NONSYMMETRIC SCHUR FUNCTIONS

Sarah Katherine Mason

A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2006

Advisor's Name
Supervisor of Dissertation

Graduate Chair's Name
Graduate Group Chairperson

Acknowledgments

My advisor, James Haglund, is everything an advisor should be. He has mastered the art of being available and encouraging while gradually pushing me toward mathematical independence. I am thankful for all his advice about jobs, publications, conferences, and much more.

I am grateful for the friendly and encouraging environment at Penn. Dennis DeTurck, Ron Donagi, Herman Gluck, Steve Shatz, and Herb Wilf are among the many wonderfully positive influences. The office staff (Paula Airey, Janet Burns, Monica Pallanti, and Robin Toney) do an amazing job and are always so much fun!

My officemates Christine Devena, Jimmy Dillies, and Bill Semus kept me smiling during my first few years. Asher Auel, Scott Corry, and Sukhendu Mehrotra have been incredibly patient with my mood swings and stress.

I would not have gone to graduate school if not for Jason Cantarella's encouragement and support. Bruce Sagan has inspired me to explore new areas of research. I am grateful to the Penn Club Swim Team for providing a wonderful avenue for my favorite form of stress release.

My parents taught me to never give up on my dreams. I want to thank my whole family and my best friend Shelene Hurst for their belief in me and for helping me to remain focused on the big picture. Jason Parsley has been my rock during a tumultous time. I am eternally grateful for all of his patience, love, support, advice, stability, and encouragement. I couldn't have done this without him by my side!

ABSTRACT

NONSYMMETRIC SCHUR FUNCTIONS

Sarah Katherine Mason

James Haglund, Advisor

We exhibit a weight-preserving bijection between semi-standard Young tableaux and skyline augmented fillings to give the first combinatorial proof that $s_\lambda(x) = \sum_\gamma NS_\gamma(x)$. The bijection involves an insertion procedure reminiscent of Schensted insertion, which leads us to an analogue of the Robinson-Schensted-Knuth Algorithm. A partial ordering on the set of composition of n into infinitely many parts is used to prove that the nonsymmetric Schur functions are a basis for all polynomials. The poset given by this ordering is isomorphic to the poset $L(n, m-1)$ of partitions which fit inside an $m \times n-1$ rectangle. We also describe a non-recursive combinatorial interpretation of the standard bases of Lascoux and Schützenberger. This construction provides a simple method to determine the right key of a semi-standard Young tableau.

Contents

List of Figures

List of Tables

Chapter 1

Introduction and Background

Symmetric functions are vital to the study of combinatorics because they provide valuable information about partitions and permutations, topics which constitute the core of the subject. Symmetric function theory also has connections to other branches of mathematics, including group theory, representation theory, Lie algebras, and algebraic geometry. The Schur functions are a symmetric function basis with many applications. They contain a vast array of information about other symmetric function bases and the underlying algebra and combinatorics. The fundamental goal of this dissertation is to develop the theory of "nonsymmetric Schur functions", which decompose the Schur functions. Many of the results are combinatorial proofs of identities which were originally observed algebraically through the study of Macdonald polynomials, a (q, t)-analogue of symmetric functions. The recent combinatorial descriptions of Macdonald polynomials [7] and nonsymmetric Macdonald polynomials [6] provide the motivation and tools necessary for this combinatorial approach.

1.1 Symmetric functions

A *symmetric function* is a formal power series $f(x)$ in the variables $X = \{x_1, x_2., , , \}$ where $f(x_{\sigma(1)}, x_{\sigma(2)}, ...) = f(x_1, x_2, ...)$ for every permutation σ of the positive integers $\mathbb{N}$. The set of all homogeneous symmetric functions of a given degree n forms a vector space over the rational numbers, denoted Λ^n. Given $f \in \Lambda^n$ and $g \in \Lambda^m$, the product (as formal power series) fg is in Λ^{m+n}. Therefore we define $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus ...$ to be the vector space direct sum of the Λ^n, considered as a graded algebra. A common theme in the theory of symmetric functions is the description of bases for the vector space Λ^n and their relationships and properties [21].

There are many different $\mathbb{Q}$-bases for symmetric functions. Sections 1.1.1 and 1.1.2 describe several of these bases. Section 1.2 introduces the combinatorial object commonly used to describe the Schur functions.

1.1.1 Monomial symmetric functions

The *monomial symmetric functions* are the functions obtained by symmetrizing a monomial. To understand this symmetrization, we need the concept of a partition.

A *partition* of n is a weakly decreasing sequence of positive integers

$\lambda = (\lambda_1, \lambda_2, \ldots \lambda_l)$ such that the sum $\lambda_1 + \lambda_2 + \ldots + \lambda_l$ equals n. The integers λ_i

are called the *parts* of λ. The notation $\lambda \vdash n$ implies that λ is a partition of n. For example, if $n = 14$, then $\lambda = (5, 3, 3, 2, 1)$ is a partition of n and we write $\lambda \vdash 14$. The *length,* $l(\lambda)$ is the number of parts of λ. Therefore, if $\lambda = (5, 3, 3, 2, 1)$, then $l(\lambda) = 5$.

A *composition* (resp. *weak composition*) of n is a sequence of positive (resp. non-negative) integers which sum to n. For example, $5 + 3 + 1$ and $3 + 1 + 5$ are both equivalent to $(5, 3, 1)$ as partitions, but as compositions $(5, 3, 1) \neq (3, 1, 5)$.

Definition 1.1.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of some integer n. Then the *monomial symmetric function* m_λ corresponding to λ is

$$m_\lambda(\mathbf{x}) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_l}^{\lambda_l}$$

where the sum is over all distinct monomials with exponents $\lambda_1, \ldots, \lambda_l$ in the variables $\mathbf{x} = \{x_1, x_2, \ldots\}$.

This amounts to beginning with the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_l^{\lambda_l}$ and permuting the indices of the x-variables in all possible distinct ways.

For example,

$$m_\emptyset = 1$$

$$m_1 = \sum_i x_i$$

$$m_2 = \sum_i x_i^2$$

$$m_{1,1} = \sum_{i<j} x_i x_j$$

$$m_{2,1} = \sum_{i \neq j} x_i^2 x_j$$

Each symmetric function can clearly be written as a sum of monomial functions with rational coefficients. This implies that the set $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ^n. Since the monomials functions are indexed by partitions, the dimension of this vector space is equal to the number of partitions of n, denoted $p(n)$.

Let Par_n be denote the set of all partitions of n and let Par denote the set of all partitions. Then the set $\{m_\lambda(\mathbf{X}) : \lambda \in Par_n\}$ is a basis for Λ^n and the set $\{m_\lambda(\mathbf{X}) : \lambda \in Par\}$ is a basis for Λ.

1.1.2 Other symmetric function bases

The *elementary symmetric functions,* e_λ can be considered as a product of certain monomial symmetric functions.

Definition 1.1.2.

$$e_0 = m_\emptyset = 1$$

$$e_n = m_{1^n} = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \qquad (n \geq 1)$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots \qquad \textit{for} \quad \lambda = (\lambda_1, \lambda_2, \ldots)$$

The set $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ^n and the set $\{e_\lambda : \lambda \in Par\}$ is a basis for Λ. For each λ, e_λ can be written as a positive sum of monomial functions.

One defines the *homogeneous symmetric functions,* h_λ by the formulas

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} x_{i_1} x_{i_2} \ldots x_{i_n}, \quad n \geq 1 \qquad (h_0 = m_\emptyset = 1)$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots \qquad \textit{for} \quad \lambda = (\lambda_1, \lambda_2, \dots).$$

Any homogeneous symmetric function can be written as a linear combination of monomial symmetric functions, with positive integer coefficients. If

$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu,$$

then $N_{\lambda\mu}$ is the number of ways to place n balls (with λ_i of these balls labeled i) into boxes so that the i^{th} box contains exactly μ_i balls [21].

The homogeneous symmetric functions form another basis for the algebra Λ of symmetric functions. They are dual to the elementary symmetric functions in the sense that there exists an endomorphism ω such that $\omega(h_\lambda) = e_\lambda$ and ω is an involution.

Definition 1.1.3. The *power sum symmetric functions,* p_λ are defined by

$$p_n = m_n = \sum_i x_i^n, \quad n \geq 1 \quad (p_0 = m_\emptyset = 1)$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \qquad \textit{for} \quad \lambda = (\lambda_1, \lambda_2, \dots).$$

The set $\{p_\lambda : \lambda \vdash n\}$ is a basis for Λ^n, so $\{p_\lambda : \lambda \in Par\}$ is a basis for Λ. Both the elementary and the homogeneous symmetric functions can be written in terms of the power sum symmetric functions. Given a partition λ, let m_1 be the number of ones in λ, m_2 be the number of twos in λ, and so on. Define

$$z_\lambda = 1^{m_1}\, m_1!\; 2^{m_2}\, m_2!\; \dots .$$

Then

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \tag{1.1.1}$$

$$e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{p_\lambda}{z_\lambda} \tag{1.1.2}$$

where $\varepsilon_\lambda = (-1)^{n-l(\lambda)}$

These equations and the definition of the involution ω imply that

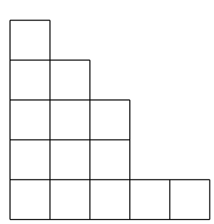
$$\omega(p_\lambda) = \varepsilon_\lambda p_\lambda.$$

Therefore p_λ is an eigenvector for ω, with eigenvalue ε_λ. For the proof of Equations 1.1.1 and 1.1.2 and a more detailed discussion of symmetric function bases, see Stanley [21].

1.2 Semi-standard Young tableaux

Each partition λ of n is associated to a collection of squares (or *cells*) called a *Ferrers diagram,* $dg(\lambda)$ or *Young diagram.* The i^{th} row of the Ferrers diagram consists of λ_i cells. Note that we use the French style of Ferrers diagram. (We often abuse notation by identifying λ with its Ferrers diagram.)

Example 1.2.1. *The Ferrers diagram of* $\lambda = (5, 3, 3, 2, 1)$



The *conjugate,* λ', of a partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_l)$ is the shape derived by transposing the diagram of λ. So the conjugate of $\lambda = (5, 3, 3, 2, 1)$ is $\lambda' = (5, 4, 3, 1, 1)$.

A *filling* of shape λ is a map $T : dg(\lambda) \to \mathbb{N}$. A *semi-standard Young tableau (SSYT)* of shape λ is a filling of λ such that T is weakly increasing along each row of λ and strictly increasing along each column. An SSYT T can be visualized as a filling of the cells of λ with positive integers so that the rows are weakly increasing and the columns are strictly increasing.

A *standard Young tableau (SYT)* of shape $\lambda \vdash n$ is a special type of SSYT, in which the map T is a bijection from $dg(\lambda)$ to the set $[n] = \{1, 2, ..., n\}$. In this case, T is a filling of the cells of λ with each of the integers from 1 to n so that the rows and columns are strictly increasing. (Similarly, a *standard filling* is a filling of the cells of λ with each of the integers from 1 to n, allowing no repeats.)

The *reading word,* $read(T)$ of an SSYT T is determined by scanning the entries of T from left to right, beginning with the bottom row and moving bottom to top, recording each entry as it is seen.

Any SSYT T is converted into a standard Young tableau through the following *standardization* process. Let $\{1^{a_1}, 2^{a_2}, ...m^{a_m}\}$ be the multiset of entries in T. In $read(T)$, convert the j^{th} 1 to the number j, the k^{th} 2 to the number $a_1 + k$, and so forth. (See Figure 1.1.)

The multiset of positive integers which appear in an SSYT T is called the *content* of T. Unless otherwise specified, we consider SSYT whose entries are

8				
7	9			
4	5	7		
2	2	4		
1	1	2	3	7

T

13				
12	14			
8	9	11		
4	5	7		
1	2	3	6	10

T'

$read(T) = 1\ 1\ 2\ 3\ 7\ 2\ 2\ 4\ 4\ 5\ 7\ 7\ 9\ 8$ $\qquad$ $read(T') = 1\ 2\ 3\ 6\ 10\ 4\ 5\ 7\ 8\ 9\ 11\ 12\ 14\ 13$

Figure 1.1: An SSYT T and its standardization T'.

less than or equal to n. It is not difficult to extend most results to the case where the entries may be greater than n. We denote the content by the tuple $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n) \in \mathbb{Z}^n$, where α_i equals the number of times the letter i appears in T. The content of the SSYT T in Figure 1.1 is $\{1, 1, 2, 2, 2, 3, 4, 4, 5, 7, 7, 7, 8, 9\}$, denoted $(2, 3, 1, 2, 1, 0, 3, 1, 1, 0, 0, 0, 0, 0)$.

Given a partition $\lambda \vdash n$ and a multiset μ of n positive integers, the *Kostka number* $K_{\lambda,\mu}$ is defined as the number of SSYT of shape λ and content μ. Much research has been devoted to the study of Kostka numbers. In particular, it would be useful to have a simple, quick formula to compute $K_{\lambda,\mu}$. The next section describes a special case in which such a formula is known.

1.2.1 The Hook Length Formula

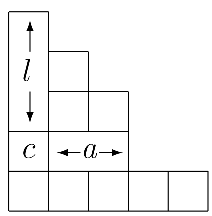
The number of standard Young tableaux of shape λ is the Kostka number $K_{\lambda,1^n}$. This number is often denoted f_λ. It can be easily calculated using the *Hook Length*

Formula of Frame, Robinson, and Thrall [2]. To state the Hook Length Formula, we need the following definitions.

Definition 1.2.2. Let c be a cell contained in a Ferrers diagram λ. Its *arm,* $a(c)$, equals the number of cells to the right of c in the same row of λ as c. Its *leg,* $l(c)$, equals the number of cells above c in the same column of λ as c.

Definition 1.2.3. Let c be a cell contained in a Ferrers diagram λ. The *hook length of* c, denoted $h(c)$, is the arm of c plus the leg of c plus 1. Thus, $h(c) = a(c)+l(c)+1$.

Example 1.2.4. $h(c) = 2 + 3 + 1 = 6$



Theorem 1.2.5. *[2] Given a partition λ of n,*

$$f_\lambda = \frac{n!}{\prod_{c\in\lambda} h(c)},$$

where $h(c)$ is the hook length of the cell c.

There are many beautiful proofs of this amazing result. The probabilistic proof of Greene, Nijenhuis, and Wilf [4] relies on a recursive formula for the number of standard Young tableaux. Several intriguing bijective proofs have been discovered including those by Zeilberger [22] and Novelli, Pak, and Stoyanovskii [16].

1.2.2 The Robinson-Schensted-Knuth Algorithm

The Robinson-Schensted-Knuth (RSK) Algorithm [19] provides a connection between permutations and standard Young tableaux. There are many useful consequences of this algorithm. The *Cauchy identity*, which relates products of Schur functions to a geometric series, can be derived using the RSK algorithm. The RSK Algorithm provides information about plane partitions and permutation enumeration.

Theorem 1.2.6 ([19]). *There exists a bijection between* $\mathbb{N}$*-matrices of finite support and pairs* (P, Q) *of SSYT of the same shape.*

The main operation involved in the RSK Algorithm is the *Schensted row insertion* $P \leftarrow k$ of a positive integer k into a semi-standard Young tableau $P = (P_{ij})$. (Here i denotes the row and j denotes the column containing the entry P_{ij}.) Let r be the largest integer such that $P_{1,r-1} \leq k$. If such an r does not exist, place k at the end of the first row and the procedure is complete. Otherwise, k 'bumps' $P_{1,r} = k'$ into the second row and the procedure is repeated in the second row for k'. Continue this procedure until an element is inserted at the end of a row (possibly as the first element of a new row). The resulting diagram is $P \leftarrow k$.

To each $\mathbb{N}$-matrix A of finite support, there is an associated two-line array,

$$w_A = \begin{pmatrix} i_1 & i_2 & \dots \\ \\ j_1 & j_2 & \dots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 1 & 4 & 2 & 3 \end{pmatrix} \xrightarrow{\text{RSK}} \underset{P}{\begin{array}{|c|c|c|c} \hline 2 & 4 & 4 & \multicolumn{1}{|c}{} \\ \hline 1 & 1 & 2 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}} \qquad \underset{Q}{\begin{array}{|c|c|c|c} \hline 2 & 2 & 3 & \multicolumn{1}{|c}{} \\ \hline 1 & 1 & 2 & \multicolumn{1}{|c|}{4} \\ \hline \end{array}}$$

Figure 1.2: The RSK algorithm

to which the entry k in the i^{th} row and j^{th} column of A contributes k copies of the pair (i, j). (The entries of A contribute to ω_A in the top-to-bottom, left-to-right order.)

Begin with a pair $(P(0), Q(0)) = (\emptyset, \emptyset)$ and let $P(t+1) = P(t) \leftarrow j_{t+1}$. Let $Q(t+1)$ be obtained from $Q(t)$ by placing i_{t+1} at the end of a row of $Q(t)$ so that $Q(t+1)$ has the same shape as $P(t+1)$. The result is the pair (P, Q), where P is the *insertion tableau* and Q is the *recording tableau*. (See Figure 1.2.)

This operation in invertible. Find the lowest row of Q in which the largest entry occurs. Delete this entry. It is the last element in the top row of ω_A. The entry c in the corresponding cell of P is the last entry to be inserted at the end of a row. Delete c and read right to left through the row below c to determine which entry bumped c. Replace this entry with c and repeat the procedure in the next row down. Continue in this manner until the entry which bumped an entry from the first row is determined. This is the last element of the lower row in the array ω_A. Continue in this manner until there are no entries left in P and Q. The resulting array is the inverse of the pair (P, Q).

Consider a matrix A which describes a permutation. Then each number from 1 to n appears exactly once in the bottom row and once in the top row of the array ω_A. Therefore the RSK algorithm sends A to a standard Young tableau. This special case of the RSK algorithm provides a bijection between permutations and pairs of standard Young tableaux of the same shape. The number of permutations of n letters is $n!$, so

$$n! = \sum_{\lambda \vdash n} f_\lambda^2.$$

The RSK algorithm has a surprising symmetry property. Let A^t be the transpose of the matrix A. Then $RSK : A \to (P,Q)$ implies that $RSK : A^t \to (Q,P)$. In the special case that A is a permutation matrix, this means that inverting the permutation switches P and Q.

1.3 Schur functions

There are several different ways to define the Schur functions. We use semi-standard Young tableaux to describe the Schur functions combinatorially. An algebraic formula for the Schur functions involving quotients of determinants can be found in [21].

1.3.1 Combinatorial definition

The Schur functions, like the other symmetric functions, are indexed by partitions. We begin with a partition λ and describe the Schur functions s_λ.

Let $SSYT(\lambda)$ be the set of all semi-standard Young tableaux of shape λ. To every semi-standard Young tableau $T \in SSYT(\lambda)$, we may associate a monomial called the *weight* of T, denoted x^T. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be the content of T. Then

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

and

$$s_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^T.$$

Notice that $s_\lambda(x)$ is a formal power series in infinitely many variables $\{x_1, x_2, \ldots\}$. To restrict to the variables $\{x_1, x_2, \ldots x_m\}$, set $x_i = 0$ for $i > m$. Since any SSYT with an entry greater than m will contribute 0 to this polynomial, this is equivalent to considering only SSYT with entries in the set $[m] = \{1, 2, \ldots, m\}$. Unless otherwise specified, if $\lambda \vdash n$, we consider s_λ as a polynomial in the variables $\{x_1, x_2, \ldots x_n\}$.

Example 1.3.1. *Let $\lambda = (2, 1)$. Then $SSYT(\lambda)$ is the collection*

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

and $s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$

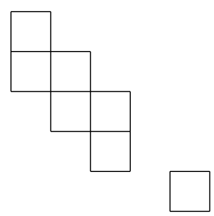
Note that the coefficient of a monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ in s_λ is the Kostka number for shape λ and content $(\alpha_1, \alpha_2, \ldots \alpha_n) = \alpha$. So

$$s_\lambda = \sum_\alpha K_{\lambda,\alpha} x^\alpha,$$

where the sum is over all weak compositions α of n.

We wish to describe a procedure for multiplying two Schur functions. To do so, we need several defintions.

Let λ and μ be two partitions such that $\mu_i \leq \lambda_i$ for all i. Then the *skew shape* λ/μ is determined by removing the first μ_i cells in the i^{th} row of the diagram of λ (for all i). For example, if $\lambda = (5, 3, 3, 2, 1)$ and $\mu = (4, 2, 1)$ then $\lambda/\mu =$



A *skew tableau* is a skew shape whose cells are filled with numbers so that the rows are weakly increasing and the columns are strictly increasing. Its *skew reading word* is the word derived by reading the entries bottom to top, left to right.

Definition 1.3.2. A word is called a *reverse lattice word* if, read backwards, any subsequence contains at least as many 1's as 2's, at least as many 2's as 3's, and so on. For instance, 2 3 1 2 1 is a reverse lattice word but 1 2 3 2 1 is not.

A *Littlewood-Richardson skew tableau* is a skew tableau whose reading word is a reverse lattice word. The *Littlewood-Richardson number, $c^{\nu}_{\lambda\mu}$* is the number of Littlewood-Richardson skew tableau of shape ν/λ and content μ.

The following identity [3] describes the product of two Schur functions.

$$s_\lambda(x_1, \ldots, x_k) \cdot s_\mu(x_1, \ldots, x_k) = \sum_{\nu} c^{\nu}_{\lambda\mu} s_\nu(x_1, \ldots, x_k),$$

where the sum is over all ν such that $\nu_i \geq \lambda_i \ \forall i$ and $|\lambda| + |\mu| = |\nu|$. This identity provides a combinatorial method for multiplying two Schur polynomials.

1.3.2 Connections to representation theory

The *symmetric group* S_n is the group of automorphisms of $[n] = \{1, 2, \ldots, n\}$, acting on the left. The symmetric group acts on standard fillings by permuting the entries. Let T be a standard filling of shape λ and say $\pi \in S_n$. Then $\pi \cdot T$ sends the entry i in T to the entry $\pi(i)$.

Each standard filling T of shape λ has a corresponding subgroup of S_n consisting of the permutations under which the entries in each row of λ are invariant. This subgroup is called the *row group,* $R(T)$, of T. If $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$, then $R(T)$ is a product of groups $S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_n}$, called a *Young subgroup* of S_n. Similarly, there is the notion of the *column group,* $C(T)$, which consists of the permutations in S_n which fix each column of λ.

A *tabloid* is an equivalence class of fillings of a shape λ, where two fillings T and T' are equivalent $\iff$ $T' = \rho(T)$ for some $\rho \in R(T)$. The symmetric group acts on the set of tabloids by acting on a representative: for $\sigma \in S_n, \ \ \sigma \cdot \{T\} = \{\sigma \cdot T\}$. Therefore the orbit of $\{T\}$ under S_n action is isomorphic to the left coset $S_n/R(T)$.

Let M^λ be the complex vector space with the tabloids of shape λ as its basis, where λ is a partition of n. Since S_n acts on the set of tabloids, M^λ is a left

$\mathbb{C}[S_n]$-module under this action. For each filling T of λ, there exists an element

$$\nu_T = \sum_{q \in C(T)} sgn(q)\{q \cdot T\},$$

where $sgn(q)$ is the ordinary sign of a permutation. Define the *Specht module,* S^λ to be the subspace of M^λ spanned by the elements ν_T.

Proposition 1.3.3 ([3]). *For each partition λ of n, S^λ is an irreducible representation of S_n. Every irreducible representation of S_n is isomorphic to exactly one S^λ.*

Let R be the free abelian group generated by equivalence classes of irreducible representations of S_n and let $[M^\lambda]$ be the equivalence class represented by M^λ. Define the map $\phi : \Lambda \to R$ by the formula

$$\phi(h_\lambda) = [M^\lambda],$$

where h_λ is the homogeneous symmetric function. It is well known that $\phi(s_\lambda) = [S^\lambda]$. Therefore, the characters of the irreducible representations of the symmetric group are in bijection with the Schur functions s_λ.

Let $GL(n, \mathbb{C})$ be the *general linear group* of all invertible $n \times n$ matrices with complex entries. The irreducible polynomial representations of the general linear group can be indexed by partitions λ so that their characters are in bijection with the Schur functions s_λ [21].

1.3.3 The Cohomology of the Grassmannian

Let E be an m-dimensional vector space ($m < \infty$) and let $\mathbb{P}(E)$ be the projective space of lines through the origin in E. Then for $0 \leq d \leq m$ let $Gr^d E$ denotes the *Grassmannian* of subspaces of E with codimension d.

One sees immediately that $Gr^{m-1}E = \mathbb{P}(E)$ and $Gr^1 E = \mathbb{P}^*(E)$, where $\mathbb{P}^*(E) = \mathbb{P}(E^*)$ is the set of lines through the origin in the dual space E^*.

Definition 1.3.4. A *complete flag* is a nested sequence of subspaces such that there is exactly one subspace of each dimension.

Given a partition λ with at most $r = m - n$ rows and n columns and a fixed complete flag

$$F_\bullet : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = E$$

of subspaces with $dim(F_i) = i$, the *Schubert variety* $\Omega_\lambda = \Omega_\lambda(F_\bullet)$ is defined by

$$\Omega_\lambda = \{V \in Gr^n E : \ \dim(V \cap F_{n+i-\lambda_i}) \geq i, \ \ 1 \leq i \leq r\}.$$

Notice that if $\lambda = 0$, then no conditions are placed on V, so Ω_0 is the whole Grassmannian.

The equivalence classes $[\Omega_\lambda]$ of Schubert varieties in the cohomology group $H^{2|\lambda|}(Gr^n E)$ are independent of the choice of fixed flag since the group $GL(E)$ (of invertible linear transformations of E) acts transitively on flags. These classes $[\Omega_\lambda]$ form a basis for $H^*(Gr^n E)$.

Next define an additive homomorphism $\Lambda \to H^*(Gr^n(\mathbb{C}^m))$ which sends a Schur polynomial s_λ to $[\Omega_\lambda]$ if λ has at most r rows and n columns and to 0 otherwise. Since we can multiply two Schur polynomials using the Littlewood-Richardson rule, the Schur polynomials completely describe the cohomology ring of the Grassmannian. (For a more detailed discussion, see Fulton [3].)

1.4 Macdonald polynomials

The Macdonald polynomials $\tilde{H}_\mu(x; q, t)$ are a special class of polynomials which are symmetric in $\mathbf{x}$ with coefficients in q and t. They have generated extensive interest since their introduction by Ian Macdonald [13] in 1988. The recent explicit combinatorial formula for Macdonald polynomials [7] broadens the theory, providing a combinatorial tool for problems once primarily approached using algebraic geometry and representation theory.

1.4.1 Combinatorial description of Macdonald polynomials

The following combinatorial statistics defined on fillings are used to describe the Macdonald polynomials [5].

Definition 1.4.1. A *descent* of σ is a pair of entries $\sigma(u) > \sigma(v)$ such that the cell u is directly above v.

If $v = (i, j)$ and $u = (i + 1, j)$ (where the first coordinate denotes the row and the second denotes the column) then u, v is a descent if and only if $\sigma(u) > \sigma(v)$. Define

$$Des(\sigma) = \{u \in dg(\lambda) : \sigma(u) > \sigma(v) \text{ is a descent}\}.$$

Example 1.4.2. $Des(\sigma) = \{(2,1),\ (3,1),\ (4,2),\ (2,3)\}$

1				
4	7			
9	4	3		
3	5	7		
1	6	4	4	7

σ

The *reading order* of a shape $dg(\lambda)$ is a total ordering of the cells given by reading from left to right, top to bottom. The word derived from a filling σ by reading the entries in reading order is called the *reading word* of σ, denoted $read(\sigma)$. (Note that this is different from the skew reading order of a skew semi-standard Young tableau.

Let three cells (a, b, and c) be situated as follows, with a and c in the same row.

a	. . .	c
b		

Then the collection $\{a, b, c\}$ is said to be a *triple*.

Define for $x, y \in \mathbb{Z}_+$

$$I(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Definition 1.4.3. Let $\{a, b, c\}$ be a triple. This triple is called an *inversion triple* if and only if $I(\sigma(a), \sigma(c)) + I(\sigma(c), \sigma(b)) - I(\sigma(a), \sigma(b)) = 1$.

In other words, if ordering the entries in a, b, and c from smallest to largest induces a counter-clockwise orientation, then $\{a, b, c\}$ is an *inverision triple.* (If two of the entries are equal, the one which comes first in reading order is considered smaller.) Define

$$Inv(\sigma) = \{\{a, b, c\} : \{a, b, c\} \text{ is an inversion triple}\}.$$

Example 1.4.4. *There are three inversions in* σ*:* $\{9, 3, 4\}$ *and* $\{4, 5, 3\}$ *in the second and third row and* $\{5, 6, 7\}$ *in the first and second row.*

1				
4	7			
9	4	3		
3	5	7		
1	6	4	4	7

σ

For a filling, σ, define its *major index* ($maj(\sigma)$) and *inversion set* ($inv(\sigma)$)

$$maj(\sigma) = \sum_{u \in Des(\sigma)} (l(u) + 1)$$

$$inv(\sigma) = |Inv(\sigma)| - \sum_{u \in Des(\sigma)} a(u),$$

where $a(u)$ is the *arm* of u and $l(u)$ is the *leg* of u.

Theorem 1.4.5. *[5] The Macdonald polynomial has the following combinatorial description.*

$$\tilde{H}_\lambda(x; q, t) = \sum_{\sigma : \lambda \to \mathbb{Z}_+} q^{inv(\sigma)} t^{maj(\sigma)} x^\sigma$$

This formula implies that the coefficients of the Macdonald polynomials are polynomials in q and t. Many well-known and important families of symmetric functions can be derived from the Macdonald polynomials by specializing the q and t parameters in various ways. For example, the *Hall-Littlewood polynomials* [11] are the case $q = 0$ and the *Jack polynomials* [8], [9] are derived by setting $t = q^\alpha$ and letting $q \to 1$.

When the Macdonald polynomials are expanded in terms of Schur functions, the coefficients are the *Kostka-Macdonald coefficients,* $\tilde{K}_{\mu,\lambda}(q,t)$, a (q,t)-analogue of the Kostka numbers. In fact, $\tilde{K}_{\mu,\lambda}(0,t)$ yields the *t-Kostka coefficients,* $\tilde{K}_{\mu,\lambda}(t)$. Lascoux and Schützenberger provide a combinatorial description for the t-Kostka coefficients in [11].

1.4.2 Compositions and rearrangements

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n into l parts. Recall that the order does not matter, so we write λ in weakly decreasing order.

A *composition* of n is an ordered sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of positive integers which sum to n. A *weak composition* is an ordered sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of non-negative integers which sum to n. Unless otherwise specified, we use the term composition to denote a weak composition of n into n parts.

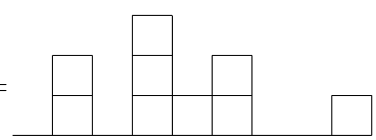
A composition γ is said to be a *rearrangement* of a partition λ if when the parts of γ are ordered in weakly decreasing order the result is the partition λ. This is denoted by $\gamma \sim \lambda$. To every partition λ, we may associate the collection R_λ of compositions which are rearrangements of λ.

For example, if $\lambda = (2,1)$ then

$$R_\lambda = \{(2,1,0),\ (2,0,1),\ (1,2,0),\ (1,0,2),\ (0,2,1),\ (0,1,2)\}.$$

Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ be a weak composition of n into n parts. (We will consider weak compositions of n into arbitrarily many parts in Section 2.2.) The *column diagram of* γ is a figure $dg'(\gamma)$ consisting of n *cells* arranged in n columns. The i^{th} column contains γ_i cells, and the number of cells in a column is called the *height* of that column. We place a horizontal line across the bottom simply to denote where the columns begin and end. This is an analogue of the Ferrers diagram of a partition λ, which consists of rows of cells such that the i^{th} row contains λ_i cells.

For example, the figure below is the column diagram of $\gamma = (0,2,0,3,1,2,0,0,1)$.

$dg'(\gamma) =$ 

If γ is a rearrangement of the partition λ, then the Ferrers diagram of λ is obtained by permuting the columns into weakly decreasing order and taking the conjugate shape.

1.4.3 Nonsymmetric Macdonald polynomials

Recall that the Macdonald polynomials are indexed by partitions λ of n. Since every partition can be considered as a collection of compositions, it is natural to ask whether the Macdonald polynomials can be decomposed into smaller polynomials indexed by compositions.

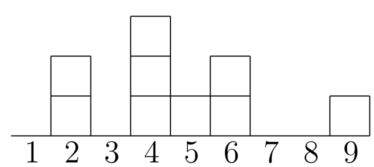
The smaller polynomials which decompose the Macdonald polynomials are referred to as the *nonsymmetric Macdonald polynomials.* The theory of the nonsymmetric Macdonald polynomials, $E_\gamma(x;q,t)$ is developed by Cherednik [1], Macdonald [14], and Opdam [17].

Haglund, Haiman, and Loehr [6] recently found a combinatorial description for the nonsymmetric Macdonald polynomials. The statistics involved in this description are analogues of the statistics *maj* and *inv* described in Section 1.4.1.

Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ be a weak composition of n into n parts. The *augmented diagram* of γ, defined by $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(i,0) : 1 \leq i \leq n\}$ is the column diagram with n extra cells adjoined in row 0. In this paper the adjoined row, called the *basement*, will always contain the numbers 1 through n in strictly increasing order. Therefore we will replace the cells with the numbers from 1 to n.

The augmented diagram for $\gamma = (0,2,0,3,1,2,0,0,1)$ is

$\widehat{dg}(\gamma) =$



An *augmented filling,* $\hat{\sigma}$, of an augmented column diagram, $\widehat{dg}(\lambda)$, is a function $\hat{\sigma} : \lambda \rightarrow \mathbb{Z}_+$, which we picture as an assignment of positive integer entries to the non-basement cells of λ. (Recall that the basement cells have already been assigned a positive integer entry.) Let $\hat{\sigma}(i)$ denote the entry in the i^{th} square of the augmented diagram encountered in reading order.

Definition 1.4.6. Two cells u and v of an augmented diagram are said to *attack* each other if either

1. they are in the same row, or
2. one is to the left of the other in the row below.

An augmented filling $\hat{\sigma} : \widehat{dg}(\gamma) \to [n]$ is called *non-attacking* if $\hat{\sigma}(u) \neq \hat{\sigma}(v)$ for every pair u, v of attacking cells.

As in [5], a *descent* of $\hat{\sigma}$ is a pair of entries $\hat{\sigma}(u) > \hat{\sigma}(v)$, where the cell u is directly above v. In other words, $v = (i, j)$ and $u = (i+1, j)$, where the i^{th} coordinate denotes the height, or row, of cell v and the j^{th} coordinate denotes the column containing v. (We include pairs u, v such that $v = (0, j)$ and $v = (1, j)$.) Define $Des(\hat{\sigma}) = \{u \in \lambda : \hat{\sigma}(u) > \hat{\sigma}(v) \text{ is a descent}\}$.

Then define

$$maj(\hat{\sigma}) = \sum_{u \in Des(\hat{\sigma})} (l(u) + 1),$$

where $l(u)$ is the leg of u.

Three cells $\{a, b, c\} \in \lambda$ form a *type A triple* if they are situated as follows

$$\begin{array}{|c|ccc|c|} \cline{1-1}\cline{5-5} a & \cdot & \cdot & \cdot & c \\ \cline{1-1}\cline{5-5} b \\ \cline{1-1} \end{array}$$

where a and c are in the same row, possibly the first row, possibly with cells between them, and the height of the column containing a and b is greater than or equal to the height of the column containing c.

Define for $x, y \in \mathbb{Z}_+$

$$I(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

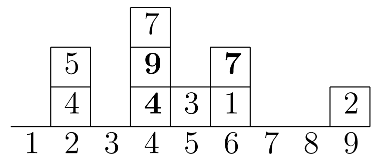
Let σ be an augmented filling and let α, β, δ be the entries of σ in the cells of a type A triple $\{a, b, c\}$.

$$\begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \end{array} \;\cdots\; \begin{array}{|c|}\hline \delta \\ \hline \end{array}$$

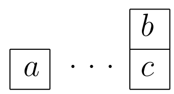
Then the triple $\{a, b, c\}$ is called a *type A inversion triple* if and only if $I(\alpha, \delta) + I(\delta, \beta) - I(\alpha, \beta) =1$.

Call an augmented filling $\hat{\sigma}$ *standard* if it is a bijection $\hat{\sigma} : \lambda \cong \{1,...,n\}$. The *standardization* of an augmented filling is the unique standard augmented filling ξ such that $\hat{\sigma} \circ \xi^{-1}$ is weakly increasing, and for each α in the image of $\hat{\sigma}$, the restriction of ξ to $\hat{\sigma}^{-1}(\{\alpha\})$ is increasing with respect to the reading order. (In other words, let a_i be the number of times the number i appears in the filling. Then the i^{th} occurence of j in reading order is sent to $i + \sum_{m=1}^{j-1} a_m$.) Therefore the triple $\{a, b, c\}$ is an inversion triple of type A if and only if after standardization, the ordering from smallest to largest of the entries in cells $\{a, b, c\}$ induces a counter-clockwise orientation. Note that for a filling of a partition, every triple is a type A triple since the lefthand column is always weakly taller. Therefore, every triple in a partition filling is a type A triple and the items called inversion triples of a partition are type A inversion triples.

Example 1.4.7. *The cells* $\{4, 7, 9\}$ *are a type A inversion triple.*

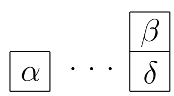


Similarly, three cells $\{a, b, c\} \in \lambda$ are a *type B triple* if they are situated as shown below,



where a and c are in the same row (possibly the basement) and the column containing b and c has greater height than the column containing a.

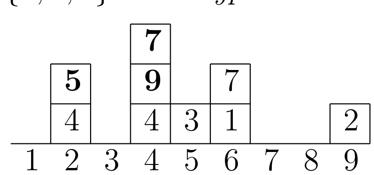
Let $\hat{\sigma}$ be an augmented filling and let α, β, δ be the entries of $\hat{\sigma}$ in the cells of a type B triple $\{a, b, c\}$.



Then the triple $\{a, b, c\}$ is called an *type B inversion triple* if and only if $I(\beta, \alpha) + I(\alpha, \delta) - I(\beta, \delta) = 1$. In other words, the triple $\{a, b, c\}$ is an inversion triple of type B if and only if after standardization, the ordering from smallest to largest of the entries in cells a, b, c induces a clockwise orientation.

We often abuse notation by identifying the entries $\{\sigma(a), \sigma(b), \sigma(c)\}$ with the triple $\{a, b, c\}$.

Example 1.4.8. *The cells* $\{5, 7, 9\}$ *are a type A inversion triple.*



For an augmented filling $\hat{\sigma}$ of a composition γ, define

$$Inv(\hat{\sigma}) = \{\text{inversion triples of } \hat{\sigma}\}, \qquad \text{and}$$

$$inv(\hat{\sigma}) = |Inv(\hat{\sigma})|.$$

Let $coinv(\hat{\sigma})$ be the number of triples which are not an inversion triple.

Theorem 1.4.9 ([6]). *The nonsymmetric Macdonald polynomials E_γ are given by the formula*

$$E_\gamma(x;q,t) = \sum_{\substack{\hat{\sigma}\,:\,\widehat{dg}(\gamma)\to[n] \\ non\text{-}attacking}} x^{\hat{\sigma}} q^{\mathrm{maj}(\hat{\sigma})} t^{\mathrm{coinv}(\hat{\sigma})} \prod_{\substack{u\in\widehat{\mathrm{dg}}(\mu) \\ \hat{\sigma}(u)\neq\hat{\sigma}(d(u))}} \frac{1-t}{1-q^{l(u)+1}t^{a(u)+1}},$$

where $x^{\hat{\sigma}} = \prod_{u\in\widehat{\mathrm{dg}}(\mu)} x_{\hat{\sigma}(u)}$ and $d(u)$ is the cell directly beneath u.

When the Macdonald polynomial is expressed in the form $P_\lambda(x;q,t)$, the nonsymmetric Macdonald polynomials decompose the Macdonald polynomials in a natural way.

Proposition 1.4.10 ([15]). *Let λ° be the rearrangement of the parts of λ in weakly increasing order. Then*

$$P_\lambda(x;q,t) = \prod_{u\in\widehat{\mathrm{dg}}(\lambda^\circ)} (1-q^{l(u)+1}t^{a(u)}) \cdot \sum_{\gamma\sim\lambda} \frac{E_\gamma(x;q^{-1},t^{-1})}{(1-q^{l(u)+1}t^{a(u)})}, \tag{1.4.1}$$

where the sum is over all rearrangements γ of λ.

It is well known that

$$s_\lambda(x) = \lim_{q,t\to 0} P_\lambda(x;q,t),$$

so letting $q,t\to\infty$ in Equation 1.4.1 yields a combinatorial expansion of the Schur functions into polynomials $E_\gamma(x;\infty,\infty)$ called *nonsymmetric Schur functions*.

Chapter 2

Combinatorial description of nonsymmetric Schur functions

Letting $q, t \to 0$ in Equation 1.4.1 yields the following combinatorial decomposition of the Schur functions [6].

$$s_\lambda(x) = \prod_{u \in \widehat{\mathrm{dg}}(\lambda^\circ)} (1 - 0^{l(u)+1} 0^{a(u)}) \cdot \sum_{\gamma \sim \lambda} \frac{E_\gamma(x; \infty, \infty)}{(1 - 0^{l(u)+1} 0^{a(u)})} \quad \Rightarrow$$

$$s_\lambda = \sum_{\gamma \sim \lambda} E_\gamma(x; \infty, \infty), \tag{2.0.1}$$

where

$$E_\gamma(x; \infty, \infty) = \sum_{\substack{\hat{\sigma} : \widehat{\mathrm{dg}}(\gamma) \to [n] \\ \text{non-attacking} \\ \mathrm{maj}(\widehat{\sigma}) = \mathrm{coinv}(\widehat{\sigma}) = 0}} x^\sigma \tag{2.0.2}$$

For ease of notation, we set $E_\gamma(x; \infty, \infty) = NS_\gamma(x)$, and call $NS_\gamma(x)$ a nonsymmetric Schur function. This chapter examines the objects which appear in this decomposition and some of their properties. Chapter 3 describes a bijection between between these objects and semi-standard Young tableaux to provide a combinatorial understanding of Equation 2.0.1.

2.1 Skyline Augmented fillings (SAF)

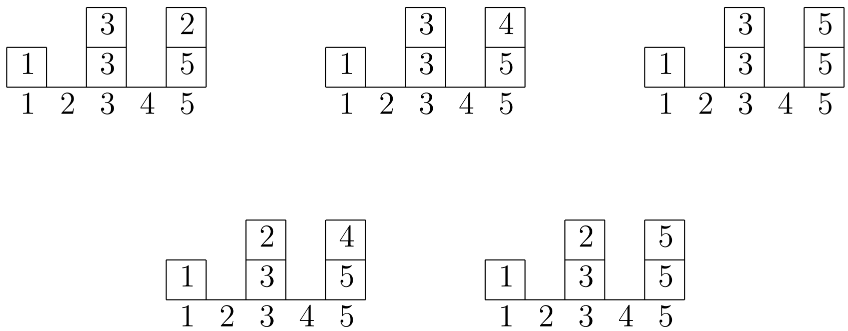
Equation 2.0.2 implies that the nonsymmetric Schur function NS_γ is a sum of the weights of certain types of augmented fillings of γ. Recall that

$$maj(\hat{\sigma}) = \sum_{u \in Des(\hat{\sigma})} (l(u) + 1),$$

and $coinv(\hat{\sigma})$ is the number of non-inversion triples. Therefore, $maj(\hat{\sigma}) = 0$ implies that there are no descents in $\hat{\sigma}$ and $coinv(\hat{\sigma}) = 0$ implies that every triple is an inversion triple.

Definition 2.1.1. A *skyline augmented filling* is (SAF) any augmented filling F such that $Des(F) = \emptyset$ and every triple is an inversion triple.

Let γ be a composition of n into n parts, where some of the parts could be equal to zero. The nonsymmetric Schur function $NS_\gamma = NS_\gamma(x)$ in the variables $x = (x_1, x_2, ..., x_n)$ is the formal power series $NS_\gamma(x) = \sum_F x^F$ summed over all skyline augmented fillings F of composition γ. Here, $x^F = \prod_{i=1}^n x_i^{\sigma_i}$ is the *weight* of σ. (See Figure 2.1.)



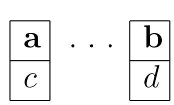
$$x_1x_2x_3^2x_5 + x_1x_3^2x_4x_5 + x_1x_3^2x_5^2 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_5^2$$

Figure 2.1: $NS_{(1,0,2,0,2)}$

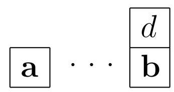
The combinatorial formula for nonsymmetric Macdonald polynomials [6] contains an additional "non-attacking" condition. Recall from Definition 1.4.6 that this condition consists of two sub-conditions. The first states that for each pair of cells a and b with a strictly to the left of b in the row directly below b, $\sigma(a) \neq \sigma(b)$. The second states that for each pair of cells a and b in the same row, $\sigma(a) \neq \sigma(b)$. (In each case, if $\sigma(a) = \sigma(b)$, then a and b are called *attacking* cells.)

Lemma 2.1.2. *If the cells a and b are in the same row of a skyline augmented filling, then $\sigma(a) \neq \sigma(b)$.*

Proof. Argue by contradiction. First, assume that a and b are in the same row (a to the left of b) and $\sigma(a) = \sigma(b)$, where the column containing $\sigma(a)$ is weakly taller than the column containing $\sigma(b)$. Assume a is directly on top of a cell c, so we have $\sigma(c) \geq \sigma(a)$. After standardization, $\{\sigma(a), \sigma(b), \sigma(c)\}$ is a type A non-inversion triple.



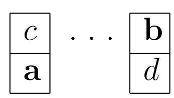
Next, assume that a and b are in the same row and $\sigma(a) = \sigma(b)$, where the column containing $\sigma(b)$ is strictly taller than the column containing $\sigma(a)$. (Again, a is to the left of b.) Then the cell d on top of b must have $\sigma(d) \leq \sigma(b)$.



Therefore after standardization we have $\sigma(d) < \sigma(a) < \sigma(b)$, which implies that $\sigma(d), \sigma(a), \sigma(b)$ is a type B non-inversion triple. □

Lemma 2.1.3. *For each pair of cells a and b in a skyline augmented filling with a to the left of b in the row directly below b, $\sigma(a) \neq \sigma(b)$.*

Proof. Assume there exist two attacking cells a and b (with a to the left of b in the row directly below b) such that $\sigma(a) = \sigma(b) = \alpha$ to get a contradiction.

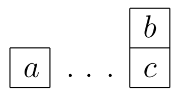


If the column containing a is taller than or equal to the column containing b, then a lies directly below a cell c which must have $\sigma(c) \leq \alpha$. When the values in these three cells are standardized, $\{c, b, a\}$ is a type A non-inversion triple. If the column containing b is taller than the column containing a, then b is directly on top of a cell d which must have $\sigma(d) \geq \alpha$. Then $\{b, a, d\}$ is a type B non-inversion triple. □

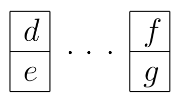
Corollary 2.1.4. *The descent and inversion conditions used to describe the skyline augmented fillings guarantee that no two cells of a skyline augmented filling are attacking.*

Lemma 2.1.5. *If $\{a, b, c\}$ is a type B triple with a and c in the same row and b directly above c, then $\sigma(a) < \sigma(c)$.*

Proof. Let a, b, c be a type B triple situated as pictured below.



To get a contradiction, assume $\sigma(a) > \sigma(c)$. (We know $\sigma(a) \neq \sigma(c)$, by Lemma 2.1.2.) In the basement row, the column containing a has a value less than the value of the column containing c. So at some intermediate row we have



with $\sigma(d) > \sigma(f)$ and $\sigma(e) < \sigma(g)$. The descent condition implies that $\sigma(d) \leq \sigma(e)$. Therefore, $\sigma(f) < \sigma(d) \leq \sigma(e) < \sigma(g)$. But then $\sigma(f) < \sigma(e) < \sigma(g)$ so $\{f, e, g\}$ is not an inversion triple. $\square$

Given any type B triple $\{a, b, c\}$ as depicted in the proof of Lemma 2.1.5, the descent condition implies that $\sigma(b) \leq \sigma(c)$. So $\sigma(a)$ must be less than $\sigma(b)$, for otherwise Lemma 2.1.5 implies that $\sigma(b) \leq \sigma(a) < \sigma(c)$, which would mean $\{b, a, c\}$ is a type B non-inversion triple. So $\sigma(a) < \sigma(b) \leq \sigma(c)$. Therefore Lemma 2.1.5 completely characterizes the relative values of the cells in a type B inversion triple.

Lemmas 2.1.2, 2.1.3, and 2.1.5 provide us with several conditions on the cells in a skyline augmented filling. They will be useful in proving facts about the insertion process.

2.2 A basis for all polynomials

We may relax the restriction on the number of parts to obtain nonsymmetric Schur functions in infinitely many variables.

Let γ be a weak composition of n into k parts, where k is any positive integer or infinity. Its column diagram consists of k columns such that the i^{th} column contains γ_i cells. As above, fill the augmented diagram with positive integers in such a way that there are no descents and every triple is an inversion triple to get a skyline augmented filling. Then $NS_\gamma(x) = \sum_F x^F$, where F ranges over all skyline augmented fillings of the augmented column diagram of γ.

We may define a nonsymmetric analogue of the monomial symmetric functions. The *nonsmmyetric monomial,* NM_γ corresponding to a weak composition γ of n into k parts is given by $NM_\gamma = \prod_{i=1}^k x_i^{\gamma_i}$. It is clear that the sum over all rearrangements of a given partition γ of the nonsymmetric monomials is equal to the monomial symmetric function m_γ.

Consider the case in which k equals infinity. Then every polynomial can be written as a sum of nonsymmetric monomials, so the nonsymmetric monomials form a basis for all polynomials.

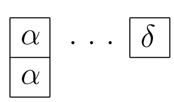
Definition 2.2.1. Let μ and γ be weak compositions of n into infinitely many parts. The *reverse dominance order* on weak compositions is defined as follows.

$$\mu \leq \gamma \iff \sum_{i=k}^{\infty} \mu_i \leq \sum_{i=k}^{\infty} \gamma_i, \qquad \forall k \geq 1.$$

If γ and μ are weak compositions of n into infinitely many parts, let $NK_{\gamma,\mu}$ denote the number of skyline augmented fillings of shape γ and type μ. We will see in Section 5.2 that the ordinary Kostka numbers are obtained as a sum of nonsymmetric Kostka numbers. In fact, $K_{\lambda,\mu} = \sum NK_{\gamma,\mu}$, where the sum is over all rearrangements γ of λ.

Theorem 2.2.2. *Suppose that γ and μ are both weak compositions of n and $NK_{\gamma,\mu} \neq 0$. Then $\mu \leq \gamma$ in the dominance order. Moreover, $NK_{\gamma,\gamma} = 1$.*

Proof. Assume that $NK_{\gamma,\mu} \neq 0$. By definition, this means that there exists a skyline augmented filling of shape γ and type μ. Assume that an entry k appears in one of the first $k-1$ columns. Then this column would contain a descent, since the basement entry is less than k. Therefore, all entries greater than or equal to k must appear after the $(k-1)^{th}$ column. So $\sum_{i=k}^{\infty} \mu_i \leq \sum_{i=k}^{\infty} \gamma_i$ for each k, as desired. Moreover, if $\mu = \gamma$, then the i^{th} column must contain only entries with value i. To see that this is indeed a skyline augmented filling, consider first a type A triple as shown below.



Here $\delta > \alpha$, so listing the entries from smallest to largest after standardization yields a counterclockwise orientation, so the triple is indeed a type A inversion triple.

Next consider a type B triple as shown below.

$$\begin{array}{ccc} & & \boxed{\beta} \\ \boxed{\alpha} & \cdots & \boxed{\beta} \end{array}$$

Since $\alpha < \beta$, listing the entries from smallest to largest after standardization yields a clockwise orientation, so the triple is a type B inversion triple.

The filling of γ in which the i^{th} column contains only the letter i is a valid skyline augmented filling, since it contains no descents and each triple is an inversion triple. We must show this is the only SAF of shape γ and type γ. Consider the last non-zero part γ_m of γ. If the value m appears to the left of this column, it creates a descent. So m can only appear in this column. But there are γ_m entries equal to m. Therefore the m^{th} column contains all of the m entries and no other entries. Assume that the last k columns have been filled so that column γ_j contains exactly γ_j j's. Consider the last non-zero column, γ_r before these columns. There are γ_r r's which cannot be placed to the left of γ_r. But there are no empty cells available to the right of γ_r. Therefore γ_r must be filled with exactly γ_r r's. Therefore any SAF of shape γ and weight γ must be the SAF in which the i^{th} column contains only the letter i, so $NK_{\gamma,\gamma} = 1$. □

Corollary 2.2.3. *The nonsymmetric Schur functions form a basis for all polynomials.*

Proof. Theorem 2.2.2 is equivalent to the assertion that the transition matrix from the nonsymmetric Schur functions to the nonsymmetric monomials (with respect to the reverse dominance order) is upper triangular with 1's on the main diagonal. Since this matrix is invertible, the nonsymmetric Schur functions are a basis for all polynomials. □

2.3 The partially ordered set $P_{n,m}$

Let γ and δ be two compositions of n into m parts. We have the dual to reverse dominance order:

$$\gamma \leq \delta \iff \sum_{i=1}^{k} \gamma_i \leq \sum_{i=1}^{k} \delta_i, \qquad \forall k \leq n.$$

This partial ordering is called the *dominance order*. We define the poset $P_{n,m}$ to be compositions of n into m parts under the dominance order. Our exploration of this poset is motivated by its natural appearance in the transition between nonsymmetric monomials and nonsymmetric Schur polynomials in the previous section.

2.3.1 Connection to the poset $L(n,m)$

Given two partitions, μ and λ, we say that $\mu \leq \lambda$ *component-wise* if and only if $\mu_i \leq \lambda_i$ for all i. Let $L(n,m)$ be the poset of partitions with at most n parts whose parts are bounded by m, ordered component-wise. Another way to think of this poset is to consider all partitions whose Young diagram fits inside an $m \times n$ rectangle, ordered by containment. (See Figure 2.2.)

Let $\lambda \in L(n,m)$, such that $l(\lambda) = k$. We consider λ to be a partition into n

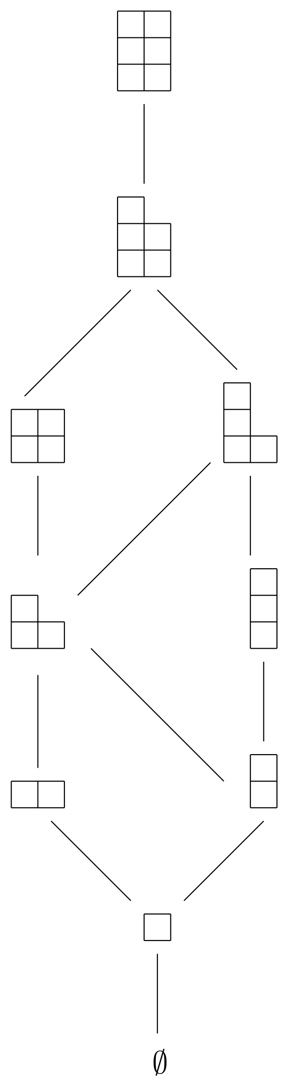


Figure 2.2: The poset $L(2,3)$

parts, so we add $n-k$ zeros to the end of λ. This poset is known to have the Sperner property [20], which states that the size of any collection of pairwise incomparable elements is bounded by the size of the largest level. $L(n, m)$ is a unimodal poset [23], which means that the size of the levels is a unimodal sequence.

Proposition 2.3.1. *The poset $P_{n,m}$ is isomorphic to $L(n, m-1)$.*

Proof. Let $\gamma \in P_{n,m}$. Then γ is a composition of n into m parts. We define a partition $f(\gamma)$ as follows. The number of parts of $f(\gamma)$ which are equal to $m-i$ is given by γ_i. Since γ is a composition of n, the number of parts of $f(\gamma)$ is at most n. Also, the largest part of $f(\gamma)$ is less than or equal to $m-1$. So $f(\gamma) \in L(n, m-1)$. We must prove that the map $f : P_{n,m} \longrightarrow L(n, m-1)$ is an order-preserving bijection.

First we describe the inverse $f^{-1} : L(n, m-1) \longrightarrow P_{n,m}$ to prove that $f : P_{n,m} \longrightarrow L(m-1, n)$ is indeed a bijection. Given $\lambda \in L(n, m-1)$, define a composition $f^{-1}(\lambda) = (f^{-1}(\lambda)_1, f^{-1}(\lambda)_2, ..., f^{-1}(\lambda)_m)$. Let $f^{-1}(\lambda)_i$ be the number of parts of p which are equal to $m-i$. Note that $\sum_{j=1...m} f^{-1}(\lambda)_j = n$, since λ has n parts including its zeros. Since $f^{-1}(\lambda)$ has m parts, $f^{-1}(\lambda)$ is a composition of n into m parts. So $f^{-1}(\lambda) \in P_{n,m}$.

Next, we show that $\gamma \leq \delta \iff f(\gamma) \leq f(\delta)$. First, assume that $\gamma \leq \delta \in P_{n,m}$. This means that for each k, $\sum_{i=1}^{k} \gamma_i \leq \sum_{i=1}^{k} \delta_i$. We know that $f(\gamma)_j$ is equal to $m-I$, for I such that $\sum_{i=1}^{I-1} \gamma_i < j$ and $\sum_{i=1}^{I} \gamma_i \geq j$. Similarly, $f(\delta)_j = m-J$, for J such that $\sum_{i=1}^{J-1} \delta_i < j$ and $\sum_{i=1}^{J} \delta_i \geq j$. Since $\sum_{i=1}^{k} \gamma_i \leq \sum_{i=1}^{k} \delta_i$ for all i,

$\sum_{i=1}^{I} \gamma_i \leq \sum_{i=1}^{I} \delta_i$. So $\sum_{i=1}^{I} \delta_i \geq j$, which implies $I \geq J$. But $I \geq J \Rightarrow m - I \leq m - J$. Therefore $f(\gamma)_j \leq f(\delta)_j$. So $\gamma \leq \delta \Rightarrow f(\gamma) \leq f(\delta)$.

Assume that $f(\gamma) \leq f(\delta) \in L(n, m-1)$. Then $f(\gamma)_i \leq f(\delta)_i$ for all i. The sum $\sum_{j=1}^{k} \gamma_j$ is equal to the number of parts of $f(\gamma)$ which are greater than or equal to $m - k$. But $f(\gamma) \leq f(\delta) \Rightarrow$ the number of parts of $f(\gamma)$ which are greater than or equal to $m-k$ is less than or equal to the number of parts of $f(\delta)$ which are greater than or equal to $m - k$. So $\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \delta_j$, and hence $\gamma \leq \delta \in P_{n,m}$. Therefore $\gamma \leq \delta \iff f(\gamma) \leq f(\delta)$ and the proof is complete. □

A poset P has a zero element, $\hat{0}$, if for every u in P, $\hat{0} \leq u$. A *chain* of a poset is a collection of pair-wise comparable elements. The *length* of a chain is the number of elements in the chain minus 1. A *maximal chain* between two comparable elements $u \leq v$ is a chain C containing u and v such that $x \in C \Rightarrow u \leq x \leq v$ and if C' is any other such chain, then $|C| \geq |C'|$. The *rank* of an element u is the length of any maximal chain between $\hat{0}$ and u. Pictorally, the rank of an element is the level of the poset in which it appears.

In $L(n, m)$, the rank of λ is the number which λ partitions, since $\hat{0}$ is the empty partition and adding a cell to the diagram increases the level of the shape by one. The zero element in $P_{n,m}$ is the element $(0, 0, \ldots, n)$. Move one cell to the column immediately to its left to increase the rank by one. Therefore each cell in the i^{th} column contributes $m - i$ to the rank, since it has been moved $m - i$ times to get to the i^{th} column. So $rank(\gamma) = \sum_i \gamma_i (m - i)$.

For example, let $\gamma = (3, 0, 1, 1, 0, 2, 4) \in P_{11,7}$. Then $rank(\gamma) = \sum_i \gamma_i(m-i) = 3 \cdot 6 + 1 \cdot 4 + 1 \cdot 3 + 2 \cdot 1 + 4 \cdot 0 = 27$. Apply the bijection to γ to get $f(\gamma) = (6, 6, 6, 4, 3, 1, 1) \in L(11, 6)$. Notice that $rank(f(\gamma)) = \sum_i f(\gamma)_i = 6 + 6 + 6 + 4 + 3 + 1 + 1 = 27$.

Chapter 3

The bijection between SSYT and SAF

Recall that letting $q, t \to \infty$ in Equation 1.4.1 implies algebraically that

$$s_\lambda(x) = \sum_\gamma NS_\gamma(x), \tag{3.0.1}$$

where the sum is over all compositions γ which rearrange λ. This chapter provides the first bijective proof of Equation 3.0.1.

In Section 3.1, we describe a simple weight-reversing bijection between SSYT of shape λ and SAF whose shape rearranges λ. This bijection provides a combinatorial interpretation of the connection between SSYT and SAF, but it is not optimal since it reverses the weights. A weight-preserving bijection is presented in Section 3.2. This bijection, while more complicated, has several applications and provides deeper insights into the nature of this connection.

3.1 A weight-reversing bijection

Described below, the bijection $f : SSYT \to SAF$ sends an SSYT with content $(a_1, a_2, \ldots, a_n)$ to an SAF with content $(a_n, a_{n-1}, \ldots a_1)$. The symmetric property of the Schur functions implies that the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is equal to the coefficient of $x_1^{a_n} x_2^{a_{n-1}} \cdots x_n^{a_1}$. Therefore, this weight-reversing bijection is a valid bijective proof of Equation 3.0.1.

3.1.1 The map f from $SSYT$'s to SAF's

Begin with an arbitrary semi-standard young tableau T of shape λ, where $\lambda \vdash n$. The cells are filled by some multiset of positive integers $\{1^{a_1}, 2^{a_2}, ..., n^{a_n}\}$. (Note that some of the a_i might equal 0.) Map the value in each cell to a new value by sending α to $n - \alpha + 1$. Call the resulting filling $rev(T)$. (Recall that a filling of shape μ is a function $\sigma : \lambda \to \mathbb{Z}_+$, as described in Section 1.2.) Notice that the column entries are now strictly decreasing and the row entries are weakly decreasing. Let $rev(T)_i$ be the set of entries in the i^{th} column of $rev(T)$.

Begin with an empty SAF and build the SAF $f(T)$ by placing the elements of $rev(T)_i$ on top of row $i - 1$ as follows. Begin with the largest member, α_1, of $rev(T)_i$. Find the left-most entry of row $i - 1$ that is greater than or equal to α_1. We know such an element exists, since the entry β_1 to the immediate left of α_1 in $rev(T)$ is greater than or equal to α_1 and was mapped to row $i - 1$ in the previous step. Place α_1 on top of this element. Next place the second-largest member, α_2

of $rev(T)_i$ in the same way. (Again, an entry greater than α_2 exists because the entry β_2 immediately to the left of α_2 in the filling is greater than or equal to α_2, as is the entry β_1. Even if α_1 has already been placed on one of these, the other is still available.) Continue in this manner until all the elements of $rev(T)_i$ have been placed. Any remaining cell of row $i-1$ has an empty cell directly above it.

Following this process for each column of $rev(T)$ produces a filling of a composition Young diagram, as in the example below.

Example 3.1.1. *Let T be a semi-standard Young tableau of shape $\lambda=(5,3,3,2,1)$ (note that $\lambda \vdash 14$) as pictured below and apply the map described above that sends each of the numbers, α, to $14-\alpha+1$.*

9				
7	7			
4	5	7		
2	2	4		
1	1	2	3	7

T

$\longrightarrow$

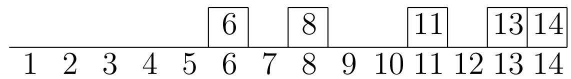
6				
8	8			
11	10	8		
13	13	11		
14	14	13	12	8

$rev(T)$

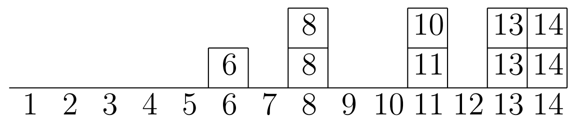
Next examine the empty composition filling.

1	2	3	4	5	6	7	8	9	10	11	12	13	14

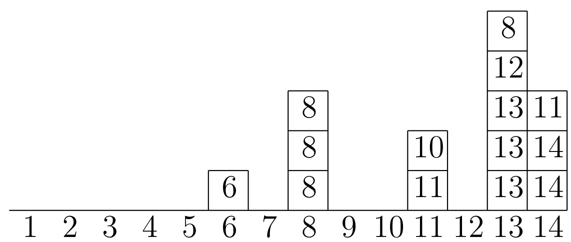
We must assign the numbers $rev(T)_1 = \{6, 8, 11, 13, 14\}$ to the first row of our augmented filling according to the map defined above. The figure below shows the placement of these numbers into the empty augmented filling.



The following figure shows the placement of the second row.



After the placement of the final three rows, we have the following figure.



We say that a cell c is *row-wise above* another cell d if c appears weakly before the cell on top of d in reading order. In the previous example, the entry 12 is row-wise above the entries 11 and 13 but not the entry 8.

Lemma 3.1.2. *Once the entries have been placed into the rows, the resulting figure is an SAF, and this placement procedure produces the only SAF containing the elements of* $rev(T)_i$ *in the* i^{th} *row.*

Proof. To see that the resulting figure is indeed a skyline augmented filling, we must prove that there are no descents and every triple is an inversion triple. There are no descents by construction, so we must check for non-inversion triples.

First check for type A non-inversion triples. They must have the cell configuration depicted below, where the column containing α and γ has height greater than or equal to the height of the column containing β.

$$\begin{array}{|c|}\hline \alpha \\ \hline \gamma \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \beta \\ \hline \delta \\ \hline \end{array}$$

Here $\alpha \leq \gamma$ by construction. Therefore, to get a non-inversion triple, we must have $\alpha \leq \beta \leq \gamma$. The elements of a row are all distinct since entries in a column of an SSYT are all distinct, so $\alpha < \beta$. But then β must have been placed before α. Since $\beta \leq \gamma$, β would have been placed on top of γ or on top of some entry to the left of γ. So this configuration would not happen. Therefore, there are no type A non-inversion triples.

Next check for type B non-inversion triples. This can occur in two ways. Either the lefthand cell in the triple has a non-empty cell on top of it (case 1) or the cell on top of the lefthand cell is empty (case 2). In either case, the column containing β and γ is strictly taller than the column containing α.

$$\underset{\text{Case 1}}{\begin{array}{|c|}\hline \delta \\ \hline \alpha \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \beta \\ \hline \gamma \\ \hline \end{array}} \qquad\qquad \underset{\text{Case 2}}{\begin{array}{c} \\ \begin{array}{|c|}\hline \alpha \\ \hline \end{array} \end{array} \cdots \begin{array}{|c|}\hline \beta \\ \hline \gamma \\ \hline \end{array}}$$

We know that $\beta \leq \gamma$ by construction. Thus, to get an non-inversion triple, we must have $\beta \leq \alpha \leq \gamma$. (Again, since column entries in an SSYT are distinct, we have $\alpha < \gamma$.)

Case 1. *There exists a non-empty cell with entry δ on top of α.*

In this case, since β is less than or equal to α and γ, we could have placed β on top of either. Since δ was placed on top of α, δ must have been placed before β. So δ must be greater than β. But then in the row immediately above the row containing α and β, any entry ϵ small enough to be placed on top of β would be placed on top of δ, unless there was already an entry κ on top of δ. Then $\kappa > \epsilon$ and we are in the same situation. Therefore, it is not possible for the column containing β and γ to be strictly taller than the column containing α.

Case 2. *The entry α is the highest cell in its column.*

In this case, since β is less than α and γ, we could have placed β on top of α instead. Placing β on top of γ means β was not placed as far to the left as it should have been. Thus Case 2 cannot occur.

Therefore our process yields a filling with no descents such that all triples are inversion triples. We conclude that our process yields a skyline augmented filling F.

Next we must show that F is the only SAF with rows $rev(T)_1, ..., rev(T)_k$. Assume there is a different skyline augmented filling with the same row entries $\{rev(T)_i\}$ to derive a contradiction. Let F' denote a different SAF whose rows contain the same entries $(rev(T)_i)$ as the rows of F but in different cells.

Find the lowest row i of F' which differs from row i of F. Consider the largest element of $rev(T)_i$ whose position in F' does not agree with its position in F. Call this element α. Now α was placed in the leftmost possible position in F after all entries greater than α had been placed. Therefore, α must lie in a position further to the right in F'.

Say α is in cell a above the entry $\beta = \sigma(b)$ in F and above the entry $\gamma = \sigma(c)$ in F'. Then this part of the skyline augmented filling looks like the picture below, where δ and ϵ might be empty cells:

F $\qquad\qquad$ F'

Since α is the largest entry of F' to occupy a cell different from the cell occupied by α in F, then ϵ must be less than α. If the column in F' containing ϵ and β were taller than the column containing α and γ, then the triple $\{\epsilon, \alpha, \beta\}$ would be a type A non-inversion triple in F'. So the column containing α and γ must be taller than the column containing ϵ and β. By Lemma 2.1.5, $\beta < \gamma$. But then $\{\alpha, \beta, \gamma\}$ is a type B non-inversion triple in F', since $\alpha \leq \beta$.

Therefore, regardless of which column is taller, F' contains at least one non-inversion triple. So F' is not a skyline augmented filling. Thus F is the unique skyline augmented filling with row entries $rev(T)_i$. $\square$

3.1.2 The inverse of f

Begin with an arbitrary skyline augmented filling, F. Select all the entries in the bottom row. Arrange them in a vertical column, sorted into descending order so that the largest entry is at the bottom of the column. Then select the entries in the second row, and arrange them in a column immediately to the right of the first column, again in descending order up the column. Continue in this manner until there are no more rows left in the skyline augmented filling. The resulting shape is clearly a Ferrers diagram, since each column of this figure has height greater than or equal to the height of the column immediately to its right.

Lemma 2.1.3 implies that the row entries of an SAF are distinct. Therefore the entries in a column of the Ferrers diagram are distinct, so the entries are strictly decreasing as one travels up the column.

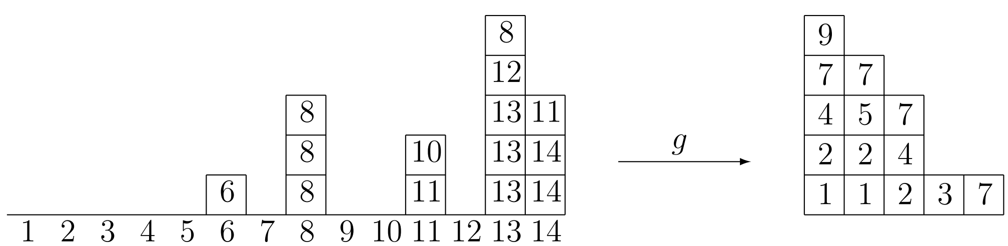
Lemma 3.1.3. *Each entry in the Ferrers diagram filling is less than or equal to the entry immediately to its left.*

Proof. The entry, α, at height i in the j^{th} column of the Young diagram filling is the i^{th} largest entry in the j^{th} row of the skyline augmented filling F. If this value is greater than the value β to its left in the Young diagram filling, at most $i-1$ entries in the $(j-1)^{st}$ row of F are greater than or equal to α while i entries in the j^{th} row of F are greater than or equal to α. Then the pigeonhole principle tells us that at least one entry in the j^{th} row is greater than the entry below it. But then we have a descent and our composition filling is not a skyline augmented filling, which is a contradiction. So each entry must be less than or equal to the entry immediately to its left. □

The cells in the Ferrers diagram filling are filled by the members of the multiset $\{1^{a_1}, 2^{a_2}, ..., k^{a_k}\}$. The number n of cells in the skyline augmented filling is equal to the number of cells in the Young diagram. Map the value in each cell of the Young diagram filling to a new value by sending α to $n - \alpha + 1$. Before the mapping, the entries were weakly decreasing by row and strictly decreasing by column. Since the map reverses the orders of the labels while preserving the fact that no repeated entries occur within a column, the labels are now weakly increasing by row and strictly increasing by column. Therefore the result, $g(F)$, is a semi-standard Young tableau.

Assume that two different skyline augmented fillings produce the same SSYT. Then these two skyline augmented fillings would contain the same set of entries in each row. But we saw in Lemma 3.1.2 that once we know the entries in a row, the placement of those entries in a skyline augmented filling is unique. So the two skyline augmented fillings must be identical. Therefore our map is injective.

Example 3.1.4. *Below we demonstrate the mapping from a skyline augmented filling to a semi-standard Young tableau.*



The maps f and g in Examples 3.1.1 and 3.1.4 are inverses, which is true in general.

Lemma 3.1.5. *The two maps f and g are inverses.*

Proof. To see this, begin with the map from a semi-standard Young tableau T to a skyline augmented filling F. This map sends the numbers in a given column to the corresponding row, changing the numbers by mapping α to $n-\alpha+1$, where the shape of the SSYT is a partition of n. When we map this skyline augmented filling back to an SSYT, first we take the numbers in each row and place them in the corresponding column in decreasing order. Then we send α to $n-\alpha+1$, which inverts the mapping in the first step of f. So the same numbers appear in each column, arranged in increasing order. Therefore we obtain the original SSYT.

Going the other way, we begin with a skyline augmented filling F and map each row to a column with the same numbers in decreasing order. We change this shape to a SSYT by mapping α to $n-\alpha+1$. When we map back to a skyline augmented filling, we first send α to $n-\alpha+1$, which inverts the reversal of weights. Next, we enter each column into its corresponding row via the unique map defined in Section 3.1.1 to produce an SAF F. Since this is the only SAF with these row entries, this is the original SAF. $\square$

Thus, the two injective maps are inverses and form a bijection between SAFs whose shape rearranges λ and SSYT of shape λ. Since the s_λ are symmetric, the number of SSYT of weight $x_1^{a_1}x_2^{a_2}...x_n^{a_n}$ is equal to the number of SSYT of weight $x_{n-1}^{a_1}x_{n-2}^{a_2}...x_1^{a_n}$. Our map sends each SSYT of weight $x_1^{a_1}x_2^{a_2}...x_n^{a_n}$ to an SAF of weight $x_{n-1}^{a_1}x_{n-2}^{a_2}...x_1^{a_n}$, so the coefficient of $\prod_{i=1}^n x_i^{\alpha_i}$ in $s_\lambda(\mathrm{x})$ is equal to the coefficient of $\prod_{i=1}^n x_i^{\alpha_i}$ in $\sum_\gamma NS_\gamma(x)$ (where the sum is over all rearrangements γ of λ), for all possible multisets $\{\alpha_1, ...\alpha_n\}$ with $0 \leq \alpha_i \leq n$, $\forall i$, and $\sum_{i=1}^n \alpha_i = n$.

This proves that the sum of the nonsymmetric Schur functions over all rearrangements of a partition λ is equal to the Schur function s_λ.

3.2 A weight-preserving bijection

Although the bijection in Section 3.1 is simple to state and easy to work with, it is not optimal due to its reversal of weights. We now describe a bijection which sends an SSYT of content $(a_1, a_2, \ldots, a_n)$ to an SAF of content $(a_1, a_2, \ldots, a_n)$.

Theorem 3.2.1. *There exists a weight-preserving bijection between SSYT of shape λ and SAF's whose shapes rearrange λ.*

The bijection in Theorem 3.2.1 makes use of an analogue of Schensted insertion, which ultimately leads to a "nonsymmetric" version of the Robinson-Schensted-Knuth Algorithm. This and other applications of the bijection are described in Chapter 4.

3.2.1 An analogue of Schensted insertion

Schensted insertion is a procedure for inserting a positive integer k into a semi-standard Young tablau T via a scanning and "bumping" technique. It is the fundamental operation of the RSK algorithm, described in Section 1.2.2. Here we define a procedure for inserting a positive integer k into a skyline augmented filling F via a similar technique. In Section 4.1 we will use this procedure to describe an analogue of the RSK algorithm for compositions.

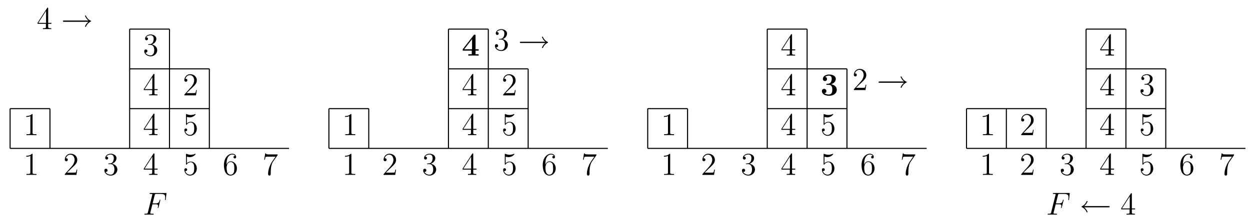


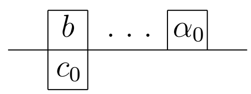
Figure 3.1: $F \leftarrow 4$

Let F be a skyline augmented filling of a weak composition γ of n into infinitely many parts. Then $F = (F(j))$, where $F(j)$ is the j^{th} cell in reading order, including the cells in the basement. We define the operation $F \leftarrow k$.

Begin with the first cell in *read*(F). Scan left to right through the reading word, stopping at the first cell c such that $F(c) \geq k$. If there is no cell directly on top of c, then place k on top of c and the resulting figure is $F \leftarrow k$. Otherwise let a be the cell directly on top of c. If $F(a) < k$ then k "bumps" $F(a)$. In other words, k replaces $F(a)$ and we now begin with the cell immediately to the right of c in *read*(F) and repeat the scanning procedure for $F(a)$. If $F(a) \geq k$ then continue scanning *read*(F) for the next cell which contains an entry greater than or equal to k. Continue this scanning and bumping process until an entry is placed on top of a column. When this occurs, the resulting diagram is $F \leftarrow k$. (See Figure 3.1.) We call the set of cells whose entries have been bumped the *insertion path* P_k. This procedure always terminates, since there are infinitely many basement entries greater than k. Let t_k denote the cell at which the procedure terminates.

Lemma 3.2.2. *When restricted to compositions of n into n parts, this procedure* *terminates.*

Proof. Suppose the procedure does not terminate to get a contradiction. This occurs only if some letter $\alpha_0 = \alpha$ reaches the last cell in the basement without finding a cell c such that $F(c) \geq \alpha$ and such that the cell b on top of c is empty or has $F(b) < \alpha$. The value α is an entry in the basement, say $F(c_0)$. The letter α_0 which is unplaced could not have been bumped from a cell to the right of c_0 in the row above c_0, for otherwise the cell containing α_0 and the cell c_0 would be attacking in F.

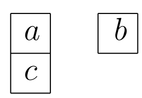


Since α_0 was not inserted on top of c_0, the entry b on top of c_0 must have $F(b) \geq \alpha$. But since F has no descents, $F(b) = \alpha$. So α_0 must have come from a higher row than this, since two occurrences of α in the same row would be attacking in F. Continuing this line of reasoning implies that the α^{th} column contains the value α at each row until a certain height h at which this column contains a cell d such that $F(d) = \beta$ is strictly smaller than α. If α_0 was bumped from row h, α must have been bumped from a cell to the right of the α^{th} column for otherwise α would bump $F(d)$. However, then α_0 and the α in row $h-1$ of the α^{th} column would be entries in attacking cells in F. By Lemma 2.1.3, there are no attacking cells in F. Therefore we have a contradiction. □

We often compare entries in the same cell during the insertion of different numbers. Therefore, we denote by $\sigma_k(a)$ the number contained in the cell a immediately following the insertion of the number k. The notation $\sigma(a)$ refers to the entry contained in the cell a in the filling F.

Proposition 3.2.3. *If F is a skyline augmented filling, then $F \leftarrow k$ is a skyline augmented filling.*

Proof. It is clear by construction that $F \leftarrow k$ has no descents. We must prove that every triple is an inversion triple. We argue by contradiction. Suppose not. Then $F \leftarrow k$ contains a triple of either type A or type B. First suppose $F \leftarrow k$ contains a type A non-inversion triple $\{a, b, c\}$ situated as shown.



Then $\sigma_k(a) \leq \sigma_k(b) \leq \sigma_k(c)$. In F, we must have different (possibly empty) entries in these cells. Because the insertion path moves along the reading word and its entries are strictly decreasing, at most one of $\sigma_k(a)$, $\sigma_k(b)$, and $\sigma_k(c)$ is different from its value in F. Examine each cell individually to derive a contradiction.

Case A1. *The cell a in $F \leftarrow k$ contains a different value, $\alpha \neq \sigma_k(a)$, in F.*

Assume the cell a was non-empty in F. Then $\{\alpha, \sigma(b), \sigma(c)\}$ was an inversion triple in F, so $\sigma(b) < \alpha \leq \sigma(c)$. Since $\sigma_k(a)$ bumped α, we know $\sigma_k(a) > \alpha$ and $\sigma_k(a) > \alpha > \sigma(b) = \sigma_k(b)$, which contradicts $\sigma_k(a) \leq \sigma_k(b)$.

If the cell a was empty in F, then b and c were part of a type B triple. Say b is directly on top of a cell d. Since $\{\sigma(b), \sigma(c), \sigma(d)\}$ is a type B inversion triple in

F, $\sigma(c) < \sigma(b) \leq \sigma(d)$ by Lemma 2.1.5. But then $\sigma_k(c) < \sigma_k(b)$, which contradicts the assumption that $\sigma_k(b) \leq \sigma_k(c)$.

Case A2. *The cell b in $F \leftarrow k$ contained a different value, $\beta \neq \sigma(b)$, in F.*

Assume first that b is non-empty in F. Since $\{\sigma(a), \beta, \sigma(c)\}$ is a type B inversion triple in F, either $\beta < \sigma(a) \leq \sigma(c)$ or $\sigma(a) \leq \sigma(c) < \beta$. If $\beta > \sigma(c)$, then $\sigma_k(b)$ bumped $\beta \Rightarrow \sigma_k(b) > \beta$. Therefore $\sigma_k(b) > \beta > \sigma(c) = \sigma_k(c)$, which contradicts $\sigma_k(b) \leq \sigma_k(c)$.

Thus $\beta < \sigma(a) \leq \sigma(c)$. Assume first that $\sigma_k(b)$ was bumped from a cell before a in reading order. Since $\sigma_k(b)$ did not bump $\sigma(a)$, we have $\sigma_k(b) \leq \sigma(a) = \sigma_k(a)$. But $\sigma_k(a) \leq \sigma_k(b)$ by assumption. Therefore $\sigma(a) = \sigma_k(b)$. Since F is a non-attacking filling, $\sigma_k(b)$ was bumped from a cell row-wise above a. Then $\sigma_k(b)$ would bump the entry in cell e on top of a unless $\sigma(e) = \sigma_k(b)$. But then $\sigma(b)$ was bumped from a cell row-wise above e. Eventually, in some higher row, the cell f in the column containing a will either be empty or contain an entry less than $\sigma_k(b)$. Therefore at this stage $\sigma_k(b)$ is inserted into cell f, a contradiction. So $\sigma_k(b)$ must have been bumped from a cell after a in reading order.

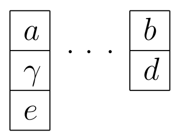
F is a non-attacking filling, so $\sigma_k(b) > \sigma(a)$. Then $\sigma_k(b) \leq \sigma(c)$ implies that the column containing $\sigma_k(b)$ must be strictly taller than the column containing a, for otherwise a, c and the cell containing $\sigma_k(b)$ would be a type A non-inversion triple. But $\sigma_k(b) \leq \sigma(c)$ contradicts Lemma 2.1.5, so the cell b must be empty in F.

If the cell b is empty in F, the same argument described above shows that

$\sigma_k(b)$ must have been bumped from after a, but the bumping from after a yields a contradiction. So this case cannot happen.

Case A3. *The cell c in $F \leftarrow k$ contained a different value, $\gamma \neq \sigma(c)$, in F.*

Since $\{\sigma(a), \gamma, \sigma(b)\}$ is a type A inversion triple in F and $\sigma(a) = \sigma_k(a) \leq \sigma_k(b) = \sigma(b)$, we have $\sigma(a) \leq \gamma < \sigma(b)$. There are no descents in F, so b is directly on top of a cell d such that $\sigma(b) \leq \sigma(d)$. So $\gamma < \sigma(d)$. Since $\sigma_k(c)$ bumps γ, γ is directly on top of a cell e such that $\sigma(e) \geq \sigma_k(c)$, as depicted below.



Since $\{\gamma, \sigma(d), \sigma(e)\}$ is a type A inversion triple in F, we have $\sigma(e) < \sigma(d)$. Therefore, $\sigma(d) > \sigma_k(c)$. So $\sigma_k(c)$ was bumped from a cell after b in the reading order, for otherwise $\sigma_k(c)$ would have bumped $\sigma(b)$, provided $\sigma_k(c) \neq \sigma(b)$. If $\sigma_k(c) = \sigma(b)$, then $\sigma_k(c)$ was bumped from a cell row-wise above b and would therefore have been inserted on top of $\sigma(b)$ unless the entry on top of $\sigma(b)$ is equal to $\sigma(b)$. So $\sigma_k(c)$ must have been bumped from a cell row-wise above the highest cell h with $\sigma(h) = \sigma(d)$ in the column containing $\sigma(d)$. But then $\sigma(c)$ would bump the entry on top of h.

Assume first that $\sigma_k(c)$ was bumped from a cell r in the same row as b. If the column containing b is weakly taller in F than the column containing $\sigma_k(c)$ then $\{\sigma(b), \sigma_k(c), \sigma(d)\}$ is a type A non-inversion triple. Otherwise, $\sigma_k(c)$ is directly on top of a cell f such that $\sigma(f) \geq \sigma_k(c)$ as shown,

$$\begin{array}{|c|}\hline a \\ \hline \gamma \\ \hline \end{array} \cdots \begin{array}{|c|}\hline b \\ \hline d \\ \hline \end{array} \cdots \begin{array}{|c|}\hline e \\ \hline f \\ \hline \end{array}$$

where the column containing e and f is strictly taller than that containing b and d.

By Lemma 2.1.5, we have $\sigma(d) < \sigma(f)$. But then $\{\sigma_k(c), \sigma(d), \sigma(f)\}$ is a type B non-inversion triple in F, since $\sigma_k(c) < \sigma(d)$. Therefore $\sigma_k(c)$ was bumped from the row containing γ.

We now know that $\sigma_k(c)$ was bumped from the row below b, to the left of γ. If the column containing γ were strictly taller than the column containing $\sigma_k(c)$, then $\sigma_k(c) > \gamma$ would contradict Lemma 2.1.5. Therefore the column containing $\sigma_k(c)$ is weakly taller in F than the column containing γ.

Let $\kappa = \sigma(h)$ be the entry which bumped $\sigma_k(c)$. Before κ bumped $\sigma_k(c) = \delta$, the figure looked like

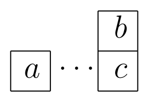
$$\begin{array}{|c|}\hline \mathrm{g} \\ \hline \delta \\ \hline \mathrm{f} \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \mathrm{a} \\ \hline \gamma \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \mathrm{b} \\ \hline \mathrm{d} \\ \hline \end{array}$$

where the column containing δ is the tallest and the column containing b is the shortest.

Since $\sigma(h) > \sigma_k(c) \geq \sigma(g)$, we have $\sigma(h) > \sigma(g)$. Also $\sigma(h) \leq \sigma(f)$, since $\sigma(h)$ is placed directly on top of the cell f. We know that $\{\sigma_k(c), \sigma(d), \sigma(f)\}$ is a type A inversion triple in F, so $\sigma(f) < \sigma(d)$. Therefore $\sigma(h) < \sigma(d)$. Also, $\sigma_k(c) \geq \sigma(b)$ implies that $\sigma(h) > \sigma(b)$. So $\sigma(h)$ must have been bumped from a cell after b in reading order, for otherwise $\sigma(h)$ would bump $\sigma(b)$.

The inequalities satisfied by $\sigma(h)$ above place $\sigma(h)$ in exactly the same situation as $\sigma_k(c)$. Therefore, the same argument used above to prove that $\sigma_k(c)$ must have been bumped from the same row as γ proves that $\sigma(h)$ must have been bumped from the same row as γ as well. Continuing in this manner, one derives an infinite number of columns to the left of the column containing γ. But there are only finitely many columns in the diagram, so this is a contradiction.

Next assume that $F \leftarrow k$ contains a type B non-inversion triple, $\{a, b, c\}$, situated as depicted.

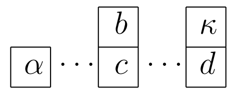


We must have $\sigma_k(b) \le \sigma_k(a) \le \sigma_k(c)$. Again, only one of the entries is different from its value in F. Examine each cell individually to derive a contradiction.

Case B1. *The cell a in $F \leftarrow k$ contains a different value, $\alpha \neq \sigma_k(a)$, in F.*

In this case, α was bumped by $\sigma_k(a)$, so $\sigma_k(a) > \alpha$. If $\sigma_k(a)$ were bumped from somewhere before the cell b in reading order, $\sigma_k(a)$ would have bumped $\sigma(b)$ unless $\sigma_k(a) = \sigma(b)$. If $\sigma_k(a) = \sigma(b)$, it would have come from a higher row than b (by Lemma 2.1.2) to the left of b (by Lemma 2.1.3) and hence been placed in a higher row of the column containing b. Therefore $\sigma_k(a)$ is bumped from somewhere after b in reading order.

Assume that $\sigma_k(a)$ is bumped from the same row as $\sigma(b)$. If the column containing $\sigma_k(a) = \kappa$ is strictly taller than the column containing b, then κ is directly on top of a cell d such that $\sigma(c) < \sigma(d)$ by Lemma 2.1.5.



But then $\sigma_k(a) = \kappa \leq \sigma(c) < \sigma(d)$ implies $\{\kappa, \sigma(c), \sigma(d)\}$ is a type B non-inversion triple in F. So the column containing $\sigma(b)$ must be weakly taller than the column containing $\sigma_k(a)$. But then $\sigma(b) \leq \sigma_k(a) \leq \sigma(c)$ implies $\{\sigma(b), \sigma_k(a), \sigma(c)\}$ is a type A non-inversion triple.

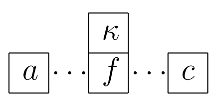
Thus $\sigma_k(a)$ must have been bumped from the same row as α, to the left of α. The column containing $\sigma_k(a) = \kappa$ must be weakly taller than the column containing b and c, for otherwise $\{\sigma_k(a), \sigma(b), \sigma(c)\}$ would be a type B non-inversion triple. Let g be the cell such that $\sigma(g) \in F$ bumped $\sigma_k(a)$ during $F \leftarrow k$. Since $\sigma(g) > \sigma_k(a)$, we must have $\sigma(g) > \sigma(b)$. Also, $\sigma(g)$ is less than or equal to the cell h beneath $\sigma_k(a)$, which has $\sigma(h) < \sigma(c)$ because $\{\sigma_k(a), \sigma(c), \sigma(h)\}$ is a type A inversion triple. So $\sigma(g) < \sigma(c)$. If $\sigma(g)$ were bumped from a cell before b in reading order, then $\sigma(g)$ would bump $\sigma(b)$. Therefore, $\sigma(g)$ was bumped from after b in reading order.

If $\sigma(g)$ is bumped from the same row as b, either $\{\sigma(b), \sigma(g), \sigma(c)\}$ is a type A non-inversion triple or $\sigma(g)$, $\sigma(c)$, and the cell below $\sigma(g)$ form a type B non-inversion triple, depending on which column is taller. Therefore $\sigma(g)$ was bumped from a cell in the same row as $\sigma_k(a)$ to the left of $\sigma_k(a)$.

Any entry δ which bumps $\sigma(g)$ must be greater than $\sigma(b)$ since $\delta > \sigma(g) > \sigma(b)$. Also, the entry ϵ directly beneath $\sigma(g)$ must be less than $\sigma(c)$ to avoid non-inversion triples. Since δ is placed on top of this cell ϵ, $\delta \leq \epsilon \Rightarrow \delta < \sigma(c)$. So we use the same argument to prove that δ is bumped from the same row as α, to the left of $\sigma(g)$. Continuing this line of reasoning implies that there is an infinite sequence of non-zero columns to the right of α. But there are only finitely many non-zero parts in a composition. Therefore we have a contradiction and Case B1 cannot occur.

Case B2. *The cell b in $F \leftarrow k$ contains a different value, $\beta \neq \sigma_k(b)$, in F.*

First assume that the cell b is empty in F. Then a and c are both at the top of their respective columns in F. Since $\sigma_k(b) \leq \sigma(a)$, the entry $\sigma_k(b)$ would have been placed on top of $\sigma(a)$ if it had been bumped from somewhere farther left in the reading word. Therefore $\sigma_k(b) = \kappa$ was bumped from somewhere to the right of a in the row immediately above a. Then κ is directly on top of a cell f such that $\sigma_k(b) \leq \sigma(f)$, as shown.



By Lemma 2.1.5, $\sigma(a) < \sigma(f)$, so $\{\sigma_k(b), \sigma(a), \sigma(f)\}$ is a type B non-inversion triple in F. Therefore the cell on top of c does contain a non-empty entry β in F. Since $\sigma(b)$ bumps β, we have $\beta < \sigma_k(b)$. But then $\beta < \sigma(a) \leq \sigma(c)$, so $\{\beta, \sigma(a), \sigma(c)\}$ is a type B non-inversion triple in F.

Case B3. *The cell c in $F \leftarrow k$ contained a different value, $\gamma \neq \sigma(c)$, in F.*

The triple $\{\sigma(a), \sigma(b), \gamma\}$ is a type B inversion triple in F. Therefore, $\sigma(a) < \sigma(b) \leq \gamma$, by Lemma 2.1.5. But this contradicts the assumption that $\sigma(b) \leq \sigma(a)$. $\square$

Lemma 3.2.4. *When multiple entries with the same value k are inserted into a semi-standard skyline filling, the order of the rows in which they appear corresponds to the order in which they were inserted, with the first one to be inserted appearing in the lowest row which contains the entry k.*

Proof. Let k_1 and k_2 have the same value, k, and assume that k_1 is inserted into the SSSF F before k_2. Then if k_2 reaches k_1 during an insertion process, k_2 is placed on top of k_1, unless the entry k' on top of k_1 has the value k as well. In that case, k_2 must have been bumped from a higher row (by Lemma 2.1.2) to the left of k' (by Lemma 2.1.3). The column C containing k_1 eventually contains a value less than k or an empty cell. Lemma 2.1.2 implies that k_2 must have been bumped from a row higher than the highest appearance of k in column C. Lemma 2.1.3 implies that if k_2 was bumped from the row directly above this highest appearance of k, then k_2 must have been bumped from a column strictly to the left of C. Therefore k_2 would have been placed on top of this occurrence of k. In either case k_2 appears in a higher row than k_1. □

3.2.2 The bijection Ψ between SSYT and SAF

Let T be a semi-standard young tableau. We may associate to T the word $col(T)$ which consists of the entries of each column of T, read top to bottom from columns left to right, as in Figure 3.2. In general, any word w can be decomposed into its maximal strictly decreasing subwords. This decomposition is called $col(w)$.

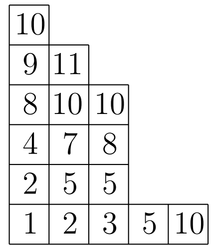
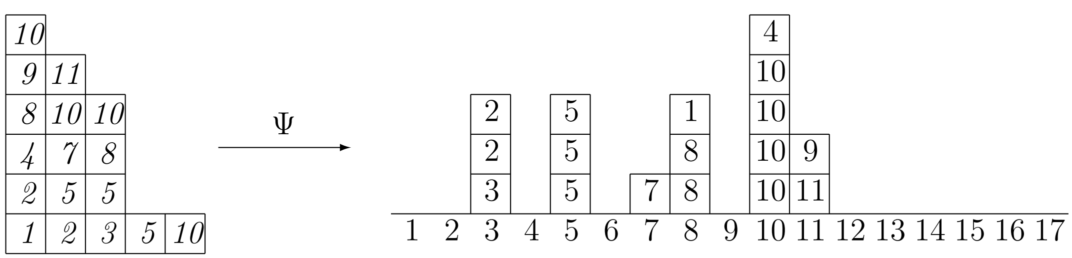


Figure 3.2: $col(T) = 10\ 9\ 8\ 4\ 2\ 1 \cdot 11\ 10\ 7\ 5\ 2 \cdot 10\ 8\ 5\ 3 \cdot 5 \cdot 10$

Example 3.2.5. *If* $w = 3\ 5\ 4\ 2\ 2\ 1$, *then* $col(w) = 3 \cdot 5\ 4\ 2 \cdot 2\ 1$.

Begin with an arbitrary SSYT T and the empty SAF ϕ with the basement row containing all letters of $\mathbb{Z}_+$. Let k be the rightmost letter in $col(T)$. Insert k into ϕ to get the SAF $(\phi \leftarrow k)$. Then let k' be the next letter in $col(T)$ reading right to left. Obtain the SAF $((\phi \leftarrow k) \leftarrow k')$. Continue in this manner until all the letters of $col(T)$ have been inserted. The resulting diagram is the SAF $\Psi(T)$.

Example 3.2.6. $\Psi(T) = F$



T F

Recall that the *insertion path* P_α for a given letter α is the set of cells whose entries are bumped during the insertion $(F \leftarrow \alpha)$ together with the cell t_α at which the procedure terminates. (The cell t_α was empty in F but in $(F \leftarrow \alpha)$ contains the last entry in P_α.)

Lemma 3.2.7. *Let $\alpha > \beta$ and assume that β is inserted immediately before α. Then the insertion path P_α for α terminates farther to the right in the reading word than the insertion path P_β for β.*

Proof. Suppose that P_α terminates before P_β in the reading order. Let $\alpha_1, ..., \alpha_j$ be the entries in P_α and let $\beta_1, ..., \beta_m$ be the entries in P_β. First assume that $\alpha = \alpha_1$ is placed strictly before the cell $\beta = \beta_1$. We know that the value $\sigma_\beta(c) = \epsilon$ bumped by α_1 is less than α_1 and is on top of a cell d such that $\sigma_\beta(d) \geq \alpha_1$. (Recall that $\sigma_\beta(d)$ refers to the entry in the cell d after the insertion of β.) Since $\beta_1 < \alpha_1$, we have $\beta_1 < \sigma_\beta(d)$. Therefore all the entries in P_β are less than $\sigma_\beta(d)$, hence $\sigma_\beta(d) \notin P_\beta$. So during the insertion of β, the cell d contained the same entry, $\sigma_\beta(d)$. If β_1 were greater than $\sigma_\beta(c)$, β_1 would have bumped $\sigma_\beta(c)$. Otherwise, β_1 would have bumped a cell higher in this column. Therefore α_1 could not be placed strictly before the cell containing β_1.

Since α_j is the last entry in P_α, then α_j is the termination point of P_α. The value α_j terminates on top of the cell c, so $\sigma_\beta(c) \geq \alpha_j$.

Since P_β did not terminate at this column in the previous step, there exists a β_k which passed over the cell c without being placed. Therefore $\beta_k > \sigma(c)$. In fact, $\beta_k \geq \sigma_\beta(c)$, for even if the entry $\sigma(c)$ in c was bumped during $F \leftarrow \beta$, the entry which bumped it must be less than or equal to β_k. This implies that

$\alpha_j \leq \sigma_\beta(c) \leq \beta_k < \beta_{k-1} < ... < \beta_1$. So α_j is smaller than all the elements of P_β which appear above it after the insertion $F \leftarrow \alpha$. In particular, $\alpha_j \leq \beta_1$, so $\alpha_j \neq \alpha_1$. So α_j appears somewhere in F during $F \leftarrow \beta$.

The value α_j cannot be in P_β, since $\alpha_j \leq \beta_k$ and β_k appears after α_j. Therefore, during the insertion $F \leftarrow \beta$, some value $\beta_i \geq \beta_k$ passed α_j without bumping α_j. By Lemma 3.2.4, if β_i passes α_j without bumping α_j then β_i is not equal to α_j. Therefore $\beta_i > \alpha_j$.

Let d be the cell directly beneath α_j during $F \leftarrow \beta$. Then $\beta_i > \sigma(d)$, for otherwise β_i would have bumped α_j since $\beta_i > \alpha_j$. But α_{j-1} bumps α_j during $F \leftarrow \alpha$. If $\sigma(d)$ is bumped by some β_l such that $l \geq i$ during $F \leftarrow \beta$, this means that $\alpha_{j-1} \leq \beta_l \leq \beta_i$. Otherwise $\alpha_{j-1} \leq \sigma_\beta(d) < \beta_i$. Either way, $\alpha_{j-1} \leq \beta_i$. Therefore α_{j-1} is less than all of the values of P_β which appear above it after the insertion $F \leftarrow \alpha$. Repeating this argument a total of j times implies that α_1 is less than all the values of P_β which appear above α_1. Recall that β_1 is above α_1. But $\alpha_1 = \alpha > \beta = \beta_1$. Therefore we have a contradiction. □

Lemma 3.2.7 implies that if C_j is a column of an SSYT consisting of the letters $a_1 > a_2 > \ldots a_k$ then t_{a_i} is after $t_{a_{i+1}}$ in reading order. If we think of ordering the columns by height (where the leftmost column comes first when two columns are the same height) then the termination points descend down these columns.

Lemma 3.2.8. *Let $\alpha \leq \beta$ and assume that β is inserted immediately before α. Then t_α is row-wise above t_β.*

Proof. Let $\alpha_1, ..., \alpha_j$ be the entries in P_α and let $\beta_1, ..., \beta_m$ be the entries in P_β. Notice that the first element (α_i) of P_α to reach the cell above β_1 must be less than or equal to $\alpha_1 = \alpha$, and hence less than or equal to $\beta = \beta_1$. The element γ_1 on top of β_1 must have $\gamma_1 \leq \beta_2$, since it was on top of β_2 before β_2 was bumped by β_1. If $\alpha_i > \gamma_1$, then α_i bumps $\gamma_1 = \alpha_{i+1}$ and α_{i+1} is inserted from a cell row-wise above β_2. Otherwise, $\alpha_i \leq \gamma_1$ which means that $\alpha_i \leq \beta_2$. In either case, we are now inserting an element row-wise above β_2 which is less than or equal to β_2.

By induction we may assume that the entry α_l of P_α which reaches the cell on top of β_k for some k has $\alpha_l \leq \beta_k$. So α_l either bumps the entry γ on top of β_k or $\alpha_l \leq \gamma \leq \beta_{k+1}$. (Since γ was on top of β_{k+1} during the previous insertion, $\gamma \leq \beta_{k+1}$.) Either way, the cell of P_α which reaches the cell on top of β_{k+1} must be less than or equal to β_{k+1}. So for any k, the entry in P_α which reaches the entry β_k in P_β must be less than or equal to β_k.

Therefore if an element α_r of P_α reaches the cell on top of the last entry β_m of P_β, then $\alpha_r \leq \beta_m$ and so α_r will be placed on top of β_m and the insertion process terminates row-wise above t_β. If no element of P_α reaches β_m, then the insertion procedure already terminated row-wise above t_β. □

Lemma 3.2.9. *Assume that $\alpha > \beta$ and α is inserted into an SAF immediately following β. Then whenever an entry α_k insertion path P_α reaches a cell $\beta_j \in P_\beta$, we have $\alpha_k > \beta_j$.*

Note that this lemma is quite similar to Lemma 3.2.7. We include it in this format largely for its role in the proof of an upcoming proposition.

Proof. Argue by induction. First consder $\beta_1 \in P_\beta$. Since $\alpha_1 > \beta_1$, α_1 reaches β_1 without bumping any cells. But $\alpha_1 > \beta_1$ implies that the base case is true.

Next, assume that the lemma is true for the first $k-1$ entries in P_β. Let α_j be the cell which reaches β_{k-1}. Assume first that α_j bumps β_{k-1}. Then $\beta_{k-1} > \beta_k$ implies that unless $\alpha_{j+1} = \beta_{k-1}$ bumps an entry between β_{k-1} and β_k, the entry of P_α to reach β_k is indeed greater than β_k. Similarly, if α_j does not bump β_{k-1}, then α_j is greater than the cell beneath β_{k-1}. This cell is greater than or equal to β_k, since β_k was previously above this cell. Again, if α_j does not bump an entry before reaching β_k, then the entry of P_α to reach β_k is greater than β_k and the proof is complete.

Therefore assume α_j bumps an entry γ between β_{k-1} and β_k to get a contradiction. This entry was not bumped by β_k. But since α_j must be less than or equal to the entry δ immediately below γ and $\beta_k \leq \alpha_j$, we must have $\beta_k \leq \delta$. Since β_j did not bump γ, we must have $\beta_j \leq \gamma$. In fact, since β_j passes γ, β_j must be less than γ by Lemma 3.2.4. So the entry bumped remains greater than β_k. Therefore whenever an entry of P_α reaches an entry of P_β, the entry of P_α is greater than the entry of P_β. □

Lemma 3.2.10. *Let α be the entry directly to the left of β in an SSYT T. If an entry α_j of the path P_α reaches the directly cell on top of a cell which contained the k^{th} entry, β_k, of P_β, then $\alpha_j \leq \beta_{k+1}$.*

We call the inequality in this lemma the *first path condition* and we say that β as above *satisfies* the first path condition. The proof will make use of an additional condition on the path P_β, called the *second path condition*. This condition states that between the insertion of β and α (for α immediately to the left of β in T) the entries above the cell P_{β_i} are less than or equal to the entry $P_{\beta_{i+1}}$.

Proof. In the following, the roman letter designates the cell containing the Greek letter during the insertion of the Greek letter. We will prove that the cell β satisfies both path conditions. To do this, we argue by induction. Consider the rightmost two columns of T. Since there is no bumping during the insertion of the rightmost column of T, the insertion path for each of these letters consists of only one entry, which is at the top of a column. Since there is no second entry, the second path condition is vacuously true for all entries in the rightmost column of T.

Similarly, there is no bumping during the insertion of the entries in the rightmost column of T. Let δ be an entry in the rightmost column of T, and ϵ the entry immediately to its left in T. It is enough to show that $\epsilon_j \leq \delta_1$ for all $\epsilon_j \in P_\epsilon$. But $\epsilon_j \leq \epsilon_1 \leq \delta_1$. Therefore the first path condition is satisfied by all entries in the rightmost column of T.

Assume that the rightmost m columns satisfy both path conditions and consider the insertion of the $(m+1)^{th}$ column from the right in T. The first entry $\beta = \beta_1$ in the $(m+1)^{th}$ column is smaller than all other entries which have been inserted so far. Therefore it is placed on top of the tallest column and is less than or equal to any other entries in the insertion path for the cell immediately to its right in T. Since there is no bumping involved in this insertion, it satisfies the first path condition vacuously. The first entry, $\alpha = \alpha_1$ in the $(m+2)^{th}$ column of T has $\alpha_1 \leq \beta_1$, and hence $\alpha_i \leq \beta_1$ for all $\alpha_i \in P_\alpha$. So β also satisfies the second path condition.

Next, assume that the first l entries of the $(m+1)^{th}$ column have been placed and satisfy both path conditions. Let $\beta = \beta_1$ be the $(l+1)^{th}$ entry of the $(m+1)^{th}$ column of T. We will first show that β satisfies the second path condition.

Assume that the entry immediately above b_1 is bumped during the insertion of some entry ϵ. First let ϵ be an entry above β in T. But $\epsilon_1 > \beta_1$, so ϵ_1 must be inserted into a cell weakly following β_1. This means that ϵ_1 and all the cells in P_ϵ must occur strictly after the cell c above b_1. So if an entry ϵ_i bumps the entry in c before the insertion of α, it must occur during the insertion of an entry ϵ in the $(m+2)^{th}$ column of T such that $\epsilon < \alpha$.

Consider the entry δ immediately to the right of ϵ in T.

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} \\ \hline \epsilon & \delta \\ \hline \end{array}$$

We know by the inductive hypothesis that if ϵ_i reaches the cell on top of an entry δ_k, then $\epsilon_i \leq \delta_{k+1}$, and hence $\epsilon_i \leq \delta_j$ for all $j \leq k+1$. Let J be the largest value such that d_J appears before β_1 in reading order. (Note that such a J exists since d_1 appears before b_1 in reading order by Lemma 3.2.8.)

Then the entry ϵ_i which bumped the entry in c must be less than or equal to δ_{J+1}, since it or an entry greater than ϵ_i had to pass the cell on top of d_J to reach the cell on top of b_1.

By successive applications of Lemma 3.2.9, if β_2 reaches d_{J+1} during $F \leftarrow \beta$, then $\beta_2 > \delta_{J+1}$. Therefore $\delta_{J+1} \geq \epsilon_i$ implies that $\beta_2 > \epsilon_i$. Otherwise, some entry $\beta_r < \beta_2$ of P_β reaches d_{J+1}, hence $\beta_r > \delta_{J+1}$. But then $\beta_2 > \beta_r > \delta_{J+1} \geq \epsilon_i$. So in either case, $\epsilon_i < \beta_2$. Therefore any entry which bumps the cell on top of b_1 is still less than or equal to β_2. So the cell on top of b_1 must remain less than or equal to β_2.

Assume that the first p entries of P_β satisfy the second path condition. Consider the cell $\beta_{p+1} \in P_\beta$. Let κ be the entry in the cell c immediately above β_{p+1} after the insertion of β. First assume that the entry κ was bumped during the insertion of an entry $\epsilon > \beta$ such that ϵ is above β in the same column of T.

When the insertion path for ϵ reaches b_{p+1}, it must be greater than β_{p+1} by successive applications of Lemma 3.2.9. When $\epsilon_i \in P_\epsilon$ reaches cell above b_{p+1}, the entry ϵ_i must be greater than or equal to the cell which reached b_{p+1}. Therefore $\epsilon_i > \beta_{p+1}$. But then ϵ_i could not bump the cell on top of β_{p+1}. Therefore the entry immediately on top of b_{p+1} could only be bumped during the insertion of an element ϵ in the $(m+2)^{th}$ column of T such that $\epsilon < \alpha$.

Let ϵ_i be the entry of P_ϵ which bumped the entry in cell c, where ϵ is in the same column as α, below α in T. Consider the entry δ immediately to the right of

ϵ in T. We know by the inductive hypothesis that if ϵ_i reaches the cell on top of an entry d_k, then $\epsilon_i \leq \delta_{k+1}$, and hence $\epsilon_i \leq \delta_j$ for all $j \leq k+1$. Let J be the largest value such that d_J appears before b_{p+1} in reading order. Note that such a J exists since d_1 appears before b_1 in reading order, and b_1 appears before b_{p+1} in reading order.

Then the entry ϵ_i which bumped the entry in c must be less than or equal to δ_{J+1}, since it or an entry greater than ϵ_i had to pass the cell on top of δ_J to reach the cell on top of β_{p+1}.

By successive applications of Lemma 3.2.9, if β_{p+2} reaches δ_{J+1}, then $\beta_{p+2} > \delta_{J+1}$. Therefore $\delta_{J+1} \geq \epsilon_i$ implies that $\beta_{p+2} > \epsilon_i$. Otherwise, some entry $\beta_r < \beta_{p+2}$ of P_β reaches δ_{J+1}, and hence $\beta_r > \delta_{J+1}$. But then $\beta_{p+2} > \beta_r > \delta_{J+1} \geq \epsilon_i$. So in either case, $\epsilon_i < \beta_{p+2}$. Therefore any entry which bumps the cell on top of β_1 is still less than or equal to β_{p+2}. So the cell on top of β_1 must remain less than or equal to β_2. Thus the entry β satisfies the second path condition.

Next we must prove that β satisfies the first path condition. Consider the $(l+1)^{th}$ entry, $\alpha = \alpha_1$, of the $(m+2)^{th}$ column of T. This entry is immediately to the left of the entry β_1. The rows of an SSYT are weakly increasing, so $\alpha_1 \leq \beta_1$. Therefore either α_1 bumps a smaller entry before reaching the cell c above b_1 or α_1 reaches c. In either case, let α_s denote the entry of P_α which reaches the cell c.

When α_s reaches the cell above b_1, we have $\alpha_s \leq \beta_1$. If α_s bumps the entry κ on top of b_1, then $\kappa \leq \beta_2$ implies that the entry of P_α to reach the cell on top of b_2

must be less than or equal to β_2. Otherwise, $\alpha_s \leq \kappa$, and hence $\alpha_s \leq \beta_2$. Again the entry to reach the cell on top of b_2 must be less than or equal to β_2.

Finally, assume that for the first p entries β_j of P_β, the entry of P_α to reach the cell on top of b_j is less than or equal to β_j. Consider the entry $\beta_{p+1} \in P_\beta$. Let α_q be the entry of P_α to reach b_{p+1}. We know by the inductive hypothesis that $\alpha_q \leq \beta_p$.

Therefore, when the entry α_q reaches the cell above b_p, the cell on top of b_p is less than or equal to β_{p+1}. If α_q bumps this cell, then the entry of P_α to reach b_{p+1} is less than or equal to β_{p+1}. Otherwise, α_q is less than or equal to the cell on top of b_p and hence $\alpha_q \leq \beta_{p+1}$. So the entry of P_α to reach b_{p+1} must be less than or equal to β_{p+1}. □

Lemmas 3.2.7 and 3.2.10 characterize the appearance of termination points. In particular, if α is directly to the left of β in T, Lemma 3.2.10 implies that if an entry α_j of P_α reaches the empty cell on top of t_β, $\alpha_j \leq \sigma(t_\beta)$. Therefore α_j would be placed in this empty cell and the procedure would terminate. This means that t_α is row-wise above t_β. Lemma 3.2.7 implies that if α is the k^{th} largest entry in a column, t_α occurs at or below the k^{th} column in reading order. (Recall that ordering the columns in reading order just means ordering their highest non-empty cells in reading order.)

Proposition 3.2.11. *The shape of $\Psi(T)$ is a rearrangement of the shape of T.*

Proof. Argue by induction on the number of columns of T. First assume that T contains only one column. The shape of T is 1^n. Then $col(T)$ is a strictly decreasing word. Therefore each letter maps to the bottom row of the semi-standard skyline filling. The resulting shape is an arrangement of zeros and ones, a rearrangement of 1^n.

Next, assume that if T contains j columns then the shape of $\Psi(T)$ is a rearrangement of the shape of T. Let T be an SSYT of shape λ which contains $j+1$ columns. After mapping the first j columns of T, the shape of the resulting figure is a rearrangement of $(\lambda_1 - 1, \lambda_2 - 1, \ldots)$. By Lemmas 3.2.7 and 3.2.10, mapping the next column into the shape adds one cell to each existing column, plus possibly several new cells if the $(j+1)^{th}$ column of T is taller than the j^{th} column. Therefore the resulting shape is a rearrangement of $(\lambda_1, \lambda_2, \ldots) = \lambda$. □

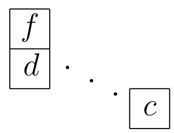
We have shown that Ψ maps an SSYT of shape λ into an SAF whose shape rearranges λ, and that Ψ preserves weight. We need only to describe the inverse of Ψ to prove that it is the desired bijection.

In fact, we know from Section 3.1 that the number of SAF whose shape rearranges λ is equal to the number of SSYT of shape λ. Since each SSYT maps to one and only one SAF, it is enough to show that given an SAF F which comes from an SSYT, there is precisely one SSYT which maps to F under Ψ.

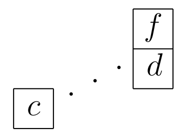
Proposition 3.2.12. *The map Ψ is invertible.*

Proof. Given a skyline augmented filling F, consider the set S of all cells which are in the top row of a column. Of these, begin with the cell c which is farthest right in the reading order. This was the last cell to be bumped into place during $\Psi : T \to F$

by Lemma 3.2.7. Scan backwards through the reading order to find the next cell, d, such that $\sigma(d) > \sigma(c)$ and d lies directly below an entry f such that $\sigma(f) \leq \sigma(c)$.



or



This is the first entry which could have bumped $\sigma(c)$, since the entries scanned between c and d were either too small to bump $\sigma(c)$ or were directly below an entry which would have created a descent with $\sigma(c)$. To see that $\sigma(d)$ did in fact bump $\sigma(c) = \alpha$, consider the cell f directly above d.

Case 1. $\sigma(f) = \sigma(c) = \alpha$.

If $\sigma(d)$ did not bump α, then α was bumped from a cell g before d in reading order. This cell could not be in the same row as d or in the same row as f for otherwise g and f would be attacking. The lowest possible row from which α could have been bumped is the first row in which the entry β in the column C_d containing d is not equal to α. If $\sigma(c)$ were bumped from a cell in this row but to the right of the column C_d, then c and the cell in C_d in the row below would be attacking. The entry β could not have bumped α, since $\beta < \alpha$. Therefore α must have been bumped from a cell to the left of the cell containing β. But then α would have bumped β, a contradiction. Therefore $\sigma(f) \neq \alpha$.

Case 2. $\sigma(f) < \alpha$.

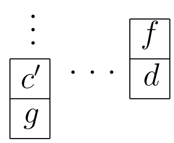
In this case, α would have bumped $\sigma(f)$ if α were bumped from a cell before f

(a) $C_d \geq C_c$ (b) $C_d < C_c$

Figure 3.3: The arrow denotes the taller column.

in reading order. Therefore, α was bumped from a cell after f. If α were bumped from a cell c' in the same row as f, then either $\{f, c', d\}$ would be a type A non-inversion triple (Figure 3.3(a)) or d, c', and the cell below c' would form a type B non-inversion triple by Lemma 2.1.5 (Figure 3.3(b)).

Therefore α was bumped from a cell c' from the same row as d.



If the column containing c' is taller than or equal to the column containing d, then the entry in the cell g below c' must be less than $\sigma(d)$ or else $\{c', d, g\}$ is a type A non-inversion triple. If the column containing d is taller, again the entry $\sigma(g)$ must be less than $\sigma(d)$, for otherwise d, g, and the cell below d would form a type B non-inversion triple by Lemma 2.1.5. In either case, the cell g directly below c' must have $\sigma(g) < \sigma(d)$. Therefore the cell c' which bumps α must have $\sigma(c') < \sigma(d)$, since $\sigma(c') \leq \sigma(g) < \sigma(d)$. Let β be the entry which bumped α. We saw that $\beta < \sigma(d)$ and $\beta > \alpha > \sigma(f)$. Therefore the same argument shows that β must have been bumped from a cell to the left of c' in the same row as c'. Continuing this argument produces an infinite sequence of columns to the left of the column containing d, which is a contradiction since there are only finitely many columns in the diagram.

Therefore the entry $\sigma(d)$ bumped $\sigma(c)$. Replace $\sigma(d)$ with $\sigma(c)$ and repeat the entire process with $\sigma(d)$, starting with the cell immediately before d in reading order. Continue this scanning procedure until there are no cells farther back in the reading order which could have bumped the selected entry, ϵ. This entry is the first letter in $col(T)$.

Choose the next element δ of S to appear in the backwards reading order. This was the termination point of the second to last entry in $col(T)$, by Lemmas 3.2.7 and 3.2.10. (If there are no other cells in S, begin with the new set S' consisting of all the cells which are in the top row of some column and repeat.) Move backwards from this element through the reading word to determine the initial element whose placement terminated with this particular element δ. Continue this procedure until the entire word $col(T)$ has been determined.

This procedure inverts the map Ψ, since it sends a skyline augmented filling F to the word $col(T)$ which maps to F under Ψ. $\square$

The map $\Psi : \text{SSYT} \rightarrow \text{SAF}$ is a weight-preserving invertible map between semi-standard Young tableaux and skyline augmented fillings. In particular, this means that the number of SSYT of shape λ with weight $\prod x_i^{a_i}$ is equal to the number of SAF with weight $\prod x_i^{a_i}$ whose shape rearranges λ. Thus the coefficient of $\prod x_i^{a_i}$ in $\sum_{\lambda'} NS_{\lambda'}$ is equal to the coefficient of $\prod x_i^{a_i}$ in s_λ. This completes the proof of

Theorem 3.2.1.

Chapter 4

Applications of the bijection

4.1 An analogue of the Robinson-Schensted-Knuth Algorithm

The insertion process utilized in the above bijection is reminiscent of Schensted insertion, the fundamental operation of the Robinson-Schensted-Knuth Algorithm.

Theorem 4.1.1. *(Robinson-Schensted-Knuth [19]) There exists a bijection between* $\mathbb{N}$*-matrices of finite support and pairs of semi-standard Young tableaux of the same shape.*

We apply the same procedure to arrive at an analogue of the RSK Algorithm for skyline augmented fillings.

Theorem 4.1.2. *There exists a bijection between* $\mathbb{N}-$ *matrices with finite support* *and pairs (F, G) of skyline augmented fillings of compositions which rearrange the same partition.*

We prove Theorem 4.1.2 by constructing this bijection.

4.1.1 A map from $\mathbb{N}$-matrices to pairs of SAF

Let $A = (a_{i,j})$ be an $\mathbb{N}$-matrix with finite support. This means that there are only finitely many non-zero entries in A. There exists a unique two-line array corresponding to A which is defined by the non-zero entries in A. Beginning at the upper lefthand corner and reading left to right, top to bottom, find the first non-zero entry $a_{i,j}$. Place an i in the top line and a j in the bottom line $a_{i,j}$ times. When this has been done for each non-zero entry, one obtains the following array.

$$w_A = \begin{pmatrix} i_1 & i_2 & \cdots \\ \\ j_1 & j_2 & \cdots \end{pmatrix}$$

Begin with an empty skyline augmented filling ϕ. Read the bottom row of w_A from right to left, inserting the entries into ϕ according to the insertion process described in Section 3.1. Each time an entry j_k from the bottom line is inserted, send the entry i_k directly above it into an SAF G which records the cell t_{j_k} at which the insertion procedure $F \leftarrow j_k$ terminated. The cell i_k is placed on top of the leftmost column of G with the same height as the column of F at which the process terminated. If t_{j_k} occurs in the bottom row of F, the corresponding entry i_k is placed on the bottom row in the i_k^{th} column of G. (If i_m were already in the i^{th}

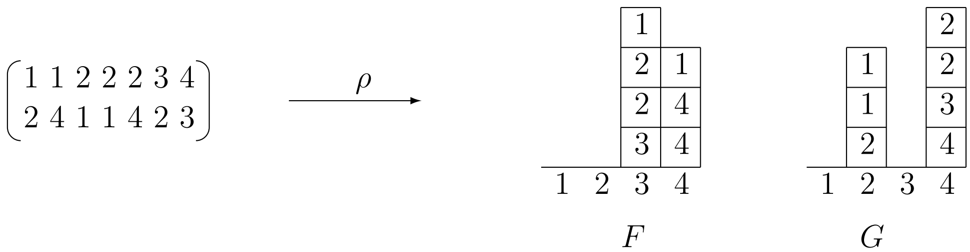


Figure 4.1: $\rho : A \longrightarrow SAF \times SAF$

column for some $m > k$, then the entries corresponding to $i_m, i_{m-1}, \ldots i_k$ inserted into F form a weakly decreasing sequence by the nature of the array. But then the cell t_{i_k} would occur in a higher row by Lemma 3.2.8. So the i^{th} column is available.) In this way the shape of G becomes a rearrangement of the shape of F. When the process is complete, the result is a pair $(F, G) = \rho(A)$ of SAF's whose shapes are rearrangements of the same partition (see Figure 4.1).

Proposition 4.1.3. *The resulting figures F and G are skyline augmented fillings.*

Proof. The map ρ produces a figure F with no descents by construction. The values in the top line of w_A are weakly decreasing as the line is read from right to left, so G contains no descents. Therefore it suffices to prove that every triple of F and every triple of G is an inversion triple.

Proposition 3.2.3 implies that F is a skyline augmented filling, since F is constructed by repeated applications of the insertion procedure $F \leftarrow k$. Use induction on the number of entries of G to prove that every triple of G is an inversion triple.

Assume that G contains only one entry, α. Then this entry is placed on top of the letter α in the basement. The only triples in the resulting figure are type B

triples consisting of a cell from the basement row to the left of α and the column containing an α in the basement plus the α in the first row. But this triple is a type B inversion triple, so if G contains only one entry then G is a skyline augmented filling.

Suppose that if G contains $i-1$ entries, then G is a skyline augmented filling. Place the next value, β, according to the placement of the corresponding entry in F. Assume that this placement of β creates a non-inversion triple to get a contradiction.

First assume that $\{\beta, \sigma(a), \sigma(c)\}$ is a type A non-inversion triple, as shown below.

β		c
a		d

Then $\beta \leq \sigma(c) \leq \sigma(a)$. Before β is placed c is directly on top of a cell d in G such that $\{a, c, d\}$ is a type B inversion triple. By Lemma 2.1.5, $\sigma(a) < \sigma(d)$. Then $\sigma(a) < \sigma(c)$, for otherwise $\{a, c, d\}$ is a type B non-inversion triple. This contradicts the assumption that $\sigma(c) \leq \sigma(a)$. So placing β could not result in a type A non-inversion triple with β on the left.

Next, assume that $\{\beta, \sigma(a), \sigma(c)\}$ is a type A non-inversion triple as shown below.

c		β
a		

Then $\sigma(c) \leq \beta \leq \sigma(a)$. But β was placed in G after $\sigma(c)$ and $\sigma(a)$ by assumption, so $\beta \leq \sigma(c)$. Therefore this situation can only occur if $\sigma(c) = \beta$. But in this case, the corresponding values $\sigma(c')$ and β' in F must have $\beta' \leq \sigma(c')$ and the letters inserted between $\sigma(c')$ and β' form a weakly decreasing sequence by the structure of the array. By Lemma 3.2.8, the insertion path for β' terminates before that of $\sigma(c')$ in reading order, which contradicts the fact that $\sigma(c)$ comes before β in the reading word of G, because the placement of $\sigma(c)$ and β mark the termination points of the corresponding insertion paths in F.

Now assume that $\{\sigma(a), \beta, \sigma(c)\}$ is a type B non-inversion triple in G, as shown below.

$$\begin{array}{cc} & \boxed{\beta} \\ \boxed{a} & \boxed{c} \end{array}$$

We must have $\beta \leq \sigma(a) \leq \sigma(c)$. However, since β was placed after $\sigma(a)$ and $\sigma(c)$, the entry β would have been placed in the leftmost possible column of G with the correct height. The column containing $\sigma(a)$ had the same height as the column containing $\sigma(c)$, so β would have been placed on top of the column containing $\sigma(a)$. So this case cannot occur.

Finally, assume that $\{\sigma(a), \beta, \sigma(c)\}$ is a type B non-inversion triple in G, as depicted below.

$$\begin{array}{cc} & \boxed{a} \\ \boxed{\beta} & \boxed{c} \end{array}$$

Since β is placed after $\sigma(a)$ which is placed after $\sigma(c)$ in G, we have $\beta \leq \sigma(a) \leq \sigma(c)$. The only way this is a type B non-inversion triple is if $\beta = \sigma(a)$. In this case, the corresponding entries β' and $\sigma(a')$ in F must have $\beta' \leq \sigma(a')$, and the elements inserted between $\sigma(a')$ and β' must form a weakly decreasing sequence. By Lemma 3.2.8, the insertion path for β' must terminate before $t_{\sigma(a')}$ in reading order. But β and $\sigma(a)$ in G mark these termination points. Since β is after $\sigma(a)$ in reading order, this is a contradiction. □

The skyline augmented filling G records the column to which a cell is added in F. Therefore F and G are clearly rearrangements of the same shape. This fact and Proposition 4.1.3 imply that ρ is indeed a map from $\mathbb{N}$-matrices to pairs of skyline augmented fillings which are rearrangements of the same shape.

4.1.2 The inverse map

Given a pair (F, G) of skyline augmented fillings whose shapes are rearrangements of the same partition λ, let G_{rs} be the first occurrence of the smallest entry of G in reading order. (Here G_{rs} is the element of G in row r and column s.) If s is the i^{th} column of height r in G, let s' be the i^{th} column of height r in F. Equal elements of G are inserted bottom to top by Lemma 3.2.4. It follows that $G_{rs} = i_1$ and $F_{rs'}$ was the last element of F to be bumped into place during the insertion of j_1.

Delete $F_{rs'}$ from F and G_{rs} from G. Scan right to left, bottom to top (backwards through the reading word) starting with the cell directly to the left of $F_{rs'}$ to determine which cell (if any) bumped $F_{rs'}$. If there exists a cell k before $F_{rs'}$ in the reading word such that $\sigma(k) > \sigma(F_{rs'})$ and the cell directly on top of k has value less than or equal to $\sigma(F_{rs'})$, this $\sigma(k)$ bumped $F_{rs'}$ by the argument from

Section 3.2. Replace $\sigma(k)$ by $\sigma(F_{rs'})$ and repeat the procedure with $\sigma(k)$ starting from the cell k. Continue working backward through the reading word until there are no more letters. The remaining entry is the letter j_1. (Notice that this is the same procedure used in the map Ψ^{-1} in Section 3.2, so we have already proved that this procedure does in fact yield the correct insertion path.)

Next find the highest occurrence of the smallest entry j_2 of the new G. Repeat the procedure to find i_2. Continue until there are no more entries in F and G. Then all of the i and j values of the array w_A have been determined, and the process is inverted.

4.2 Standard Bases for Schubert polynomials

The Schubert polynomials were introduced by Lascoux and Schützenberger [11] as a combinatorial tool for attacking problems in algebraic geometry. The Schubert polynomials can be described as a sum of key polynomials [18], which are decomposed into *standard bases*, $\mathfrak{U}(\pi, \lambda)$, for a permutation π and a partition λ. Lascoux and Schützenberger [12] define an action of the symmetric group on the free algebra wwhich can be used to build the standard bases inductively. We provide a non-inductive description of the standard bases.

4.2.1 Key polynomials

A *key* is a semi-standard Young tableau such that the set of entries in the $(j+1)^{th}$ column are a subset of the set of entries in the j^{th} column, for all j. There is a bijection [18] between compositions and keys given by $\gamma \mapsto key(\gamma)$ where $key(\gamma)$ is the key whose first γ_j columns contain the letter j. To invert this map, send the key T to the composition with j^{th} part equal to the number of times j occurs in T. (Recall that this composition is called the *content* of T.)

Example 4.2.1. *Let* $\gamma = (2, 1, 1, 4, 0, 3)$. *Then* $key(\gamma) =$

6			
4			
3	6		
2	4	6	
1	1	4	4

Let w be any word such that applying the RSK algorithm to w yields the pair of tableaux (P, Q). Then w is said to be *Knuth equivalent* [10] to $col(P)$, denoted $w \sim P$. (There is precisely one word which is the column form of an SSYT in each equivalence class.) This equivalence can also be described by the following set of relations on subwords.

$$xzy \sim zxy \quad \text{for} \quad x \le y < z,$$

$$yxz \sim yzx \quad \text{for} \quad x < y \le z.$$

Let λ' be the conjugate shape of a partition λ, obtained by reflecting the Ferrers diagram of λ across the line $x = y$. Let w be an arbitrary word such that $w \sim T$

for T of shape λ. Then *colform*(w) is the composition consisting of the lengths of the subwords in *col*(w). The word w is said to be *column-frank* if *colform*(w) is a rearrangement of the nonzero parts of λ'.

Example 4.2.2. *Let* $w = 3\ 5\ 4\ 2\ 2\ 1$*. Then colform*$(w) = (1, 3, 2)$*. But* $w \sim T$*, where* $T =$

$$\begin{array}{|c|c} \hline 5 & \\ \cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Therefore $\lambda' = (4, 2)$*, so* w *is not column-frank since* $(1, 3, 2)$ *is not a rearrangement of* $(4, 2)$*.*

Let $v = 4\ 2\ 5\ 3\ 2\ 1$*. Then colform*$(v) = (2, 4)$*. In this case,* $v \sim P$*, where* $P =$

$$\begin{array}{|c|c} \hline 4 & \\ \cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Again, $\lambda' = (4, 2)$*, so in this case* v *is column-frank.*

The set $SSYT(\lambda)$ of semi-standard Young tableaux of shape λ admits a partial ordering. Let $S, T \in SSYT(\lambda)$. Then $S \leq T \iff$ each entry in S is less than or equal to the corresponding entry in T.

Let T be a semi-standard Young tableau of shape λ. The *right key* of T, denoted $K_+(T)$ is the key of shape λ whose j^{th} column is given by the rightmost column of any column-frank word v such that $v \sim T$ and *colform*(v) is of the form $(\ldots, \lambda'_j)$ [18]. (For $T = 5\ 3\ 2\ 1 \cdot 4\ 2$ above, we have $K_+(T) = 5\ 4\ 2\ 1 \cdot 4\ 2$.)

Given an arbitrary partition λ and permutation π, there exists an associated key $\mathrm{K}(\pi,\lambda)$ defined as follows. Take the subword of π consisting of the first λ_1 letters and reorder the letters in decreasing order. This is the first column of $\mathrm{K}(\pi,\lambda)$. Then take the first λ_2 letters of π and order these in decreasing order to get the second column of $\mathrm{K}(\pi,\lambda)$. Continuing this way, one derives the word $col(\mathrm{K}(\pi,\lambda))$ as in [12]. For example, let $\pi = 241635$ and $\lambda = (4,2,2,1)$. Then $\mathrm{K}(\pi,\lambda) = 6\ 4\ 2\ 1 \cdot 4\ 2 \cdot 4\ 2 \cdot 2$.

To any composition γ, there is an associated *key polynomial,* κ_γ. The original algebraic definition of the key polynomial involves operators on reduced words, but Lascoux and Schutzenberger [12] provide the following combinatorial description.

$$\kappa_\gamma = \sum_{\substack{T \in SSYT(\gamma') \\ K_+(T) \leq \mathrm{key}(\gamma')}} x^T,$$

where γ' is the partition such that γ rearranges γ' and $SSYT(\gamma')$ is the collection of SSYT of shape γ'.

The Schubert polynomials [11] help bridge the gap between combinatorics and algebraic geometry. Reiner and Shimozono [18] introduce many beautiful connections between key polynomials, Schur polynomials, and Schubert polynomials.

Definition 4.2.3. The standard basis $\mathfrak{U}(\pi,\lambda)$ is the sum of the weights of all SSYT having right key $\mathrm{K}(\pi,\lambda)$.

This definition does not provide a method for construction of the standard bases, since it is not at all obvious which SSYT have a given right key.

4.2.2 Constructing the standard bases

Lascoux and Schützenberger [12] define an action of the symmetric group on polynomials to construct the standard bases inductively. Each permutation in the symmetric group can be decomposed into a series of elementary transpositions, so it suffices to define an action for a simple transposition, σ_i, which permutes i and $i+1$. The operator $\overline{\pi}_i = \overline{\pi}_{\sigma_i}$ is

$$f \longrightarrow \frac{(f^{\sigma_i} - f)}{(1 - x_i/x_{i+1})} = \overline{\pi}_i f,$$

where f^{σ_i} denotes the transposition action of σ_i on the indices of the variables in f. For example, if $f = x_1^2 x_2 x_3$, then $\overline{\pi}_1 f = \frac{(x_1 x_2^2 x_3 - x_1^2 x_2 x_3)}{(1 - x_1/x_2)} = x_1 x_2^2 x_3$

The operators $\overline{\pi}_i$ satisfy the Coxeter relations $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i = \overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1}$ and $\overline{\pi}_i \overline{\pi}_j = \overline{\pi}_j \overline{\pi}_i$ for $\|j - i\| > 1$ [12]. We can lift the operator $\overline{\pi}_i$ to an operator θ_i on the free algebra by the following process. Given i and a word w in the alphabet $X = x_1, x_2, \ldots$, let m be the number of times the letter x_{i+1} occurs in w and let $m + k$ be the number of times the letter x_i occurs in w. Then if $k \geq 0$, w and w^{σ_i} differ by the exchange of a subword x_i^k into x_{i+1}^k. The case where $k < 0$ is not needed in this paper. When $k \geq 0$, define $w\theta_i$ to be the sum of all words in which the subword x_i^k has been changed respectively into $x_i^{k-1} x_{i+1}, \;\; x_i^{k-2} x_{i+1}^2, \;\; \ldots, \;\; x_{i+1}^k$.

Every partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ has a corresponding *dominant monomial*,

$$x^\lambda = (x_{\lambda_1} \ldots x_2 x_1)(x_{\lambda_2} \ldots x_2 x_1) \ldots$$

which equals the weight of the *super tableau*. (The super tableau is the SSYT which contains only the entry i in the i^{th} row.)

Theorem 4.2.4. *(Lascoux-Schützenberger [12]) Let x^λ be the dominant monomial corresponding to λ and $\sigma_i\sigma_j...\sigma_k$ be any reduced decomposition of a permutation π. Then $\mathfrak{U}(\pi,\lambda) = \theta_i\theta_j...\theta_k x^\lambda$.*

Theorem 4.2.4 provides an inductive method for constructing the standard basis $\mathfrak{U}(\pi,\lambda)$. Begin with $\mathfrak{U}(id,\lambda)$ and apply θ_i to determine $\mathfrak{U}(\sigma_i,\lambda)$. Then apply θ_j to $\mathfrak{U}(\sigma_i,\lambda)$ to get $\mathfrak{U}(\sigma_j\sigma_i,\lambda)$. Continue this process until the desired standard basis is obtained. We abuse notation by identifying the standard basis, a sum of monomials, with the set of diagrams (or their weights) which constitute the sum. Figure 4.2 depicts all the standard bases for the partition $(2,1)$.

4.2.3 A non-inductive construction of the standard bases

The set of standard bases for the partition λ can be considered as a decomposition of the Schur function s_λ. For any partition λ of n, we have ([12])

$$\sum_{\pi\in S_n} \mathfrak{U}(\pi,\lambda) = s_\lambda.$$

Since the nonsymmetric Schur functions are also a decomposition of the Schur functions, it is natural to determine their relationship to the standard bases.

Theorem 4.2.5. $\mathfrak{U}(\pi,\lambda) = NS_{\pi(\lambda)}$, *where $\pi(\lambda)$ denotes the action of π on the parts of λ.*

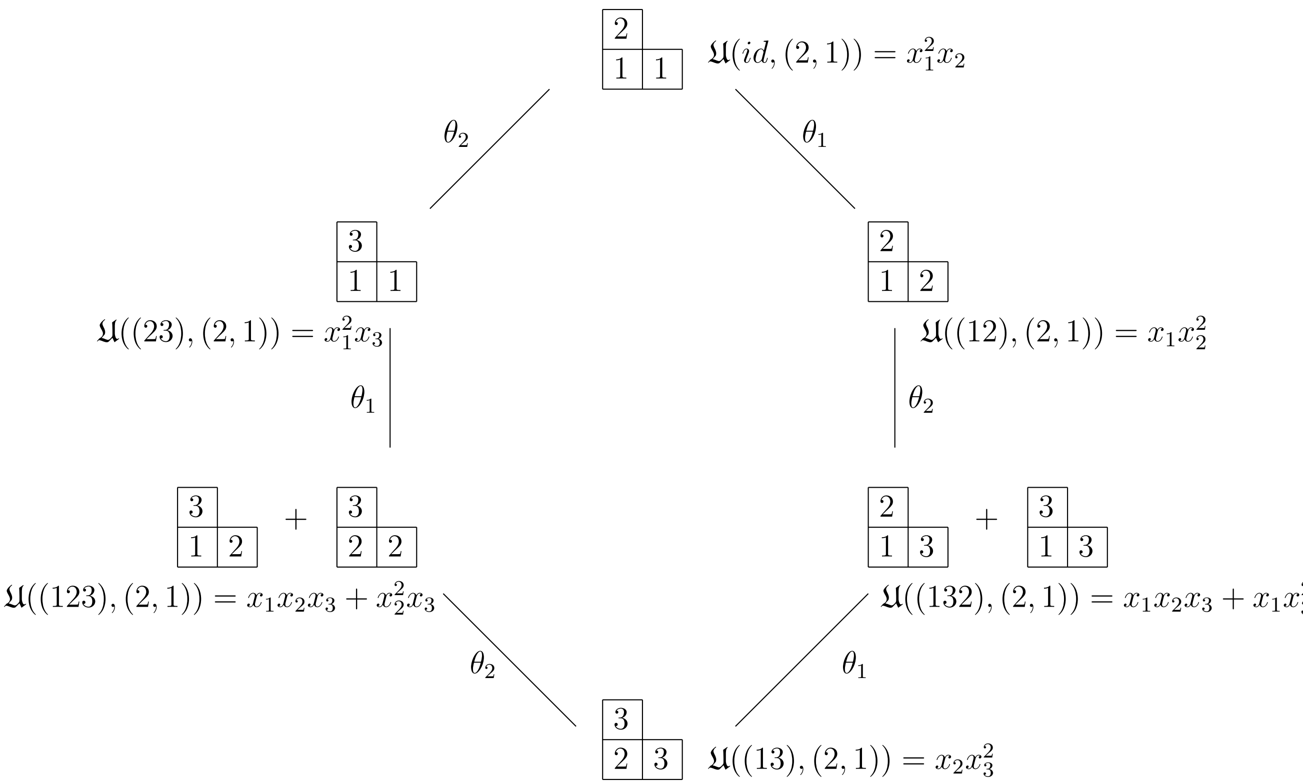


Figure 4.2: The standard bases for $\lambda = (2, 1)$.

To prove Theorem 4.2.5 combinatorially, we need a few lemmas. Recall that Lemma 3.2.4 implies the following:

Let col(T) be the column reading word of a SSYT T. Label the occurrences of the entry α in $col(T)$ by $\alpha_1, \alpha_2, ..., \alpha_k$ in increasing order starting from the right. Then $i < j \Rightarrow$ the i^{th} occurrence of α (α_i) in $\Psi(T)$ is in a lower row than α_j, the j^{th} occurrence of α.

Lemma 4.2.6. *Given an arbitrary skyline augmented filling F with row entries $R_1, R_2, ...R_k$ (where $k = max_i\{\gamma_i\}$), then F is the only SAF with these row entries.*

Proof. Given the row entries $R_1, R_2, ..., R_k$, map them into a skyline augmented filling as follows. Let α_1 be the largest entry in R_1. Place α_1 as far left as possible in the first row of an empty SAF without creating a descent. Next place the second largest entry, α_2, of R_1 into the leftmost empty cell c in the first row such that the entry directly beneath c is greater than or equal to α_2. Continue placing the elements of R_1 in this manner until the smallest element of R_1 has been placed.

Next choose the largest entry of R_2. Place it as far left as possible without creating a descent in the second row of the partially constructed SAF. Continue this procedure until the smallest entry of R_2 has been placed. Do this for each of the k rows. Notice that by Lemma 3.1.2 the resulting figure F is indeed a skyline augmented filling, the only SAF with row entries $R_1, R_2, ..., R_k$. □

Let $\tilde{\theta}_i$ be the action of θ_i on an individual semi-standard Young tableau. This action can be described by a matching procedure. Let $col(T)$ be the column word of T and let $(i+1)_1$ be the first occurrence of $i+1$ in $col(T)$. Match $(i+1)_1$ with the leftmost occurrence of i which lies to the right of $(i+1)_1$ in $col(T)$. If there is no such i, the matching procedure is complete. Otherwise, continue with the next $i+1$ until there are no more occurrences of $i+1$.

For example, if $col(T) = 4\ 2\ 1 \cdot 5\ 3\ 1 \cdot 4\ 2 \cdot 3$ and $i = 2$, then the first 3 in $col(T)$ (read left to right) is matched to the second 2 in $col(T)$.

When the matching procedure is complete, send the rightmost unmatched i to $i+1$. The resulting word is $\tilde{\theta}_i(T) = T'$. If $T \in \mathfrak{U}(\pi, \lambda)$ then either $\tilde{\theta}_i(T) \in \mathfrak{U}(\pi, \lambda)$ or $\tilde{\theta}_i(T) \in \mathfrak{U}(\sigma_i\pi, \lambda)$.

Proposition 4.2.7. *There exists a map* Θ_i :*SAF*$\longrightarrow$*SAF such that the following diagram commutes.*

$$\begin{array}{ccc} T & \xrightarrow{\tilde{\theta}_i} & T' \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ F & \xrightarrow{\Theta_i} & F' \end{array}$$

Proof. Let F be an arbitrary skyline augmented filling and let $read(F)$ be the reading word obtained by reading F left to right, top to bottom, keeping track of the rows. Find the first occurence $(i+1)_1$ of $i+1$ in this word. If there exists an occurrence of i in the same row as $(i+1)_1$, match $(i+1)_1$ to this i. Otherwise match this entry $i+1$ to the first unmatched i which occurs after $(i+1)_1$ in $read(F)$. If there is no such i, $(i+1)_1$ is unmatched and the matching process is complete. Continue this matching procedure for each $i+1$ until either an unmatched $i+1$ is reached or there are no more entries $i+1$.

Pick the rightmost unmatched i. Change it to $i+1$. (If there is none, then $\Theta_i(F) = F$.) The result is a collection of rows which differ from $read(F)$ in precisely one entry. Lemma 4.2.6 provides a procedure for mapping this collection of rows to a unique SAF. (This is the same procedure used in Section 3.1.) This SAF is $\Theta_i(F) = F'$. We must show that $\Theta_i(\Psi(T)) = \Psi(\tilde{\theta}_i(T))$.

Given an arbitrary SSYT T, consider the SAF $F = \Psi(T)$. We claim that sending the rightmost unmatched i in T to an $i+1$ affects the row of F which contains the rightmost i in $read(F)$.

To see this, begin with the word $col(T)$ and let i_0 be the rightmost unmatched i in $col(T)$. Let m be the number of matched i's to the right of i_0 in $col(T)$.

Let $i = i_1$ be the first value equal to i to be inserted into F. If i_1 is matched to some $i+1 = (i+1)_1$, then $(i+1)_1$ is inserted after i_1. If $(i+1)_1$ reaches i_1 during some insertion stage and i_1 is not in the bottom row, $(i+1)_1$ will bump i_1 since i_1 must be on top of a cell with entry greater than or equal to $i+1$. If i_1 is on the bottom row, $i+1$ will be placed immediately to the right of i in the $(i+1)^{th}$ column. In either situation, i_1 is still matched to $(i+1)_1$ in F.

Assume that the rightmost $k-1$ matched appearances of i in $col(T)$ are still matched to the same $i+1$ in F. Consider the k^{th} matched $i = i_k$. It is matched to some $i+1 = (i+1)_k$ which is inserted after i_k. The only way for $(i+1)_k$ to pass i_k is if i_k is immediately on top of a cell containing the value $i = i_{k-1}$. But then by the inductive hypothesis, i_{k-1} is matched to an $i+1 = (i+1)_{k-1}$ appearing either in the same row or at some higher row. Since $(i+1)_k$ was not placed on top of $(i+1)_{k-1}$, $(i+1)_{k-1}$ must appear after i_{k-1} in the same row. But then $(i+1)_k$ is placed on top of $(i+1)_{k-1}$ if not before. Hence $(i+1)_k$ appears on the same row as i_k, and so the two remain matched to each other.

Therefore, the first m matched values i remain matched to the same values $i+1$ in F. So the lowest candidate for a row containing an unmatched i in F is the row containing i_0. However, it is possible that i_0 is matched in F to some $i+1 = (i+1)_2$ which was matched to some value $i_2 \neq i_0$ in $col(T)$.

If this is the case, then $(i+1)_2$ passed i_2 in F during some insertion process without bumping i_2. But then i_2 is on top of some value less than $i+1$, and hence i_2 is on top of an entry i. This means that the column containing i_2 contains only i's between the row containing i_2 and the row containing i_0. Therefore, $(i+1)_2$ is placed in the same row as i_0.

Assume there are q such i's in this column. Then there must be at most $q-1$ $(i+1)$'s which are matched to these i's in F, for otherwise there are q $(i+1)$'s to the right of $q-1$ matched i's and one unmatched i. But then one of the $(i+1)$'s would have been matched to i_0 in $col(T)$.

Therefore the top i in the column with q i's is the unmatched i in F which is converted to an $i+1$ during $\Theta_i(F)$. However, in $\tilde{\theta}_i(\Psi(T))$, when i_0 is converted to an $i+1 = (i+1)_0$, this entry fills the position occupied by $(i+1)_2$ in $\Psi(T)$. Then the other values $i+1$ are shifted up by one row. Similarly, the second i in the column of i's now occupies the position filled by i_0 in $\Psi(T)$. So the i's are shifted down by one row. Hence the row which appears different in $\tilde{\theta}_i(\Psi(T))$ is the row which contains the q^{th} i of this column of i's in $\Psi(T)$. This row contained an i in $\Psi(T)$ but contains an $i+1$ in $\tilde{\theta}_i(\Psi(T))$. But this is precisely the row which contained the rightmost unmatched i in $read(\Psi(T))$, and hence the row in which an i is converted to an $i+1$. So the rows affected by $\tilde{\theta}_i$ and Θ_i are the same.

The only way the rows of $\Psi(T)$ and $\Psi(\tilde{\theta}_i(T))$ could differ other than this one change would be if some letter α which is inserted on top of $i+1$ in $\Psi(\tilde{\theta}_i(T))$ cannot be placed on top of i in $\Psi(T)$. This could only happen if the letter α is equal to $i+1$. But if $i+1$ is inserted in the row directly above $i+1$ in $\Psi(\tilde{\theta}_i(T))$, then this $i+1$ would be matched in $col(T)$ to the i which was changed to $i+1$. This is a contradiction; the rows of $\Psi(\tilde{\theta}_i(T))$ are the same as the rows of $\Psi(T)$ except for the altering of the rightmost unmatched i.

But Θ_i sends $\Psi(T)$ to the unique SAF whose rows are identical to those of $\Psi(T)$ other than the altering the same row which is altered in $\Theta_i(\Psi(T))$. Therefore $\Theta_i(\Psi(T)) = \Psi(\tilde{\theta}_i(T))$ and the diagram commutes. □

Lemma 4.2.8. *For $F \in NS_\gamma$, either $\Theta_i(F) \in NS_\gamma$ or $\Theta_i(F) \in NS_{\sigma_i\gamma}$. In particular, $\Theta_i(F) \in NS_{\sigma_i\gamma}$ precisely when there are no unmatched $(i+1)$'s to the right of the rightmost unmatched i in $read(F)$.*

Proof. Assume that $F \in NS_\gamma$. If $\Theta_i(F) = F$, then $\Theta_i(F) \in NS_\gamma$. We must show that in the case where an unmatched i is sent to $i+1$, the resulting skyline augmented filling is either in NS_γ or in $NS_{\sigma_i\gamma}$. But the map Θ_i shifts the lowest (row-wise) unmatched i, denoted i_0, in F. If there are no occurrences of the letter i in the row directly below the row r containing the shifted i, then $i+1$ is mapped to the same position as i_0 in F. If there are no $i+1$'s in the row $r+1$ above r, then the entries in $r+1$ are sent to the same positions as in F, and so the shape is still γ.

If there exists an $i+1 = (i+1)_{r+1}$ in row $r+1$, then there must be an i in row $r+1$, for otherwise the $i+1$ would be matched to the i_0 in row r. If $(i+1)_{r+1}$ lies to the left of i_0 in F, then it is mapped to the same position in $\Theta_i(F)$ and everything remains the same. Otherwise, it will be placed on top of the $i+1 = (i+1)_0$ which replaced i_0. Then the $i = i_{r+1}$ in row $r+1$ will be placed in the position previously occupied by $i+1$. The only entries in row $r+2$ affected by this change are i and $i+1$. Again, if there is an $i+1 = (i+1)_{r+2}$ in row $r+2$ then there must be an $i = i_{r+2}$ in row $r+2$. By similar reasoning, the positions of the i and the $i+1$ will switch, but all other entries remain in the same position as in F. Therefore the heights of the columns containing i and $i+1$ remain the same. The same reasoning implies that the shape of the result is γ regardless of whether or not an $i+1$ is in a row above i_0.

If there exists an $i = i_{r-1}$ in the row $r-1$ directly below i_0, their must also be an $i+1 = (i+1)_{r-1}$ in row $r-1$, since i_{r-1} is matched. If $(i+1)_{r-1}$ appears to the left of i_{r-1}, then i_0 could not have been on top of i_{r-1} in F, because i_0 would reach the empty cell on top of $(i+1)_{r-1}$ first. Therefore, $(i+1)_0$ is mapped to the same cell in $\Theta_i(F)$ as i_0 in F. In this case, we are back in the previous situation.

Finally, assume that i_{r-1} appears to the left of $(i+1)_{r-1}$ in F. If the column containing $(i+1)_{r-1}$ is strictly taller than the column containing i_{r-1}, then the entry on top of $(i+1)_{r-1}$ must be equal to $i+1$ to ensure a type B inversion triple. But this contradicts the fact that there are no $i+1$ entries in row r.

Therefore the column containing i is weakly taller than the column containing $i+1$. The entry below i must be less than $i+1$ to ensure that these three cells are a type A inversion triple. But this entry must also be greater than or equal to i. Hence it must be equal to i, denoted i_{r-2}. There must be an $i+1 = (i+1)_{r-2}$ in row $r-2$, since i_{r-2} is matched in F. Since F is a non-attacking filling, $(i+1)_{r-2}$ must appear weakly to the right of $(i+1)_{r-1}$. But the entry α immediately below $(i+1)_{r-1}$ must have $\alpha \geq i+1$. By Lemma 2.1.5, the column containing $(i+1)_{r-1}$ must be weakly taller than the column containing $(i+1)_{r-2}$, which means that the column containing i_{r-2} is weakly taller than the column containing $(i+1)_{r-2}$. This again implies that the entry below i_{r-2} is i. Continuing in this manner, we have a column consisting only of i's, so this must be the i^{th} column. Since there must be an $i+1$ in each of these rows, there must be an $i+1$ in the first row. But any entry in the bottom row lies on top of its corresponding entry in the basement row. Therefore the $i+1$ in the bottom row is in the $(i+1)^{th}$ column. This entry is weakly to the right of the other $(i+1)$'s. Since it is immediately to the right of the i^{th} column, this means that the $(i+1)^{th}$ column contains only the entries $i+1$ until the r^{th} row.

The entry i_0 cannot lie strictly to the right of i_{r-1} since F is non-attacking. But it cannot lie strictly to the left of the i^{th} column without creating a descent. Therefore i_0 is in the i^{th} column. Sending i_0 to $i+1 = (i+1)_0$ moves it from the i^{th} column into the $(i+1)^{th}$ column. There are no unmatched $(i+1)$'s in the row $r+1$ above r, so if there is only an i in the row above, it is placed on top of $(i+1)_0$. If both i and $i+1$ appear, $i+1$ is placed on top of $(i+1)_0$ and i is placed in the position occupied by $i+1$ in F. If neither appears, then the rest of the column containing $(i+1)_0$ is the same as the rest of the i_0 column in F. This implies that Θ_i permutes the i^{th} and $(i+1)^{th}$ column, resulting in shape $\sigma_i\gamma$. This case occurs only when there are no unmatched $(i+1)$s after the rightmost unmatched i (in reading order), for each $(i+1)$ which appears in a lower row is matched to the i in its same row. □

We are now ready to describe the connection between the standard bases of Lascoux and Schützenberger and the nonsymmetric Schur functions.

Recall that Theorem 4.2.5 states that $\mathfrak{U}(\pi, \lambda) = NS_{\pi(\lambda)}$, where $\pi(\lambda)$ denotes the action of π on the parts of λ.

Proof. (of Theorem 4.2.5)

Fix a partition λ and argue by induction on the length of the permutation π in $\mathfrak{U}(\pi, \lambda)$. First let π be the identity. Then $\mathfrak{U}(\pi, \lambda)$ is the dominant monomial.

Consider λ as a composition of n into n parts by adding zeros to the right if necessary. Each cell a in λ_1 must have $\sigma(a) = 1$, for otherwise there would be a descent. If the second column contained a cell b such that $\sigma(b) = 1$, this cell and the cell beside it in first column would be attacking. To avoid descents, each cell b in the second column must have $\sigma(b) \leq 2$. Therefore each cell b in the second column has $\sigma(b) = 2$. Continuing inductively, we see that each cell c in the i^{th} column must have $\sigma(c) = i$. To see that this is indeed an SAF, we need only to check type A triples. But if the two cells in the lefthand column are equal and less than the cell in the righthand column, the result is a type A inversion triple. Therefore, the $NS_\lambda = \mathfrak{U}(\pi, \lambda)$.

Next assume that $\mathfrak{U}(\pi, \lambda) = NS_{\pi(\lambda)}$, where $\pi(\lambda)$ is the permutation π applied to the columns of λ when λ is considered as a composition of n into n parts. The monomials in $\mathfrak{U}(\sigma_i\pi, \lambda)$ are the images of monomials of $\mathfrak{U}(\pi, \lambda)$ whose image under (possibly multiple applications of) $\tilde{\theta}_i$ is not a monomial of $\mathfrak{U}(\pi, \lambda)$. Pick some such monomial of $\mathfrak{U}(\pi, \lambda)$ which maps to $\mathfrak{U}(\sigma_i\pi, \lambda)$, represented by the SSYT T. By Proposition 4.2.7, $\Psi(\tilde{\theta}_i(T)) = \Theta_i(\Psi(T))$. Since $\Psi(T) \in NS_{\pi(\lambda)}$, Lemma 4.2.8 implies that $\Theta_i(\Psi(T)) \in NS_{\pi(\lambda)}$ or $\Theta_i(\Psi(T)) \in NS_{\sigma_i\pi(\lambda)}$. If $\Theta_i(\Psi(T)) \in NS_{\pi(\lambda)}$, then $\Psi(\tilde{\theta}_i(T)) \in NS_{\pi(\lambda)}$, so $\tilde{\theta}_i(T) \in \mathfrak{U}(\pi, \lambda)$ because $\mathfrak{U}(\pi, \lambda) = NS_{\pi(\lambda)}$ by the inductive hypothesis. But this contradicts the assumption that $\tilde{\theta}_i(T) \in \mathfrak{U}(\sigma_i\pi, \lambda)$, so $\Theta_i(\Psi(T)) \in NS_{\sigma_i\pi(\lambda)}$. Therefore, $\mathfrak{U}(\sigma_i\pi, \lambda) \subseteq NS_{\sigma_i\pi(\lambda)}$.

Let F be a monomial in $NS_{\sigma_i\pi(\lambda)}$. Consider *read*$(F)$. Match the occurrences

i and $i+1$ of F as in Proposition 4.2.7 above. Find the leftmost unmatched $i+1 = (i+1)_0$. Send it to i and map the row entries back in according to the procedure in Lemma 4.2.6. The resulting reading word is the reading word of an SAF F' where this $i = i_0$ is the rightmost unmatched i. (If there were any unmatched values of i to the right of i_0, they would have been matched to $(i+1)_0$ in $read(F)$. If i_0 were matched to some $i+1$ to its left, that would be an unmatched $i+1$ in $read(F)$ farther left than the selected $i+1$.) Therefore $\Theta_i(F') = F$. If $F' \in NS_{\pi(\lambda)}$, then $\Psi^{-1}(F) = T$ is the semi-standard Young tableau in $\mathfrak{U}(\pi, \lambda)$ such that $\tilde{\theta}_i(T)$ maps to F. Otherwise, repeat the procedure on F' until an element of $NS_{\pi(\lambda)}$ is reached. This happens eventually since by Lemma 4.2.8, an element of $NS_{\pi(\lambda)}$ is sent by Θ_i to an element of $NS_{\sigma_i\pi(\lambda)}$ precisely when there are no unmatched $(i+1)$s to the right of the rightmost unmatched i in $read(F)$. F' satisfies this condition when the leftmost unmatched $i+1$ in F is also the rightmost unmatched $i+1$. Since the number of i's must be less than or equal to the number of $(i+1)$'s, this will eventually occur. Therefore $NS_{\sigma_i\pi(\lambda)} \subseteq \mathfrak{U}(\sigma_i\pi, \lambda)$. So $NS_{\sigma_i\pi(\lambda)} = \mathfrak{U}(\sigma_i\pi, \lambda)$. □

Theorem 4.2.5 provides us with a non-inductive construction of the standard bases. In particular, given a partition $\lambda \vdash n$ and a permutation $\pi \in S_n$, first consider λ as a composition of n into n parts by adding zeros to the right if necessary. Then apply the permutation π to the columns of I to get the shape $\pi(\lambda)$. Finally, determine all skyline augmented fillings of the shape $\pi(\lambda)$. The monomials given by the weights of these SAFs are the monomials of $\mathfrak{U}(\pi, \lambda)$

Recall that the standard basis $\mathfrak{U}(\pi, \lambda)$ is equal to the sum of the weights of all SSYT with right key $K(\pi, \lambda)$. Therefore all of the SSYT which map to an SAF of shape $\pi(\lambda)$ have the same right key, $K(\pi, \lambda)$. The *lock* (denoted $lock(\gamma)$) of a composition γ is the SAF of shape γ whose i^{th} column contains only the entries i.

Theorem 4.2.5 provides a simple method for determining the right key of a semi-standard Young tableau.

Corollary 4.2.9. *Given an arbitrary SSYT T, let γ be the shape of $\Psi(T)$. Then $K_+(T) = key(\gamma)$.*

Proof. We must show that the map $\Psi : SSYT \to SAF$ sends a key T to $lock(\gamma)$, where γ is the composition $content(T)$.

We prove this by induction on the number of columns of T. If T has only one column, $C_1 = \alpha_1\ \alpha_2\ \ldots \alpha_j$, then this column maps to a filling F with one row such that the α_i^{th} column contains the entry α_i. This is precisely $lock(content(T))$.

Next, assume that $\Psi(T) = lock(content(T))$ for all keys T with less than or equal to $m-1$ columns. Let S be a key with m columns. After the insertion of the last $m-1$ columns, the figure is $lock(content(S \setminus C_1))$ by the inductive hypothesis. We must show that inserting the first column produces $lock(content(S))$.

Let $C_1 = \alpha_1\ \alpha_2\ \ldots \alpha_l$. The last entry, α_1 to be inserted, must be placed on top of the α_1^{th} column, since all other columns contain strictly smaller entries. Therefore if α_2 had been placed here during a previous step, α_2 is bumped to the α_2^{th} column.

If not, then α_2 was already on top of the α_2^{th} column, since all other columns contain entries which are strictly smaller than α_2. Continuing in this manner we see that after all elements of C_1 have been inserted, the element α_i must lie on top of the α_i^{th} column. Therefore $\Psi(S) = lock(content(S))$.

Since $lock(\pi(\lambda)) \in \mathfrak{U}(\pi, \lambda)$, all SSYT which map to SAFs of shape $\pi(\lambda)$ have the same right key as $lock(\pi(\lambda))$. Therefore, if $T \in \mathfrak{U}(\pi, \lambda)$, then $key(T) = key(\pi(\lambda))$.

$\square$

For example, the SSYT T of Example 3.2.6 has right key

11				
10	11			
8	10	10		
7	8	8		
5	5	5		
3	3	3	10	10

since it maps to an SAF of shape $(0, 0, 3, 0, 3, 0, 1, 3, 0, 5, 2, 0, 0, 0, 0, 0, 0)$.

This is a simple and quick procedure for calculating the right key of any SSYT which facilitates the computation of key polynomials and Schubert polynomials.

Chapter 5

Symmetries and Kotska numbers

In this chapter, we describe several properties of skyline augmented fillings and nonsymmetric Schur functions. These properties are potentially useful for answering difficult questions about Schur functions.

5.1 Symmetries of nonsymmetric Schur functions

The nonsymmetric Schur functions are not symmetric in general. However, the polynomial NS_γ is invariant under the action of a certain subgroup of S_n.

Theorem 5.1.1. *Let $H = \{i : \gamma_i = \gamma_{i+1}\}$ and let $G = \langle \sigma_i : i \in H \rangle$ be the subgroup of S_n generated by the elementary transpositions $\sigma_i = (i, i+1)$ for $i \in H$. Then for $\pi \in S_n$, we have $\pi(NS_\gamma) = NS_\gamma \iff \pi \in G$.*

Proof. Recall that Theorem 4.2.5 states that $\mathfrak{U}(\pi, \lambda) = NS_{\pi(\lambda)}$, where $\pi(\lambda)$ denotes the action of π on the parts of λ. So for any composition γ, $NS_\gamma = \mathfrak{U}(\pi, \lambda)$ for some permutation π of the partition λ which rearranges to γ. (This means that $\gamma = \pi(\lambda)$ as compositions.)

If $\gamma_i = \gamma_{i+1}$, then $\sigma_i(\gamma) = \gamma$. Let $\prod_{i=1}^n x_i^{a_i}$ be the weight of an SAF F of shape γ in NS_γ. Say $a_i - a_{i+1} = b$. The monomial determined by applying σ_i to the weight of $F \in \mathfrak{U}(\pi, \lambda)$ is equal to the weight of $\Theta_i^b(F)$. But Proposition 4.2.7 states that $\Theta_i^b(F) = \tilde{\theta}_i^b(T)$ for some $T \in SSYT(\lambda)$. The map $\tilde{\theta}_i^b$ sends an SSYT T in $\mathfrak{U}(\pi, \lambda)$ to an SSYT T' in $\mathfrak{U}(\sigma_i\pi, \lambda)$ such that the number of i's and $(i+1)$'s are interchanged.

We saw in Section 4.2.2 that $\tilde{\theta}_i^b$ is a bijection between SSYT of shape λ in $\mathfrak{U}(\pi, \lambda)$, content $\{1^{a_1}, 2^{a_2}, \ldots i^{a_i}, (i+1)^{a_{i+1}}, \ldots n^{a_n}\}$ and SSYT of shape λ in $\mathfrak{U}(\sigma_i\pi, \lambda)$, content $\{1^{a_1}, 2^{a_2}, \ldots i^{a_{i+1}}, (i+1)^{a_i}, \ldots n^{a_n}\}$. Therefore $\tilde{\Theta}_i^b$ is a bijection between SAF of shape $\pi(\lambda) = \gamma$, content $\{1^{a_1}, 2^{a_2}, \ldots i^{a_i}, (i+1)^{a_{i+1}}, \ldots n^{a_n}\}$ and SAF of shape $\sigma_i(\gamma)$, content $\{1^{a_1}, 2^{a_2}, \ldots i^{a_{i+1}}, (i+1)^{a_i}, \ldots n^{a_n}\}$.

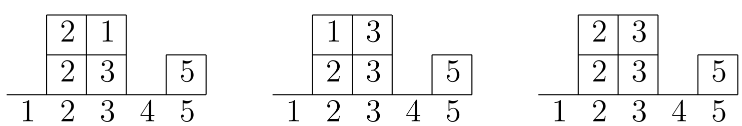
We know that $\sigma_i(\gamma) = \gamma$, so the bijection is between SAF's of shape γ and content $\{1^{a_1}, 2^{a_2}, \ldots i^{a_i}, (i+1)^{a_{i+1}}, \ldots n^{a_n}\}$ and SAF's of shape γ and content $\{1^{a_1}, 2^{a_2}, \ldots i^{a_{i+1}}, (i+1)^{a_i}, \ldots n^{a_n}\}$. Therefore NS_γ is invariant under σ_i if $\gamma_i = \gamma_{i+1}$.

It remains to show that if $\gamma_j \neq \gamma_{j+1}$, then $\sigma_j(NS_\gamma) \neq NS_\gamma$. But consider the SAF $lock(\gamma)$ in which the i^{th} column contains only the entry i. First assume $\gamma_j > \gamma_{j+1}$. There are γ_j j's in $lock(\gamma)$. The only way to place more $(j+1)$'s in the shape γ is to place them in a column C_k to the right of the $(j+1)^{th}$ column. But this would eliminate some of the entries k. Therefore there is no SAF of shape γ

with content $\{1^{\gamma_1}, 2^{\gamma_2}, \ldots j^{\gamma_{j+1}}, (j+1)^{\gamma_j}, \ldots n^{\gamma_n}\}$.

Finally, assume that $\gamma_j < \gamma_{j+1}$. Again consider $lock(\gamma)$. Any SAF with content $\{1^{\gamma_1}, 2^{\gamma_2}, \ldots j^{\gamma_{j+1}}, (j+1)^{\gamma_j}, \ldots n^{\gamma_n}\}$ must have the property that the m^{th} column contains all of the γ_m m's for $m > j+1$. Therefore any j must appear either in the j^{th} column or the $(j+1)^{th}$ column. But there must be γ_j entries equal to $j+1$ in the first γ_j rows of the $(j+1)^{th}$ column. This means that the γ_j entries in the j^{th} column are equal to j and the remainder of the j's appear in the $(j+1)^{th}$ column, beginning at row $\gamma_j + 1$ and continuing up the column. But then the j in row $\gamma_j + 1$ of the $(j+1)^{th}$ column and the j at the top of the j^{th} column would be attacking, which is a contradiction. So $\sigma_j(NS_\gamma) \neq NS_\gamma$ when $\gamma_j \neq \gamma_{j+1}$. $\square$

For example, let $\gamma = (0,2,2,0,1)$. Then $NS_\gamma = x_1 x_2^2 x_3 x_5 + x_1 x_2 x_3^2 x_5 + x_2^2 x_3^2 x_5$, as shown below.



The polynomial NS_γ is fixed under the permutation σ_2 and no other permutation. So the subgroup of S_n that fixes NS_γ is $\langle (2,3) \rangle$.

5.2 Nonsymmetric Kostka numbers

Recall that the Kostka number $K_{\lambda\mu}$ is the number of semi-standard Young tableaux of shape λ and weight μ. The nonsymmetric Kostka number $NK_{\gamma\delta}$ is the number of skyline augmented fillings of shape γ and weight δ. Equation 3.0.1 implies that

$$K_{\lambda\mu} = \sum_{\gamma \sim \lambda} NK_{\gamma\mu}$$

since $K_{\lambda\mu}$ is the coefficient of x^μ in s_λ and $NK_{\gamma\mu}$ is the coefficient of x^μ in NS_γ. Therefore, a simple formula for $NK_{\gamma\mu}$ would provide a simple formula for $K_{\lambda\mu}$.

The Hook Length Formula [2] described in Chapter 1.2 is an example of a computational formula for the Kostka number $K_{\lambda 1^n}$. The probabilistic proof of Greene, Nijenhuis, and Wilf [4] utilizes a recursive formula for counting the number of standard Young tableaux. We describe a recursive formula for counting the number of standard skyline augmented fillings (SSAF) with the hope that this formula will lead to an analogue of the Hook Length Formula in the nonsymmetric setting.

Definition 5.2.1. An *attic* is a cell c in $\widehat{dg}(\gamma)$ satisfying the following conditions.

1. c is the highest cell in its column.

2. If the column containing c has height k, then there are no columns of height $k-1$ to the left of c.

3. For each column j to the left of the column containing c, we have

$$\sum_{i=1}^{j} \gamma_i < j.$$

Let γ^c be the figure obtained by deleting the cell c. Note that if c is an attic, γ^c is still a composition. Let NF_γ be the number of SSAF of shape γ.

Proposition 5.2.2. *For any composition γ,*

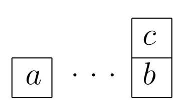
$$NF_\gamma = \sum_c NF_{\gamma^c},$$

where the sum is over all attics c.

Proof. Let γ be a composition. We must prove that for any SSAF of shape γ, the entry 1 must appear in an attic. We must also show that the number of SSAF of shape γ^c is equal to the number of SSAF of shape γ with the entry 1 in c.

Let F be an arbitrary SSAF of shape γ. Let c be the cell containing the entry 1. Then c must appear at the top of a column, for otherwise c and the cell on top of c would create a descent. So c satisfies the first attic condition.

Assume that the column C_0 containing c has height k. To get a contradiction on the second attic condition, assume that there exists a column C_1 of height $k-1$ to the left of C_0. Let a be the cell in the $(k-1)^{th}$ row of C_1 and let b be the cell in the $(k-1)^{th}$ row of C_0 as depicted below.



Then $\sigma(a) < \sigma(b)$ by Lemma 2.1.5. But then $1 = \sigma(c) < \sigma(a) < \sigma(b)$ implies that $\{a, b, c\}$ is a type B non-inversion triple. So c must satisfy the second attic condition.

To prove that c satisfies the third attic condition, again argue by contradiction. Let γ_j be a part of γ to the left of the column containing c. Suppose that $\sum_{i=1}^{j} \gamma_i \geq j$. This means that in the first j columns, there are at least j cells. None of these

Figure 5.1: Shape γ maps to shape γ^c

cells can contain an entry greater than j, for that would create a descent. Therefore each of the numbers 1 through j must appear somewhere in the first j columns. So the entry 1 appears in one of the first j columns. Because c is not contained in one of these columns, the entry 1 could not be contained in the cell c. This is a contradiction, so c satisfies the third attic condition.

This argument shows that c must be an attic if c contains the entry 1. Next we must prove that the number of SSAF of shape γ^c is equal to the number of SSAF of shape γ with the entry 1 in c.

Begin with an SSAF of shape γ and let c be the cell containing the entry 1. Delete c and delete the leftmost basement entry. Send each entry α to the entry $\alpha - 1$. The resulting figure is a standard filling of the shape $\widehat{dg}(\gamma^c)$. (See Figure 5.1.) We must prove that this filling is an SAF.

Sending each cell α to $\alpha - 1$ does not create any non-inversion triples, since this changes the entries but not their order. We must check for non-inversion triples created by the deletion of cell c.

Let C_0 be the column containing c. Consider an arbitrary column C_i to the left of C_0. If C_i is taller than or equal to C_0 in γ, the triples involving these two columns are type A triples. After deleting c, the column C_i will still be taller than or equal to C_0, so the triples involving these two columns will remain type A triples. Therefore no non-inversion triples are created.

If C_i is shorter than C_0 in γ, then the second attic condition implies that C_i is at least two rows shorter than C_0. So the triples involving these two columns are type B triples and remain type B triples after c is deleted. Therefore no non-inversion triples are created with columns to the left of C_0.

Let C_j be a column to the right of C_0. If C_0 is strictly shorter than or strictly taller than C_j, the type of triple involving these two columns remains the same after c is deleted. However, if C_0 is the same height as C_j, then we must check to see if any type B non-inversion triples were created. Consider the type A triple $\{c, a, e\}$ shown below.

$$\begin{array}{|c|c|c|}\cline{1-1}\cline{3-3} c & & e \\ \hline a & \multicolumn{1}{c|}{\cdots} & b \\ \cline{1-1}\cline{3-3}\end{array}$$

Since $\sigma(c) = 1$ and $\{c, a, e\}$ is a type A triple, $\sigma(c) < \sigma(a) < \sigma(e)$. Therefore $\sigma(a) < \sigma(e) < \sigma(b)$ so $\{a, e, b\}$ is a type B inversion triple after the deletion of c.

Next we must show that all type B triples involving cells in lower rows are inversion triples. Consider d in a cell of C_0 somewhere below c and f in a cell of C_1 somewhere below b such that d and f are in the same row r of F, the highest row such that $\sigma(f) < \sigma(d)$. Let h be the cell immediately above d and let g be the cell immediately above f, as shown.

$$\begin{array}{|c|}\hline h \\ \hline d \\ \hline \end{array} \; \cdots \; \begin{array}{|c|}\hline g \\ \hline f \\ \hline \end{array}$$

Then $\sigma(h) < \sigma(g) < \sigma(f) < \sigma(d)$. But then $\{h, g, d\}$ would be a type A non-inversion triple in F. So for every pair of cells d, f in the columns C_0 and C_1 in the same row of F, the entry on the left is smaller.

Consider two cells, h, g in F as depicted.

$$\begin{array}{|c|}\hline h \\ \hline d \\ \hline \end{array} \; \cdots \; \begin{array}{|c|}\hline g \\ \hline f \\ \hline \end{array}$$

We know that $\sigma(h) < \sigma(g)$. Since $\{h, d, g\}$ is a type A inversion triple in F, $\sigma(d) < \sigma(g)$. Therefore when c is removed, $\sigma(d) < \sigma(g) < \sigma(f)$. So $\{d, g, f\}$ is a type B inversion triple. This implies that every type B triple created by the deletion of c is an inversion triple. Therefore deleting c does not create any non-inversions. So this mapping of an SSAF of shape γ to a filling of shape γ^c does indeed produce an SSAF.

Similarly, we show that any SSAF of shape γ^c remains an SSAF when its entries α are mapped to $\alpha + 1$ and the cell c is added containing the entry 1.

Given an arbitrary SSAF G of shape γ^c consider the column C_0 which will contain c and a column C_1 to its left. If C_0 is strictly taller than C_1, the triples involving the two columns are type B inversion triples. Adding the cell c does not change this, and c will be at least two rows higher than any entry of C_1. So no non-inversion triples are created. If C_1 is strictly taller than C_0, the column C_1 is still weakly taller than C_0 after adding the cell c. So the existing triples remain the same, and we must check that the cells d and b of C_1 in the row containing c and row $r-1$ do not form an inversion triple with c.

$$\begin{array}{|c|}\hline d \\ \hline b \\ \hline\end{array} \cdots \begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array}$$

We know that $\sigma(d) < \sigma(b)$ since there are no descents in G. But $\sigma(c) = 1$ so $\sigma(c) < \sigma(d) < \sigma(b)$ implies that $\{c, d, b\}$ is a type A inversion triple. If C_1 and C_0 were the same height, c would not be added onto C_0 by the second attic condition. Therefore adding the cell c onto the column C_0 does not create any new non-inversion triples to the left.

Consider a column C_2 to the right of C_0. If C_0 is weakly taller than C_2, adding c to C_0 does not change the type of existing triples, nor does it add an additional triple. If the height of C_0 is one less than the height of C_2, adding c changes the triples from type B triples to type A triples. But by Lemma 2.1.5, we know that for every pair h, g of cells in the same row of these two columns, $\sigma(h) < \sigma(g)$. Consider the type A triple $\{h, g, d\}$ depicted below.

$$\begin{array}{|c|}\hline h \\ \hline d \\ \hline\end{array} \cdots \begin{array}{|c|}\hline g \\ \hline f \\ \hline\end{array}$$

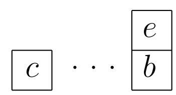
We must prove that $\{h, g, d\}$ is a type A inversion triple. But since $\{d, g, f\}$ was a type B inversion triple, $\sigma(d) < \sigma(g) < \sigma(f)$. So $\sigma(h) < \sigma(d) < \sigma(g)$ implies that $\{h, d, g\}$ is indeed a type A inversion triple.

In this case, the cell c is added to column C_0 on top of a cell a as depicted below.

$$\begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} \cdots \begin{array}{|c|}\hline e \\ \hline b \\ \hline\end{array}$$

We already know that $\sigma(a) < \sigma(e) < \sigma(b)$, since $\{a, e, b\}$ is a type B inversion triple in G. But $\sigma(c) = 1$ implies that $\sigma(c) < \sigma(a) < \sigma(e)$, so $\{c, a, e\}$ is a type A inversion triple in $G \cup c$.

Finally, if C_2 is at least two rows taller than C_0, adding c does not change the type of triples involving the two columns. We must check that if c is placed in the same row as b, with e directly on top of b as depicted below, then the triple $\{c, e, b\}$ is a type B inversion triple.



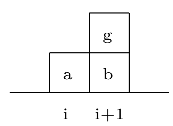
But $\sigma(e) < \sigma(b)$ and $\sigma(c) = 1$, so $\sigma(c) < \sigma(e) < \sigma(b)$. So $\{c, e, b\}$ is indeed a type B inversion triple.

Therefore if c is an attic, adding the cell c with $\sigma(c) = 1$ to an SSAF of shape γ^c and increasing the entries by one creates an SSAF of shape γ. Therefore, the number of SSAF of shape γ with the entry 1 in the cell c is exactly the number of SSAF of shape γ^c. $\square$

One significant difference between standard skyline augmented fillings and standard Young tableaux is that every partition admits at least one SYT, but there exist compositions which do not admit a standard SAF.

The complicating factor in this situation is the parts of the composition equal to zero. It is clear that if γ admits a standard SAF, the last part γ_m of γ must be non-zero. (Otherwise the entry m would create a descent wherever it is placed.) If all of the zero parts of γ appear at the beginning of γ, then the shape looks like a composition shifted to the right. If the non-zero columns are weakly decreasing, then the attics are precisely the corners in the second row through the top row. These corners are precisely the places where an n could appear if the shape (excluding the first row) were an SYT. Therefore $NF_\gamma = f_{\gamma'}$, where γ' is the partition derived by subtracting 1 from each non-zero part of γ and then discarding the zero parts.

If the columns are not weakly decreasing, then there exist two columns C_i and C_{i+1} such that C_{i+1} is immediately to the right of C_i and C_{i+1} is strictly taller than C_i. Then the cells a and b in the first row (containing i and $i+1$) together with the cell g immediately on top of B form a type B triple as shown.



But $\sigma(g)$ must be less than i, since there are no descents and i and $i+1$ already appear in the entries a and b. Then $\{g, a, b\}$ is a type B non-inversion triple, a contradiction. Therefore there are no standard SAFs of this shape.

Chapter 6

Conclusion

The Schur functions, which are described combinatorially through semi-standard Young tableaux (SSYT), are a useful basis for symmetric functions. They can be derived from a specialization of Macdonald polynomials. Recently Haglund, Haiman, and Loehr found a combinatorial description of the Macdonald polynomials involving new statistics on fillings of partitions.

Nonsymmetric Macdonald polynomials decompose the Macdonald polynomials into components indexed by compositions rather than partitions. By extending the partition statistics to compositions Haglund, Haiman, and Loehr provide a combinatorial description of the nonsymmetric Macdonald polynomials. The nonsymmetric Schur functions are derived as a specialization of nonsymmetric Macdonald polynomials and hence have a combinatorial description given by the statistics on compositions. We demonstrate that the nonsymmetric Schur functions in arbitrarily many variables form a basis for all polynomials.

The specialization $q = t = 0$ implies algebraically that the nonsymmetric Schur functions decompose the ordinary Schur functions. This thesis provides the first combinatorial proof of this fact through a bijection between SSYT and new objects called skyline augmented fillings (SAF).

The bijection between SSYT and SAF involves an insertion process similar to Schensted insertion and therefore leads to a nonsymmetric analogue of the Robinson-Schensted-Knuth algorithm. This version describes a bijection between $\mathbb{N}$-matrices of finite support and pairs of SAF's which rearrange the same partition.

Schubert polynomials are a combinatorial tool used to approach problems in algebraic geometry. The standard bases of Lascoux and Schützenberger are the building blocks of Schubert polynomials. We prove that the standard basis $\mathfrak{U}(\pi, \lambda)$ is equal to the nonsymmetric Schur function $NS_{\pi(\lambda)}$. This provides a non-inductive method for constructing standard bases.

To each SSYT, there is an associated key, which can be used to construct a key polynomial. Key polynomials are sums of standard bases and have many applications to Schubert polynomials and Schubert varieties. We demonstrate that the key associated to an arbitrary semi-standard Young tableau can be determined by mapping the SSYT to its associated SAF. This provides a simple method for constructing the key of an SSYT.

Appendix A

Tables of nonsymmetric Schur functions

Table A.1: $n = 2$

γ	$\mathbf{NS}_\gamma$
(2,0)	x_1^2
(0,2)	$x_1x_2 + x_2^2$
(1,1)	x_1x_2

Table A.2: $n = 3$

γ	$\mathbf{NS}_\gamma$
(3,0,0)	x_1^3
(2,1,0)	$x_1^2x_2$
(2,0,1)	$x_1^2x_3$
(1,2,0)	$x_1x_2^2$
(1,1,1)	$x_1x_2x_3$
(0,3,0)	$x_1^2x_2 + x_1x_2^2 + x_2^3$
(1,0,2)	$x_1x_2x_3 + x_1x_3^2$
(0,2,1)	$x_1x_2x_3 + x_2^2x_3$
(0,1,2)	$x_2x_3^2$
(0,0,3)	$x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_3^3$

Table A.3: $n = 4 \quad (\gamma_i \leq 2, \quad \forall i)$

γ	$\mathbf{NS}_\gamma$
(2,2,0,0)	$x_1^2x_2^2$
(2,1,1,0)	$x_1^2x_2x_3$
(2,1,0,1)	$x_1^2x_2x_4$
(2,0,2,0)	$x_1^2x_2x_3 + x_1^2x_3^2$
(1,2,1,0)	$x_1x_2^2x_3$
(2,0,1,1)	$x_1^2x_3x_4$
(1,1,2,0)	$x_1x_2x_3^2$
(1,2,0,1)	$x_1x_2^2x_4$
(2,0,0,2)	$x_1^2x_2x_4 + x_1^2x_3x_4 + x_1^2x_4^2$
(1,1,1,1)	$x_1x_2x_3x_4$
(0,2,2,0)	$x_1x_2x_3^2 + x_1x_2^2x_3 + x_2^2x_3^2$
(1,1,0,2)	$x_1x_2x_3x_4 + x_1x_2x_4^2$
(1,0,2,1)	$x_1x_2x_3x_4 + x_1x_3^2x_4$
(0,2,1,1)	$x_1x_2x_3x_4 + x_2^2x_3x_4$
(1,0,1,2)	$x_1x_3x_4^2$
(0,2,0,2)	$x_1x_2x_3x_4 + x_1x_2x_4^2 + x_1x_2^2x_4 + x_2^2x_3x_4 + x_2^2x_4^2$
(0,1,2,1)	$x_2x_3^2x_4$
(0,1,1,2)	$x_2x_3x_4^2$
(0,0,2,2)	$x_1x_3x_4^2 + x_1x_2x_3x_4 + x_2x_3x_4^2 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_3^2x_4^2$

Table A.4: $n = 4 \quad (\exists i \quad s.t. \ \gamma_i > 2)$

γ	$\mathbf{NS}_\gamma$
(4,0,0,0)	x_1^4
(3,1,0,0)	$x_1^3x_2$
(3,0,1,0)	$x_1^3x_3$
(3,0,0,1)	$x_1^3x_4$
(1,3,0,0)	$x_1^2x_2^2 + x_1x_2^3$
(0,4,0,0)	$x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4$
(0,3,1,0)	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_2^3x_3$
(1,0,3,0)	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_1x_3^3$
(0,3,0,1)	$x_1^2x_2x_4 + x_1x_2^2x_4 + x_2^3x_4$
(0,1,3,0)	$x_1x_2x_3^2 + x_2^2x_3^2 + x_2x_3^3$
(0,0,4,0)	$x_1^3x_3 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_2^3x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_2^2x_3^2 + x_1x_3^3 +$ $x_2x_3^3 + x_3^4$
(1,0,0,3)	$x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_1x_3^2x_4 + x_1^2x_4^2 + x_1x_2x_4^2 +$ $x_1x_3x_4^2 + x_1x_4^3$
(0,0,3,1)	$x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_3^3x_4$
(0,1,0,3)	$x_1x_2x_3x_4 + x_2^2x_3x_4 + x_2x_3^2x_4 + x_1x_2x_4^2 + x_2^2x_4^2 + x_2x_3x_4^2 + x_2x_4^3$
(0,0,1,3)	$x_1x_3x_4^2 + x_2x_3x_4^2 + x_3^2x_4^2 + x_3x_4^3$
(0,0,0,4)	$x_1^3x_4 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_2^3x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4 +$ $x_1x_3^2x_4 + x_2x_3^2x_4 + x_3^3x_4 + x_1^2x_4^2 + x_1x_2x_4^2 + x_2^2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2 +$ $x_3^2x_4^2 + x_1x_4^3 + x_2x_4^3 + x_3x_4^3 + x_4^4$

Bibliography

[1] Ivan Cherednik. Nonsymmetric Macdonald polynomials. *Internat. Math. Res. Notices*, 10:483–515, 1995.

[2] J. S. Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric group. *Canad. J. Math.*, 6:316–325, 1954.

[3] W. Fulton. *Young Tableaux with Applications to Representation Theory and Geometry*. New York: Cambridge University Press, 1997.

[4] C. Greene, A. Nijenhuis, and H. S. Wilf. A probabilistic proof of a formula for the number of Young tableaux of a given shape. *Adv. in Math.*, 31:104–109, 1979.

[5] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *Jour. Amer. Math. Soc.*, 18:735–761, 2005.

[6] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Nonsymmetric Macdonald polynomials. *pre-print*, 2005.

[7] J. Haglund, M. Haiman, and N. Loehr. Combinatorial theory of Macdonald polynomials I: Proof of Haglund's formula. *Proc. Nat. Acad. Sci. U.S.A.*, 102(8):2690–2696, 2005.

[8] H. Jack. A class of symmetric polynomials with a parameter. *Proc. R. Soc. Edinburgh (A)*, 69:1–18, 1970.

[9] Friedrich Knop and Siddhartha Sahi. A recursion and a combinatorial formula for Jack polynomials. *Invent. Math.*, 128(1):9–22, 1997.

[10] D. E. Knuth. Permutations, matrices, and generalized young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

[11] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de H. O. Foulkes. *C. R. Acad. Sci. Paris Sér. A-B*, 286:A323–A324, 1978.

[12] A. Lascoux and M.-P. Schützenberger. Keys and standard bases. In *"Invariant Theory and Tableaux," IMA Volumes in Math and its Applications (D. Stanton, ED.)*, volume 19, pages 125–144. Southend on Sea, UK, 1990.

[13] I. G. Macdonald. A new class of symmetric polynomials. *Actes du* 20^e *Séminaire Lotharingien, Publ. Inst. Rech. Math. Av.*, 372, 1988.

[14] I. G. Macdonald. Affine Hecke algebras and orthogonal polynomials. *Séminaire Bourbaki 1994/95*, 237(797):189–207, 1996.

[15] Dan Marshall. Symmetric and nonsymmetric Macdonald polynomials. *Ann. Comb.*, 3(2-4):385–415, 1999.

[16] J.-C. Novelli, I. Pak, and A. V. Stoyanovskii. A direct bijective proof of the hook-length formula. *Discrete Mathematics and Theoretical Computer Science*, 1:53–67, 1997.

[17] Eric M. Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.*, 175(1), 1995.

[18] V. Reiner and M. Shimozono. Key polynomials and a flagged littlewood-richardson rule. *Journal of Combinatorial Theory*, 70:107–143, 1995.

[19] C. E. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[20] R. P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic and Discrete Methods*, 1:164–184, 1980.

[21] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, United Kingdom, 1999.

[22] D. Zeilberger. A short hook-length bijection inspired by the Greene-Nijenhuis-Wilf proof. *J. Algorithms*, 3:101–108, 1982.

[23] D. Zeilberger. Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials. *Amer. Math. Monthly*, 96(7):590–602, 1989.