BOTT-CHERN CHARACTERISTIC FORMS AND INDEX THEOREMS FOR COHERENT SHEAVES ON COMPLEX MANIFOLDS

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ABSTRACT

BOTT-CHERN CHARACTERISTIC FORMS AND INDEX THEOREMS FOR COHERENT SHEAVES ON COMPLEX MANIFOLDS

Hua Qiang

Jonathan Block

In the paper [Blo10], Block constructed a dg-category $\mathcal{P}_{\mathcal{A}^{0,\bullet}}$ using cohesive modules which is a dg-enhancement of $D^b_{\mathrm{Coh}}(X)$, the bounded derived category of complexes of analytic sheaves with coherent cohomology. This enables us to study coherent sheaves using global differential geometric constructions.

In the first part of my thesis, we construct natural superconnections in the sense of Quillen [Qui85] on cohesive modules and use them to define characteristic classes with values in Bott-Chern cohomology. In addition, we generalize the double transgression formulas in [BGS88a] [BC65] [Don87] and prove the invariance of these characteristic classes under derived equivalences. This provides an extension of Bott-Chern characteristic classes to coherent sheaves on complex manifolds and answers the question raised in [Bis13].

In the second part of my thesis, we define the generalized Dolbeault-Dirac operator on the generalized Dolbeault complex for a cohesive module. We identify it with a generalized Dirac operator in the sense of Clifford modules and Clifford superconnections as in [BGV91]. Applying the heat kernel method and a theorem of

Getzler in [Get91], we first derive a generalization of the Hirzebruch-Riemann-Roch formula to compute the Euler characteristic. Then, generalizing Bismut's proof of the family index theorem, we derive a generalization of the classical Grothendieck-Riemann-Roch formula with values in Bott-Chern cohomology in special cases.

Contents

Chapter 1

Introduction

Traditionally, the complex structure of a complex manifold X is encoded in the sheaf of holomorphic functions $\mathcal{O}_X$. For applications to noncommutative geometry, such local constructions are not available and we are forced to use global differential geometric constructions. When the manifold is projective, every coherent sheaf $\mathscr{S}$ admits a global resolution by holomorphic vector bundles:

$$0 \to E^n \to E^{n-1} \to ... \to E^1 \to E^0 \to \mathscr{S} \to 0$$

and we can apply the theory of holomorphic vector bundles to study $\mathscr{S}$. However, for a general compact complex manifold that is not projective, there may not exist a global resolution by holomorphic vector bundles and alternative methods are required.

Even though the sheaf of holomorphic sections of $\mathscr{S}$ does not admit global resolution by holomorphic vector bundles, the underlying sheaf of real analytic sections $\mathscr{S} \otimes_{\mathcal{O}_X} \mathcal{R}_X$ does admit a global resolution by real analytic vector bundles, see [AF61] for its application in topological K-theory.

In [Blo10][Blo06], Block constructed higher order differentials on such resolutions and the resulting geometric objects are called cohesive modules. He further showed that they form a dg-category $\mathcal{P}_{A^{0,\bullet}(X)}$ whose homotopy category $\mathrm{Ho}\mathcal{P}_{A^{0,\bullet}(X)}$ is equivalent to $D^b_{\mathrm{Coh}}(X)$, the bounded derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. Using the theory of superconnections defined in [Qui85], it's natural to extend the classical geometric constructions for holomorphic bundles to cohesive modules.

In chapter 2, we begin by review the theory of cohesive modules and then construct the analog of Chern connections on a Hermitian cohesive module in the following sense.

Proposition. *Given a cohesive module $(E^\bullet, \mathbb{E}'')$ with a Hermitian structure h_E, there exist an unique ∂^X-superconnection $\mathbb{E}''$ of total degree -1 such that the superconnection $\mathbb{E} = \mathbb{E}'' + \mathbb{E}'$ is unitary. That is, $\mathbb{E}$ satisfies the graded Leibniz rule:*

$$(-1)^{|s|} d^X h_E(s,t) = -h_E(\mathbb{E}s, t) + h_E(s, \mathbb{E}t)$$

for all $s, t \in \mathcal{A}^\bullet(X, E^\bullet)$.

We then study the basic properties of the curvature $\mathcal{R}^{\mathbb{E}}$ of the Chern superconnection $\mathbb{E}$. Though it is possibly no longer a $(1,1)$ form on X unless $\mathbb{E}$ is a single holomorphic vector bundle, if we define a new grading on $\mathcal{A}^{p,q}(X, \mathrm{End}^d(E^\bullet))$ by

$-p+q+d$, then the curvature $\mathcal{R}$ is of exotic degree zero. Applying the Chern-Weil theory for superconnections, we obtain characteristic forms with values in Bott-Chern cohomology, which is a refinement of deRham cohomology. Though it is well known in literature, we prove the deRham cohomology classes of a cohesive module only depends on the $\mathbb{Z}_2$-graded topological bundle structure by transgressing the characteristic forms defined by Chern superconnection to those defined by its connection component.

In chapter 3, we prove the characteristic classes in Bott-Chern cohomology are independent of the Hermitian metric by establishing several transgression formulas. These formulas were first obtained by Bott and Chern in [BC65]. To generalize them to cohesive modules, we study the universal cohesive module $\widetilde{E}$ on the space of Hermitian metrics $\mathcal{M}$ on E.

Theorem (Bott-Chern transgression formula)**.** *For any convergent power series $f(T)$ in T, the $\mathcal{M}$-directional derivative of the Bott-Chern characteristic form $\mathrm{Tr}_s f(\mathcal{R}_h)$ is given by:*

$$d^{\mathcal{M}}\mathrm{Tr}_s f(\mathcal{R}_h) = \partial^X \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)$$

where θ is the Mauer-Cartan form on $\mathcal{M}$ defined by $\theta = h^{-1}\cdot d^{\mathcal{M}}h$.

The forms $\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)$ appeared in the above double transgression formula is the holomorphic analog of Chern-Simons forms. In [Don87], Donaldson studied these secondary characteristic forms their relation to stability. Following Donaldson, the technical computations in chapter 3 establishes the following formula that generalizes Donaldson's result.

Theorem (Donaldson transgression formula for secondary class)**.** *If $g(T)$ is a convergent power series in T, the $\mathcal{M}$-directional derivative of the secondary forms* $\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta)$ *is given by*

$$d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta) = \frac{1}{2}\bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])\cdot\theta) - \frac{1}{2}\partial^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta)$$

Donaldson observed a close relationship between these secondary characteristic forms and stability conditions on holomorphic vector bundles, we hope these results can be properly generalized to cohesive modules.

In chapter 4, we prove the invariance of characteristic classes under homotopy equivalences between cohesive modules. By a criteria proved by Block in [Blo10], two cohesive modules E, F are homotopy equivalent if and only if there is a degree zero closed morphism $\phi \in \mathcal{P}^0_{\mathcal{A}^{0,\bullet}(X)}(E,F)$ which induces a quasi-isomorphism between $(E^\bullet, \mathbb{E}''_0)$ and $(F^\bullet, \mathbb{F}''_0)$. Using this result, we prove the invariance in two steps. First, we show that characteristic classes of an acyclic cohesive module are trivial based on a rescaling technique that is well known in literature.

Proposition. *Assume $(E^\bullet, \mathbb{E}'')$ is a cohesive module such that $(E^\bullet, \mathbb{E}''_0)$ is an acyclic complex. Let $\mathbb{E}''_t = \sum_k t^{(1-k)/2}\mathbb{E}''_k$ be the rescaled cohesive structure with parameter $t \in \mathbb{R}^+$ and associated curvature $\mathcal{R}_t$. Let N_E further be the grading operator on E,* *then the integral*

$$\mathcal{I}_E = \int_1^\infty \mathrm{Tr}_s\{\exp(-\mathcal{R}_t) \cdot N_E \frac{dt}{t}\}$$

is finite and the Chern character $\mathrm{ch}(E)$ *is* $\partial^X \bar{\partial}^X$*-exact as follows:*

$$\mathrm{ch}(E) = \mathrm{Tr}_s \exp(-\mathcal{R}) = \partial^X \bar{\partial}^X \mathcal{I}_E$$

This formula is of independent interest as it is related to the theory of analytic torsion studied in [RS73][BGS88a][BGS88b][BGS88c].

Then we show that the characteristic classes are additive with respect to short exact sequences of mapping cone in the following sense.

Proposition. *If* $0 \to E \xrightarrow{\phi} F \to \mathrm{Cone}(\phi)$ *is the mapping cone sequence for a morphism* $\phi \in \mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}(E, F)$*, then the Bott-Chern cohomology classes are additive in the sense that the equality*

$$f(E^\bullet, \mathbb{E}'') - f(F^\bullet, \mathbb{F}'') + f(\mathrm{Cone}^\bullet(\phi), \mathbb{C}_\phi) = 0$$

holds in Bott-Chern cohomology.

We prove this by explicitly constructing the form that provide the $\partial\bar{\partial}$-transgression among special representatives of the characteristic classes.

Combining these with the results obtained in the previous chapter, we see that the characteristic classes defined for cohesive modules descend to $D^b_{\mathrm{Coh}}(X)$ and extend Bott-Chern cohomology to coherent sheaves. This answers a question raised in [Bis13].

Theorem. *If $\mathcal{S}$ is an object in $D^b_{Coh}(X)$ and $(E^\bullet, \mathbb{E}'')$ is a cohesive module that resolves $\mathcal{S}$. Then for any convergent power series $f(T)$ in T, the characteristic class in Bott-Chern cohomology defined by $\mathrm{Tr}_s f(\mathcal{R}^{\mathbb{E}}_h)$ for a choice of hermitian metric h on $E^\bullet$ is independent of the representative cohesive module and the metric h on it.*

From chapter 5, we move on to applications of the characteristic classes defined in previous chapters. Our main result is an Atiyah-Singer type index formula for cohesive modules.

The classical Hirzebruch-Riemann-Roch theorem for holomorphic vector bundles can be viewed a consequence of Atiyah-Singer Index theorem for the Dirac operator $\bar{\partial}^X + \bar{\partial}^{X,*}$. For cohesive modules, we have a natural generalization of such an operator: $\mathbb{E}'' + \mathbb{E}'',^*$. We call it the generalized Dolbeault-Dirac operator and relate it to the theory of Clifford module and generalized Dirac operators in chapter 5. Using the heat kernel method for Clifford superconnection presented in [BGV91], we obtain the following index formula for the Euler characteristic of cohesive modules that extends classical the Hirzebruch-Riemann-Roch formula.

Theorem. *If $E = (E^\bullet, \mathbb{E}'', h)$ is a Hermitian cohesive module over X and*

$$D^E = \sqrt{2}(\mathbb{E}'' + \mathbb{E}'',^*)$$

is the generalized Dolbeault-Dirac operator, then its index $\mathrm{Ind}(D^E)$ computes the Euler characteristic $\chi(E)$ of the generalized Dolbeault complex $(\mathcal{A}^{0,\bullet}(X, E^\bullet), \mathbb{E}'')$.

In addition, $\mathrm{Ind}(D^E)$ *is computed by*

$$\mathrm{Ind}(D^E) = (2\pi i)^{-\dim X} \int_X \mathrm{Todd}(X) \cdot \mathrm{ch}(E)$$

This is a local index formula if X *is further assumed to be Kähler. In general, the integrand is only cohomologous to the terms appearing in the heat kernel asymptotic expansions.*

In chapter 6, we consider a holomorphic submersion $\pi : M \to B$ between compact complex manifolds and relate cohesive modules on them. In [Blo10], a push-forward operation $\pi_!$ was defined for quasi-cohesive modules $(E^\bullet, \mathbb{E}'')$. We complete the push-forward to a natural superconnection $\pi_!\mathbb{E}$ on the push-forward quasi-cohesive module whose degree zero component is the family of generalized Dolbeault-Dirac operators studied in the previous chapter. This is the infinite dimensional version of the construction of Chern superconnection, completed from a cohesive structure with a choice of hermitian metric. Here the metric is the fiberwise L^2-metric.

Using the family index theorem for Clifford superconnections established by Getzler in [Get91], we obtain the following generalization of Grothendieck-Riemann-Roch formula to cohesive modules.

Theorem. *Assume* $(M, B, \pi, P, g_{M/B})$ *is a holomorphic submersion with horizontal splitting* P *and vertical metric* $g_{M/B}$ *that are compatible with the complex structures. For any Hermitian cohesive module* E *on* M*, the Chern character of the push-*

forward quasi-cohesive module $\pi_! E$ is computed by

$$\mathrm{ch}(\pi_! E) = (2\pi i)^{-\dim M/B} \int_{M/B} \mathrm{Todd}(M/B) \cdot \mathrm{ch}(E)$$

If the submersion is a Kähler fibration in the sense of [BGS88b], the above formula is local.

Chapter 2

Cohesive Modules and Chern superconnections

In this chapter, we first recall the theory of cohesive module developed by Block in [Blo10]. Once we fix a hermitian metric on a cohesive module, we can complete the cohesive structure into a superconnection in the sense of Quillen in [Qui85]. We call it the Chern superconnection since it reduces to the usual notion of a Chern connection when the cohesive module is a holomorphic hermitian vector bundle. Some basic properties like Bianchi identity of its curvature are established. Though it is no longer a $(1,1)$ form, we introduce an exotic grading on the space of complex differential forms with values in cohesive module such that the curvature is of exotic degree 0. This is the crucial property that enables most of the arguments later.

2.1 Dg-category of cohesive modules

Let X be a complex compact manifold, and let $(\mathcal{A}^{0,\bullet}(X), \bar{\partial}^X)$ be its Dolbeault differential graded algebra (dga). $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ is the dg-category of cohesive modules over $(\mathcal{A}^{0,\bullet}(X), \bar{\partial}^X)$. We recall the definition of this dg-category from [Blo10] below. Through this article, we work with double or triple $\mathbb{Z}$-graded objects and we write $|\bullet|$ for the total degree. The commutators and traces are taken in the sense of superspaces.

Definition 2.1.1. A cohesive module $E = (E^\bullet, \mathbb{E}'')$ on X consists of two data:

1. A finite dimensional $\mathbb{Z}$-graded complex vector bundle $E^\bullet$.
2. A flat $\mathbb{Z}$-graded $\bar{\partial}^X$-superconnection $\mathbb{E}''$ on $E^\bullet$. That is a $\mathbb{C}$-linear map:

$$\mathbb{E}'' : \mathcal{A}^{0,\bullet}(X, E^\bullet) \to \mathcal{A}^{0,\bullet}(X, E^\bullet)$$

   of total degree one which satisfies both the $\bar{\partial}^X$-Leibniz formula:

$$\mathbb{E}''(e \otimes \omega) = (\mathbb{E}''e) \wedge \omega + (-1)^{|e|} e \otimes \bar{\partial}^X \omega \tag{2.1.1}$$

   for all $\omega \in \mathcal{A}^{0,\bullet}(X), e \in \mathcal{A}^0(X, E^\bullet)$, and the flatness equation:

$$\mathbb{E}'' \circ \mathbb{E}'' = 0 \tag{2.1.2}$$

Remark 2.1.2. In equation (2.1.1), we view the space of smooth sections of $E^\bullet$ as a right module over $\mathcal{A}^0(X)$. If we consider it as a left module by the isomorphism

$I(e \otimes \omega) = (-1)^{|e||\omega|}\omega \otimes e$, the induced superconnection $I(\mathbb{E}'')$ satisfies the usual Leibniz formula:

$$I(\mathbb{E}'')(\omega \otimes e) = \bar{\partial}^X \omega \otimes e + (-1)^{|\omega|}\omega \wedge I(\mathbb{E}'')e \tag{2.1.3}$$

for all $\omega \in \mathcal{A}^{0,\bullet}(X), e \in \mathcal{A}^0(X, E^\bullet)$. We continue to work with the right module convention as in [Blo10] so that the shift operation on the dg-category is simply taking $(E^\bullet, \mathbb{E})$ to $(E^{\bullet+1}, -\mathbb{E})$.

Definition 2.1.3. The objects in the dg-category $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ are cohesive modules defined above. The degree k morphisms $\mathcal{P}^k_{\mathcal{A}^{0,\bullet}(X)}(E, F)$ between two cohesive modules $E = (E^\bullet, \mathbb{E}'')$, $F = (F^\bullet, \mathbb{F}'')$ are $\mathcal{A}^{0,\bullet}(X)$-linear maps $\phi : \mathcal{A}^{0,\bullet}(X, E^\bullet) \to \mathcal{A}^{0,\bullet}(X, F^\bullet)$ of total degree k. The differential $d : \mathcal{P}^k_{\mathcal{A}^{0,\bullet}(X)}(E, F) \to \mathcal{P}^{k+1}_{\mathcal{A}^{0,\bullet}(X)}(E, F)$ is defined by the commutator:

$$d(\phi) = \mathbb{F}'' \circ \phi - (-1)^k \phi \circ \mathbb{E}'' \tag{2.1.4}$$

It's simple to verify $d^2 = 0$ and therefore $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ is a dg-category.

Example 2.1.4. If $(E^\bullet, \delta)$ is a complex of holomorphic vector bundles on X, it defines a cohesive module with cohesive structure given by setting $\mathbb{E}''_0 = \delta$, $\mathbb{E}''_1 = (-1)^\bullet \bar{\partial}^{E^\bullet}$ and $\mathbb{E}''_k = 0$ for $k > 1$. The flatness condition is equivalent to the following set of equations:

$$\delta \circ \delta = 0 \qquad (\delta \text{ defines a complex of vector bundles}) \tag{2.1.5}$$

$$\delta \circ \bar{\partial}^{E^\bullet} = \bar{\partial}^{E^{\bullet+1}} \circ \delta \qquad (\delta \text{ are holomoprhic homomorphisms}) \tag{2.1.6}$$

$$\bar{\partial}^{E^\bullet} \circ \bar{\partial}^{E^\bullet} = 0 \qquad (\bar{\partial}^{E^\bullet} \text{ defines a holomorphic structure on } E^\bullet) \tag{2.1.7}$$

Remark 2.1.5. More generally, Block proved that the homotopy category of $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ is equivalent to $D^b_{\text{Coh}}(X)$ in [Blo10]. That is, any complex of sheaves of $\mathcal{O}_X$-modules with coherent cohomology is represented by a cohesive module. In particular, any coherent sheaf is represented by a cohesive module $(E^\bullet, \mathbb{E}'')$ such that the underlying complex $(E^\bullet, \mathbb{E}''_0)$ is exact except at the end.

Definition 2.1.6. A Hermitian form h on a cohesive module E is a Hermitian form on the $\mathbb{Z}$-graded bundle $E^\bullet$ such that the E^k is orthogonal to E^l if $k \neq l$. We denote by $(E, \mathbb{E}'', h)$ a Hermitian cohesive module.

We define a conjugate linear involution on differential forms that is different from complex conjugation. It is motivated by the involution on Clifford algebras and we will see this is the adjoint operation in an appropriate sense in chapter 5.

Definition 2.1.7. For $\omega \in \mathcal{A}^k(X)$ a complex differential form of degree k, we define

$$\omega^* = (-1)^{\frac{(k+1)k}{2}} \overline{\omega} \tag{2.1.8}$$

Lemma 2.1.8. *The $*$-operation is the unique operation on $\mathcal{A}^\bullet(X)$ such that $f^* = \bar{f}$ if f is a smooth function, $\omega^* = -\overline{\omega}$ if ω is one form, and in general,*

$$(\omega \wedge \eta)^* = \eta^* \wedge \omega^*, \forall \omega, \eta \in \mathcal{A}^\bullet(X) \tag{2.1.9}$$

Definition 2.1.9. We extend the $*$-operation to $\mathcal{A}^\bullet(X, \text{End}^\bullet E)$ by

$$(L \otimes \omega)^* = (-1)^{|L||\omega|} L^* \otimes \omega^* \tag{2.1.10}$$

for all $L \in \mathcal{A}^0(X, \text{End}^\bullet E)$ and $\omega \in \mathcal{A}^\bullet(X)$. Here L^* is the ordinary Hilbert space adjoint.

Lemma 2.1.10. *The operator $*$ extends to a conjugate linear involution on the algebra $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$.*

Proof. Without loss of generality, we take homogeneous elements S, T, ω, η and we have:

$$(S \otimes \omega)^* = (-1)^{|S||\omega|} S^* \otimes \omega^*$$

$$(T \otimes \eta)^* = (-1)^{|T||\eta|} T^* \otimes \eta^*$$

so, $(T \otimes \eta)^* \cdot (S \otimes \omega)^* = (-1)^{|S||\omega|+|T||\eta|+|S||\eta|} T^* S^* \otimes \eta^* \wedge \omega^*$. On the other hand, by the involution property of $*$-operation on $\mathrm{End}^\bullet E$ and $\mathcal{A}^\bullet(X)$ respectively, we have:

$$\begin{aligned}(S \otimes \omega \cdot T \otimes \eta)^* &= (-1)^{|T||\omega|} (ST \otimes \omega \wedge \eta)^* \\ &= (-1)^{|T||\omega|+|ST||\omega \wedge \eta|} T^* S^* \otimes \eta^* \wedge \omega^* \\ &= (T \otimes \eta)^* \cdot (S \otimes \omega)^*\end{aligned}$$

This proves our claim. □

Definition 2.1.11. If h is a Hermitian structure on the cohesive module $E^\bullet$, then we extend h to $\mathcal{A}^\bullet(X, E^\bullet)$ by the formula:

$$h(e \otimes \omega, f \otimes \eta) = \omega^* \wedge h(e, f) \wedge \eta \tag{2.1.11}$$

where $e, f \in \mathcal{A}^0(X, E^\bullet)$ and $\omega, \eta \in \mathcal{A}^\bullet(X)$.

The following result shows that $*$ defines the adjoint operation with respect to the hermitian metric h.

Lemma 2.1.12. *If* $A \in \mathcal{A}^{0,\bullet}(X, \mathrm{End}^\bullet E)$*, then* $A^* \in \mathcal{A}^{\bullet,0}(X, \mathrm{End}^{-\bullet} E)$ *and for any* $s, t \in \mathcal{A}^\bullet(X, E^\bullet)$*, we have*

$$h(As, t) = h(s, A^* t) \tag{2.1.12}$$

Proof. Assume $A = L \otimes \tau, s = e \otimes \omega, t = f \otimes \eta$ and $|e| + |L| = |f|$ for otherwise both side of the equation are zero.

$$\begin{aligned} h(As, t) =& h(L \otimes \tau(e \otimes \omega), f \otimes \eta) = (-1)^{|\tau||e|} h(Le \otimes \tau \wedge \omega, f\eta) \\ =& (-1)^{|\tau||e|} (\tau \wedge \omega)^* h(Le, f)\eta = (-1)^{|\tau||e|} \omega^* \wedge \tau^* h(e, L^* f)\eta \\ =& (-1)^{|\tau||e|} \omega^* h(e, L^* f)\tau^* \wedge \eta = (-1)^{|\tau|(|e|+|f|)} \omega^* h(e, L^* \otimes \tau^*(f \otimes \eta)) \end{aligned}$$

By assumption $|e| + |L| = |f|$, we have $|\tau|(|e| + |f|) = |\tau||L| \mod 2$. Finally, since $A^* = (-1)^{|L||\tau|} L^* \otimes \tau^*$, we have

$$h(As, t) = (-1)^{2|L||\tau|} h(s, A^* t) = h(s, A^* t) \tag{2.1.13}$$

□

2.2 Chern superconnection of a Hermitian cohesive module

The holomorphic Chern connection ∇^E for a holomorphic Hermitian vector bundle E is the unique unitary connection on E with $\bar{\partial}^X$-component given by the Dolbeault differential $\bar{\partial}^E$. We generalize this construction in this section to Hermitian cohesive modules.

Lemma 2.2.1. *For a complex Hermitian vector bundle (E, h) with an $\bar{\partial}^X$-connection ∇'' which is not necessarily flat, there exist an unique ∂^X-connection ∇' such that $\nabla = \nabla' + \nabla''$ is unitary with respect to h. That is, for any $e, f \in \mathcal{A}^\bullet(X, E)$, ∇ satisfies:*

$$d^X h(e, f) = -h(\nabla e, f) + h(e, \nabla f) \tag{2.2.1}$$

Proof. The problem is local, so it suffices to construct ∇ and prove its uniqueness locally. Choose a local frame $s = (s_1, s_2, ...s_n)$ on E, let $\Theta \in \mathcal{A}^{0,1}(X, \mathrm{End}E)$ be the connection $(0,1)$-form such that $\nabla'' s = s \otimes \Theta$. With respect to the frame s, the Hermitian metric h is represented by the Hermitian matrix valued function $\mathcal{H} = h(s, s)$. Since any $(1,0)$-connection ∇' is locally represented by its connection $(1,0)$-form $\Omega \in \mathcal{A}^{1,0}(X, \mathrm{End}E)$ such that $\nabla' s = s \otimes \Omega$, if we set $\nabla = \nabla' + \nabla''$, the condition for ∇ being unitary is:

$$d^X \mathcal{H} = -(\Theta + \Omega)^* \mathcal{H} + \mathcal{H}(\Theta + \Omega) \tag{2.2.2}$$

Comparing the $(1,0)$ and $(0,1)$ component of equation (2.2.2) above, then (2.2.1) is equivalent to

$$\partial^X \mathcal{H} = -\Theta^* \mathcal{H} + \mathcal{H}\Omega \tag{2.2.3}$$

$$\bar{\partial}^X \mathcal{H} = -\Omega^* \mathcal{H} + \mathcal{H}\Theta \tag{2.2.4}$$

Using (2.2.3), we can solve for Ω as

$$\Omega = \mathcal{H}^{-1}\partial^X \mathcal{H} + \mathcal{H}^{-1}\Theta^* \mathcal{H} \tag{2.2.5}$$

which shows the uniqueness of ∇'. But we can also use (2.2.5) as the definition of Ω. It remains to verify that with Ω so defined, equation (2.2.4) is satisfied. Since $\mathcal{H}$ is a Hermitian matrix, $\mathcal{H}^* = \mathcal{H}$, we can now compute Ω^* as

$$\begin{aligned}\Omega^* =& (\partial^X \mathcal{H})^* (\mathcal{H}^{-1})^* + \mathcal{H}^* \Theta^{**} (\mathcal{H}^{-1})^* \\ =& -\bar{\partial}^X \mathcal{H}^* (\mathcal{H}^*)^{-1} + \mathcal{H}^* \Theta (\mathcal{H}^*)^{-1} \\ =& -\bar{\partial}^X \mathcal{H}\mathcal{H}^{-1} + \mathcal{H}\Theta\mathcal{H}^{-1} \end{aligned} \tag{2.2.6}$$

Multiplying $\mathcal{H}$ on the right, we get (2.2.4). □

Remark 2.2.2. The above equation (2.2.1) differs by a minus sign from the ordinary equation for unitary connections:

$$d^X h(e, f) = h(\nabla e, f) + h(e, \nabla f) \tag{2.2.7}$$

This is caused by our extension of h to $\mathcal{A}^\bullet(X, E^\bullet)$. In the ordinary case, if $s \otimes \Omega$ is a one form with values in E, then $h(s \otimes \Omega, t) = \overline{\Omega} h(s, t)$. However, by our definition, $h(s \otimes \Omega, t) = \Omega^* h(s, t) = -\overline{\Omega} h(s, t)$.

Definition 2.2.3. We define an exotic grading on the spaces $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ and $\mathcal{A}^\bullet(X, E^\bullet)$. For an element $A = L \otimes \tau$, where $L \in \mathrm{End}^d(E^\bullet)$ or $L \in E^d$, and $\tau \in \mathcal{A}^{p,q}(X)$, we define its exotic degree by $\deg(A) = -p + q + d$. $\mathcal{A}^\bullet(X)$ inherit an exotic grading as well given by $\deg(\tau) = -p + q$. We denote by $\mathcal{G}^k$ the subspaces of exotic degree k. In particular, the subspace $\mathcal{G}^0$ in $\mathcal{A}^\bullet(X)$ consists of forms with bi-degree (p, p).

Lemma 2.2.4. *With respect to the exotic grading* deg*,* $\mathcal{A}^\bullet(X)$ *is a* $\mathbb{Z}$*-graded algebra;* $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ *is a* $\mathbb{Z}$*-graded algebra and* $\mathbb{Z}$*-graded module over* $\mathcal{A}^\bullet(X)$*;* $\mathcal{A}^\bullet(X, E^\bullet)$ *is both a* $\mathbb{Z}$*-graded module over* $\mathcal{A}^\bullet(X)$ *where the action is exterior product, and a* $\mathbb{Z}$*-graded module over* $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ *where the action is evaluation. In addition, the* $*$*-operator defined before on* $\mathcal{A}^\bullet(X)$ *and* $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ *maps* $\mathcal{G}^\bullet$ *to* $\mathcal{G}^{-\bullet}$*.*

Proof. It's simple to verify that the subspaces $\mathcal{G}^\bullet$ respects all these structures in the sense

$$\mathcal{G}^i \cdot \mathcal{G}^j \subseteq \mathcal{G}^{i+j} \tag{2.2.8}$$

whenever the composition is defined by the algebra multiplication or module action. For the $*$-operation, the claim follows from the fact that $*$-operator interchanges forms of bi-degrees (p, q) and (q, p). □

Proposition 2.2.5 (Chern superconnection)**.** *Let* $E = (E^\bullet, \mathbb{E}'', h)$ *be a Hermitian cohesive module. There exist an unique* ∂^X*-superconnection* $\mathbb{E}' : \mathcal{A}^0(X, E^\bullet) \to \mathcal{A}^{\bullet,0}(X, E^\bullet)$ *of exotic degree* -1 *such that the superconnection* $\mathbb{E} = \mathbb{E}' + \mathbb{E}''$ *is unitary. That is, for any* $s, t \in \mathcal{A}^\bullet(X, E^\bullet)$*, we have*

$$(-1)^{|s|} d^X h(s, t) = -h(\mathbb{E}s, t) + h(s, \mathbb{E}t) \tag{2.2.9}$$

Proof. There is a decomposition of a $\bar{\partial}-$ superconnection into homogeneous components: $\mathbb{E}'' = \mathbb{E}''_1 + \sum_{k \neq 1} \mathbb{E}''_k$ where $\mathbb{E}''_1$ is an ordinary $\bar{\partial}^X$-connection on $E^\bullet$ and the other terms $\mathbb{E}''_k \in \mathcal{A}^{0,k}(X, \mathrm{End}^{1-k} E)$ are linear over $\mathcal{A}^\bullet(X)$.

By lemma 2.2.1, there is an unique ∂^X-connection $\mathbb{E}'_1$ on $E^\bullet$ such that $\nabla^{E^\bullet} = \mathbb{E}''_1 + \mathbb{E}'_1$ is unitary on the graded Hermitian complex vector bundle $E^\bullet$. If we set

$\mathbb{E}'_k = (\mathbb{E}''_k)^*$ for $k \neq 1$, by lemma 2.1.12, $\mathbb{E}_k = \mathbb{E}''_k + \mathbb{E}'_k$ is unitary. Adding $\nabla^{E^\bullet}$ and all $\mathbb{E}_k$ together, the resulting d^X-superconnection $\mathbb{E}$ is unitary. Uniqueness is obvious from the construction and equation (2.2.9). □

Remark 2.2.6. We can always write a homogeneous term $A = L \otimes \tau$ in such a way that τ is a real form of degree k. Since A is odd, A^* is just $L^* \otimes \tau^*$ and therefore $A = A^*$ is equivalent to $L = (-1)^{\frac{k(k+1)}{2}} L^*$. In other words, L satisfies the following conditions:

$$\begin{cases} L \text{ is Hermitian} & k = 0, 3 \pmod 4 \\ L \text{ is skew Hermitian} & k = 1, 2 \pmod 4 \end{cases}$$

In particular, this is compatible with the well known fact that an unitary connection has skew Hermitian connection matrix.

2.3 Chern-Weil construction of characteristic forms

Definition 2.3.1. Let $\mathbb{E}$ be the superconnection for a Hermitian cohesive module $(E^\bullet, \mathbb{E}'', h)$ given by Proposition 2.2.5. We call it the Chern superconnection. The curvature of $\mathbb{E}$ is defined by the usual formula

$$\mathcal{R} = \mathbb{E}^2 = \frac{1}{2}[\mathbb{E}, \mathbb{E}] \tag{2.3.1}$$

In the classical case of Hermitian holomorphic vector bundles, the curvature of the Chern connection is a $(1,1)$-form. In the case of Hermitian cohesive modules, we have similar properties of Chern superconnections for cohesive modules.

Lemma 2.3.2. *If $\mathbb{E}$ is the Chern superconnection for a Hermitian cohesive module $(E^\bullet, \mathbb{E}, h)$ and $\mathcal{R}$ is its curvature, then it satisfies the following properties:*

$$\mathcal{R}^* = \mathcal{R} \tag{2.3.2}$$

$$\mathbb{E}' \circ \mathbb{E}' = 0 \tag{2.3.3}$$

$$\mathcal{R} = [\mathbb{E}', \mathbb{E}''] \tag{2.3.4}$$

Consequently, the curvature $\mathcal{R}$ is of exotic degree zero: $\mathcal{R} \in \mathcal{G}^0$.

Proof. By definition, the curvature is

$$\begin{aligned}
\mathcal{R} =& \mathbb{E} \circ \mathbb{E} = (\mathbb{E}' + \mathbb{E}'') \circ (\mathbb{E}' + \mathbb{E}'') \\
=& (\mathbb{E}')^2 + [\mathbb{E}', \mathbb{E}''] + (\mathbb{E}'')^2 \\
=& (\mathbb{E}')^2 + [\mathbb{E}', \mathbb{E}'']
\end{aligned}$$

where the term $(\mathbb{E}'')^2$ vanishes by our flatness assumption on $\mathbb{E}''$. Since the exotic degree of $\mathbb{E}'$ is -1 and that of $[\mathbb{E}', \mathbb{E}'']$ is 0, $(\mathbb{E}')^2 \in \mathcal{G}^{-2}$ and $[\mathbb{E}', \mathbb{E}''] \in \mathcal{G}^0$. We use the defining equation (2.2.9) repeatedly to get:

$$\begin{aligned}
0 =& (d^X)^2 h(s,t) = (-1)^{|s|+1} d^X h(\mathbb{E}s, t) + (-1)^{|s|} d^X h(s, \mathbb{E}t) \\
=& - h(\mathcal{R}s, t) + h(\mathbb{E}s, \mathbb{E}t) - h(\mathbb{E}s, \mathbb{E}t) + h(s, \mathcal{R}t) \\
=& - h(\mathcal{R}s, t) + h(s, \mathcal{R}t)
\end{aligned}$$

Since $\mathcal{R}$ is linear, by Lemma 2.1.12, the curvature $\mathcal{R}$ is self adjoint: $\mathcal{R} = \mathcal{R}^*$. Since $*$ maps $\mathcal{G}^{-2}$ to $\mathcal{G}^2$, we see $(\mathbb{E}')^2 = 0$. The last equality follows from this. □

Lemma 2.3.3 (Bianchi Identity)**.** $[\mathbb{E}, \mathcal{R}] = 0$

Proof. By definition, $\mathcal{R} = \frac{1}{2}[\mathbb{E}, \mathbb{E}]$. Using either the graded Jacobi identity or simply by expanding the expression, we have

$$[\mathbb{E}, \mathcal{R}] = [\mathbb{E}, \mathbb{E}^2] = \mathbb{E} \cdot \mathbb{E}^2 - \mathbb{E}^2 \cdot \mathbb{E} = 0 \tag{2.3.5}$$

□

For a cohesive module E, the Chern superconnection depends on a choice of hermitian form h. When we want to emphasize the dependence, we will write $\mathbb{E}_h$ for the superconnection and $\mathcal{R}_h$ for its curvature. The ordinary Chern-Weil theory of characteristic classes had been extended to superconnections and take value in deRham cohomology. We recall the explicit construction from which we observe the resulting forms lie in the refined Bott-Chern cohomology.

Definition 2.3.4. Assume (E, h) is a Hermitian cohesive module with Chern superconnection $\mathbb{E}_h$ and curvature $\mathcal{R}_h$. Let $f(T)$ be a convergent power series in T, we define the characteristic form of (E, h) associated to $f(T)$ by

$$f(E^\bullet, \mathbb{E}'', h) = \mathrm{Tr}_s f(\mathcal{R}_h) \tag{2.3.6}$$

where Tr_s is the supertrace.

Remark 2.3.5. The power series is required to be convergent since $\mathcal{R}$ is no longer concentrated in degree $(1, 1)$. For our application, the ordinary Chern class, Chern character, Todd genus, $\hat{A}$-genus are defined via convergent series.

Proposition 2.3.6. *The characteristic forms are closed:*

$$d^X f(E^\bullet, \mathbb{E}'', h) = 0 \tag{2.3.7}$$

Proof. Without loss of generality, we assume $f(T) = T^n$ is a monomial. Then we have

$$d^X \mathrm{Tr}_s(\mathcal{R}_h^n) = \sum_i \mathrm{Tr}_s(\mathcal{R}_h^{i-1}[\mathbb{E}_h, \mathcal{R}_h]\mathcal{R}_h^{n-i}) \tag{2.3.8}$$

Since each term in the summation is zero by Bianchi identity, $\mathrm{Tr}_s(\mathcal{R}_h^n)$ is closed. □

Consequently, the characteristic forms defines deRham cohomology classes. We will show that these cohomology classes only dependent on the connection component $\nabla^E = \mathbb{E}_1$ in $\mathbb{E}_h$. Though the result is well known in literature, to illustrate the technique of transgression which will be used throughout the next few sections, we present the argument here.

Let $\mathbb{E}_t = (1-t)\nabla^E + t\mathbb{E}$ be a one parameter family of superconnections that joins the connection term $\nabla^E = \mathbb{E}_1$ to $\mathbb{E}$. If we write $A = \mathbb{E} - \mathbb{E}_1 = \sum_{k\neq 1} \mathbb{E}_k$ for the linear terms, then $\mathbb{E}_t = \nabla^E + tA$. Let $\mathcal{R}_t$ be the corresponding curvatures for $\mathbb{E}_t$.

Lemma 2.3.7. *The deformation of curvature is computed by*

$$\frac{d}{dt}\mathcal{R}_t = [\mathbb{E}_t, A] \tag{2.3.9}$$

Proof. Since $\mathbb{E}_t = \nabla^E + tA$, we have $\mathcal{R}_t = \mathcal{R}_0 + t[\nabla^E, A] + t^2A^2$. We can then compute

$$\frac{d}{dt}\mathcal{R}_t = [\nabla^E, A] + 2tA^2 \tag{2.3.10}$$

since $2tA^2 = t[A, A] = [tA, A]$, the right side of the above equality is $[\mathbb{E}_t, A]$. □

Corollary 2.3.8. *If we define* $f(E, \mathbb{E}_t)$ *by the same formula* $\mathrm{Tr}_s f(\mathcal{R}_t)$*, we have*

$$f(E, \mathbb{E}) - f(E, \nabla^E) = d^X \int_0^1 \mathrm{Tr}_s\{A \cdot f'(\mathcal{R}_t)\} dt \tag{2.3.11}$$

Consequently, $f(E, \mathbb{E})$ *is cohomologous to* $f(E, \nabla^E)$ *in deRham cohomology.*

It is well-known that $f(E, \nabla^E)$ only depends on the topological vector bundle structure of E as a $\mathbb{Z}_2$-graded vector bundle, so the characteristic class in deRham cohomology defined by $f(E^\bullet, h, \mathbb{E}'')$ is independent of h.

Definition 2.3.9. Let $\mathcal{A}^{p,p}(X)$ be the space of forms of bi-degree (p, p) and $Z^p(X)$ be the subspace of d^X-closed forms. We set $B^p(X)$ the subspace of $Z^p(X)$ that is the image of $\mathcal{A}^{p-1,p-1}(X)$ under $\partial^X \bar{\partial}^X$, then the p-th Bott-Chern cohomology $H^p_{BC}(X)$ is defined by:

$$H^p_{BC}(X) = \frac{Z^p(X)}{B^p(X)}$$

By Lemma 2.3.2, $f(E, h) \in \mathcal{G}^0$ and hence the forms defines Bott-Chern cohomology classes. We will prove in the next chapter that these refined cohomology classes are independent of the Hermitian structure h. In the fourth chapter, we will show they are further invariant under homotopy equivalences between cohesive modules.

Chapter 3

Transgression formulas for characteristic forms

As we have seen in the previous chapter, the characteristic forms define cohomology classes in deRham cohomology that are independent of the hermitian metric h. In the first two sections, we extend the result to prove that the characteristic forms define metric independent cohomology classes in Bott-Chern cohomology, which is a refinement of deRham cohomology.

The space of hermitian metrics $\mathcal{M}$ is a star-shaped topological space and hence connected. We prove the independence by computing the infinitesimal deformation of the characteristic forms over $\mathcal{M}$ and establishes a $\partial^X\bar{\partial}^X$-transgression formula instead of a d^X-transgression formula. While constructing the explicit elements that provide the transgression, we observe such elements are canonical and can be appropriately defined as secondary characteristic forms. The main technical computation in section 3.3 is motivated by the results in [Don87] where the secondary forms were first studied and shows the forms are well-defined secondary classes.

3.1 Universal cohesive module and Maurer-Cartan equation

We fix a cohesive module $E = (E^\bullet, \mathbb{E}'')$ on X and let $\mathcal{M}$ be the space of Hermitian structures on $E^\bullet$. We endow $\mathcal{M}$ with the topology of uniform $\mathcal{C}^\infty$-convergence on compact subsets of X. If $h \in \mathcal{M}, x \in X$, let A^h_x be the subset of $\mathrm{End}^0 E_x$ whose elements are Hermitian endomorphisms with respect to h_x. A^h_x forms a bundle A^h over X, and the tangent space of $\mathcal{M}$ at h, $T_h\mathcal{M}$, can be identified with the linear space of sections of A^h over X.

Definition 3.1.1. Consider the projection $\pi : \mathcal{M} \times X \to X$, we define $\widetilde{E}^\bullet$ as the pull-back bundle $\pi^* E^\bullet$ over $\mathcal{M} \times X$. There is an universal cohesive structure $\widetilde{\mathbb{E}}''$ on $\widetilde{E}^\bullet$ defined by

$$\widetilde{\mathbb{E}}'' = \bar{\partial}^{\mathcal{M}} + \mathbb{E}'' \tag{3.1.1}$$

Finally, we write $\tilde{h}$ for the universal Hermitian form on $\widetilde{E}^\bullet$. We denote by $\widetilde{E}$ the universal Hermitian cohesive module $(\widetilde{E}^\bullet, \widetilde{\mathbb{E}}'', \tilde{h})$.

Remark 3.1.2. The bundle $\widetilde{E}^\bullet$ is flat in the $\mathcal{M}$-direction with flat connection $d^{\mathcal{M}}$. In the above formula, $\bar{\partial}^{\mathcal{M}}$ acts on an element $e \otimes \omega$ by $(-1)^{|e|} e \otimes \bar{\partial}^{\mathcal{M}}\omega$.

Lemma 3.1.3. $\widetilde{\mathbb{E}}''$ *defines a* $\mathbb{Z}$*-graded flat* $\bar{\partial}_{\mathcal{M}\times X}$*-connection on* $\widetilde{E}^\bullet$*.*

Proof. Note that the vertical cohesive structure $\mathbb{E}''$ is constant along $\mathcal{M}$-direction, so $d^{\mathcal{M}}\mathbb{E}'' = 0$, we have

$$(\widetilde{\mathbb{E}}'')^2 = (\bar{\partial}^{\mathcal{M}} + \mathbb{E}'')^2 = (\bar{\partial}^{\mathcal{M}})^2 + \bar{\partial}^{\mathcal{M}}(\mathbb{E}'') + (\mathbb{E}'')^2 = 0 \tag{3.1.2}$$

□

Definition 3.1.4. The Maurer-Cartan one form $\theta \in \mathcal{A}^1(\mathcal{M} \times X, \mathrm{End}^0 \widetilde{E})$ on $\mathcal{M}$ with values in A is the one form defined by

$$\theta = h^{-1} d^{\mathcal{M}} h \tag{3.1.3}$$

where d^M is the exterior differential on $\mathcal{M}$.

Lemma 3.1.5. *θ is a one form in the $\mathcal{M}$-direction with values in Hermitian endomorphisms, $\theta \in \mathcal{A}^1(\mathcal{M}, \mathrm{End}(E, h))$.*

Proof. We choose a frame $(e_1, e_2, ..., e_n)$ for $E^\bullet$ locally on X. Any Hermitian form h is represented by a smooth function with values in Hermitian matrix by

$$\mathcal{H} : x \in X \to [h_x(e_i, e_j)]_{ij} \tag{3.1.4}$$

If $\mathcal{H}_t(x)$ is a smooth family of Hermitian matrix valued functions with parameter t, we can differentiate $\mathcal{H}_t(X)$ with respect to t, then $\mathcal{H}_t^{-1}\dot{\mathcal{H}}_t$ is a Hermitian with respect to $\mathcal{H}_t$. □

Lemma 3.1.6. *θ satisfies the Maurer-Cartan equation:*

$$d^{\mathcal{M}}\theta = -\theta^2 \tag{3.1.5}$$

Proof. This is straightforward calculation:

$$d^{\mathcal{M}}\theta = d^{\mathcal{M}}(h^{-1}d^{\mathcal{M}}h) = d^{\mathcal{M}}(h^{-1})d^{\mathcal{M}}h + h^{-1}d^{\mathcal{M}}(d^{\mathcal{M}}h) \tag{3.1.6}$$

$$= -h^{-1}(d^{\mathcal{M}}h)h^{-1}d^{\mathcal{M}}h = -\theta^2 \tag{3.1.7}$$

□

Since $\widetilde{E}$ is flat along the $\mathcal{M}$-direction, by the explicit expression of θ with respect to a local frame on X, we see θ measures the deformation of h over $\mathcal{M}$ in the following sense.

Lemma 3.1.7. *The flat $\mathcal{M}$-directional exterior derivative $d^{\mathcal{M}}$ has the following compatibility relation with the universal Hermitian form $\tilde{h}$*

$$(-1)^{|s|}d^{\mathcal{M}}\tilde{h}(s,t) = -\tilde{h}(d^{\mathcal{M}}s,t) + \tilde{h}(s,\theta t) + \tilde{h}(s,d^{\mathcal{M}}t) \tag{3.1.8}$$

for all $s,t \in \mathcal{A}^0(\mathcal{M}\times X, \widetilde{E})$.

The above lemma together with equation (2.2.9) for each $h \in \mathcal{M}$ shows that $\widetilde{\mathbb{E}} = d^{\mathcal{M}} + \mathbb{E}_h$ differs from the Chern superconnection of $(\widetilde{E}^{\bullet}, \tilde{h}, \widetilde{\mathbb{E}}'')$ by the Maurer-Cartan form θ.

Proposition 3.1.8. *The universal superconnection $\widetilde{\mathbb{E}}$ satisfies the following equation*

$$(-1)^{|s|}d^{\mathcal{M}\times X}\tilde{h}(s,t) = -\tilde{h}(\widetilde{\mathbb{E}}s,t) + \tilde{h}(s,\widetilde{\mathbb{E}}t) + \tilde{h}(s,\theta t) \tag{3.1.9}$$

Even though the previous proposition shows $\widetilde{\mathbb{E}}$ is not the Chern superconnection, we will use $\widetilde{\mathbb{E}}$ to study $\widetilde{E}^{\bullet}$. Denote by $\widetilde{\mathcal{R}}$ the curvature of $\widetilde{\mathbb{E}}$, it is given by

$$\widetilde{\mathcal{R}} = (d^{\mathcal{M}} + \mathbb{E}_h)\circ(d^{\mathcal{M}} + \mathbb{E}_h) = d^{\mathcal{M}}\mathbb{E}_h + \mathcal{R}_h \tag{3.1.10}$$

Proposition 3.1.9. *The $\mathcal{M}$-directional derivative of the Chern superconnection $\mathbb{E}_h$ is given by*

$$d^{\mathcal{M}}\mathbb{E}_h = -[\mathbb{E}'_h, \theta] \tag{3.1.11}$$

and the $\mathcal{M}$-directional derivative of curvature $\widetilde{\mathcal{R}}$ is given by

$$d^{\mathcal{M}}\widetilde{\mathcal{R}} = d^{\mathcal{M}}\mathcal{R}_h \tag{3.1.12}$$

Proof. Since $\mathbb{E}_h = \mathbb{E}'_h + \mathbb{E}''$ and $\mathbb{E}''$ is independent of h, we have

$$d^{\mathcal{M}}\mathbb{E}_h = d^{\mathcal{M}}\mathbb{E}'_h \tag{3.1.13}$$

By the explicit construction of $\mathbb{E}'_h$ from $\mathbb{E}''$ and h, we can write

$$\mathbb{E}'_h = h^{-1} \circ \mathbb{E}'' \circ h \tag{3.1.14}$$

Taking exterior differential in $\mathcal{M}$-variable, we have

$$\begin{aligned} d^{\mathcal{M}}\mathbb{E}'_h &= d^{\mathcal{M}}h^{-1} \circ \mathbb{E}' \circ h - h^{-1} \circ \mathbb{E}' \circ d^{\mathcal{M}}h \\ &= -h^{-1} \circ d^{\mathcal{M}}h \circ h^{-1} \circ \mathbb{E}' \circ h - h^{-1} \circ \mathbb{E}' \circ h \circ h^{-1} \circ d^{\mathcal{M}}h \\ &= -\theta \circ \mathbb{E}'_h - \mathbb{E}'_h \circ \theta = -[\mathbb{E}'_h, \theta] \end{aligned}$$

□

3.2 Bott-Chern double transgression formula

Recall the characteristic forms defined in the previous section $f(E^\bullet, \mathbb{E}'', h) = \mathrm{Tr}_s f(\mathcal{R}_h)$ for a convergent power series $f(T)$, the following transgression formula computes the deformation of the characteristic forms over $\mathcal{M}$.

Proposition 3.2.1 (First Transgression Formula).

$$d^{\mathcal{M}}\mathrm{Tr}_s f(\mathcal{R}_h) = -\bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot[\mathbb{E}'_h,\theta]) \tag{3.2.1}$$

$$\partial^X \mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot[\mathbb{E}'_h,\theta]) = 0 \tag{3.2.2}$$

Proof. On the universal Hermitian cohesive module $\widetilde{E}^\bullet$, by proposition 2.3.6 and equation (3.1.10), we have

$$0 = d^{\mathcal{M}\times X}\mathrm{Tr}_s f(\widetilde{\mathcal{R}}) = d^{\mathcal{M}\times X}\mathrm{Tr}_s f(\mathcal{R}_h + d^{\mathcal{M}}\mathbb{E}_h) \tag{3.2.3}$$

Without loss of generality, we assume $f(T) = T^n$ and we expand (3.2.3) by the multilinear property of $\mathrm{Tr}_s f(T)$ to get:

$$0 = d^X\mathrm{Tr}_s\mathcal{R}_h^n + d^{\mathcal{M}}\mathrm{Tr}_s\mathcal{R}_h^n + d^X\sum_i \mathrm{Tr}_s(\mathcal{R}_h^i\cdot d^{\mathcal{M}}\mathbb{E}_h\cdot\mathcal{R}_h^{n-1-i})\} + ... \tag{3.2.4}$$

The first term is zero by proposition 2.3.6 and the omitted terms are at least degree 2 forms in $\mathcal{M}$-variables. If we collects the forms of degree one in the $\mathcal{M}$-variables, we have

$$0 = d^{\mathcal{M}}\mathrm{Tr}_s(\mathcal{R}_h^n) + n\cdot d^X\mathrm{Tr}_s(\mathcal{R}_h^{n-1}\cdot d^{\mathcal{M}}\mathbb{E}_h) \tag{3.2.5}$$

where we commute $\mathcal{R}_h^i\cdot d^{\mathcal{M}}\mathbb{E}_h$ and $\mathcal{R}_h^{n-1-i}$ under Tr_s. Using lemma 3.1.9, $d^{\mathcal{M}}\mathbb{E}_h = -[\mathbb{E}'_h,\theta]$, we have the following equality:

$$d^{\mathcal{M}}\mathrm{Tr}_s f(\mathcal{R}_h) = d^X\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot[\mathbb{E}'_h,\theta]) \tag{3.2.6}$$

Now we compare both sides of (3.2.6) and the consider the subspaces $\mathcal{G}^\bullet$ defined by the exotic degree. On the left side of (3.2.6), we have a one form on $\mathcal{M}$ with values in $\mathcal{G}^0$ since $\mathcal{R}_h \in \mathcal{G}^0$. On the right side, $\mathcal{R}_h$ is in $\mathcal{G}^0$ while $[\mathbb{E}_h', \theta]$ is in $\mathcal{G}^{-1}$ since $\mathbb{E}_h'$ is of exotic degree -1. Finally, since ∂^X increases the exotic degree by -1 while $\bar{\partial}^X$ increases the exotic degree by 1, we get equation (3.2.1) by comparing the $\mathcal{G}^0$ component and equation (3.2.2) by the $\mathcal{G}^{-2}$ component. □

Our next goal is to express $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}_h', \theta])$ in equation (3.2.2) as the image of ∂^X. To do this, we first introduce a notation.

Definition 3.2.2. If $g(T) = T^n$, for a pair $(A; B)$ of variables, we define

$$g(A; B) = \sum_{i=1}^{n} A^{i-1} B A^{n-i} \tag{3.2.7}$$

In general for a convergent power series $g(T)$, we define $g(A; B)$ by the previous formula for its homogeneous components and take the sum. A simple norm estimates shows the convergence.

Proposition 3.2.3 (Second Transgression formula)**.**

$$\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}_h', \theta]) = \partial^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \tag{3.2.8}$$

Proof. For each $\mathbb{E}_h$ along the vertical fiber X, we have

$$d^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) = \mathrm{Tr}_s([\mathbb{E}_h, f'(\mathcal{R}_h) \cdot \theta]) \tag{3.2.9}$$

By the graded Leibniz formula of $\mathbb{E}_h$, we can expand the equation and compute

$$\begin{aligned} d^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) &= \mathrm{Tr}_s([\mathbb{E}_h, f'(\mathcal{R}_h)] \cdot \theta) + \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}_h, \theta]) \\ &= \mathrm{Tr}_s(f'(\mathcal{R}_h; [\mathbb{E}_h, \mathcal{R}_h]) \cdot \theta) + \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}_h, \theta]) \end{aligned}$$

By Bianchi identity, the first term is zero, hence we have

$$d^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) = \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta]) + \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}''_h, \theta]) \tag{3.2.10}$$

Note as before, $[\mathbb{E}'_h, \theta] \in \mathcal{G}^{-1}$, $[\mathbb{E}''_h, \theta] \in \mathcal{G}^1$ and $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \in \mathcal{G}^0$, comparing the $\mathcal{G}^{-1}$ component we have (3.2.9). □

Combining the first transgression formula (3.2.1) and second transgression formula (3.2.8), we established the double transgression formula for characteristic forms on cohesive modules.

Corollary 3.2.4 (Bott-Chern formula)**.** *The $\mathcal{M}$-directional derivative of the characteristic form $f(E, \mathbb{E}'', h)$ at h is given by:*

$$d^{\mathcal{M}} \mathrm{Tr}_s f(\mathcal{R}_h) = \partial^X \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \tag{3.2.11}$$

This generalizes the classical formula obtained by Bott and Chern in [BC65] for holomorphic vector bundles. If we view the previous equality in Bott-Chern cohomology, the right hand side is zero and we established the invariance of characteristic classes under metric deformation.

Corollary 3.2.5. *The characteristic forms $f(E, \mathbb{E}'', h)$ in the Bott-Chern cohomology are independent of the choice of Hermitian metric h.*

In the last section, we will show the characteristic classes only dependent on the homotopy class of the cohesive module.

3.3 Transgression formula for secondary classes

Our next goal is to study the differential forms $\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)$ that appear in the double transgression formula. As we will prove later, they are the secondary characteristic classes and define holomorphic analog of the Chern-Simons forms. We will use them to study infinite determinant bundle and stability in future research. For their applications in holomorphic vector bundles, see [BGS88a][BGS88b][BGS88c].

Lemma 3.3.1. *The curvature tensors $\mathcal{R}_h$ has $\mathcal{M}$-directional derivative given by:*

$$d^{\mathcal{M}}(\mathcal{R}_h) = [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \tag{3.3.1}$$

Proof. By Bianchi identity for the universal cohesive module and use Lemma 3.1.9, we have:

$$0 = [\widetilde{\mathbb{E}}, \widetilde{\mathcal{R}}] = [d^{\mathcal{M}} + \mathbb{E}_h, \mathcal{R}_h - [\mathbb{E}'_h, \theta]] \tag{3.3.2}$$

Expand the terms and use lemma (3.1.9) again, we have

$$0 = d^{\mathcal{M}}\mathcal{R}_h + [\mathbb{E}_h, \mathcal{R}_h] - [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] + (d^{\mathcal{M}})^2(\mathbb{E}_h) \tag{3.3.3}$$

By Bianchi identity for $\mathbb{E}_h$, the middle term in (3.3.3) vanishes and we get (3.3.1).

□

We will start to prove the main result of this section, namely the form $\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)$ appearing in the double transgression formula is itself a well-defined secondary characteristic form. Like the double transgression formula of Bott and Chern, the goal is to compute the $\mathcal{M}$-directional derivative of $\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)$ and show it's in the image of ∂^X and $\bar{\partial}^X$. We break the lengthy computation into several lemmas.

Lemma 3.3.2.

$$d^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) = \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot [\mathbb{E}_h, \theta]) \tag{3.3.4}$$

Proof. Without loss of generality, we assume $g(T) = T^n$. By our definition, $g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]])$ is a summation of the form:

$$g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]) = \sum_{i+j=n-1} \mathcal{R}_h^i [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \mathcal{R}_h^j \tag{3.3.5}$$

If we consider the following differential, keeping in mind when passing d^X over Tr_s, we act via $\mathbb{E}_h$ and follows the Leibniz rule

$$\begin{aligned} d^X \sum_i \mathrm{Tr}_s(\mathcal{R}_h^i [\mathbb{E}'_h, \theta] \mathcal{R}_h^{n-i-1} \theta) &= \sum_i d^X \mathrm{Tr}_s(\mathcal{R}_h^i [\mathbb{E}'_h, \theta] \mathcal{R}_h^{n-i-1} \theta) \\ = \sum_{i+j+k=n-2} \mathrm{Tr}_s(\mathcal{R}_h^i [\mathbb{E}_h, \mathcal{R}_h] \mathcal{R}_h^j [\mathbb{E}'_h, \theta] \mathcal{R}_h^k \theta) &+ \sum_{i=1}^n \mathrm{Tr}_s(\mathcal{R}_h^{i-1} [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \mathcal{R}_h^{n-i} \theta) \\ + \sum_{i+j+k=n-2} \mathrm{Tr}_s(\mathcal{R}_h^i [\mathbb{E}'_h, \theta] \mathcal{R}_h^j [\mathbb{E}_h, \mathcal{R}_h] \mathcal{R}_h^k \theta) &+ \sum_{i=1}^n \mathrm{Tr}_s(\mathcal{R}_h^{i-1} [\mathbb{E}'_h, \theta] \mathcal{R}_h^{n-i} [\mathbb{E}_h, \theta]) \end{aligned} \tag{3.3.6}$$

The terms in the first and third summations are zero by Bianchi identity $[\mathbb{E}_h, \mathcal{R}_h] = 0$. Since the second and last summations are just $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]) \cdot \theta)$ and $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot [\mathbb{E}_h, \theta])$ respectively, the result follows. □

Lemma 3.3.3.

$$d^{\mathcal{M}} \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) = \bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) - \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, \theta]) \cdot [\mathbb{E}'', \theta]) \tag{3.3.7}$$

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot [\mathbb{E}'_h, \theta]) = 0 \tag{3.3.8}$$

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot [\mathbb{E}'', \theta]) = 0 \tag{3.3.9}$$

Proof. Without loss of generality, we assume $g(T) = T^n$ is a monomial. By Leibniz formula, we get

$$d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) = \mathrm{Tr}_s(g(\mathcal{R}_h; d^{\mathcal{M}}\mathcal{R}_h) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot d^{\mathcal{M}}\theta) \tag{3.3.10}$$

We can substitute $d^{\mathcal{M}}\mathcal{R}_h$ and $d^{\mathcal{M}}\theta$ in the above equation by previous two lemmas 3.1.6 and 3.3.1. Then we have

$$d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) = \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]) \cdot \theta) - \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) \tag{3.3.11}$$

Lemma 3.3.2 together with equation (3.3.11) shows that

$$\begin{aligned} & d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) \\ =& d^X\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) - \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}_h, \theta]) \end{aligned} \tag{3.3.12}$$

In equation (3.3.12), the left hand side is a 2-form on $\mathcal{M}$ with value in $\mathcal{G}^0$. For the right hand side of the equation, keeping in mind that $\mathbb{E}'_h$ increases exotic degree by -1 while $\mathbb{E}''$ increases it by 1, we can decompose $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}_h, \theta])$ as a sum of its $\mathcal{G}^{-2}$ and $\mathcal{G}^0$ components respectively as:

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}_h, \theta]) = \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}'_h, \theta]) \tag{3.3.13}$$

$$+ \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}'', \theta]) \tag{3.3.14}$$

Comparing the $\mathcal{G}^{-2}$ components in equation (3.3.12), we get equation (3.3.8). Taking its adjoint, we get (3.3.9). Finally if we compare the $\mathcal{G}^0$ components in equation (3.3.12), we get

$$\begin{aligned} d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta) + \mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta^2) =& \bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])\cdot\theta) \\ & - \mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])[\mathbb{E}'',\theta]) \end{aligned}$$

Adding the zero term $\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot[\mathbb{E}'',\theta])$ to it, we get equation (3.3.7). □

Lemma 3.3.4. *The last term* $\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}_h,\theta])\cdot[\mathbb{E}'',\theta])$ *in equation* (3.3.7) *is given by:*

$$\begin{aligned} \mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}_h,\theta])\cdot[\mathbb{E}'',\theta]) =& \partial^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) \\ & + \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)\cdot[\mathbb{E}'_h,[\mathbb{E}'',\theta]]) \end{aligned} \tag{3.3.15}$$

And the term $\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta)$ *satisfies:*

$$\bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) + \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)\cdot[\mathbb{E}'',[\mathbb{E}'',\theta]]) = 0 \tag{3.3.16}$$

Proof. We consider the following differential and compute it as in lemma 3.3.2. Again we assume without loss of generality that $g(T) = T^n$.

$$\begin{aligned} & d^X\sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}[\mathbb{E}'',\theta]) \\ =& \sum_{i+j+k=n-2}\mathrm{Tr}_s(\mathcal{R}_h^{i}[\mathbb{E}_h,\mathcal{R}_h]\mathcal{R}_h^{j}\theta\mathcal{R}_h^{k}[\mathbb{E}'',\theta]) + \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}[\mathbb{E}_h,\theta]\mathcal{R}_h^{n-i}[\mathbb{E}'',\theta]) \\ -& \sum_{i+j+k=n-2}\mathrm{Tr}_s(\mathcal{R}_h^{i}\theta\mathcal{R}_h^{j}[\mathbb{E}_h,\mathcal{R}_h]\mathcal{R}_h^{k}[\mathbb{E}'',\theta]) - \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}[\mathbb{E}_h,[\mathbb{E}'',\theta]]) \end{aligned} \tag{3.3.17}$$

Again the terms in the first and third summations are zero by Bianchi identity. So we get

$$\begin{aligned}\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}_h,\theta])\cdot[\mathbb{E}'',\theta]) =& d^X\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)\cdot[\mathbb{E}'',\theta]) \\ &+\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)\cdot[\mathbb{E}_h,[\mathbb{E}'',\theta]])\end{aligned} \tag{3.3.18}$$

Substitute (3.3.18) into (3.3.7) in lemma 3.3.3, we have

$$\begin{aligned}& d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta)+\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta^2) \\ =&\bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])\cdot\theta)-d^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) \\ &-\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)\cdot[\mathbb{E}_h,[\mathbb{E}'',\theta]])\end{aligned} \tag{3.3.19}$$

Note again the left hand side is a 2-form on $\mathcal{M}$ with values in $\mathcal{G}^0$, we get the first equality by comparing $\mathcal{G}^0$ components and the second equality in by comparing $\mathcal{G}^2$ components. □

Combining the formulas we proved so far, we are ready to derive the following main theorem.

Theorem 3.3.5. *Let $g(T)$ be a convergent power series in T, then the $\mathcal{M}$-directional derivative of $\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta)$ is given by the following formula:*

$$d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta)=\frac{1}{2}\bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])\cdot\theta)-\frac{1}{2}\partial^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) \tag{3.3.20}$$

Proof. We write $\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathbb{E}_h,[\mathbb{E}'',\theta]])$ as the sum of its $\mathcal{G}^0$ and $\mathcal{G}^2$ components

$$\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathbb{E}'_h,[\mathbb{E}'',\theta]])+\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathbb{E}'',[\mathbb{E}'',\theta]]) \tag{3.3.21}$$

then by Jacobi identity and flatness of $\mathbb{E}''$, we have

$$[\mathbb{E}'', [\mathbb{E}'', \theta]] = \frac{1}{2}[[\mathbb{E}'', \mathbb{E}''], \theta] = 0 \tag{3.3.22}$$

Similarly, we compute

$$[\mathbb{E}'_h, [\mathbb{E}'', \theta]] = [[\mathbb{E}'_h, \mathbb{E}''], \theta] - [\mathbb{E}'', [\mathbb{E}'_h, \theta]] \tag{3.3.23}$$

By equation (3.3.8), we can add $0 = [\mathbb{E}'_h, [\mathbb{E}'_h, \theta]]$ to the above equation and note that $\mathcal{R}_h = [\mathbb{E}'_h, \mathbb{E}'']$, we have

$$[\mathbb{E}'_h, [\mathbb{E}'', \theta]] = [\mathcal{R}_h, \theta] - [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \tag{3.3.24}$$

By lemma 3.3.1, we have $d^{\mathcal{M}}(\mathcal{R}_h) = [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]$ and therefore we can rewrite the above formula as

$$[\mathbb{E}_h, [\mathbb{E}'', \theta]] = [\mathbb{E}'_h, [\mathbb{E}'', \theta]] = [\mathcal{R}_h, \theta] - d^{\mathcal{M}}\mathcal{R}_h \tag{3.3.25}$$

so we have

$$\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathbb{E}_h, [\mathbb{E}'', \theta]]) = \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathcal{R}_h, \theta]) - \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)d^{\mathcal{M}}\mathcal{R}_h) \tag{3.3.26}$$

By the property of Tr_s and $\mathcal{R}_h$ has even total degree, we have

$$\begin{aligned}\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)d^{\mathcal{M}}\mathcal{R}_h) =& \sum_{i=1}^{n} \mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}d^{\mathcal{M}}\mathcal{R}_h) \\ =& \sum_{i=1}^{n} \mathrm{Tr}_s(\theta\mathcal{R}_h^{n-i}d^{\mathcal{M}}\mathcal{R}_h\mathcal{R}_h^{i-1}) \\ =& \mathrm{Tr}_s(\theta g(\mathcal{R}_h; d^{\mathcal{M}}\mathcal{R}_h))\end{aligned} \tag{3.3.27}$$

Using this equation, we can rewrite the last term in equation (3.3.15) in lemma 3.3.4 as:

$$\begin{aligned}
\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathbb{E}'_h,[\mathbb{E}'',\theta]]) =& \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta]]) - \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)d^{\mathcal{M}}\mathcal{R}_h) \\
=& \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta]]) + \mathrm{Tr}_s(g(\mathcal{R}_h;d^{\mathcal{M}}\mathcal{R}_h)\theta) \\
=& \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta]]) + d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\theta) \\
&+ \mathrm{Tr}_s(g(\mathcal{R}_h)\theta^2)
\end{aligned} \tag{3.3.28}$$

Finally we plug the equation (3.3.28) into equation (3.3.19), after collecting terms, we get

$$\begin{aligned}
&2d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta) + 2\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta^2) \\
&= \bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])\cdot\theta) - \partial^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) - \mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta])
\end{aligned} \tag{3.3.29}$$

We expand the last term $\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta])$ explicitly to get

$$\begin{aligned}
\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta]]) &= \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}\mathcal{R}_h\theta) - \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}\theta\mathcal{R}_h) \\
&= \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i+1}\theta) - \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i}\theta\mathcal{R}_h^{n-i}\theta) \\
&= \mathrm{Tr}_s(\theta\mathcal{R}_h^{n}\theta) - \mathrm{Tr}_s(\mathcal{R}_h^{n}\theta^2) = -2\mathrm{Tr}_s(\mathcal{R}_h^{n}\theta^2) \\
&= -2\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta^2)
\end{aligned} \tag{3.3.30}$$

Plug equation (3.3.30) into (3.3.29), we have

$$2d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta) = \bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'_h,\theta])\cdot\theta) - \partial^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) \tag{3.3.31}$$

Divide both sides by 2, we get the desired formula. □

We are now ready to define secondary Bott-Chern classes for cohesive modules with Hermitian structures. Recall $\mathcal{G} = \mathcal{G}^0$ is the space of (p, p) forms, let $\mathcal{G}'$ be the subspace of $\mathcal{G}$ defined by $\mathrm{Im}\partial^X + \mathrm{Im}\bar{\partial}^X$. We will define the secondary classes as elements in $\mathcal{G}^0/\mathcal{G}' \cap \mathcal{G}^0$.

Definition 3.3.6. Assume k_1, k_2 be two Hermitian metrics on a cohesive module E. For a convergent power series $f(T)$, we define the secondary Bott-Chern form $\tilde{f}(k_1, k_2)$ associated to k_1, k_2 as an element in $\mathcal{G}^0/\mathcal{G}' \cap \mathcal{G}^0$ with a representatives $\tilde{f}(k_1, k_2; \gamma)$ in $\mathcal{G}^0$ given by:

$$\tilde{f}(k_1, k_2; \gamma) = \int_\gamma \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) d\gamma \tag{3.3.32}$$

where $\gamma(t)$ is a curve on $\mathcal{M}$ that connects k_1 to k_2. The following proposition shows that this is well-defined.

Proposition 3.3.7. *The equivalence class of $\tilde{f}(k_1, k_2; \gamma)$ in $\mathcal{G}^0/\mathcal{G}' \cap \mathcal{G}^0$ is independent of γ.*

Proof. Let τ be a another path connecting k_1 to k_2, then by convexity of $\mathcal{M}$, the loop $\eta = \gamma - \tau$ is the boundary of a smooth 2-simplex σ in $\mathcal{M}$. By Stokes formula, we have

$$\tilde{f}(k_1, k_2; \gamma) - \tilde{f}(k_1, k_2; \tau) = \int_{\partial\sigma} \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) d\eta = \int_\sigma d^{\mathcal{M}} \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) d\sigma \tag{3.3.33}$$

By theorem 3.3.5, the integrand is an element in $\mathcal{G}'$, the result follows. □

Corollary 3.3.8. *The primary characteristic form is related to the secondary form via the following equation:*

$$f(E, h_2) - f(E, h_1) = -\bar{\partial}^X \partial^X \tilde{f}(h_1, h_2; \gamma) \tag{3.3.34}$$

for any path γ that connects h_1 to h_2.

Proof. By the Bott-Chern formula (3.2.11), we have

$$f(E, h_2) - f(E, h_1) = \int_\gamma d^{\mathcal{M}} \mathrm{Tr}_s(f(\mathcal{R}_h)) d\gamma \tag{3.3.35}$$

$$= -\mathrm{Tr}_s\{\int_\gamma \bar{\partial}^X \partial^X f'(\mathcal{R}_h) \cdot \theta d\gamma\} \tag{3.3.36}$$

the result follows from the definition of $\tilde{f}(h_1, h_2; \gamma)$. □

Remark 3.3.9. In [Don87] where similar formulas were derived for holomorphic vector bundles, the secondary characteristic classes were related to Yang-Mills functionals on the space of connections whose critical points correspond to Hermitian-Yang-Mills connections. We generalized them to cohesive modules and hope to relate them to stability theories in future research.

Chapter 4

Invariance of Bott-Chern classes under quasi-isomorphism

In this chapter, we study the Bott-Chern forms under deformation of the cohesive structures. Unlike the situation of Hermitian metrics, the characteristic classes will depend on the cohesive structures. In fact, this is why we want to refine the characteristic classes to take value in Bott-Chern cohomology since their image in deRham cohomology only depend on the underlying topological complex vector bundle structure and therefore are independent of the cohesive structure.

The main goal is to prove the invariance of characteristic classes under derived equivalences. We first prove that if the underlying complex is acyclic, the Bott-Chern cohomology classes are zero by explicitly constructing the $\partial\bar{\partial}$-transgression elements. Then by suitably choosing metrics on the mapping cone short exact sequence, we derive an additive property of characteristic forms. Together with a criteria of homotopy equivalence established in [Blo10], the main theorem will be proved.

4.1 Invariance under gauge transformation

Definition 4.1.1. For a $\mathbb{Z}$-graded Hermitian vector bundle $(E^\bullet, h)$, define $\mathcal{E}$ to be the space of $\bar{\partial}^X$-superconnections of total degree 1 and define $\mathcal{E}''$ to be subspace of $\mathcal{E}$ whose elements satisfy the flatness condition.

Definition 4.1.2. Recall the subspaces $\mathcal{G}^\bullet$ defined by exotic degrees in chapter 2. For each $k \in \mathbb{Z}$, we define $\mathcal{G}''^{,k}$ to be the subspace of $\mathcal{G}^k$ whose elements are forms of type $(0, \bullet)$. Similarly we define $\mathcal{G}'^{,k}$ to be the subspace of $\mathcal{G}^k$ whose elements are forms of type $(\bullet, 0)$.

Example 4.1.3. If $E^\bullet$ is concentrated in degree 0 and we set $\mathcal{A}$ to be the space of $\bar{\partial}^X$-connections and $\mathcal{A}''$ to be the subspace of $\bar{\partial}^X$-flat connections, then by Koszul-Malgrange theorem, $\mathcal{A}''$ is the space of holomorphic structures on E. If the underlying complex manifold X is a Riemann surface, then every $\bar{\partial}^X$-connection is automatically flat and $\mathcal{A}'' = \mathcal{A}$ is an affine space modeled on $\mathcal{A}^{0,1}(X, \mathrm{End}E)$.

We assume $\mathcal{E}''$ is non-empty and it has a marked point $\mathbb{E}''$ such that $E = (E^\bullet, \mathbb{E}'', h)$ is a Hermitian cohesive module. The following lemma is a straightforward consequence of the explicit construction of Chern superconnections in Proposition 2.2.5.

Lemma 4.1.4. *$\mathcal{E}$ is an affine space modeled on $\mathcal{G}'',^1$. $\mathcal{E}''$ is the subspace whose elements $\alpha'' \in \mathcal{G}'',^1$ satisfy the Maurer-Cartan equation:*

$$\mathbb{E}''(\alpha'') + \frac{1}{2}[\alpha'', \alpha''] = 0 \tag{4.1.1}$$

The $*$ operation interchanges $\mathcal{G}'',^\bullet$ and $\mathcal{G}',^{-\bullet}$. If $\mathbb{E}$ is the Chern superconnection of the Hermitian cohesive module E, then the space of Chern superconnection is in one-to-one correspondence with $\mathcal{E}''$. The correspondence is given by:

$$\mathbb{F} \xrightarrow{\bar{\partial}^X-\text{component}} \mathbb{F}'' \tag{4.1.2}$$

whose inverse is given by:

$$\mathbb{E} + \alpha' + \alpha'' \xleftarrow{\alpha' = (\alpha'')^*} \mathbb{E}'' + \alpha'' \tag{4.1.3}$$

Definition 4.1.5. By the above lemma, the tangent space $T_{\mathbb{E}''}\mathcal{E}''$ of $\mathcal{E}''$ at a point $\mathbb{E}''$ can be identified with the space of solutions in $\mathcal{G}'',^1$ to the equation:

$$\mathbb{E}''(\alpha'') = 0 \tag{4.1.4}$$

In particular, the tangent space is a subspace in $\mathcal{G}'',^1$. We define a one form δ'' on $\mathcal{E}''$ with values in $\mathcal{G}'',^1$ by the formula:

$$\delta''(\alpha'') = \alpha'', \ \forall \alpha'' \in T_{\mathbb{E}''}\mathcal{E}'' \tag{4.1.5}$$

By duality, we also define the one form γ' on $\mathcal{E}''$ with values in $\mathcal{G}',^{-1}$ by the formula:

$$\delta'(\alpha'') = (\alpha'')^* = \alpha' \tag{4.1.6}$$

Finally, we set $\delta = \delta' + \delta''$ to be an one form with values in $\mathcal{G}'',^1 \oplus \mathcal{G}',^{-1}$.

Lemma 4.1.6. *If $\mathbb{E}_t$ is a family of Chern superconnections such that $\mathbb{E}''|_{t=0} = \mathbb{E}''$ and $\mathbb{E}''_t = \mathbb{E}'' + A''_t$, then the deformation of the curvatures $\mathcal{R}_t$ is given by:*

$$\frac{d}{dt}\mathcal{R}_t = [\mathbb{E}_t, \frac{d}{dt}\mathbb{E}_t] \tag{4.1.7}$$

where $\mathbb{E}_t t$ is the Chern superconnection of $(E^\bullet, \mathbb{E}''_t, h)$. Using the one form δ defined above, we can rewrite this formulas as

$$d^{\mathcal{E}}\mathcal{R} = -[\mathbb{E}, \delta] \tag{4.1.8}$$

Proof. We write $\alpha_t = \alpha'_t + \alpha''_t$ for the tangent vectors $\frac{d}{dt}\mathbb{E}_t$ for short. For each t, α''_t satisfies:

$$[\mathbb{E}''_t, \alpha''_t] = 0 \tag{4.1.9}$$

and by duality, α'_t satisfies

$$[\mathbb{E}'_t, \alpha'_t] = 0 \tag{4.1.10}$$

By definition, $\mathcal{R}_t = [\mathbb{E}''_t, \mathbb{E}'_t]$, so if we take its derivative, we have

$$\frac{d}{dt}\mathcal{R}_t = [\alpha''_t, \mathbb{E}''_t] + [\alpha'_t, \mathbb{E}'_t] \tag{4.1.11}$$

Adding up the above equations, we get the desired equality. □

Proposition 4.1.7. *Let $f(T)$ be a convergent power series in T, the $\mathcal{E}$-directional differential of the Bott-Chern characteristic form $f(E, \mathbb{E}'', h)$ at $\mathbb{E}'' \in \mathcal{E}''$ is given by:*

$$d^{\mathcal{E}} f(\mathcal{R}) = -\mathrm{Tr}_s f'(\mathcal{R} \cdot [\mathbb{E}, \delta]) = -d^X \mathrm{Tr}_s f'(\mathcal{R} \cdot \delta) \tag{4.1.12}$$

Proof. The first equality is a direct consequence of previous lemma. The second equality follows from Bianchi identity $[\mathbb{E}, \mathcal{R}] = 0$ as before. $\square$

Corollary 4.1.8. *We compare the $\mathcal{G}^\bullet$ components of both sides of the equality* (4.1.12)*. We get the following identities:*

$$d^{\mathcal{E}}\mathrm{Tr}_s f(\mathcal{R}) = -\partial^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'') - \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta') \tag{4.1.13}$$

$$\partial^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta') = 0 \tag{4.1.14}$$

$$\bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'') = 0 \tag{4.1.15}$$

Equation (4.1.13) is the first transgression formula over $\mathcal{E}''$. Motivated by the results we established in the previous section, equation (4.1.14) and (4.1.15) are expected to admit a further transgression. However this is not the case in general for otherwise we would have proved $\mathrm{Tr}_s f(\mathcal{R})$ is even independent of the cohesive structure. We will show the terms $\mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta')$ and $\mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'')$ admit a further transgression formula when restricted to certain subspaces in $\mathcal{E}''$.

Definition 4.1.9. We define the generalized Dolbeault complex associated to a cohesive module E by $(\mathcal{G}''^{,\bullet}, \mathbb{E}'')$. By the flatness condition on $\mathbb{E}''$, it defines a differential. The cohomology of the complex is defined to be the Dolbeault cohomology of the cohesive module E.

Definition 4.1.10. Let $\mathbb{E}''_t$ be a one parameter family of cohesive structures. The tangent vectors $\alpha''_t = \dot{\mathbb{E}}''_t$ satisfies $\mathbb{E}''_t(\alpha''_t) = 0$ so they are pointwise closed with respect to the generalized Dolbeault operator $\mathbb{E}''_t$. We say the family is exact if there exist a smooth section γ_t'' with values in $\mathcal{G}''^{,0}$ such that

$$[\mathbb{E}_t'', \gamma_t''] = \delta_t'', \forall t \tag{4.1.16}$$

That is to say, the tangent vectors α_t'' defines zero cohomology class in the first Dolbeault cohomology groups and in addition, we can find a smooth lift of them.

Example 4.1.11. Consider the group $\mathrm{GL}(E)$ whose elements are of the form $f = \sum_{k=0}^{\dim X} f_k$ where $f_k \in \mathcal{A}^{0,k}(X, \mathrm{End}^{-k} E)$ and f_0 is invertible. Since $\mathcal{A}^{\bullet>0}$ is nilpotent and we required f_0 to be invertible, $\mathrm{GL}(E)$ forms a group. $\mathrm{GL}(E)$ acts on $\mathcal{E}$ and preserves the subspace $\mathcal{E}''$ via gauge transformation:

$$(\mathbb{E}'')^f = f^{-1} \circ \mathbb{E}'' \circ f = f^{-1} \circ [\mathbb{E}'', f] + \mathbb{E}'' \tag{4.1.17}$$

If f_t is a one parameter family of gauge group elements in $\mathrm{GL}(E)$ such that $f_0 = \mathrm{Id}_E$, then for any choice of $\mathbb{E}'' \in \mathcal{E}''$, we claim the one parameter family of cohesive structures $\mathbb{E}_t'' = (\mathbb{E}'')^{f_t}$ is exact.

To see this, we simply take $\gamma_t'' = f_t^{-1} \frac{d}{dt} f_t$ with values in $\mathcal{G}''^{,1}$ and we can compute

$$\frac{d}{dt}\mathbb{E}_t'' = \frac{d}{dt}(f_t^{-1} \circ [\mathbb{E}'', f_t] + \mathbb{E}'') \tag{4.1.18}$$

$$= -\gamma_t'' \circ f_t^{-1} \circ [\mathbb{E}'', f_t] + f_t^{-1} \circ [\mathbb{E}'', f_t \circ \gamma_t''] \tag{4.1.19}$$

$$= -\gamma_t'' \circ \mathbb{E}_t'' + \gamma_t'' \circ \mathbb{E}'' + \mathbb{E}_t'' \circ \gamma_t'' - \gamma_t'' \circ \mathbb{E}'' \tag{4.1.20}$$

$$= [\mathbb{E}_t'', \gamma_t''] \tag{4.1.21}$$

This shows that the family $\mathbb{E}_t''$ obtained by applying a family of gauge transformations is exact.

Remark 4.1.12. If E is again just a holomorphic vector bundle concentrated in degree 0, then $\mathrm{GL}(E)$ is just the group of invertible linear automorphisms. Two holomorphic vector bundle structure on E are equivalent if and only if they differ by a gauge transformation.

Definition 4.1.13. If $\mathcal{S}$ is a submanifold of $\mathcal{E}''$ for which the restriction δ'' is exact and admits a smooth lift $\gamma'' \in \mathcal{G}'',^0$ such that $[\mathbb{E}'', \gamma''] = \delta''$, we say $\mathcal{S}$ is exact.

Proposition 4.1.14. *If $\mathcal{S}$ is exact with a lift γ'' of δ'' and we set $\gamma' = (\gamma'')^*$, then we have the following identities:*

$$\bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R} \cdot \gamma'')) = \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'') \tag{4.1.22}$$

$$\partial^X \mathrm{Tr}_s(f'(\mathcal{R} \cdot \gamma')) = -\mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta') \tag{4.1.23}$$

Proof. By Bianchi identity, we have

$$d^X \mathrm{Tr}_s(f'(\mathcal{R} \cdot \gamma'')) = \mathrm{Tr}_s(f'(\mathcal{R}) \cdot [\mathbb{E}, \gamma'']) \tag{4.1.24}$$

We compare the $\mathcal{G}^2$ component of the equation and we get the first equation (4.1.22). If we replace γ'' by $\gamma' = (\gamma'')^*$, it satisfies

$$[\mathbb{E}', \gamma'] = -\delta' \tag{4.1.25}$$

by taking adjoints of $[\mathbb{E}'', \gamma''] = \delta''$. The same argument as above shows the second equation (4.1.23). □

Combining Proposition 4.1.7 and Proposition 4.1.14, we have the following double transgression formula for Bott-Chern forms over an exact submanifold.

Proposition 4.1.15. *With the same assumption as above, if we set* $\gamma = \gamma' + \gamma'' \in \mathcal{G}^0$*, the* $\mathcal{S}$*-directional differential of Bott-Chern forms is given by:*

$$d^{\mathcal{S}}\mathrm{Tr}_s f(\mathcal{R}) = -\partial^X \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \gamma) \tag{4.1.26}$$

As a corollary, the Bott-Chern characteristic classes remains invariant over S. We shall now apply this proposition to prove the invariance of Bott-Chern cohomology classes under homotopy equivalences.

For two cohesive modules $(E^\bullet, \mathbb{E}'')$ and $(F^\bullet, \mathbb{F}'')$, we have the following criteria for homotopy equivalence. For its proof, we refer to [Blo10].

Proposition 4.1.16. $(E^\bullet, \mathbb{E}'')$ *and* $(F^\bullet, \mathbb{F}'')$ *are equivalent in the homotopy category* $\mathrm{Ho}(\mathcal{P}_{\mathcal{A}})$ *if and only if there is a degree zero closed morphism* $\phi \in \mathcal{P}^0_{\mathcal{A}}$ *which is a quasi-isomorphism between the complexes* $(E^\bullet, \mathbb{E}''_0)$ *and* $(F^\bullet, \mathbb{F}''_0)$*.*

Using this criteria, the invariance of Bott-Chern classes will be proved in two steps. First we show that if the underlying complex $(E^\bullet, \mathbb{E}''_0)$ is acyclic, the characteristic classes are zero. Next we show that the Bott-Chern characteristic classes are additive with respect to short exact sequences of mapping cones and reduce to the acyclic case.

4.2 Transgression formula for acyclic complex

We start by define a rescaling operation on superconnections. If $t \in \mathbb{R}^+$ is a positive real constant and $\mathbb{E}$ is a superconnection, define

$$\mathbb{E}_t = \sum_k t^{1-k}\mathbb{E}_k \tag{4.2.1}$$

It's easy to verify that $\mathbb{E}''_t = (\mathbb{E}'')_t$ and $\mathbb{E}''_t$ are flat. So $\mathbb{E}''_t$ forms a smooth family of cohesive structures.

Definition 4.2.1. We define the grading operator N_E for the cohesive module E as an element in $\mathcal{A}^0(X, \mathrm{End}^0(E^\bullet))$ such that it acts by:

$$N_E(A \otimes \omega) = |A| \cdot A \otimes \omega \tag{4.2.2}$$

Lemma 4.2.2. *With N_E so defined, we choose $\gamma''_t = \frac{1}{t}N_E \in \mathcal{G}^0$ and we have*

$$[\mathbb{E}''_t, \gamma''_t] = \frac{1}{t}\sum_k (1-k)(\mathbb{E}''_{k,t}) = \frac{d}{dt}\mathbb{E}''_t \tag{4.2.3}$$

Proof.

$$\frac{d}{dt}\mathbb{E}''_t = \sum_k (1-k)t^{-k}\mathbb{E}''_k = \frac{1}{t}\sum_k (1-k)(\mathbb{E}''_k)_t \tag{4.2.4}$$

On the other hand, we may take $s \in E^d$ so we have

$$\mathbb{E}''_t \circ N_E(s) = d\sum_k t^{1-k}\mathbb{E}''_k(s) \tag{4.2.5}$$

and note that $\mathbb{E}''_k(s) \in \mathcal{A}^{0,k}(X, E^{d+1-k})$, so we have

$$N_E \circ \mathbb{E}''_{k,t}(s) = N_E \sum_k t^{1-k}\mathbb{E}''_k(s) = \sum_k t^{1-k}(d+1-k)\mathbb{E}''_k(s) \tag{4.2.6}$$

Taking the difference of the above two equalities and divide both sides by t, we get (4.2.3). □

Since N_E is clearly self adjoint, we have $\gamma' = \gamma'' = \frac{1}{t}N_E$ and we have the following corollary.

Corollary 4.2.3. *For the family of rescaled cohesive structures, we have*

$$\frac{d}{dt}\mathrm{Tr}_s(e^{-\mathcal{R}_t}) = -\frac{2}{t}\partial^X\bar{\partial}^X\mathrm{Tr}_s(e^{-\mathcal{R}_t}\cdot N_E) \tag{4.2.7}$$

Finally, by the same arguments in [BGS88a], if $(E^\bullet, \mathbb{E}_0'')$ is acyclic, then the degree zero component Δ_E of its curvature

$$\mathcal{R}_t^{0,0} = t^2(\mathbb{E}_0'' + \mathbb{E}_0')^2 = t^2\Delta_E$$

is a strictly positive element and the characteristic form of the Chern character

$$\mathrm{Tr}_s\exp(-\mathcal{R}_t) = \mathrm{Tr}_s\exp(-t^2\Delta_E + \mathcal{O}(t))$$

decays exponentially fast uniformly on X when t approaches ∞. Applying this result and let $t\to\infty$, we transgressed the characteristic forms to zero as desired.

Theorem 4.2.4. *If $(E^\bullet, \mathbb{E}'')$ is a cohesive module such that $(E^\bullet, \mathbb{E}_0'')$ is an acyclic complex, then the integral*

$$\mathcal{I}_E = \int_1^\infty \mathrm{Tr}_s(\exp(-\mathcal{R}_t)\cdot N_E)\frac{dt}{t} \tag{4.2.8}$$

is finite and

$$\mathrm{Tr}_s\exp(-\mathcal{R}) = \partial^X\bar{\partial}^X\mathcal{I}_E \tag{4.2.9}$$

4.3 Additive properties of Bott-Chern classes

We begin by defining the mapping cone of a closed morphism. Let $\phi \in \mathcal{P}^0_{\mathcal{A}}(E, F)$ be a closed degree 0 morphism between two cohesive modules $(E^\bullet, \mathbb{E}'')$ and $(F^\bullet, \mathbb{F}'')$.

Definition 4.3.1. The mapping cone $\mathrm{Cone}(\phi)$ is the cohesive module $(\mathrm{Cone}(\phi)^\bullet, \mathbb{C}''_\phi)$ whose underlying complex vector bundle is defined by:

$$\mathrm{Cone}(\phi)^\bullet = F^\bullet \oplus E^{\bullet+1}$$

and with respect to this decomposition, the cohesive structure $\mathbb{C}''_\phi$ is given by:

$$\mathbb{C}''_\phi = \begin{pmatrix} \mathbb{F}'' & \phi \\ 0 & -\mathbb{E}'' \end{pmatrix}$$

Consider the one parameter family of morphisms $\phi_t = t\phi$ with $t \in [0, 1]$ such that $\phi_0 = 0$ and $\phi_1 = \phi$. The corresponding mapping cones have the same underlying bundle $\mathrm{Cone}(\phi)^\bullet$ and cohesive structures $\mathbb{C}''_t$. It's a simple calculation that

$$\frac{d}{dt}\mathbb{C}''_{\phi_t} = \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix} = \alpha'' \tag{4.3.1}$$

is constant. If we set for $t > 0$, $\gamma''_t \in \mathcal{G}''^{,0}$ by the formula:

$$\gamma''_t = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{t} \cdot \mathrm{Id}_F \end{pmatrix} = \frac{1}{t}\gamma''$$

then it satisfies that

$$[\mathbb{C}''_t, \gamma''_t] = \frac{d}{dt}\mathbb{C}''_t \tag{4.3.2}$$

So we can apply the previous Proposition 4.1.15 for all nonzero value of t. Note again in this case $\gamma' = \gamma''$, we have the following equality of characteristic forms.

$$f(\mathrm{Cone}(\phi), \mathcal{C}_1'') - f(\mathrm{Cone}(\phi), \mathcal{C}_s'') = -\partial^X \bar{\partial}^X [\int_s^1 2\mathrm{Tr}_s f'(\mathcal{R}_t) \cdot \gamma'' \frac{dt}{t}] \tag{4.3.3}$$

It's clear that we can't let $t \to 0$ in the above formula due to the singularity. Instead, motivated by the construction in [BC65], we will modify the integrand to remove the singularity.

To do this, we calculate $\mathcal{R}_t$ explicitly as

$$\mathcal{R}_t = \begin{pmatrix} \mathcal{R}_F + t^2\phi\phi^* & t(\mathbb{F}'\phi - \phi\mathbb{E}') \\ t(\phi^*\mathbb{F}'' - \mathbb{E}''\phi^*) & \mathcal{R}_E + t^2\phi^*\phi \end{pmatrix} = \mathcal{R}_0 + tA_t \tag{4.3.4}$$

where $\mathcal{R}_0$ is the curvature computed by $\mathbb{C}_0''$ using the direct sum Hermitian form

$$\mathcal{R}_0 = \begin{pmatrix} \mathcal{R}_F & 0 \\ 0 & \mathcal{R}_E \end{pmatrix} \tag{4.3.5}$$

and A_t is the reminder term

$$A_t = \begin{pmatrix} t\phi\phi^* & \mathbb{F}'\phi - \phi\mathbb{E}' \\ \phi^*\mathbb{F}'' - \mathbb{E}''\phi^* & t\phi^*\phi \end{pmatrix} \tag{4.3.6}$$

If we evaluate a convergent power series $g(T)$ on $\mathcal{R}_t$, we have

$$g(\mathcal{R}_t) = g(\mathcal{R}_0 + tA_t) = g(\mathcal{R}_0) + tR_g(t, \mathcal{R}_0, A_t) \tag{4.3.7}$$

for some reminder term R_g that is a power series in t with coefficients that are polynomials in $\mathcal{R}_0$ and A_t.

We have already shown that $\mathrm{Tr}_s f(\mathcal{R}_0)$ is d^X-closed by Proposition 2.3.6, therefore it is also $\partial^X\bar{\partial}^X$-closed. So we can subtract it from the above equation and derive the following equation:

$$f(\mathrm{Cone}(\phi), \mathcal{C}_1'') - f(\mathrm{Cone}(\phi), \mathcal{C}_s'') = -2\partial^X\bar{\partial}^X \int_s^1 (f'(\mathcal{R}_t)\gamma'' - f'(\mathcal{R}_0)\gamma'')\frac{dt}{t} \quad (4.3.8)$$

Now we substitute equation (4.3.7) with $g = f'$ into the above equality, the singular term $\frac{1}{t}$ cancels out and we get

$$f(\mathrm{Cone}(\phi), \mathcal{C}_1'') - f(\mathrm{Cone}(\phi), \mathcal{C}_s'') = -2\partial^X\bar{\partial}^X \int_s^1 R_{f'}(t, A_t, \mathcal{R}_0)dt \quad (4.3.9)$$

Now the we can let $s \to 0$ since the integrand is bounded and we established the following proposition.

Proposition 4.3.2. *The characteristic form $f(\mathrm{Cone}(\phi), \mathcal{C}_1'')$ coincide with $f(\mathrm{Cone}(\phi), \mathcal{C}_0'')$ in Bott Chern cohomology.*

Under the direct sum Hermitian metric $h_E \oplus h_F$ on the mapping cone $\mathrm{Cone}^\bullet$, we have the following simple equality of differential forms:

$$f(\mathrm{Cone}(\phi)^\bullet, \mathbb{C}_0'', h_\phi) = f(F^\bullet, \mathbb{F}'', h_F) - f(E^\bullet, \mathbb{E}'', h_E) \quad (4.3.10)$$

where the minus sign comes from the shift in degree of $E^\bullet$ in the mapping cone. Together with the previous corollary, we established the following proposition.

Proposition 4.3.3. *The Bott Chern cohomology classes of an exact triangle*

$$0 \to (E^\bullet, \mathbb{E}'') \xrightarrow{\phi} (F^\bullet, \mathbb{F}'') \xrightarrow{i_F} (\mathrm{Cone}^\bullet(\phi), \mathbb{C}_\phi)$$

is additive in the sense

$$f(E^\bullet, \mathbb{E}'') - f(F^\bullet, \mathbb{F}'') + f(\text{Cone}^\bullet(\phi), \mathbb{C}_\phi) = 0 \tag{4.3.11}$$

in Bott-Chern cohomology

If ϕ is a homotopy equivalence, it was shown in [Blo10] that this is equivalent to $\phi_0 : (E^\bullet, \mathbb{E}''_0) \to (F^\bullet, \mathbb{F}''_0)$ being a quasi-isomorphism. Therefore the mapping cone Cone(ϕ) is acyclic so proposition 4.2.4 shows that the left side of equation (4.3.10) is zero in Bott-Chern cohomology. Combining these, we established our main result below.

Theorem 4.3.4. *If two cohesive modules* $(E^\bullet, \mathbb{E}'')$ *and* $(F^\bullet, \mathbb{F}'')$ *are homotopy equivalent, then they have the same Bott-Chern cohomology classes.*

As a corollary, we can define characteristic classes for an object $\mathscr{S}$ in $D^b_{\text{Coh}}(X)$ as follows. We choose a cohesive module representative $(E^\bullet, \mathbb{E}'')$ and equip it with some Hermitian structure h_E. Then for any convergent power series $f(T)$, we define the associated class $f(\mathscr{S})$ as the class of $f(E^\bullet, \mathbb{E}, h_E)$.

In [Bis13], Bismut obtained a Grothendieck-Riemann-Roch theorem with values in Bott-Chern cohomology for holomorphic submersions between compact complex manifolds. The objects are holomorphic vector bundles and one requires the push-forward to have constant dimension, that is, the push-forward remains to be holomorphic vector bundle. In addition, he raised the question of defining the Bott-Chern classes for general coherent sheaves and extend his arguments. In the next two chapters, we will follow his analytical approach to obtain a Grothendieck-Riemann-Roch type theorem in Bott-Chern cohomology for submersions that are Kähler fibration. It is of future interest to try to combine the analytic techniques of Bismut and our treatment of characteristic classes for coherent sheaves to prove the Grothendieck-Riemann-Roch theorem in Bott-Chern cohomology in full generality.

Chapter 5

Generalized Dolbeault-Dirac operator on cohesive module

In the previous three chapters, we defined and studied in detail the characteristic forms on cohesive modules induced by the cohesive structure and an arbitrary hermitian metric. We see much of the classical theory concerning holomorphic vector bundles can be generalized to cohesive modules. From this chapter, we study the analytical objects naturally attached to cohesive modules.

Using Dolbeault resolution, the Dolbeault complex $(\mathcal{A}^{0,\bullet}(X,E),\bar{\partial}^E)$ computes the cohomology of a holomorphic vector bundle. By Hodge theory, the $\bar{\partial}^E$-harmonic forms computes the Dolbeault cohomology. The cohesive structure $\mathbb{E}''$ is a natural generalization of the $\bar{\partial}$-operator and we use the heat kernel approach method to compute the $\mathbb{E}''$-harmonic forms.

In this chapter we study the operator $D^E = \sqrt{2}(\mathbb{E}'' + \mathbb{E}'',*)$ on a cohesive module E. It is a natural generalization of the classical Dolbeault-Dirac operator $\bar{\partial}^E + \bar{\partial}^{E,*}$ associated with a holomorphic Hermitian vector bundle E whose index is the Euler characteristic of E.

We first identify D^E as a generalized Dirac operator explicitly by constructing the Clifford superconnection $\mathbb{A}^E$ associated to it. It is well-known that for a holomorphic vector bundle E on a Kähler manifold X, $\mathbb{A}^E$ is just the induced Levi-Civita connection ∇^{LC}. For holomorphic vector bundles on non-Kähler manifolds, $\mathbb{A}^E$ is a superconnection which involves the torsion of Chern connection. We refer to [MM06] for a detailed treatment. After we explicitly identified D^E, we apply a theorem of Getzler in [Get91] to obtain the local index formula.

5.1 Clifford module and Dirac operators

We begin by reviewing the concepts of Clifford module, Clifford superconnection, and generalized Dirac operator. For a more detailed presentation of these material and proof of the results cited, we refer to [BGV91].

Let X be a compact even dimensional manifold with Riemannian metric g. We also denote the induced metric on the cotangent bundle T^*X by g. Let $\mathcal{A}^\bullet(X)$ be the space of differential forms on X. The bundle of Clifford algebras is denoted by $C(X)$ whose fiber at $x \in X$ is the Clifford algebra $C(T^*_xX, g_x)$ of the cotangent vector space at x. Let ∇^{LC} be the Levi-Civita on TX and we use the same notation for its induced connection on T^*X. Since ∇^{LC} preserves g, it lifts to a connection on $C(X)$.

Definition 5.1.1. A Clifford module of $C(M)$ is a $\mathbb{Z}_2$-graded vector bundle $E^{\pm}$ on M with an even action of $C(M)$ on it.

Example 5.1.2. Let $\Lambda^{\bullet}T^*X$ be the bundle of exterior algebras with $C(M)$ action induced by

$$c(v) \cdot \eta = v \wedge \eta - \iota(v^*)\eta \tag{5.1.1}$$

where $v \in T^*X$ and $v^* \in TX$ is the dual tangent vector under metric g.

Definition 5.1.3. Let σ be the symbol map $\sigma : C(X) \otimes \mathbb{C} \to \Lambda^{\bullet}T^*X \otimes \mathbb{C}$ defined by the formula:

$$\sigma(a) = c(a) \cdot 1 \tag{5.1.2}$$

where the Clifford action is the one in example 5.1.2. Then σ is an isomorphism of $\mathbb{Z}_2$-graded vector bundles. Its inverse σ^{-1} is called the quantization map.

Example 5.1.4. For a complex manifold X with Hermitian metric h and underlying Riemannian metric g, there is a canonical polarization of (T^*X, g) by forms of type $(1,0)$ and $(0,1)$:

$$T^*X \otimes \mathbb{C} = T^*_{\mathbb{C}}X = T^{*1,0}X \oplus T^{*0,1}X \tag{5.1.3}$$

Let $\Lambda^{\bullet}T^{*0,1}X$ be the subbundle of anti-holomorphic forms, then it is a Clifford module with $C(X)$ action induced by

$$c(v) \cdot \eta = \sqrt{2}(v^{0,1} \wedge \eta - \iota(v^{*1,0})\eta) \tag{5.1.4}$$

where $v = v^{1,0} + v^{0,1}$ is the decomposition and $v^{*1,0}$ is a tangent vector of type $(0,1)$ dual to $v^{1,0}$ under the complexified Riemannian metric.

Definition 5.1.5. A connection $\nabla^{\mathcal{E}}$ on a Clifford module $\mathcal{E}$ is called a Clifford connection if for any vector field Z on X and $a \in \mathcal{A}^0(X, C(X))$, we have:

$$[\nabla^{\mathcal{E}}_Z, c(a)] = c(\nabla^{LC}_Z a) \tag{5.1.5}$$

A superconnection $\mathbb{A}$ on a Clifford module $\mathcal{E}$ is called a Clifford superconnection if for any $a \in \mathcal{A}^0(X, C(X))$, we have:

$$[\mathbb{A}, c(a)] = c(\nabla^{LC} a) \tag{5.1.6}$$

where $\nabla^{LC} a$ is in $\mathcal{A}^1(X, C(X))$ and both sides are viewed as odd endomorphisms on $\mathcal{A}^\bullet(X, E^\pm)$.

Locally on X, we can choose a Spin structure and form the spinor bundle $\mathscr{S}$ and every Clifford module E is of the form:

$$E = \mathscr{S} \otimes W \tag{5.1.7}$$

for some vector bundle W, called the twisting bundle. If we choose a orthonormal frame e_i on TX with dual frame e^i, and let Γ^i_{jk} be the Christoffel symbols associated with the Levi-Civita connection ∇^{LC}, then the following formula

$$\nabla^{\mathscr{S}} = d + e^i \otimes \frac{1}{4}\omega^i_{jk} c(e^j) c(e^k) \tag{5.1.8}$$

defines a Clifford connection on $\mathscr{S}$. We then have the following simple result concerning Clifford superconnections on a general Clifford module $\mathcal{E}$.

Proposition 5.1.6. *If $\mathcal{E} = \mathscr{S} \otimes W$, there is a one-to-one correspondence between superconnections $\mathbb{A}^W$ on W and Clifford superconnections on $\mathcal{E}$, induced by the correspondence:*

$$\mathbb{A}^W \longrightarrow \nabla^{\mathscr{S}} \otimes 1 + 1 \otimes \mathbb{A}^W \tag{5.1.9}$$

If we can construct Clifford superconnections on a Clifford module E locally, we can apply partition of unity to glue local Clifford superconnections into a global one. The previous proposition, together with the existence of superconnections locally, implies the existence of Clifford superconnections on Clifford modules in general.

Definition 5.1.7. For a Clifford module $\mathcal{E}$ with Clifford superconnection $\mathbb{A}$, we define the associated generalized Dirac operator $\not{D}^{\mathbb{A}}$ by the following composition of maps:

$$\mathcal{A}^0(X, \mathcal{E}) \xrightarrow{\mathbb{A}} \mathcal{A}^\bullet(X, \mathcal{E}) \xrightarrow{\sigma^{-1}} \mathcal{A}^0(X, C(X) \otimes \mathcal{E}) \xrightarrow{c} \mathcal{A}^0(X, \mathcal{E})$$

Since $\mathbb{A}$ is an odd operator while both quantization map and Clifford action are even, $\not{D}^{\mathbb{A}}$ is an odd operator that interchanges sections of $\mathcal{E}^\pm$.

When the superconnection $\mathbb{A}$ is in fact a connection, we recovered the classical notion of twisted Dirac operators on Spin manifolds. In general, these generalized Dirac operators are square roots of generalized Laplacian operators in the following sense.

Proposition 5.1.8. *If D is a first order differential operator of odd parity on a $\mathbb{Z}_2$-graded vector bundle $V^\pm$ such that D^2 is a generalized Laplacian, that is, D^2*

satisfies the equation

$$[[D^2, f], f] = -2g(df, df) \tag{5.1.10}$$

for any smooth function $f \in \mathcal{A}^0(X)$*, then the following formula defines a Clifford module structure on* $V^{\pm}$

$$c(df) = [D, f] \tag{5.1.11}$$

and there exist an unique Clifford superconnection $\mathbb{A}$ *on* E *such that* $D = \not{D}^{\mathbb{A}}$

5.2 Spin Dirac operators

We assume X is an n-dimensional compact complex hermitian manifold with underlying Riemannian metric g. By example 5.1.4, the bundle of anti-holomorphic forms is a Clifford module. If X is spin, then we have an isomorphism

$$\Lambda^\bullet T^{*0,1}X = \mathscr{S} \otimes K^{-1/2} \tag{5.2.1}$$

where K is the canonical line bundle $\det T^{*1,0}X$ and a choice of spin structure is the same as a choice of square root of K. We will use the decomposition at least locally on X where a spin structure exist.

We are mainly interested in the Dolbeault-Dirac operator

$$D^X = \bar{\partial}^X + \bar{\partial}^{X*} \tag{5.2.2}$$

It's clear that D^X satisfies the conditions of proposition 5.1.8 and the induced Clifford module structure is the previous one considered in example 5.1.4, our first goal is to explicitly describe the associated Clifford superconnection.

Definition 5.2.1. Let $w^i, \bar{w}^i$ be local unitary basis for $T^{*1,0}X$ and $T^{*0,1}X$ respectively, and we let $w_i, \bar{w}_i$ be the dual basis. Set

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j) \qquad\qquad e_{2j} = \frac{i}{\sqrt{2}}(w_j - \bar{w}_j)$$

then e_k is an orthogonal basis with dual basis e^k given by similar formulas in terms of $w^j, \bar{w}^j$. With these local frames, $\Lambda^\bullet T^{*0,1}X$ has basis

$$\bar{w}^{i_1} \wedge \bar{w}^{i_2} ... \wedge \bar{w}^{i_s}, i_1 < i_2 < ... < i_s, s \leq n$$

and we write $\bar{w}^I$ for the previous expression with increasing multi-index $I = (i_1, i_2, ..., i_s)$. We also obtain a frame $w^1 \wedge w^2 ... \wedge w^n$ for the canonical bundle K.

Using these local frame and our explicit formula for the Clifford connection on spinor bundles (5.1.8), the following proposition is clear from our local decomposition (5.2.1).

Proposition 5.2.2. *If ω^{LC} is the connection one form for the Levi-Civita connection under the frame $\{e_k, k \leq 2n\}$, and $\omega^{\det}$ is the connection one form for the holomorphic Chern connection on K under the frame $w^1 \wedge w^2 ... \wedge w^n$, then the following expression defines a connection ∇^{Cl} on $\Lambda^\bullet T^{*0,1}X$ under the frame $\{\bar{w}^I, |I| \leq n\}$*

$$\nabla^{Cl} = d + \frac{1}{4} g(\omega^{LC} e_i, e_j) c(e^i) c(e^j) - \frac{1}{2}\omega^{\det} \tag{5.2.3}$$

∇^{Cl} is independent of the choice of basis and is a Clifford connection.

Definition 5.2.3. Denote by ∇^C the Chern connection of TX, viewed as the holomorphic tangent bundle, canonically isomorphic to $T^{1,0}X$. Let

$$S = \nabla^C - \nabla^{LC} \tag{5.2.4}$$

as an element in $\mathcal{A}^1(X, \mathrm{End}(TX))$, the torsion T is then computed by

$$T(U,V) = S(U)V - S(V)U \tag{5.2.5}$$

We define the anti-symmetrization of T as the 3-form on X by

$$T_{as} = \sum_{i,j,k} \frac{1}{2} g(T(e_i, e_j), e_k) e^i \wedge e^j \wedge e^k \tag{5.2.6}$$

Since the Chern connection preserves the almost complex structure and metric, using the well known-formula $d = \nabla + \iota(T)$, we have the following simple lemma.

Lemma 5.2.4. *If Θ is the fundamental form of the hermitian metric defined by*

$$\Theta(U,V) = g(JU,V) \tag{5.2.7}$$

where J is the almost complex structure. Then

$$T_{as} = -i(\partial^X - \bar{\partial}^X)\Theta \tag{5.2.8}$$

The following proposition was proved in [Bis89], for a detailed proof, see [MM06].

Proposition 5.2.5. *The Dolbeault-Dirac operator D^X and the generalized Dirac operator $\not{D}^{\nabla^{Cl}}$ are related by the torsion form as*

$$D^X = \not{D}^{\nabla^{Cl}} + \frac{1}{4} c(T_{as}) \tag{5.2.9}$$

Definition 5.2.6. Since T_{as} is a real 3-form on X, we may regard $d + \frac{1}{4}T_{as}$ as a Hermitian superconnection on the trivial twisting bundle and modify ∇^{Cl} by setting

$$\mathbb{A}^T = \nabla^{Cl} + \frac{1}{4} T_{as} \tag{5.2.10}$$

then $\mathbb{A}^T$ is a Clifford superconnection and the associated generalized Dirac operator coincide with D^X.

If E is a holomorphic hermitian complex vector bundle on X, there is a canonical $\bar{\partial}^X$-connection $\bar{\partial}^E$on $\mathcal{A}^{0,\bullet}(X,E)$ and we can form the Dolbeault-Dirac operator

$$D^E = \sqrt{2}(\bar{\partial}^E + \bar{\partial}^{E*}) \tag{5.2.11}$$

If we denote by $\nabla^{Cl,E}$ the tensor product connection $\nabla^{Cl} \otimes 1 + 1 \otimes \nabla^E$ where ∇^E is the holomorphic hermitian Chern connection on E, we have the following generalization of Proposition 5.2.5.

Proposition 5.2.7. $\nabla^{Cl,E}$ *is a Clifford connection on the Clifford module* $\mathcal{A}^{0,\bullet}(X,E)$ *and*

$$D^E = \not{D}^{\nabla^{Cl,E}} + \frac{1}{4}c(T_{as}) \tag{5.2.12}$$

Therefore if we modify the Clifford connection by

$$\mathbb{A}^{E,T} = \mathbb{A}^T \otimes 1 + 1 \otimes \nabla^E \tag{5.2.13}$$

where A^T *was defined above, then* $\mathbb{A}^{E,T}$ *is a Clifford superconnection on* $\mathcal{A}^{0,\bullet}(X,E)$ *with associated generalized Dirac operator* D^E*.*

5.3 Generalized Dolbeault-Dirac operators

For a cohesive module E with hermitian form h, we can form the generalized Dolbeault-Dirac operator

$$D^E = \sqrt{2}(\mathbb{E}'' + \mathbb{E}''^{,*}) \tag{5.3.1}$$

as an differential operator on the space of sections $\mathcal{A}^{0,\bullet}(X, E^\bullet)$ of odd parity. It is obvious that this operator coincide with the classical Dolbeault-Dirac operator when E is just a single holomorphic vector bundle.

To express our generalized Dolbeault-Dirac operator as a generalized Dirac operator associated with certain Clifford superconnection, we first prove the following two simple lemmas which relate the linear terms in D^E to Clifford actions.

Lemma 5.3.1. *If ω is a form of type $(0, \bullet)$, then on the Clifford module $\mathcal{A}^{0,\bullet}(X)$, the Clifford action coincide with exterior product*

$$c(\sigma^{-1}\omega)s = \omega \wedge s \tag{5.3.2}$$

Proof. The statement is clearly local and we can work with unitary frames $\omega_I, \bar{\omega}_I$ defined in 5.2.1. Note that σ^{-1} is uniquely determined by:

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \cdot e_{i_k} \xrightarrow{\sigma^{-1}} e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k} \tag{5.3.3}$$

we have the following extension to complexified algebras

$$\sigma^{-1}(\bar{\omega}_{i_1} \wedge \bar{\omega}_{i_2} \wedge \ldots \wedge \bar{\omega}_{i_k}) = \bar{\omega}_{i_1} \cdot \bar{\omega}_{i_2} \cdot \ldots \cdot \bar{\omega}_{i_k} \tag{5.3.4}$$

and the result follows. □

Lemma 5.3.2. *If η is a form of type $(\bullet, 0)$ and let $\omega = \eta^*$ be its adjoint defined in 2.1.7, the on the Clifford module $\mathcal{A}^{0,\bullet}(X)$, the Clifford action of η coincide with the Hodge dual of the Clifford action of ω*

$$< c(\sigma^{-1}\omega)s, t >=< s, c(\sigma^{-1}\eta)t > \tag{5.3.5}$$

Proof. We can either perform a local computation as above, or use the result in 2.1.12 and the fact that the Clifford module $\mathcal{A}^{0,\bullet}(X)$ is unitary. In fact, this is the motivation to define the star operation at the beginning. $\square$

If we work locally on X, we may assume the vector bundles $E^\bullet$ are given a holomorphic structures and we can construct the holomorphic Chern connections $\nabla^{E^\bullet}$, we have by Proposition 5.2.7

$$\sqrt{2}(\bar{\partial}^{E^\bullet} + \bar{\partial}^{E^\bullet *}) = c \circ \sigma^{-1} \circ (\mathbb{A}^T \otimes 1 + 1 \otimes \nabla^{E^\bullet}) \tag{5.3.6}$$

Using the previous two lemmas and the explicit local decomposition in Proposition 2.2.5, for the terms $\mathbb{E}''_k, \mathbb{E}'_k$ of homogeneous bi-degree $(0, k, 1-k)$ and $(k, 0, k-1)$, we have

$$c \circ \sigma^{-1}(1 \otimes \mathbb{E}''_k + 1 \otimes \mathbb{E}'_k) = \sqrt{2}(\mathbb{E}''_k + \mathbb{E}''^*_k) \tag{5.3.7}$$

Adding these equations (one for each k) together, we established the following result.

Proposition 5.3.3. *If we define the modified superconnection by $\mathbb{A}^{E,T}$ by*

$$\mathbb{A}^{E,T} = \mathbb{A}^T \otimes 1 + 1 \otimes \mathbb{E} \tag{5.3.8}$$

Then $\mathbb{A}^{E,T}$ is a Clifford superconnection on the Clifford module $\mathcal{A}^{0,\bullet}(X, E^\bullet)$ and the associated generalized Dirac operator is the generalized Dolbeault-Dirac operator for the hermitian cohesive module $(E, \mathbb{E}'', h)$.

5.4 Local index formulas

For an arbitrary superconnection $\mathbb{A}$ on a super bundle E and positive real number s, we recall the scaled superconnection $\mathbb{A}_s$ defined at the beginning of section 4.2,

$$\mathbb{A}_s = \sum_k s^{(1-k)/2} \mathbb{A}_k \tag{5.4.1}$$

If E is a Clifford module and $\mathbb{A}$ is a Clifford superconnection, then it is easy to verify that the family of scaled superconnections $\mathbb{A}_s$ are Clifford superconnections and we denote by $D_s^{\mathbb{A}} = \not{D}^{\mathbb{A}_s}$ the generalized Dirac operators associated to them. When the superconnections are clear from the context, we write D_s for short.

Definition 5.4.1. It follows from the general theory of elliptic operators that the heat semigroup $\exp(-tD_s^2)$ associated to the generalized Laplacian D_s^s are smoothing kernels. We define the integral kernels $p_{t,s}(x, y)$ associated with the operators D_s such that for any test section $\phi \in \mathcal{A}^{0,\bullet}(X, E^\bullet)$, we have

$$\exp(-tD_s^2)\phi(x) = \int_X p_{t,s}(x, y)\phi(y)dy \tag{5.4.2}$$

Definition 5.4.2. We define the chirality operator Γ on the Clifford algebra by

$$\Gamma = i^n e^1 e^2 ... e^{2n} \tag{5.4.3}$$

For a Clifford module $\mathcal{E}$, we define the relative supertrace $\mathrm{Str}_{\mathcal{E}/\mathscr{S}}$ on forms with values in $C(X)$-endomorphism bundle $\mathcal{A}^\bullet(X, \mathrm{End}_{C(X)}(\mathcal{E}))$ by the formula

$$\mathrm{Str}_{\mathcal{E}/\mathscr{S}}(a) = 2^{-n/2} \mathrm{Str}_{\mathcal{E}}(c(\Gamma)a) \tag{5.4.4}$$

In the local decomposition (5.1.7), we have an induced isomorphism

$$\mathrm{End}(\mathscr{S} \otimes W) = C(X) \otimes \mathrm{End}(W)$$

and $\mathrm{End}_{C(X)}(\mathscr{S} \otimes W) = \mathrm{End}(W)$, the relative supertrace coincide with Str_W.

Definition 5.4.3. We define a supertrace Tr_s on the Clifford algebra by

$$\mathrm{Tr}_s(a) = \mathrm{Tr}_{\mathscr{S}}(\Gamma a) \tag{5.4.5}$$

It is related to the symbol map and Berezin integral T by the formula

$$\mathrm{Tr}_s(a) = (-2i)^n T \circ \sigma(a) \tag{5.4.6}$$

Recall that both the exterior algebra and Clifford algebra are $\mathbb{Z}$-filtered algebra and the symbol map preserves the filtration. It follows that $\mathrm{Tr}_s(a)$ vanishes on the elements of Clifford filtration strictly less that $2n$.

We have the following theorem due to Getzler in [Get91] concerning the small t asymptotic expansion of $p_{t,s}(x, y)$ along the diagonal.

Proposition 5.4.4. *The heat kernel $p_{t,s/t}(x, x)$ has an asymptotic expansion of the form*

$$p_{t,s}(x, x) = (4\pi t)^{-n} \sum_{i \geq 0}^{N} a_i(x, s) t^i + O(t^{N-n+1/2}) \tag{5.4.7}$$

where $a_i(x, s) \in \mathrm{End}(\mathcal{E}_x)$ vary smoothly in x, s. Under the canonical isomorphism

$$\mathrm{End}(\mathcal{E}) = C(X) \otimes \mathrm{End}_{C(X)}(\mathcal{E}) \tag{5.4.8}$$

we have $a_i(x,s) \in C_{2i}(X) \otimes \mathrm{End}_{C(X)}(\mathcal{E})$ *where* $C_{2i}(X)$ *is the Clifford filtration of degree* $2i$*. If we apply the symbol map* σ *to kernel functions and set* $s = 1/t$*, we get an asymptotic expansion*

$$\sigma(p_{t,1/t}) \sim (4\pi t)^{-n} \sum_{i \geq 0} A_i(x) t^i \tag{5.4.9}$$

where $A_i(x) \in \oplus_{j \leq 2i} \mathcal{A}^j(X, \mathrm{End}_{C(X)}\mathcal{E})$*.*

Definition 5.4.5. The full symbol of the heat kernel p is defined by

$$\sigma(p) = \sum_{i=0}^{n} [A_i]_{(2i)} \tag{5.4.10}$$

where $[A_i]_{(2i)}$ is the degree $2i$ form component of A_i.

The following result is well-known in the heat kernel proof of the classical Atiyah-Singer index formula, for a proof of the general case, we refer to [Get91] and [BGV91].

Theorem 5.4.6. *If* $\mathcal{R}^{\mathcal{E}/\mathscr{S}}$ *is the relative curvature of the Clifford module* $\mathcal{E}/\mathscr{S}$*, then the full symbol is given by*

$$\sigma(p) = \det{}^{1/2}\left(\frac{\mathcal{R}^{LC}/2}{\sinh(\mathcal{R}^{LC}/2)}\right) \cdot \exp(-\mathcal{R}^{\mathcal{E}/\mathscr{S}}) \tag{5.4.11}$$

As we remarked before, the modification term T_{as} is a three form on X and $\nabla^T = d + \frac{1}{4}T_{as}$ can be viewed as a superconnection on the trivial twisting bundle.

Lemma 5.4.7. *Let* $\mathcal{R}^T$ *be the curvature of* ∇^T*, it is computed by*

$$\mathcal{R}^T = \frac{1}{4} dT_{as} = \frac{i}{2} \partial^X \bar{\partial}^X \Theta \tag{5.4.12}$$

consequently, it is an exact form in both deRham cohomology and Bott-Chern cohomology.

Lemma 5.4.8. *If we apply the relative supertrace to the full symbol, and under the local decomposition*

$$\Lambda^\bullet T^{*0,1}X = \mathscr{S} \otimes E^\bullet \otimes \mathbb{C} \otimes K^{-1/2}$$

we have

$$\mathrm{Str}_{\mathcal{E}/\mathscr{S}}\sigma(p) = \hat{A}(\nabla^{LC}) \cdot \mathrm{ch}(\mathbb{E}) \cdot \mathrm{ch}(\nabla^T) \cdot \mathrm{ch}(\nabla^{K^{-1/2}}) \tag{5.4.13}$$

Consequently, if we apply the supertrace on $\mathrm{End}\mathcal{E}$ *by first applying supertrace on* $C(X)$*, we have*

$$\mathrm{Str}_{C(X)} p_{t,1/t}(x,x) \sim (4\pi t)^{-n} \sum_{i\geq n} \mathrm{Str}_{C(X)} A_i(x) t^i \tag{5.4.14}$$

so there is no pole at $t = 0$*.*

Proposition 5.4.9. *We have the following McKean-Singer formula for generalized Dirac operators.*

$$\mathrm{Ind}(D_s) = \mathrm{Tr}_s \exp(-tD_s^2) = \int_X \mathrm{Tr}_s p_{t,s}(x,x)dx \tag{5.4.15}$$

As an corollary of the McKean-Singer formula, we have the following index theorems for cohesive modules.

Corollary 5.4.10. *In deRham cohomology, the following differential forms are cohomologous:*

$$\hat{A}(\nabla^{LC}) \cdot \mathrm{ch}(\nabla^{K^{-1/2}}) = \mathrm{Todd}(X) \tag{5.4.16}$$

Therefore the generalized Dolbeault-Dirac operator D^E has its index given by the following generalization of Hirzebruch-Riemann-Roch formula:

$$\mathrm{Ind}(D^E) = (2\pi i)^{-n} \int_X \mathrm{Todd}(X) \cdot \mathrm{ch}(E) \tag{5.4.17}$$

Corollary 5.4.11. *If X is equipped with a Kälher metric, then Θ is closed and $T_{as} = 0$. In addition, the Chern connection ∇^C coincide with the Levi-Civita connection ∇^{LC}. Therefore, as differential forms on X, we have:*

$$\hat{A}(\nabla^{LC}) \cdot \mathrm{ch}(\nabla^{K^{-1/2}}) = \mathrm{Todd}(\nabla^C) \tag{5.4.18}$$

Consequently, the local index density is given by the Bott-Chern form

$$\mathrm{ind} D^E = (2\pi i)^{-n} [\mathrm{Todd}(\nabla^C) \cdot \mathrm{ch}(\mathbb{E})]_{(2n)} \tag{5.4.19}$$

Remark 5.4.12. Since we are working with superconnections rather than connections, the usual formula for Chern character of a connection ∇

$$\mathrm{ch}(\nabla) = \exp(-\frac{\nabla^2}{2\pi i}) \tag{5.4.20}$$

needs modification. Instead of dividing by $2\pi i$ of the curvature, we define a map ϕ on $\mathcal{A}^\bullet(X)$ by

$$\phi(\omega) = (2\pi i)^{-|\omega|/2} \omega \tag{5.4.21}$$

for homogeneous forms ω, the Chern character is then defined by the formula

$$\mathrm{ch}(E, \mathbb{E}) = \phi(\mathrm{Tr}_s \exp(-\mathbb{E}^2)) \tag{5.4.22}$$

Chapter 6

Family index theorem for cohesive modules

In the last chapter, we extend the index theorem for a single operator to a family index theorem. For a holomorphic submersion $\pi : M \to B$ between compact complex manifolds and fix a cohesive module E on M, the restriction of E to fibers M/B provide a family of cohesive modules. In [Blo06], a quasi-cohesive module is constructed as the push forward of E. We perform an infinite dimensional Chern superconnection construction for the quasi-cohesive module $\pi_! E$ and the resulting superconnection is adapted to the family of generalized Dolbeault-Dirac operators. The final index formula is a generalization of the classical Grothendieck-Riemann-Roch formula and takes value in Bott-Chern cohomology when the submersion is Kähler in the sense of [BGS88b].

6.1 Riemannian fibre bundles

Let $\pi : M \to B$ be a submersion between compact manifolds and we choose a splitting $TM = T(M/B) \oplus T_H M$ such that π induces an isomorphism $T_H M = \pi^* TB$. Let P be the projection from TM to $T(M/B)$ with respect to the splitting. We obtain a splitting of the dual bundles as well and have the canonical decomposition

$$\mathcal{A}^\bullet(M) = \mathcal{A}^\bullet(M/B) \otimes \pi^* \mathcal{A}^\bullet(B) \tag{6.1.1}$$

where the tensor product should be understood as the projective tensor product of Fréchet algebras over M.

For any connection $\nabla^{M/B}$ for the vertical tangent bundle $T(M/B)$ on M, we define the following geometric tensors.

Definition 6.1.1. We define the second fundamental form S as an element in $\mathcal{A}^1_H(M, \mathrm{End}T(M/B))$ by

$$S(Z) \cdot U = \nabla^{M/B}_Z U - P[Z, U], Z \in T_H M, U \in T(M/B) \tag{6.1.2}$$

The mean curvature k is the horizontal one form given by the trace of S

$$k(Z) = \mathrm{tr}_{M/B} S_Z \tag{6.1.3}$$

The curvature Ω of the splitting is an element in $\mathcal{A}^2_H(M, T(M/B))$ defined by

$$\Omega(Z, W) = -P[Z, W], \forall Z, W \in T_H M \tag{6.1.4}$$

Definition 6.1.2. If $T(M/B)$ is equipped with a Riemannian metric $g_{M/B}$, we choose an arbitrary Riemannian metric g_B on B which lifts to a metric on $T_H M$.

We define the Levi-Civita connection $\nabla^{M/B}$ of the vertical tangent bundle by

$$\nabla^{M/B} = P \circ \nabla^{LC, g_B \oplus g_{M/B}} \tag{6.1.5}$$

then $\nabla^{M/B}$ is independent of g_B and coincide with the Levi-Civita connections along the fibre when restricted.

Therefore we may work with the Levi-Civita connection ∇^g with $g = g_B \oplus g_{M/B}$ or the direct sum connection $\nabla^{\oplus} = \pi^* \nabla^B \oplus \nabla^{M/B}$ where ∇^B is the Levi-Civita connection on B. The following tensor captures the difference of ∇^g and $\nabla^{\oplus}$.

Definition 6.1.3. Let $\omega \in \mathcal{A}^1(M, \Lambda^2 T^* M)$ be the one form defined by the formula

$$\omega(X)(Y, Z) = g(S(X)Y, Z) - g(S(X)Z, Y) + \frac{1}{2} g(\Omega(X, Z), Y) \tag{6.1.6}$$

$$- \frac{1}{2} g(\Omega(X, y), Z) + \frac{1}{2} g(\Omega(Y, Z), X) \tag{6.1.7}$$

Proposition 6.1.4. *The Levi-Civita connection ∇^g is related to the connection $\nabla^{\oplus}$ by the formula*

$$g(\nabla^g_X Y, Z) = g(\nabla^{\oplus}_X Y, Z) + \omega(X)(Y, Z) \tag{6.1.8}$$

6.2 Clifford modules over a degenerate Clifford algebra

For the fibre bundle $(M, B, \pi, P, g_{M/B})$, we may consider the vertical bundle of Clifford algebras $C(M/B)$ whose fibre at $m \in M$ is the Clifford algebra of the

Euclidean vector space $(T^*_m M/B, g_{M/B})$. For the horizontal direction, we use the degenerate metric and the Clifford algebra $C(T^*_H)$ reduces to the exterior algebra $\pi^*\Lambda^\bullet T^*B$.

Definition 6.2.1. The degenerate bundle of Clifford algebra $C_0(M)$ on the fibre bundle $(M, B, \pi, P, g_{M/B})$ is defined by

$$C_0(M) = \pi^*\Lambda^\bullet T^*B \otimes C(M/B) \tag{6.2.1}$$

To define the natural connections on the degenerate Clifford algebra, we use the method of adiabatic limits. For a positive real parameter u, let g_u be the metric on T^*M defined by

$$g_u = g_{M/B} \oplus u g_B \tag{6.2.2}$$

and we denote by $C_u(M)$ the bundle of Clifford algebras on M defined by the metric g_u.

Definition 6.2.2. We define a map τ^u from $\Lambda^2 T^*M$ to $\mathrm{End}(T^*M)$ by

$$\tau^u(v_1 \wedge v_2)\xi = 2(g_u(v_1, \xi)v_2 - g_u(v_2, \xi)v_1) \tag{6.2.3}$$

Then $\tau^0 = \lim_{u\to 0} \tau^u$ exists and we define a connection on T^*M by

$$\nabla^{T^*M,0} = \nabla^{\oplus} + \frac{1}{2}\tau^0(\omega) \tag{6.2.4}$$

where ω was defined in 6.1.3.

Definition 6.2.3. A Clifford module $\mathcal{E}$ for the vertical Clifford algebra $C(M/B)$ is a $\mathbb{Z}_2$-graded vector bundle with an even Clifford action c^π. A connection $\nabla^{\mathcal{E}}$ is a

Clifford connection if it satisfies

$$[\nabla^{\mathcal{E}}_Y, c^\pi(a)] = c^\pi(\nabla^{M/B}_Y a) \tag{6.2.5}$$

for any $Y \in TM, a \in C(M/B)$. A superconnection $\mathbb{A}^{\mathcal{E}}$ is a Clifford superconnection if it satisfies

$$[\mathbb{A}^{\mathcal{E}}, c^\pi(a)] = c^\pi(\nabla^{M/B} a) \tag{6.2.6}$$

For any Clifford module $\mathcal{E}$ for the vertical Clifford algebra $C(M/B)$, the vector bundle

$$\mathscr{E} = \pi^* \Lambda^\bullet T^* B \otimes \mathcal{E} \tag{6.2.7}$$

defines a Clifford module for the degenerate Clifford algebra $C_0(M)$ with horizontal actions given by wedge product and vertical actions given by c^π.

Proposition 6.2.4. *The connection* $\nabla^{\mathscr{E},0}$ *defined by*

$$\nabla^{\mathscr{E},0} = \pi^* \nabla^B \otimes 1 + 1 \otimes \nabla^{\mathcal{E}} + \frac{1}{2} c_0(\omega) \tag{6.2.8}$$

is a Clifford connection for $C_0(M)$ *on* $\mathscr{E}$

$$[\nabla^{\mathscr{E},0}, c_0(a)] = c_0(\nabla^{T^*M,0} a) \tag{6.2.9}$$

Lemma 6.2.5. *If* f_α *is a horizontal basis and* e_i *is a vertical basis with dual basis* f^α, e^i *then*

$$c_0(\omega) = k(f_\alpha) f^\alpha - \frac{1}{2} g_{M/B}(\Omega(f_\alpha, f_\beta), e_i) f^\alpha \wedge f^\beta \cdot c^\pi(e^i) \tag{6.2.10}$$

6.3 Push forward superconnection

We assume M, B are compact complex manifolds and π is a holomorphic submersion. In addition, we assume the splitting is compatible with the almost complex structures such that

$$T_H^*M = \pi^*T^*B \tag{6.3.1}$$

is an isomorphism of complex vector bundles. The Riemannian metric $g_{M/B}$ induces hermitian metrics on M/B defined by the usual formula

$$h_{M/B}(X,Y) = g_{M/B}(X,Y) + ig_{M/B}(JX,Y) \tag{6.3.2}$$

Similarly if we choose a Riemannian metric g_B on B, we obtain a hermitian metric h_B on B.

Let $(E, E^\bullet)$ be a hermitian cohesive module over M with Chern superconnection $\mathbb{E}$. We define its push-forward $\pi_!E$ as a quasi-cohesive module in the sense of [Blo06].

$\pi_!E$ is an infinite dimensional bundle over B whose fiber over b is the space of smooth sections $\mathcal{A}^{0,\bullet}(X_b, E^\bullet)$ where $X_b = \pi^{-1}(b)$ is the fibre. We view it as a Clifford module of $C(M/B)$ with space of sections $\mathcal{A}^{0,\bullet}(M/B, E)$, the vertical antiholomorphic forms with values in cohesive module E.

Definition 6.3.1. Let $\mathscr{E}$ be the Clifford module defined in previous section, then we have the identification

$$\mathcal{A}^\bullet(B, \pi_!E) = \mathcal{A}^0(M, \mathscr{E})$$

A superconnection $\mathbb{A}$ adapted to a family of differential operators D_b is a differential operator on the bundle $\mathscr{E}$ over M of odd parity such that

$$\mathbb{A}(vs) = d_B(v) \cdot s + (-1)^{|v|} v \mathbb{A} s \tag{6.3.3}$$

for all $v \in \mathcal{A}^\bullet(B), s \in \mathcal{A}^0(M, \mathscr{E})$ and it decomposes as

$$\mathbb{A} = D + \sum_{i=1}^{\dim B} \mathbb{A}_i \tag{6.3.4}$$

where $\mathbb{A}_i$ maps $\mathcal{A}^\bullet(B, \pi_! E)$ to $\mathcal{A}^{\bullet+i}(B, \pi_! E)$.

Definition 6.3.2. We define the push-forward superconnection $\pi_! \mathbb{E}$ as the Dirac operator for degenerate Clifford module:

$$\pi_! \mathbb{E} = c_0 \circ \nabla^{\mathscr{E},0} \tag{6.3.5}$$

which is a composition of maps

$$\mathcal{A}^{0,\bullet}(M/B, E^\bullet) \xrightarrow{\nabla^{\mathscr{E},0}} \mathcal{A}^\bullet(M) \otimes \mathcal{A}^{0,\bullet}(M/B, E) \xrightarrow{\sigma^{-1}}$$

$$\pi^* \mathcal{A}^\bullet(B) \otimes C(M/B) \otimes \mathcal{A}^{0,\bullet}(M/B, E) \xrightarrow{c_0} \pi^* \mathcal{A}^\bullet(B) \otimes \mathcal{A}^{0,\bullet}(M/B, E)$$

Since ∇^B is torsion free, it follows that $\pi_! \mathbb{E}$ is a superconnection for $\pi_! E$.

Proposition 6.3.3. *$\pi_! \mathbb{E}$ is a superconnection adapted to the family of spin^c Dirac operators along the fibre.*

We still need to modify the superconnection by the torsion three form. Let h be the hermitian metric with fundamental form Θ on M, we define the three form T on M by

$$T(\Theta) = \frac{i}{4}(\partial^{M/B} - \bar{\partial}^{M/B})\Theta \tag{6.3.6}$$

and regard $\nabla^T = d + T(\Theta)$ as a superconnection as before, then if we set

$$\mathbb{E}^\Theta = \mathbb{E} \otimes 1 + 1 \otimes \nabla^T(\Theta) \tag{6.3.7}$$

then $\pi_! \mathbb{E}^T$ is a superconnection on $\pi_!$ adapted to the family of generalized Dolbeault-Dirac operators D^E.

Definition 6.3.4. The fibre bundle is called a Kälher fibration if there exist a closed $(1,1)$ form τ on M such that τ restricts to Kälher forms to each fibre and the splitting is orthogonal with respect to τ.

As is clear from our argument, we may choose any global $(1,1)$ form that restricts to the fiberwise fundamental form, so instead of using Θ, we may define $T(\tau)$ and twist the Clifford modules by the trivial bundle with superconnection $d + T(\tau)$, the resulting superconnection $\mathbb{E}^\tau$ has its push-forward $\pi_! \mathbb{E}^\tau$ is a superconnection for $\pi_! E$ adapted to the family of modified Dirac operators

$$D_b^{\tau,E} = D_b^{Cl,E} + c^\pi \circ T(\tau) \tag{6.3.8}$$

We define the scaled superconnections $(\pi_! \mathbb{E}^\tau)_s$ as before and extend supertrace and relative supertrace in this context.

Proposition 6.3.5. *The heat semigroup* $\exp(-t\pi_! \mathbb{E}_s)$ *is smoothing for* $t > 0$ *and are represented by smooth kernel functions* $p_{t,s}(x,y)$ *on* M*. The asymptotic expansion along the diagonal takes the form*

$$p_{t,1/t}(x,x) \sim (4\pi t)^{-\dim M/B} \sum_{i \geq 0} t^i A_i(x) \tag{6.3.9}$$

where the coefficients A_i lies in $\mathcal{A}^{\leq 2i}(M, \mathrm{End}_{C(M/B)}\mathcal{E})$ and the full symbol

$$\sigma(p) = \sum_{i=0}^{\dim M} \sigma_{2i}(A_i) \tag{6.3.10}$$

is given by the formula

$$\sigma(p) = \det{}^{1/2}\{\frac{\mathcal{R}^{M/B}/2}{\sinh \mathcal{R}^{M/B}/2}\} \exp(-\mathcal{R}^{\mathcal{E}/\mathscr{S}}) \tag{6.3.11}$$

where $\mathcal{R}^{\mathcal{E}/\mathscr{S}}$ is the relative curvature of the vertical Clifford module $\mathcal{E}$ with respect to the vertical Clifford algebra $C(M/B)$.

Corollary 6.3.6. *Locally on M/B we may decompose the vertical Clifford module $\mathcal{E} = \mathcal{A}^{0,\bullet}(M/B, E)$ by*

$$\Lambda^\bullet T^{*0,1}(M/B) \otimes E = \mathscr{S}(M/B) \otimes K^{-1/2} \otimes \mathbb{C} \otimes E$$

where they have the spin connection ∇^S, Chern connection $\nabla^{K^{-1/2}}$, modified superconnection ∇^T and Chern superconnection $\mathbb{E}$ respectively. Applying the relative supertrace, we have

$$\mathrm{Str}_{\mathcal{E}/\mathscr{S}}\sigma(p) = \hat{A}(\mathcal{R}^{M/B}) \exp(-\mathcal{R}^K/2) \exp(-dT) \exp(-\mathcal{R}^{\mathbb{E}}) \tag{6.3.12}$$

consequently, we have the integral formula for the Chern character in deRham cohomology of $\pi_! E$ by McKean-Singer formula

$$\lim_{t\to 0} \mathrm{ch}(\pi_! \mathbb{E}_t) = (2\pi i)^{-\dim M/B} \int_{M/B} \mathrm{Todd}(M/B)\mathrm{ch}(E) \tag{6.3.13}$$

Corollary 6.3.7. *If the fibration is Kälher, then $T = 0$ and $\nabla^{M/B}$ is the holomorphic Chern connection for $T(M/B)$ so along the fiber, we have*

$$\hat{A}(\mathcal{R}^{M/B}) \exp(-\mathcal{R}^K/2) = \mathrm{Todd}(\mathcal{R}^{M/B}) \tag{6.3.14}$$

as an identity of differential forms.

Remark 6.3.8. In [Bis13], Bismut had already extend the earlier results in [BGS88b] and [BGS88c] by relaxing the Kähler fibration condition. His main results were limited to complexes of holomorphic vector bundles and in addition, he required that the index bundle is locally of constant rank. We hope his analytical techniques can be properly modified to incorporate the family of generalized Dolbeault-Dirac operators and derive a Grothendieck-Riemann-Roch formula in full generality. This will be studied in future research, right now, our result is limited to Kähler fibration.

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