Moduli Problems in Derived Noncommutative Geometry

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ABSTRACT

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We study moduli spaces of boundary conditions in 2D topological field theories. To a compactly generated linear ∞-category $\mathcal{X}$, we associate a moduli functor $\mathcal{M}_{\mathcal{X}}$ parametrizing compact objects in $\mathcal{X}$. Using the Barr-Beck-Lurie monadicity theorem, we show that $\mathcal{M}_{\mathcal{X}}$ is a flat hypersheaf, and in particular an object in the ∞-topos of derived stacks. We find that the Artin-Lurie representability criterion makes manifest the relation between finiteness conditions on $\mathcal{X}$, and the geometricity of $\mathcal{M}_{\mathcal{X}}$. If $\mathcal{X}$ is fully dualizable (smooth and proper), then $\mathcal{M}_{\mathcal{X}}$ is geometric, recovering a result of Toën-Vacqiue from a new perspective. Properness of $\mathcal{X}$ does not imply geometricity in general: perfect complexes with support is a counterexample. However, if $\mathcal{X}$ is proper and perfect (symmetric monoidal, with "compact = dualizable"), then $\mathcal{M}_{\mathcal{X}}$ is geometric.

The final chapter studies the moduli of oriented 2D topological field theories (Noncommutative Calabi-Yau Spaces). The Cobordism Hypothesis, Deligne's conjecture, and formal $\mathbb{E}_n$-geometry are used to outline an approach to proving the unobstructedness of this space and constructing a Frobenius structure on it.

Contents

Chapter 1

Introduction

The subject of this thesis, derived noncommutative geometry, is the natural coming together of two fundamental paradigm shifts: one within mathematics, and the other in physics. The first is a movement away from mathematics based on sets, to a mathematics where the primitive entities are shapes. The second, is the radical idea in physics that the notion of space-time is not intrinsic to a physical theory. Sections §1.1 and §1.2 are devoted to a cursory overview of these two incipient revolutions in the way we perceive reality. In section §1.3, we sketch in quick, broad strokes the emerging contours of derived noncommutative geometry. Our purpose in including these sections is to place this thesis within the broader context into which it naturally fits, and to lend some perspective to the results proven here. Section §1.4 details, in a semi-informal tone, the main results of this article, and outlines the organizational structure of the document.

1.1 Brave New Mathematics

For several centuries, *scientific* thought has been conflated with *reductionist* paradigm. The tremendous advances in human knowledge during the aforementioned era stand testimony, no doubt, to the power and practical utility of reductionism. Nevertheless, as with any approach that has thought as its basis, it is limited. What follows is a discussion of a particular limitation.

One glaring manifestation of the reductionist approach is the fact that modern mathematics is based on an axiomatic framework where the primordial entities are sets.

1.2 Brane Theory

1.3 Derived Noncommutative Geometry

The author feels that identifying the act of doing DNG, with "replacing a scheme X by its derived category of sheaves", is decidedly undemocratic, and somewhat limiting, placing little emphasis, for instance, on the symplectic point of view on the Elephant. Furthermore, it obscures the physical origins of DNG.

1.4 About this work

1.5 Background and Notation

Throughout this work, we will assume familiarity with the language of homotopical mathematics as developed by Lurie in [Lur09, Lur11b, Lur04]. Specifically, we will assume that the reader has at least a fleeting acquaintance with the rudiments of topology, algebra and algebraic geometry in the ∞-categorical context. Having said that, we would like to emphasize that an intimate knowledge of the inner workings of the theory in *loc. cit.* is not needed in order to read this paper.

An attempt has been made to keep the statements of the results and the proofs devoid of references to a particular model for ∞-categories. Any equivalent (in a suitable sense) model will suffice. In particular, the reader who is more comfortable with the parlance of Toën/Toën-Vezzosi [Toë07, TV05, TV08], should encounter little difficulty in translating most results of this paper into that language. There is one caveat: for statements that involve functor categories, monads and the Barr-Beck-Lurie theorem, model categories must be replaced by a more flexible notion such as Segal Categories, as is done, for instance, in [TV02].

To a large extent, the notation used in this paper is consistent with the notation in [Lur09, Lur11b, Lur04]. The following is a list of some frequently used notation.

Notation 1.5.1 (*Bibliographical Convention*)**.** We will use the letter

- "T" to refer to the book Higher Topos Theory [Lur09].

- "A" to refer to the book, Higher Algebra [Lur11b].

- "G" to refer to the thesis Derived Algebraic Geometry [Lur04].

Thus, for example, T.3.2.5.1. refers to [Lur09, Remark 3.2.5.1.], while A.6.3.6.10. refers to [Lur11b, Theorem 6.3.6.10].

Notation 1.5.2 (*Spaces*)**.** We will denote by $\mathcal{S}$ (resp. $\widehat{\mathcal{S}}$) the ∞-category of small (resp. large) spaces (T.1.2.1.6.), and by $\mathcal{S}_\infty := \mathrm{Stab}(\mathcal{S})$ the stable ∞-category of spectra. For an ∞-category $\mathcal{C}$, $\mathrm{Stab}(\mathcal{C})$ is its stabilization (A.1.4.)

Notation 1.5.3 (∞-*categories*)**.** Throughout, κ will denote an arbitrary regular cardinal, and ω is the smallest one. We will denote by

- Cat_∞ (resp. $\widehat{\mathrm{Cat}}_\infty$) the ∞-category of small (resp. large) ∞-categories.

- $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ (resp. $\mathrm{Cat}_\infty^{\vee}$) the subcategory of Cat_∞ consisting of small stable (resp. idempotent complete stable) ∞-categories and exact functors.

- $\mathcal{P}r^L$ (resp. $\mathcal{P}r^R$) the subcategory of $\widehat{\mathrm{Cat}}_\infty$ consisting of presentable ∞-categories and left adjoints (resp. accessible right adjoints). See T.5.5.

- $\mathcal{P}r^L_\kappa$ (resp. $\mathcal{P}r^R_\kappa$) the subcategory of $\mathcal{P}r^L$ (resp. $\mathcal{P}r^R$) consisting of κ-compactly generated ∞-categories and functors that preserve κ-compact objects (resp. are κ-accessible). See T.5.5.7.

- $\mathcal{P}r^L_{st}$ the subcategory of $\mathcal{P}r^L$ consisting of stable ∞-categories.

Notation 1.5.4 (*Functor categories*)**.** For $\mathcal{C}$, $\mathcal{D}$ in Cat_∞, $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ denotes the ∞-category of functors $\mathcal{C} \to \mathcal{D}$. We will denote by

- $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ consisting of functors that preserve all small colimits (resp. are accessible, and preserve all small limits).

- $\mathrm{Fun}^{\mathrm{LAd}}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{RAd}}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ consisting of functors that have right adjoints (resp. have left adjoints).

- $\mathrm{Fun}^{\mathrm{L}}_{\kappa}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}_{\kappa}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ consisting of functors that preserve all small colimits and κ-compact objects (resp. are κ-accessible, and preserve all small limits).

By the adjoint functor theorem (T.5.5.2.9.), if $\mathcal{C}$ and $\mathcal{D}$ are presentable, then we have natural equivalences $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{LAd}}(\mathcal{C},\mathcal{D})$ and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{RAd}}(\mathcal{C},\mathcal{D})$.

Notation 1.5.5 (*Categorical hom and tensor*)**.** The categories $\mathcal{P}r^L$ and $\mathcal{P}r^L_\kappa$ are symmetric monoidal, and the inclusion functor $\mathcal{P}r^L_\kappa \subseteq \mathcal{P}r^L$ is symmetric monoidal (A.6.3.) with unit $\mathcal{S}$. We will denote by $\otimes$ the tensor product on $\mathcal{P}r^L$. This is not to be confused with the Cartesian monoidal structure "$\times$"on $\widehat{\mathrm{Cat}}_\infty$. The functors $\mathrm{Fun}^{\mathrm{L}}(-,-)$ (resp. $\mathrm{Fun}^{\mathrm{L}}_{\kappa}(-,-)$) defines an internal hom on $\mathcal{P}r^L$ (resp. $\mathcal{P}r^L_\kappa$).

Notation 1.5.6 (*Algebras and modules*)**.** Let $\mathcal{O}^\otimes$ be an ∞-operad, $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be an $\mathcal{O}$-monoidal category and let $\mathcal{M}$ be an ∞-category tensored over $\mathcal{C}$ (A.4.2.1.9.). We will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the ∞-category of $\mathcal{O}$-algebra objects in $\mathcal{C}$. For A in

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, we will write $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{M})$ for the ∞-category of A-modules in $\mathcal{M}$. When $\mathcal{O}$ is the commutative operad $\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, and $\mathrm{Mod}_A(\mathcal{M}) := \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{M})$. When $\mathcal{O}$ is the associative operad (A.4.1.1.), $\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We will use the abbreviations $\mathrm{CAlg}_A := \mathrm{CAlg}(\mathrm{Mod}_A(\mathcal{S}_\infty))$ for any A in $\mathrm{CAlg}(\mathcal{S}_\infty)$ and $\mathrm{Mod}_A := \mathrm{Mod}_A(\mathcal{S}_\infty)$.

Notation 1.5.7 (*1-Categories*)**.** We will denote by Cat the ∞-category of 1-categories. In the quasicategorical model, this is the simplicial nerve of the Dwyer-Kan localization of the 1-category of categories along the subcategory of weak equivalences. We will denote by

- $\mathrm{N}(-)$ the natural inclusion $\mathrm{Cat} \to \mathrm{Cat}_\infty$. In the quasicategorical model, this is the nerve functor.

- $h : \mathrm{Cat}_\infty \to \mathrm{Cat}$ the left adjoint to $\mathrm{N}(-)$. We will refer to $h\mathcal{C}$ as the homotopy category of $\mathcal{C}$.

Notation 1.5.8 (*Ground ring*)**.** Throughout, we will fix an $\mathbb{E}_\infty$-ring $\mathbf{k}$. We will assume that $\mathbf{k}$ is a Derived G-ring in Chapter 4.

Notation 1.5.9 (*Algebraic geometry*)**.** . We will denote by $\mathrm{Aff}_{\mathbf{k}}$ the category of derived affine schemes. By definition $\mathrm{Aff}_{\mathbf{k}} := \mathrm{CAlg}^{op}$. We will denote by $\mathrm{Spec} : \mathrm{CAlg}^{op} \to \mathrm{Aff}_{\mathbf{k}}$ and $\mathcal{O} : \mathrm{Aff}_{\mathbf{k}} \to \mathrm{CAlg}$ the tautological equivalences. We will denote by $\mathcal{S}t_{\mathbf{k}}$ the ∞-topos of derived stacks over $\mathbf{k}$.

Notation 1.5.10 (*Diagrams and limits*)**.** For K in Cat_∞, $K^{\triangleleft}$ (resp. $K^{\triangleright}$) will denote the category obtained from K by adjoining an initial (resp. final) object $\{\infty\}$. For x in K we will usually denote by ψ_x the unique morphism $\infty \to x$ (resp. $x \to \infty$). Our terminology regarding limits follows T.4.

Chapter 2

Brane Moduli

$\mathcal{V}$

2.1 Commutative spaces

2.2 Noncommutative spaces

2.3 Moduli of perfect branes

There is a functor $\mathcal{M} : \mathrm{CAlg}_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_\infty$ whose action on objects and 1-morphisms can be described as follows. To an object A in $\mathrm{CAlg}_{\mathbf{k}}$, $\mathcal{M}$ assigns the ∞-category of A-modules, Mod_A. The action of $\mathcal{M}$ on 1-morphisms $f : A \to B$ in $\mathrm{CAlg}_{\mathbf{k}}$ is given by base change (left Kan extension). In symbols:

$$\mathcal{M}(A) := \mathrm{Mod}_A$$

$$\mathcal{M}(f) := B \otimes_A (-)$$

The existence of the ∞-functor $\mathcal{M}$ can be established in several ways. In the language of [Lur11b, §6.6.3., §6.3.5.9.], $\mathcal{M}$ is the composite:

$$\mathrm{CAlg}_{\mathbf{k}} \longrightarrow \mathrm{Alg}(\mathrm{Mod}_{\mathbf{k}}) \longrightarrow \widehat{\mathrm{Cat}}_\infty^{\mathrm{alg}} \xrightarrow{\widehat{\Theta}} \widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

Recall that, roughly speaking, the ∞-category $\widehat{\mathrm{Cat}}_\infty^{\mathrm{alg}}$ consists of pairs $(\mathcal{C}^\otimes, A)$, where where $\mathcal{C}^\otimes$ is a (not necessarily small) symmetric monoidal ∞-category and A is an object in $\mathrm{Alg}(\mathcal{C})$. Similarly, the ∞-category $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}$ consists of pairs $(\mathcal{C}^\otimes, \mathcal{N})$, where $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category, and $\mathcal{N}$ is a (not necessarily small) ∞-category tensored over $\mathcal{C}^\otimes$. In the diagram above, the first arrow is the forgetful functor from $\mathbb{E}_\infty$-algebras to $\mathbb{E}_1$-algebras, and the last arrow is the forgetful functor that sends $(\mathcal{C}^\otimes, \mathcal{N})$ to the underlying ∞-category $\mathcal{N}$. The functor $\mathrm{Alg}(\mathrm{Mod}_{\mathbf{k}}) \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{alg}}$ is the inclusion of the subcategory consisting of pairs $(\mathcal{C}^\otimes, A)$, where $\mathcal{C}^\otimes \simeq \mathrm{Mod}_{\mathbf{k}}^\otimes$, and the morphisms are equivalent to the identity on $\mathcal{C}^\otimes$. Roughly speaking, $\widehat{\Theta}$ associates to $(\mathcal{C}^\otimes, A)$ the pair $(\mathcal{C}^\otimes, \mathrm{RMod}_A(\mathcal{C}))$.

Remark 2.3.1. The ∞-category $\mathcal{M}(A) = \mathrm{Mod}_A$ is presentable. This follows, for instance, from A.4.2.3.7., and the fact the ∞-category of spectra is presentable. Furthermore, for any $f : A \to B$ in $\mathrm{CAlg}_{\mathbf{k}}$, the functor $\mathcal{M}(f)$ has a right adjoint, namely, the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$.

Remark 2.3.2. More is true: the right adjoint $\mathcal{M}(B) \to \mathcal{M}(A)$ preserves all colimits. In particular, it is ω-accessible. It follows that $\mathcal{M}(f) : \mathcal{M}(A) \to \mathcal{M}(B)$ preserves ω-compact objects.

Remark 2.3.3. The categories Mod_A have a symmetric monoidal structure induced by the symmetric monoidal structure on $\mathcal{S}_\infty$. Thus, $\mathcal{M}(A)$ can be viewed as commutative algebra object in $\mathcal{P}r^L$. In particular, it $\mathcal{M}(A)$ is a module over itself; i.e., it can viewed as an object in $\mathrm{Mod}_{\mathcal{M}(A)}(\mathcal{P}r^L) =: \mathcal{P}r^L_A$.

For A in $\mathrm{CAlg}_{\mathbf{k}}$, functor $\mathcal{M}(\theta_A) : \mathcal{M}(\mathbf{k}) \to \mathcal{M}(A)$ induced by the structure morphism $\theta_A : \mathbf{k} \to A$ is symmetric monoidal (A.4.4.3.1), i.e., it is a morphism in $\mathrm{CAlg}(\mathcal{P}r^L)$. Restricting the action of $\mathcal{M}(A)$ along $\mathcal{M}(\theta_A)$, we get an induced $\mathcal{M}(\mathbf{k})$-module structure on $\mathcal{M}(A)$. Morphisms in $\mathrm{CAlg}_{\mathbf{k}}$ commute with the structure maps $\theta_{(-)}$ by definition; this immediately implies that the functors $\mathcal{M}(f)$ are $\mathcal{M}(\mathbf{k})$-linear.

Recall that $\mathcal{P}r^L_\omega$ (resp. $\mathcal{P}r^L_{\omega,\mathbf{k}}$) denotes the subcategory of $\mathcal{P}r^L$ (resp. $\mathcal{P}r^L_{\mathbf{k}}$) consisting of all compactly generated ∞-categories (resp. all $\mathrm{Mod}_{\mathbf{k}}$-linear compactly generated ∞-categories), and morphisms that preserve small colimits and ω-compact objects. The preceding three remarks are summarized by the following lemma:

Lemma 2.3.4. *There exists a functor* $\mathfrak{M}_{\mathbf{1}} : \mathrm{CAlg}_{\mathbf{k}} \to \mathcal{P}r^L_{\omega,\mathbf{k}}$ *such that the diagram below is (homotopy) commutative:*

$$\begin{tikzcd}
& \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[r] \arrow[d] & \mathcal{P}r^L_{\mathbf{k}} \arrow[d] \\
& \mathcal{P}r^L_{\omega} \arrow[r] & \mathcal{P}r^L \arrow[d] \\
& \mathrm{CAlg}_{\mathbf{k}} \arrow[r, "\mathcal{M}"'] \arrow[u, bend left=60, dotted, shift left=8, "\mathfrak{M}_{\mathbf{1}}"] & \widehat{\mathrm{Cat}}_\infty
\end{tikzcd}$$

Notation 2.3.5. Let $\mathrm{QC}^{\mathrm{aff}}$ denote the composite

$$\mathrm{Aff}_{\mathbf{k}}^{op} \xrightarrow{\mathcal{O}} \mathrm{CAlg}_{\mathbf{k}} \xrightarrow{\mathfrak{M}_1} \mathcal{P}r^L_{\omega,\mathbf{k}} \longrightarrow \mathcal{P}r^L_{\mathbf{k}}$$

For a derived affine scheme X in $\mathrm{Aff}_{\mathbf{k}}$, $\mathrm{QC}^{\mathrm{aff}}(X)$ is the ∞-category of quasicoherent sheaves on X. We would like to extend this functor to arbitrary derived stacks.

Let $j : \mathrm{Aff}_{\mathbf{k}} \to \mathcal{P}(\mathrm{Aff}_{\mathbf{k}})$ denote the Yoneda embedding. By the universal property of categories of presheaves, left Kan extension defines an equivelance $\mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathrm{Aff}_{\mathbf{k}}), \mathcal{C})$, for any $\mathcal{C}$ that admits all small colimits. Take $\mathcal{C} = (\mathcal{P}r^L_{\mathbf{k}})^{op}$, and let $\widetilde{\mathrm{QC}}$ denote that image of $\mathrm{QC}^{\mathrm{aff}}$ under the induced equivalence $\mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathcal{P}r^L_{\mathbf{k}}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathrm{Aff}_{\mathbf{k}})^{op}, \mathcal{P}r^L_{\mathbf{k}})$.

Notation 2.3.6. Let $a : \mathcal{P}(\mathrm{Aff}_{\mathbf{k}}) \to \mathcal{S}t_{\mathbf{k}}$ be the localization functor, with right adjoint i, and let QC denote the composite $\widetilde{\mathrm{QC}} \circ i^{op}$. We will often implicitly identify $\mathcal{S}t_{\mathbf{k}}$ with the essential image of the fully faithful functor i.

Definition 2.3.7. For a derived stack X over $\mathbf{k}$, the ∞-category $\mathrm{QC}(X)$ is called the ∞-*category of quasicoherent sheaves* on X.

Remark 2.3.8. The ∞-category $\mathrm{QC}(X)$, is stable. This follows, for instance, from the fact that $\mathrm{Mod}_{\mathbf{k}}$-linear ∞-categories are stable [].

Remark 2.3.9. Let X be a discrete scheme. Then the relationship between the ∞-category QC and the abelian category $\mathrm{Qcoh}(X)$ of quasicoherent sheaves on X is as follows: there is a t-structure on QC such that $\mathrm{QC}^{\heartsuit} \simeq \mathrm{Qcoh}(X)$, and we have an equivalence $h\mathrm{QC} \simeq D(\mathrm{Qcoh}(X))$.

Remark 2.3.10. The ètale topology is subcanonical, so for A in $\mathrm{CAlg}_{\mathbf{k}}$, $\mathrm{Spec}(A)$ is a derived stack. Furthermore, we have $\mathrm{QC}^{\mathrm{aff}}(\mathrm{Spec}(A)) = \mathrm{QC}(\mathrm{Spec}(A))$.

Remark 2.3.11. Let $\mathcal{F} \in \mathcal{P}(\mathrm{Aff}_{\mathbf{k}})$, and $\Phi : (j/\mathcal{F}) \to \mathcal{P}(\mathrm{Aff}_{\mathbf{k}})$ be the functor that carries $\mathrm{Spec}(A) \to \mathcal{F}$ to $\mathrm{Spec}(A)$. Then we have a natural equivalence $\mathcal{F} \simeq \mathrm{colim}\ \Phi$. Take $\mathcal{F} = i(X)$ for some derived stack X. Using the preceding remark and the fact that $\widetilde{\mathrm{QC}}$ preserves limits, we have

$$\mathrm{QC}(X) \simeq \lim(\widetilde{\mathrm{QC}} \circ \Phi^{op}) \simeq \lim_{\mathrm{Spec}(A)\to X} \mathrm{QC}(\mathrm{Spec}(A))$$

The diagram $\Phi : (j/\mathcal{F}) \to \mathcal{P}(\mathrm{Aff}_{\mathbf{k}})$ is large, and consequently the description of QC in 2.3.11 is not very useful in practice. In the category $\mathcal{S}t_{\mathbf{k}}$, one often has small diagrams taking values in (the essential image of) $\mathrm{Aff}_{\mathbf{k}}$ whose colimit is a given derived stack X. For example, if $U_{\bullet} \to X$ is an ètale (or flat) hypercover then we have $\mathrm{colim}_n\ U_n \simeq X$. However, since i^{op} does not preserve limits, one has to work much harder to show that $\mathrm{QC}(X) \simeq \lim_n \mathrm{QC}(U_n)$. The following proposition is the homotopical/derived analogue of flat descent for quasicoherent sheaves on ordinary schemes:

Proposition 2.3.12. *The functor* $\mathrm{QC} : \mathcal{S}t_{\mathbf{k}}^{op} \to \mathcal{P}r_{\mathbf{k}}^{L}$ *is a sheaf for the flat hyper-topology.*

Proof. This is known to the experts; see e.g. [Lur04, Example 4.2.5] and [TV08, Theorem 1.3.7.2]. It also follows from Theorem 3.1.1; indeed, it is the special case of that theorem when $\mathcal{X} \simeq \hat{\mathbf{k}}$. □

Remark 2.3.13. The inclusion $\mathcal{P}r_{\omega,\mathbf{k}}^{L} \subseteq \mathcal{P}r_{\mathbf{k}}^{L}$ does not reflect limits in general: the limit in $\mathcal{P}r_{\mathbf{k}}^{L}$ (or equivalently in $\widehat{\mathrm{Cat}}_{\infty}$) of a diagram of compactly generated categories need not be compactly generated. Following [BZFN10], we make the following definition:

Definition 2.3.14. A derived stack X is *perfect* if $\mathrm{QC}(X)$ is an ω-compactly generated ∞-category. Let $\mathcal{S}t_{\mathbf{k}}^{\mathrm{perf}}$ denote the full subcategory of $\mathcal{S}t_{\mathbf{k}}$ consisting of perfect stacks.

Proposition 2.3.15 ([Toë07],[BZFN10])**.** *The cartesian symmetric monoidal structure on* $\mathcal{S}t_{\mathbf{k}}$ *restricts to a symmetric monoidal structure on* $\mathcal{S}t_{\mathbf{k}}^{\mathrm{perf}}$*. Furthermore, the restriction of* QC *to* $\mathcal{S}t_{\mathbf{k}}^{\mathrm{perf}}$ *is symmetric monoidal. In other words, if* X *and* Y *are perfect stacks over* $\mathbf{k}$*, then* $X \times_{\mathbf{k}} Y$ *is perfect and we have a natural equivalence:*

$$\mathrm{QC}(X) \otimes_{\mathrm{Mod}_{\mathbf{k}}} \mathrm{QC}(Y) \simeq \mathrm{QC}(X \times_{\mathbf{k}} Y)$$

Furthermore, we have a natural equivalence

$$\mathrm{QC}(X \times_{\mathbf{k}} Y) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(\mathrm{QC}(X), \mathrm{QC}(Y))$$

One can associate with a derived stack X a functor:

$$\mathfrak{M}^{\mathrm{QC}}_X : \mathrm{Aff}^{op}_{\mathbf{k}} \to \mathcal{P}r^L_{\mathbf{k}}$$

$$S \mapsto \mathrm{QC}(X \times_{\mathbf{k}} S)$$

Definition 2.3.16. The induced functor $\mathcal{M}^{\mathrm{QC}}_X : \mathrm{Aff}^{op}_{\mathbf{k}} \to \widehat{\mathcal{S}}$ defined by $\mathcal{M}^{\mathrm{QC}}_X(S) := (\mathfrak{M}^{\mathrm{QC}}_X)^{\simeq}$ is the *moduli of quasicoherent sheaves on* X.

The drawback of $\mathcal{M}^{\mathrm{QC}}_X$ is that it takes values in the category $\widehat{\mathcal{S}}$ of *large* spaces. Since the category of derived stacks is as a localization of the category of presheaves on $\mathrm{Aff}_{\mathbf{k}}$ taking values in *small* spaces, it is more convenient to have a moduli functor that *a priori* takes values in small spaces. Fortunately, if X is a *perfect* stack, i.e., if $\mathrm{QC}(X)$ is ω-compactly generated, then the large ∞-category $\mathrm{QC}(X)$ is determined by the small ∞-category $\mathrm{Perf}(X) := \mathrm{QC}(X)^{\omega}$: we have $\mathrm{QC}(X) = \mathrm{Ind}_{\omega}(\mathrm{Perf}(X))$. This suggests that one should restrict attention to perfect stacks and replace the functor $\mathfrak{M}^{\mathrm{QC}}_X$ by the functor $\mathfrak{M}^{\mathrm{pe}}_X$:

$$\begin{tikzcd} \mathrm{Aff}^{op}_{\mathbf{k}} \arrow[r, "\mathfrak{M}^{\mathrm{QC}}_X"'] \arrow[rrr, bend left, "\mathfrak{M}^{\mathrm{pe}}_X"] & \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[r] & \mathcal{P}r^L_{\omega} \arrow[r, "(-)^{\omega}"'] & \mathrm{Cat}_{\infty} \end{tikzcd}$$

Here $(-)^{\omega}$ is the functor that associates to compactly generated ∞-category the full subcategory of ω-compact objects. Explicitly, we have $\mathfrak{M}^{\mathrm{pe}}_X(S) := \mathcal{M}(X \times_{\mathbf{k}} S)^{\omega} =: \mathrm{Perf}(X \times_{\mathbf{k}} S)$.

Definition 2.3.17. The *moduli of perfect complexes* on X is the functor $\mathcal{M}_X^{\mathrm{pe}}$: $\mathrm{Aff}_{\mathbf{k}}^{op} \to \mathcal{S}$ defined by $\mathcal{M}_X^{\mathrm{pe}}(S) := (\mathfrak{M}_X^{\mathrm{pe}}(S))^{\simeq} \simeq \mathrm{Perf}(X \times_{\mathbf{k}} S)^{\simeq}$.

According to Proposition 2.3.15, we have the following alternate description of the functor $\mathfrak{M}_X^{\mathrm{QC}}$, when X is perfect:

$$\mathfrak{M}_X^{\mathrm{QC}}(S) \simeq \mathrm{QC}(X) \otimes_{\mathrm{QC}(\mathrm{Spec}(\mathbf{k}))} \mathrm{QC}(S)$$

Thus, $\mathfrak{M}_X^{\mathrm{QC}}$ and $\mathfrak{M}_X^{\mathrm{pe}}$ are manifestly invariants of the *noncommutative shadow* $\mathrm{QC}(X)$ of the commutative space X. Furthermore, it suggests that for an arbitrary noncommutative space $\mathcal{X}$, the functor $\mathcal{M}_{\mathcal{X}}^{\mathrm{perf}} : \mathrm{Aff}_{\mathbf{k}}^{op} \to \mathcal{S}$ defined by $\mathcal{M}_{\mathcal{X}}^{\mathrm{perf}}(S) := ((\mathcal{X} \otimes_{\mathbf{k}} \mathrm{QC}(S))^{\omega})^{\simeq}$ is a *commutative reflection* of $\mathcal{X}$, which one might think of as the *moduli of perfect branes* on $\mathcal{X}$. As above, it will be useful to keep track of somewhat more refined information. To this end, let $\tilde{\mathfrak{M}}$ denote the composite functor:

$$\begin{tikzcd} \mathcal{P}r^L_{\omega,\mathbf{k}} \times \mathrm{Aff}_{\mathbf{k}}^{op} \arrow[r, "id\times\mathcal{O}"] \arrow[rrr, bend left, "\tilde{\mathfrak{M}}"] & \mathcal{P}r^L_{\omega,\mathbf{k}} \times \mathrm{CAlg}_{\mathbf{k}} \arrow[r, "id\times\mathfrak{M}_{\mathbf{1}}"] & \mathcal{P}r^L_{\omega,\mathbf{k}} \times \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[r, "\otimes_{\mathbf{k}}"] & \mathcal{P}r^L_{\omega,\mathbf{k}} \end{tikzcd}$$

Notation 2.3.18. Composition with the functor $(-)^{\omega}$ defines a functor $\mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathcal{P}r^L_{\omega,\mathbf{k}}) \to \mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathrm{Cat}_{\infty})$. Define $\mathfrak{M}^{\mathrm{perf}}$ to be the composite of $\mathfrak{M}$ with this functor:

$$\begin{tikzcd} \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[r, "\mathfrak{M}"] \arrow[rr, bend left, "\mathfrak{M}^{\mathrm{perf}}"] & \mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathcal{P}r^L_{\omega,\mathbf{k}}) \arrow[r] & \mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathrm{Cat}_{\infty}) \end{tikzcd}$$

Likewise, the functors $(-)^{\simeq} : \mathcal{P}r^L_{\omega,\mathbf{k}} \to \widehat{\mathcal{S}}$ and $(-)^{\simeq} : \mathrm{Cat}_{\infty} \to \mathcal{S}$ which carry an ∞-category to the underlying ∞-groupoid induce functors $\mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathcal{P}r^L_{\omega,\mathbf{k}}) \to \hat{\mathcal{P}}(\mathrm{Aff}_{\mathbf{k}})$ and $\mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{op}, \mathrm{Cat}_{\infty}) \to \mathcal{P}(\mathrm{Aff}_{\mathbf{k}})$. Let $\mathcal{M}$ and $\mathcal{M}^{\mathrm{perf}}$ be the functors which make the following diagrams commutative:

$$\begin{tikzcd} \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[r, "\mathfrak{M}"] \arrow[rr, bend left, "\mathcal{M}"] & \mathrm{Fun}(\mathrm{Aff}^{op}_{\mathbf{k}}, \mathcal{P}r^L_{\omega,\mathbf{k}}) \arrow[r] & \hat{\mathcal{P}}(\mathrm{Aff}_{\mathbf{k}}) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[r, "\mathfrak{M}^{\mathrm{perf}}"] \arrow[rr, bend left, "\mathcal{M}^{\mathrm{perf}}"] & \mathrm{Fun}(\mathrm{Aff}^{op}_{\mathbf{k}}, \mathrm{Cat}_\infty) \arrow[r] & \mathcal{P}(\mathrm{Aff}_{\mathbf{k}}) \end{tikzcd}$$

We will write $\mathfrak{M}_{\mathcal{X}}(A)$ for $\mathfrak{M}(\mathcal{X})(A)$. For S in $\mathrm{Aff}_{\mathbf{k}}$ and $\mathcal{X}$ in $\mathcal{P}r^L_{\omega,\mathbf{k}}$, one has the following explicit formulae summarizing the above definitions:

$$\mathfrak{M}_{\mathcal{X}}(S) = \mathcal{X} \otimes_{\mathbf{k}} \mathrm{QC}(S)$$

$$\mathcal{M}_{\mathcal{X}}(S) = (\mathcal{X} \otimes_{\mathbf{k}} \mathrm{QC}(S))^{\simeq}$$

$$\mathfrak{M}^{\mathrm{perf}}_{\mathcal{X}}(S) = (\mathcal{X} \otimes_{\mathbf{k}} \mathrm{QC}(S))^{\omega}$$

$$\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}(S) = ((\mathcal{X} \otimes_{\mathbf{k}} \mathrm{QC}(S))^{\omega})^{\simeq}$$

Definition 2.3.19. Let $\mathcal{X}$ be a derived noncommutative space, i.e., an object of $\mathcal{P}r^L_{\omega,\mathbf{k}}$. The *moduli of perfect branes* on $\mathcal{X}$ is the object $\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}$ in $\mathcal{P}(\mathrm{Aff}_{\mathbf{k}})$.

Remark 2.3.20. According to Theorem 3.1.1, the presheaf $\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}$ is in fact a derived stack.

Remark 2.3.21. It immediately from the definitions that for a commutative space X we have:

$$\mathfrak{M}^{\mathrm{QC}}_X \simeq \mathfrak{M}_{\mathrm{QC}(X)}$$

$$\mathfrak{M}^{\mathrm{pe}}_X \simeq \mathfrak{M}^{\mathrm{perf}}_{\mathrm{QC}(X)}$$

Of course, there are similar statements for $\mathcal{M}_X$ and $\mathcal{M}_X^{\text{pe}}$.

Remark 2.3.22. Using A.6.3.4.1 and A.6.3.4.6, and the fact that for a commutative algebra A, left modules are canonically identified with right modules (upto a contracticble ambiguity), one sees that we have natural equivalences:

$$\mathfrak{M}_{\mathcal{X}}(\mathrm{Spec}(A)) \simeq \mathcal{X} \otimes_{\mathrm{Mod}_{\mathbf{k}}} \mathrm{Mod}_A \simeq \mathrm{Mod}_A(\mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Mod}_{\mathbf{k}}}(\mathrm{Mod}_A, \mathcal{X})$$

Chapter 3

Brane Descent

3.1 Perfect Branes Descend

Theorem 3.1.1. *For any $\mathcal{X}$ in $\mathcal{P}r^L_{\omega,\mathbf{k}}$, the functor $\mathfrak{M}_{\mathcal{X}} : \mathrm{Aff}^{op}_{\mathbf{k}} \to \mathcal{P}r^L_{\omega,\mathbf{k}}$ is a sheaf for the flat hypertopology.*

Corollary 3.1.2. *The presheaf $\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}$ is a sheaf for the flat hypertopology on $\mathrm{Aff}_{\mathbf{k}}$. In particular, it defines an object in $\mathcal{S}t^{\wedge}_{\mathbf{k}}$, the hypercompletion of the ∞-topos $\mathcal{S}t_{\mathbf{k}}$.*

Before we proceed to outline the proof of the theorem, we make a simple observation that will play a central role in the way we organize our efforts:

Lemma 3.1.3. *Let $(\mathcal{C}, \tau)$ be an ∞-site and let $\mathcal{D}$, $\mathcal{E}$ be ∞-categories. Let $\mathcal{F}$ be a $\mathcal{D}$ valued presheaf on $\mathcal{C}$, and let $\mathcal{D} \to \mathcal{E}$ be a functor.*

1. *Assume that $\mathcal{F}$ is a sheaf for the τ-hypertopology, and that f preserves products*

and limits of cosimplicial objects. Then $f \circ \mathcal{F}$ is an $\mathcal{E}$-valued sheaf for the τ-hypertopology.

2. *Assume that $f \circ \mathcal{F}$ is a sheaf for the τ-hypertopology, and that f reflects products and limits of cosimplicial objects. Then $\mathcal{F}$ is a sheaf for the τ-hypertopology.*

Proof. Follows immediately from the definitions. □

The proof of Theorem 3.1.1 will occupy the rest of this chapter. We will proceed in several steps:

1. We will see that the forgetful functor $\mathcal{P}r^L_{\omega,\mathbf{k}} \to \mathcal{P}r^L_{\mathbf{k}}$ reflects products and limits of cosimplicial objects (Lemmas 3.2.6 and 3.2.7), while the forgetful functor $\mathcal{P}r^L_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_\infty$ preserves and reflects all limits (Lemma 3.2.2).

2. Step 1. and Lemma 3.1.3 reduce the problem to showing that the composite functor $\mathcal{M}^\sharp_{\mathcal{X}}$:

$$\mathrm{Aff}^{op}_{\mathbf{k}} \xrightarrow{\mathfrak{M}_{\mathcal{X}}} \mathcal{P}r^L_{\omega,\mathbf{k}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

   is a sheaf for the flat hypertopology. Standard techniques from the theory of cohomological descent (Theorem 3.3.3) allow us to further reduce the problem to showing that $\mathcal{M}^\sharp_{\mathcal{X}}$ is a sheaf for the flat topology; i.e., it will suffice to consider only *1-coskeletal* hypercovers.

3. We will appeal to a corollary (Proposition 3.4.2) of the Barr-Beck-Lurie theorem (Theorem 3.4.1) to show that $\mathcal{M}^\natural_{\mathcal{X}}$ is a sheaf for the flat topology (Propo-

sition 3.5.2). To apply this corollary, we must show that certain "Beck-Chevalley" conditions (see *loc. cit*) are satisfied: this follows from a base change lemma for branes (Proposition 3.5.3).

3.2 Gluing Compact Objects

The fact that compact objects do not glue in general, or equivalently, the fact that the functor $\mathcal{P}r^L_\omega \subseteq \widehat{\mathrm{Cat}}_\infty$ does not preserve and reflect limits, will be a recurring theme through this paper. Our purpose in this section is to note this fact, and to point out conditions on a diagram category K that ensure that the inclusion $\mathcal{P}r^L_\omega \subseteq \mathcal{P}r^L$ *does* preserve and reflect limits of shape K. In particular we will see that the forgetful functor $\mathcal{P}r^L_{\omega,\mathbf{k}} \to \mathcal{P}r^L_{\mathbf{k}}$ reflects products and limits of cosimplicial objects (Lemmas 3.2.6 and 3.2.7)

We begin with the some observations about the relationship between linear structures and limits, which will be used in the sequel, and which, together with the lemmas mentioned above, complete Step 1 in the proof of Theorem 3.1.1 as outline in §3.1.

Lemma 3.2.1. *The forgetful functor* $\mathcal{P}r^L_{\omega,\mathbf{k}} := \mathrm{Mod}_{\mathrm{Mod}_k}(\mathcal{P}r^L_\omega) \to \mathcal{P}r^L_\omega$ *preserves and reflects all small limits.*

Proof. This follows from the general statement that the forgetful functor from a module category to the underlying category preserves and reflects all small limits

A.4.2.3.3. ∎

Lemma 3.2.2. *The forgetful functor $i^L_{\mathbf{k}} : \mathcal{P}r^L_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_\infty$ preserves and reflects all limits.*

Proof. We have $i^L_{\mathbf{k}} = i^L \circ \pi^L_{\mathbf{k}}$, where $i^L : \mathcal{P}r^L \to \widehat{\mathrm{Cat}}_\infty$ is the natural inclusion, and $\pi^L_{\mathbf{k}} : \mathcal{P}r^L_{\mathbf{k}} := \mathrm{Mod}_{\mathrm{Mod}_{\mathbf{k}}}(\mathcal{P}r^L) \to \mathcal{P}r^L$ is the forgetful functor. The functor $\pi^L_{\mathbf{k}}$ preserves and reflects all limits by A.4.2.3.1. According to T.5.5.3.13., the categories $\mathcal{P}r^L$ and $\widehat{\mathrm{Cat}}_\infty$ admits all small limits and i^L preserves all small limits. The fact that i^L is conservative, together with the following lemma, implies that i^L reflects all small limits. ∎

Lemma 3.2.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and let K be a simplicial set. Assume that $\mathcal{C}$ admits limits of diagrams of shape K, that f preserves these limits, and that f is conservative. Then f reflects limits of shape K.*

Proof. Let $\phi : K^{\triangleleft} \to \mathcal{C}$ be a diagram, and suppose that $f \circ \phi : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram. Since $\mathcal{C}$ admits limits of diagrams of shape K, there exists a limit diagram $\psi : K^{\triangleleft} \to \mathcal{C}$ with $\psi_{|K} \simeq \phi_{|K}$. By the definition of limits, there is a morphism $\alpha : \phi \to \psi$ in $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{C})$. We will complete the proof by showing that α is an equivalence. Since f is conservative, it will suffice to show that $f(\alpha)$ is an equivalence.

Since f preserves K-limit diagrams, $f \circ \psi$ is also a limit diagram. Furthermore, we have $(f \circ \psi)_{|K} = (f \circ \phi)_{|K}$. Since the restriction $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{D}) \to \mathrm{Fun}_K(K, \mathcal{D}) \simeq$

$\{f \circ \phi\}$ is a trivial Kan fibration, we have a natural equivalence $\beta : f \circ \psi \to f \circ \phi$ in $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{D})$. Using the fact that $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{D}) \to \mathrm{Fun}_K(K, \mathcal{D})$ is a trivial fibration again, we conclude that $\beta \circ f(\alpha)$ is an equivalence. By the two out of three property, $f(\alpha)$ is an equivalence. $\square$

Lemma 3.2.4. *Let K be a simplicial set, and let $\nu : K^{\triangleleft} \to \mathcal{P}r^L_\omega$ be a diagram classifying a coCartesian fibration $\nu^\flat : \mathcal{V} \to K^{\triangleleft}$. Let $\psi_x : \mathcal{V}_\infty \to \mathcal{V}_k$ be the natural functor (see....). Let $i : \mathcal{P}r^L_\omega \to \widehat{\mathrm{Cat}}_\infty$ be the natural inclusion. Assume that :*

(i) $i \circ \nu$ is a limit diagram.

(ii) An object X in $\mathcal{V}_\infty$ is compact if and only if $\psi_x(X)$ is compact for all x in K.

Then the following hold:

(1) ν is a limit diagram.

(2) The induced diagram $(-)^\omega \circ \nu : K^{\triangleleft} \to \mathrm{Cat}_\infty$ is a limit diagram.

Proof. Assume that the hypotheses (i) and (ii) are satisfied. We will now prove (1). The limit $\mathcal{V}_\infty$ of $\nu_{|K}$ in $\mathcal{P}r^L_{\mathbf{k}}$ is charaterized upto equivalence by $\mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(\mathcal{W}, \mathcal{V}_\infty) \simeq \lim_{x \in K} \mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(\mathcal{W}, \mathcal{V}_x)$, for any $\mathcal{W}$ in $\mathcal{P}r^L_{\mathbf{k}}$. Our hypothesis that ν takes values in $\mathcal{P}r^L_{\omega,\mathbf{k}}$ implies that each of the functors ψ_x preserves ω-compact objects, and therefore this equivalence restricts to a fully faithful functor $\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_\infty) \to \lim_{x \in K} \mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_x)$, where $\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(-,-) \subseteq \mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(-,-)$ denotes the full subcategory of functors that preserve ω-compact objects. We will show tha this functor is essentially surjective.

Now let $\mathcal{W} \in \mathcal{P}r^L_{\omega,\mathbf{k}}$, and let $w^\flat : \mathcal{W}^\sharp \to K^\triangleleft$ be a cocartesian fibration classified by the constant functor $K^\triangleleft \to \mathcal{P}r^L_{\omega,\mathbf{k}}$ that sends every object to $\mathcal{W}$, and let $\sigma_x : \mathcal{W}^\sharp_\infty \to \mathcal{W}^\sharp_x$ denote the functor (equivalence) induced by the unique morphism $\infty \to x$. Let $X \in \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_x)$, and let $\chi : \mathcal{W}^\sharp_{|K} \to \mathcal{V}_{|K}$ be the corresponding cocartesian section. Note that $\chi_x : \mathcal{W}^\sharp_x \to \mathcal{V}_x$ preserves ω-compact objects for all x in K. The equivalence $\mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(\mathcal{W}, \mathcal{V}_\infty) \simeq \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(\mathcal{W}, \mathcal{V}_x)$ implies that χ extends to a map $\chi : \mathcal{W}^\sharp \to \mathcal{V}$ defined by a cocartesian section such that $\chi_\infty \in \mathrm{Fun}^{\mathrm{L}}_{\mathbf{k}}(\mathcal{W}_\infty, \mathcal{V}_\infty)$. We have natural equivalences $\chi_x \circ \sigma_x \simeq \psi_x \circ \chi_x$, since χ is cocartesian.

Let $X \in \mathcal{W}^\sharp_\infty$ be a compact object. Then we have an equivalence $\chi_0(\sigma_0(X)) \simeq \psi_0(\chi_\infty(X))$ in $\mathcal{V}_0$. Since σ_0 is an equivalence and σ_0 preserves compact objects, we conclude that $\psi_0(\chi_\infty(X))$ is compact.Condition (ii) now implies that $\chi_\infty(X)$ is compact. Thus $\chi_\infty \in \mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}^\sharp_\infty, \mathcal{V}_\infty)$, so χ defines an element X' in $\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_\infty)$ that maps to $\mathcal{X}$. This proves essential surjectivity of the natural functor mapping $\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_\infty)$ to $\lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_x)$.

So we have $\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_\infty) \simeq \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}_{\omega,\mathbf{k}}(\mathcal{W}, \mathcal{V}_x)$. This equivalence characterizes $\mathcal{V}_\infty$ as a limit of $\nu_{|K}$ in $\mathcal{P}r^L_{\omega,\mathbf{k}}$, so $\phi : K^\triangleleft \to \mathcal{P}r^L_{\omega.\mathbf{k}}$ is a limit diagram. This proves (1). $\square$

The following lemma is the only simple observation that one needs to add to the results of "T" to prove that $\mathcal{P}r^L_{\omega,\mathbf{k}} \to \mathcal{P}r^L_{\mathbf{k}}$ reflects limits of cosimplicial objects.

Lemma 3.2.5. *Let K be an ∞-category and let $\nu : K^\triangleleft \to \mathcal{P}r^L_{\omega,\mathbf{k}}$ be a diagram, classifying a coCartesian fibration $\nu^\flat : \mathcal{V} \to K^\triangleleft$. Assume that:*

(i) The induced diagram $\nu : K^{\triangleleft} \to \mathcal{P}r^L_{\mathbf{k}}$ is a limit diagram.

(ii) There is an object 0 in K such that for every object x in K there is an edge $f_x : 0 \to x$.

Then the following are equivalent for an object X in $\mathcal{V}_\infty$:

(1) X is a compact object of $\mathcal{V}_{-\infty}$.

(2) $\psi_x(X)$ is a compact object of $\mathcal{V}_x$ for all objects x in K.

(3) $\psi_0(X)$ is a compact object of $\mathcal{V}_0$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial: indeed, the hypothesis that ν takes values in $\mathcal{P}r^L_{\omega,\mathbf{k}}$ implies that for any x, the functor ψ_x preserves colimits and ω-compact objects. We will complete the proof by showing that (3) $\Rightarrow$ (1).

Suppose that $\psi_0(X)$ is compact. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a set of objects in $\mathcal{V}_\infty$, and let $A = \coprod_{\lambda\in\Lambda} A_\lambda$. Since ν is a limit diagram in $\mathcal{P}r^L_{\mathbf{k}}$ (or equivalently, in $\widehat{\mathrm{Cat}}_\infty$ by Lemma 3.2.2), we may identify $\mathcal{V}_\infty$ with cocartesian sections of $\nu^\flat$. Let χ, α_λ, α in $\mathrm{Fun}_{K^{\triangleleft}}(K^{\triangleleft}, \mathcal{V})$ be cocartesian sections with $\chi_\infty = X$, $\alpha_{\lambda,\infty} = A_\lambda$ and $\alpha_\infty = A$. By definition of the ψ_x's, we have $\psi_x(X) = \chi_x$, $\psi_x(A_\lambda) = \alpha_{\lambda,\infty}$ and $\psi_x(A) = \alpha_x$. Note that since the functors ψ_x preserve all colimits, we have $\alpha_x \simeq \coprod_{\lambda\in\Lambda} \alpha_{x,\lambda}$, where the coproduct is computed in $\mathcal{V}_x$.

Let $t : X \to A$ be a morphism in $\mathcal{V}_\infty$, and let $\theta : \chi \to \alpha$ be the corresponding morphism of cocartesian sections. Let $\phi_x : \mathcal{V}_0 \to \mathcal{V}_x$ be the functor induced by $f_x : 0 \to x$. Since χ, α and α_λ, $\lambda \in \Lambda$ are cocartesian, and since ϕ_x preserves colimits (recall that $\nu^\flat$ is classified by a diagram $K^\triangleleft \to \mathcal{P}r^L_{\mathbf{k}}$), we have a commutative diagrams:

$$\begin{tikzcd}
\phi_x(\chi_0) \arrow[r, "\phi_x(\theta_0)"] \arrow[d, "\wr"] & \phi_x(\alpha_0) \arrow[d, "\wr"] & \coprod_{\lambda\in\Lambda} \phi_x(\alpha_{x,0}) \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
\chi_x \arrow[r, "\theta_x"] & \alpha_x & \coprod_{\lambda\in\Lambda} \alpha_{x,\lambda} \arrow[l, "\sim"']
\end{tikzcd}$$

Since χ_0 is a compact object of $\mathcal{V}_0$ by hypothesis, there exists an ω-small set $\Lambda^\omega \subset \Lambda$ such that $\theta_0 : \chi_0 \to \alpha_0 \simeq \coprod_{\lambda\in\Lambda} \alpha_{0,\lambda}$ factors through $\coprod_{\lambda\in\Lambda^\omega} \alpha_{0,\lambda}$. From the diagram above, we see that this implies that θ_x factors through $\coprod_{\lambda\in\Lambda^\omega} \alpha_{x,\lambda}$ for all x. Thus, θ factors through $\coprod_{\lambda\in\Lambda^\omega} \alpha_\lambda$, or equivalently, $t : X \to \coprod_{\lambda\in\Lambda} A_\lambda$ factors through $\coprod_{\lambda\in\Lambda^\omega} A_\lambda$. The category $\mathcal{V}_\infty$ is stable (because it is admits a $\mathbf{k}$-linear structure, for instance). Therefore (A.1.4.5.1), this proves that X is compact. $\square$

Lemma 3.2.6. *Let K be an ∞-category satisfying the hypothesis (ii) of Lemma 3.2.5. Then forgetful functor $\pi : \mathcal{P}r^L_{\omega,\mathbf{k}} \to \mathcal{P}r^L_{\mathbf{k}}$ reflects limits of diagrams of shape K. In particular, π reflects limits of cosimplicial objects.*

Proof. This follows immediately from Lemma 3.2.1 and Lemma 3.2.5, since $\mathrm{N}(\Delta)$ satisfies the hypotheses of Lemma 3.2.5. $\square$

Lemma 3.2.7. *The functor $\pi : \mathcal{P}r^L_{\omega,\mathbf{k}} \to \mathcal{P}r^L_{\mathbf{k}}$ reflects ω-small products.*

Proof. This follows from Lemma 3.2.1, Lemma 3.2.4 and the following fact: If $\{\mathcal{C}_\alpha\}$ is a finite family of ∞-categories with product $\mathcal{C}$, then an object X in $\mathcal{C}$ is ω-compact as soon as its image in each $\mathcal{C}_\alpha$ is ω-compact (T.5.3.4.10). $\square$

3.3 Bootstrapping: from Covers to Hypercovers

Notation 3.3.1. Let $\mathcal{X}$ be an object in $\mathcal{P}r^L_{\omega,\mathbf{k}}$. The functor $\mathcal{M}^\sharp_{\mathcal{X}} : \mathrm{Aff}^{op}_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_\infty$ is defined by the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Aff}^{op}_{\mathbf{k}} \arrow[r, "\mathfrak{M}_{\mathcal{X}}"] \arrow[dr, "\mathcal{M}^\sharp_{\mathcal{X}}"'] & \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[d] \\ & \widehat{\mathrm{Cat}}_\infty \end{tikzcd}$$

Definition 3.3.2. Let $U^\bullet : \mathrm{N}(\Delta^{op}_+) \to \mathrm{Aff}_{\mathbf{k}}$ be an augmented simplicial derived affine scheme. Put $U := U^{-1}$. We will say that $U^\bullet$ is *of cohomological descent* if the natural map $\mathcal{M}^\sharp_{\mathcal{X}}(U) \to \lim \mathcal{M}^\sharp(U^\bullet)$ is an equivalence for every $\mathcal{X}$ in $\mathcal{P}r^L_{\omega,\mathbf{k}}$. We will say that $U^\bullet$ is *universally* of cohomological descent, if any base change of $U^\bullet$ is of cohomological descent.

For a map $f : U^0 \to U$ in $\mathrm{Aff}_{\mathbf{k}}$, the Čech nerve of f, denoted $\check{C}(f)$, is the 0-coskeleton of f computed in $(\mathrm{Aff}_{\mathbf{k}})_{/U}$. Let $\mathbb{P}$ denote the class of morphisms in $\mathrm{Aff}_{\mathbf{k}}$ whose Čech nerve is *universally* of cohomological descent.

Theorem 3.3.3. *$\mathbb{P}$-hypercovers are universally of cohomological descent.*

3.4 The Barr-Beck-Lurie Theorem

The involution on the $(\infty, 2)$-category of ∞-categories that takes an ∞-category to its opposite, interchanges left adjoints with right adjoints, and monads with comonads. Consequently, every theorem about monads has a dual comonadic analogue. In particular, we have the following comonadic analogue of the Barr-Beck-Lurie theorem.

Theorem 3.4.1 (Lurie [Lur11b, Theorem 6.2.2.5])**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be an ∞-functor that admits a right adjoint $g : \mathcal{D} \to \mathcal{C}$. Then the following are equivalent:*

1. *f exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$.*

2. *f satisfies the following two conditions:*

   (a) f is conservative, i.e., it reflects equivalences.

   (b) Let U be a cosimplicial object in $\mathcal{C}$, which is f-split. Then U has a limit in $\mathcal{C}$, and this limit is preserved by f.

In practice, we will use the following consequence of the comonadic Barr-Beck-Lurie theorem, which is the dual version of A.6.2.4.3:

Proposition 3.4.2. *Let $\mathcal{C}^\bullet : \mathrm{N}(\Delta_+) \to \widehat{\mathrm{Cat}}_\infty$ be a coaugmented cosimplicial ∞-category, and set $\mathcal{C} := \mathcal{C}^{-1}$. Let $f : \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that:*

1. *The ∞-category $\mathcal{C}$ admits totalizations of f-split cosimplicial objects, and those totalizations are preserved by f.*

2. (Beck-Chevalley conditions) *For a morphism $\alpha : [m] \to [n]$ in Δ_+, let $\tilde{\alpha}$ be the morphism defined by $\tilde{\alpha}(0) = 0$ and $\tilde{\alpha}(i) = \alpha(i-1)$ for $1 \leq i \leq m$. Then*

for every α, the diagram

$$\begin{tikzcd}
\mathcal{C}^m \arrow[r, "d^0"] \arrow[d, "\alpha"] & \mathcal{C}^{m+1} \arrow[d, "\tilde{\alpha}"] \\
\mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1}
\end{tikzcd}$$

is right adjointable.

Then the canonical map $\theta : \mathcal{C} \to \lim_{\Delta} \mathcal{C}^{\bullet}$ admits a fully faithful right adjoint. If f is conservative, then θ is an equivalence.

3.5 Flat Hyperdescent: the proof

In this section, we will complete Step 4. by showing that $\mathfrak{M}_{\mathcal{X}}$ defines a $\widehat{\mathrm{Cat}}_{\infty}$-valued hypersheaf (Proposition 3.5.2). Finally, we will combine this with the results of the previous sections to prove the main theorem of this chapter (Theorem 3.1.1).

Notation 3.5.1. Let $\mathcal{X}$ be an object in $\mathcal{P}r^L_{\omega,\mathbf{k}}$. The functor $\mathcal{M}^{\sharp}_{\mathcal{X}} : \mathrm{Aff}^{op}_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_{\infty}$ is defined by the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{Aff}^{op}_{\mathbf{k}} \arrow[r, "\mathfrak{M}_{\mathcal{X}}"] \arrow[dr, "\mathcal{M}^{\sharp}_{\mathcal{X}}"'] & \mathcal{P}r^L_{\omega,\mathbf{k}} \arrow[d] \\
& \widehat{\mathrm{Cat}}_{\infty}
\end{tikzcd}$$

The functor $\mathcal{M}^{\natural}_{\mathcal{X}} : \mathrm{CAlg}_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_{\infty}$ is defined to be the composite $\mathcal{M}^{\natural}_{\mathcal{X}} = \mathcal{M}^{\sharp}_{\mathcal{X}} \circ \mathcal{O}$, where $\mathcal{O} : \mathrm{Aff}^{\mathrm{op}}_{\mathbf{k}} \to \mathrm{CAlg}_{\mathbf{k}}$ is the tautological equivalence. Explicitly, we have $\mathcal{M}^{\natural}_{\mathcal{X}}(A) \simeq \mathcal{X} \otimes_{\mathrm{Mod}_{\mathbf{k}}} \mathrm{Mod}_A \simeq \mathrm{Mod}_A(\mathcal{X})$ (see Remark 2.3.22).

Proposition 3.5.2. *Let $\mathcal{X}$ be a compactly generated $\mathbf{k}$-linear ∞-category. The $\widehat{\mathrm{Cat}}_\infty$ valued presheaf $\mathcal{M}^\sharp_{\mathcal{X}}$ on $\mathrm{Aff}_{\mathbf{k}}$ defined in Notation 3.5.1 is a sheaf for the flat hypertopology.*

The proof of the proposition will be given at the end of this section. In light of Theorem 3.3.3, the essential point is to verify that $\mathcal{M}^\natural_{\mathcal{X}}$ carries the Čech nerve of a faithfully flat morphism $f : A \to A^0$ to a limit diagram in $\widehat{\mathrm{Cat}}_\infty$. For this, we will appeal to Proposition 3.4.2. We begin by collecting together some preliminary results that will allow us the verify the hypotheses of that Proposition.

The lemma that follows facilitates the verification of the "Beck-Chevalley conditions" of Proposition 3.4.2:

Lemma 3.5.3. *(Base change). For any $\mathcal{X}$ in $\mathcal{P}r^L_{\omega,\mathbf{k}}$, the functor $\mathcal{M}^\natural_{\mathcal{X}} : \mathrm{CAlg}_{\mathbf{k}} \to \widehat{\mathrm{Cat}}_\infty$ of Notation 3.5.1 carries cocartesian squares to right adjointable squares.*

Proof. The proof is essentially the same as [TV08, Prop 1.1.0.8]. Let

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r, "p'"'] & B' \end{tikzcd}$$

be a cocartesian square in $\mathrm{CAlg}_{\mathbf{k}}$, and let

$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{X}) \arrow[r, "p^*"] \arrow[d, "f^*"] & \mathrm{Mod}_{A'}(\mathcal{X}) \arrow[d, "f'^*"] \arrow[l, bend right, dotted, "p_*"'] \\ \mathrm{Mod}_B(\mathcal{X}) \arrow[r, "p'^*"'] & \mathrm{Mod}_{B'}(\mathcal{X}) \arrow[l, bend right, dotted, "p'_*"'] \end{tikzcd}$$

be the diagram in $\widehat{\mathrm{Cat}}_\infty$ by induced by $\mathcal{M}^\natural_{\mathcal{X}}$. Here, for a morphism $p : A \to A'$ in $\mathrm{CAlg}_{\mathbf{k}}$, $p^* := \mathcal{M}^\natural_{\mathcal{X}}(p) = \mathfrak{M}_X(p) = A \otimes_{A'} (-)$, and $p_* : \mathrm{Mod}_{A'}(\mathcal{X}) \to \mathrm{Mod}_A(\mathcal{X})$ is the forgetful functor, which is right adjoint to p^* (see Remark **??**).

Let $M \in \mathrm{Mod}_{A'}(\mathcal{X})$. To prove the lemma, we must show that the natural morphism $\nu_M : f^* p_* M \to p'_* f'^* M$ is an equivalence. This follows from the following peculiarity of *commutative* algebras: pushouts coincide with tensor products in $\mathrm{CAlg}_{\mathbf{k}}$, i.e., we have $B' \simeq A' \coprod_A B \simeq A' \otimes_A B$. Consequently, we have a chain of equivalences:

$$M \otimes_{A'} B' \xrightarrow{\sim} M \otimes_{A'} (A' \otimes_A B) \xrightarrow{\sim} M \otimes_A B$$

which, as the reader will readily check, is a homotopy inverse of ν_M. □

Let $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathbf{k}}$ be the Čech nerve of a faithfully flat morphism $f : A \to A^0$, and let $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathbf{k}}$. In order to deduce from the base change lemma that the cosimplicial ∞-category $\mathcal{M}^\natural_{\mathcal{X}}(A^\bullet)$ satisfies the Beck-Chevalley conditions, we need following simple observation:

Lemma 3.5.4. *Let $f : A \to A^0$ be a morphism in $\mathrm{CAlg}_{\mathbf{k}}$, and let $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathbf{k}}$ be the Čech nerve $\check{C}(f) := cosk_0(f)$. For a morphism α in Δ_+, let $\tilde{\alpha}$ be the morphism defined in Proposition 3.4.2. Then for each $\alpha : [m] \to [n]$ in Δ_+, the following diagram is right adjointable:*

$$\begin{tikzcd} A^m \arrow[r, "d^0"] \arrow[d, "A(\alpha)"'] & A^{m+1} \arrow[d, "A(\tilde{\alpha})"] \\ A^n \arrow[r, "d^0"] & A^{n+1} \end{tikzcd} \tag{\dagger}$$

Proof. Since $A^\bullet$ is the 0-coskeleton of f, we have $A^n \simeq \otimes_A^{n+1} A^0 \simeq \coprod_A^{n+1} A^0$, and d^0 is the inclusion of the summand $\coprod_A^{n+1} A^0 \to A^0 \coprod_A (\coprod_A^{n+1} A^0)$. It follows immediately that the square (†) is cocartesian for any $\alpha : [m] \to [n]$. □

Lemma 3.5.6 almost says that if $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathbf{k}}$ is the Čech nerve of a flat morphism $f : A \to A^0$ in $\mathrm{CAlg}_{\mathbf{k}}$, then $\mathcal{M}^\natural_{\mathcal{X}}(A^\bullet)$ satisfies condition 1. of Proposition 3.4.2. In preparation for the proof of Lemma 3.5.6, we prove the following special case of that lemma:

Lemma 3.5.5. *Let $f : A \to B$ be a morphism in $\mathrm{CAlg}_{\mathbf{k}}$, and let $f^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be the functor defined by $f^*(M) := B \otimes_A M$. Assume that f is flat. Then f^* preserves totalizations of f^*-split cosimplicial objects.*

Proof. Let $M^\bullet \in \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{Mod}_A)$ be an f^*-split cosimplicial module. We wish to show that the natural map

$$B \otimes_A |M^\bullet| \longrightarrow |B \otimes_A M^\bullet| \qquad (*)$$

is an equivalence. Since the forgetful functor, $\mathrm{Mod}_B \to \mathcal{S}_\infty$ is conservative, Whitehead's theorem implies that it will suffice to show that the induced morphism on π_n is an isomorphism for all $n \in \mathbb{Z}$. We will use the Bousfield-Kan spectral sequence to compute the homotopy groups, and show that we have an isomorphism on the E_2 page.

There is a Bousfield-Kan spectral sequence with $E_2^{p,q} = \pi^{-p}\pi_q M^\bullet$, and $E_\infty^{p+q} = \pi_{p+q}|M^\bullet|$. Here $\pi^{-p}\pi_q M^\bullet$ is the $(-p)$th cohomotopy group of the cosimplicial abelian group $\pi_q M^\bullet$. Since B is a flat A-module, $\pi_0 B$ is a flat $\pi_0 A$-module. So we have an induced spectral sequence with $E_2^{p,q} = \pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M^\bullet$ and $E_\infty^{p+q} = \pi_0 B \otimes_{\pi_0 A} \pi_{p+q}|M^\bullet|$. Finally, since B is flat over A, we have $\pi_{p+q}(B \otimes_A |M^\bullet|) \simeq \pi_0 B \otimes_{\pi_0 A} \pi_{p+q}|M^\bullet|$. So, in summary, we have a spectral sequence

$$E_2^{p,q} = \pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M^\bullet \Longrightarrow \pi_{p+q}(B \otimes_A |M^\bullet|)$$

Similarly, we have a Bousfeld-Kan spectral sequence for the right hand side of $(*)$, with $E_2^{p,q} = \pi^{-p}\pi_q(B \otimes_A M^\bullet)$ and $E_\infty^{p+q} = \pi_{p+q}|B \otimes_A M^\bullet|$. Using the flatness of B over A again, we have $\pi^{-p}\pi_q(B\otimes_A M^\bullet) \simeq \pi^{-p}(\pi_0 B \otimes_{\pi_0 A} \pi_q M^\bullet) \simeq \pi_0 B \otimes_{\pi_o A} \pi^{-p}\pi_q M^\bullet$. So the spectral sequence becomes

$$E_2^{p,q} = \pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M^\bullet \Longrightarrow \pi_{p+q}|B \otimes_A M^\bullet|$$

Thus, the E_2 pages of the spectral sequences for the left and right hand sides of $(*)$ coincide. To complete the proof, it will suffice to show that these spectral sequences degenerate. Let $N \to N^\bullet$ be a split coaugmented cosimplicial B-module. Then $\pi_q N \to \pi_q N^\bullet$ is a split coaugmented cosimplical abelian group for all q, and so we have $\pi^{-p}\pi_q N^\bullet = 0$ for $p \neq 0$, and $\pi^0 \pi_q N^\bullet = \pi_q N$. Applying this to $N^\bullet := B \otimes_A M^\bullet$, and $N = |B \otimes_A M^\bullet|$, we see that both the spectral sequences above degenerate at the E_2 page. □

Lemma 3.5.6. *Let $f : A \to B$ be a flat morphism in $\mathrm{CAlg}_{\mathbf{k}}$, and let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\omega,\mathbf{k}}$. Then the category $\mathcal{M}^\natural_{\mathcal{X}}(A)$ admits all small limits, and the functor $\mathcal{M}^\natural_{\mathcal{X}}(f) : \mathcal{M}^\natural_{\mathcal{X}}(A) \to \mathcal{M}^\natural_{\mathcal{X}}(B)$ preserves totalizations of $\mathcal{M}^\natural_{\mathcal{X}}(f)$-split cosimplicial* *objects.*

Proof. The first statement is clear: the ∞-category $\mathcal{M}^{\natural}_{\mathcal{X}}(A) \simeq \mathcal{X} \otimes_{\mathrm{Mod}_{\mathbf{k}}} \mathrm{Mod}_A$ is presentable (2.3.1), and in particular admits all small limits and colimits.

Since $\mathcal{X}$ is a compactly generated $\mathbf{k}$-linear ∞-category, Proposition **??** says that the restricted Yoneda embedding gives an equivalence $\mathcal{X} \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_{\mathbf{k}})$. Furthermore, for any A in $\mathrm{CAlg}_{\mathbf{k}}$, we have $\mathcal{M}^{\natural}_{\mathcal{X}}(A) \simeq \mathrm{Mod}_A(\mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_{\mathbf{k}})) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_A)$.

Let $X \in \mathcal{X}^{\omega}$ be an object classified by a morphism of small $\mathbf{k}$-linear ∞-categories $\psi_X : \mathfrak{B}\mathbf{k} \to \mathcal{X}^{\omega}$. Using the natural identifications $\mathrm{Fun}_{\mathbf{k}}(\mathfrak{B}\mathbf{k}, \mathrm{Mod}_A) \simeq \mathrm{Mod}_{\mathbf{k}\otimes A} \simeq \mathrm{Mod}_A$, we see that pullback along ψ_X defines a functor $\psi^*_{X,A} : \mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_A) \to \mathrm{Mod}_A$.

Let $f : A \to B$ be a morphism in $\mathrm{CAlg}_{\mathbf{k}}$. Under the identification $\mathcal{M}^{\natural}_{\mathcal{X}}(A) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_A)$, the functor $\mathcal{M}^{\natural}_{\mathcal{X}}(f)$ corresponds to the functor $f^* \circ (-)$, where $f^* := \mathcal{M}^{\natural}_{\mathbf{1}}(f) = B \otimes_A (-)$. Furthermore, for every X in $\mathcal{X}^{\omega}$, we have a homotopy commutative diagram in $\widehat{\mathrm{Cat}}_{\infty}$

$$\begin{tikzcd}
\mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_A) \arrow[r, "f^*\circ(-)"] \arrow[d, "\psi^*_{X,A}"'] & \mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_B) \arrow[d, "\psi^*_{X,B}"] \\
\mathrm{Mod}_A \arrow[r, "f^*"'] & \mathrm{Mod}_B
\end{tikzcd}$$

Now suppose that $f : A \to B$ is a flat morphism. Let $M^{\bullet}$ be a cosimplicial object in $\mathrm{Fun}_{\mathbf{k}}(\mathcal{X}^\omega, \mathrm{Mod}_A)$, for which the induced cosimplicial object $f^*M^\bullet$ is split. To complete the proof of the lemma, it will suffice to show that the natural morphism $\nu_{M^\bullet} : f^*(\lim M^\bullet) \to \lim f^*(M^\bullet)$ is an equivalence.

Since the family of functors $\{\psi^*_{X,B}\}_{X\in\mathcal{X}^\omega}$ is jointly conservative, it is enough to show that for each X in $\mathcal{X}^\omega$, the morphism $\psi^*_{X,B}(\nu_{M^\bullet})$ is an equivalence. The commutativity of the diagram above, together with the fact that the functors $\psi^*_{X,-}$ commute with all limits, implies that this equivalent to showing that the natural morphism $\nu_{\psi^*_{X,A}(M^\bullet)} : f^*(\lim \psi^*_{X,A}) \to \lim f^*(\psi^*_{X,A}(M^\bullet))$ is an equivalence for every object X in $\mathcal{X}^\omega$.

Note that the cosimplicial B-module $f^*(\psi^*_{X,A}(M^\bullet))$ is split, being the image under $\psi^*_{X,B}$ of the split cosimplicial object $f^*(M^\bullet)$. Applying Lemma 3.5.5 to the A-module $\psi^*_{X,A}(M^\bullet)$, we see that the morphism $\nu_{\psi^*_{X,A}(M^\bullet)}$ is an equivalence for every X is $\mathcal{X}^\omega$. □

Lemma 3.5.7. *Let $f : A \to B$ be a faithfully flat morphism in* $\mathrm{CAlg}_{\mathbf{k}}$*, and let $\mathcal{X}$ be an object in $\mathcal{P}r^L_{\omega,\mathbf{k}}$. Then the functor $\mathcal{M}^\natural_X(f) : \mathcal{M}^\natural_{\mathcal{X}}(A) \to \mathcal{M}^\natural_{\mathcal{X}}(A)$ is conservative.*

Proof. We will retain the notation from Lemma 3.5.6. Since the family $\{\psi^*_{X,B}\}_{X\in\mathcal{X}^\omega}$ is jointly conservative, $\mathcal{M}^\natural_{\mathcal{X}}(f)$ is conservative if and only if $\{\psi^*_{X,B} \circ \mathcal{M}^\natural_{\mathcal{X}}(f)\}_{X\in\mathcal{X}^\omega} = \{f^* \circ \psi^*_{X,A}\}_{X\in\mathcal{X}^\omega}$ is a jointly conservative family. Using the fact that $\{\psi^*_{X,A}\}_{X\in\mathcal{X}^\omega}$ is jointly conservative, we see that this is equivalent to asking that $f^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is conservative. Since Mod_B is stable, this is equivalent to asking that f^* reflects zero objects. But this is what is means for a flat morphism to be *faithfully* flat. □

Lemma 3.5.8. *Let $\mathcal{X}$ be an object in $\mathcal{P}r^L_{\omega,\mathbf{k}}$. The functor $\mathcal{M}^\natural_{\mathcal{X}}$ preserves finite products.*

Proof. This is formal. Let A_i, $i = 1, 2$, be commutative k-algebras, and let $A := A_1 \times A_2$. Consider the adjunction

$$\begin{tikzcd} \mathcal{M}^\natural_{\mathcal{X}}(A) \arrow[r, bend left] & \mathcal{M}^\natural_{\mathcal{X}}(A_1) \times \mathcal{M}^\natural_{\mathcal{X}}(A_2) \arrow[l] \end{tikzcd}$$

The left adjoint, which is the natural morphism $\mathcal{M}^\natural_{\mathcal{X}}(A) \to \lim \mathcal{M}^\natural_{\mathcal{X}}(A_i)$, carries $M \in \mathrm{Mod}_A(\mathcal{X})$ to $(M \otimes_A A_1, M \otimes_A A_2)$. The right adjoint carries (M_1, M_2) to $p_{1*}M_1 \times p_{2*}M_2$, where $p_i : A \to A_i$ is the natural projection, and $p_{i*} : \mathrm{Mod}_{A_i}(\mathcal{X}) \to \mathrm{Mod}_A(\mathcal{X})$ is the forgetful functor. We will show that the unit and counit of this adjunction are equivalences.

The ∞-category $\mathrm{Mod}_A(\mathcal{X})$ is stable, and therefore we have natural equivalences $M \oplus N \simeq M \times N$ for M, N in $\mathrm{Mod}_A(\mathcal{X})$. Using this, together with the projection formula, we have, for M in $\mathrm{Mod}_A(\mathcal{X})$:

$$\begin{aligned} p_{1*}(M \otimes_A A_1) \times p_{2*}(M \otimes_A A_2) &\simeq M \otimes_A p_{1*}A_1 \times M \otimes_A p_{2*}A_2 \\ &\simeq (M \otimes_A p_{1*}A_1) \oplus (M \otimes_A p_{2*}A_2) \\ &\simeq M \otimes_A (p_{1*}A_1 \oplus p_{2*}A_2) \\ &\simeq M \otimes_A A \\ &\simeq M \end{aligned}$$

One checks that the composite morphism $p_{1*}(M \otimes_A A_1) \times p_{2*}(M \otimes_A A_2) \to M$ is inverse to the unit of the adjunction, proving that the unit is an equivalence.

For M_i in $\mathrm{Mod}_{A_i}(\mathcal{X})$, we have natural equivalences $p_{i*}M_i \otimes_A A_i \simeq M_i$ and $p_{i*}M_i \otimes_A A_j \simeq 0$ for $i \neq j$. From this, one immediately deduces that the natural maps $(p_{1*}M_1 \times p_{2*}M_2) \otimes_A A_i \simeq (p_{1*}M_1 \otimes_A A_i) \oplus (p_{2*}M_2 \otimes_A A_i) \to M_i$ are equivalences. This shows that the counit is an equivalence. □

We are now in a position to prove the main proposition of this section.

Proof of Proposition 3.5.2. Let $\mathcal{X}$ be in $\mathcal{P}r^L_{\omega,\mathbf{k}}$. We must show that $\mathcal{M}^\sharp_\mathcal{X}$ preserves finite products and carries flat hypercover to limit diagrams. By virtue of Lemma 3.5.8, only the second statement remains to be proved. In view of Theorem 3.3.3, it will suffice to show that $\mathcal{M}^\sharp_\mathcal{X}$ carries the Čech nerve of a flat morphism in $\mathrm{Aff}_\mathbf{k}$ to a limit diagram in $\widehat{\mathrm{Cat}}_\infty$.

Let $U^\bullet : \mathrm{N}(\Delta_+^{op}) \to \mathrm{Aff}_\mathbf{k}$ be a flat hypercover, and let $A^\bullet := \mathcal{O}(U^\bullet) : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_\mathbf{k}$. Put $A := A^{-1}$. Lemma 3.5.6 says that the associated diagram $\mathcal{M}^\natural_\mathcal{X}(A^\bullet) : \mathrm{N}(\Delta_+^{op}) \to \widehat{\mathrm{Cat}}_\infty$, satisifies condition 1. of the corollary of the Barr-Beck-Lurie theorem, Proposition 3.4.2. The base change lemma for branes (Lemma 3.5.3), together with Lemma 3.5.4, implies that $\mathcal{M}^\natural_\mathcal{X}(A^\bullet)$ satisfies condition 2. of Proposition 3.4.2. Finally, Lemma 3.5.7 tells us that the natural map $\mathcal{M}^\natural_\mathcal{X}(A) \to \mathcal{M}^\natural_\mathcal{X}(A^0)$ is conservative. Thus, by Proposition 3.4.2, the natural map $\mathcal{M}^\natural_\mathcal{X}(A) \to \lim \mathcal{M}^\natural{}_\mathcal{X}(A^n)$ is an equivalence. □

Finally, we can now prove the main theorem of this chapter:

Proof of Theorem 3.1.1. By virtue of Lemmas 3.2.2, 3.2.6 and 3.2.7, the natural inclusion $\mathcal{P}r^L_{\omega,\mathbf{k}} \to \widehat{\mathrm{Cat}}_\infty$ reflects finite products and limits of cosimplicial objects. The theorem now follows from Proposition 3.5.2 and Lemma 3.1.3. $\square$

We need to simple observation before we can write down the proof of Corollary 3.1.2:

Lemma 3.5.9. *The functors $(-)^{\simeq} : \widehat{\mathrm{Cat}}_\infty \to \widehat{\mathcal{S}}$ and $(-)^{\simeq} : \mathrm{Cat}_\infty \to \mathcal{S}$, which carry an ∞-category to the maximal ∞-groupoid that it contains, preserve all limits.*

Proof. The functor $(-)^{\simeq}$ is a right adjoint, and therefore preserves all limits: the natural inclusion $\pi_{\leq\infty}$ of spaces into ∞-categories is left adjoint to $(-)^{\simeq}$. $\square$

Proof of corollary.3.1.2. The flat hypertopology is finer than the étale hypertopology. Thus, the second statement follows immediately from the first statement and the fact that the hypercomplete objects in an ∞-topos of sheaves on an ∞-site $(\mathcal{C}, \tau)$ are precisely those presheaves that are sheaves for the τ-hypertopology [Lur11a, Prop. 5.12].

Let $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathbf{k}}$ be a flat hypercover. By virtue of Propostion 3.5.2, the induced functor $\mathfrak{M}_{\mathcal{X}}(A^\bullet) : \mathrm{N}(\Delta_+) \to \mathcal{P}r^L_{\omega,\mathbf{k}}$ satisfies the hypotheses of Lemma 3.2.4. It follows that the induced diagram $(\mathfrak{M}_{\mathcal{X}}(A^\bullet))^\omega$ is a limit diagram in Cat_∞. Applying Lemma 3.5.9, we see that $\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}(A^\bullet) := ((\mathfrak{M}_{\mathcal{X}}(A^\bullet))^\omega)^{\simeq}$ is a limit diagram. Thus, $\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}$ carries flat hypercovers to limit diagrams in $\mathcal{S}$. The proof that $\mathcal{M}^{\mathrm{perf}}_{\mathcal{X}}$ preserves finite products is similar. $\square$

Chapter 4

Geometricity

4.1 Geometric Stacks

4.2 The Artin-Lurie Criterion

4.3 Infinitesimal theory: Brane Jets

4.4 Dualizability implies Geometricity

4.5 A Proper Counterexample

Chapter 5

Moduli of 2D-TFTs

5.1 The Cobordism Hypothesis

5.2 Moduli of Noncommutative Calabi-Yau Spaces

5.3 Geometricity

5.4 Unobstructedness

5.5 Frobenius Structures: from TFTs to CohFTs

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