mathcal{RJ}_5$, $\mathcal{RC}^0$ and $\partial^I\overline{\mathcal{RA}}_5^t$ actually have positive local degree (see next chapter) and so are not blown down.

The bielliptic loci are ruled out by [Sho82, Deb88a, Bea89], which show that for a Prym to be a Jacobian, the base curve must be hyperelliptic, trigonal, or a plane quintic.

The locus $\mathcal{RC}^1$ is not in $\mathcal{RS}_5$, as any point of $\mathcal{RC}^1$ will only be theta related to other points of $\mathcal{RC}^1$. Specifically, a curve over $(X, \mu) \in \mathcal{RC}^1$ is must be a plane quintic Q and an odd point of order two ν with $P(Q, \nu) = X$, but also the lifts of μ must be odd, so all three points in any instance of theta symmetry give points of $\mathcal{RC}^1$.

In Donagi's survey [Don88], it is shown that $\partial^{III}\overline{\mathcal{RA}}_5^t$, the Beauville Pryms, $\mathcal{RA}_4 \times \mathcal{A}_1$ and $\mathcal{RA}_4 \times \mathcal{RA}_1$ are not in the Schottky locus, and all other product loci other than $\mathcal{RA}_1 \times \mathcal{A}_4$ are of two small dimension. However, $\mathcal{RA}_1 \times \mathcal{A}_4$ is contracted by β_5.

The only remaining component is $\partial^{II}\overline{\mathcal{RA}}_5^t$, but we know this locus is mapped to $\mathbb{P}^7/G_3$, and so any points of it in $\mathcal{RS}_5$ must be the limits of points of other components. □

Chapter 6

Schottky-Jung Relations

Finally, we prove our main theorem:

Theorem 6.1. *In $\overline{\mathcal{RA}}_5^t$, we have $\overline{\mathcal{RS}}_5 = \overline{\mathcal{RJ}}_5 \cup \overline{\mathcal{RC}^0} \cup \partial^I \overline{\mathcal{RA}}_5^t \cup \mathcal{A}_4 \times \mathcal{RA}_1$.*

Proof. As the general fiber of β is a connected Q_6^--configuration, we have that $\deg \beta = 119$. Then proposition 5.2 tells us that the only blowndown component of $\mathcal{RS}_5$ is $\mathcal{A}_4 \times \mathcal{RA}_1$, so it remains to identify the components with positive local degree.

Izadi [Iza91] showed that β_5 restricts to a birational map $\overline{\mathcal{RC}^0} \to \alpha(\overline{\mathcal{A}}_4)$, and thus has local degree one.

The Schottky-Jung relations imply that the restriction to $\overline{\mathcal{RJ}}_5$ is birational to the Prym map $\mathcal{RM}_5 \to \mathcal{A}_4$. Thus, over $A \in \mathcal{A}_4$, we get a double cover of the Fano surface of a cubic threefold. However, when we blow up to ge ta finite map, we get a double cover of the 27 lines on a cubic surface, and so we have local degree 54.

In [vGvdG86], van Geemen and van der Geer compute the lcoal degree on $\partial^I \overline{\mathcal{RA}}_5^t$ as follows: fix $X \in \mathcal{A}_4$. Then the part of the fiber of β_5 is $K(X)$ and by blowing up, they showed that the degree of the map is the same as that of the map $K(X) \to \mathbb{P}^4$ given by the linear system $\Gamma_{00} = \{s \in \Gamma(X, 2\Theta) | \operatorname{mult}_0 s \geq 4\}$. We blow X up at 0, call the exceptional divisor E, and then this linear system is precisely $2\Theta - 4E$ on the abelian variety. Thus, the local degree is $\frac{1}{2}(2\Theta - 4E)^4 = 64$.

This gives us degree $1 + 54 + 64 = 119$, and so there are no other components with positive local degree. □

And so, as noted in [Don87b], this implies

Corollary 6.2. $\mathcal{S}_5^{\text{small}} = \mathcal{J}_5$

Proof. The points of $\mathcal{S}_5^{\text{small}}$ are just the abelian varieties A such that $(A, \mu) \in \mathcal{RS}_5$ for all nonzero points of order two $\mu \in A$. Among the components above, Jacobians have all nonzero points of order two, but the intermediate Jacobians of cubic threefolds only have even points of order two, the boundary component has only the vanishing cycle, and the product locus only has pullbacks of the points of order two on the elliptic curve. □

Bibliography

[ACGH85] Enrico Arbarello, M. Cornalba, Phillip A. Griffiths, and Joseph Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[AM67] Aldo Andreotti and A. L. Mayer. On period relations for abelian integrals on algebraic curves. *Ann. Scuola Norm. Sup. Pisa (3)*, 21:189–238, 1967.

[Bea77] Arnaud Beauville. Prym varieties and the Schottky problem. *Invent. Math.*, 41(2):149–196, 1977.

[Bea82] Arnaud Beauville. Sous-variétés spéciales des variétés de Prym. *Compositio Math.*, 45(3):357–383, 1982.

[Bea89] Arnaud Beauville. Prym varieties: a survey. In *Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987)*, volume 49 of *Proc.*

Sympos. Pure Math., pages 607–620. Amer. Math. Soc., Providence, RI, 1989.

[CM08] Sebastian Casalaina-Martin. Singularities of theta divisors in algebraic geometry. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 25–43. Amer. Math. Soc., Providence, RI, 2008.

[CvdG00] Ciro Ciliberto and Gerard van der Geer. The moduli space of abelian varieties and the singularities of the theta divisor. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 61–81. Int. Press, Somerville, MA, 2000.

[Deb88a] Olivier Debarre. Sur les variétés abéliennes dont le diviseur theta est singulier en codimension 3. *Duke Math. J.*, 57(1):221–273, 1988.

[Deb88b] Olivier Debarre. Sur les variétés de Prym des courbes tétragonales. *Ann. Sci. École Norm. Sup. (4)*, 21(4):545–559, 1988.

[Deb90] Olivier Debarre. Variétés de Prym et ensembles d'Andreotti et Mayer. *Duke Math. J.*, 60(3):599–630, 1990.

[Don81] Ron Donagi. The tetragonal construction. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):181–185, 1981.

[Don87a] Ron Donagi. Big Schottky. *Invent. Math.*, 89(3):569–599, 1987.

[Don87b] Ron Donagi. Non-Jacobians in the Schottky loci. *Ann. of Math. (2)*, 126(1):193–217, 1987.

[Don88] Ron Donagi. The Schottky problem. In *Theory of moduli (Montecatini Terme, 1985)*, volume 1337 of *Lecture Notes in Math.*, pages 84–137. Springer, Berlin, 1988.

[Don92] Ron Donagi. The fibers of the Prym map. In *Curves, Jacobians, and abelian varieties (Amherst, MA, 1990)*, volume 136 of *Contemp. Math.*, pages 55–125. Amer. Math. Soc., Providence, RI, 1992.

[DS80] Ron Donagi and Roy Campbell Smith. The degree of the Prym map onto the moduli space of five-dimensional abelian varieties. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 143–155. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[DS81] Ron Donagi and Roy Campbell Smith. The structure of the Prym map. *Acta Math.*, 146(1-2):25–102, 1981.

[GSM07] Samuel Grushevsky and Riccardo Salvati Manni. Singularities of the theta divisor at points of order two. *Int. Math. Res. Not. IMRN*, (15):Art. ID rnm045, 15, 2007.

[Igu81] Jun-ichi Igusa. On the irreducibility of Schottky's divisor. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):531–545 (1982), 1981.

[Iza91] Elham Izadi. *On the moduli space of four dimensional principally polarized abelian varieties*. PhD thesis, Univ. of Utah, 1991.

[JS09] H. Jung and F. Schottky. Neue sätze über symmetralfunctionen und die abelschen funktionen. *S.-B. Berlin Akad. Wiss.*, 1909.

[Kri10] Igor Moiseevich Krichever. Characterizing Jacobians via trisecants of the Kummer Variety, arXiv. org: math. *AG/0605625*, 2010.

[Mum74] David Mumford. Prym varieties. I. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 325–350. Academic Press, New York, 1974.

[PP08] Giuseppe Pareschi and Mihnea Popa. Castelnuovo theory and the geometric Schottky problem. *J. Reine Angew. Math.*, 615:25–44, 2008.

[Rec74] Sevin Recillas. Jacobians of curves with g^1_4's are the Prym's of trigonal curves. *Bol. Soc. Mat. Mexicana (2)*, 19(1):9–13, 1974.

[RF74] Harry E. Rauch and Hershel M. Farkas. *Theta functions with applications to Riemann surfaces*. The Williams & Wilkins Co., Baltimore, Md., 1974.

[Sch88] F. Schottky. Zur theorie der abelschen funktionen von vier variablen. *J. Reine und Angew. Math.*, (102):304–352, 1888.

[Shi86] Takahiro Shiota. Characterization of Jacobian varieties in terms of soliton equations. *Invent. Math.*, 83(2):333–382, 1986.

[Sho82] V. V. Shokurov. Distinguishing Prymians from Jacobians. *Invent. Math.*, 65(2):209–219, 1981/82.

[SM94] Riccardo Salvati Manni. Modular varieties with level 2 theta structure. *Amer. J. Math.*, 116(6):1489–1511, 1994.

[vGvdG86] Bert van Geemen and Gerard van der Geer. Kummer varieties and the moduli spaces of abelian varieties. *Amer. J. Math.*, 108(3):615–641, 1986.