ROOT NUMBERS OF ABELIAN VARIETIES AND REPRESENTATIONS OF THE WEIL-DELIGNE GROUP

Maria Sabitova

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Ted Chinburg
Supervisor of Dissertation

David Harbater
Graduate Group Chairperson

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ABSTRACT

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Maria Sabitova

Ted Chinburg

We generalize a theorem of D. Rohrlich concerning root numbers of elliptic curves over the field of rational numbers. Our result applies to abelian varieties over number fields. Namely, under certain conditions which naturally extend the conditions used by D. Rohrlich, we show that the root number $W(A, \tau)$ associated to an abelian variety A over a number field F and a complex finite-dimensional irreducible representation τ of $\mathrm{Gal}(\overline{F}/F)$ with real-valued character is equal to 1. In the case where the ground field is $\mathbb{Q}$, we show that our result is consistent with a refined version of the conjecture of Birch and Swinnerton-Dyer. We also give a description of unitary, orthogonal, and symplectic admissible representations of the Weil-Deligne group of a local non-Archimedean field.

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Chapter 1

Introduction

One of the main objects of study in this thesis is the root number $W(A, \tau)$ associated to an abelian variety A of dimension g over a number field F and a continuous irreducible complex finite-dimensional representation τ of $\mathrm{Gal}(\overline{F}/F)$ with real-valued character. The root number $W(A, \tau)$ is a complex number of absolute value 1. Assume for simplicity that $F = \mathbb{Q}$. Then $W(A, \tau)$ appears in the following conjectural functional equation:

$$\Lambda(A, \tau, s) = W(A, \tau) \cdot \Lambda(A, \tau^*, 2 - s), \qquad (1.0.1)$$

where $s \in \mathbb{C}$, τ^* is the contragredient of τ, and

$$\Lambda(A, \tau, s) = C^s \cdot \Gamma(s)^{g \dim \tau} \cdot L(A, \tau, s)$$

for some positive constant C and the twisted L-function $L(A, \tau, s)$ which is a meromorphic function of s defined in a right half-plane. This function is conjectured to have an analytic continuation to the entire complex plane. Since τ has real-valued character, $\tau \cong \tau^*$. Assuming (1.0.1) and considering the power series expansion of $L(A, \tau, s)$ about $s = 1$, we get:

$$W(A, \tau) = (-1)^{\mathrm{ord}_{s=1} L(A,\tau,s)}. \tag{1.0.2}$$

In this thesis we generalize a result by D. Rohrlich for elliptic curves ([Ro2], p. 313, Prop. E) to abelian varieties. We prove the following theorem:

Theorem 1.0.1. *Let F be a number field, L a finite Galois extension of F, and τ an irreducible complex finite-dimensional representation of* $\mathrm{Gal}(L/F)$ *with real-valued character. Let g be a fixed positive integer and assume that the decomposition subgroups of* $\mathrm{Gal}(L/F)$ *at all the places of F lying over all the primes less or equal to $2g + 1$ are abelian. If the Schur index $m_{\mathbb{Q}}(\tau)$ is 2 then $W(A, \tau) = 1$ for every abelian variety A of dimension g over F.*

If $F = \mathbb{Q}$ then Theorem 1.0.1 is predicted by the conjectures of Birch-Swinnerton-Dyer and Deligne-Gross. Namely, the conjectures of Birch-Swinnerton-Dyer and Deligne-Gross imply

$$\mathrm{ord}_{s=1} L(A, \tau, s) = \langle \sigma_A, \tau \rangle, \tag{1.0.3}$$

where σ_A is the natural representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{C} \otimes_{\mathbb{Z}} A(\overline{\mathbb{Q}})$ and $\langle \sigma_A, \tau \rangle$ is the multiplicity of τ in σ_A ([Ro3], p. 127, Prop. 2). Thus, we get from (1.0.2) and (1.0.3):

$$W(A, \tau) = (-1)^{\langle \sigma_A, \tau \rangle}.$$

Since σ_A is realizable over $\mathbb{Q}$ and τ is irreducible, $m_{\mathbb{Q}}(\tau)$ divides $\langle \sigma_A, \tau \rangle$. Thus, if $m_{\mathbb{Q}}(\tau) = 2$ then $W(A, \tau) = 1$ for every abelian variety A over $\mathbb{Q}$ if (1.0.3) is true (cf.

[Ro2], p. 313).

To prove Theorem 1.0.1 we use the following formula:

$$W(A,\tau) = \prod_v W(A_v, \tau_v),$$

where v runs through all the places of F, $A_v = A \times_F F_v$, F_v denotes the completion of F with respect to v, and τ_v is the restriction of τ to $\mathrm{Gal}(\overline{F_v}/F_v) \hookrightarrow \mathrm{Gal}(\overline{F}/F)$. To define $W(A_v, \tau_v)$ for every place v let σ'_v denote the representation of the Weil-Deligne group $\mathcal{W}'(\overline{F}_v/F_v)$ associated to the first cohomology of A_v. Then $W(A_v, \tau_v) = W(\sigma'_v \otimes \tau_v)$, where τ_v is viewed as a representation of $\mathcal{W}'(\overline{F}_v/F_v)$. We will in fact show the following stronger result:

Theorem 1.0.2 (Theorem A). $W(A_v, \tau_v) = 1$ *for all* v *under the hypotheses of Theorem 1.0.1.*

First, we describe $W(A_v, \tau_v)$ when τ_v is a complex finite-dimensional continuous representation of $\mathrm{Gal}(\overline{F_v}/F_v)$ with real-valued character. If v is an infinite place then σ'_v is associated to the components of $H^1(A_v(\mathbb{C}), \mathbb{C})$ in the Hodge decomposition. We show in Lemma 3.1.1 that

$$W(A_v, \tau_v) = (-1)^{g \dim \tau_v}. \tag{1.0.4}$$

If v is a finite place, then

$$W(\sigma'_v \otimes \tau_v) = \frac{\epsilon(\sigma'_v \otimes \tau_v, \psi_v, dx_v)}{|\epsilon(\sigma'_v \otimes \tau_v, \psi_v, dx_v)|},$$

where ψ_v is a nontrivial additive character of F_v and dx_v is a Haar measure on F_v. Here σ'_v is isomorphic to the representation of $\mathcal{W}'(\overline{F}_v/F_v)$ afforded by $H^1_l(A_v)$, where

l is a rational prime different from the residual characteristic of F_v. It is known that $H^1_l(A_v) \cong V_l(A_v)^*$ as $\mathrm{Gal}(\overline{F}_v/F_v)$-modules over $\mathbb{Q}_l$, where $V_l(A_v) = T_l(A_v) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, $T_l(A_v)$ is the l-adic Tate module of A_v, and $V_l(A_v)^*$ denotes the contragredient of $V_l(A_v)$. Thus, we can assume that σ'_v is the representation of $\mathcal{W}'(\overline{F}_v/F_v)$ associated to $V_l(A_v)^*$. Clearly, $W(\sigma'_v \otimes \tau_v)$ does not depend on the choice of dx_v and it turns out that $W(\sigma'_v \otimes \tau_v)$ does not depend on the choice of ψ_v either. Moreover, $W(\sigma'_v \otimes \tau_v) = \pm 1$ (see Section 2.1).

We consider two cases: A_v is an abelian variety with potential good reduction and the general case. If A_v has potential good reduction, it follows from Néron-Ogg-Šhafarevič criterion that σ'_v is actually a representation of the Weil group $\mathcal{W}(\overline{F}_v/F_v)$. If the characteristic of the residue class field k_v of F_v is greater than $2g+1$, we use the theory of Serre-Tate together with methods of the representation theory to describe the class of $\sigma'_v \otimes \omega_v^{1/2}$ in the Grothendieck group of virtual representations of $\mathcal{W}(\overline{F}_v/F_v)$ (Corollary 2.2.7, Formula (2.2.4)). Here ω_v is the one-dimensional representation of $\mathcal{W}(\overline{F}_v/F_v)$ given by

$$\omega_v|_{I_v} = 1, \quad \omega_v(\Phi_v) = q_v^{-1},$$

where I_v is the inertia subgroup of $\mathrm{Gal}(\overline{F}_v/F_v)$, Φ_v is an inverse Frobenius element of $\mathrm{Gal}(\overline{F}_v/F_v)$, and $q_v = \mathrm{card}(k_v)$. Since the root number of representations of $\mathcal{W}(\overline{F}_v/F_v)$ is multiplicative in short exact sequences, this result enables us to prove the following formula for $W(\sigma'_v \otimes \tau_v)$ when $\mathrm{char}(k_v) > 2g+1$ (cf. Proposition 2.2.9):

$$W(\sigma'_v \otimes \tau_v) = \det \tau_v(-1)^{l_1} \cdot \beta^{\dim \tau_v} \cdot \gamma^{l_2} \cdot (-1)^{\langle \nu_v, \tau_v \rangle}, \tag{1.0.5}$$

where $l_1 \in \mathbb{Z}$, $\beta = \pm 1$, $\gamma = \pm 1$, $l_2 = \langle 1, \tau_v \rangle + \langle \eta_v, \tau_v \rangle$, η_v is the unramified quadratic character of $F_v^\times$, and ν_v is a representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ realizable over $\mathbb{Q}$ (cf. [Ro2], p. 318, Thm. 1).

In the general case we use the theory of uniformization of abelian varieties. According to this theory there exists a semi-abelian variety G_v over F_v and a discrete subgroup Y_v of G_v such that, in terms of rigid geometry, A_v is isomorphic to the quotient G_v/Y_v. The semi-abelian variety G_v fits into an exact sequence

$$0 \longrightarrow T_v \longrightarrow G_v \xrightarrow{f_v} B_v \longrightarrow 0, \tag{1.0.6}$$

where B_v is an abelian variety over F_v with potential good reduction, T_v is a torus over F_v of dimension r; Y_v is an étale sheaf of free abelian groups over $\mathrm{Spec}(F_v)$ of rank r. To describe σ'_v in this case we use a formula of M. Raynaud ([Ra], p. 314) which gives the action of the inertia group I_v on the l^n-torsion points of an abelian variety over a non-Archimedean local field in the case when the uniformization data splits. We need this formula to show that in this case

$$\sigma'_v \cong \kappa_v \oplus (\chi_v \otimes \omega_v^{-1} \otimes \mathrm{sp}(2)), \tag{1.0.7}$$

where κ_v is the representation of $\mathcal{W}'(\overline{F}_v/F_v)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ on $V_l(B_v)^*$,

$$\chi_v : \mathrm{Gal}(\overline{F}_v/F_v) \longrightarrow \mathrm{GL}_r(\mathbb{Z})$$

is the representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ corresponding to the Galois module $Y_v(\overline{F}_v)$, and $\mathrm{sp}(2)$ is given by (2.1.1) (see Proposition 2.3.1). Since the root number of a direct sum of representations of $\mathcal{W}'(\overline{F}_v/F_v)$ equals the product of the root numbers of the summands, we get from (1.0.7)

$$W(\sigma'_v \otimes \tau_v) = W(\kappa_v \otimes \tau_v) \cdot W(\chi_v \otimes \omega_v^{-1} \otimes \tau_v \otimes \mathrm{sp}(2)). \tag{1.0.8}$$

If $\mathrm{char}(k_v) > 2g+1$ then (1.0.5) can be applied to κ_v, i.e.,

$$W(\kappa_v \otimes \tau_v) = \det \tau_v(-1)^{l_1} \cdot \beta^{\dim \tau_v} \cdot \gamma^{l_2} \cdot (-1)^{\langle \nu_v, \tau_v \rangle}, \tag{1.0.9}$$

where l_1, β, γ, l_2, and ν_v are as in (1.0.5) when σ'_v is replaced by κ_v.

The rest of the proof of Theorem 1.0.2 is analogous to one of Proposition E ([Ro2], p. 347). Namely, it follows from Lemma on p. 339 and Lemma on p. 347 in [Ro2] that $\dim \tau$ is even. Hence we get from (1.0.4) that $W(A_v, \tau_v) = 1$ for infinite places. If v is a finite place then the assumption $m_{\mathbb{Q}}(\tau) = 2$ implies $W(\chi_v \otimes \omega_v^{-1} \otimes \tau_v \otimes \mathrm{sp}(2)) = 1$ ([Ro2], p. 327, Prop. 6), hence we have from (1.0.8)

$$W(\sigma'_v \otimes \tau_v) = W(\kappa_v \otimes \tau_v). \tag{1.0.10}$$

If v is a finite place such that $\mathrm{char}(k_v) > 2g+1$ then (1.0.9) holds which, together with the assumption $m_{\mathbb{Q}}(\tau) = 2$, implies $W(\kappa_v \otimes \tau_v) = 1$, hence $W(\sigma'_v \otimes \tau_v) = 1$.

If v is a finite place such that $\mathrm{char}(k_v) \leq 2g+1$ then the conditions on bad primes in Theorem 1.0.1 imply that τ_v is symplectic ([Ro2], Lemma on p. 347). Also, $\kappa_v \otimes \omega_v^{1/2}$ is symplectic, because κ_v comes from an abelian variety (see Section 2.1). Since real powers of ω_v do not change the root number,

$$W(\kappa_v \otimes \tau_v) = W(\kappa_v \otimes \omega_v^{1/2} \otimes \tau_v) = 1 \tag{1.0.11}$$

as the root number of the tensor product of two symplectic representations of $\mathcal{W}(\overline{F}_v/F_v)$ ([Ro2], p. 319, Prop. 2 and the remark after it). Thus, in this case we also have $W(\sigma'_v \otimes \tau_v) = 1$ by (1.0.10) and (1.0.11).

If U is a complex finite-dimensional vector space and $\lambda : D \longrightarrow \mathrm{GL}(U)$ is a representation of a group D on U, then by $\check{\lambda} : D \longrightarrow \mathrm{GL}(\check{U})$ we denote the representation of D on $\check{U}$, where $\check{U}$ is a $\mathbb{C}[D]$-module with the underlying D-module U^* and multiplication by constants defined as follows:

$$a \cdot \phi = \overline{a}\phi, \quad a \in \mathbb{C},\ \phi \in U^*.$$

We say that U is *unitary* if U admits a nondegenerate invariant hermitian form (not necessarily positive definite). In this thesis we also study unitary, orthogonal, and symplectic representations of the Weil-Deligne group $\mathcal{W}'(\overline{K}/K)$ of a local non-Archimedean field K. Namely, we prove the following theorem:

Theorem 1.0.3 (Theorem B). *Let σ' be a minimal unitary, orthogonal, or symplectic admissible representation of $\mathcal{W}'(\overline{K}/K)$ (i.e., a unitary, orthogonal, or symplectic representation respectively that cannot be written as an orthogonal sum of nonzero invariant subrepresentations). Let U be a representation space of σ' and $\langle\cdot\,,\cdot\rangle$ a non-degenerate invariant form on U. Then either σ' is indecomposable or $U \cong V \oplus \tilde{V}$, where V is an indecomposable submodule of U, $\tilde{V} = V^*$ if $\langle\cdot\,,\cdot\rangle$ is bilinear, and $\tilde{V} = \check{V}$ if $\langle\cdot\,,\cdot\rangle$ is sesquilinear. Moreover, if λ is the isomorphism of $V \oplus \tilde{V}$ onto U and $\langle\cdot\,,\cdot\rangle'$ is the form on $V \oplus \tilde{V}$ given by*

$$\langle x, y\rangle' = \langle\lambda(x), \lambda(y)\rangle, \quad x, y \in V \oplus \tilde{V},$$

then $\langle \cdot\,, \cdot \rangle'|_V$ *and* $\langle \cdot\,, \cdot \rangle'|_{\tilde{V}}$ *are degenerate,* $\langle \cdot\,, \cdot \rangle' : V \times \tilde{V} \longrightarrow \mathbb{C}$ *is the standard form given by*

$$\langle u, f \rangle' = f(u), \quad u \in V,\, f \in \tilde{V}.$$

This thesis is organized in the following way. In Chapter 2 we study the root number $W(\sigma' \otimes \tau)$, where τ is a complex finite-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$ with real-valued character, K is a local non-Archimedean field of characteristic zero, and σ' is the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)^*$, where A is an abelian variety over K. Section 2.1 contains general facts and notation. In Section 2.2 we study the case of an abelian variety with potential good reduction. Section 2.3 deals with the general case. In Chapter 3 we give the proof of Theorem 1.0.2 (Theorem A) and discuss two special cases of the theorem when local calculations are especially easy. Chapter 4 is devoted to Theorem 1.0.3 (Theorem B). In Section 4.1 we give a proof of the theorem and in Section 4.2 we use it instead of Raynaud's result mentioned above to give an elementary proof of (1.0.7) in a special case when in (1.0.6) the image of Y_v under f_v is finite.

We put proofs of the results of Section 2.2 in Appendix A. Appendix B contains a lemma needed for the proof of the main result of Section 2.3 (Proposition 2.3.1). Appendix C contains an example of an orthogonal complex finite-dimensional irreducible representation of a finite group with Schur index 2 over the rationals. In Appendix D we give a description of the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)^*$ in the case when A is the quotient of a torus by a discrete subgroup. This result will be used in Section 4.2.

Unless stated otherwise, we assume that all the representations under consideration are complex and finite-dimensional.

Chapter 2

Root numbers of abelian varieties over local non-Archimedean fields of characteristic zero

2.1 General facts and notation

Let K be a non-Archimedean local field of characteristic zero with residue class field k and a uniformizer ϖ. Let $\overline{K}$ be a fixed algebraic closure of K and let K^{unr} be the maximal unramified extension of K contained in $\overline{K}$. Let $I = \mathrm{Gal}(\overline{K}/K^{unr})$ be the inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$ and let Φ be an inverse Frobenius element of $\mathrm{Gal}(\overline{K}/K)$, i.e., Φ is a preimage of the inverse of the Frobenius automorphism under the decomposition map

$$\pi : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{k}/k).$$

By a representation σ of the Weil group $\mathcal{W}(\overline{K}/K)$ we mean a continuous homomorphism

$$\sigma : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}(U),$$

where U is a finite-dimensional complex vector space (for the definition of $\mathcal{W}(\overline{K}/K)$ see [Ro1], §1). Let $\omega : \mathcal{W}(\overline{K}/K) \longrightarrow \mathbb{C}^\times$ be the one-dimensional representation of $\mathcal{W}(\overline{K}/K)$ given by

$$\omega|_I = 1, \quad \omega(\Phi) = q^{-1},$$

where $q = \mathrm{card}(k)$. For a finite extension F of K contained in $\overline{K}$, we identify by local class field theory the one-dimensional representations of $\mathcal{W}(\overline{K}/F)$ with characters of $F^\times$ (i.e., continuous homomorphisms from $F^\times$ into $\mathbb{C}^\times$). Also, if ϕ is a representation of $\mathcal{W}(\overline{K}/F)$, the representation of $\mathcal{W}(\overline{K}/K)$ induced by ϕ will be denoted by $\mathrm{Ind}_K^F\phi$. Analogously, if ψ is a representation of $\mathcal{W}(\overline{K}/K)$, then the restriction of ψ to $\mathcal{W}(\overline{K}/F)$ will be denoted by $\mathrm{Res}_K^F\psi$.

By a representation σ' of the Weil-Deligne group $\mathcal{W}'(\overline{K}/K)$ we mean a continuous homomorphism

$$\sigma' : \mathcal{W}'(\overline{K}/K) \longrightarrow \mathrm{GL}(U),$$

where U is a finite-dimensional complex vector space and the restriction of σ' to the subgroup $\mathbb{C}$ of $\mathcal{W}'(\overline{K}/K)$ is complex analytic (for the definition of $\mathcal{W}'(\overline{K}/K)$ see [Ro1], §3). It is known that there is a bijection between representations of $\mathcal{W}'(\overline{K}/K)$

and pairs (σ, N), where $\sigma : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}(U)$ is a representation of $\mathcal{W}(\overline{K}/K)$ and N is a nilpotent endomorphism on U such that

$$\sigma(g)N\sigma(g)^{-1} = \omega(g)N, \quad g \in \mathcal{W}(\overline{K}/K).$$

In what follows we identify σ' with the corresponding pair (σ, N) and write $\sigma' = (\sigma, N)$. Also, a representation σ of $\mathcal{W}(\overline{K}/K)$ is identified with the representation $(\sigma, 0)$ of $\mathcal{W}'(\overline{K}/K)$ ([Ro1], §§1–3).

For a positive integer n let $\mathrm{sp}(n) = (\sigma, N)$ denote the special representation of dimension n, i.e., the representation of $\mathcal{W}'(\overline{K}/K)$ on $\mathbb{C}^n$ (with the standard basis $e_0, \ldots, e_{n-1}$) given by the following formulas:

$$\begin{aligned} \sigma(g)e_i &= \omega(g)^i e_i, \quad 0 \le i \le n-1,\, g \in \mathcal{W}(\overline{K}/K), \\ Ne_j &= e_{j+1}, \qquad 0 \le j \le n-2, \\ Ne_{n-1} &= 0. \end{aligned} \tag{2.1.1}$$

We say that a representation $\sigma' = (\sigma, N)$ of $\mathcal{W}'(\overline{K}/K)$ is *admissible* if σ is semisimple ([Ro1], p. 132, §5).

Let A be an abelian variety over K. For a rational prime l different from $p = \mathrm{char}(k)$ let $T_l(A)$ be the l-adic Tate module of A. It is a free $\mathbb{Z}_l$-module of rank $2g$, where $g = \dim A$. Put $V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and let

$$\sigma_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l(A)^*)$$

denote the contragredient of the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)$. We are interested in the representation $\sigma' = (\sigma, N)$ of $\mathcal{W}'(\overline{K}/K)$ associated to σ_l by the standard procedure (see e.g., [Ro1], §4). Let $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ be a field embedding. Then $\sigma : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l(A)^* \otimes_\imath \mathbb{C})$ is a representation of $\mathcal{W}(\overline{K}/K)$ (which is not necessarily obtained from the restriction of σ_l to $\mathcal{W}(\overline{K}/K)$ by extending scalars via $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$) and $N \in \mathrm{End}(V_l(A)^* \otimes_\imath \mathbb{C})$ is a nilpotent endomorphism (see [Ro1], p. 130, §4 for more detail). A priori, σ' depends on the choice of l and $\imath$, but by abuse of notation we write σ' instead of $\sigma'_{l,\imath}$. We will prove later that in our context σ' does not depend on the choice of l and $\imath$. Let τ be a representation of $\mathrm{Gal}(\overline{K}/K)$ with real-valued character. Our goal in this chapter is to compute the root number $W(\sigma' \otimes \tau)$.

Note that there is a nondegenerate, skew-symmetric, $\mathrm{Gal}(\overline{K}/K)$-equivariant pairing

$$\langle -, - \rangle : V_l(A) \times V_l(A) \longrightarrow \mathbb{Q}_l \otimes \omega_l,$$

where ω_l is the l-adic cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$. Indeed, let $A^\vee$ be the dual abelian variety to A and let

$$e_l : T_l(A) \times T_l(A^\vee) \longrightarrow \mathbb{Z}_l \otimes \omega_l$$

be the Weil pairing, which is nondegenerate and $\mathrm{Gal}(\overline{K}/K)$-equivariant ([M], p. 131, §16). Let $\mathscr{L}$ be an ample invertible sheaf on A ([M], p. 114, Cor. 7.2). Then $\varphi_{\mathscr{L}} : A \longrightarrow A^\vee$ is an isogeny ([M], p. 119, §10) and the pairing

$$e_l^{\mathscr{L}} : T_l(A) \times T_l(A) \longrightarrow \mathbb{Z}_l \otimes \omega_l$$

defined for $a, a' \in T_l(A)$ by $e_l^{\mathscr{L}}(a, a') = e_l(a, \varphi_{\mathscr{L}}(a'))$ is skew-symmetric ([M], p. 134, Prop. 16.6). Clearly, the pairing on $V_l(A)$ obtained from $e_l^{\mathscr{L}}$ by extending scalars to $\mathbb{Q}_l$ is nondegenerate and $\mathrm{Gal}(\overline{K}/K)$-equivariant.

Having $\langle -,-\rangle$, it is easy to show that $\sigma' \otimes \omega^{1/2}$ is symplectic (cf. [Ro1], p. 150, §16). Then $\sigma' \otimes \omega^{1/2} \otimes \tau$ is self-contragredient and of trivial determinant, hence $W(\sigma' \otimes \omega^{1/2} \otimes \tau)$ does not depend on the choice of a nontrivial additive character of K and $W(\sigma' \otimes \omega^{1/2} \otimes \tau) = \pm 1$ ([Ro2], p. 315). Since $W(\sigma' \otimes \tau) = W(\sigma' \otimes \omega^{1/2} \otimes \tau)$, the same conclusion holds for $W(\sigma' \otimes \tau)$.

One of the main theories we are using to find a formula for $W(\sigma' \otimes \tau)$ is the theory of uniformization of abelian varieties. According to this theory there exists a semi-abelian variety G over K and a discrete subgroup Y of G such that, in terms of rigid geometry, A is isomorphic to the quotient G/Y. The semi-abelian variety G fits into an exact sequence

$$0 \longrightarrow T \longrightarrow G \xrightarrow{f} B \longrightarrow 0, \tag{2.1.2}$$

where B is an abelian variety over K with potential good reduction, T is a torus over K of dimension r; Y is an étale sheaf of free abelian groups over $\mathrm{Spec}(K)$ of rank r.

2.2 Case of an abelian variety with potential good reduction.

We keep the notation of Section 2.1. Let B be an abelian variety over K with potential good reduction and let

$$\kappa_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l(B)^*)$$

denote the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(B)^*$. First, note that the representation $\kappa' = (\kappa, S)$ of $\mathcal{W}'(\overline{K}/K)$ associated to κ_l is actually a representation of $\mathcal{W}(\overline{K}/K)$, i.e., $S = 0$. Indeed, κ' is a representation of $\mathcal{W}(\overline{K}/K)$ if and only if κ_l is trivial on an open subgroup of I ([Ro1], p. 131, Prop.(i)). Let

$$\psi_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(T_l(B))$$

denote the representation corresponding to the $\mathrm{Gal}(\overline{K}/K)$-module $T_l(B)$. Since B has potential good reduction, the image by ψ_l of I is finite ([S-T], p. 496, Thm. 2(i)), which implies that the image by κ_l of I is finite, hence κ_l is trivial on an open subgroup of I (cf. [Ro1], p. 148).

Lemma 2.2.1. *A complex finite-dimensional representation of a group is semisimple if and only if its restriction to a subgroup of finite index is semisimple.*

Proof. It is known that a complex finite-dimensional representation λ of a group is semisimple if and only if its restriction to a normal subgroup of finite index is semisimple ([Che], p. 82, Prop. 1 and [Ro1], p. 148). Moreover, since every subgroup of finite index contains a normal subgroup of finite index, this implies that λ is semisimple if and only if its restriction to a subgroup of finite index is semisimple. $\square$

Lemma 2.2.2. *κ is semisimple.*

Proof. Since the image by κ of I is finite, by Lemma 2.2.1 it is enough to show that $\kappa(\Phi)$ is diagonalizable. Also, if $L \subset \overline{K}$ is a finite extension of K over which B acquires good reduction then again by Lemma 2.2.1, κ is semisimple if and only if its restriction to $\mathcal{W}(\overline{K}/L)$ is semisimple. Thus, we can assume that B has good reduction (cf. [Ro1], p. 148). Let B_0 be the Néron minimal model of B and $\tilde{B} = B_0 \times_{\mathcal{O}} k$ the special fiber of B_0. Since B has good reduction, the reduction map defines a $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism of $T_l(B)$ onto $T_l(\tilde{B})$, where $\mathrm{Gal}(\overline{K}/K)$ acts on $T_l(\tilde{B})$ via the decomposition map π ([S-T], p. 495, Lem. 2). Thus,

$$V_l(B) \cong V_l(\tilde{B}) \tag{2.2.1}$$

as $\mathrm{Gal}(\overline{K}/K)$-modules.

Lemma 2.2.3. *Let D be a group and let U be a finite-dimensional representation of D over a field ℓ. Then U is semisimple if the subalgebra of $\mathrm{End}_\ell(U)$ generated by the image of D is semisimple.*

Proof. Obvious. $\square$

Since the subalgebra of $\mathrm{End}_{\mathbb{Q}_l}(V_l(\tilde{B}))$ generated by the automorphisms of $V_l(\tilde{B})$ defined by elements of $\mathrm{Gal}(\overline{k}/k)$ is semisimple ([T1], p. 138), the natural l-adic representation β_l of $\mathrm{Gal}(\overline{k}/k)$ on $V_l(\tilde{B})$ is semisimple by Lemma 2.2.3. Since $\mathrm{Gal}(\overline{k}/k)$

is abelian, β_l is a direct sum of one-dimensional representations, hence $\beta_l(\pi(\Phi))$ is diagonalizable, consequently, $\kappa_l^*(\Phi)$ is diagonalizable, because $\kappa_l^*(\Phi)$ is equivalent to $\beta_l(\pi(\Phi))$ via (2.2.1). This proves that $\kappa(\Phi)$ is diagonalizable, because $\kappa(\Phi)$ is just $\kappa_l(\Phi)$ considered as an element of $\mathrm{GL}(V_l(B)^* \otimes_\imath \mathbb{C})$. □

Corollary 2.2.4. *The representation κ does not depend on the choice of l and $\imath$.*

Proof. [Ro1], p. 148 and Lemma 2.2.2. □

Since B has potential good reduction, by the theory of Serre-Tate there exists a minimal finite subextension L/K^{unr} of $\overline{K}/K^{unr}$ over which B acquires good reduction. It is a Galois extension and it is tamely ramified if $p > 2m+1$, where $m = \dim B$. Moreover, $\mathrm{Gal}(\overline{K}/L)$ is contained in the kernel of the representation ψ_l ([S-T], p. 497, Cor. 2 and p. 498, Cor. 3). Thus, κ and, consequently $\kappa \otimes \omega^{1/2}$, can be considered as representations of the group

$$\mathcal{W}(L/K) = \mathcal{W}(\overline{K}/K)/\mathrm{Gal}(\overline{K}/L) \cong \mathrm{Gal}(L/K^{unr}) \rtimes \langle \Phi \rangle,$$

where $\langle \Phi \rangle$ is the infinite cyclic group generated by Φ (cf. [Ro2], p. 331). Throughout this section we assume that $p > 2m+1$. Then, under this assumption $E = \mathrm{Gal}(L/K^{unr})$ is a finite cyclic group of order not divisible by p and $\kappa \otimes \omega^{1/2}$ is a semisimple (by Lemma 2.2.2), symplectic (see Section 2.1) representation of the semidirect product $G = E \rtimes \langle \Phi \rangle$ of finite and infinite cyclic groups. Using Corollary on p. 499 in [S-T], it is immediate that κ has $\mathbb{Q}$-valued character. Since ω is trivial on I, it follows that $\mathrm{Res}_E^G(\kappa \otimes \omega^{1/2})$ has $\mathbb{Q}$-valued character. The following results give a description of such a representation, i.e., a semisimple symplectic representation λ of a semi-direct product of a finite cyclic group E and an infinite cyclic group such that the restriction of λ to E has $\mathbb{Q}$-valued character. They will be used later to generalize a formula for the root number obtained by D. Rohrlich.

Proposition 2.2.5. *Let $C = \langle c \rangle$ be an infinite cyclic group generated by an element c and let $E = \langle e \rangle$ be a finite cyclic group of order n generated by an element e. Let $G = E \rtimes C$ be a semi-direct product, where C acts on E via $c^{-1}ec = e^k$ for some $k \in (\mathbb{Z}/n\mathbb{Z})^\times$. Denote by s the order of k in $(\mathbb{Z}/n\mathbb{Z})^\times$. Then every irreducible symplectic representation λ of G factors through the group $H = G/\langle c^{2s} \rangle$ and as a representation of H it has the following form*

$$\lambda = Ind_{E \rtimes \Gamma}^{H} \phi,$$

where Γ is a subgroup of $C/\langle c^{2s} \rangle$ generated by an element c^x and ϕ is a one-dimensional representation of $E \rtimes \Gamma$ satisfying the following conditions:

- $\phi(e) = \xi$ *for an n-th root of unity ξ of order d $(d \neq 1, 2)$*
- *x is the order of k in $(\mathbb{Z}/d\mathbb{Z})^\times$*
- *x is even*
- $\phi(c^x) = -1$
- $1 + k^{\frac{x}{2}} \equiv 0 \, (mod \, d)$.

Conversely, every representation of this form is symplectic and irreducible.

Proof. See Appendix A. □

In the notation of Proposition 2.2.5 let $\lambda = \mathrm{Infl}_H^G \mathrm{Ind}_{E\rtimes\Gamma}^H \phi$ be a symplectic irreducible representation of G and θ the one-dimensional representation of $E \rtimes \Gamma$ such that $\theta(c^x) = -1$, $\theta(e) = 1$. Let

$$\hat{\lambda} = \mathrm{Infl}_H^G \mathrm{Ind}_{E\rtimes\Gamma}^H (\phi \otimes \theta). \tag{2.2.2}$$

Whereas λ is symplectic, $\hat{\lambda}$ is realizable over $\mathbb{R}$, as can be checked using Proposition 39 ([S], p. 109).

For a group D let $R(D)$ denote the Grothendieck group of the abelian category of finite-dimensional representations of D over $\mathbb{C}$. If ρ is such a representation we denote by $[\rho]$ the corresponding element of $R(D)$.

Proposition 2.2.6. *Let $G = E \rtimes C$ be a semi-direct product as in Proposition* 2.2.5 *and λ a semisimple symplectic representation of G. If $Res_E^G \lambda$ has $\mathbb{Q}$-valued character, then in $R(G)$ we have*

$$[\lambda] = [\mu] + [\mu^*] + 2 \cdot ([\mu_0] - [\mu_0']) + [\mu_1] + \cdots + [\mu_a], \tag{2.2.3}$$

where μ is a representation of G, μ^* is the contragredient of μ, μ_0 and μ_0' are symplectic representations of G with finite images, $\mu_1, \ldots, \mu_a$ are irreducible symplectic subrepresentations of λ with finite images, $\hat{\mu}_1, \ldots, \hat{\mu}_a$ are representations with finite images given by (2.2.2) *such that $\hat{\mu}_1 \oplus \cdots \oplus \hat{\mu}_a$ is realizable over $\mathbb{Q}$.*

Proof. See Appendix A. □

Corollary 2.2.7. *Let κ be the representation of $\mathcal{W}(\overline{K}/K)$ corresponding to $V_l(B)^*$, $m = \dim B$, and $p > 2m+1$. Then in $R(\mathcal{W}(\overline{K}/K))$ we have*

$$[\kappa \otimes \omega^{1/2}] = [\mu] + [\mu^*] + 2 \cdot ([\mu_0] - [\mu_0']) + [\mu_1] + \cdots + [\mu_a], \qquad (2.2.4)$$

where μ is a representation of $\mathcal{W}(\overline{K}/K)$, μ^* is the contragredient of μ, μ_0 and μ_0' are symplectic representations of $\mathcal{W}(\overline{K}/K)$ with finite images, $\mu_1, \ldots, \mu_a$ are irreducible symplectic subrepresentations of $\kappa \otimes \omega^{1/2}$ with finite images, $\hat{\mu}_1, \ldots, \hat{\mu}_a$ are representations with finite images given by (2.2.2) such that $\hat{\mu}_1 \oplus \cdots \oplus \hat{\mu}_a$ is realizable over $\mathbb{Q}$.

Let τ be a representation of $\mathrm{Gal}(\overline{K}/K)$ with real-valued character. To compute the root number $W(\kappa \otimes \tau)$ we generalize the following result by D. Rohrlich ([Ro2], p. 318, Thm. 1) :

Theorem 2.2.8. *Let K be a local non-Archimedean field of characteristic zero. Let τ be a representation of $\mathrm{Gal}(\overline{K}/K)$ with real-valued character. Then*

$$W(\lambda \otimes \tau) = \det \tau(-1) \cdot \varphi(u_{H_2/K})^{\dim \tau} \cdot (-1)^{\langle 1,\tau\rangle + \langle \eta,\tau\rangle + \langle \hat{\lambda},\tau\rangle},$$

were λ is a two-dimensional irreducible, symplectic representation of $\mathrm{Gal}(\overline{K}/K)$ of the form $\lambda = Ind_K^{H_2}\phi$, H_2 is the unramified quadratic extension of K, ϕ is a tame character of $H_2^\times$; η is the unramified quadratic character of $K^\times$; $\hat{\lambda} = Ind_K^{H_2}(\phi \otimes \theta)$, where θ is the unramified quadratic character of $H_2^\times$, and $\varphi(u_{H_2/K}) = \pm 1$.

We prove the following generalization of Theorem 2.2.8:

Proposition 2.2.9. *Let κ be the representation of $\mathcal{W}(\overline{K}/K)$ corresponding to $V_l(B)^*$, $m = \dim B$, and $p > 2m+1$. Let τ be a representation of $\mathrm{Gal}(\overline{K}/K)$ with real-valued character. In the notation of Corollary 2.2.7 we have*

$$W(\kappa \otimes \tau) = \det \mu(-1)^{\dim \tau} \cdot \det \tau(-1)^{l_1} \cdot \alpha^{\dim \tau} \cdot (-1)^{l_2},$$

where $l_1 = \dim \mu + \frac{1}{2}(\dim \mu_1 + \cdots + \dim \mu_a)$, $\alpha = \pm 1$, $l_2 = a \cdot \langle 1, \tau \rangle + a \cdot \langle \eta, \tau \rangle + \langle \hat{\mu}_1 \oplus \cdots \oplus \hat{\mu}_a, \tau \rangle$, and η is the unramified quadratic character of $K^\times$.

Proof. See Appendix A. □

2.3 General case

We keep the notation of Section 2.1. Let $\sigma' = (\sigma, N)$ be the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)^*$, κ the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(B)^*$, and

$$\chi : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}_r(\mathbb{Z})$$

the representation corresponding to the $\mathrm{Gal}(\overline{K}/K)$-module $Y(\overline{K})$. It is known that there is a finite Galois extension $L \subset \overline{K}$ of K such that $\mathrm{Gal}(\overline{K}/L)$ acts trivially on $Y(\overline{K})$, hence χ has finite image. Here κ is actually a representation of $\mathcal{W}(\overline{K}/K)$ (see Section 2.2) and we identify κ with the representation $(\kappa, 0)$ of $\mathcal{W}'(\overline{K}/K)$. Also, we identify χ with the representation $(\mathrm{Res}^{\mathrm{Gal}(\overline{K}/K)}_{\mathcal{W}(\overline{K}/K)} \chi, 0)$ of $\mathcal{W}'(\overline{K}/K)$.

The main result of this section is the following proposition:

Proposition 2.3.1.

$$\sigma' \cong \kappa \oplus (\chi \otimes \omega^{-1} \otimes \mathrm{sp}(2)).$$

To prove Proposition 2.3.1 we will need the following lemmas.

Lemma 2.3.2. *Let $\lambda' = (\lambda, R)$ be the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(T)^*$. Then $R = 0$ and*

$$\lambda \cong \chi \otimes \omega^{-1}.$$

Proof. ¿From the exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence (2.1.2) we get the following exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$0 \longrightarrow T(\overline{K}) \longrightarrow G(\overline{K}) \longrightarrow B(\overline{K}) \longrightarrow 0.$$

Since $T(\overline{K})$ is a divisible group, the last sequence induces an exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence of l-adic Tate modules:

$$0 \longrightarrow T_l(T) \longrightarrow T_l(G) \longrightarrow T_l(B) \longrightarrow 0.$$

By tensoring the above sequence with $\mathbb{Q}_l$ over $\mathbb{Z}_l$ and taking duals over $\mathbb{Q}_l$ afterwards, we get the exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$0 \longrightarrow V_l(B)^* \longrightarrow V_l(G)^* \longrightarrow V_l(T)^* \longrightarrow 0. \tag{2.3.1}$$

Let X be the character group of T. Then $T(\overline{K}) \cong \mathrm{Hom}_{\mathbb{Z}}(X(\overline{K}), \overline{K}^{\times})$ as $\mathrm{Gal}(\overline{K}/K)$-modules over $\mathbb{Z}$, hence we have the following sequence of isomorphisms of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$\begin{aligned} V_l(T) &= T_l(T) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong \mathrm{Hom}_{\mathbb{Z}}(X(\overline{K}), T_l(\overline{K}^{\times})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \\ &\cong (X(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}_l)^* \otimes_{\mathbb{Q}_l} V_l(\overline{K}^{\times}). \end{aligned} \tag{2.3.2}$$

It is known that there is an injective homomorphism $\phi : Y \longrightarrow X$ with finite cokernel ([F-C], p. 58), consequently

$$Y(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong X(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$$

as $\mathrm{Gal}(\overline{K}/K)$-modules over $\mathbb{Q}_l$. Thus, we get from (2.3.2)

$$V_l(T)^* \cong (Y(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} V_l(\overline{K}^{\times})^*. \tag{2.3.3}$$

Let $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ be a field embedding and let $F_{l,\imath}$ be the functor which associates to an l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ a representation of $\mathcal{W}'(\overline{K}/K)$. Clearly, the image of the representation of $\mathrm{Gal}(\overline{K}/K)$ on $Y(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ under $F_{l,\imath}$ is χ and the image of the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(\overline{K}^{\times})^*$ under $F_{l,\imath}$ is ω^{-1}. Since $F_{l,\imath}$ respects tensor products, by (2.3.3) the image λ' of the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(T)^*$ under $F_{l,\imath}$ is isomorphic to $\chi \otimes \omega^{-1}$. Thus, λ' is a representation of $\mathcal{W}(\overline{K}/K)$, i.e., $R = 0$. □

Lemma 2.3.3. *Let $\rho' = (\rho, P)$ be the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of* $\mathrm{Gal}(\overline{K}/K)$ *on $V_l(G)^*$. Then $P = 0$ and*

$$\rho \cong \kappa \oplus (\chi \otimes \omega^{-1}).$$

Proof. Sequence (2.3.1) induces an exact sequence of corresponding representations of $\mathcal{W}'(\overline{K}/K)$, i.e.,

$$0 \longrightarrow V_l(B)^* \otimes_\imath \mathbb{C} \xrightarrow{h} V_l(G)^* \otimes_\imath \mathbb{C} \xrightarrow{g} V_l(T)^* \otimes_\imath \mathbb{C} \longrightarrow 0 \tag{2.3.4}$$

is an exact sequence of $\mathcal{W}'(\overline{K}/K)$-modules, where $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ is a field embedding, $(\kappa, 0)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(B)^* \otimes_\imath \mathbb{C}$, $\rho' = (\rho, P)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(G)^* \otimes_\imath \mathbb{C}$, and by Lemma 2.3.2, $(\chi \otimes \omega^{-1}, 0)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(T)^* \otimes_\imath \mathbb{C}$. In particular, (2.3.4) is an exact sequence of $\mathcal{W}(\overline{K}/K)$-modules and it splits if ρ is semisimple, which implies that $\rho \cong \kappa \oplus (\chi \otimes \omega^{-1})$. Thus, it is enough to show that $P = 0$ and ρ is semisimple.

Let $L \subset \overline{K}$ be a finite Galois extension of K such that $T \times_K L$ splits and $B \times_K L$ has good reduction. Since ρ is semisimple if and only if its restriction to a subgroup of finite index is semisimple (Lemma 2.2.1) and $\mathrm{Res}_{\mathcal{W}'(\overline{K}/L)}\rho' = (\mathrm{Res}_K^L \rho, P)$ ([Ro1], p. 130), to prove that $P = 0$ and ρ is semisimple we can assume that T splits over K and B has good reduction over K. Then it follows from Lemma 2.3.2 that χ is trivial. Also, since the image of I under ρ is finite, by Lemma 2.2.1 to prove that ρ is semisimple it is enough to prove that $\rho(\Phi)$ is diagonalizable.

Taking into account that χ is trivial, from (2.3.4) we obtain that in a suitable basis $\rho(\Phi)$ has the following form:

$$\rho(\Phi) = \begin{pmatrix} \kappa(\Phi) & * \\ 0 & qE_r \end{pmatrix}, \tag{2.3.5}$$

where E_r is the $r \times r$-identity matrix. Let

$$\kappa_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(T_l(B))$$

be the l-adic representation corresponding to the Galois module $T_l(B)$. It is known that the absolute values of the eigenvalues of $\kappa_l(\Phi)$ are equal to $q^{1/2}$ ([S-T], Corollary on p. 499). Then the absolute values of the eigenvalues of $\kappa(\Phi)$ are equal to $q^{-1/2}$, since the eigenvalues of $\kappa(\Phi)$ are the inverses of the eigenvalues of $\kappa_l(\Phi)$. It follows that none of the eigenvalues of $\kappa(\Phi)$ is equal to q. Since $\kappa(\Phi)$ is diagonalizable by Lemma 2.2.2, formula (2.3.5) shows that $\rho(\Phi)$ is diagonalizable, hence ρ is semisimple and ρ' is admissible.

Let us show now that $P = 0$. Since (2.3.4) is an exact sequence of $\mathcal{W}'(\overline{K}/K)$-modules, we have

$$\begin{aligned} Ph(x) &= 0, \quad \forall x \in V_l(B)^* \otimes_\imath \mathbb{C}, \\ g(Py) &= 0, \quad \forall y \in V_l(G)^* \otimes_\imath \mathbb{C}, \end{aligned}$$

which implies that $P^2 = 0$. On the other hand, since ρ' is admissible, it has the following form:

$$\rho' \cong \bigoplus_{i=1}^{s} \alpha_i \otimes \mathrm{sp}(n_i),$$

where each α_i is a representation of $\mathcal{W}(\overline{K}/K)$, each n_i is a positive integer, and we can assume that $n_i \neq n_j$ if $i \neq j$ ([Ro1], p. 133, Cor. 2). Since $P^2 = 0$, it follows that each n_i is 1 or 2, i.e., without loss of generality we can assume that

$$\rho' \cong \alpha_1 \oplus (\alpha_2 \otimes \mathrm{sp}(2)). \tag{2.3.6}$$

We will show that $\alpha_2 = 0$. Assume that $\alpha_2 \neq 0$. ¿From (2.3.6) we have

$$\rho \cong \alpha_1 \oplus \alpha_2 \oplus (\alpha_2 \otimes \omega).$$

On the other hand, since ρ is semisimple, the exact sequence (2.3.4) of $\mathcal{W}(\overline{K}/K)$-modules splits, i.e.,

$$\rho \cong \kappa \oplus (\omega^{-1})^{\oplus r}.$$

Thus, combining the last two isomorphisms, we get

$$\alpha_1 \oplus \alpha_2 \oplus (\alpha_2 \otimes \omega) \cong \kappa \oplus (\omega^{-1})^{\oplus r}. \tag{2.3.7}$$

By assumption, B has good reduction, hence by Néron-Ogg-Šhafarevič criterion ([S-T], p. 493, Thm. 1) the inertia group I acts trivially on $V_l(B)^*$. Since by Lemma 2.2.2, κ is semisimple it implies that $\kappa \cong \bigoplus_{i=1}^{2m} \kappa_i$, where $m = \dim B$ and $\kappa_1, \ldots, \kappa_{2m}$ are one-dimensional representations of $\mathcal{W}(\overline{K}/K)$. Thus, it follows from (2.3.7) that α_2 is a sum of one-dimensional representations. Let α_0 be one of them. Using the uniqueness of decomposition of a semisimple module into simple modules we have from (2.3.7):

$$\alpha_0 \cong \omega^{-1} \quad \text{or} \quad \alpha_0 \cong \kappa_i$$

for some κ_i, hence

$$\alpha_0 \otimes \omega \cong 1 \quad \text{or} \quad \alpha_0 \otimes \omega \cong \kappa_i \otimes \omega.$$

In particular, the absolute value of $\alpha_0 \otimes \omega(\Phi)$ is 1 or $q^{-3/2}$, because the absolute value of $\kappa_i(\Phi)$ is $q^{-1/2}$ for each i (see above). This implies that $\alpha_0 \otimes \omega$ is neither ω^{-1} nor κ_j for any j which contradicts (2.3.7). Thus, $\alpha_2 = 0$ and ρ' is a representation of $\mathcal{W}(\overline{K}/K)$. □

Lemma 2.3.4. *σ' is admissible.*

Proof. One has the following exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence ([Ra], p. 312):

$$0 \longrightarrow G(\overline{K})_{l^n} \longrightarrow A(\overline{K})_{l^n} \longrightarrow Y(\overline{K})/l^n Y(\overline{K}) \longrightarrow 0,$$

where $G(\overline{K})_{l^n}$ denotes $\mathrm{Hom}(\mathbb{Z}/l^n\mathbb{Z}, G(\overline{K}))$ and $A(\overline{K})_{l^n}$ denotes $\mathrm{Hom}(\mathbb{Z}/l^n\mathbb{Z}, A(\overline{K}))$. Clearly, $G(\overline{K})$ is divisible. (It follows from the definition of a semi-abelian scheme together with the fact that the groups of points over $\overline{K}$ of a torus or of an abelian variety are divisible.) Since $Y(\overline{K})$ is a free group of rank r and $G(\overline{K})$ is divisible, we have the following exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$0 \longrightarrow T_l(G) \longrightarrow T_l(A) \longrightarrow \chi \otimes \mathbb{Z}_l^r \longrightarrow 0.$$

By tensoring the above sequence with $\mathbb{Q}_l$ over $\mathbb{Z}_l$ and taking duals over $\mathbb{Q}_l$ afterwards, we get:

$$0 \longrightarrow \chi \otimes \mathbb{Q}_l^r \longrightarrow V_l(A)^* \longrightarrow V_l(G)^* \longrightarrow 0, \tag{2.3.8}$$

because $\chi \cong \chi^*$ as a representation with finite image, realizable over $\mathbb{Z}$.

As in the proof of Lemma 2.3.3, by Lemma 2.2.1 we can assume that B has good reduction over K, T splits over K, and hence χ is trivial. Also, by Lemma 2.2.1 to prove that σ' is admissible it is enough to prove that $\sigma(\Phi)$ is diagonalizable.

Sequence (2.3.8) induces an exact sequence of corresponding representations of $\mathcal{W}'(\overline{K}/K)$, i.e.,

$$0 \longrightarrow (\chi \otimes \mathbb{Q}_l^r) \otimes_\imath \mathbb{C} \longrightarrow V_l(A)^* \otimes_\imath \mathbb{C} \longrightarrow V_l(G)^* \otimes_\imath \mathbb{C} \longrightarrow 0 \tag{2.3.9}$$

is an exact sequence of $\mathcal{W}'(\overline{K}/K)$-modules. Moreover, χ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $(\chi \otimes \mathbb{Q}_l^r) \otimes_\imath \mathbb{C}$, $\sigma' = (\sigma, N)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(A)^* \otimes_\imath \mathbb{C}$, and by Lemma 2.3.3, $\kappa \oplus (\chi \otimes \omega^{-1})$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(G)^* \otimes_\imath \mathbb{C}$. Taking into account that χ is trivial and (2.3.9) is an exact sequence of $\mathcal{W}(\overline{K}/K)$-modules, we obtain that in a suitable basis $\sigma(\Phi)$ has the following form:

$$\sigma(\Phi) = \begin{pmatrix} E_r & * & * \\ 0 & qE_r & * \\ 0 & 0 & \kappa(\Phi) \end{pmatrix}. \tag{2.3.10}$$

Here $\kappa(\Phi)$ is diagonalizable by Lemma 2.2.2. Since the absolute values of the eigenvalues of $\kappa(\Phi)$ are equal to $q^{-1/2}$ (see above), none of the eigenvalues of $\kappa(\Phi)$ is equal to 1 or q. Thus, (2.3.10) shows that $\sigma(\Phi)$ is diagonalizable, hence σ is semisimple, and σ' is admissible. □

Proof of Proposition 2.3.1. Since σ' is admissible by Lemma 2.3.4 and the representations of the Weil-Deligne group $\mathcal{W}'(\overline{K}/K)$ on $(\chi \otimes \mathbb{Q}_l^r) \otimes_\imath \mathbb{C}$ and $V_l(G)^* \otimes_\imath \mathbb{C}$ are actually representations of the Weil group $\mathcal{W}(\overline{K}/K)$, the same argument as in the proof of (2.3.6) in Lemma 2.3.3 applied to (2.3.9) gives that σ' has the following form:

$$\sigma' \cong \gamma \oplus (\delta \otimes \mathrm{sp}(2)), \tag{2.3.11}$$

where γ and δ are representations of $\mathcal{W}(\overline{K}/K)$. Hence

$$\sigma \cong \gamma \oplus \delta \oplus (\delta \otimes \omega).$$

On the other hand, since σ is semisimple by Lemma 2.3.4, the exact sequence (2.3.9) of $\mathcal{W}(\overline{K}/K)$-modules splits, i.e.,

$$\sigma \cong \chi \oplus \kappa \oplus (\chi \otimes \omega^{-1}).$$

Thus, combining the last two isomorphisms, we get

$$\gamma \oplus \delta \oplus (\delta \otimes \omega) \cong \chi \oplus \kappa \oplus (\chi \otimes \omega^{-1}). \tag{2.3.12}$$

Note that χ is isomorphic to a subrepresentation of $\gamma \oplus (\delta \otimes \omega)$, because by (2.3.9)

$$\chi \hookrightarrow \ker N \cong \gamma \oplus (\delta \otimes \omega).$$

Thus, δ is isomorphic to a subrepresentation of $\kappa \oplus (\chi \otimes \omega^{-1})$ by the uniqueness of decomposition of a semisimple module into simple modules. We claim that δ is isomorphic to a subrepresentation of $\chi \otimes \omega^{-1}$. Indeed, suppose there is an irreducible subrepresentation δ_0 of δ which is isomorphic to a subrepresentation of κ. Since the absolute values of the eigenvalues of $\kappa(\Phi)$ are equal to $q^{-1/2}$ (see above), the eigenvalues of $\delta_0(\Phi)$ are of absolute value $q^{-1/2}$. Hence the eigenvalues of $\delta_0 \otimes \omega(\Phi)$ are of absolute value $q^{-3/2}$. On the other hand, it follows from (2.3.12) that $\delta_0 \otimes \omega$ is isomorphic to a subrepresentation of χ, κ, or $\chi \otimes \omega^{-1}$, which is a contradiction because the eigenvalues of $\chi(\Phi)$, $\kappa(\Phi)$, and $\chi \otimes \omega^{-1}(\Phi)$ are of absolute values 1, $q^{-1/2}$, and q respectively. Thus, δ is isomorphic to a subrepresentation of $\chi \otimes \omega^{-1}$. Since $\dim \delta = r$ by Lemma B.0.8 (see Appendix B), we have $\delta \cong \chi \otimes \omega^{-1}$, hence $\gamma \cong \kappa$ by (2.3.12) and $\sigma' \cong \kappa \oplus (\chi \otimes \omega^{-1} \otimes \mathrm{sp}(2))$ by (2.3.11). □

Corollary 2.3.5. *The representation σ' does not depend on the choice of l and $\imath$.*

Proof. The statement is a consequence of Corollary 2.2.4 and Proposition 2.3.1. □

Corollary 2.3.6. *Let τ be a representation of $\mathrm{Gal}(\overline{K}/K)$ with real-valued character. Then*

$$W(\sigma' \otimes \tau) = W(\kappa \otimes \tau) \cdot \det \tau(-1)^r \cdot \det \chi(-1)^{\dim \tau} \cdot (-1)^{\langle \chi, \tau \rangle}. \qquad (2.3.13)$$

Moreover, when $p > 2g + 1$ we have

$$W(\sigma' \otimes \tau) = \det \mu(-1)^{\dim \tau} \cdot \det \chi(-1)^{\dim \tau} \cdot \det \tau(-1)^{r+l_1} \cdot \alpha^{\dim \tau} \cdot (-1)^{\langle \chi, \tau \rangle + l_2}, \quad (2.3.14)$$

where μ, l_1, α, and l_2 are as in Proposition 2.2.9.

Proof. Since the root number of a direct sum of representations of $\mathcal{W}'(\overline{K}/K)$ equals the product of the root numbers of the summands, we get from Proposition 2.3.1

$$W(\sigma' \otimes \tau) = W(\kappa \otimes \tau) \cdot W(\chi \otimes \omega^{-1} \otimes \mathrm{sp}(2) \otimes \tau),$$

where by Proposition 6 ([Ro2], p. 327)

$$W(\chi \otimes \omega^{-1} \otimes \mathrm{sp}(2) \otimes \tau) = \det \tau(-1)^r \cdot \det \chi(-1)^{\dim \tau} \cdot (-1)^{\langle \chi, \tau \rangle},$$

which proves (2.3.13).

Formula (2.3.14) is a consequence of (2.3.13) together with Proposition 2.2.9. □

Chapter 3

Root numbers of abelian varieties over number fields (Theorem A)

3.1 Proof of Theorem A

We keep the notation of the introduction.

Lemma 3.1.1. *Let A be an abelian variety of dimension g over a number field F and τ a representation of* $\mathrm{Gal}(\overline{F}/F)$ *with real-valued character. Then at every infinite place v of F we have*

$$W(A_v, \tau_v) = (-1)^{g \dim \tau}.$$

Proof. To define $W(A_v, \tau_v)$ let σ'_v denote the representation of the Weil-Deligne group $\mathcal{W}'(\overline{F}_v/F_v)$ associated to the components of $H^1(A_v(\mathbb{C}), \mathbb{C})$ in the Hodge decomposition, then $W(A_v, \tau_v) = W(\sigma'_v \otimes \tau_v)$, where τ_v is viewed as a representation of

$\mathcal{W}'(\overline{F}_v/F_v)$. If v is an infinite place such that $F_v \cong \mathbb{C}$, then the representation $\sigma'_v = \sigma_v$ of $\mathcal{W}'(\mathbb{C}/\mathbb{C}) = \mathcal{W}(\mathbb{C}/\mathbb{C}) = \mathbb{C}^\times$ has the following form:

$$\sigma_v = (\varphi_{1,0} \otimes H^{1,0}) \oplus (\varphi_{0,1} \otimes H^{0,1}),$$

where $\varphi_{p,q} : \mathcal{W}(\mathbb{C}/\mathbb{C}) \longrightarrow \mathbb{C}^\times$ $(p, q \in \mathbb{Z})$ are given by

$$\varphi_{p,q}(z) = z^{-p}\overline{z}^{-q},$$

$H^{1,0}$ and $H^{0,1}$ are the components of $H^1(A_v(\mathbb{C}), \mathbb{C})$ in the Hodge decomposition:

$$H^1(A_v(\mathbb{C}), \mathbb{C}) = H^{1,0} \oplus H^{0,1}.$$

Here $H^{1,0}$ and $H^{0,1}$ are endowed with the trivial action of $\mathcal{W}(\mathbb{C}/\mathbb{C})$, hence

$$\sigma_v = (\varphi_{1,0} \oplus \varphi_{0,1})^{\oplus g}. \tag{3.1.1}$$

Let v be an infinite place such that $F_v \cong \mathbb{R}$. We have

$$\mathcal{W}'(\mathbb{C}/\mathbb{R}) = \mathcal{W}(\mathbb{C}/\mathbb{R}) = \mathbb{C}^\times \cup J\mathbb{C}^\times,$$

where $J^2 = -1$ and $JzJ^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. Here $\mathcal{W}(\mathbb{C}/\mathbb{C})$ is identified with the subgroup $\mathbb{C}^\times$ of $\mathcal{W}(\mathbb{C}/\mathbb{R})$. In this case the representation $\sigma'_v = \sigma_v$ of $\mathcal{W}(\mathbb{C}/\mathbb{R})$ has the following form:

$$\sigma_v = \mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}} \varphi_{0,1} \otimes H^{0,1},$$

where $\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}} \varphi_{0,1}$ denotes the representation of $\mathcal{W}(\mathbb{C}/\mathbb{R})$ induced from $\varphi_{0,1}$. As in the complex case, $H^{0,1}$ is endowed with the trivial action of $\mathcal{W}(\mathbb{C}/\mathbb{R})$, hence

$$\sigma_v = (\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}} \varphi_{0,1})^{\oplus g} \tag{3.1.2}$$

([Ro1], p. 155, §20).

It follows from the proof of Theorem 2(i) ([Ro2], p. 329) that

$$\begin{aligned} W((\varphi_{1,0} \oplus \varphi_{0,1}) \otimes \tau_v) &= (-1)^{\dim \tau} \quad \text{if} \quad F_v \cong \mathbb{C} \quad \text{and} \\ W((\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}} \varphi_{0,1}) \otimes \tau_v) &= (-1)^{\dim \tau} \quad \text{if} \quad F_v \cong \mathbb{R}. \end{aligned}$$

Now the statement follows from these formulas together with formulas (3.1.1) and (3.1.2). □

Lemma 3.1.2 ([Ro2], Lemma on p. 347). *Let G be a finite group, $D \subseteq G$ an abelian subgroup, and τ an irreducible representation of G with real-valued character. If $m_{\mathbb{Q}}(\tau) = 2$ then $Res_D^G \tau$ is symplectic.*

Lemma 3.1.3. *Let G be a finite group and τ an irreducible representation of G with real-valued character. If $m_{\mathbb{Q}}(\tau) = 2$ then $\dim \tau$ is even and $\det \tau$ is trivial.*

Proof. By Lemma on p. 339 in [Ro2] if τ has odd dimension or nontrivial determinant, then there is a cyclic subgroup D of G such that $\mathrm{Res}_D^G \tau$ is not symplectic, which contradicts Lemma 3.1.2. □

Proof of Theorem A (Theorem 1.0.2). By Lemmas 3.1.1 and 3.1.3, $W(A_v, \tau_v) = 1$ at every infinite place v of F.

Let v be a finite place of F lying over a prime number p. Let σ_v' be the representation of $\mathcal{W}'(\overline{F}_v/F_v)$ associated to the first cohomology of A_v. Since by Lemma 3.1.3 $\det \tau$ is trivial and $\dim \tau$ is even, (2.3.13) implies

$$W(A_v, \tau_v) = W(\kappa_v \otimes \tau_v) \cdot (-1)^{\langle \chi_v, \tau_v \rangle}, \tag{3.1.3}$$

where χ_v is a representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ realizable over $\mathbb{Z}$ (see Section 2.3 for the definition of χ_v). Moreover, when $p > 2g+1$ from (2.3.14) we have

$$W(A_v, \tau_v) = (-1)^{a\langle 1,\tau_v\rangle + a\langle \eta_v,\tau_v\rangle + \langle \lambda,\tau_v\rangle + \langle \chi_v,\tau_v\rangle}, \tag{3.1.4}$$

where η_v is the unramified quadratic character of $F_v^\times$, and $\lambda = \hat{\mu}_1 \oplus \cdots \oplus \hat{\mu}_a$ is a representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ realizable over $\mathbb{Q}$.

The rest of the proof is analogous to the argument given by D. Rohrlich in [Ro2]. Let $K \subset \overline{F}$ be a finite Galois extension of F such that τ factors through the group $G = \mathrm{Gal}(K/F)$ and χ_v factors through the decomposition subgroup H of G at v. Then

$$\langle \chi_v, \tau_v\rangle = \langle \mathrm{Ind}_H^G \chi_v, \tau\rangle$$

by Frobenius reciprocity. Since χ_v is realizable over $\mathbb{Q}$, $\mathrm{Ind}_H^G\chi_v$ is realizable over $\mathbb{Q}$, hence $\langle \mathrm{Ind}_H^G \chi_v, \tau\rangle$ is divisible by $m_{\mathbb{Q}}(\tau)$. By assumption $m_{\mathbb{Q}}(\tau) = 2$, hence $\langle \chi_v, \tau_v\rangle$ is even. Analogously, $\langle 1, \tau_v\rangle$, $\langle \eta_v, \tau_v\rangle$, and $\langle \lambda, \tau_v\rangle$ are even, hence $W(A_v, \tau_v) = W(\kappa_v \otimes \tau_v)$ by (3.1.3) and when $p > 2g+1$ we have $W(A_v, \tau_v) = 1$ by (3.1.4). When $p \leq 2g+1$ the decomposition subgroup of $\mathrm{Gal}(L/F)$ at v is abelian by assumption, hence τ_v is symplectic by Lemma 3.1.2. Also, $\kappa_v \otimes {\omega_v}^{1/2}$ is symplectic, because κ_v comes from an abelian variety (see Section 2.1). Since κ_v is a representation of $\mathcal{W}(\overline{F}_v/F_v)$ (see Section 2.2) and real powers of ω_v do not change the root number,

$$W(\kappa_v \otimes \tau_v) = W(\kappa_v \otimes {\omega_v}^{1/2} \otimes \tau_v) = 1$$

by Proposition 2 and the remark after it on p. 319 in [Ro2]. □

3.2 Special cases of Theorem A

We keep the notation of the introduction. In this section we discuss two special cases of Theorem A when the local calculations of the root number under consideration become especially easy. The first case is when the conductor of A is prime to the conductor of τ (Proposition 3.2.1) and the second case is when τ is symplectic (Proposition 3.2.3).

Proposition 3.2.1. *Let A be an abelian variety over a number field F of dimension g and conductor N. Let τ be a continuous complex finite-dimensional representation of $\mathrm{Gal}(\overline{F}/F)$ with real-valued character, of even dimension and conductor f. For each place v of F let τ_v denote the restriction of τ to the decomposition subgroup of $\mathrm{Gal}(\overline{F}/F)$ at v and let $m_v(A)$ be the exponent of N at v. Assume that f is prime to N. Then for the local root number $W(A_v, \tau_v)$ associated to $A_v = A \times_F F_v$ and τ_v one has the following formula:*

$$W(A_v, \tau_v) = \begin{cases} 1 & \text{if} \quad v \nmid fN \text{ or } v = \infty \\ \det \tau_v(\varpi_v)^{m_v(A)} & \text{if} \quad v | N \\ \det \tau_v(-1)^g & \text{if} \quad v | f \end{cases}$$

where ϖ_v is a uniformizer of F_v. (The statement of this proposition was suggested by B. Gross.)

Proof. If $v = \infty$ then $W(A_v, \tau_v) = 1$ by Lemma 3.1.1. Suppose $v < \infty$. If v does not divide N then A_v has good reduction over F_v, hence by the criterion of Néron-Ogg-

Šhafarevič σ'_v is actually a representation of $\mathcal{W}(\overline{F}_v/F_v)$ trivial on I_v. Since $\sigma'_v \otimes \omega_v^{1/2}$ is symplectic (see Section 2.1), this implies that

$$\sigma'_v \otimes \omega_v^{1/2} \cong \alpha \oplus \alpha^*$$

for some representation α of $\mathcal{W}(\overline{F}_v/F_v)$. Thus, taking into account that real powers of ω_v do not change the root number, τ_v has finite image and real-valued character, we have

$$W(A_v, \tau_v) = W(\sigma'_v \otimes \omega_v^{1/2} \otimes \tau_v) = W(\alpha \otimes \tau_v) W((\alpha \otimes \tau_v)^*) = \tag{3.2.1}$$

$$= \det(\alpha \otimes \tau_v)(-1) = \det \alpha(-1)^{\dim \tau} \cdot \det \tau_v(-1)^{\dim \alpha}.$$

Since $\dim \tau$ is even and $\dim \alpha = g$, (3.2.1) gives

$$W(A_v, \tau_v) = \det \tau_v(-1)^g.$$

Let v do not divide f. Then τ_v is unramified. Let V be a representation space of τ_v, W a representation space of σ'_v, and $\sigma'_v = (\sigma_v, M)$, where σ_v is a representation of $\mathcal{W}(\overline{F}_v/F_v)$ and M is a nilpotent endomorphism on W. Denote $U = W \otimes V$ and $U_{M\otimes 1}^{I_v} = (\ker(M \otimes 1))^{I_v}$. We have

$$W(\sigma'_v \otimes \tau_v) = W(\sigma_v \otimes \tau_v) \cdot \frac{\delta(\sigma'_v \otimes \tau_v)}{|\delta(\sigma'_v \otimes \tau_v)|}, \tag{3.2.2}$$

where $\delta(\sigma'_v \otimes \tau_v) = \det(-\Phi_v|_{U^{I_v}/U^{I_v}_{M\otimes 1}})$ (see [Ro1], §11). Since τ_v is an unramified representation of $\mathcal{W}(\overline{F}_v/F_v)$, we have $U^{I_v} \cong W^{I_v} \otimes V$ and $U^{I_v}_{M\otimes 1} \cong W^{I_v}_M \otimes V$, where $W^{I_v}_M = (\ker M)^{I_v}$. Hence

$$\delta(\sigma'_v \otimes \tau_v) = \det(-\Phi_v|_{W^{I_v}/W^{I_v}_M})^{\dim \tau} \cdot \det(\Phi_v|_V)^{\dim W^{I_v} - \dim W^{I_v}_M} = \tag{3.2.3}$$

$$= \delta(\sigma'_v)^{\dim \tau} \cdot \det \tau_v(\varpi_v)^{\dim W^{I_v} - \dim W^{I_v}_M}.$$

Also, since τ_v is unramified and of finite image, for a nontrivial additive character ψ_v of F_v by (3.4.6) ([T2], p. 15) we have

$$W(\sigma_v \otimes \tau_v) = W(\sigma_v)^{\dim \tau} \cdot \det \tau_v(\varpi_v)^{a(\sigma_v)+2gn(\psi_v)}. \tag{3.2.4}$$

Putting (3.2.2), (3.2.3), and (3.2.4) together and taking into account that the determinant of τ_v is ± 1 (because τ_v is of finite image and real-valued character) and

$$a(\sigma'_v) = a(\sigma_v) + \dim W^{I_v} - \dim W_M^{I_v},$$

we get

$$W(\sigma'_v \otimes \tau_v) = W(\sigma'_v)^{\dim \tau} \cdot \det \tau_v(\varpi_v)^{a(\sigma'_v)}.$$

Since $W(\sigma'_v) = W(\sigma'_v \otimes \omega_v^{1/2}) = \pm 1$ (as the root number of a symplectic representation), $\dim \tau$ is even, and $a(\sigma'_v) = m_v(A)$, this implies

$$W(A_v, \tau_v) = W(\sigma'_v \otimes \tau_v) = \det \tau_v(\varpi_v)^{m_v(A)}$$

and the proposition follows. □

Remark 3.2.2. It might happen that the conductor of A is not coprime to the conductor of τ. Indeed, there exist elliptic curves and irreducible representations τ of $\mathrm{Gal}(\overline{F}/F)$ with real-valued character, of even dimension and trivial determinant such that $W(E,\tau) = -1$ (see e.g, [Ro2], p. 312, Prop. B). It follows from Proposition 3.2.1 that the conductors of such E and τ are not coprime.

Proposition 3.2.3. *Let K be a local non-Archimedean field and let $\overline{K}$ be a fixed separable algebraic closure of K. If σ' and τ' are admissible symplectic representations* *of* $\mathcal{W}'(\overline{K}/K)$ *then*

$$W(\sigma' \otimes \tau') = 1.$$

Proof. Let I be the inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$, Φ an inverse Frobenius element of $\mathrm{Gal}(\overline{K}/K)$, and let ω be the unramified character of $K^\times$ equal to the cardinality of the residue class field of K on a uniformizer. It follows from Theorem 1.0.3 (Theorem B) that

$$\sigma' \cong \rho' \oplus (\rho')^* \oplus (\pi_1 \otimes \mathrm{sp}(n_1)) \oplus \cdots \oplus (\pi_k \otimes \mathrm{sp}(n_k)),$$

where ρ' is a representation of $\mathcal{W}'(\overline{K}/K)$, each π_i is an irreducible representation of $\mathcal{W}(\overline{K}/K)$ and each n_i is a positive integer such that $\pi_i \otimes \mathrm{sp}(n_i)$ is symplectic. Then

$$W((\rho' \oplus (\rho')^*) \otimes \tau') = \det(\rho' \otimes \tau')(-1) = 1,$$

because τ' is symplectic. Clearly, this argument is symmetric in σ' and τ', hence it is enough to prove Proposition 3.2.3 when σ' and τ' have the following forms:

$$\begin{aligned} \sigma' &= \alpha \otimes \mathrm{sp}(n), \\ \tau' &= \beta \otimes \mathrm{sp}(m), \end{aligned}$$

where $n \geq m$ are positive integers and α, β are irreducible representations of $\mathcal{W}(\overline{K}/K)$ such that $\alpha \otimes \mathrm{sp}(n)$ and $\beta \otimes \mathrm{sp}(m)$ are symplectic. Note that $\alpha \otimes \omega^{\frac{n-1}{2}}$ is either orthogonal or symplectic. In fact, since σ' is symplectic, we have

$$\alpha \otimes \mathrm{sp}(n) \cong (\alpha \otimes \mathrm{sp}(n))^* \cong \alpha^* \otimes \omega^{-(n-1)} \otimes \mathrm{sp}(n).$$

By the uniqueness of decomposition of an admissible reperesentation of $\mathcal{W}'(\overline{K}/K)$ into indecomposables ([Ro1], p. 133, Cor. 2) this implies $\alpha \cong \alpha^* \otimes \omega^{-(n-1)}$ or equivalently

$\alpha\otimes\omega^{\frac{n-1}{2}} \cong (\alpha\otimes\omega^{\frac{n-1}{2}})^{*}$. Since $\alpha\otimes\omega^{\frac{n-1}{2}}$ is an irreducible representation of $\mathcal{W}(\overline{K}/K)$, $\alpha\otimes\omega^{\frac{n-1}{2}} \cong \rho\otimes\omega^{s}$ for some irreducible representation ρ of $\mathcal{W}(\overline{K}/K)$ with finite image and $s \in \mathbb{C}$ ([Ro1], Prop. on p. 127). Thus, $\rho\otimes\omega^{s} \cong \rho^{*}\otimes\omega^{-s}$ and hence ω^{s} has finite image (it can been seen e.g., by taking the determinant). Consequently, $\alpha\otimes\omega^{\frac{n-1}{2}}$ has finite image and since it is self-dual, it is either orthogonal or symplectic. Also, if n is a positive integer then

$$\omega^{-(\frac{n-1}{2})}\otimes \mathrm{sp}(n) = \begin{cases} \text{orthogonal}, & \text{if } n \text{ is odd,} \\ \\ \text{symplectic}, & \text{if } n \text{ is even} \end{cases}$$

([Ro1], p. 136).

Since the real powers of ω do not change the root number, without loss of generality we can assume that α (as well as β) is either orthogonal or symplectic. Thus, we have the following four cases:

1) n and m are even, α and β are orthogonal;

2) n and m are odd, α and β are symplectic;

3) n is odd, m is even, α is symplectic, and β is orthogonal;

4) n is even, m is odd, α is orthogonal, and β is symplectic.

Lemma 3.2.4. *For positive integers m and n such that $m \leq n$ we have*

$$\mathrm{sp}(m)\otimes\mathrm{sp}(n) \cong \bigoplus_{i=0}^{m-1}(\omega^{i}\otimes\mathrm{sp}(n+m-2i-1)). \tag{3.2.5}$$

Proof. Clearly, (3.2.5) is equivalent to

$$(\omega^{-(\frac{m-1}{2})} \otimes \mathrm{sp}(m)) \otimes (\omega^{-(\frac{n-1}{2})} \otimes \mathrm{sp}(n)) \cong \bigoplus_{i=0}^{m-1} (\omega^{-(\frac{n+m-2i-2}{2})} \otimes \mathrm{sp}(n+m-2i-1)). \tag{3.2.6}$$

In the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ we choose the following basis over $\mathbb{C}$:

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For a positive integer k let $\mathbb{C}^k$ be the representation space of $\omega^{-(\frac{k-1}{2})} \otimes \mathrm{sp}(k) = (\nu, M)$ with the standard basis $e_0, e_1, \ldots, e_{k-1}$. Define an action of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^k$ as follows:

$$\begin{aligned} X_- &= N, \\ X_0 e_j &= (k-1-2j)e_j, \qquad 0 \le j \le k-1, \\ X_+ e_j &= j(k-j)e_{j-1}, \qquad 1 \le j \le k-1, \\ X_+ e_0 &= 0. \end{aligned} \tag{3.2.7}$$

This yields the unique irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension k ([K], §18). We claim that any $\mathfrak{sl}(2, \mathbb{C})$-submodule of $(\omega^{-(\frac{m-1}{2})} \otimes \mathrm{sp}(m)) \otimes (\omega^{-(\frac{n-1}{2})} \otimes \mathrm{sp}(n))$ is also a $\mathcal{W}'(\overline{K}/K)$-submodule. Indeed, it follows from the fact that an element of this tensor product is an eigenvector for X_0 with eigenvalue ρ if and only if it is an eigenvector for $\mathcal{W}(\overline{K}/K)$ with weight $\omega^{-p/2}$; and N just acts as X_-. The lemma follows easily from the claim together with the decomposition of the tensor product of two irreducible representations of $\mathfrak{sl}$ into irreducibles ([K], §18). □

By Lemma A.0.6 (see Appendix A)

$$\alpha \otimes \beta \cong \pi \oplus \pi^* \oplus \mu_1 \oplus \cdots \oplus \mu_a, \tag{3.2.8}$$

where π is a representation of $\mathcal{W}(\overline{K}/K)$, $\mu_1, \ldots, \mu_a$ are irreducible orthogonal representations of $\mathcal{W}(\overline{K}/K)$ in cases 1) and 2) and $\mu_1, \ldots, \mu_a$ are irreducible symplectic representations of $\mathcal{W}(\overline{K}/K)$ in cases 3) and 4).

Let $\sigma' \otimes \tau' = (\lambda, N)$. Then

$$W(\sigma' \otimes \tau') = W(\lambda) \cdot \Delta(\sigma' \otimes \tau'),$$

where $W(\lambda) = 1$ (by Prop. 2 and the remark after it in [Ro2], p. 319) and

$$\Delta(\sigma' \otimes \tau') = \frac{\delta(\sigma' \otimes \tau')}{|\delta(\sigma' \otimes \tau')|}.$$

Thus, it is enough to show that $\Delta(\sigma' \otimes \tau') = 1$. Note that $\Delta((\pi \oplus \pi^*) \otimes \omega^r \otimes \mathrm{sp}(k)) = 1$ for any positive integer k, $r \in \mathbb{R}$, and any representation π of $\mathcal{W}(\overline{K}/K)$. It follows from the fact the real powers of ω do not change Δ (see (3.2.3)) together with Lemma (ii) ([Ro1], p. 144). Lemma 3.2.4 together with (3.2.8) imply

$$\Delta(\sigma' \otimes \tau') = \prod_{i=0}^{m-1} \prod_{j=1}^{a} \Delta(\mu_j \otimes \mathrm{sp}(n + m - 2i - 1)). \tag{3.2.9}$$

Let j be fixed and let V_j denote a representation space of μ_j. It follows from the definition that for each i we have

$$\Delta(\mu_j \otimes \mathrm{sp}(n + m - 2i - 1)) = (-1)^{(n+m)\cdot \dim V_j^I} \cdot \frac{\det(\Phi|_{V_j^I})^{n+m-2i-2}}{|\det(\Phi|_{V_j^I})|^{n+m-2i-2}}. \tag{3.2.10}$$

Since μ_j is self-dual, $\det \mu_j = \pm 1$. Moreover, V_j^I is either $\{0\}$ or V_j. Hence (3.2.10) gives

$$\Delta(\mu_j \otimes \mathrm{sp}(n + m - 2i - 1)) = (-1)^{(n+m)\cdot \dim V_j^I} \cdot \det(\Phi|_{V_j^I})^{n+m}. \tag{3.2.11}$$

In cases 1) and 2), $n+m$ is even, hence (3.2.9) and (3.2.11) imply $\Delta(\sigma' \otimes \tau') = 1$. In cases 3) and 4), $\det(\Phi|_{V_j^I}) = 1$ and $\dim V_j^I$ is even (because μ_j is symplectic), hence (3.2.9) and (3.2.11) imply $\Delta(\sigma' \otimes \tau') = 1$. □

Remark 3.2.5. If τ is symplectic then $m_{\mathbb{Q}}(\tau) = 2$ but not vice versa: there are examples of irreducible orthogonal complex representations of finite groups with the Schur index over the rationals equal to 2 (see Appendix C).

Chapter 4

Representations of the Weil-Deligne group (Theorem B)

4.1 Theorem B

We keep the notation of Section 2.1 except that K does not have to be of characteristic zero and $\overline{K}$ denotes a separable algebraic closure of K. If U is a complex finite-dimensional vector space and $\lambda : D \longrightarrow \mathrm{GL}(U)$ is a representation of a group D on U, then by $\check{\lambda} : D \longrightarrow \mathrm{GL}(\check{U})$ we denote the representation of D on $\check{U}$, where $\check{U}$ is a $\mathbb{C}[D]$-module with the underlying D-module U^* and multiplication by constants defined as follows:

$$a \cdot \phi = \overline{a}\phi, \quad a \in \mathbb{C},\ \phi \in U^*.$$

We say that U is *unitary* if U admits a nondegenerate invariant hermitian form (not necessarily positive definite).

Proposition 4.1.1. *Let σ' be an admissible representation of $\mathcal{W}'(\overline{K}/K)$. Then it can be written in the following form:*

$$\sigma' \cong \bigoplus_{i=1}^{k} \pi_i \otimes \mathrm{sp}(n_i),$$

where each π_i is a representation of $\mathcal{W}(\overline{K}/K)$, n_i is a positive integer, and $n_i \neq n_j$ whenever $i \neq j$ ([Ro1], p. 133, Cor. 2). If σ' is unitary, orthogonal, or symplectic with respect to a corresponding invariant nondegenerate form $\langle \cdot, \cdot \rangle$ then each $\pi_i \otimes \mathrm{sp}(n_i)$ is unitary, orthogonal, or symplectic respectively with respect to the restriction of $\langle \cdot, \cdot \rangle$.

Proof. Let U be a representation space of σ' and U_i a representation space of $\pi_i \otimes \mathrm{sp}(n_i)$, $1 \leq i \leq k$, so that $U \cong \bigoplus_{i=1}^{k} U_i$. Let $\langle \cdot\,, \cdot \rangle$ be a nondegenerate invariant form on U and let $\tilde{U}$ be a $\mathcal{W}'(\overline{K}/K)$-module over $\mathbb{C}$ such that $\tilde{U} = U^*$ if $\langle \cdot\,, \cdot \rangle$ is bilinear and $\tilde{U} = \check{U}$ if $\langle \cdot\,, \cdot \rangle$ is sesquilinear. Let $\phi : U \longrightarrow \tilde{U}$ be the isomorphism of $\mathcal{W}'(\overline{K}/K)$-modules induced by $\langle \cdot\,, \cdot \rangle$, and let $\psi : \tilde{U} \cong (\bigoplus_{i=1}^{k} U_i)^{\sim} \longrightarrow \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k$ denote the usual isomorphism. It is easy to show that for any n we have $\mathrm{sp}(n)^{\sim} \cong \omega^{-(n-1)} \otimes \mathrm{sp}(n)$, hence $\tilde{U}_i \cong V_i$, where V_i denotes a representation space of $\tilde{\pi}_i \otimes \omega^{-(n_i-1)} \otimes \mathrm{sp}(n_i)$. Denote by $\lambda : \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k \longrightarrow V_1 \oplus \cdots \oplus V_k$ the corresponding isomorphism. For each i let $\rho_i : U_i \longrightarrow V_i$ be defined by the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\lambda \circ \psi \circ \phi} & V_1 \oplus \cdots \oplus V_k \\ \uparrow & & \downarrow \pi_i \\ U_i & \xrightarrow{\rho_i} & V_i \end{array}$$

where π_i is the projection onto i-th factor. To prove that $\langle \cdot\,,\cdot\rangle|_{U_i}$ is nondegenerate for each i is equivalent to proving that ρ_i is an isomorphism for each i, which follows from Lemma 4.1.2 below. $\square$

Lemma 4.1.2. *Let $\alpha' = (\alpha, N)$ and $\beta' = (\beta, M)$ be two isomorphic admissible representations of $\mathcal{W}'(\overline{K}/K)$. Then they can be written in the following forms:*

$$\alpha' \cong \bigoplus_{i=1}^{k} \alpha_i \otimes \mathrm{sp}(n_i) \quad \textit{and} \quad \beta' \cong \bigoplus_{i=1}^{k} \beta_i \otimes \mathrm{sp}(n_i),$$

where each α_i and β_i is a representation of $\mathcal{W}(\overline{K}/K)$, $n_i \neq n_j$ whenever $i \neq j$. Let U (resp. V) be a representation space of α' (resp. β') and for each i let U_i (resp. V_i) be a representation space of $\alpha_i \otimes \mathrm{sp}(n_i)$ (resp. $\beta_i \otimes \mathrm{sp}(n_i)$). Let $\phi : U \longrightarrow V$ be an isomorphism of $\mathcal{W}'(\overline{K}/K)$-modules and $\psi_i : U_i \longrightarrow V_i$, $1 \leq i \leq k$, defined by the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\ \phi\ } & V \\ \uparrow & & \downarrow{\scriptstyle \pi_i} \\ U_i & \xrightarrow{\ \psi_i\ } & V_i \end{array}$$

where π_i is the projection onto i-th factor. Then each ψ_i is an isomorphism of $\mathcal{W}'(\overline{K}/K)$-modules.

Proof. We will prove the lemma by induction on k. Clearly, it holds when $k = 1$. Let k be arbitrary and let $e_0, \ldots, e_{n_k-1}$ be the standard basis of $\mathbb{C}^{n_k}$. Without loss of generality we can assume that $n_k > n_i$ for any i. Let U_k° be a representation space of α_k, so $U_k = \bigoplus_{j=0}^{n_k-1}(U_k^\circ \otimes e_j)$. Then

$$\begin{aligned} U &= \ker N^{n_k-1} \oplus (U_k^\circ \otimes e_0) \quad \text{and} \\ N^j U &= N^j(\ker N^{n_k-1}) \oplus (U_k^\circ \otimes e_j), \quad 0 \leq j \leq n_k - 1. \end{aligned} \tag{4.1.1}$$

Since ϕ is an isomorphism of $\mathcal{W}'(\overline{K}/K)$-modules, we have from (4.1.1):

$$M^j V = M^j(\ker M^{n_k-1}) \oplus \phi(U_k^\circ \otimes e_j), \quad 0 \le j \le n_k - 1. \tag{4.1.2}$$

On the other hand, (4.1.1) holds in V, i.e.,

$$M^j V = M^j(\ker M^{n_k-1}) \oplus (V_k^\circ \otimes e_j), \quad 0 \le j \le n_k - 1, \tag{4.1.3}$$

where V_k° denotes a representation space of β_k. We have the following filtration of V:

$$\begin{aligned} &V \supseteq \ker M^{n_k-1} \supseteq MV \supseteq M(\ker M^{n_k-1}) \supseteq \cdots \\ &\cdots \supseteq M^{n_k-1}V = \phi(U_k^\circ \otimes e_{n_k-1}) = V_k^\circ \otimes e_{n_k-1}. \end{aligned} \tag{4.1.4}$$

Since V is a semisimple $\mathcal{W}(\overline{K}/K)$-module, taking into account (4.1.2), we get from (4.1.4):

$$V = (\bigoplus_{j=0}^{n_k-2} A_j) \oplus \phi(U_k), \tag{4.1.5}$$

where each A_j is a complement of $M^{j+1}V$ in $M^j(\ker M^{n_k-1})$. Analogously, taking into account (4.1.3), we get from (4.1.4):

$$V = (\bigoplus_{j=0}^{n_k-2} A_j) \oplus V_k. \tag{4.1.6}$$

Combining (4.1.5) and (4.1.6), we see that $\pi_k \circ \phi(U_k) = V_k$, hence ψ_k is an isomorphism. To be able to apply the inductive step, note that A_j's can be chosen in such a way that

$$\bigoplus_{j=0}^{n_k-2} A_j = \bigoplus_{i=0}^{k-1} V_i.$$

Indeed, this follows from the following formulas:

$$\begin{aligned}
\ker M^{n_k-1} &= MV \oplus (\bigoplus_{i=0}^{n_k-1}(V_i^\circ \otimes e_0)) \quad \text{and} \\
M^j(\ker M^{n_k-1}) &= M^{j+1}V \oplus (\bigoplus_{i=0}^{n_k-1}(V_i^\circ \otimes e_j)), \quad 0 \le j \le n_k - 1,
\end{aligned}$$

where each V_i° is a representation space of β_i. Thus, by (4.1.5)

$$V = (\bigoplus_{i=0}^{k-1} V_i) \oplus \phi(U_k) = \phi(\bigoplus_{i=0}^{k-1} U_i) \oplus \phi(U_k),$$

which implies that the projection of $\phi(\bigoplus_{i=0}^{k-1} U_i)$ onto $\bigoplus_{i=0}^{k-1} V_i$ is an isomorphism, hence by induction $\psi_1, \ldots, \psi_{k-1}$ are isomorphisms. □

Proof of Theorem B. Since σ' is minimal, it follows from Proposition 4.1.1, that $\sigma' \cong \alpha \otimes \mathrm{sp}(n)$, where α is a representation of $\mathcal{W}(\overline{K}/K)$. Since σ' is admissible, α is semisimple, hence $\alpha = \bigoplus_{i=1}^{k} \alpha_i$, where each α_i is an irreducible subrepresentation of α. For each i let U_i be a representation space of $\alpha_i \otimes \mathrm{sp}(n)$, so that $U = U_1 \oplus \cdots \oplus U_k$. Let $\phi : U \longrightarrow \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k$ be the composition of the isomorphism induced by $\langle \cdot , \cdot \rangle$ with the usual isomorphism of $(U_1 \oplus \cdots \oplus U_k)^{\sim}$ onto $\tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k$. For each i and j let $\phi_{ij} : U_i \longrightarrow \tilde{U}_j$ be defined by the following diagram:

$$\begin{array}{ccc}
U & \xrightarrow{\phi} & \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k \\
\uparrow & & \downarrow \pi_j \\
U_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j
\end{array}$$

where π_j is the projection onto j-th factor. We claim that for any i there exists $j = j(i)$ such that ϕ_{ij} is an isomorphism. Indeed, let U_i° be a representation space of α_i so that $U_i = U_i^\circ \otimes \mathbb{C}^n$, where $\mathbb{C}^n$ is the representation space of $\mathrm{sp}(n)$. Let $W = \mathbb{C}$ be the representation space of $\omega^{-(n-1)}$ and $\psi : U \longrightarrow \bigoplus_{i=1}^{k}(\tilde{U}_i^\circ \otimes W \otimes \mathbb{C}^n)$ the composition of ϕ with the usual isomorphism induced by $\alpha_i \otimes \mathrm{sp}(n)^\sim \cong \tilde{\alpha}_i \otimes \omega^{-(n-1)} \otimes \mathrm{sp}(n)$, $1 \leq i \leq k$. For each i and j let $\psi_{ij} : U_i^\circ \otimes \mathbb{C}^n \longrightarrow \tilde{U}_j^\circ \otimes W \otimes \mathbb{C}^n$ be defined by the following diagram:

$$\begin{array}{ccc}
U & \xrightarrow{\psi} & (\tilde{U}_1^\circ \otimes W \otimes \mathbb{C}^n) \oplus \cdots \oplus (\tilde{U}_k^\circ \otimes W \otimes \mathbb{C}^n) \\
\uparrow & & \downarrow \pi_j \\
U_i^\circ \otimes \mathbb{C}^n & \xrightarrow{\psi_{ij}} & \tilde{U}_j^\circ \otimes W \otimes \mathbb{C}^n.
\end{array}$$

Let $e_0, \ldots, e_{n-1}$ be the standard basis of $\mathbb{C}^n$. If for each i there exists $j = j(i)$ such that the projection of $\psi(U_i^\circ \otimes e_0)$ onto $\tilde{U}_j^\circ \otimes W \otimes e_0$ is nonzero, then $U_i^\circ \cong \tilde{U}_j^\circ \otimes W$ (because each U_i° is irreducible), hence $U_i \cong \tilde{U}_j$ and $\phi_{ij} \neq 0$. Then it follows from Schur's lemma for indecomposable representations of $\mathcal{W}'(\overline{K}/K)$ ([Ro1], p. 133, Cor. 1) that ϕ_{ij} is an isomorphism.

Assume now that there exists i such that the projection of $\psi(U_i^\circ \otimes e_0)$ onto $\tilde{U}_j^\circ \otimes W \otimes e_0$ is zero for any j. Let N (resp. M) be the nilpotent endomorphism of U (resp. of $(\tilde{U}_1^\circ \otimes W \otimes \mathbb{C}^n) \oplus \cdots \oplus (\tilde{U}_k^\circ \otimes W \otimes \mathbb{C}^n)$). Then $\psi(U_i^\circ \otimes e_0) \subseteq X$, where $X = \bigoplus_{t \geq 1; s}(\tilde{U}_s^\circ \otimes W \otimes e_t)$ and $X \subseteq \ker M^{n-1}$. Since $U_i^\circ \otimes e_0 \not\subseteq \ker N^{n-1}$, we get a contradiction with ψ being an isomorphism.

Thus, in particular, there exists some j such that ϕ_{1j} is an isomorphism. If $j = 1$ then $\langle \cdot\,, \cdot \rangle|_{U_1}$ is nondegenerate, hence U_1 and its orthogonal complement are invariant subspaces of U. Since U is minimal, it implies that $U = U_1$ and U is indecomposable. If $j \neq 1$ then without loss of generality we can assume that $j = 2$, $\langle \cdot\,, \cdot \rangle|_{U_1}$ and $\langle \cdot\,, \cdot \rangle|_{U_2}$ are degenerate. Let us show that $\langle \cdot\,, \cdot \rangle|_{U_1 \oplus U_2}$ is nondegenerate. Suppose it is degenerate, i.e., $K = \ker(\langle \cdot\,,\cdot\rangle|_{U_1 \oplus U_2})$ is nonzero. Let R_1 (resp. R_2) be the nilpotent endomorphism of U_1 (resp. U_2). Then $R = R_1 \oplus R_2$ is the nilpotent endomorphism of $U_1 \oplus U_2$. We claim that $K \bigcap \ker R \neq 0$. Indeed, let $x \in K$ and $x \neq 0$. Then there exists i $(0 \leq i \leq n-1)$ such that $R^i x \in \ker R$ and $R^i x \neq 0$. Also,

$$\langle R^i x, y\rangle = (-1)^i \cdot \langle x, R^i y\rangle = 0 \quad \text{for any } y \in U_1 \oplus U_2,$$

hence $R^i x \in K$. Let $x \in K \bigcap \ker R$ and $x \neq 0$, i.e., $x = x_1 + x_2$, where $x_i \in \ker R_i$, $i = 1, 2$. Without loss of generality we can assume that $x_1 \neq 0$. Since ϕ_{12} is an isomorphism, there exists $y_2 \in U_2$ such that $\langle x_1, y_2\rangle \neq 0$. By assumption, $\langle \cdot\,,\cdot\rangle|_{U_2}$ is degenerate, hence $K_2 = \ker(\langle \cdot\,,\cdot\rangle|_{U_2})$ is nonzero. Then by the same argument as above $K_2 \bigcap \ker R_2 \neq 0$. Since $\ker R_2 = U_2^\circ \otimes e_{n-1}$, it is irreducible, consequently $\ker R_2 \subseteq K_2$. In particular, $\langle x_2, y_2\rangle = 0$. Hence

$$\langle x_1 + x_2, y_2\rangle = \langle x_1, y_2\rangle.$$

Since $\langle x_1, y_2\rangle \neq 0$ by the choice of y_2, we get a contradiction with $x_1 + x_2 \in K$. Thus, $\langle \cdot\,,\cdot\rangle|_{U_1 \oplus U_2}$ is nondegenerate. Since U is minimal the same argument as above implies that $U = U_1 \oplus U_2 \cong \tilde{U}_2 \oplus U_2$. □

4.2 An application of Theorem B

We keep the notation of Section 2.3.

In this section we apply Theorem B to prove a special case of Proposition 2.3.1 when the image of $Y \hookrightarrow G$ under f in (2.1.2) is finite (see Proposition 4.2.4 below).

We give an elementary proof of this case which, as far as the uniformization theory is concerned, uses only exact sequence (2.1.2), the fact that $Y \hookrightarrow G$ is a free discrete subgroup, and that there is a $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism $T_l(G(\overline{K})/Y(\overline{K})) \cong T_l(A)$.

Let $\rho' = (\rho, S)$ be the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(T(\overline{K})/\Lambda)^*$, where $\Lambda = T(\overline{K}) \bigcap Y(\overline{K})$ is a free discrete subgroup of $T(\overline{K})$ of rank s ($s \leq r$). Let $L \subset \overline{K}$ be a finite Galois extension of K such that $\mathrm{Gal}(\overline{K}/L)$ acts trivially on $Y(\overline{K})$. Thus, $Y(\overline{K})$ can be considered as a $\mathrm{Gal}(L/K)$-module. Let

$$\chi : \mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}_r(\mathbb{Z})$$

denote the corresponding representation. Thus, from (2.1.2) we have the following exact sequence of $\mathrm{Gal}(L/K)$-modules:

$$0 \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow Y(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow C \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow 0,$$

where $C = f(Y(\overline{K}))$. Let $\chi_1 : \mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}_s(\mathbb{Z})$ denote the representation of $\mathrm{Gal}(L/K)$ on Λ and $\chi_2 : \mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}_{r-s}(\mathbb{Z})$ denote the representation of $\mathrm{Gal}(L/K)$ on $C \otimes_{\mathbb{Z}} \mathbb{C}$. Then

$$\chi \cong \chi_1 \oplus \chi_2.$$

Proposition 4.2.1.

$$\rho' \cong (\chi_2 \otimes \omega^{-1}) \oplus (\chi_1 \otimes \omega^{-1} \otimes \mathrm{sp}(2)).$$

Proof. See Appendix D. □

Corollary 4.2.2. *If the image F of $Y \hookrightarrow G$ under $f : G \to B$ in* (2.1.2) *is an étale sheaf of finite abelian groups over* $\mathrm{Spec}(K)$*, then*

$$\rho' \cong \chi \otimes \omega^{-1} \otimes \mathrm{sp}(2).$$

Proposition 4.2.3. *If the image F of $Y \hookrightarrow G$ under $f : G \to B$ in* (2.1.2) *is an étale sheaf of finite abelian groups over* $\mathrm{Spec}(K)$*, then one has an exact sequence of* $\mathrm{Gal}(\overline{K}/K)$*-modules:*

$$0 \longrightarrow V_l(B)^* \longrightarrow V_l(A)^* \longrightarrow V_l(T(\overline{K})/\Lambda)^* \longrightarrow 0, \tag{4.2.1}$$

where $V_l(T(\overline{K})/\Lambda) = T_l(T(\overline{K})/\Lambda) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

Proof. ¿From the exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence (2.1.2) we get the following exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$0 \longrightarrow T(\overline{K})/\Lambda \longrightarrow G(\overline{K})/Y(\overline{K}) \longrightarrow B(\overline{K})/F(\overline{K}) \longrightarrow 0.$$

Since $T(\overline{K})$ is a divisible group, the last sequence induces an exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence of l-adic Tate modules:

$$0 \longrightarrow T_l(T(\overline{K})/\Lambda) \longrightarrow T_l(G/Y) \longrightarrow T_l(B/F) \longrightarrow 0,$$

where $T_l(G/Y)$ denotes $T_l(G(\overline{K})/Y(\overline{K}))$ and $T_l(B/F)$ denotes $T_l(B(\overline{K})/F(\overline{K}))$. We claim that there is a $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism $G(\overline{K})/Y(\overline{K}) \cong A(\overline{K})$. Indeed, for any finite Galois extension $L \subset \overline{K}$ of K such that the degeneration data for A splits over L we have a natural isomorphism

$$A(L) \cong G(L)/Y(L) \tag{4.2.2}$$

([Cha], p. 720, Prop. 3.1). Since $A(\overline{K})$ (resp. $G(\overline{K})$, resp. $Y(\overline{K})$) is naturally isomorphic to the direct limit of $A(L)$ (resp. $G(L)$, resp. $Y(L)$) when L runs over finite extensions of K contained in $\overline{K}$, the claim follows from (4.2.2) together with the fact that the direct limit is an exact functor. Hence $T_l(G/Y) \cong T_l(A)$ as $\mathrm{Gal}(\overline{K}/K)$-modules and we have

$$0 \longrightarrow T_l(T(\overline{K})/\Lambda) \longrightarrow T_l(A) \longrightarrow T_l(B/F) \longrightarrow 0. \tag{4.2.3}$$

By tensoring the above sequence with $\mathbb{Q}_l$ over $\mathbb{Z}_l$, we get an exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$0 \longrightarrow V_l(T(\overline{K})/\Lambda) \longrightarrow V_l(A) \longrightarrow V_l(B/F) \longrightarrow 0, \tag{4.2.4}$$

where $V_l(B/F)$ denotes $T_l(B/F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Thus, it suffices to show that $V_l(B/F) \cong V_l(B)$ as $\mathrm{Gal}(\overline{K}/K)$-modules. Applying $\mathrm{Hom}(\mathbb{Z}/l^n\mathbb{Z}, -)$ to the exact sequence $0 \longrightarrow F(\overline{K}) \longrightarrow B(\overline{K}) \longrightarrow B(\overline{K})/F(\overline{K}) \longrightarrow 0$ and taking into account that $B(\overline{K})$ is a divisible group, we get an exact sequence

$$0 \longrightarrow F_{l^n} \longrightarrow B_{l^n} \longrightarrow (B/F)_{l^n} \longrightarrow \mathrm{Ext}^1(\mathbb{Z}/l^n\mathbb{Z}, F(\overline{K})) \longrightarrow 0,$$

where $(B/F)_{l^n}$ denotes $(B(\overline{K})/F(\overline{K}))_{l^n}$. Here $\mathrm{Ext}^1(\mathbb{Z}/l^n\mathbb{Z}, F(\overline{K})) \cong F(\overline{K})/l^nF(\overline{K})$, as can be seen by applying the functor $\mathrm{Hom}(-, F(\overline{K}))$ to the standard projective resolution of $\mathbb{Z}/l^n\mathbb{Z}$. Thus, we get

$$0 \longrightarrow F_{l^n} \longrightarrow B_{l^n} \longrightarrow (B/F)_{l^n} \xrightarrow{\alpha_n} F(\overline{K})/l^nF(\overline{K}) \longrightarrow 0.$$

This sequence is equivalent to the following two short exact sequences

$$0 \longrightarrow F_{l^n} \longrightarrow B_{l^n} \longrightarrow \ker \alpha_n \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow \ker \alpha_n \longrightarrow (B/F)_{l^n} \longrightarrow F(\overline{K})/l^n F(\overline{K}) \longrightarrow 0.$$

Applying the left exact functor $\varprojlim(-)$ to these sequences and denoting $\varprojlim(\ker \alpha_n)$ by X and $\varprojlim(F(\overline{K})/l^n F(\overline{K}))$ by Z, we get

$$0 \longrightarrow T_l(F) \longrightarrow T_l(B) \longrightarrow X \quad \text{and}$$

$$0 \longrightarrow X \longrightarrow T_l(B/F) \longrightarrow Z.$$

Since $F(\overline{K})$ is finite, $T_l(F)$ and Z are torsion groups, hence $T_l(F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = Z \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = 0$. Thus, it follows that the sequences

$$0 \longrightarrow V_l(B) \longrightarrow X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \quad \text{and}$$

$$0 \longrightarrow X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \longrightarrow V_l(B/F) \longrightarrow 0$$

are exact, consequently, $V_l(B) \hookrightarrow V_l(B/F)$. On the other hand, from (4.2.4)

$$\dim_{\mathbb{Q}_l} V_l(B/F) = \dim_{\mathbb{Q}_l} V_l(A) - \dim_{\mathbb{Q}_l} V_l(T(\overline{K})/\Lambda),$$

where $\dim_{\mathbb{Q}_l} V_l(A) = 2 \cdot \dim A$ and $\dim_{\mathbb{Q}_l} V_l(T(\overline{K})/\Lambda) = 2 \cdot \dim T$. (The last assertion follows from (D.0.6), since $s = r$.) Hence

$$\dim_{\mathbb{Q}_l} V_l(B/F) = 2 \cdot (\dim A - \dim T) = 2 \cdot \dim B.$$

Since $\dim_{\mathbb{Q}_l} V_l(B) = 2 \cdot \dim B$, we have $\dim_{\mathbb{Q}_l} V_l(B/F) = \dim_{\mathbb{Q}_l} V_l(B)$. Thus, $V_l(B/F) \cong V_l(B)$. Clearly, they are isomorphic as $\mathrm{Gal}(\overline{K}/K)$-modules and the proposition follows. $\square$

Proposition 4.2.4. *If the image F of $Y \hookrightarrow G$ under $f : G \to B$ in* (2.1.2) *is an étale sheaf of finite abelian groups over* $\mathrm{Spec}(K)$*, then*

$$\sigma' \cong \kappa \oplus (\chi \otimes \omega^{-1} \otimes \mathrm{sp}(2)). \tag{4.2.5}$$

Proof. Except for the slight variations we proceed as in the proof of Lemma 2.3.3. Sequence (4.2.1) induces an exact sequence of corresponding representations of $\mathcal{W}'(\overline{K}/K)$, i.e.,

$$0 \longrightarrow V_l(B)^* \otimes_\imath \mathbb{C} \longrightarrow V_l(A)^* \otimes_\imath \mathbb{C} \longrightarrow V_l(T(\overline{K})/\Lambda)^* \otimes_\imath \mathbb{C} \longrightarrow 0 \tag{4.2.6}$$

is an exact sequence of $\mathcal{W}'(\overline{K}/K)$-modules, where $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ is a field embedding, $(\kappa, 0)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(B)^* \otimes_\imath \mathbb{C}$, $\sigma' = (\sigma, N)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(A)^* \otimes_\imath \mathbb{C}$, and by Corollary 4.2.2, $\chi \otimes \omega^{-1} \otimes \mathrm{sp}(2)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(T(\overline{K})/\Lambda)^* \otimes_\imath \mathbb{C}$.

The same argument as in the proof of (2.3.6) in Lemma 2.3.3 applied to (4.2.6) proves that σ' has the following form:

$$\sigma' \cong \alpha \oplus (\beta \otimes \mathrm{sp}(2)) \oplus (\gamma \otimes \mathrm{sp}(3)), \tag{4.2.7}$$

where α, β, and γ are some representations of $\mathcal{W}(\overline{K}/K)$. First, we will prove that $\gamma = 0$. Assume $\gamma \neq 0$. ¿From (4.2.7) we have

$$\sigma \cong \alpha \oplus \beta \oplus (\beta \otimes \omega) \oplus \gamma \oplus (\gamma \otimes \omega) \oplus (\gamma \otimes \omega^2).$$

On the other hand, since σ is semisimple, the exact sequence (4.2.6) of $\mathcal{W}(\overline{K}/K)$-modules splits, i.e.,

$$\sigma \cong \kappa \oplus (\chi \otimes \omega^{-1}) \oplus \chi.$$

Thus, combining the last two isomorphisms, we get

$$\kappa \oplus (\chi \otimes \omega^{-1}) \oplus \chi \cong \alpha \oplus \beta \oplus (\beta \otimes \omega) \oplus \gamma \oplus (\gamma \otimes \omega) \oplus (\gamma \otimes \omega^2). \tag{4.2.8}$$

Moreover, since κ is isomorphic to a subrepresentation of $\ker N$ and $\ker N \cong \alpha \oplus (\beta \otimes \omega) \oplus (\gamma \otimes \omega^2)$, κ is isomorphic to a subrepresentation of $\alpha \oplus (\beta \otimes \omega) \oplus (\gamma \otimes \omega^2)$. Thus, by the uniqueness of decomposition of a semisimple module into simple modules we get from (4.2.8) that γ is isomorphic to a subrepresentation of $(\chi \otimes \omega^{-1}) \oplus \chi$. Since χ has finite image, this implies that for any irreducible component γ_0 of γ either γ_0 or $\gamma_0 \otimes \omega$ has finite image. In particular, the absolute value of each eigenvalue of $\gamma_0(\Phi)$ equals either 1 or q. Since $\sigma' \otimes \omega^{1/2}$ is symplectic, Theorem B together with the uniqueness of decomposition of an admissible representation of $\mathcal{W}'(\overline{K}/K)$ into indecomposable representations imply that for any irreducible component γ_0 of γ the representation $(\gamma_0 \otimes \omega^{1/2} \otimes \mathrm{sp}(3))^*$ is an irreducible component of $\gamma \otimes \omega^{1/2} \otimes \mathrm{sp}(3)$. In particular, this implies that $\gamma_0^* \otimes \omega^{-3}$ is an irreducible component of γ. Thus, either $\gamma_0^* \otimes \omega^{-3}$ or $\gamma_0^* \otimes \omega^{-2}$ has finite image, hence the absolute value of each eigenvalue of $\gamma_0^*(\Phi)$ equals either q^{-3} or q^{-2} and we get a contradiction with the previous statement about the absolute values of eigenvalues of $\gamma_0(\Phi)$. Thus, $\gamma = 0$ and $\sigma' \cong \alpha \oplus (\beta \otimes \mathrm{sp}(2))$.

The same argument can be used to show that $\beta \cong \chi \otimes \omega^{-1}$. Namely, taking into account that κ is isomorphic to a subrepresentation of $\alpha \oplus (\beta \otimes \omega)$, it follows from (4.2.8) that β is isomorphic to a subrepresentation of $\chi \oplus (\chi \otimes \omega^{-1})$. Hence

$$\beta \cong \beta_1 \oplus (\beta_2 \otimes \omega^{-1}) \tag{4.2.9}$$

for some subrepresentations β_1, β_2 of χ. Since $\sigma' \otimes \omega^{1/2}$ is symplectic, Theorem

B together with the uniqueness of decomposition of an admissible representation of $\mathcal{W}'(\overline{K}/K)$ into indecomposable representations imply that $\beta \otimes \omega^{1/2} \otimes \mathrm{sp}(2)$ is symplectic. In particular, $\beta \cong \beta^* \otimes \omega^{-2}$ which together with (4.2.9) gives

$$\beta_1 \oplus (\beta_2 \otimes \omega^{-1}) \cong (\beta_1^* \otimes \omega^{-2}) \oplus (\beta_2^* \otimes \omega^{-1}).$$

By taking the determinant of both sides of this congruence, we get

$$\det \beta_1 \cdot \det \beta_2 \cdot \omega^{-n_2} = \det \beta_1^* \cdot \det \beta_2^* \cdot \omega^{-2n_1-n_2}, \tag{4.2.10}$$

where $n_1 = \dim \beta_1$ and $n_2 = \dim \beta_2$. Since β_1 and β_2 have finite images as subrepresentations of χ and ω does not have a finite image, (4.2.10) gives $n_1 = 0$. Thus, β is isomorphic to a subrepresentation of $\chi \otimes \omega^{-1}$. It follows from (4.2.6) that $\dim \beta \geq \dim \chi$, because $\dim \beta = \operatorname{rank} N$ and $\dim \chi = \operatorname{rank} S$, hence $\beta \cong \chi \otimes \omega^{-1}$. Then $\kappa \cong \alpha$ by (4.2.8) with $\gamma = 0$ and the uniqueness of decomposition of a semisimple representation into simple subrepresentations. $\square$

Appendix A

Lemma A.0.5. *Let $C = \langle c \rangle$ be an infinite cyclic group generated by an element c and let $E = \langle e \rangle$ be a finite cyclic group of order n generated by an element e. Let $G = E \rtimes C$ be a semi-direct product, where C acts on E via $c^{-1}ec = e^k$ for some $k \in (\mathbb{Z}/n\mathbb{Z})^\times$. Denote by s the order of k in $(\mathbb{Z}/n\mathbb{Z})^\times$. Then every irreducible representation λ of G has the following form:*

$$\lambda = \lambda_0 \otimes \phi,$$

where λ_0 is an irreducible representation of G trivial on the subgroup of C generated by c^s and ϕ is a one-dimensional representation of G.

Proof. Since c^s is contained in the center of G and λ is an irreducible complex representation, by Schur's lemma $\lambda(c^s)$ is equal to a scalar $a \in \mathbb{C}^\times$. Define a one-dimensional representation ϕ of G as follows: $\phi(e) = 1$ and $\phi(c)$ equals an s-th root of a. Then $\lambda_0 = \lambda \otimes \phi^{-1}$ is trivial on $\langle c^s \rangle$ and $\lambda = \lambda_0 \otimes \phi$. □

Proof of Proposition 2.2.5. Let λ be an irreducible symplectic representation of G. Then by Lemma A.0.5, $\lambda = \lambda_0 \otimes \phi$, where λ_0 is an irreducible representation of G

trivial on the subgroup of C generated by c^s and ϕ is a one-dimensional representation of G. Since λ is symplectic, λ and its contragredient representation have the same character, which implies that for any $g \in G$ we have

$$\phi(g) \cdot \operatorname{tr} \lambda_0(g) = \phi(g)^{-1} \cdot \operatorname{tr} \lambda_0(g^{-1}).$$

Taking into account that λ_0 is trivial on $\langle c^s \rangle$, the above equation for $g = c^s$ gives $\phi(c^{2s}) = 1$, i.e., λ can be considered as an irreducible symplectic representation of the finite group $H = G/\langle c^{2s} \rangle \cong E \rtimes C/\langle c^{2s} \rangle$. By abuse of notation we will denote the image of c in $C/\langle c^{2s} \rangle$ also by c, then $c^{2s} = 1$ and $c^{-1}ec = e^k$ in H. As an irreducible representation of the semi-direct product H, λ can be constructed from a one-dimensional representation ψ_1 of E in the following way. Let $\psi_1(e) = \xi$ for some n-th root of unity ξ of order d in $\mathbb{C}^\times$. Let $\Gamma = \langle c^x \rangle$, where $x = |k|$ in $(\mathbb{Z}/d\mathbb{Z})^\times$, and ψ_2 be a one-dimensional representation of Γ. Then ψ_1 and ψ_2 can be extended to representations of $E \rtimes \Gamma$ via

$$\begin{aligned} \psi_1(c^{xv}e^t) &= \psi_1(e^t), \\ \psi_2(c^{xv}e^t) &= \psi_2(c^{xv}). \end{aligned}$$

Then $\lambda = \operatorname{Ind}_{E \rtimes \Gamma}^{H}(\psi_1 \otimes \psi_2)$ ([S], p. 62, Prop. 25). Let W be a representation space of H corresponding to λ, $V = \mathbb{C}b \subseteq W$ be a one-dimensional subrepresentation of $\operatorname{Res}_{E \rtimes \Gamma}^{H} \lambda$ isomorphic to $\psi_1 \otimes \psi_2$ and spanned by a nonzero vector $b \in V$ over $\mathbb{C}$. Then $W = V \oplus cV \oplus c^2V \oplus \cdots \oplus c^{x-1}V$ and λ has the following form in the basis

$\{b, cb, c^2b, \ldots, c^{x-1}b\}$:

$$\lambda(e) = \begin{pmatrix} \xi & 0 & 0 & \ldots & 0 \\ 0 & \xi^k & 0 & \ldots & 0 \\ 0 & 0 & \xi^{k^2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \xi^{k^{x-1}} \end{pmatrix}, \quad \lambda(c) = \begin{pmatrix} 0 & 0 & 0 & \ldots & \psi_2(c^x) \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Since λ is symplectic, $x = \dim \lambda$ is even and $\det \lambda = 1$, hence $\det \lambda(c) = -\psi_2(c^x) = 1$, which implies that $\psi_2(c^x) = -1$. Denote by χ the character of λ. By Proposition 39 ([S], p. 109), λ is symplectic if and only if

$$-1 = \frac{1}{|H|} \cdot \sum_{y \in H} \chi(y^2). \tag{A.0.1}$$

Let $y = c^v e^t$, consequently, $y^2 = c^{2v} e^{t(1+k^v)}$. Clearly, $\chi(y^2) = 0$ if $y^2 \notin E \rtimes \Gamma$ and $y^2 \in E \rtimes \Gamma$ if and only if x divides $2v$ and, since x is even, if and only if $\frac{x}{2}$ divides v. Let $v = \frac{x}{2}m$, then we have

$$\begin{aligned} \sum_{y \in H} \chi(y^2) &= \sum_{\substack{y \in H \\ y^2 \in E \rtimes \Gamma}} \chi(y^2) = \sum_{t,m} \chi(c^{mx} e^{t(1+k^v)}) \\ &= \sum_{\substack{m,t \\ m \text{ even}}} \chi(e^{t(1+k^v)}) - \sum_{\substack{m,t \\ m \text{ odd}}} \chi(e^{t(1+k^v)}). \end{aligned} \tag{A.0.2}$$

Let $S_1 = \sum_{m \text{ even}} \sum_t \chi(e^{t(1+k^v)})$ and $S_2 = \sum_{m \text{ odd}} \sum_t \chi(e^{t(1+k^v)})$.

If m is even, then $v = x(\frac{m}{2})$ and, since $x = |k|$ in $(\mathbb{Z}/d\mathbb{Z})^\times$, $1 + k^v \equiv 2 \pmod{d}$. Since $\chi(e^t) = \xi^t + \xi^{kt} + \cdots + \xi^{k^{x-1}t}$ and $\xi^d = 1$, we have $\chi(e^{t(1+k^v)}) = \chi(e^{2t})$ and $S_1 = \sum_{m \text{ even}} \sum_t \chi(e^{2t})$. We will show that $\sum_t \chi(e^{2t}) = 0$. First, note that if $d = 1, 2$, then λ is one-dimensional, hence cannot be symplectic.

If $r \in \mathbb{Z}$ and $r \equiv 0 \pmod{d}$ then

$$\sum_{t=0}^{n-1} \chi(e^{rt}) = \sum_{t=0}^{n-1} \sum_{j=0}^{x-1} \xi^{rtk^j} = nx.$$

If $r \in \mathbb{Z}$ and $r \not\equiv 0 \pmod{d}$ then

$$\sum_{t=0}^{n-1} \chi(e^{rt}) = \sum_{t=0}^{n-1} \sum_{j=0}^{x-1} \xi^{rtk^j} = \sum_{j=0}^{x-1} \frac{1-\xi^{rnk^j}}{1-\xi^{rk^j}} = 0.$$

Thus

$$\sum_{t=0}^{n-1} \chi(e^{rt}) = \begin{cases} nx, & r \equiv 0 \pmod{d}; \\ 0, & r \not\equiv 0 \pmod{d}. \end{cases} \tag{A.0.3}$$

Since $d \neq 1, 2$, formula (A.0.3) implies that $S_1 = 0$.

If m is odd, then $k^v \equiv k^{\frac{x}{2}} \pmod{d}$, hence $\chi(e^{t(1+k^v)}) = \chi(e^{t(1+k^{\frac{x}{2}})})$. Thus

$$S_2 = \sum_{m \text{ odd}} \sum_{t=0}^{n-1} \chi(e^{t(1+k^v)}) = \frac{2s}{x} \cdot \sum_{t=0}^{n-1} \chi(e^{t(1+k^{\frac{x}{2}})})$$

and by (A.0.3) we have

$$S_2 = \begin{cases} 2sn, & 1+k^{\frac{x}{2}} \equiv 0 \pmod{d}; \\ 0, & 1+k^{\frac{x}{2}} \not\equiv 0 \pmod{d}. \end{cases} \tag{A.0.4}$$

Hence by (A.0.2)

$$\frac{1}{|H|} \cdot \sum_{y \in H} \chi(y^2) = \frac{1}{2sn} \cdot (S_1 - S_2) = -\frac{S_2}{2sn},$$

which together with (A.0.1) and (A.0.4) proves the proposition. □

Let D be a group, U a finite-dimensional $\mathbb{C}[D]$-module, and U^* the contragredient of U. Let $\check{U}$ denote the vector space over $\mathbb{C}$ with the underlying abelian group U^*

and multiplication by constants defined as follows:

$$a \cdot \phi = \overline{a}\phi, \quad a \in \mathbb{C}, \, \phi \in U^*,$$

where $\overline{a}$ is the complex conjugate of a. Clearly, the $\mathbb{C}[D]$-module structure on U^* makes $\check{U}$ into a $\mathbb{C}[D]$-module. In what follows by $\check{U}$ we mean a $\mathbb{C}[D]$-module with this structure. We say that U is *unitary* if U admits a nondegenerate invariant hermitian form (not necessarily positive definite).

Lemma A.0.6. *Every semisimple unitary, orthogonal, or symplectic representation λ of a group D has the following form*

$$\lambda \cong \nu \oplus \tilde{\nu} \oplus \lambda_1^{z_1} \oplus \cdots \oplus \lambda_t^{z_t},$$

where ν is a representation of D, $\tilde{\nu} = \nu^*$ if λ is orthogonal or symplectic and $\tilde{\nu} = \check{\nu}$ if λ is unitary, $\lambda_1, \ldots, \lambda_t$ are pairwise nonisomorphic irreducible unitary, orthogonal, or symplectic representations of D respectively.

Proof of Lemma A.0.6. We say that a unitary, orthogonal, or symplectic representation is *minimal* if it cannot be written as an orthogonal sum of nonzero invariant subspaces. Clearly, every unitary, orthogonal, or symplectic representation is an orthogonal sum of minimal unitary, orthogonal, or symplectic representations respectively. Thus, it is enough to prove that if λ is a semisimple minimal unitary, orthogonal, or symplectic representation of D, then either λ is irreducible or $\lambda \cong \nu \oplus \tilde{\nu}$ for some irreducible representation ν of D. Let U be a representation space of D corresponding to λ, $\tilde{U} = U^*$ if λ is orthogonal or symplectic and $\tilde{U} = \check{U}$ if λ is unitary. Since λ is semisimple, $U = V_1 \oplus \cdots \oplus V_n$, where $V_1, \ldots, V_n$ are nonzero simple $\mathbb{C}[D]$-submodules of U. Let $\langle \cdot\,, \cdot \rangle$ be a nondegenerate invariant form on U. It defines a $\mathbb{C}[D]$-module isomorphism ϕ between U and $\tilde{U}$ via $\phi(u) = \langle u\,, \cdot \rangle$, $u \in U$. Let $\psi : \tilde{U} \longrightarrow \tilde{V_1} \oplus \cdots \oplus \tilde{V_n}$ denote the usual isomorphism between $\tilde{U} = (V_1 \oplus \cdots \oplus V_n)^{\sim}$ and $\tilde{V_1} \oplus \cdots \oplus \tilde{V_n}$. For each i and j let $\alpha_{ij} : V_i \longrightarrow \tilde{V_j}$ be a $\mathbb{C}[D]$-module homomorphism defined by the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\psi \circ \phi} & \tilde{V_1} \oplus \cdots \oplus \tilde{V_n} \\ \uparrow & & \downarrow{\scriptstyle \pi_j} \\ V_i & \xrightarrow{\alpha_{ij}} & \tilde{V_j} \end{array}$$

where π_j is the projection onto j-th factor. Since $\psi \circ \phi$ is an isomorphism, there exists some $\tilde{V_i}$ such that $\alpha_{1i} \neq 0$, which implies that α_{1i} is an isomorphism, since $V_1, \ldots, V_n$ are simple. If $i = 1$, then it follows that $\langle \cdot\,, \cdot \rangle|_{V_1}$ is nondegenerate, hence V_1 and its orthogonal complement are invariant subspaces of U. Since U is minimal, it implies that $U = V_1$ and U is irreducible. Thus, we can assume that for each j we have $\alpha_{jj} = 0$, which is equivalent to $\langle V_j\,, V_j \rangle = 0$. Without loss of generality we can assume that $\alpha_{12} \neq 0$. Then $\alpha_{21} \neq 0$. Indeed, if $\alpha_{12} \neq 0$, then there is some $u \in V_1$ such that $\langle u\,, \cdot \rangle|_{V_2} \neq 0$, i.e., there is some $v \in V_2$ such that $\langle u\,, v \rangle \neq 0$, hence $\langle v\,, u \rangle \neq 0$, which is equivalent to $\alpha_{21} \neq 0$. Let us prove now that $\langle \cdot\,, \cdot \rangle|_{V_1 \oplus V_2}$ is nondegenerate. Let $u + v \in V_1 \oplus V_2$ and $\langle u + v\,, x + y \rangle = 0$ for any $x + y \in V_1 \oplus V_2$. We have $\langle u + v\,, x + y \rangle = \langle u\,, y \rangle + \langle v\,, x \rangle = 0$, because $\langle V_1\,, V_1 \rangle = \langle V_2\,, V_2 \rangle = 0$. Take $x = 0$ in this equation, then $\langle u\,, y \rangle = 0$ for any $y \in V_2$, hence $u = 0$, because $\alpha_{12}(V_1) = \tilde{V_2}$. Analogously, $v = 0$. Since U is minimal, the same argument as above implies that

$U = V_1 \oplus V_2 \cong V_1 \oplus \tilde{V}_1.$ □

Proof of Proposition 2.2.6. By Lemma A.0.6

$$\lambda \cong \nu \oplus \nu^* \oplus \lambda_1^{z_1} \oplus \cdots \oplus \lambda_t^{z_t}, \tag{A.0.5}$$

where ν is a representation of G and $\lambda_1, \ldots, \lambda_t$ are pairwise nonisomorphic irreducible symplectic representations of G. Let $\nu = \nu_1^{l_1} \oplus \cdots \oplus \nu_r^{l_r}$, where $\nu_1, \ldots, \nu_r$ are pairwise nonisomorphic irreducible representations of G. By Lemma A.0.5 for each i we have $\nu_i = \nu_i^0 \otimes \phi_i$, where ϕ_i is a one-dimensional representation of G and ν_i^0 is an irreducible representation of G trivial on $\langle c^s \rangle$. It follows that ν_i^0 can be considered as a representation of $H = G/\langle c^{2s} \rangle$ and as a representation of H it can be written in the following form $\nu_i^0 = \mathrm{Ind}_{E \rtimes \Gamma_i}^H \psi_i$, where ψ_i is a one-dimensional representation of $E \rtimes \Gamma_i$, $\psi_i(e) = \xi_i$ for an n-th root of unity ξ_i of order d_i, $x_i = |k|$ in $(\mathbb{Z}/d_i\mathbb{Z})^\times$, and $\Gamma_i = \langle c^{x_i} \rangle$ (see the proof of Proposition 2.2.5 and [S], p. 62, Prop. 25). Thus

$$\nu_i^0(e) = \begin{pmatrix} \xi_i & 0 & \ldots & 0 \\ 0 & \xi_i^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \xi_i^{k^{x_i-1}} \end{pmatrix},$$

$$\nu_i(e) = \nu_i^0 \otimes \phi_i(e) = \begin{pmatrix} \xi_i\phi_i(e) & 0 & \ldots & 0 \\ 0 & (\xi_i\phi_i(e))^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (\xi_i\phi_i(e))^{k^{x_i-1}} \end{pmatrix}.$$

In the second matrix we used the relation $\phi_i(e)^{k-1} = 1$, which follows from the fact that ϕ_i is a one-dimensional representation of G and $c^{-1}ec = e^k$. By Proposition 2.2.5 each $\lambda_i = \mathrm{Ind}_{E \rtimes L_i}^{H} \rho_i$, where ρ_i is a one-dimensional representation of $E \rtimes L_i$, $\rho_i(e) = \eta_i$ for an n-th root of unity η_i of order u_i, $y_i = |k|$ in $(\mathbb{Z}/u_i\mathbb{Z})^\times$, $L_i = (c^{y_i})$, and $\rho_i(c^{y_i}) = -1$.

We will need the following lemma:

Lemma A.0.7. *Let $d_1, \ldots, d_m$ be pairwise distinct natural numbers. For each d_i let $p_i(X) \in \mathbb{C}[X]$ be a monic polynomial, all the roots of which are some primitive d_i-th roots of unity and let $p(X) = p_1(X) \cdots p_m(X)$. If $p(X) \in \mathbb{Q}[X]$, then each $p_i(X)$ is a power of the d_i-th cyclotomic polynomial $\Phi_{d_i}(X)$.*

Proof. The statement follows from considering the factorization of $p(X)$ into irreducibles in $\mathbb{Q}[X]$. □

Since the characteristic polynomial p of $\lambda(e)$ has coefficients in $\mathbb{Q}$, by Lemma A.0.7 we can assume that $\xi_1\phi_1(e), \ldots, \xi_r\phi_r(e), \eta_1, \ldots, \eta_t$ are primitive roots of unity of the same order d and that $p = \Phi_d^v$ for some v, where Φ_d is the d-th cyclotomic polynomial. Indeed, λ can be written as a sum of semisimple symplectic representations of G which have this property and it is enough to show that for each of them (2.2.3) holds.

Let $x = |k|$ in $(\mathbb{Z}/d\mathbb{Z})^\times$ and $\Gamma = \langle c^x \rangle$. If $\lambda \cong \nu \oplus \nu^*$ then there is nothing to prove. Thus, we assume that there is λ_1 in (A.0.5). Since λ_1 is symplectic, x is even, $d \neq 1, 2$, and $k^{\frac{x}{2}} \equiv -1 \pmod{d}$ by Proposition 2.2.5. Note, that x divides each x_i. Indeed, $(\xi_i\phi_i(e))^{k^{x_i}} = \xi_i\phi_i(e)$, hence $k^{x_i} \equiv 1 \pmod{d}$, because $\xi_i\phi_i(e)$ is a primitive d-th root of unity by assumption. For each i denote by p_{ν_i} the characteristic polynomial of $\nu_i(e)$ and by $p_{\nu_i^*}$ the characteristic polynomial of $\nu_i^*(e)$. Then $p_{\nu_i} = p_{\nu_i^*}$. This is true because x divides x_i, x is even, $k^{\frac{x_i}{2}} \not\equiv 1 \pmod{d}$, and $k^{\frac{x}{2}} \equiv -1 \pmod{d}$, hence each root of p_{ν_i} appears in p_{ν_i} with its complex conjugate. Thus

$$p = p_{\nu_1}^{2l_1} \cdots p_{\nu_r}^{2l_r} p_{\lambda_1}^{z_1} \cdots p_{\lambda_t}^{z_t},$$

where for each i we denote by p_{λ_i} the characteristic polynomial of $\lambda_i(e)$.

For each primitive d-th root of unity ξ write $q(\xi) = (X-\xi)(X-\xi^k)\cdots(X-\xi^{k^{x-1}})$, where $x = |k|$ in $(\mathbb{Z}/d\mathbb{Z})^\times$. Clearly, all $\xi, \xi^k, \ldots, \xi^{k^{x-1}}$ are distinct and for two primitive d-th roots of unity ξ and ξ' either $q(\xi) = q(\xi')$ or $q(\xi)$ and $q(\xi')$ have no common roots. In this notation $p_{\lambda_i} = q(\eta_i)$ and $p_{\nu_i} = q(\xi_i\phi_i(e))^{\alpha_i}$, where $\alpha_i = \frac{x_i}{x}$. Since $\lambda_1, \ldots, \lambda_t$ are irreducible, symplectic, and pairwise nonisomorphic, it follows from Proposition 2.2.5 that $q(\eta_i) \neq q(\eta_j)$ for $i \neq j$. Without loss of generality we can assume that p has the following form:

$$p = q(\xi_1\phi_1(e))^{2m_1} \cdots q(\xi_f\phi_f(e))^{2m_f} q(\eta_1)^{z_1} \cdots q(\eta_t)^{z_t}, \tag{A.0.6}$$

where $f \leq r$, $m_1, \ldots, m_f$ are positive integers, and $q(\xi_1\phi_1(e)), \ldots, q(\xi_f\phi_f(e))$ have no common roots. There are two possibilities:

1. there exists some $q(\xi_i\phi_i(e))$ which is not equal to any of $q(\eta_1), \ldots, q(\eta_t)$. Without loss of generality we can assume that $i = 1$;

2. each $q(\xi_i\phi_i(e))$ equals some $q(\eta_j)$.

(1) In this case, since $p = \Phi_d^v$, it follows from (A.0.6) that for each j we have $z_j + 2 \cdot \alpha(j) = 2m_1$, where $\alpha(j) = m_\beta$ if $q(\eta_j)$ equals some $q(\xi_\beta \phi_\beta(e))$ and $\alpha(j) = 0$ otherwise. Thus, in this case all $z_1, \ldots, z_t$ are even and $[\lambda] = [\nu] + [\nu^*] + 2 \cdot [\mu_0]$, where $\mu_0 = \lambda_1^{\frac{z_1}{2}} \oplus \cdots \oplus \lambda_t^{\frac{z_t}{2}}$ is symplectic of finite image because all $\lambda_1, \ldots, \lambda_t$ are symplectic of finite images. (2) In this case, since $p = \Phi_d^v$, it follows from (A.0.6) that for each j we have $z_j + 2 \cdot \alpha(j) = v$, where $\alpha(j) = m_\beta$ if $q(\eta_j)$ equals some $q(\xi_\beta \phi_\beta(e))$ and $\alpha(j) = 0$ otherwise. Moreover, it follows that $q(\eta_1) \cdots q(\eta_t) = \Phi_d$. Thus

$$[\lambda] = [\nu] + [\nu^*] - 2 \cdot [\mu_0'] + v \cdot [\lambda_1] + \cdots + v \cdot [\lambda_t],$$

where $\mu_0' = \lambda_1^{\alpha(1)} \oplus \cdots \oplus \lambda_t^{\alpha(t)}$ is symplectic of finite image and it is enough to show that $\hat{\lambda}_1 \oplus \cdots \oplus \hat{\lambda}_t$ is realizable over $\mathbb{Q}$. Recall that for each i, $\hat{\lambda}_i = \mathrm{Ind}_{E \rtimes \Gamma}^H \varphi_i$, where $\varphi_i(e) = \xi_i$ for some primitive d-th root of unity ξ_i, $x = |k|$ in $(\mathbb{Z}/d\mathbb{Z})^\times$, $\Gamma = \langle c^x \rangle$, and $\varphi_i(c^x) = 1$ (see Proposition 2.2.5 and (2.2.2)). Since the representations of this form are completely defined by a root of unity ξ, we will denote them by $\Theta(\xi)$. For any r dividing d the cyclic group $\langle k \rangle$ acts on the set of all primitive r-th roots of unity via $\xi \longmapsto \xi^k$. Let $\{\xi_r^1, \ldots, \xi_r^{w_r}\}$ be the set of representatives for this action and let

$$\Theta(r) = \bigoplus_{i=1}^{w_r} \Theta(\xi_r^i).$$

Then the characteristic polynomial of $\Theta(r)(e)$ is just Φ_r. Since the characteristic polynomial of $(\hat{\lambda}_1 \oplus \cdots \oplus \hat{\lambda}_t)(e)$ is $q(\eta_1) \cdots q(\eta_t) = \Phi_d$, it follows that $\hat{\lambda}_1 \oplus \cdots \oplus \hat{\lambda}_t = \Theta(d)$. By induction on d we will prove that each $\Theta(d)$ is realizable over $\mathbb{Q}$.

Clearly, $\Theta(d)$ is realizable over $\mathbb{Q}$ when $d = 1$, because in this case $\Theta(d) = 1$. Let $L = \langle e^d \rangle \rtimes C$ and $\pi = \mathrm{Ind}_L^H 1$. Then the characteristic polynomial of $\pi(e)$ is $x^d - 1$, consequently,

$$\pi \cong \bigoplus_{r|d} \Theta(r) \quad \text{on} \quad E.$$

We will prove that this is true on the whole group H. Observe, that for any r all $\Theta(\xi_r^1), \ldots, \Theta(\xi_r^{w_r})$ are irreducible over $\mathbb{C}$ and $\Theta(\xi_r^i) \cong \Theta(\xi_{r'}^{i'})$ only if $i = i'$ and $r = r'$. Let χ_r^i be the character of $\Theta(\xi_r^i)$. Then, using Frobenius reciprocity, we have

$$\langle \pi, \Theta(\xi_r^i) \rangle = \langle \mathrm{Ind}_L^H 1, \Theta(\xi_r^i) \rangle = \langle 1, \mathrm{Res}_L^H \Theta(\xi_r^i) \rangle =$$
$$\frac{d}{2ns} \sum_{u,v} \chi_r^i(e^{du} c^v) = \frac{d}{2ns} \sum_{u,v} \chi_r^i(c^v) = 1,$$

hence $\pi \cong \bigoplus_{r|d} \Theta(r)$ on H. Since π is realizable over $\mathbb{Q}$ and $\Theta(r)$ is realizable over $\mathbb{Q}$ for any $r < d$ by induction, $\Theta(d) = \pi - \bigoplus_{r|d, r \neq d} \Theta(r)$ is realizable over $\mathbb{Q}$. □

Proof of Proposition 2.2.9. Let $\lambda = \kappa \otimes \omega^{1/2}$. Then $W(\kappa \otimes \tau) = W(\lambda \otimes \tau)$, because real powers of ω do not change the root number. Since the root number of representations of $\mathcal{W}(\overline{K}/K)$ is multiplicative in short exact sequences, there is a unique homomorphism

$$\alpha : R(\mathcal{W}(\overline{K}/K)) \longrightarrow \mathbb{C}^\times$$

such that $\alpha([\lambda]) = W(\lambda)$ for any representation λ of $\mathcal{W}(\overline{K}/K)$. Thus, it follows from Corollary 2.2.7 that

$$W(\lambda \otimes \tau) = W(\mu \otimes \tau) \cdot W(\mu^* \otimes \tau) \cdot \frac{W(\mu_0 \otimes \tau)^2}{W(\mu_0' \otimes \tau)^2} \cdot W(\mu_1 \otimes \tau) \cdots W(\mu_a \otimes \tau). \quad \text{(A.0.7)}$$

Since τ has finite image and real-valued character, we have

$$\begin{aligned} W(\mu \otimes \tau) \cdot W(\mu^* \otimes \tau) &= W(\mu \otimes \tau) \cdot W((\mu \otimes \tau)^*) \\ &= \det(\mu \otimes \tau)(-1) = \det \mu(-1)^{\dim \tau} \cdot \det \tau(-1)^{\dim \mu}. \end{aligned}$$

Also, since μ_0 and μ_0' are symplectic and of finite images,

$$W(\mu_0 \otimes \tau) = \pm 1, \quad W(\mu_0' \otimes \tau) = \pm 1$$

([Ro2], p. 315), hence from (A.0.7) we get

$$W(\lambda \otimes \tau) = \det \mu(-1)^{\dim \tau} \cdot \det \tau(-1)^{\dim \mu} \cdot W(\mu_1 \otimes \tau) \cdots W(\mu_a \otimes \tau). \qquad \text{(A.0.8)}$$

Thus, we need to compute $W(\mu_1 \otimes \tau), \ldots, W(\mu_a \otimes \tau)$. Let γ be an irreducible symplectic subrepresentation of λ. Let L/K^{unr} be a minimal subextension of $\overline{K}/K^{unr}$ over which B acquires good reduction. Then, as was discussed in Section 2.2, λ and, consequently, γ can be considered as representations of $G = E \rtimes \langle \Phi \rangle$, where $E = \mathrm{Gal}(L/K^{unr})$ is a finite cyclic group (because $p > 2m + 1$) and $\langle \Phi \rangle$ is an infinite cyclic group. Let $x = \dim \gamma$. Then by Proposition 2.2.5, as a representation of G, γ is induced from a one-dimensional representation of $E \rtimes \langle \Phi^x \rangle$. Hence, as a representation of $\mathcal{W}(\overline{K}/K)$, γ is induced from a one-dimensional representation ϕ of $\mathcal{W}(\overline{K}/H_x)$, where H_x is the unramified extension of K of degree x, i.e., $\gamma = \mathrm{Ind}_K^{H_x} \phi$. Since γ is symplectic, x is even. Let $x = 2y$ and let H_y be the unramified extension of K of degree y, hence $K \subseteq H_y \subseteq H_x$. Let $\gamma' = \mathrm{Ind}_{H_y}^{H_x} \phi$, $\tau' = \mathrm{Res}_K^{H_y} \tau$, then $\gamma = \mathrm{Ind}_K^{H_y} \gamma'$ and by Formula (1.4) ([Ro2], p. 316) we have

$$W(\gamma \otimes \tau) = W(\mathrm{Ind}_K^{H_y}(\gamma' \otimes \tau')) = W(\gamma' \otimes \tau') W(\mathrm{Ind}_K^{H_y} 1_{H_y})^{2 \dim \tau}. \qquad \text{(A.0.9)}$$

Let us prove first that $W(\mathrm{Ind}_K^{H_y} 1_{H_y})^{2 \dim \tau} = 1$. Let ϖ be a uniformizer of K. It is easy to check that $\mathrm{Ind}_K^{H_y} 1_{H_y} = \bigoplus_{i=0}^{y-1} \chi_i$, where $\chi_0, \ldots, \chi_{y-1}$ are all the distinct unramified characters of $K^\times$, satisfying $\chi_i(\varpi)^y = 1$. Hence $W(\mathrm{Ind}_K^{H_y} 1_{H_y}) = \prod_{i=0}^{y-1} W(\chi_i)$. By

Formula (ϵ3) ([Ro1], p. 142) for each i we have $W(\chi_i) = \xi_i^{n(\psi)}$, where $n(\psi) \in \mathbb{Z}$, each ξ_i is a y-th root of unity, and $\xi_i \neq \xi_j$ if $i \neq j$. Hence $\prod_{i=0}^{y-1} W(\chi_i) = \prod_{i=0}^{y-1} \xi_i^{n(\psi)} = \pm 1$ and

$$W(\mathrm{Ind}_K^{H_y} 1_{H_y})^{2\dim\tau} = 1. \tag{A.0.10}$$

To compute $W(\gamma' \otimes \tau')$ we will show that Theorem 2.2.8 can be applied to H_y, γ', and τ'. Indeed, τ' is a representation of $\mathrm{Gal}(\overline{K}/H_y)$ with real-valued character and $\gamma' = \mathrm{Ind}_{H_y}^{H_x}\phi$ is a two-dimensional representation of $\mathcal{W}(\overline{K}/H_y)$ induced from a character ϕ of finite image (by Proposition 2.2.5), hence γ' is a representation of $\mathrm{Gal}(\overline{K}/H_y)$. Since $\mathrm{Ind}_K^{H_y}\gamma' = \gamma$ is irreducible, γ' is irreducible too. Since $\dim\gamma' = 2$, γ' is symplectic if and only if $\det\gamma'$ is trivial, because $\mathrm{Sp}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$ ([Ro2], p. 317). From Proposition 2.2.5 we find that as a representation of $E \rtimes (\Phi^y)$, γ' has the following form

$$\gamma'(e) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \gamma'(\Phi^y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where e is a generator of E, ξ is a root of unity. It follows immediately that $\det\gamma' = 1$. Thus to be able to apply Theorem 2.2.8, we need only to check that ϕ is a tame character of $H_x^\times$. It follows from the fact that ϕ is trivial on $\mathrm{Gal}(\overline{K}/L)$ and L/K^{unr} is tamely ramified, because $p > 2m+1$. By Theorem 2.2.8

$$W(\gamma' \otimes \tau') = \det\tau'(-1) \cdot \varphi^{\dim\tau} \cdot (-1)^{\langle 1,\tau'\rangle + \langle \eta',\tau'\rangle + \langle \hat{\gamma}',\tau'\rangle}, \tag{A.0.11}$$

where η' is the unramified quadratic character of $H_y^\times$, $\hat{\gamma}' = \mathrm{Ind}_{H_y}^{H_x}(\phi \otimes \theta)$, θ is the unramified quadratic character of $H_x^\times$, and $\varphi = \pm 1$.

Since $\tau' = \mathrm{Res}_K^{H_y}\tau$, we have $\det\tau' = \det\tau \circ N_{H_y/K}$, hence

$$\det\tau'(-1) = \det\tau(-1)^{[H_y:K]} = \det\tau(-1)^y. \qquad \text{(A.0.12)}$$

By Frobenius reciprocity

$$\langle 1, \tau'\rangle + \langle \eta', \tau'\rangle = \langle 1_{H_y} \oplus \eta', \tau'\rangle = \langle 1_{H_y} \oplus \eta', \mathrm{Res}_K^{H_y}\tau\rangle = \langle \mathrm{Ind}_K^{H_y}(1_{H_y} \oplus \eta'), \tau\rangle.$$

As was mentioned above, $\mathrm{Ind}_K^{H_y} 1_{H_y} = \bigoplus_{i=0}^{y-1}\chi_i$, where $\chi_0, \ldots, \chi_{y-1}$ are all the distinct unramified characters of $K^\times$, satisfying $\chi_i(\varpi)^y = 1$. Analogously, $\mathrm{Ind}_K^{H_y}\eta' = \bigoplus_{i=y}^{2y-1}\chi_i$, where $\chi_y, \ldots, \chi_{2y-1}$ are all the distinct unramified characters of $K^\times$ satisfying $\chi_i^y(\varpi) = -1$ $(y \le i \le 2y-1)$. Thus

$$\mathrm{Ind}_K^{H_y}(1_{H_y} \oplus \eta') = \bigoplus_{i=0}^{2y-1}\chi_i,$$

where $\chi_0, \ldots, \chi_{2y-1}$ are all the distinct unramified characters of $K^\times$ satisfying $\chi_i(\varpi)^{2y} = 1$, and

$$\langle 1, \tau'\rangle + \langle \eta', \tau'\rangle = \sum_{i=0}^{2y-1}\langle \chi_i, \tau\rangle.$$

Since τ has a real-valued character, for each χ_i of order greater than 2, $\langle \chi_i, \tau\rangle$ will appear in this sum twice, i.e.,

$$\langle 1, \tau'\rangle + \langle \eta', \tau'\rangle \equiv \langle 1, \tau\rangle + \langle \eta, \tau\rangle \pmod 2. \qquad \text{(A.0.13)}$$

Finally, by Frobenius reciprocity,

$$\langle \hat{\gamma}', \tau'\rangle = \langle \mathrm{Ind}_{H_y}^{H_x}(\phi \otimes \theta), \mathrm{Res}_K^{H_y}\tau\rangle = \langle \mathrm{Ind}_K^{H_x}(\phi \otimes \theta), \tau\rangle = \langle \hat{\gamma}, \tau\rangle. \qquad \text{(A.0.14)}$$

Now formulas (A.0.9) – (A.0.14) imply

$$W(\gamma \otimes \tau) = \det\tau(-1)^y \cdot \varphi^{\dim\tau} \cdot (-1)^{\langle 1,\tau\rangle + \langle \eta,\tau\rangle + \langle \hat{\gamma},\tau\rangle}. \qquad \text{(A.0.15)}$$

Applying (A.0.15) to $\mu_1, \ldots, \mu_a$ and substituting the result into (A.0.8) we get the statement of the proposition. $\square$

Appendix B

We keep the notation of Section 2.1.

Lemma B.0.8. *Let* $\sigma' = (\sigma, N)$ *be the representation of* $\mathcal{W}'(\overline{K}/K)$ *associated to the natural* l*-adic representation of* $\mathrm{Gal}(\overline{K}/K)$ *on* $V_l(A)^*$ *and let* $\sigma' \cong \gamma \oplus (\delta \otimes \mathrm{sp}(2))$ *for some representations* γ *and* δ *of* $\mathcal{W}(\overline{K}/K)$*. Then* $\dim \delta = r$*.*

Proof. Since $\dim \delta = \mathrm{rank}\, N$ and for any finite extension $L \subset \overline{K}$ of K we have $\mathrm{Res}_{\mathcal{W}'(\overline{K}/L)} \sigma' = (\mathrm{Res}^L_K \sigma, N)$ ([Ro1], p. 130), we can assume that T splits over K and B has good reduction over K.

We have the following exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules ([Ra], p. 312):

$$0 \longrightarrow G(\overline{K})_{l^n} \longrightarrow A(\overline{K})_{l^n} \xrightarrow{\phi_{l^n}} Y(\overline{K})/l^n Y(\overline{K}) \longrightarrow 0. \tag{B.0.1}$$

Since $G(\overline{K})$ is divisible, sequence (B.0.1) induces an exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence of l-adic Tate modules:

$$0 \longrightarrow T_l(G) \longrightarrow T_l(A) \longrightarrow Z \longrightarrow 0,$$

where $Z = \varprojlim (Y(\overline{K})/l^n Y(\overline{K}))$ with the maps being the natural quotient maps. By tensoring the above sequence with $\mathbb{Q}_l$ over $\mathbb{Z}_l$ we get the following exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence:

$$0 \longrightarrow V_l(G) \longrightarrow V_l(A) \stackrel{\phi}{\longrightarrow} Z \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \longrightarrow 0. \tag{B.0.2}$$

For a positive integer n let μ_n denote the group of n-th roots of unity in $\overline{K}$, $T_l(\mu) = \varprojlim \mu_{l^n}$ with the l-th power maps. Let K_{l^n} be the tamely ramified extension of K^{unr} of degree l^n and let $t_{l^n} : I \longrightarrow \mu_{l^n}$ be the composition of the restriction map onto $\mathrm{Gal}(K_{l^n}/K^{unr})$ with the isomorphism $\mathrm{Gal}(K_{l^n}/K^{unr}) \cong \mu_{l^n}$.

Let $x_{l^n} \in A(\overline{K})_{l^n}$, $i \in I$, and $\phi_{l^n}(x_{l^n}) = [y]$ for some $y \in Y(\overline{K})$ and ϕ_{l^n} given by (B.0.1). Then a formula on p. 314 in [Ra] yields:

$$i(x_{l^n}) = x_{l^n} + \nu_{l^n}(y \otimes t_{l^n}(i)), \tag{B.0.3}$$

where $\nu_{l^n} : Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n} \longrightarrow T(\overline{K})_{l^n}$ is the following composition of $\mathrm{Gal}(\overline{K}/K)$-module homomorphisms:

$$Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X(\overline{K}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mu_{l^n} \stackrel{\sim}{\longrightarrow} T(\overline{K})_{l^n},$$

where X is the character group of T and the first map is induced by the geometric monodromy

$$\mu_0 : Y \times X \longrightarrow \mathbb{Z};$$

finally, $\nu_{l^n}(y \otimes t_{l^n}(i)) \in T(\overline{K})_{l^n}$ is considered as an element of $A(\overline{K})_{l^n}$ via the inclusions

$$T(\overline{K})_{l^n} \hookrightarrow G(\overline{K})_{l^n} \hookrightarrow A(\overline{K})_{l^n}.$$

For each $i \in I$ we have the following maps $\alpha_n(i) : Y(\overline{K})/l^nY(\overline{K}) \longrightarrow T(\overline{K})_{l^n}$ given by the following composition:

$$Y(\overline{K})/l^nY(\overline{K}) \xrightarrow{\psi_n(i)} Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n} \xrightarrow{\nu_{l^n}} T(\overline{K})_{l^n},$$

where $\psi_n(i)([y]) = y \otimes t_{l^n}(i)$, $y \in Y(\overline{K})$. It is easy to show that $\{\alpha_n(i)\}$ induce the homomorphism

$$\alpha(i) = (\alpha_n(i)) : Z \longrightarrow T_l(T),$$

where $Z = \varprojlim(Y(\overline{K})/l^nY(\overline{K}))$. By extending scalars to $\mathbb{Q}_l$ we get

$$\alpha'(i) : Z \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \longrightarrow V_l(T). \qquad \text{(B.0.4)}$$

Let $\beta_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l(A))$ be the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)$. Then (B.0.3) and (B.0.4) imply:

$$\beta_l(i) = \mathrm{id} + \alpha'(i) \circ \phi, \quad i \in I,$$

where $\mathrm{id} : V_l(A) \longrightarrow V_l(A)$ is the identity map and ϕ is given by (B.0.2). On the other hand,

$$\beta_l(i) = \exp(a_l(i)R_l),$$

where i is in some open subgroup J of I, $a_l : I \longrightarrow \mathbb{Q}_l$ is a nontrivial continuous homomorphism, and R_l is a nilpotent endomorphism on $V_l(A)$ ([Ro1], Prop. on p. 131). Since $\sigma' = (\sigma, N)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ associated to $\beta_l^* : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l(A)^*)$, it follows that N is obtained from $-R_l^t$ by extending scalars via a field embedding $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. Thus, $\mathrm{rank}\, R_l = \mathrm{rank}\, N = \dim \delta$ and

$R_l^2 = N^2 = 0$ by assumption. Thus,

$$\alpha'(i) \circ \phi = a_l(i) R_l, \quad i \in J,$$

and, since ϕ is surjective, it is enough to show that there exists $i_0 \in I$ such that $\alpha'(i_0)$ is surjective.

There exists $i_0 \in I$ such that $t_{l^n}(i_0)$ is a generator of μ_{l^n} for each n. It implies that $\psi_n(i_0)$ is an isomorphism for each n, hence it is enough to show that the map

$$\nu' : \varprojlim (Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \longrightarrow V_l(T)$$

induced by (ν_{l^n}) is surjective. Since μ_0 is nondegenerate ([F-C], p. 52, Remark 6.3), we have the following exact sequence:

$$0 \longrightarrow Y(\overline{K}) \xrightarrow{g} \mathrm{Hom}_{\mathbb{Z}}(X(\overline{K}), \mathbb{Z}) \longrightarrow M \longrightarrow 0,$$

where $g(y) = \mu_0(y, \cdot)$ and M is finite, since $Y(\overline{K})$ and $\mathrm{Hom}_{\mathbb{Z}}(X(\overline{K}), \mathbb{Z})$ are free abelian groups of the same rank r. Applying the functor $(-) \otimes_{\mathbb{Z}} \mu_{l^n}$ to the above sequence, we get:

$$Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n} \xrightarrow{\nu_{l^n}} T(\overline{K})_{l^n} \longrightarrow M \otimes_{\mathbb{Z}} \mu_{l^n} \longrightarrow 0,$$

hence the exact sequence

$$0 \longrightarrow \mathrm{im}\, \nu_{l^n} \longrightarrow T(\overline{K})_{l^n} \longrightarrow M/l^n M \longrightarrow 0.$$

Since $Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n}$ is a finite group, $\mathrm{im}\, \nu_{l^n}$ is a finite group, hence $\{\mathrm{im}\, \nu_{l^n}\}$ satisfies the Mittag-Leffler condition and we have the following exact sequence:

$$0 \longrightarrow \varprojlim (\mathrm{im}\, \nu_{l^n}) \longrightarrow T_l(T) \longrightarrow \varprojlim (M/l^n M) \longrightarrow 0. \qquad \text{(B.0.5)}$$

Here $\varprojlim(\operatorname{im}\nu_{l^n}) \cong \operatorname{im}\nu$, where

$$\nu = (\nu_{l^n}) : \varprojlim(Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n}) \longrightarrow T_l(T).$$

Indeed, let $S_n = Y(\overline{K}) \otimes_{\mathbb{Z}} \mu_{l^n}$, then we have an exact sequence

$$0 \longrightarrow \ker\nu_{l^n} \longrightarrow S_n \longrightarrow \operatorname{im}\nu_{l^n} \longrightarrow 0,$$

where the maps from S_n to $\operatorname{im}\nu_{l^n}$ are induced by ν_{l^n}. Since $\ker\nu_{l^n}$ is finite for each n, $\{\ker\nu_{l^n}\}$ satisfies the Mittag-Leffler condition, hence one has the following exact sequence

$$0 \longrightarrow \varprojlim(\ker\nu_{l^n}) \longrightarrow \varprojlim S_n \longrightarrow \varprojlim(\operatorname{im}\nu_{l^n}) \longrightarrow 0, \tag{B.0.6}$$

which together with (B.0.5) implies $\varprojlim(\operatorname{im}\nu_{l^n}) \cong \operatorname{im}\nu$.

Thus, applying the exact functor $(-) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ to (B.0.5) and taking into account that M is finite, we get:

$$0 \longrightarrow (\operatorname{im}\nu) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \longrightarrow V_l(T) \longrightarrow 0,$$

which implies

$$\operatorname{im}\nu' \cong (\operatorname{im}\nu) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong V_l(T),$$

hence ν' is surjective. □

Appendix C

The following example was suggested by R. Gow. Let Q be the quaternion group and let $A = X \rtimes Y$ be the semidirect product of a cyclic group $X = \langle x \rangle$ of order 3 generated by an element x and a cyclic group $Y = \langle y \rangle$ of order 4 generated by an element y, with X normal. Let $G = Q \times A$.

Proposition C.0.9. *G has an irreducible complex finite-dimensional orthogonal representation with Schur index 2 over the rationals.*

Proof. Let $\phi : Q \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be a representation of Q given by the following formulas on the generators i and j of Q:

$$\phi(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \phi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that ϕ is irreducible and has $\mathbb{Q}$-valued character. Let

$$\psi : A \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

be a representation of A given by the following formulas on the generators x and y of

A:

$$\psi(x) = \begin{pmatrix} \xi & 0 \\ 0 & \bar{\xi} \end{pmatrix}, \quad \psi(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\xi = \exp(\frac{2\pi i}{3})$ and $\bar{\xi}$ denotes the complex conjugate of ξ. In fact, $\psi = \mathrm{Ind}^{A}_{X \rtimes \langle y^2 \rangle} \alpha$, where α is a 1-dimensional representation of $X \rtimes \langle y^2 \rangle$ given by $\alpha(x) = \xi$ and $\alpha(y^2) = -1$. It is easy to check that ψ is irreducible and has $\mathbb{Q}$-valued character. Thus, $\sigma = \phi \otimes \psi : G \longrightarrow \mathrm{GL}_4(\mathbb{C})$ is irreducible and has $\mathbb{Q}$-valued character. It is easy to check by Frobenius-Schur method that σ is orthogonal, i.e., has Schur index 1 over the reals. We claim that σ has Schur index 2 over the rationals. Since σ has $\mathbb{Q}$-valued character, by Brauer-Speiser theorem it is enough to show that σ is not realizable over $\mathbb{Q}$. To do so, we first find the decomposition of the group algebra $\mathbb{Q}[G]$ into simple factors and then show that neither of them corresponds to σ.

It is known that there is the following isomorphism of algebras over $\mathbb{Q}$:

$$\mathbb{Q}[G] \cong \mathbb{Q}[Q] \otimes_{\mathbb{Q}} \mathbb{Q}[A].$$

Thus, we need to find the decompositions of $\mathbb{Q}[Q]$ and $\mathbb{Q}[A]$.

Since $\{\pm 1\}$ is a normal subgroup of Q and $Q/\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, Q has 4 distinct 1-dimensional representations realizable over $\mathbb{Q}$, which together with ϕ are all the irreducible complex representations of Q (up to isomorphism). On the other hand, the natural embedding of Q into the quaternion ring $\mathbb{H}_{\mathbb{Q}}$ over $\mathbb{Q}$ defines a surjective homomorphism $\mathbb{Q}[Q] \longrightarrow \mathbb{H}_{\mathbb{Q}}$, hence

$$\mathbb{Q}[Q] \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{H}_{\mathbb{Q}}. \tag{C.0.1}$$

Thus, Q has a unique irreducible representation over $\mathbb{Q}$ of degree 4 which must be $\phi \oplus \phi$.

It is easy to check that A has 5 conjugacy classes of cyclic subgroups, hence A has 5 irreducible representations over $\mathbb{Q}$ (up to isomorphism). Since $A/X \cong \mathbb{Z}/4\mathbb{Z}$, A has 2 distinct 1-dimensional representations realizable over $\mathbb{Q}$ and one 2-dimensional representation λ over $\mathbb{Q}$ given by

$$\lambda(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that λ is irreducible over $\mathbb{Q}$. Also, A has the following irreducible (over $\mathbb{C}$) representation $\mu = \mathrm{Ind}^{A}_{X \rtimes \langle y^2 \rangle} \beta$, where $\beta(x) = \xi$, $\beta(y^2) = 1$. It is easy to check that $\mu \cong \nu$, where ν is given by

$$\nu(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \nu(y) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

which implies that μ is realizable over $\mathbb{Q}$. Since the simple factors of $\mathbb{Q}[A]$ corresponding to λ and μ are $\mathbb{Q}(i)$ and $M_2(\mathbb{Q})$ respectively, we get

$$\mathbb{Q}[A] = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}(i) \times M_2(\mathbb{Q}) \times U,$$

where U is a simple algebra corresponding to the last 5-th irreducible representation η of A over $\mathbb{Q}$. It follows that $\dim \eta = 4$ and $\eta \cong \psi \oplus \psi$. We have

$$\eta \cong \psi \oplus \psi \cong \mathrm{Ind}^G_H \alpha \oplus \mathrm{Ind}^G_H \alpha^* \cong \mathrm{Ind}^G_H (\alpha \oplus \alpha^*),$$

where $H = A \rtimes \langle y^2 \rangle$, α^* is the contragredient of α, and $\alpha \oplus \alpha^*$ is isomorphic to the representation γ of H given by

$$\gamma(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \gamma(y^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since all the simple algebras in the decomposition (C.0.1) of $\mathbb{Q}[Q]$ are central and simple, the simple algebras in the decomposition of $\mathbb{Q}[G]$ will be isomorphic to the tensor products of the simple algebras appearing in the decomposition of $\mathbb{Q}[Q]$ with the simple algebras appearing in the decomposition of $\mathbb{Q}[A]$. This implies that the irreducible representations of G over $\mathbb{Q}$ are isomorphic to the tensor products of the irreducible representations of Q over $\mathbb{Q}$ with the irreducible representations of A over $\mathbb{Q}$. Thus, G has the following list of irreducible representations over $\mathbb{Q}$:

- 8 1-dimensional representations;
- 8 2-dimensional representations;
- 6 4-dimensional representations, namely $(\phi \oplus \phi) \otimes \pi_i$, where π_1, π_2 are 1-dimensional representations of A realizable over $\mathbb{Q}$ and $\omega_i \otimes (\psi \oplus \psi)$, where $\omega_1, \ldots, \omega_4$ are 1-dimensional representations of Q realizable over $\mathbb{Q}$;
- 2 8-dimensional representations;
- 1 16-dimensional representation.

Since σ is a 4-dimensional representation of G irreducible over $\mathbb{C}$, it follows that σ is not from this list, hence σ is not realizable over $\mathbb{Q}$. $\square$

Appendix D

Proof of Proposition 4.2.1. Let $\Gamma = T(\overline{K})/\Lambda$. We have the following exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$0 \longrightarrow \Lambda \longrightarrow T(\overline{K}) \longrightarrow \Gamma \longrightarrow 0. \tag{D.0.1}$$

Since $\Lambda \cong \mathbb{Z}^s$ and $T(\overline{K})$ is a divisible group, this sequence induces the following exact $\mathrm{Gal}(\overline{K}/K)$-equivariant sequence of l-adic Tate modules:

$$0 \longrightarrow T_l(T) \longrightarrow T_l(\Gamma) \longrightarrow \chi_1 \otimes \mathbb{Z}_l^s \longrightarrow 0, \tag{D.0.2}$$

where $T_l(T)$ denotes $T_l(T(\overline{K}))$. Let $L \subset \overline{K}$ be a finite Galois extension of K over which T splits. Since $T_l(T)$ is a free $\mathbb{Z}_l$-module of rank r it follows from (D.0.2) that $T_l(\Gamma)$ is a free $\mathbb{Z}_l$-module of rank $s + r$, hence by Proposition D.0.10 below we have

$$\mathrm{Res}_K^L \rho' \cong (\omega_L^{-1})^{\oplus(r-s)} \oplus (\omega_L^{-1} \otimes \mathrm{sp}(2))^{\oplus s}, \tag{D.0.3}$$

where $\omega_L = \mathrm{Res}_K^L \omega$. The rest of the proof is similar to the proof of Proposition 2.3.1. Since $\mathrm{Res}_{\mathcal{W}'(\overline{K}/L)}\rho' = (\mathrm{Res}_K^L \rho, S)$ and $\mathrm{Res}_K^L \rho$ is semisimple by (D.0.3), ρ' is admissible by Lemma 2.2.1. Hence it has the following form:

$$\rho' \cong \bigoplus_{i=1}^{t} \alpha_t \otimes \mathrm{sp}(n_t), \tag{D.0.4}$$

where each α_i is a representation of $\mathcal{W}(\overline{K}/K)$ and each n_i is a positive integer ([Ro1], p. 133, Cor. 2). Also, it follows from (D.0.3) that $S^2 = 0$ and $\operatorname{rank} S = s$. Thus, each n_i in (D.0.4) is 1 or 2 and

$$\rho' \cong \alpha \oplus (\beta \otimes \mathrm{sp}(2)), \tag{D.0.5}$$

where α is a representation of $\mathcal{W}(\overline{K}/K)$ of dimension $r - s$ and β is a representation of $\mathcal{W}(\overline{K}/K)$ of dimension s.

Applying the exact functor $(-) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ to (D.0.2) and taking duals afterwards, we get

$$0 \longrightarrow \chi_1 \otimes \mathbb{Q}_l^s \longrightarrow V_l(\Gamma)^* \longrightarrow V_l(T)^* \longrightarrow 0, \tag{D.0.6}$$

where $V_l(T) = V_l(T(\overline{K}))$ and $\chi_1 \cong (\chi_1)^*$, since χ_1 is a representation of finite image, realizable over $\mathbb{Z}$. Sequence (D.0.6) induces an exact sequence of corresponding representations of $\mathcal{W}'(\overline{K}/K)$, i.e.,

$$0 \longrightarrow (\chi_1 \otimes \mathbb{Q}_l^s) \otimes_\imath \mathbb{C} \longrightarrow V_l(\Gamma)^* \otimes_\imath \mathbb{C} \longrightarrow V_l(T)^* \otimes_\imath \mathbb{C} \longrightarrow 0 \tag{D.0.7}$$

is an exact sequence of $\mathcal{W}'(\overline{K}/K)$-modules. Moreover, χ_1 is the representation of $\mathcal{W}'(\overline{K}/K)$ on $(\chi_1 \otimes \mathbb{Q}_l^s) \otimes_\imath \mathbb{C}$, $\rho' = (\rho, S)$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(\Gamma)^* \otimes_\imath \mathbb{C}$, and by Lemma 2.3.2, $\chi \otimes \omega^{-1}$ is the representation of $\mathcal{W}'(\overline{K}/K)$ on $V_l(T)^* \otimes_\imath \mathbb{C}$. Since ρ is semisimple, the exact sequence (D.0.7) of $\mathcal{W}(\overline{K}/K)$-modules splits, i.e.,

$$\rho \cong \chi_1 \oplus (\chi \otimes \omega^{-1}).$$

On the other hand, from (D.0.5) we have:

$$\rho \cong \alpha \oplus \beta \oplus (\beta \otimes \omega).$$

Thus, combining the last two congruences, we get

$$\alpha \oplus \beta \oplus (\beta \otimes \omega) \cong \chi_1 \oplus (\chi \otimes \omega^{-1}). \tag{D.0.8}$$

We claim that $\beta \otimes \omega$ is isomorphic to a subrepresentation of χ_1. Suppose there is an irreducible component β_0 of β such that $\beta_0 \otimes \omega$ is isomorphic to a subrepresentation of $\chi \otimes \omega^{-1}$, i.e.,

$$\beta_0 \otimes \omega \cong x \otimes \omega^{-1} \tag{D.0.9}$$

for some irreducible component x of χ. It follows from (D.0.8) that β_0 is isomorphic to a subrepresentation of $\chi \otimes \omega^{-1}$ or χ_1, which is impossible, because x, χ, and χ_1 have finite images, whereas ω does not. Indeed, suppose $\beta_0 \cong y \otimes \omega^{-1}$ or $\beta_0 \cong z$, where y is an irreducible component of χ and z is an irreducible component of χ_1. From (D.0.9) we get

$$\beta_0 \cong x \otimes \omega^{-2},$$

hence

$$x \otimes \omega^{-2} \cong y \otimes \omega^{-1} \quad \text{or} \quad x \otimes \omega^{-2} \cong z.$$

By taking determinants of both sides in each case, we get

$$\frac{\det x}{\det y} = \omega^{-m+2k} \quad \text{or} \quad \frac{\det x}{\det z} = \omega^{2k}, \tag{D.0.10}$$

where $m = \dim y$, $k = \dim x$. Since x, y, and z have finite images (as being subrepresentations of χ or χ_1) and ω has infinite image, (D.0.10) gives a contradiction.

Thus, $\beta \otimes \omega$ is isomorphic to a subrepresentation of χ_1. Since $\beta \otimes \omega$ and χ_1 have the same dimension s, we have $\beta \otimes \omega \cong \chi_1$, hence $\beta \cong \chi_1 \otimes \omega^{-1}$. By the uniqueness of decomposition of a semisimple module into simple modules, we conclude from (D.0.8) that $\alpha \cong \chi_2 \otimes \omega^{-1}$. □

Proposition D.0.10. *Let $\Lambda \subset (K^\times)^r$ be a free discrete subgroup of rank s ($s \leq r$) and denote $(\overline{K}^\times)^r/\Lambda$ by Γ. Let $\rho' = (\rho, S)$ be the representation of $\mathcal{W}'(\overline{K}/K)$ associated to the l-adic representation of* $\mathrm{Gal}(\overline{K}/K)$ *on $V_l(\Gamma)^*$. Let $T_l(\Gamma)$ be a free $\mathbb{Z}_l$-module of rank $s + r$. Then*

$$\rho' \cong (\omega^{-1})^{\oplus(r-s)} \oplus (\omega^{-1} \otimes \mathrm{sp}(2))^{\oplus s}.$$

Proof. Let $p_1, \ldots, p_s \in \Lambda$ be a basis of Λ, satisfying the assertion of Lemma D.0.12 below. First, let us choose a $\mathbb{Q}_l$-basis for $V_l = V_l(\Gamma)$. Let $f_1 = (f_1(n)), \ldots, f_s = (f_s(n)) \in T_l(\Gamma)$, where $f_1(n), \ldots, f_s(n)$ as elements of $(\overline{K}^\times)^r$ have the following form:

$$f_1(n)^{l^n} = p_1, \quad \ldots \quad , f_s(n)^{l^n} = p_s \quad \text{and}$$

$$f_1(n+1)^l = f_1(n), \quad \ldots \quad , f_s(n+1)^l = f_s(n).$$

Let $\xi = (\xi(n))$, where each $\xi(n) \in \overline{K}^\times$ is a primitive l^n-th root of unity and $\xi(n+1)^l = \xi(n)$. Let $f_{s+1} = (f_{s+1}(n)), \ldots, f_{s+r} = (f_{s+r}(n)) \in T_l(\Gamma)$, where $f_{s+1}(n), \ldots, f_{s+r}(n)$

as elements of $(\overline{K}^{\times})^r$ satisfy the following properties:

$$\begin{aligned} f_{s+1}(n) &= (\xi(n), 1, \ldots, 1), \\ f_{s+2}(n) &= (1, \xi(n), \ldots, 1), \\ &\ldots \\ f_{s+r}(n) &= (1, 1, \ldots, \xi(n)). \end{aligned}$$

Then $f_1, \ldots, f_{s+r}$ is a basis of V_l. Indeed, it is easy to check that $f_1, \ldots, f_{s+r}$ are linearly independent over $\mathbb{Z}_l$. Since $T_l(\Gamma)$ is a free $\mathbb{Z}_l$-module of rank $s+r$, it follows that $f_1, \ldots, f_{s+r}$ is a basis of V_l.

Let $\rho_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l)$ be the l-adic representation associated to the $\mathrm{Gal}(\overline{K}/K)$-module V_l. Then the matrix representation of ρ_l with respect to the basis $f_1, \ldots, f_{s+r}$ has the following form:

$$\rho_l(\Phi) = \begin{pmatrix} E_s & 0 \\ * & q^{-1} \cdot E_r \end{pmatrix}, \quad \rho_l(i) = \begin{pmatrix} E_s & 0 \\ B(i) & E_r \end{pmatrix}, \tag{D.0.11}$$

where E_r and E_s are the identity matrices, $i \in I$, and $B(i) \in \mathrm{Mat}_{r \times s}(\mathbb{Q}_l)$. It is known that there exists a nilpotent endomorphism S_l of V_l^* such that S is obtained from S_l by extending of scalars via a field embedding $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$; moreover, S_l is a unique nilpotent endomorphism such that

$$\rho_l^*(i) = \exp(t_l(i) S_l), \tag{D.0.12}$$

where $t_l : I \longrightarrow \mathbb{Q}_l$ is a nontrivial continuous homomorphism and i belongs to an open subgroup of I. Furthermore, for any $g \in \mathcal{W}(\overline{K}/K)$ we have

$$\rho(g) = \rho_l^*(g)\exp(-t_l(i)S_l), \tag{D.0.13}$$

where $\rho_l^*(g)\exp(-t_l(i)S_l)$ is considered as an element of $\mathrm{GL}(V_l^* \otimes_\imath \mathbb{C})$ via $\imath$ ([Ro1], p. 131, Prop.(i), (ii)). Formula (D.0.11) for $\rho_l(\Phi)$ implies that, considered as a matrix over $\mathbb{C}$ via $\imath$, it is diagonalizable. It follows from Formula (D.0.13) that $\rho(\Phi)$ is diagonalizable, hence ρ is semisimple and ρ' is admissible by Lemma 2.2.1.

Let ϖ be a uniformizer of K and let $\Pi = (\varpi(n))$, where each $\varpi(n) \in \overline{K}^\times$ has the following property:

$$\varpi(n)^{l^n} = \varpi \quad \text{and} \quad \varpi(n+1)^l = \varpi(n).$$

There exists $i_0 \in I$ such that

$$i_0(\Pi) = (i_0(\varpi(n))) = (\varpi(n)\xi(n)^{\alpha(n)}) = \xi^\alpha \Pi,$$

where $\alpha = (\alpha(n)) \in \mathbb{Z}_l$. By Lemma D.0.11 below $\alpha \neq 0$.

Lemma D.0.11. *Let $g \in \mathcal{O}$ and let $g_n \in \overline{K}$ denote a root of $x^{l^n} - g = 0$. Then $i(g_n) = g_n$ for any $i \in I$ and $n \in \mathbb{N}$ if and only if $g \in \mathcal{O}^\times$.*

Proof. Clearly, $i(g_n) = g_n$ for any $i \in I$ and $n \in \mathbb{N}$ if and only if $K(g_n)$ is unramified over K for any n.

Let $g \in \mathcal{O}^\times$. Then the assertion follows from the fact that $x^{l^n} - g$, considered as a polynomial in $k[x]$, has no multiple root ([L], p. 48, Prop. 7).

Conversely, since $g_n^{l^n} = g$, the valuation of g in $K(g_n)$ is divisible by l^n. Since $K(g_n)$ is unramified over K for any n, the valuation v_g of g in K coincides with the valuation of g in $K(g_n)$, hence v_g is divisible by l^n for any n, which implies that v_g must be zero and $g \in \mathcal{O}^\times$. □

Let $p_k = (p_{kj})$, $p_{kj} \in K^\times$, $1 \le k \le s$, $1 \le j \le r$. It follows from Lemma D.0.12 below that without loss of generality we can assume that $p_{kk} \notin \mathcal{O}^\times$ for any k and that $p_{kj} \in \mathcal{O}^\times$ whenever $k > j$, $1 \le j \le r$. Thus there exist $u_k \in \mathcal{O}^\times$ and $m_k \in \mathbb{Z}^\times$ such that $p_{kk} = u_k \cdot \varpi^{m_k}$. Let $(u_k(n))$ be a sequence in $\overline{K}^\times$ such that

$$u_k(n)^{l^n} = u_k \quad \text{and} \quad u_k(n+1)^l = u_k(n).$$

For $f_k(n) \in (\overline{K}^\times)^r$ write $f_k(n) = (f_{kj}(n))$, where $f_{kj}(n) \in \overline{K}^\times$, $1 \le k \le s$, and $1 \le j \le r$. Then as $f_{kk}(n)$ we can take $u_k(n) \cdot \varpi(n)^{m_k}$. For i_0 we have

$$i_0(f_1) = (i_0(f_1(n))) = (i_0(f_{11}(n)), i_0(f_{12}(n)), \ldots, i_0(f_{1r}(n))),$$

where by Lemma D.0.11 we have:

$$i_0(f_{11}(n)) = u_1(n) \cdot \varpi(n)^{m_1} \cdot \xi(n)^{\alpha(n)m_1} = f_{11}(n) \cdot \xi(n)^{\alpha(n)m_1}.$$

Analogously, using Lemma D.0.11, we get the following formulas:

$$\begin{aligned} i_0(f_1) &= f_1 \cdot f_{s+1}^{\alpha m_1} \cdot f_{s+2}^{a_2} \cdots f_{s+r}^{a_r}, \\ i_0(f_2) &= f_2 \cdot f_{s+2}^{\alpha m_2} \cdot f_{s+3}^{b_3} \cdots f_{s+r}^{b_r}, \\ &\cdots \\ i_0(f_s) &= f_s \cdot f_{2s}^{\alpha m_s} \cdot f_{2s+1}^{c_{s+1}} \cdots f_{s+r}^{c_r} \end{aligned}$$

for some $a_i, b_j, \dots, c_k \in \mathbb{Z}_l$. This implies that $B(i_0)$ has the following form:

$$B(i_0) = \begin{pmatrix} \alpha m_1 & 0 & \dots & 0 \\ * & \alpha m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & \alpha m_s \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

where $\alpha, m_1, \dots, m_s \in \mathbb{Z}_l^\times$, hence $\operatorname{rank} B(i_0) = s$.

Since ρ' is admissible,

$$\rho' \cong \bigoplus_{j=1}^{k} \pi_j \otimes \operatorname{sp}(n_j), \tag{D.0.14}$$

where $\pi_1, \dots, \pi_k$ are representations of $\mathcal{W}(\overline{K}/K)$ ([Ro1], p. 133, Cor. 2). Since $\rho_l^*(i) = \exp(t_l(i)S_l)$ by (D.0.12) and $(\rho_l(i) - E_{s+r})^2 = 0$ from (D.0.11), it follows that $(S_l)^2 = 0$, i.e., each n_j in (D.0.14) is 1 or 2, hence

$$\rho' \cong \alpha \oplus (\beta \otimes \operatorname{sp}(2)),$$

where α and β are representations of $\mathcal{W}(\overline{K}/K)$. Since $\operatorname{rank} B(i_0) = s$, the equation (D.0.12) implies that $\operatorname{rank} S = \operatorname{rank} S_l = s$, hence $\dim \beta = s$ and $\dim \alpha = r - s$.

Let us prove now that $\alpha \cong \bigoplus_{r-s} \omega^{-1}$ and $\beta \cong \bigoplus_s \omega^{-1}$. It can be easily verified that (D.0.11) – (D.0.13) imply

$$\rho(g) = \begin{pmatrix} E_s & * \\ 0 & (\omega^{-1})^{\oplus r} \end{pmatrix}, \quad g \in \mathcal{W}(\overline{K}/K), \tag{D.0.15}$$

hence there is a complete flag of subrepresentations

$$(0) \neq W_1 \subset \dots \subset W_{s+r} = V_l^* \otimes_\iota \mathbb{C}$$

of ρ. Since ρ is semisimple, it implies that ρ is a direct sum of one-dimensional subrepresentations, hence from (D.0.15)

$$\alpha \cong \bigoplus_{r-s} \omega^{-1} \quad \text{and} \quad \beta \cong \bigoplus_{s} \omega^{-1}.$$

□

Lemma D.0.12. *Let K be a non-Archimedean local field with ring of integers $\mathcal{O}$. Let $\Lambda \subset (K^\times)^r$ be a free discrete subgroup of rank s ($s \leq r$). There exist a basis $p_1, \ldots, p_s$ of Λ and natural numbers $n_1, \ldots, n_s$ ($1 \leq n_1 < n_2 < \cdots < n_s \leq r$) with the following property: if $p_k = (p_{kj})$, $1 \leq k \leq s$, $1 \leq j \leq r$, and $p_{kj} \in K^\times$, then $p_{in_i} \notin \mathcal{O}^\times$ for any i and $p_{ln_i} \in \mathcal{O}^\times$ whenever $l > i$.*

Proof. First, note that $(\mathcal{O}^\times)^r \cap \Lambda = \{1\}$. Indeed, if $x \in (\mathcal{O}^\times)^r \cap \Lambda$ and $x \neq 1$, then (x^n) is an infinite sequence in $(\mathcal{O}^\times)^r \cap \Lambda$, hence it has a limit point, because $(\mathcal{O}^\times)^r$ is compact, which contradicts the assumption that Λ is discrete.

Let ϖ be a uniformizer of K. The map $\mathcal{O}^\times \times \mathbb{Z} \longrightarrow K^\times$ given by

$$(u, n) \mapsto u\varpi^n$$

is an isomorphism of topological groups. For every positive integer r it induces an isomorphism $(K^\times)^r \cong (\mathcal{O}^\times)^r \times \mathbb{Z}^r$. Let $\pi : (K^\times)^r \longrightarrow \mathbb{Z}^r$ be the projection onto $\mathbb{Z}^r$ and $t_1, \ldots, t_s$ be a basis of Λ. Since $(\mathcal{O}^\times)^r \cap \Lambda = \{1\}$, $\pi(t_1), \ldots, \pi(t_s)$ form a basis of $\pi(\Lambda)$. Indeed, otherwise, there exist $m_1, \ldots, m_s \in \mathbb{Z}$, not all of which are zeros, such that

$$m_1\pi(t_1) + \cdots + m_s\pi(t_s) = 0.$$

Then $1 \neq t_1^{m_1} \cdots t_s^{m_s} \in (\mathcal{O}^\times)^r \cap \Lambda$. Thus, $\pi(\Lambda) \subseteq \mathbb{Z}^r$ is a subgroup of rank s and it is enough to prove the following sublemma:

Sublemma D.0.13. *Let $G \subseteq \mathbb{Z}^r$ be a subgroup of rank s ($s \leq r$). There exist a basis $g_1, \ldots, g_s$ of G and natural numbers $n_1, \ldots, n_s$ ($1 \leq n_1 < n_2 < \cdots < n_s \leq r$) with the following property: if $g_k = (g_{kj})$, $1 \leq k \leq s$, $1 \leq j \leq r$, and $g_{kj} \in \mathbb{Z}$, then $g_{in_i} \neq 0$ for any i and $g_{ln_i} = 0$ whenever $l > i$.*

Indeed, if we assume Sublemma D.0.13, then there is a basis $g_1, \ldots, g_s$ of $\pi(\Lambda)$ with the property described in Sublemma D.0.13. Since $\pi(t_1), \ldots, \pi(t_s)$ is a basis of $\pi(\Lambda)$, there is a matrix $D = (d_{ij}) \in \mathrm{GL}_s(\mathbb{Z})$ such that

$$g_i = \sum_j d_{ij} \pi(t_j), \quad 1 \leq i \leq s.$$

Then $p_i = \prod_j t_j^{d_{ij}}$, $1 \leq i \leq s$, will be a basis of Λ with the required property. □

Proof of Sublemma D.0.13. Suppose $r = s$. We will prove the sublemma in this case by induction on r. Clearly, it holds when $r = 1$. Let r be arbitrary and $e_1, \ldots, e_r$ be the standard basis of $\mathbb{Z}^r$. There exists $k \in \mathbb{Z}^\times$ such that $G \cap (e_r) = (ke_r)$, because $me_r \in G$, where $m = |B|$ and $B = \mathbb{Z}^r/G$. Then $G/(ke_r) \subseteq \mathbb{Z}^{r-1}$ is a subgroup of rank $r-1$. By induction, there exist $g_1, \ldots, g_{r-1} \in G$ such that in $G/(ke_r)$ we have:

$$\bar{g}_i = \sum_{j=1}^{r-1} a_{ij} e_j, \quad 1 \leq i \leq r-1,$$

for some $a_{ij} \in \mathbb{Z}$ such that $a_{ii} \neq 0$ for any i and $a_{ij} = 0$ whenever $i > j$. Then $g_1, \ldots, g_{r-1}, g_r = ke_r$ will be a basis of G with the required property.

Suppose now that $s \neq r$. Let $q_1, \ldots, q_s$ be a basis of G. Then $q_i = \sum_j b_{ij} e_j$, where $B = (b_{ij}) \in \mathrm{Mat}_{s\times r}(\mathbb{Z})$. Since $q_1, \ldots, q_s$ is a basis, $\operatorname{rank} B = s$, i.e., there exists an $s \times s$-submatrix B_0 of B such that $\det B_0 \neq 0$. Let B_0 have the following form:

$$B_0 = \begin{pmatrix} b_{1n_1} & b_{1n_2} & \ldots & b_{1n_s} \\ b_{2n_1} & b_{2n_2} & \ldots & b_{2n_s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{sn_1} & b_{sn_2} & \ldots & b_{sn_s} \end{pmatrix}.$$

Then $p_i = \sum_j b_{in_j} e_{n_j}$, $1 \leq i \leq s$, are linearly independent, hence generate a free subgroup H of rank s in $\mathbb{Z}e_{n_1} \oplus \cdots \oplus \mathbb{Z}e_{n_s}$. By the case $r = s$ above there is a matrix $C \in \mathrm{GL}_s(\mathbb{Z})$ such that

$$\sum_i c_{ki} p_i = \sum_j h_{kn_j} e_{n_j},$$

where $h_{kn_j} \in \mathbb{Z}$, $h_{in_i} \neq 0$ for any i and $h_{ln_i} = 0$ whenever $l > i$. Then $g_k = \sum_i c_{ki} q_i$, $1 \leq k \leq s$, will be a basis of G with the required property. $\square$

Bibliography

[Cha] C. Chai, *Néron models for semiabelian varieties: congruence and change of base field*, Asian J. Math. **4** (2000), no. 4, 715–736.

[Che] C. Chevalley, "Algebra," Springer-Verlag, New-York, 1988.

[Chi] T. Chinburg, B. Erez, G. Pappas, and M. Taylor, *ε-constants and the Galois structure of de Rham cohomology*, Ann. of Math. **146** (1997), 411–473.

[F-C] G. Faltings and C. Chai, "Degeneration of abelian varieties," Springer-Verlag, Berlin, 1990.

[K] A. A. Kirillov, "Elements of the theory of representations," Springer-Verlag, Berlin-New York, 1976.

[L] S. Lang, "Algebraic number theory," 2nd ed., Springer-Verlag, New York, 1994.

[M] J. S. Milne, *Abelian varieties*, "Arithmetic Geometry," Springer-Verlag, New York, 1986, 103–150.

[Ra] M. Raynaud, *1-Motifs et monodromie géométrique*, Astérisque **223** (1994), 295–319.

[Ro1] D.E. Rohrlich, *Elliptic curves and the Weil-Deligne group*, Elliptic Curves and Related Topics, CRM Proceedings & Lecture Notes **4** (1994), Amer. Math. Soc., Providence, 125–157.

[Ro2] D.E. Rohrlich, *Galois theory, elliptic curves, and root numbers*, Compos. Math. **100** (1996), 311–349.

[Ro3] D.E. Rohrlich, *The vanishing of certain Rankin-Selberg convolutions*, Automorphic Forms and Analytic Number Theory, Les publications CRM, Montréal, 1990, 123–133.

[S] J.-P. Serre, "Linear representations of finite groups," Springer-Verlag, New York-Heidelberg, 1977.

[S-T] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. Math. **88** (1968), 492–517.

[T1] J. Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

[T2] J. Tate, *Number theoretic background*, Automorphic forms, Representations, and L-Functions, Proc. Symp. Pure Math. **33** – Part 2, Amer. Math. Soc., Providence (1979), 3–26.