# QUANTUM FAMILY ALGEBRAS

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ABSTRACT

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We study algebras of invariants of an action of a semi-simple Lie algebra in the tensor product of the universal enveloping algebra and the endomorphisms of given representation. These algebras were intorduced by A. A. Kirillov and are called *family algebras*. We prove the criterion of commutativity of these algebras, compute examples of family algebras and the related generalized exponents. We study characteristic identities and prove the formula for quantum eigenvalues in the case of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

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# Chapter 1

# Introduction

In [12], [13] A. A. Kirillov introduced a new class of associative algebras, related to simple complex Lie algebras. These algebras were called *family algebras*. The initial motivation for the study of the family algebras is a problem of computing of generalized exponents and the $q$-analog of the zero-weight multiplicity of a given representation.

**Generalized exponents and $q$-multiplicity of a weight of representation.** Recall Kostant's decomposition of symmetric algebra $S(\mathfrak{g}) = I(\mathfrak{g}) \otimes H(\mathfrak{g})$, where $H(\mathfrak{g})$ is the graded space of harmonic polynomials, and $I(\mathfrak{g})$ is a subalgebra of invariants in $S(\mathfrak{g})$. The decomposition of $H(\mathfrak{g})$ into irreducible components has the property that each representation $\pi_\xi$ occurs in this decomposition with the multiplicity equal to the multiplicity of the zero-weight of this representation. The degrees $k$, such that $\pi_\xi \subset H^k(\mathfrak{g})$, are called *generalized exponents* of $\pi_\xi$. For the adjoint representations these are just regular exponents. In [18] G. Lusztig defined a $q-$analog $m_\mu^\nu(q)$ of the multiplicity of the weight $\nu$ in $\pi_\mu$. The particular case of the zero-weight gives the polynomial

$$m_\mu^0(q) = \sum_m \dim(\mathrm{Hom}_G(\pi_\mu, H^m(\mathfrak{g})))q^m.$$

The powers of $q$ are exactly the generalized exponents of $\pi_\mu$. A beautiful formula for $m_\mu^\nu(q)$ was found independently by W. Hesselink and D. Peterson, [8]:

$$m_\mu^\nu(q) = \sum_{w\in W} (-1)^w P_q(w(\mu+\rho) - \nu - \rho)$$

where $\rho$ is the sum of fundamental weights of $\mathfrak{g}$, $W$ is the Weyl group, and $P_q$ is the $q$-analogue of Kostant's multiplicity function, defined by the generating function

$$\sum_\nu P_q(\nu)e^\nu = \prod_{\alpha>0}(1-qe^\alpha)^{-1}$$

(the product is over all positive roots of $\mathfrak{g}$). Unfortunately, Hesselink-Peterson formula is impractical for real computations, because it involves summation over the elements of a Weyl group. An interesting combinatorial interpretation of the $q-$multiplicity of zero weight, and of the Poincare series of family algebras in particular, follows from the works of R. Gupta and S. Kato ( [10], [9], [11]). Namely, R. Gupta constructed two dual bases of the extended character ring. One of them consists of generalized Hall-Littlewood functions $\{P_\lambda\}$. Let $\{E_\lambda\}$ be the dual basis. Then the $q$-multiplicity $m_\lambda^\nu(q)$ is a scalar product of the character $\chi_\lambda$ of $\pi_\lambda$ and the element $E_\nu$.

But in general, the computation of the generalized exponents of a particular representation remains a difficult problem. The explicit construction of the classical family algebra over the invariants can give another way to compute the generalized exponents of certain representations. Indeed, let $\pi_1, ..., \pi_k$ be the irreducible components of End $V_\lambda$. Then one can construct a special basis of a classical family algebra over $I(g)$ with the following property: roughly speaking, the elements of this basis are matrices with coefficients in harmonic polynomials, and the degrees of these polynomials are the generalized exponents of the representations $\pi_1, ..., \pi_k$. So the structure of a family algebra gives the information about the $q$-multiplicities of zero-weights of some representations. In Chapter 4 we describe this construction in full details. In Chapters 6-10, we illustrate it on some examples.

**The structure of family algebras.** Our main goal is to study the structure of classical family algebras over the rings of invariants, and the quantum family algebras over the center of the universal enveloping algebra of the corresponding Lie algebra.

In Chapter 3 we describe the $Z(\mathfrak{g})$-module structure on $Q_\lambda(\mathfrak{g})$. But the linear isomorphism between quantum family algebra and direct sum of matrix algebras, constructed in Proposition 3.1, does not preserve the algebraic structure. The example of $\mathfrak{sl}_3(\mathbb{C})$ in Chapter 6 illustrates that in general these algebras are not isomorphic.

In [13] A. Kirillov proved that one can get an algebra isomorphism for classical family algebra $C_\lambda(\mathfrak{g})$ and direct sum of matrix algebras after the extension of the ring of scalars $I(\mathfrak{g})$ to the fraction field of the symmetric algebra $S(\mathfrak{h})$ of the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$. In Chapter 5 we describe the *algebraic* structure of commutative quantum (and classical ) family algebras over a smaller field – the fraction field of the center (or the ring of invariants respectively).

Namely, we define an element $M_\lambda$ of a family algebra and prove the following criterion of commutativity:

**Theorem 1.1.** *The following statements are equivalent:*

*1)* $Q_\lambda(\mathfrak{g})$ *is commutative;*

*2) The representation* $\pi_\lambda$ *has simple spectrum;*

*3) The element* $M_\lambda$ *generates* $Q_\lambda(\mathfrak{g})$ *over* $K(\mathfrak{g})$ *=Frac* $Z(\mathfrak{g})$*;*

*4)* $C_\lambda(\mathfrak{g})$ *is commutative.*

In Chapters 8-9 we study a nice case of the exceptional Lie algebra $\mathfrak{g}_2$. This is a good example to illustrate the techniques of harmonic basis and computing of q-analogues of zero-multiplicities using family algebras. In Chapter 10 we compute Poincare series of family algebras of spinor representations.

**Polynomial identities.** In Chapter 5 we consider a special element $M_\lambda$ in $Q_\lambda(\mathfrak{g})$. It has many interesting properties. In particular, it satisfies a polynomial identity with coefficients in $Z(\mathfrak{g})$:

$$\sum_i a_i(\lambda) M_\lambda^i = 0, \qquad a_i \in Z(\mathfrak{g}).$$

M. D. Gould, H. S. Green and A J. Bracken in [1], [6], [7] investigated the properties of similar identities and computed examples for different $\lambda$. The formula of the polynomial identity for $M_\lambda$ is known only in some particular cases.

For example, the polynomial identities for standard representation follow from the results of many papers. The famous work by A.M. Perelomov and V.S. Popov [21] can be reinterpreted to obtain the identity for the element $M_\lambda$ for the standard representation. In [13] another form of this result is given by A.Kirillov. A.Molev in [20] uses the homomorphism from the Yangian $Y(n)$ to the Lie algebra $gl_n(\mathbb{C})$ (and from the twisted Yangians in the other cases) to obtain the elements of the center $Z(\mathfrak{g})$ and some polynomial identities for the case of the standard representation. In Chapter 6 we give the uniform formula for all representations of $\mathfrak{sl}_2(\mathbb{C})$:

**Theorem 1.2.** *The element $M_n$ of the quantum family algebra $Q_n(\mathfrak{g})$ satisfies*

$$\prod_{j=0}^{n} (M_n - b_j) = 0, \tag{1.1.1}$$

*where*

$$b_j = \left(j^2 - \frac{n}{2} - jn\right) + \left(\frac{n}{2} - j\right)\sqrt{(2C+1)}, \qquad j = 0, ..., n.$$

*Here $C \in Z(\mathfrak{sl}_2(\mathbb{C}))$ – is the Casimir element, the generator of the center $Z(\mathfrak{sl}_2(\mathbb{C}))$ of the universal enveloping algebra.*

# Chapter 2

# Main definitions

**1.** Let $\mathfrak{g}$ be a simple complex Lie algebra, $\pi_\lambda$ – its irreducible representation with the highest weight $\lambda$, $d = \dim\, \pi_\lambda$, and $V_\lambda$ – the space of $\pi_\lambda$. Consider the universal enveloping algebra $U(\mathfrak{g})$ and the symmetric algebra $S(\mathfrak{g})$. Recall that $U(\mathfrak{g})$ and $S(\mathfrak{g})$ have isomorphic $\mathfrak{g}$-module structures, coming from the adjoint representation *ad* of the Lie algebra $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$, and let $I(\mathfrak{g})$ be the ring of invariants in $S(\mathfrak{g})$.

We can define a $\mathfrak{g}$-module structure on $\operatorname{End} V_\lambda$ by the formula

$$X \cdot A = [\pi_\lambda(X), A], \qquad A \in \operatorname{End} V_\lambda, \quad X \in \mathfrak{g}.$$

Let $G$ be connected and simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. The representations $\pi_\lambda$ and *ad* of the Lie algebra $\mathfrak{g}$ give the rise to the corresponding representations $\Pi_\lambda$ and *Ad* of the Lie group $G$. Then the action of the Lie group $G$ on

$\mathrm{End}V_\lambda$ is

$$g \cdot A = \Pi_\lambda(g)\, A\, \Pi_\lambda(g)^{-1}, \qquad A \in \mathrm{End}\, V_\lambda, \quad g \in G.$$

Our goal is to study the following algebras of invariants, introduced by A. A. Kirillov ([12], [13]):

$$C_\lambda(\mathfrak{g}) = (\mathrm{End}V_\lambda \otimes S(\mathfrak{g}))^G \qquad (\textit{classical family algebra}),$$

$$Q_\lambda(\mathfrak{g}) = (\mathrm{End}V_\lambda \otimes U(\mathfrak{g}))^G \qquad (\textit{quantum family algebra}).$$

We can treat the elements of classical (quantum) family algebras as matrices $X = (x_{ij})$ with the coefficients in $S(\mathfrak{g})$ (respectively, $U(\mathfrak{g})$ ), satisfying

$$Ad(g)x_{ij} = (\Pi_\lambda^{-1}(g) X \Pi_\lambda(g))_{ij}, \qquad \text{for} \quad g \in G. \tag{2.2.1}$$

**2.** One can see that the quantum family algebra is a deformation of the classical family algebra. Namely, consider $U_t(\mathfrak{g})$ – a factor-algebra of the tensor algebra $T(\mathfrak{g})$ by the ideal, generated by the elements

$$X \otimes Y - Y \otimes X - t\,[X, Y]\,, \qquad X, Y \in \mathfrak{g}.$$

For $t \neq 0$ all the algebras $U_t(\mathfrak{g})$ are isomorphic to $U(\mathfrak{g})$, and $U_0(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$. Accordingly, we could define $Q_\lambda(\mathfrak{g})_t = (\mathrm{End}\, V_\lambda \otimes U_t(\mathfrak{g}))^G$. Then $Q_\lambda(\mathfrak{g})_t \simeq$

$Q_\lambda(\mathfrak{g})$ for $t \neq 0$, and $Q_\lambda(\mathfrak{g})_0 \simeq C_\lambda(\mathfrak{g})$. We will use this relation between family algebras to deduce characteristic identity for classical family algebra of $\mathfrak{sl}_2(\mathbb{C})$ from the characteristic identity of quatnum family algebra.

# Chapter 3

# $Q_\lambda(\mathfrak{g})$ as a $Z(\mathfrak{g})$-module

**1.** The $Z(\mathfrak{g})$-module structure of $Q_\lambda(\mathfrak{g})$ follows from the works of B. Kostant.

**Proposition 3.1.** *There is an isomorphism of $Z(\mathfrak{g})$-modules*

$$Q_\lambda(\mathfrak{g}) = \bigoplus_{i=1}^{k} Mat_{d_i}(Z(\mathfrak{g})),$$

*where $\{d_1, ..., d_k\}$ are the multiplicities of the weights $\{\mu_1, ..., \mu_k\}$ of $\pi_\lambda$.*

*Proof.* Let $\nu$ be a dominant weight. Following [15], define a *strongly commutative ring*

$$R_{\nu\lambda} = (\operatorname{End} V_\lambda \otimes \pi_\nu (U(\mathfrak{g})))^{\mathfrak{g}}. \tag{3.3.1}$$

There is a natural homomorphism of $\mathbb{C}$-algebras $\phi_\nu : Q_\lambda(\mathfrak{g}) \to R_{\nu\lambda}$, defined by

$$\phi_\nu(A \otimes u) = A \otimes \pi_\nu(u), \qquad u \in U(\mathfrak{g}), \quad A \in \text{ End}V_\lambda.$$

We will write $\nu \gg \lambda$, if $\pi_\lambda$ is *totally subordinate* to $\pi_\nu$ (for the definition see [17]).

**Lemma 3.2.** *( B. Kostant, [15]) Let $\nu \gg \lambda$. Then there is an isomorphism of $\mathbb{C}$-algebras*

$$R_{\nu\lambda} \to \bigoplus_{i=1}^{k} Mat_{d_i}(\mathbb{C}),$$

*where $\{d_1, ..., d_k\}$ are the multiplicities of the weights $\{\mu_1, ..., \mu_k\}$ of the representation $\pi_\lambda$. In particular, there exists a highest weight $\nu_0$, such that one has an isomorphism for all highest weights $\nu \geq \nu_0$.*

Note that

$$R_{\nu\lambda} = (\text{End}V_\nu \otimes \text{End}V_\lambda)^{\mathfrak{g}} = \left(\bigoplus l_\lambda(\eta) l_\nu(\eta^*) \pi_\eta \otimes \pi_{\eta^*}\right). \tag{3.3.2}$$

On the other hand, by Kostant's decomposition, $U(\mathfrak{g}) = Z(\mathfrak{g}) \otimes E(\mathfrak{g})$, where $E(\mathfrak{g})$ consists of harmonic elements. Let $E(\mathfrak{g}) = \oplus l(\xi)\pi_\xi$ be the decomposition of the harmonic part into irreducible components with multiplicities $l(\xi)$. It is well-known that the multiplicity $l(\xi)$ of $\pi_\xi$ in this decomposition equals the multiplicity of the zero-weight in $\pi_\xi$. Put also $\text{End}V_\lambda = \oplus l_\lambda(\eta)\pi_\eta$, where $l_\lambda(\eta)$ is the multiplicity of $\pi_\eta$ in this decomposition.

We have a linear isomorphism:

$$Q_\lambda(\mathfrak{g}) = (Z(\mathfrak{g}) \otimes E(\mathfrak{g}) \otimes \mathrm{End}V_\lambda)^{\mathfrak{g}} = Z(\mathfrak{g}) \otimes \left( \bigoplus l_\lambda(\eta) l(\eta^*) \pi_\eta \otimes \pi_{\eta^*} \right),$$

where $\eta^*$ is the highest weight of the representation, dual to $\pi_\eta$.

**Lemma 3.3.** *( B. Kostant, [15]) With the notations as above:*

*$l_\nu(\eta)$ – the multiplicity of $\pi_\eta$ in End $V_\nu$,*

*$l(\eta)$ – the multiplicity of the zero-weight in $\pi_\eta$,*

*the following is true:*

$$l_\nu(\eta) \le l(\eta).$$

*The equality takes place for $\nu \gg \eta$.*

Take $\nu$ such that all $\pi_\eta$ in the decomposition of End $V_\lambda$, and $\pi_\lambda$ itself would be totally subordinate to $\pi_\nu$. Then from (3.3.2) and the Lemma 3.3 one has

$$R_{\nu\lambda} = \bigoplus l_\lambda(\eta) l(\eta^*) \pi_\eta \otimes \pi_{\eta^*},$$

so there is an isomorphism of $Z(\mathfrak{g})$-modules

$$Q_\lambda(\mathfrak{g}) = Z(\mathfrak{g}) \otimes R_{\nu\lambda} = \bigoplus_{i=1}^{k} \mathrm{Mat}_{d_i}(Z(\mathfrak{g})).$$

□

# Chapter 4

# Harmonic basis of the family algebra and generalized exponents

**1.** Recall the Kostant's decomposition $S(\mathfrak{g}) = I(\mathfrak{g}) \otimes H(\mathfrak{g})$, where $H(\mathfrak{g})$ is a graded space of harmonic polynomials. The decomposition of $H(\mathfrak{g})$ into irreducible components has the property that each representation $\pi_\xi$ occurs with the multiplicity equal to the multiplicity $l(\xi)$ of the zero-weight of this representation. The degrees $k$, such that $\pi_\xi \subset H^k(\mathfrak{g})$ are called *generalized exponents* of $\pi_\xi$. There are several wonderful papers on computing of generalized exponents and the $q$-analogue of weight multiplicities ( see, for example, [8], [9], [10], [11], and others). But the computation of the exponents of a particular representation remains a difficult problem.

The explicit construction of the classical family algebra over the invariants can give another way to compute generalized exponents of certain representations. Below we describe a linear basis of $C_\lambda(\mathfrak{g})$ over $I(\mathfrak{g})$. This basis, roughly speaking, consists of matrices with homogeneous coefficients in $H(\mathfrak{g})$, and the degrees of these coefficients are the exponents of the irreducible components in the decomposition of $\mathrm{End}V_\lambda$.

**2.** First we introduce some notations. Let

$$\mathrm{End}(V_\lambda) = \pi_0 \oplus ... \oplus \pi_N$$

be a decomposition into irreducible components. Note that this is *not* a decomposition into isotipic components – each $\pi_i$ is irreducible, and it can be isomorphic to some other $\pi_j$ for $i \neq j$. For any irreducible subspace $\pi_i$ we put:

$D_i = \dim \pi_i$;

$l_i$ – the multiplicity of the zero-weight space of $\pi_i$;

$\{w_i(1), ..., w_i(D_i)\}$ – a basis of $\pi_i$ in End $V_\lambda$, such that $w_i(1)$ is the highest weight vector of $\pi_i$;

$\{M_i^1, ..., M_i^{l_i}\}$ – the generalized exponents of $\pi_i$ (i.e. $\pi_i \subset H^{M_i^k}(\mathfrak{g})$, $k = 1, ..., l_i$). This is also a list with possible repetitions;

$h_i^k$ – the lowest weight vector of $\pi_i$ in the realization in $H^{M_i^k}(\mathfrak{g})$.

Then any element $X$ of the classical family algebra can be written as

$$X = \sum_{k,i} w_i(k) \otimes x_i^k, \quad x_i^k \in S(\mathfrak{g}).$$

In this case we write $X(w_i(k)) = x_i^k$. The condition (2.2.1) implies that the element

$X$ is completely determined by its values $\{X(w_i(1))\}_{i=0,\dots,N}$ on the highest weight vectors. Therefore, we will use the following notation: we say that

$$X = [x_0, ..., x_N], \quad \text{if} \quad X(w_i(1)) = x_i \quad \text{for} \quad i = 0, ..., N.$$

Moreover, the equivariance condition (2.2.1) and Kostant's decomposition give us that for any element $X \in C_\lambda(\mathfrak{g})$ the value of $X(w_i(1))$ is a sum of the lowest weight harmonic polynomials with the coefficients in $I(\mathfrak{g})$:

$$X(w_i(1)) = \sum_{i=0}^{N} \sum_{k=1}^{l_i} c_i^k h_i^k,$$

where $c_i^k \in I(\mathfrak{g})$. Hence, the elements $\{H_{i,k}\}$, satisfying the property

$$H_{i,k}(w_j(1)) = \delta_j^i h_i^k,$$

or, equivalently,

$$H_{i,k} = [0, \dots, \underset{i}{(h_i^k)}, \dots, 0],$$

form a basis of $C_\lambda(\mathfrak{g})$ over $I(\mathfrak{g})$.

**3. Example.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and let $e, f, h$ be the basis in $\mathfrak{sl}_2(\mathbb{C})$ with the commutation relations $[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$. Consider the $(n+1)$-dimensional irreducible representation $\pi_n$ of $\mathfrak{sl}_2(\mathbb{C})$. The space $\mathrm{Mat}_{n+1}(\mathbb{C})$ splits into the following components:

$$\mathrm{Mat}_{n+1}(\mathbb{C}) = \bigoplus_{k=0}^{n} \pi_{2k}.$$

Each component $\pi_{2k}$ is an irreducible representation with the highest weight $\lambda = 2k$. Let $w_{2k}$ be the highest weight vector of $\pi_{2k}$ in $\mathrm{Mat}_{n+1}(\mathbb{C})$. The element $w_{2k}$ is a matrix with 1 on the $k$-th diagonal and other entries equal to zero:

$$w_{2k} = \begin{pmatrix} 0 & ... & 1 & \dots & 0 \\ 0 & ... & ... & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & ... & ... & 1 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The only exponent of each $(2k+1)$-dimensional representation $\pi_{2k}$ is $k$, and the corresponding harmonic polynomial of the lowest weight is $f^k$. The elements

$$\left\{ H_i = [0, ..., \underset{i}{f^i}, ...0] \right\}_{i=0,\,...,\,n}$$

form the described above basis of $C_n(\mathfrak{g})$ over $I(\mathfrak{g})$. In Chapter 7 we construct a harmonic basis of the classical family algebra of the adjoint representation of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$.

# Chapter 5

# Commutativity of family algebras

In [13] A. A. Kirillov proved that classical family algebra is commutative if and only if representation $\pi_\lambda$ has simple spectrum. Here we extend the result to quantum family algebras and describe the generators of commutative family algebras.

Let $K(\mathfrak{g})$ be the fraction field of $Z(\mathfrak{g})$. We fix a basis $\{x_i\}$ of the Lie algebra $\mathfrak{g}$, and let $\{x^i\}$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form. Consider the diagonal map

$$\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}),$$

defined on the elements of the Lie algebra $\mathfrak{g}$ by the formula

$$\delta(x) = 1 \otimes x + x \otimes 1, \qquad x \in \mathfrak{g}.$$

We use the modification of $\delta$ to get a map $\delta_\lambda : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathrm{End} V_\lambda$:

$$\delta_\lambda(x) = 1 \otimes \pi_\lambda(x) + x \otimes 1, \qquad x \in \mathfrak{g}.$$

It is clear that $\delta_\lambda(Z(\mathfrak{g}))$ is a subset of the center of the quantum family algebra $Q_\lambda(\mathfrak{g})$. For example, consider the Casimir element $C_2 = \sum x_i x^i$ in $Z(\mathfrak{g})$, and put

$$M_\lambda = \sum \pi_\lambda(x_i) \otimes x^i, \qquad M_\lambda \in Q_\lambda(\mathfrak{g}).$$

It is easy to see that $M_\lambda$ has the following relation to $C_2$:

$$M_\lambda = \frac{1}{2}(\delta_\lambda(C_2) - 1 \otimes C_2 - \pi_\lambda(C_2) \otimes 1).$$

The element $M_\lambda$ plays an important role in the study of family algebras.

**Definition.** We say that an irreducible representation $\pi_\lambda$ of a semi-simple Lie algebra $\mathfrak{g}$ has *simple spectrum*, if all the weights of this representation occur with the multiplicity one.

**Theorem 5.1.** *The following statements are equivalent:*

*1) $Q_\lambda(\mathfrak{g})$ is commutative;*

*2) The representation $\pi_\lambda$ has simple spectrum;*

*3) The element $M_\lambda$ generates $Q_\lambda(\mathfrak{g})$ over $K(\mathfrak{g})$ =Frac $Z(\mathfrak{g})$;*

*4) $C_\lambda(\mathfrak{g})$ is commutative.*

*In this case* $rk_{Z(\mathfrak{g})}Q_\lambda(\mathfrak{g}) = dim\,\pi_\lambda$, *and* $\{1,\, M_\lambda,\, ...,\, M_\lambda^{d-1}\}$ *with* $d = dim\,\pi_\lambda$ *is the linear basis of* $Q_\lambda(\mathfrak{g})$ *over* $K(\mathfrak{g})$.

*Proof.* $1 \Rightarrow 2$. Suppose $Q_\lambda(\mathfrak{g})$ is commutative. Then $R_{\nu\lambda}$, defined in the Chapter 3, is also commutative for all $\nu$. By Lemma 3.2, for $\nu \gg \lambda$, the algebra $R_{\nu\lambda}$ is isomorphic to the direct sum of matrix algebras, those sizes $d_i$ are the multiplicities of the weights of the representation $\pi_\lambda$. The commutativity implies that $d_i = 1$ for all the matrix algebras in this sum. So $\pi_\lambda$ has simple spectrum.

$2 \Rightarrow 3$. In [6] it is shown that the element $M_\lambda$ satisfies some polynomial identity $P(M_\lambda) = 0$, where $P$ is a polynomial with the coefficients in the center $Z(\mathfrak{g})$. Moreover, let $P_0$ be the polynomial with the property $P_0(M_\lambda) = 0$, and for any other polynomial $P$ with the property $P(M_\lambda) = 0$, we have $\deg P_0 \leq \deg P$. Then we call $P_0$ a *minimal polynomial* of $M_\lambda$. In the same paper it is proved that deg $P_0$ is equal to the number of distinct weights in representation $\pi_\lambda$.

Assume that $\pi_\lambda$ has simple spectrum. Then all its weights are different, and hence, $\deg P_0 = \dim \pi_\lambda$. Then $\{1, M_\lambda, ..., M_\lambda^{d-1}\}$ are linearly independent over $Z(\mathfrak{g})$, where $d = \dim \pi_\lambda$.

On the other side, the rank of $Q_\lambda(g)$ over $Z(\mathfrak{g})$ is equal to the zero-weight multiplicity in $\pi_\lambda \otimes \pi_\lambda^*$. All the weights of $\pi_\lambda$ are different, so the multiplicity of the zero-weight in $\pi_\lambda \otimes \pi_\lambda^*$ is equal to $\dim \pi_\lambda$, and $\{1, M_\lambda, ..., M_\lambda^{d-1}\}$ is the basis of $Q_\lambda(g)$ over $K(\mathfrak{g})$, $d = \dim \pi_\lambda$.

$4 \Rightarrow 2$ is proved in [13], and $3 \Rightarrow 1$, $1 \Rightarrow 4$ are obvious implications,. $\square$

It turns out that there are not so many irreducible representations with simple spectrum. The complete list of the highest weights of such representations is given in the Table 1. [1].

| $\mathfrak{g}$ | $\lambda$ – the highest weight |
|---|---|
| $A_n$ | $\omega_k, \quad k = 1, ..., n$<br>$k\omega_1, \quad k = 1, 2...$<br>$k\omega_n, \quad k = 1, 2...$ |
| $B_n$ | $\omega_1$<br>$\omega_n$ (spin-representation) |
| $C_n$ | $\omega_1$ |
| $D_n$ | $\omega_1$<br>$\omega_{n-1}, \omega_n$ (spin-representations) |
| $G_2$ | $\omega_1$ (dim $= 7$) |
| $E_6$ | $\omega_1$ (dim $= 27$)<br>$\omega_6$ (dim $= 27$) |
| $E_7$ | $\omega_7$ (dim $= 56$) |

**Table 1. Representations with simple spectrum.**

**3.** Using the same definition of $M_\lambda$, one can construct a linear basis of $C_\lambda(\mathfrak{g})$ over $K_0(\mathfrak{g})$ =Frac $I(\mathfrak{g})$ for the representations with simple spectrum. The Examples 3 and 4 show that we can not substitute the field $K(\mathfrak{g})$ just by the ring $Z(\mathfrak{g})$ (or the

[1] I would like to thank prof. J. Stembridge and prof. A. Zelivinsky for this information.

field $K_0(\mathfrak{g})$ by the ring $I(\mathfrak{g})$) — in general, the element $M_\lambda$ does not generate the corresponding family algebra over the ring.

**Examples.** Put $M = M_\lambda$.

1. For any irreducible representation $\pi_n$ of $\mathfrak{sl}_2(\mathbb{C})$ the set $\{1, M, \dots, M^n\}$ *is* a linear basis of the free $Z(\mathfrak{sl}_2(\mathbb{C}))$ - module $Q_n(\mathfrak{sl}_2(\mathbb{C}))$ (see [24] for details).

2. Consider the standard representation ($\lambda = \omega_1$) of the Lie algebra $\mathfrak{g}$ of the series $A_n$, $B_n$ or $C_n$. The set $\{1, M, \dots, M^d\}$ *forms* a linear basis of the free $Z(\mathfrak{g})$-module $Q_n(\mathfrak{g})$. Here $d + 1 = \dim \pi_{\omega_1}$.

3. But for the Lie algebra $\mathfrak{g}$ of the series $D_n$ the linear basis of $C_{\omega_1}(\mathfrak{g})$ over $I(\mathfrak{g})$ is $\{1, M, \dots, M^{2n-2}, N\}$, where

$$N = M^{-1}\text{Pfaf}(M), \quad \text{Pfaf}(M) = \sqrt{\det(M)}.$$

(See [12] for the details).

4. Put $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda = (2, 0)$, and $M = M_{(2,0)}$. Then $I(\mathfrak{sl}_3(\mathbb{C})) = \mathbb{C}[C_2, C_3]$, where $C_2 = \frac{1}{10}\,\text{trace}(M^2)$, $C_3 = \frac{1}{7}\,\text{trace}(M^3)$. The following statement can be proved.

**Proposition 5.2.** *The family algebra $C_{(2,0)}(\mathfrak{sl}_3(\mathbb{C}))$ is generated over $I(\mathfrak{sl}_3(\mathbb{C}))$ by two elements $M, N$ (deg $M = 1$, deg $N = 2$) with the relations*

$$MN = C_3, \quad N^2 = \frac{1}{72}\left(36C_2N_2 + 36C_2^2 - 21C_3M - 15C_2M^2 + M^4\right).$$

*Remark.* The element $N$ in the proposition can be defined by

$$N = \frac{1}{72C_3}\left((3M)^5 - 15C_2(3M)^3 - 21C_3(3M)^2 + 36C_2^2(3M) + 36C_2C_3\right).$$

It is a matrix with polynomial coefficients in $\mathfrak{g}$, so $N \in C_{(2,0)}(\mathfrak{sl}_3(\mathbb{C}))$. But $N$ can not be expressed as a sum of $M^2$ and $C_2$ with constant coefficients. So $\{1, M, M^2, M^3, M^4, M^5\}$ is not a linear basis of $C_{(2,0)}(\mathfrak{sl}_3(\mathbb{C}))$ over $I(\mathfrak{sl}_3(\mathbb{C}))$.

# Chapter 6

# Characteristic identities and quantum eigenvalues of $\mathfrak{sl}_2(\mathbb{C})$.

**1.** In both cases, classical and quantum, the family algebras of $\mathfrak{sl}_2(\mathbb{C})$ are generated by one element $M_n$. It is interesting to know the decomposition of the higher powers of this element as linear combinations of the first $n$ powers.

**Examples.** Take $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, choose the basis as above. Put a parameter $t$ into the commutation relations: $[e, f] = th, \quad [h, e] = 2te, \quad [h, f] = -2tf$. Define the element $M = M_n$ as above: $M = eF + fE + \frac{h}{2}H$, $E = \pi_n(e)$, $F = \pi_n(f)$, $H = \pi_n(h)$. The value $t = 0$ corresponds to $M \in C_n(\mathfrak{sl}_2(\mathbb{C}))$ and the value $t = 1$ corresponds to $M \in Q_n(\mathfrak{sl}_2(\mathbb{C}))$. Put $C = \frac{h^2}{2} + 2ef$ in the classical case, and put $C = \frac{h^2}{2} + ef + fe$ in the quantum case. Then we can write the following characteristic identities:

$$\dim\pi = 2, \qquad M^2 = -tM + \frac{C}{2};$$

$$\dim\pi = 3, \qquad M^3 = -4tM^2 + (2C - 4t^2)M + 4tC;$$

$$\dim\pi = 4,$$

$$M^4 = -10tM^3 + (5C - 33t^2)M^2 + (33Ct - 36t^3)M - \frac{9}{4}C^2 + 54t^2C;$$

$$\dim\pi = 5,$$

$$M^5 = -20tM^4 + (-148t^2 + 10C)M^3 + (148tC - 480t^3)M^2 +$$
$$+(-16C^2 + 720t^2C - 576t^4)M + 1152t^3C - 96tC^2.$$

**2.** The examples above are the polynomial identities, satisfied by a matrix with entries from the universal enveloping algebra. In [6] the properties of such identities are studied and some low-dimensional examples are given. Here we derive a uniform formula for all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and let $M = M_n$ be the defined above element of $Q_n(\mathfrak{sl}_2(\mathbb{C}))$ for the fixed representation $\pi_n$. We will look for the characteristic identity in the following form:

$$\prod_{j=0}^{n}(M-b_j)=0,$$

where $b_j$ are the elements of the $W_\tau$-extension of $Z(\mathfrak{sl}_2(\mathbb{C}))$, $W_\tau$ is the translated Weyl group. We say that $\{b_j\}$ are *quantum eigenvalues* of the representation $\pi_n$.

**Theorem 6.1.** *The element $M$ of the quantum family algebra $Q_n(\mathfrak{sl}_2(\mathbb{C}))$ satisfies the identity*

$$\prod_{j=0}^{n}(M-b_j)=0, \tag{6.6.1}$$

*where* $b_j=\left(j^2-\frac{n}{2}-jn\right)+\left(\frac{n}{2}-j\right)\sqrt{(2C+1)}, \qquad j=0,...,n.$

*Proof.* We follow the idea of the proof of the existence theorem in [6] (see also [15]). We apply to $M$ a map $\phi_m$, defined in the Chapter 3. It sends $U(\mathfrak{sl}_2(\mathbb{C}))$ by the means of the representation $\pi_m$ to End $V_m$. Then $\phi_m(M)$ is a diagonalizable operator on $V_n\otimes V_m$, and in the case of $\mathfrak{sl}_2(\mathbb{C})$ it is not difficult to compute the eigenvalues of this operator. As for $m$ was chosen arbitrarily, it allows to write the quantum eigenvalues for $M$.

Below are the details of this program. For a pair of irreducible representations $\pi_m$ and $\pi_n$ of $\mathfrak{sl}_2(\mathbb{C})$ with the highest weights $m$ and $n$ respectively, we construct the element of End $(V_m\otimes V_n)$:

$$M_{m,n}=\sum_i \pi_m(x_i)\pi_n(x^i),$$

where $\{x_i\}$ is some basis of $\mathfrak{g}$, and $\{x^i\}$ is the dual basis with respect to the Killing form. Note that $M_{m,n} = \frac{1}{2}((\pi_m \otimes \pi_n)(C) - \pi_m(C) \otimes 1 - 1 \otimes \pi_n(C))$, where $C = \frac{h^2}{2} + ef + fe$.

**Lemma 6.2.** *The element $M_{m,n} \in End(V_m \otimes V_n)$ satisfies the following identity:*

$$\prod_{i=0}^{min(m,n)} (M_{m,n} - f_i) = 0,$$

*where* $f_i = \frac{1}{2}\left((m+1)(n-2i) + (2i^2 - 2ni - n)\right), \qquad i = 0, ..., min(m,n).$

*Proof.* The operators $\pi_m(C) \otimes 1$ and $1 \otimes \pi_n(C)$ are scalar on $V_m \otimes V_n$ with the eigenvalues $\frac{m^2}{2} + m$ and $\frac{n^2}{2} + n$ respectively. The decomposition

$$\pi_m \otimes \pi_n = \bigoplus_{i=0}^{\min(m,n)} \pi_{m+n-2i}$$

follows that $(\pi_m \otimes \pi_n)(C)$ has eigenvalues $\frac{(m+n-2i)^2}{2} + (m+n-2i)$ with the multiplicities $m+n-2i+1$. Therefore, $M_{m,n}$ has eigenvalues

$$f_i = \frac{1}{2}\left(\frac{(m+n-2i)^2}{2} + (m+n-2i) - \left(\frac{n^2}{2} + n\right) - \left(\frac{m^2}{2} + m\right)\right) =$$

$$= \frac{1}{2}\left((m+1)(n-2i) + (2i^2 - 2ni - n)\right),$$

and satisfies

$$\prod_{i=0}^{\min(m,n)} (M_{m,n} - f_i)^{(n+m-2i+1)} = 0.$$

But $(\pi_m \otimes \pi_n)(C)$ is diagonalizable, so is $M_{m,n}$, and we can get rid of the multiplicities:

$$\prod_{i=0}^{\min(m,n)} (M_{m,n} - f_i) = 0.$$

$\square$

To prove the identity (6.6.1), we want to rewrite it in the equivalent form with the coefficients in $Z(\mathfrak{sl}_2(\mathbb{C}))$ (not the $W_\tau$-extension of $Z(\mathfrak{sl}_2(\mathbb{C}))$). Note that (6.6.1) is equivalent to

$$\prod_{j=0}^{n}(M - b_j) = B \prod_{j=0}^{\left[\frac{n-1}{2}\right]} (M^2 - B_j M + A_j), \tag{6.6.2}$$

where $B = \left(M + \frac{n^2+2n}{4}\right)$, if $n$ is even, and $B = 1$, if $n$ is odd,

$$A_j = b_j b_{n-j} = \left(j^2 - \frac{n}{2} - jn\right)^2 - \left(\frac{n}{2} - j\right)^2 (2C+1),$$

$$B_j = b_j + b_{n-j} = 2\left(j^2 - \frac{n}{2} - jn\right).$$

Suppose that the identity (6.6.1) is not true: $\prod_{j=0}^{n}(M - b_j) = T \neq 0$.

From (6.6.2), the element $T$ belongs to $\mathrm{End}(V_n) \otimes U(\mathfrak{sl}_2(\mathbb{C}))$, and from our assumption, at least one matrix element $T_{ij} \in U(\mathfrak{sl}_2(\mathbb{C}))$ is non-zero.

**Lemma 6.3.** *(See [4].) Let $\mathfrak{g}$ be semi-simple Lie algebra. Then for any non-zero element $u$ of $U(\mathfrak{g})$ there exists an irreducible representation $\pi$ such that $\pi(u) \neq 0$.*

Let $\pi_m$ be a representation of $\mathfrak{sl}_2$ such that $\pi_m(T_{ij}) \neq 0$. We apply $\pi_m$ to (6.6.2):

$$\pi_m(B) \prod_{j=0}^{\left[\frac{n-1}{2}\right]} (\pi_m(M)^2 - B_j \pi_m(M) + \pi_m(A_j)) = \pi_m(T) \neq 0.$$

But $\pi_m(C) = \frac{m^2}{2} + m$, $\pi_m(M) = M_{m,n}$, so

$$\pi_m(M)^2 - B_j \pi_m(M) + \pi_m(A_j) = M_{m,n}^2 - (f_j + f_{n-j}) M_{m,n} + f_j f_{n-j}.$$

Therefore,

$$\pi_m(B) \prod_{j=0}^{\left[\frac{n-1}{2}\right]} (M_{m,n}^2 - (f_j + f_{n-j}) M_{m,n} + f_j f_{n-j}) \neq 0.$$

The last inequality is equivalent to

$$\prod_{i=0}^{n} (M_{m,n} - f_i) \neq 0,$$

which is impossible because of the Lemma 6.2. □

**Corollary 6.4.** *The element $M$ in $C_n(\mathfrak{sl}_2(\mathbb{C}))$ satisfies the identity*

$$\prod_{j=0}^{n} (M - b_j) = 0, \tag{6.6.3}$$

*where $b_j = (\frac{n}{2} - j)\sqrt{2C}$, $\qquad j = 0, \dots, n$.*

*Proof.* Recall that $Q_n(\mathfrak{sl}_2(\mathbb{C}))$ is a deformation of $C_n(\mathfrak{sl}_2(\mathbb{C}))$. Hence, all the statements about $Q_n(\mathfrak{sl}_2(\mathbb{C}))$ could be rewritten for $Q_n(\mathfrak{sl}_2(\mathbb{C}))_t$, introducing the parameter $t$. Then the quantum eigenvalues from the Proposition 6.1 for $Q_n(\mathfrak{sl}_2(\mathbb{C}))_t$ look like

$$(b_j)_t = \left(j^2 - \frac{n}{2} - jn\right) t + \left(\frac{n}{2} - j\right) \sqrt{(2C + t^2)}, \qquad j = 0, ..., n.$$

Put $t = 0$ to get the eigenvalues of $C_n(\mathfrak{sl}_2(\mathbb{C}))$. $\square$

# Chapter 7

# Classical family algebra of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.

**1.** We illustrate some ideas from [13] on the example of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$ and compute harmonic basis of the classical family algebra. Note that the adjoint representation is not a representation with simple spectrum.

Recall some general facts about $C_\lambda(\mathfrak{g})$ (see [13] for the details). Any element $A \in C_\lambda(\mathfrak{g})$ can be interpreted as a polynomial map $A : \mathfrak{g}^* \to \mathrm{End}V_\lambda$, equivariant under the action of $G$:

$$A(\mathrm{Ad}gX) = \Pi_\lambda(\mathfrak{g})A(X)\Pi_\lambda(\mathfrak{g})^{-1}. \qquad (7.7.1)$$

From now on we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. The set of regular semi-simple elements in $\mathfrak{g}$, which are Ad-equivalent to regular elements of Cartan subalgebra $\mathfrak{h}$, is an open dense subset of $\mathfrak{g}$. The covariance property (7.7.1) implies that any element $A \in C_\lambda(\mathfrak{g})$ is completely determined by its restriction $A|_\mathfrak{h}$. Let $Wt(\lambda) = \{\mu_1, ..., \mu_N\}$ be the set of weights of the representation $\pi_\lambda$ with corresponding multiplicities $\{d_1, ..., d_N\}$. The operators $A|_\mathfrak{h} \in \mathrm{End}V_\lambda \otimes S(\mathfrak{h})$ have a block-diagonal form with blocks of sizes $d_i$. The map $A \to A|_\mathfrak{h}$ is an injection, so $C_\lambda(\mathfrak{g})$ is realized as a subalgebra of

$$B_\lambda(\mathfrak{g}) = \bigoplus_{\mu_i \in Wt(\lambda)} \mathrm{Mat}_{d_i}(S(\mathfrak{h})).$$

Then we can think of $B_\lambda(\mathfrak{g})$ as an algebra of polynomial maps

$$B : \mathfrak{h} \to \bigoplus_{\mu_i \in Wt(\lambda)} \mathrm{Mat}_{d_i}(\mathbb{C}).$$

Moreover, let $\tilde{w} \in G$ be a representative of an element $w$ of the Weyl group $W$ in the normalizer in $G$ of the torus $T$. Let $B_\lambda^W(\mathfrak{g})$ be a subalgebra of $B_\lambda(\mathfrak{g})$, which consists of elements $B$ with the property

$$B(\mathrm{Ad}\tilde{w}H) = \Pi_\lambda(\tilde{w})B(H)\Pi_\lambda(\tilde{w})^{-1}, \quad H \in \mathfrak{h}, \quad w \in W.$$

Then we have the following inclusion of subalgebras:

$$C_\lambda(\mathfrak{g}) \subset B_\lambda^W(\mathfrak{g}) \subset B_\lambda(\mathfrak{g}).$$

It turns out that the localizations of $C_\lambda(\mathfrak{g})$ and $B_\lambda^W(\mathfrak{g})$ coincide. Let $K_0$ = Frac $I(\mathfrak{g})$. Recall that $I(\mathfrak{g})$ is naturally isomorphic to $I(\mathfrak{h}) = S(\mathfrak{h})^W$ – the algebra of $W$-invariant elements of $S(\mathfrak{h})$. In [13] the isomorphism of algebras

$$C_\lambda(\mathfrak{g}) \otimes_{I(\mathfrak{g})} K_0 = B_\lambda^W(\mathfrak{g}) \otimes_{I(\mathfrak{g})} K_0,$$

is proved. Here we illustrate the construction on the example of the classical family algebra of the adjoint representation of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$.

**2.** The weights of $V_{ad}$ are $\{\pm\alpha, \pm\beta, \pm\gamma, 0\}$. The algebra $B_{ad}(\mathfrak{sl}_3(\mathbb{C}))$ can be realized as the $8 \times 8$-matrices $B$ of the form

$$B = \begin{pmatrix} B_+ & 0 & 0 \\ 0 & B_0 & 0 \\ 0 & 0 & B_- \end{pmatrix},$$

where $B_\pm$ are $3 \times 3$ diagonal matrices with the entries in $S(\mathfrak{h})$:

$$B_\pm = \begin{pmatrix} b_{\pm\alpha} & 0 & 0 \\ 0 & b_{\pm\beta} & 0 \\ 0 & 0 & b_{\pm\gamma} \end{pmatrix}.$$

They correspond to positive and negative roots respectively. $B_0$ is a $2 \times 2$-matrix, that corresponds to the zero-weight of $V_{ad}$:

$$B_0 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \qquad b_i \in S(\mathfrak{h}).$$

Take the standard realization of $\mathfrak{sl}_3(\mathbb{C})$ by $3 \times 3$ matrices and choose the basis $\{H_+, H_-\}$ of the Cartan subalgebra $\mathfrak{h}$:

$$H_+ = 1/(\sqrt{3})(\varepsilon E_{11} + E_{22} + \varepsilon^2 E_{33}),$$

$$H_- = 1/(\sqrt{3})(\varepsilon^2 E_{11} + E_{22} + \varepsilon E_{33}).$$

Here $\varepsilon = e^{\frac{2\pi i}{3}}$. This basis of $\mathfrak{h}$ is convenient for the explicit calculations. Then $I(\mathfrak{sl}_3(\mathbb{C})) = \mathbb{C}[C_2, C_3]$ , where $C_2, C_3$ have the following restrictions to $\mathfrak{h}$:

$$C_2|_{\mathfrak{h}} = \Delta_2 = H_+H_-, \qquad C_3|_{\mathfrak{h}} = \Delta_3 = H_+^3 + H_-^3.$$

Any element of $S(\mathfrak{h})$ can be written as a linear combination of the elements

$$1, \quad H_\pm, \quad H_\pm^2, \quad H_+^3 - H_-^3$$

with the coefficients in $I(\mathfrak{h}) = S(\mathfrak{h})^W = \mathbb{C}[\Delta_2, \Delta_3]$.

The covariance condition (7.7.1) implies that all the entries of $B_\pm$ for $B \in B_{ad}^W(\mathfrak{sl}_3(\mathbb{C}))$ are completely determined by the value of $b_\gamma \in S(\mathfrak{h})$. The stabilizer of $\gamma$ in $W$ is trivial, so $b_\gamma$ can be any polynomial in $\mathfrak{h}$. From the same condition we get

$$\begin{aligned} b_1 &= P + Q(H_-^3 - H_+^3), \quad b_2 = RH_- + SH_+^2, \\ b_3 &= (SH_-^2 + RH_+), \quad b_4 = P - Q(H_-^3 - H_+^3), \end{aligned} \tag{7.7.2}$$

where $P$, $Q$, $R$, $S \in \mathbb{C}[\Delta_1, \Delta_2]$. Hence, we proved

**Proposition 7.1.** *Every element $B \in B_{ad}^W(\mathfrak{sl}_3(\mathbb{C}))$ is completely defined by a pair $(b_\gamma, B_0)$, where $b_\gamma \in S(\mathfrak{h})$, and $B_0$ is the $2\times 2$-matrix those entries satisfy (7.7.2).*

The basis of $B_{ad}^W(\mathfrak{sl}_3(\mathbb{C}))$ over $I(\mathfrak{h})$ can be written as

$$\mathcal{R}_1 = \{(0, id), (0, s_1), (0, s_2), (0, s_3), (1, 0), (H_\pm, 0), (H_\pm^2, 0), (H_+^3 - H_-^3, 0)\}.$$

Here

$$id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & H_- \\ H_+ & 0 \end{pmatrix},$$

$$s_2 = \begin{pmatrix} 0 & H_+^2 \\ H_-^2 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} H_-^3 - H_+^3 & 0 \\ 0 & -H_-^3 + H_+^3 \end{pmatrix}.$$

**3.** We construct the harmonic basis of $C_{ad}(\mathfrak{sl}_3(\mathbb{C}))$ (see Chapter 4). The space $\mathrm{End}V_{ad}$ is decomposed into five irreducible components:

$$\mathrm{End}V_{ad} = \pi_{(0,0)} \oplus \pi_{(1,1)_s} \oplus \pi_{(1,1)_s} \oplus \pi_{(2,2)} \oplus \pi_{(0,3)} \oplus \pi_{(3,0)}.$$

We distinguish the two copies of the adjoint representation. One of them is represented by symmetric matrices, and the other – by antisymmetric ones. Accordingly, we use the notation $\pi_{(1,1)_s}$, $\pi_{(1,1)_a}$. The generalized exponents of the representations in the decomposition are given in the Table 2.

| | |
|---|---|
| $\pi_{(1,1)}$ | 1,2 |
| $\pi_{(2,2)}$ | 2,3,4 |
| $\pi_{(0,3)}$ | 3 |
| $\pi_{(3,0)}$ | 3 |

**Table 2. Generalized exponents.**

Hence, there exists a harmonic basis of $C_a d(\mathfrak{sl}_3(\mathbb{C}))$

$$\mathcal{R}_0 = \{1, N_1, N_2, M_1, M_2, L_2, L_3, L_4, P_3, Q_3\}$$

over $I(\mathfrak{sl}_3(\mathbb{C}))$ with the following properties:

$$\begin{aligned} 1) \quad & N_i \in \pi_{(1,1)_s} \otimes H(\mathfrak{g}), \qquad i = 1, 2, \\ & M_i \in \pi_{(1,1)_a} \otimes H(\mathfrak{g}), \qquad i = 1, 2, \\ & L_i \in \pi_{(2,2)} \otimes H(\mathfrak{g}), \qquad i = 1, 2, 3 \\ & P_3 \in \pi_{(0,3)} \otimes H(\mathfrak{g}), \\ & Q_3 \in \pi_{(3,0)} \otimes H(\mathfrak{g}). \end{aligned}$$

2) the indices of the elements of this basis indicate their degrees (e.g. $N_2$ is the

matrix those entries are homogeneous polynomials in $\mathfrak{g}$ of the second degree).

We describe the restrictions of the elements of $\mathcal{R}_0$ to the Cartan subalgebra $\mathfrak{h}$. As for $C_{ad}(\mathfrak{sl}_3(\mathbb{C})) \subset B^W_{ad}(\mathfrak{sl}_3(\mathbb{C}))$, the restriction of any element $A \in C_{ad}(\mathfrak{sl}_3(\mathbb{C}))$ is a pair $(a_\gamma, A_0)$, where $a_\gamma \in S(\mathfrak{h})$, and $A_0$ is the $2 \times 2$-matrix, those entries satisfy (7.7.2). The list of the restrictions of the elements of $\mathcal{R}_0$ is given in the Table 3.

*Remark.* Some of these elements have been obtained by pure technical computations, and some of them have nice interpretation, which we would like to mention. Let $\{x_i\}$ be a basis of $sl_3(\mathbb{C})$, and $\{x^i\}$ be the dual basis with respect to the Killing form.

1. $M_1 = \sum x_i \otimes \pi_{ad}(x^i)$. It is the element $M_\lambda$, defined in Chapter 5. It has the property

$$(M_1)_{i,j} = [x_i, x^j].$$

2. $N_1$ has the similar property

$$(N_1)_{i,j} = [x_i, x^j]^0_+,$$

where $[a, b]_+ = ab + ba$, for $a, b \in$Mat$_3(\mathbb{C})$, and $(A_{ij})^0 = A_{ij} - \frac{1}{8}\delta_{ij}\text{trace}(A)$.

3. $M_2 = M_1 N_1$

4. $L_2 = \frac{5M_1^2+3N_1^2}{4}$

5. $N_2 =$const$_1 \cdot DN_1$, $L_3 =$const$_2 \cdot DL_2$, $L_4 =$const$_3 \cdot DL_3$, where $D$ is a differential operator, defined in [12].

**Table 3. Harmonic basis of $C_{ad}(\mathfrak{sl}_3(\mathbb{C}))$ .**

| $A$ | $a_\gamma$ | $A_0$ |
|---|---|---|
| $1$ | $1$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $M_1$ | $H_- - H_+$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ |
| $N_1$ | $H_+ + H_-$ | $\begin{pmatrix} 0 & -2H_- \\ -2H_+ & 0 \end{pmatrix}$ |
| $M_2$ | $H_+^2 - H_-^2$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ |
| $N_2$ | $(H_+^2 + H_-^2)$ | $\begin{pmatrix} 0 & -2H_+^2 \\ -2H_-^2 & 0 \end{pmatrix}$ |
| $L_2$ | $-2(H_+^2 + H_-^2) + \Delta_2$ | $-3\Delta_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $L_3$ | $\Delta_3 - 4\Delta_2(H_- + H_+)$ | $-3\Delta_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $L_4$ | $-2(H_+^4 + H_-^4) - 4\Delta_2(H_-^2 + H_+^2) + 3\Delta_2^2$ | $-9\Delta_2^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $P_3$ | $H_+^3 - H_-^3$ | $i\sqrt{3}(H_-^3 - H_+^3) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |
| $Q_3$ | $H_+^3 - H_-^3$ | $i\sqrt{3}(H_-^3 - H_+^3) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ |

Let $I_D$ be the localization of $I(\mathfrak{h}) = S(\mathfrak{h})^W$ by $D = \Delta_3^2 - 4\Delta_2^3$. Using the bases $\mathcal{R}_1$, $\mathcal{R}_0$, one can prove

**Proposition 7.2.**

$$C_{ad}(\mathfrak{sl}_3(\mathbb{C})) \otimes_{I(\mathfrak{h})} I_D = B_{ad}^W(\mathfrak{sl}_3(\mathbb{C})) \otimes_{I(\mathfrak{h})} I_D.$$

The isomorphism $I(\mathfrak{g}) = I(\mathfrak{h})$ implies

$$C_{ad}(\mathfrak{sl}_3(\mathbb{C})) \otimes_{I(\mathfrak{g})} K_0(\mathfrak{g}) = B_{ad}^W(\mathfrak{sl}_3(\mathbb{C})) \otimes_{I(\mathfrak{g})} K_0(\mathfrak{g}).$$

# Chapter 8

# The exceptional Lie algebra $\mathfrak{g}_2$

**1.** Let us describe a structure of the exceptional Lie algebra $\mathfrak{g}_2$ . As a vector space, it is isomorphic to the direct sum $\mathfrak{g}_2 = V_+ \oplus \mathfrak{g}_0 \oplus V_-$, where $\mathfrak{g}_0 = \mathfrak{sl}_3(\mathbb{C})$, $V_+$ and $V_-$ are 3-dimensional representations of $\mathfrak{g}_0$. The subalgebra $\mathfrak{g}_0$ acts on $V_+$ and $V_-$ by standard and dual to standard representations respectively.

Let $w_+, v_+ \in V_+$, $X, Y \in \mathfrak{sl}_3(\mathbb{C})$, $w_-, v_- \in V_-$. The commutation relations in $\mathfrak{g}_2$ are given in the following table.

| | $v_+$ | $X$ | $v_-$ |
|---|---|---|---|
| $w_+$ | $-2w_+ \wedge v_+$ | $-Xw_+$ | $w_+ * v_-$ |
| $Y$ | $Yv_+$ | $[Y, X]$ | $Yv_-$ |
| $w_-$ | $-v_+ * w_-$ | $-Xw_-$ | $2w_- \wedge v_-$ |

**Table 4. Commutation relations in $\mathfrak{g}_2$**

Here $(v * \phi)(w) = 3\phi(w)v - \phi(v)w$, $v$, $w \in V_+$, $\phi \in V_-$ (see [5]).

Let $x$ be an element of $\mathfrak{g}_2$. Then $x = (v, X, f)$, where

$$v = (v_1, v_2, v_3)^T \in V_+, \qquad X \in sl_3(\mathbb{C}), \qquad f = (f_1, f_2, f_3) \in V_-.$$

We introduce a check-operation from $\mathrm{Mat}_{n,m}(\mathbb{C})$ to $\mathrm{Mat}_{m,n}(\mathbb{C})$ by the formula

$$(\check{A})_{k,l} = A_{m-l,n-k}.$$

Then the 7-dimensional representation of $\mathfrak{g}_2$ can be realized as follows:

$$\pi(x) = \begin{pmatrix} X & v & B(f) \\ f & 0 & -\check{v} \\ B(-\check{v}) & -\check{f} & -\check{X} \end{pmatrix}, \tag{8.8.1}$$

where

$$B(f) = \frac{1}{\sqrt{2}} \begin{pmatrix} -f_2 & f_3 & 0 \\ f_1 & 0 & -f_3 \\ 0 & -f_1 & f_2 \end{pmatrix}. \tag{8.8.2}$$

The matrix $B(f)$ has some useful properties:

1) for any $Y \in sl_3(\mathbb{C})$,

$$YB(f) + B(f)\check{Y} = -B(fY);$$

2) for any $v \in V_+$, $f \in V_-$

$$B(f)B(\check{v}) = \frac{< f, v > *Id - vf}{2}, \qquad < f, v >= \sum f_i v_i.$$

**2.** The Lie algebra $\mathfrak{g}_2$ contains the Lie subalgebra $\mathfrak{sl}_3(\mathbb{C})$. Therefore, some questions can be answered by restricting the representations of $\mathfrak{g}_2$ to $\mathfrak{sl}_3(\mathbb{C})$. For example, we can compute ranks of family algebras of $\mathfrak{g}_2$, using the branching rules and the multiplicities of zero-weights in representations of $\mathfrak{sl}_3(\mathbb{C})$.

Let $\Pi_{(K,\,L)}$ be an irreducible representation of $\mathfrak{g}_2$ with the highest weight $(K,\, L)$. Then $\Pi_{(K,\,L)}|_{\mathfrak{g}_0} = \bigoplus_{k,\,l} a(k,\, l)\pi_{(k,\,l)}$, where $\pi_{(k,\,l)}$ is an irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight $(k,\, l)$, and $a(k,\, l)$ is the multiplicity of $\pi_{(k,\,l)}$ in this decomposition.

In [3] a generating function for the branching of representations of $\mathfrak{g}_2$ to the representations of $\mathfrak{sl}_3(\mathbb{C})$ is computed. Let $P_{(K,\,L)} = \sum a(k,\, l)X^{(k,\,l)}$ be the branching polynomial of the representation $\Pi_{(K,\,L)}$. Then

$$\sum P_{(K,\,L)}Y^{(K,\,L)} = \frac{A}{B},$$

where

$$A = 1 - X^{(1,1)}Y^{(1,1)},$$

$$B = (1 - X^{(1,0)}Y^{(1,0)})(1 - X^{(0,1)}Y^{(1,0)})(1 - X^{(0,0)}Y^{(1,0)})*$$

$$(1 - X^{(1,0)}Y^{(0,1)})(1 - X^{(0,1)}Y^{(0,1)})(1 - X^{(1,1)}Y^{(0,1)}).$$

One can easily deduce the following branching rule for these algebras.

**Proposition 8.1.** *The representation of* $\mathfrak{sl}_3(\mathbb{C})$ *with the highest weight* $(k, l)$ *is contained in the decomposition* $\Pi_{(K,\,L)}|_{\mathfrak{g}_0} = \bigoplus_{k,\,l} a(k,\,l)\pi_{(k,\,l)}$ *if and only if the point* $(K, L)$ *is contained in the polygon, bounded by the following lines:*

$$k + l = K,\ k = 0,\ l = 0,$$

$$k + l = 2K + L,\ k = K + L,\ l = K + L.$$

*The multiplicities* $a(k, l)$ *equal* $1$ *on the boundaries of the polygon, and grow proportionally from the boundaries of the polygon to the center with the slope* $\pi/4$. *In the central triangle, bounded by the lines*

$$k = K,\ l = K,\ k + l = K + L,$$

*the values of* $a(k, l)$ *are constant and equal to* $K + 1$ *.*

The polygon and the values of multiplicities $a(k, l)$ are shown on the Figure 8.1.

Figure 8.1: Branching of representations of $\mathfrak{g}_2$

Let $M(k,\,l)$ be the multiplicity of zero-weight in $\Pi_{(K,\,L)}$, and let $m(k,\,l)$ be the multiplicity of the zero-weight in $\pi_{(k,\,l)}$. We have:

$$m(k,\,l) = \begin{cases} min(k,l)+1, & \text{if} \quad k = l \bmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, to compute $M(k,\,l)$, we have to sum $a(k,\,l)m(k,\,l)$ over the polygon, described in Proposition 8.1:

$$M(k,\,l) = \sum_{k,\,l} m(k,\,l)a(k,\,l). \tag{8.8.3}$$

The Proposition 8.1 and the sum (8.8.3) give a convenient way to compute the number of generalized exponents for representations of $\mathfrak{g}_2$, and, in particular, the ranks of family algebras.

# Chapter 9

# Classical family algebra $C_{(0,1)}(\mathfrak{g}_2)$.

**1.** Let us consider 7-dimensional representation $\pi = \Pi_{(1,0)}$ of $\mathfrak{g}_2$. This is a representation with simple spectrum, so it is generated by the element $M$ over $\operatorname{Frac} I(\mathfrak{g}_2)$. We can write this element in a matrix form and prove that it generates the family algebra even over the ring $I(\mathfrak{g})$ (see the discussion after the Theorem 5.1). The Proposition 9.2 and its colloraries illustrate the usage of family algebras in computation of generalized exponents.

We choose a convenient basis in the Lie algebra $\mathfrak{g}_2$.

Let $\{X_{\pm\alpha}, \quad X_{\pm\beta}, \quad X_{\pm\gamma}, \quad H_\alpha, \quad H_\beta\}$ be a standard basis in $\mathfrak{sl}_3(\mathbb{C})$, corresponding to the roots of $\mathfrak{sl}_3(\mathbb{C})$. Choose a basis $\{e_1, e_2, e_3\}$ of weight vectors of the standard representation $V_+$ of $\mathfrak{sl}_3(\mathbb{C})$, and the dual basis $\{g_1, g_2, g_3\}$ of weight vectors in the dual to standard representation $V_-$ of $\mathfrak{sl}_3(\mathbb{C})$. Consider the basis of $\mathfrak{g}_2$

$$\{H_-, H_+, X_{\pm\alpha}, X_{\pm\beta}, X_{\pm\gamma}, e_i, g_i\} \quad (i = 1, 2, 3).$$

Here $H_- = H_A + \mu H_B, \quad H_+ = -H_A - \bar{\mu} H_B, \quad \mu = \frac{1}{2} + \frac{i\sqrt{3}}{6}$, $A$ is a short simple root, and $B$ is a long simple root of $\mathfrak{g}_2$.

This basis has a nice duality property with respect to the form $Tr(\pi(x)\,\pi(y))$ on $\mathfrak{g}_2$:

$$e_i^* = \frac{g_i}{3}, \quad X_\nu^* = \frac{X_{-\nu}}{2}, \quad H_+^* = \frac{H_-}{2}, \quad g_i^* = \frac{e_i}{3}, \quad H_-^* = \frac{H_+}{2}.$$

In this basis the element $M$ has the form

$$M = \begin{pmatrix} M'/2 & e/3 & B(g/3) \\ g/3 & 0 & -\check{e}/3 \\ B(-\check{e}/3) & -\check{g}/3 & -\check{M}'/2 \end{pmatrix}. \tag{9.9.1}$$

Here $e = (e_1, e_2, e_3)^T$, $g = (g_1, g_2, g_3)$, $\varepsilon = \frac{-1+i\sqrt{3}}{2}$, and $M'$ is a generating element of the classical family algebra of the standard representation of $\mathfrak{sl}_3(\mathbb{C})$:

$$M' = \begin{pmatrix} H_- - H_+ & X_{-\alpha} & X_{-\gamma} \\ X_\alpha & \varepsilon H_- - \varepsilon^2 H_+ & X_{-\beta} \\ X_\gamma & X_\beta & \varepsilon^2 H_- - \varepsilon H_+ \end{pmatrix}.$$

**2.** As in Chapter 7, we describe here the algebras $B_{(0,1)}(\mathfrak{g}_2)$ and $B^W_{(0,1)}(\mathfrak{g}_2)$.

The weights of the representation $\Pi_{(1,0)}$ are

$$\lambda_0 = 0, \quad \lambda_{\pm 1} = \mp(A+B), \quad \lambda_{\pm 2} = \mp A, \quad \lambda_{\pm 3} = \pm(2A+B).$$

Hence, the algebra $B_{(0,1)}(\mathfrak{g}_2)$ can be realized as $7 \times 7$ diagonal matrices with polynomial entries:

$$B = \mathrm{diag}(b_3, b_2, b_1, b_0, b_{-1}, b_{-2}, b_{-3}), \qquad B \in B_{(0,1)}(\mathfrak{g}_2), \quad b_i \in S(\mathfrak{g}_2)$$

The algebra $I(\mathfrak{g}_2)$ is generated by two elements. Their restrictions to $\mathfrak{h}$ are

$$\Delta_2 = H_+H_-, \quad \Delta_6 = H_+^6 + H_-^6.$$

Any element of $S(\mathfrak{h})$ can be written as a linear combination of polynomials

$$\{1, \quad H_\pm, \quad H_\pm^2, \quad H_\pm^3, \quad H_\pm^4, \quad H_\pm^5, \quad H_-^6 - H_+^6.\}$$

with coefficients in $I[\mathfrak{h}] = \mathbb{C}[\Delta_2, \Delta_6]$.

Weyl group $W$ of $\mathfrak{g}_2$ is generated by two fundamental reflections $S_A$, $S_B$. Their action in the basis of $H_+, H_-$ is represented by the following matrices:

$$S_A = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon^2 & 0 \end{pmatrix}, \quad S_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Acting by the Weyl group $W$ on the elements of $B_{(0,1)}(\mathfrak{g}_2)$ we get, that any element of $B^W_{(0,1)}(\mathfrak{g}_2)$ is uniquely determined by the values of ( $b_3$, $b_0$). Moreover, $b_0$ must be an invariant polynomial, and $b_3$ must be invariant under the action of $S_B$. Therefore,

$$b_3 = P + Q(H_+ - H_-) + R(H_+^2 + H_-^2) + \\ +S(H_+^3 - H_-^3) + T(H_+^4 + H_-^4) + U(H_+^5 - H_-^5), \tag{9.9.2}$$

where $P$, $Q$, $R$, $S$, $T$, $U \in I[\mathfrak{h}]$

We obtained the basis of $B^W_{(0,1)}(\mathfrak{g}_2)$ over $\mathbb{C}[\Delta_2, \Delta_6]$ :

$$u_0 = (1, 0), \quad u_1 = (H_+ - H_-, 0), \quad u_2 = (H_+^2 + H_-^2, 0), \quad u_3 = (H_+^3 - H_-^3, 0),$$

$$u_4 = (H_+^4 + H_-^4, 0), \quad u_5 = (H_+^5 - H_-^5, 0), \quad u_6 = (0, 1),$$

Note that

$$u_0 = \frac{(8M^3 + 6\Delta_2 M)^2}{\Delta_6 - 2\Delta_2^3}, \qquad u_1 = -2M,$$

$$u_2 = 4M^2 + 2\Delta_2 u_0, \qquad u_3 = -8M^3 + 3\Delta_2 u_1,$$

$$u_4 = 16M^4 + 4\Delta_2 u_2 - 6\Delta_2^2, \qquad u_5 = -32M^5 + 5\Delta_2 u_3 - 10\Delta_2^2 u_1,$$

$$u_6 = 1 - u_0.$$

Therefore,

**Proposition 9.1.** *The powers of $M$ form a basis of $B^W_{(0,1)}(\mathfrak{g}_2)$ over localization of the ring $I[\mathfrak{h}]$ by $D = \Delta_6 - 2\Delta_2^3$.*

**3.** Now we construct harmonic basis of $\mathbb{C}_{(0,1)}(\mathfrak{g}_2)$ and compute the generalized exponents of the constituents of End$V_{(0,1)}$. We have: $\Pi_{(0,1)} \otimes \Pi_{(0,1)} = \Pi_{(0,0)} \oplus \Pi_{(1,0)} \oplus \Pi_{(0,1)} \oplus \Pi_{(2,0)}$. The dimensions and the zero-weight multiplicities of these representations are given in the Table 6.

| $\pi$ | dim | dim<br>0−space |
|---|---|---|
| $\Pi_{(0,0)}$ | 1 | −− |
| $\Pi_{(1,0)}$ | 7 | 1 |
| $\Pi_{(0,1)}$ | 14 | 2 |
| $\Pi_{(2,0)}$ | 27 | 3 |

**Table 5. Irreducible components.**

**Proposition 9.2.** *There exists harmonic basis* $\{1, P_3, R_2, R_4, R_6, Q_1, Q_5, \}$ *of family algebra* $C_{(0,1)}(\mathfrak{g}_2)$ *such that*

*1) every element of this basis is a linear combination of the powers of* $M$ *with coefficients in* $I(\mathfrak{g}_2)$*;*

*2)* $P_3 \in \Pi_{(1,0)} \otimes H(\mathfrak{g})$, $Q_i \in \Pi_{(0,1)} \otimes H(\mathfrak{g})$, $i = 1, 5$ $R_j \in \Pi_{(2,0)} \otimes H(\mathfrak{g})$, $j = 2, 4, 6$

*Proof.* One can explicitly find the irreducible components of the action of $\mathfrak{g}_2$ on End$V_{(1,0)}$. Namely,

$(0,0)$. Irreducible subspace of the trivial representation $\Pi_{(0,0)}$ consists of scalar matrices $\{a{*}\mathrm{Id}_7\}$.

$(1,0)$. Irreducible subspace of representation $\Pi_{(1,0)}$ consists of matrices of the form

$$\begin{pmatrix} a * \mathrm{Id}_3 & s & B(-r) \\ r & 0 & -s \\ B(-\check{s}) & -\check{r} & -a * \mathrm{Id}_3 \end{pmatrix}.$$

Here $r$, $s^T \in \mathbb{C}^3$, $a \in \mathbb{C}$. Note that this is check-antisymmetric matrix.

$(0,1)$. Irreducible subspace of representation $\Pi_{(0,1)}$ consists of matrices of the form

$$\begin{pmatrix} X & v & B(f) \\ f & 0 & -\check{v} \\ B(-\check{v}) & -\check{f} & -\check{X} \end{pmatrix},$$

$X \in \mathfrak{sl}_3(\mathbb{C})$, $v^T, f \in \mathbb{C}^3$

$(2,0)$. Irreducible subspace of the representation $\Pi_{(2,0)}$ consists of traceless check-symmetric matrices.

Let $C_2 = TrM^2$, $C_2 \in I(\mathfrak{g}_2)$. Using the irreducible components above, one can show that $Q_1 = M$, $R_2 = M^2 - \frac{C_2}{7}$, $P_3 = M^3 - \frac{C_2}{8}M$ are harmonic elements, satisfying conditions of the Proposition.

Notice that $M^4 - \frac{Tr(M^4)}{7} \in \Pi_{(2,0)} \otimes S^4(\mathfrak{g}_2)$, and $Tr(M^4) = \frac{C_2^2}{4} \in I(\mathfrak{g}_2)$.

One can write

$$M^4 - \frac{Tr(M^4)}{7} = R_4 + b * C_2 * R_2,$$

where $b \in \mathbb{C}$, $R_2$ is as above and $R_4$ - is some element in $\Pi_{(2,0)} \otimes H^4(\mathfrak{g})$. Let us show that $R_4 \neq 0$. Then the non-zero element

$$R_4 = M^4 - \frac{C_2^2}{28} - C_2 * R_2$$

satisfies both conditions of the Proposition.

Indeed, recall that we can treat elements of $S(\mathfrak{g}) \otimes \text{End } V_\lambda$ as polynomial maps from $\mathfrak{g}$ to $\text{End}V_\lambda$. Take $x_0 = (v, X, f) \in \mathfrak{g}_2$ such that $v = (1,0,0)^T$, $f = (0,0,0)$, $X \in \mathfrak{sl}_3(\mathbb{C})$. Then $M^4|_{x_0} \neq 0$, but $C_2|_{x_0} = 0$, so $R_4 \neq 0$ as well.

$M^5$ is check-antisymmetric matrix, so $M^5 \in (\Pi_{(1,0)} + \Pi_{(0,1)}) \otimes S^5(\mathfrak{g}_2)$. Note that $\Pi_{(1,0)}$ has only one exponent, and we already know that it is "3". Hence,

$$M^5 = a * C_2 P_3 + b * C_2^2 Q_1 + c * C_2 Q_3 + Q_5,$$

where $a, b, c, d \in \mathbb{C}$, the elements $P_3$, $Q_1$ are defined above, and $Q_3$, $Q_5$ are some elements in $\Pi_{(0,1)} \otimes H^3(\mathfrak{g}_2)$ and $\Pi_{(0,1)} \otimes H^5(\mathfrak{g}_2)$ respectively. In the same way we prove that $Q_5 \neq 0$. Then "1" and "5" are exponents of $\Pi_{(0,1)}$, and from the Table 6, there are only two of them. So "3" is not exponent, and $Q_3 = 0$. Therefore,

$$Q_5 = M^5 - a * C_2 R_3 - b * C_2^2 * Q_1$$

and the harmonic element $Q_5$ satisfies the conditions of the Proposition.

We have:

$$M^6 - Tr(M^6) = a * C_2^2 R_2 + b * C_2 R_4 + R_6,$$

where $R_2$, $R_4$ - as above, $a$, $b \in \mathbb{C}$, and $R_6$ - is some element in $\Pi_{(2,0)} \otimes H^6(\mathfrak{g}_2)$. Note that $C_6 = Tr(M^6) \in I[\mathfrak{g}^2]$. As before, by the restriction of polynomial maps $\mathfrak{g}_2 \to \mathrm{End} V_{(0,1)}$ to the suitable elements of $\mathfrak{g}_2$, one can get that $R_6 \neq 0$. So $R_6$ satisfies the conditions of the Proposition, and "$2, 4, 6$" are the exponents of $\Pi_{(2,0)}$.

The harmonic elements $\{1, P_3, R_2, R_4, R_6, Q_1, Q_5, \}$ are independent over $I(\mathfrak{g})$ by the construction and create the basis of $C_{(0,1)}(\mathfrak{g}_2)$. □

**Corollary 9.3.** *The q-analogues of zero-multiplicities are*

| | |
|---|---|
| $\Pi_{(1,0)}$ | $q^3$, |
| $\Pi_{(0,1)}$ | $q + q^5$, |
| $\Pi_{(2,0)}$ | $q^2 + q^4 + q^6$. |

**Corollary 9.4.** $\{1, M, ..., M^6\}$ *is a linear basis of* $C_\lambda(\mathfrak{g})$ *over the ring* $I(\mathfrak{g}_2)$.

# Chapter 10

# Family algebras of spinor representations

**1.** Among representations with simple spectrum there are so-called *minuscule* representations. The weights of minuscule representations form a single orbit of the Weyl group. For classical Lie algebras the minuscule representations are just the standard representations and the spinor representations of special orthogonal Lie algebras. The family algebras of standard representations of classical family algebras are described in [12] - see the Examples after the Theorem 5.1.

It turns out that, as in the case of standard representation of special orthogonal Lie algebras $\mathfrak{so}_{2n}(\mathbb{C})$, for spinor representations the element $M_\lambda$ does not generate the family algebras over the ring of invariants. This will follow from the formulas for q-analogues of zero-multiplicities of the irreducible constituents of $\mathrm{End}(V_\lambda)$ for the spinor representation $V_\lambda$. The constituents in this decomposition are so-called *small* representations. Representation is called small if none of its weights is twice a root. Some nice properties of small representations allow to compute easily their generalized exponents, without applying Hesselink-Peterson formula.

**2.** The key-point is the result of B. Broer, [2]. We also used the information on small representations from [22], [23]. Let $H^m(\mathfrak{g})$ be $G$-harmonic polynomials of degree $m$ on $\mathfrak{g}$, and let $H^m(\mathfrak{h})$ be $W-$harmonic polynomials of degree $m$ on the Cartan subalgebra $\mathfrak{h}$. Let $V$ be a $\mathfrak{g}$-representation and let $V_0$ be its zero-weight space.

**Proposition 10.1.** *(B. Broer) If $V$ is small, then*

$$Hom_G(V, H^m(\mathfrak{g})) = Hom_W(V_0, H^m(\mathfrak{h}))$$

*for every $m$.*

And the generating functions $\sum_m \mathrm{Hom}_W(V_0, H^m(\mathfrak{h}))q^m$ for exponents of representations of classical Weyl groups easily follow from the results of A. Kirillov in [14]. (See also [25], [19]).

**3.** Let $\mathfrak{g}$ be a Lie algebra of the series $B_n$. The Weyl group $W = \mathbb{Z}_2^n \rtimes S_n$ is a group of permutations of $n$ variables and all sign flips. The representations of W are labeled by pairs of Young diagrams $(\lambda, \mu)$ with the property $|\lambda| + |\mu| = n$. The spin-representation has the highest weight $\omega_n$ and the dimension $2^n$. It is self-dual, so

$$\mathrm{End}(V_{\omega_n}) = \omega_n \otimes \omega_n = \pi_0 \oplus \pi_{2\omega_n} \oplus \bigoplus_{i=1}^{n-1} \pi_{\omega_i} \tag{10.10.1}$$

Let $V = V_{\omega_1}$ The corresponding to (10.10.1) representations $(\lambda, \mu)$ of Weyl groups in the zero-weight spaces are labeled by diagrams

$$\lambda = (n-p), \quad \mu = (p), \quad \text{for} \quad V_{\omega_{2p}} = \Lambda^{2p}(V), \quad 0 < p < \frac{n}{2},$$
$$\lambda = (p), \quad \mu = (n-p), \quad \text{for} \quad V_{\omega_{2p+1}} = \Lambda^{2p+1}(V), \quad 0 < p < \frac{n-1}{2},$$
$$\lambda = (n/2), \quad \mu = (n/2), \quad \text{for} \quad V_{2\omega_n} = \Lambda^n(V), \text{ if } n \text{ is even},$$
$$\lambda = ((n-1)/2), \quad \mu = ((n+1)/2), \quad \text{for} \quad V_{2\omega_n} = \Lambda^n(V), \text{ if } n \text{ is odd}.$$

Using from [14] the formulas of Poincare series for isotipic components of $W$ in polynomial algebra, we obtain

**Proposition 10.2.** *The $q$-multiplicities of zero-weights of representations in the decomposition (10.10.1) are*

$$m^0_{\omega_{2p}}(q) = q^p \binom{n}{p}_{q^2}, \quad m^0_{\omega_{2p+1}}(q) = q^{n-p} \binom{n}{p}_{q^2},$$

$$m^0_{2\omega_n}(q) = \begin{cases} q^{n/2}\binom{n}{\frac{n}{2}}_{q^2}, & if n \text{ is even}, \\ \\ q^{\frac{n+1}{2}}\binom{n}{\frac{n+1}{2}}_{q^2}, & if n \text{ is odd}. \end{cases}$$

Here we use a notation

$$\binom{n}{p}_{q^2} = \frac{(1-q^2)...(1-q^{2n})}{(1-q^2)...(1-q^{2p})(1-q^2)...(1-q^{2n-2p})}.$$

**4.** Let $\mathfrak{g}$ be the Lie algebra of series $D_n$. The Weyl grop $W = \mathbb{Z}_2^{n-1} \rtimes S_n$ is a group of permutations of $n$ variables with an even number of sign flips. The representations of W are labeled by pairs of Young diagrams $(\lambda, \mu)$ with the property $|\lambda| + |\mu| = n$, and for even $n$ also by the pairs $(\lambda, \varepsilon)$ with $\varepsilon = \pm 1$, $|\lambda| = n/2$. So the situation varies a little for $n$ even and odd, and we consider these cases separately.

Let $n = 2k$. Then the spin-representations with the highest weights $\omega_n$ and $\omega_{n-1}$ are self-dual, and

$$\mathrm{End}(V_{\omega_n}) = \pi_0 \oplus \pi_{2\omega_n} \oplus \bigoplus_{p=1}^{k-1} \pi_{\omega_{2p}}, \tag{10.10.2}$$

$$\mathrm{End}(V_{\omega_{n-1}}) = \pi_0 \oplus \pi_{2\omega_{n-1}} \oplus \bigoplus_{p=1}^{k-1} \pi_{\omega_{2p}}. \tag{10.10.3}$$

Let $V = V_{\omega_1}$. Then we have the following representations of Weyl group.

$$\lambda = (n-p), \quad \mu = (p), \quad \text{for} \quad V_{\omega_{2p}} = \Lambda^{2p}(V), \quad p = 1, \dots, k-1,$$

$$\lambda = (k), \quad \varepsilon = \pm 1, \quad \text{for} \quad V_{2\omega_{n-1}}, V_{2\omega_n}.$$

Accordingly, we obtain

**Proposition 10.3.** *For even $n$ the $q$-analoges of zero-weight multiplicities of constituents in (10.10.2), (10.10.3) are*

$$m^0_{\omega_{2p}}(q) = \frac{(q^{n-p} + q^p)}{(1+q^n)} \binom{n}{p}_{q^2}, \quad p = 1, ..., (k-1)$$

$$m^0_{2\omega_n}(q) = m^0_{2\omega_{n-1}}(q) = \frac{q^{n/2}}{(1+q^n)} \binom{n}{\frac{n}{2}}_{q^2}.$$

In the case of $n = 2k+1$ the representations with the highest weights $\omega_n$ and $\omega_{n-1}$ are dual to each other, so

$$\mathrm{End}(V_{\omega_n}) = \mathrm{End}(V_{\omega_{n-1}}) = \pi_0 \oplus \pi_{\omega_n + \omega_{n-1}} \oplus \bigoplus_{i=1}^{k-1} \pi_{\omega_{2i}}. \tag{10.10.4}$$

Note that as before,

$$\lambda = (n-p), \quad \mu = (p), \quad \text{for} \quad V_{\omega_{2p}} = \Lambda^{2p}(V), \quad 0 \le p < k,$$

$$\lambda = \left(\frac{n+1}{2}\right), \quad \mu = \left(\frac{n-1}{2}\right), \quad \text{for} \quad V_{\omega_n+\omega_{n-1}} = \Lambda^{n-1}(V).$$

and we obtain

**Proposition 10.4.** *For odd $n$ the $q$-analoges of zero-weight multiplicities of constituents in (10.10.4) are*

$$m^0_{\omega_{2p}}(q) = \frac{(q^{n-p}+q^p)}{(1+q^n)} \binom{n}{p}_{q^2}, \quad p = 1, ..., (k-1)$$

$$m^0_{\omega_n+\omega_1}(q) = \frac{q^{(n-1)/2}+q^{(n+1)/2}}{(1+q^n)} \binom{n}{\frac{n-1}{2}}_{q^2}.$$

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