MIXED ZETA FUNCTIONS

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ABSTRACT

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We examine Dirichlet series which combine the data of a distance function, u, a homogeneous degree zero function, φ, and a multivariable Dirichlet series, K. By using an integral representation and Cauchy's residue formula, we show that under certain conditions on K, such functions extend to meromorphic functions on $\mathbb{C}$, or to some region strictly larger than the domain of absolute convergence, and have real poles and polynomial growth in vertical strips. When $\varphi = 1$, we also do this for u which come from completely nonvanishing polynomials on $\mathbb{R}^n_{>0}$. Using standard Tauberian results, this allows us to deduce estimates for counting functions of points in expanding regions. We show that some of these results can be generalized to multivariable mixed zeta functions, and we use these to prove relations between coefficients of Laurent series of different Dirichlet series at $s = 0$.

Contents

Chapter 1

Introduction

Our object of study is a family of generalized Dirichlet series which combine the data of certain homogeneous functions with multivariable Dirichlet series. We wish to determine whether these functions have meromorphic extensions to $\mathbb{C}$, and if so, describe the nature of the poles and growth rates along vertical strips. Once this has been achieved, we can deduce asymptotics for an associated counting function.

In the simplest case, if u is a continuous n-dimensional distance function (that is, if $u : \mathbb{R}^n \to \mathbb{R}$ satisfies $u(\lambda \mathbf{x}) = \lambda u(\mathbf{x})$ for all $\lambda \geq 0$, $\mathbf{x} \in \mathbb{R}^n$, and $u(\mathbf{x}) > 0 \iff \mathbf{x} \neq \mathbf{0}$), we may define the Dirichlet series

$$\zeta_u(s) \quad := \quad \sum_{\mathbf{n} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}} u(\mathbf{n})^{-s} \tag{1.0.1}$$

for $\Re(s) > n$. We call this the zeta function of u, despite the fact that in general it is not known to have a functional equation or product expansion.

In chapter 2, we give a simple proof that for an arbitrary distance function which is smooth away from the origin, the Dirichlet series (1.0.1), modified by a smooth weight function, φ, extends to a meromorphic function on $\mathbb{C}$ (theorem 2.1.4). This method of proof also applies to distance zeta functions where the summation is over n-tuples of positive integers, and gives expressions for the residues of its poles, all of which are simple.

The heart of this thesis is chapter 3, where the Dirichlet series we consider are defined by summation over more general sets, with arbitrary weighting functions, provided an associated multivariable Dirichlet series K has good properties (see p. 16). We call these mixed zeta functions. Depending on the strength of our hypotheses on K, we show that mixed zeta functions extend to meromorphic functions on $\mathbb{C}$ or some halfplane $\mathbb{C}_{>\kappa}$, where they have real poles, and polynomial growth along vertical strips (theorems 3.4.2 and 3.4.1). This is done using the Mellin transform to express the zeta function in terms of iterated contour integrals involving K and an associated 'beta' function (proposition 3.2.1). This generalizes results of Essouabri [8].

In chapter 4, we show that similar results hold when the weight function φ is identically 1, and u comes from a completely nonvanishing polynomial whose Newton polyhedron is of full dimension (see sections 4.2 and 4.3 for definitions). In this case, u is not necessarily a distance function. We demonstrate how this can be used to recover certain results of Sargos [31] as a special case.

Chapter 5 is very short, and simply shows how the results of chapters 3 and 4 allow us to deduce growth rates of certain counting functions, using know results.

In chapter 6, we consider multivariable mixed zeta functions, and use them to prove relations between the coefficients in the Laurent expansions of certain one-variable Dirichlet series at $s = 0$. As special cases, we recover theorem 1.1 of Friedman and Pereira [9], and theorem 1.1 of Castillo-Garate and Friedman [2].

Note to the reader: In the course of reading this dissertation, should you find yourself confused, know that this manuscript is in need of several days of editing. (It's me, not you).

1.1 Notation and conventions

1. If $a \in S$, where $S = \mathbb{Z}$ or $\mathbb{R}$, then $S_{\geq a}$ denotes the set $\{b \in S \mid b \geq a\}$. If $S = \mathbb{C}$, however, we define $\mathbb{C}_{>a} := \{s \in \mathbb{C} \mid \Re(s) > \Re(a)\}$. The sets $S_{>a}$, $S_{\leq a}$, and $S_{<a}$ are defined analogously.

2. For $\Omega \subseteq \mathbb{R}^r$, we define $\Omega_{\mathbb{C}} = \{\mathbf{z} \in \mathbb{C}^n \mid \Re(\mathbf{z}) \in \Omega\}$.

3. Let T be a finite set of cardinality m, which we will regard as indexing the coordinates in a copy of $\mathbb{C}^m$, in the sense that if we choose a bijection $T \to [m] := \{1, \ldots, m\}$, we have a corresponding isomorphism $\mathbb{C}^T \cong \mathbb{C}^m$. We generally write elements of $\mathbb{C}^T$ using boldface letters, and for $i \in T$, write the i-th coordinate using the plain font version with i as a subscript, so that, for example, $\mathbf{a} = (a_i)_{i \in T} \in \mathbb{C}^T$. However, if $T = \{*\}$ is a singleton, we do not

distinguish between $\mathbf{a} \in \mathbb{C}^T$ and $a_* \in \mathbb{C}$.

4. Let $I \subseteq T$. If $\mathbf{x} = (x_i)_{i \in T} \in \mathbb{C}^T$, set $\mathbf{x}_I := (x_i)_{i \in I} \in \mathbb{C}^I$. For $\mathbf{y} \in \mathbb{C}^I$ and $\mathbf{z} \in \mathbb{C}^{T \smallsetminus I}$, define their concatenation $\mathbf{y}\!:\!\mathbf{z} \in \mathbb{C}^T$ by
$$(\mathbf{y}\!:\!\mathbf{z})_i = \begin{cases} y_i & i \in I \\ z_i & i \notin I. \end{cases}$$
If we write $(\mathbf{y}, \mathbf{z})$ for $\mathbf{y} \in \mathbb{C}^a$, $\mathbf{z} \in \mathbb{C}^b$, we mean $\mathbf{y} : \tilde{\mathbf{z}}$, where $\tilde{\mathbf{z}} \in \mathbb{C}^{[a+b] \smallsetminus [a]}$ corresponds to $\mathbf{z}$ under the bijection $[b] \cong [a+b] \smallsetminus [a] : n \leftrightarrow a+n$. (This is, of course, the usual meaning).

5. If $\mathbf{x} \in \mathbb{C}^T$ and $\mathbf{k} \in \mathbb{Z}^T_{\geq 0}$, we define $\mathbf{x}^{\mathbf{k}} = \prod_{i \in T} x_i^{k_i}$, $\mathbf{k}! = \prod_{i \in T} k_i!$, $(\mathbf{x})^+_{\mathbf{k}} = \prod_{i \in T} (x_i)^+_{k_i}$, where $(x)^+_k$ is the rising factorial, and where the empty product is 1.

6. If $A \in M_{n \times m}(\mathbb{Z}_{\geq 0})$ and $\mathbf{x} \in \mathbb{C}^n$, then $\mathbf{x}^A \in \mathbb{C}^m$ is the vector whose i-th component is $\mathbf{x}^{A_i}$, where A_i is the i-th column of A. Note that this implies $(\mathbf{x}^A)^B = \mathbf{x}^{AB}$ whenever either side is defined.

7. For $\mathbf{k} \in \mathbb{Z}^T_{\geq 0}$ and a smooth function ψ defined on a subset of $\mathbb{R}^T$, the partial derivative $\prod_{i \in T} \left(\frac{\partial}{\partial x_i}\right)^{k_i} \psi(\mathbf{x})$ will be denoted by either $\psi^{(\mathbf{k})}$ or $\partial_{\mathbf{k}} \psi$. For $j \in T$, we let $\mathbf{e}_j$ be the j-th vector in the standard basis for $\mathbb{R}^T$, and write $\partial_j = \partial_{\mathbf{e}_j} = \frac{\partial}{\partial x_j}$. If f is any function defined on a subset of $\mathbb{R}^T$, and $S \subseteq T$, define $f|_S(\mathbf{x}) := f(\mathbf{x}\!:\!\mathbf{0})$ for $\mathbf{x} \in \mathbb{R}^S$ such that $\mathbf{x}\!:\!\mathbf{0}$ is in the domain of f.

8. Set $|\mathbf{x}| = \sum_{i \in T} x_i$, where the empty sum is 0. Although we use the same symbol for the absolute value of a complex number, this should not cause confusion. The Euclidean norm on $\mathbb{R}^n$ will be denoted by $\| \ \|$.

9. We let $\mathbf{0}_T$ and $\mathbf{1}_T$ be the elements of $\mathbb{Z}^T$, all of whose components are 0 and 1 respectively. When T is clear from the context, we sometimes omit the subscript. Note that T may be empty.

10. To avoid a proliferation of set-theoretic complements when dealing with sums indexed by power sets, it will be convenient to define
$$\mathcal{C}(T) = \{(I, T \smallsetminus I) \mid I \subseteq T\}.$$

11. For a function $F : \mathbb{R}_{>0} \to \mathbb{C}$, $\mathcal{M}F(s) := \int_0^\infty F(t)t^{s-1}dt$ denotes its Mellin transform, provided the integral converges.

12. For $c \in \mathbb{R}$ and F a function defined on a domain in $\mathbb{C}$ which contains the line $\Re(s) = c$, we write $\int_{(c)} F(s)ds$ for $\frac{1}{2\pi i}$ times the integral of F along $\Re(s) = c$, assuming this exists. Note that this differs from the usual convention, which does not include the factor $\frac{1}{2\pi i}$. We include the factor to simplify expressions involving the inverse Mellin transform.

13. If f and g are complex valued functions on a set X, we use the 'big O' notation $f = O(g)$ on $Y \subseteq X$ if there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in Y$. We also use the alternative notation $f \ll g$ on Y. If f and g depend on a parameter λ which affects the choice of C, we write $f = O_\lambda(g)$ and $f \ll_\lambda g$ on Y.

14. By 'a sequence in X', we will mean a function $c : \mathcal{A} \to X$, where $\mathcal{A}$ is any countable set. We write $c(\alpha) = c_\alpha$ for $\alpha \in \mathcal{A}$, and also denote the sequence

by $c = (c_\alpha)_{\alpha \in \mathcal{A}}$. Note that we do not require that $\mathcal{A}$ has an ordering, so when $X = \mathbb{C}$, $\sum_{\alpha \in \mathcal{A}} c_\alpha$ is in general not well-defined. However, if $\sum_{\alpha \in \mathcal{A}} |c_\alpha| < \infty$ under some identification $\mathcal{A} \cong \mathbb{Z}_{>0}$, then $\sum_{\alpha \in \mathcal{A}} c_\alpha$ is well-defined.

Chapter 2

Distance zeta functions

We let the group of positive reals act on $\mathbb{R}^n$ by scalar multiplication. Let $E \subseteq \mathbb{R}^n$ be $\mathbb{R}_{>0}$-invariant, and put $E' = E \smallsetminus \{\mathbf{0}\}$. We say that a function $f : E' \to \mathbb{C}$ is homogeneous of degree $d \in \mathbb{R}$ if $f(\lambda \mathbf{x}) = \lambda^d f(\mathbf{x})$ for all $\lambda > 0$, $\mathbf{x} \in E'$. The set of homogeneous degree d functions on E' will be denoted by $\mathcal{H}_d(E)$.

We say that $u : E' \to \mathbb{R}$ is a *distance function* on E if

- u is homogeneous of degree 1,
- $\inf\{u(\mathbf{x}) \mid \mathbf{x} \in E,\ \|\mathbf{x}\| = 1\} > 0$.

If we extend the domain of u to E by setting $u(\mathbf{0}) = 0$, we still call u a distance function. Thus we recover the definition in the introduction when $E = \mathbb{R}^n$. We write $\mathcal{D}(E)$ for the set of distance functions on E. If $u \in \mathcal{D}(E)$, define its unit ball by $\mathcal{B}(u) := \{\mathbf{x} \in E \mid u(\mathbf{x}) \leq 1\}$.

Let $\mathcal{Z}_{a,b}(E)$ be the set of pairs (φ, u), where $\varphi \in \mathcal{H}_a(E)$ is bounded on $\mathcal{B}(\|\ \|) \cap E$, and $u^{1/b} \in \mathcal{D}(E)$. Then for $(\varphi, u) \in \mathcal{Z}_{a,b}(E)$, the generalized Dirichlet series

$$\zeta_{\varphi,u}(s) \quad := \quad \sum_{\mathbf{n} \in \mathbb{Z}^n \cap E} \varphi(\mathbf{n}) u(\mathbf{n})^{-s} \tag{2.0.1}$$

converges for $s \in \mathbb{C}_{>(n+a)/b}$, as one sees by comparison with the series $\sum_{\mathbf{n} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}} \|\mathbf{n}\|^{-\Re(s)}$.

Note that it is enough to consider $(\varphi, u) \in \mathcal{Z}(E) := \mathcal{Z}_{0,1}(E)$, since if $(\varphi, u) \in \mathcal{Z}_{a,b}(E)$, the functions $\tilde{u} = u^{1/b}$ and $\tilde{\varphi} = u^{-a/b}\varphi$ give $(\tilde{\varphi}, \tilde{u}) \in \mathcal{Z}(E)$, and $\zeta_{\varphi,u}(s) = \zeta_{\tilde{\varphi},\tilde{u}}(bs - a)$.

We say $f : E' \to \mathbb{C}$ is smooth if it extends to a smooth function on an open neighbourhood of $E' \subset \mathbb{R}^n$. We write $\mathcal{H}_a^\infty(E)$ and $\mathcal{D}^\infty(E)$ for the smooth functions in $\mathcal{H}_a(E)$ and $\mathcal{D}(E)$ respectively, and set $\mathcal{Z}^\infty(E) = \mathcal{H}_0^\infty(E) \times \mathcal{D}^\infty(E) \subset \mathcal{Z}(E)$.

2.1 Meromorphic continuation of $\zeta_{\varphi,u}$

In the case where u is a distance function on $\mathbb{R}^n$ which is smooth away from the origin, Herglotz [14] has shown that when $n = 2$, $\zeta_u(s) := \zeta_{1,u}(s)$ extends to a meromorphic function on $\mathbb{C}$ with a single pole at $s = 2$, which is simple and has residue $2\operatorname{vol}(\mathcal{B}(u))$.

For general n, Hlawka ([16],[17]) studied the Fourier transforms $\hat{\varphi}_\delta$ of the functions

$$\varphi_\delta(\mathbf{x}) = \frac{1}{\Gamma(\delta+1)} \begin{cases} (1-u(\mathbf{x})^2)^\delta & u(\mathbf{x}) \le 1 \\ 0 & \text{otherwise.} \end{cases} , \qquad (\delta = 0, 1, \ldots)$$

where u is smooth away from the origin, and the unit sphere $\{\mathbf{x} \in \mathbb{R}^n \mid u(\mathbf{x}) = 1\}$ is convex with positive Gaussian curvature everywhere (when $\delta = 0$, Herz [15] has independently obtained similar results). Based on the asymptotic expansions for $\hat{\varphi}_\delta$ in [17], he shows [18] that $\zeta_u(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = n$, with residue $n\,\mathrm{vol}(\mathcal{B}(u))$. Although this is true, it appears that the asymptotic expansions in [17] are incorrect for large δ (see section 3.5).

While it may be possible to correct the results of [17], we instead adapt the method of Zagier in [35] to show that for $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n)$, $\zeta_{\varphi,u}$ extends to a meromorphic function on $\mathbb{C}$. This will turn out to be a special case of the results we prove in chapter 3, but the method here is simpler.

Lemma 2.1.1. *Suppose* $(\varphi, u) \in \mathcal{Z}_{a,1}(\mathbb{R}^n_{\ge 0})$, *where* $a > -n$. *For any* $F : \mathbb{R}_{\ge 0} \to \mathbb{C}$ *satisfying*

$$F(t) = \begin{cases} O(t^{-n-a'}) & t \to 0^+, \\ O(t^{-a''}) & t \to \infty \end{cases}$$

for some $a' < a$ *and all* $a'' > 0$, *we have*

$$\int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x}) F(u(\mathbf{x})) d\mathbf{x} \;=\; (n+a)\mathcal{M}F(n+a) \int_{\mathcal{B}(u)} \varphi(\mathbf{x}) d\mathbf{x}. \qquad (2.1.1)$$

Proof. In the calculation below, we use the change of variables $x_i \to y_i x_n$ ($i =$

$1, \ldots, n-1)$ followed by $x_n \to \frac{w}{u(y_1,\ldots,y_{n-1},1)}$.

$$\begin{aligned}
&\int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x} \\
&\quad = \int_0^\infty \int_{\mathbb{R}^{n-1}} \varphi(x_1,\ldots,x_n)F(u(x_1,\ldots,x_n))d\mathbf{x}\ dx_n \\
&\quad = \int_0^\infty \int_{\mathbb{R}^{n-1}_{>0}} \varphi(y_1,\ldots,y_{n-1},1)F(x_n u(y_1,\ldots,y_{n-1},1))x_n^{n+a-1}d\mathbf{y}\ dx_n \\
&\quad = \int_{\mathbb{R}^{n-1}_{>0}} \int_0^\infty \varphi(y_1,\ldots,y_{n-1},1)F(w)w^{n+a-1}u(y_1,\ldots,y_{n-1},1)^{-n-a}dw\ d\mathbf{y} \\
&\quad = \mathcal{M}F(n+a)\int_{\mathbb{R}^{n-1}_{>0}} \varphi(y_1,\ldots,y_{n-1},1)u(y_1,\ldots,y_{n-1},1)^{-n-a}d\mathbf{y}. \qquad (2.1.2)
\end{aligned}$$

If $\mathcal{M}F(n+a) = 0$, we are done, and if $\mathcal{M}F(n+a) \neq 0$, the ratio $\frac{\int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x}}{\mathcal{M}F(n+a)}$ is independent of F, so we may replace F by $\chi_{[0,1]}$, the characteristic function of the unit interval, to obtain

$$\frac{\int_{\mathbb{R}^{n-1}_{>0}} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x}}{\mathcal{M}F(a+n)} = \frac{\int_{\mathbb{R}^{n-1}_{>0}} \varphi(\mathbf{x})\chi_{[0,1]}(u(\mathbf{x}))d\mathbf{x}}{\mathcal{M}\chi_{[0,1]}(a+n)} = (a+n)\int_{\mathcal{B}(u)} \varphi(\mathbf{x})d\mathbf{x}.$$

□

Remark 2.1.2. If $(\varphi, u) \in \mathcal{Z}_{a,1}(\mathbb{R}^n)$, then under the same hypotheses, (2.1.1) remains true with $\mathbb{R}^n_{\geq 0}$ replaced by $\mathbb{R}^n$.

Corollary 2.1.3. *Suppose $(\varphi, u) \in \mathcal{Z}_{a,1}(\mathbb{R}^n)$. For any $b > -n-a$, and any $F : \mathbb{R}_{\geq 0} \to \mathbb{C}$ satisfying*

$$F(t) = \begin{cases} O(t^{-n-a'}) & t \to 0^+, \\ O(t^{-a''}) & t \to \infty \end{cases}$$

for some $a' < a$ and all $a'' > 0$, we have

$$\int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x} = (n+a+b)\mathcal{M}F(n+a)\int_{\mathcal{B}(u)} \varphi(\mathbf{x})u(\mathbf{x})^b d\mathbf{x}. \qquad (2.1.3)$$

Proof. Define $\tilde{F}(x) = x^{-b}F(x)$, and apply lemma 2.1.1 to $(\varphi u^b, u)$, using $\tilde{F}$ instead of F:

$$\begin{aligned}\int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x} &= \int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x})u(\mathbf{x})^b\tilde{F}(u(\mathbf{x}))d\mathbf{x} \\ &= (n+a+b)\mathcal{M}\tilde{F}(n+a+b)\int_{\mathcal{B}(u)} \varphi(\mathbf{x})u(\mathbf{x})^b d\mathbf{x} \\ &= (n+a+b)\mathcal{M}F(n+a)\int_{\mathcal{B}(u)} \varphi(\mathbf{x})u(\mathbf{x})^b d\mathbf{x}.\end{aligned}$$

□

In the proof of theorem 2.1.4 below, it will be convenient to introduce normalized periodic Bernoulli functions, defined as follows. If P_k be the k-th Bernoulli polynomial, then $B_k(x) := \frac{P_k(\{x\})}{k!}$, where $\{x\}$ is the fractional part of x. Thus if ϕ is a smooth function on $[a,b] \subset \mathbb{R}$, the Euler-Maclaurin summation formula can be written as

$$\begin{aligned}\sum_{n=\lfloor a\rfloor+1}^{\lceil b\rceil-1} \phi(n) &= \int_a^b \phi(x)dx + \sum_{r=1}^{N} (-1)^r B_r(x)\phi^{(r-1)}(x)\Big|_{x=a^+}^{b^-} \\ &\quad +(-1)^{N-1}\int_a^b B_N(x)\phi^{(N)}(x)dx.\end{aligned}$$

for any $N \in \mathbb{Z}_{>0}$. For $\mathbf{k} \in \mathbb{Z}^T_{\geq 0}$, and $\mathbf{x} \in \mathbb{R}^T$, we set $B_{\mathbf{k}}(\mathbf{x}) := \prod_{i\in T} B_{k_i}(x_i)$.

Theorem 2.1.4. *If $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n)$, then $\zeta_{\varphi,u}(s)$ extends to a meromorphic function, analytic away from $s = n$, where it has a simple pole of residue $n\int_{\mathcal{B}(u)} \varphi(\mathbf{x})d\mathbf{x}$.*

Proof. Let ψ be a Schwartz function on $\mathbb{R}^n$. Choose $N \in \mathbb{Z}_{>0}$. By iterating the Euler-Maclaurin summation formula n times, we find

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} \psi(\mathbf{n}) = \int_{\mathbb{R}^n} \psi(\mathbf{x})d\mathbf{x} + \sum_{\varnothing\neq I\subseteq\{1,\ldots,n\}} (-1)^{(N-1)\#I}\int_{\mathbb{R}^n} \psi^{(N_I)}(\mathbf{x})\prod_{i\in I} B_N(x_i)d\mathbf{x}.$$

If we replace ψ by $\mathbf{x} \mapsto \psi(t\mathbf{x})$, $(t > 0)$, this gives

$$\begin{aligned}
&\sum_{\mathbf{n}\in\mathbb{Z}^n} \psi(t\mathbf{n}) \\
&= \int_{\mathbb{R}^n} \psi(t\mathbf{x})d\mathbf{x} + \sum_{\varnothing\neq I\subseteq\{1,\dots,n\}} (-1)^{(N-1)\#I} \int_{\mathbb{R}^n} t^{N\#I}\psi^{(N_I)}(t\mathbf{x}) \prod_{i\in I} B_N(x_i)d\mathbf{x} \\
&= t^{-n}\int_{\mathbb{R}^n} \psi(\mathbf{x})d\mathbf{x} + \sum_{\varnothing\neq I\subseteq\{1,\dots,n\}} (-1)^{(N-1)\#I} t^{N\#I-n} \int_{\mathbb{R}^n} \psi^{(N_I)}(\mathbf{x}) \prod_{i\in I} B_N(x_i/t)d\mathbf{x} \\
&= t^{-n}\int_{\mathbb{R}^n} \psi(\mathbf{x})d\mathbf{x} + O(t^{N-n}), \qquad (t\to 0^+).
\end{aligned}$$

Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function satisfying:

$$F \textit{ is smooth, and for all } k\in\mathbb{Z}_{\geq 0} \textit{ and } a\in\mathbb{Z},\ F^{(k)}(x) = O_k(x^a) \textit{ on } \mathbb{R}_{>0}. \tag{2.1.4}$$

For example,

$$F_1(x) \quad := \quad \begin{cases} e^{-x-1/x} & x > 0 \\ 0 & x = 0 \end{cases} \tag{2.1.5}$$

is one such function. Note that the Mellin transform of F is an entire function.

If we set $\varphi(\mathbf{0}) = u(\mathbf{0}) = 0$, then $\psi(\mathbf{x}) = \varphi(\mathbf{x})F(u(\mathbf{x}))$ is a Schwartz function on $\mathbb{R}^n$. Therefore

$$\Theta^F_{\varphi,u}(t) \quad := \quad \sum_{\mathbf{n}\in\mathbb{Z}^n\smallsetminus\{0\}} \varphi(\mathbf{n})F(tu(\mathbf{n})) \ = \ t^{-n}\int_{\mathbb{R}^n} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x} + O(t^{N-n}),$$

as $t \to 0^+$. By taking the Mellin transform, we obtain, for some function $A^F_{\varphi,u}(s)$ which is analytic on $\Re(s) > n - N$,

$$\begin{aligned}
\frac{1}{s-n}\int_{\mathbb{R}^n} \varphi(\mathbf{x})F(u(\mathbf{x}))d\mathbf{x} + A^F_{\varphi,u}(s) &= \int_0^\infty \Theta^F_{\varphi,u}(t)t^{s-1}dt \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^n\smallsetminus\{\mathbf{0}\}} \varphi(\mathbf{n}) \int_0^\infty F(tu(\mathbf{n}))t^{s-1}dt \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^n\smallsetminus\{\mathbf{0}\}} \varphi(\mathbf{n})u(\mathbf{n})^{-s} \int_0^\infty F(\tau)\tau^{s-1}d\tau \\
&= \zeta_{\varphi,u}(s)\mathcal{M}F(s).
\end{aligned}$$

Since N is arbitrary, $A^F_{\varphi,u}(s)$ extends to an entire function, and we see that $\zeta_{\varphi,u}(s)$ extends to a meromorphic function with a simple pole at $s = n$, which has residue $n \int_{\mathcal{B}(u)} \varphi(\mathbf{x}) d\mathbf{x}$ by remark 2.1.2.

All other poles are contained in the set $Z(\mathcal{M}F)$ of zeros of $\mathcal{M}F$. This is true for any function F as above; in particular, it holds for the function $F_\lambda(x) := F_1(x^\lambda)$, where $\lambda > 0$. Since $\mathcal{M}F_\lambda(s) = \mathcal{M}F_1(s/\lambda)/\lambda$, we have $Z(\mathcal{M}F_\lambda) = \lambda Z(\mathcal{M}F_1)$, and because the zero set is discrete, $\cap_{\lambda>0} Z(\mathcal{M}F_\lambda)$ either contains only 0, or is empty. But since $\mathcal{M}F_1(s) > 0$ on the real axis, the intersection is empty, and we conclude that there are no poles of $\zeta_{\varphi,u}(s)$ other than at $s = n$. $\square$

Next, we examine $\zeta_{\varphi,u}$ for $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{\geq 0})$.

Theorem 2.1.5. *If $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{>0})$, then $\zeta_{\varphi,u}(s)$ extends to a meromorphic function with at most simple poles, contained in $\mathbb{Z}_{\leq n}$. For $p \in \mathbb{Z}_{\leq n}$, the residue of $\zeta_{\varphi,u}(s)$ at $s = p$ is*

$$\sum_{(I,J)\in\mathcal{C}([n])} \sum_{\mathbf{k}\in\{0,\ldots,n-p\}^J, \#I-|\mathbf{k}|=p} (-1)^{|\mathbf{k}|} B_{\mathbf{k+1}}(\mathbf{0}) \int_{\mathcal{B}(u|_I)} [u\, \partial^{\mathbf{0}:\mathbf{k}}(\varphi u^{-p})]|_I(\mathbf{x}) d\mathbf{x}. \qquad (2.1.6)$$

Proof. With the notation above, the Euler-Maclaurin formula gives, for $N \in \mathbb{Z}_{>0}$,

$$\begin{aligned} \Theta^F_{\varphi,u}(t) \quad &:= \quad \sum_{\mathbf{n}\in\mathbb{Z}^n_{>0}} \psi(t\mathbf{n}) \\ &= \quad \sum_{(I,J)\in\mathcal{C}([n])} \sum_{\mathbf{k}\in\{0,\ldots,N\}^J} t^{|\mathbf{k}|-\#I} (-1)^{|\mathbf{k}|} B_{\mathbf{k+1}}(\mathbf{0}) \int_{\mathbb{R}^I_{>0}} \psi^{(\mathbf{0}:\mathbf{k})}(\mathbf{x}\!:\!\mathbf{0}) d\mathbf{x} \\ &\qquad + O(t^{N-n}), \end{aligned}$$

as $t \to 0^+$. As before, $\zeta_{\varphi,u}(s)\mathcal{M}F(s)$ is the Mellin transform of this expression, so $\zeta_{\varphi,u}(s)$ extends to a meromorphic function with at most simple poles, contained in

$\mathbb{Z}_{\leq n}$. For $p \in \mathbb{Z}_{\leq n}$, the residue of $\zeta_{\varphi,u}(s)$ at $s = p$ is

$$\frac{1}{\mathcal{M}F(p)} \sum_{(I,J)\in\mathcal{C}([n])} \sum_{\mathbf{k}\in\{0,\ldots,n-p\}^J, \#I-|\mathbf{k}|=p} (-1)^{|\mathbf{k}|} B_{\mathbf{k}+\mathbf{1}}(\mathbf{0}) \int_{\mathbb{R}^I_{>0}} \psi^{(\mathbf{0}:\mathbf{k})}(\mathbf{x}:\mathbf{0})d\mathbf{x}, \tag{2.1.7}$$

for any F satisfying hypotheses (2.1.4) and such that $\mathcal{M}F(p) \neq 0$.

Fix $p \in \mathbb{Z}_{\leq n}$, and set $\tilde{F}(x) = x^p F(x)$, so that $\psi(\mathbf{x}) = \varphi(\mathbf{x})u(\mathbf{x})^{-p}\tilde{F}(u(\mathbf{x}))$. For $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$, we can write $\psi^{(\mathbf{k})} = \sum_{j=0}^{|\mathbf{k}|} \phi_{\mathbf{k},j}\tilde{F}^{(j)} \circ u$, where $\phi_{\mathbf{k},j}$ is smooth and homogeneous of degree $j - |\mathbf{k}| - p$. The functions $\phi_{\mathbf{k},j}$ can be defined inductively by

$$\phi_{\mathbf{0},0} = \varphi u^{-p}, \qquad \phi_{\mathbf{k}+\mathbf{e}_i,j} = \phi_{\mathbf{k},j}^{(\mathbf{e}_i)} + \phi_{\mathbf{k},j-1}u^{(\mathbf{e}_i)},$$

where $\phi_{\mathbf{k},j} = 0$ if $j < 0$ or $j > |\mathbf{k}|$. In particular, $\phi_{\mathbf{k},0} = \partial^{\mathbf{k}}(\varphi u^{-p})$.

If $(I, J) \in \mathcal{C}([n])$, $\mathbf{k} \in \{0, \ldots, n-p\}^J$ and $\#I - |\mathbf{k}| = p$, then

$$\begin{aligned}
&\int_{\mathbb{R}^I_{>0}} \psi^{(\mathbf{0}:\mathbf{k})}(\mathbf{x}:\mathbf{0})d\mathbf{x} \\
&= \sum_{j=0}^{|\mathbf{k}|} \int_{\mathbb{R}^I_{>0}} (\phi_{\mathbf{0}:\mathbf{k},j}\tilde{F}^{(j)} \circ u)|_I(\mathbf{x})d\mathbf{x} \\
&= \mathcal{M}\tilde{F}(0) \int_{\mathcal{B}(u|_I)} (\phi_{\mathbf{0}:\mathbf{k},0}u)|_I(\mathbf{x})d\mathbf{x} + \sum_{j=1}^{|\mathbf{k}|} \mathcal{M}(\tilde{F}^{(j)})(j)j \int_{\mathcal{B}(u|_I)} \phi_{\mathbf{0}:\mathbf{k},j}|_I(\mathbf{x})d\mathbf{x} \\
&= \mathcal{M}F(p) \int_{\mathcal{B}(u|_I)} [u\,\partial^{\mathbf{0}:\mathbf{k}}(\varphi u^{-p})]|_I(\mathbf{x})d\mathbf{x},
\end{aligned} \tag{2.1.8}$$

where in the second line, we have used corollary 2.1.3 for the first term, and lemma 2.1.1 for the rest. Substituting (2.1.8) in (2.1.7) gives (2.1.6). □

We give expressions for the first few residues below.

Corollary 2.1.6. *For $i \neq j \in [n]$, let $S(j) = [n] \smallsetminus \{j\}$ and $S(i,j) = [n] \smallsetminus \{i,j\}$.*

The residue of $\zeta_{\varphi,u}(s)$ *is*

$$\begin{aligned} n\int_{\mathcal{B}(u)} \varphi(\mathbf{x})d\mathbf{x} \qquad & \textit{at } s=n, \\ -\frac{n-1}{2}\sum_{j=1}^{n}\int_{\mathcal{B}(u|_{S(j)})} \varphi|_{S(j)}(\mathbf{x})d\mathbf{x} \qquad & \textit{at } s=n-1. \end{aligned}$$

At $s=n-2$*, it is*

$$\begin{aligned} -\frac{1}{12}\sum_{j=1}^{n}\int_{\mathcal{B}(u|_{S(j)})} & [\varphi^{(\mathbf{e}_j)}u-(n-2)\varphi u^{(\mathbf{e}_j)}]|_{S(j)}(\mathbf{x})d\mathbf{x} \\ & +\frac{n-2}{4}\sum_{i<j}\int_{\mathcal{B}(u|_{S(i,j)})} \varphi|_{S(i,j)}(\mathbf{x})d\mathbf{x}. \end{aligned}$$

Corollary 2.1.7. *When* $\varphi \equiv 1$*,* $\zeta_{\varphi,u}(s)$ *is regular at* $s=0$*.*

Proof. If $\mathbf{k}\neq\mathbf{0}$, the integrand in (2.1.6) vanishes, and if $\mathbf{k}=\mathbf{0}$ then $I=\varnothing$, so the integral is also zero. $\square$

Chapter 3

Mixed zeta functions

Mixed zeta functions are Dirichlet series which combine the data of a pair $(\varphi, u) \in \mathscr{Z}^{\infty}(\mathbb{R}_{\geq 0})$ and a multivariable Dirichlet series K, which we now define.

Such multivariable series will be defined by pairs of sequences $(\mathbf{m}, c)$, where $\mathbf{m} = (\mathbf{m}_\alpha)_{\alpha \in \mathcal{A}}$ for $\mathbf{m}_\alpha \in \mathbb{R}^n_{>0}$, and $c = (c_\alpha)_{\alpha \in \mathcal{A}}$ for $c_\alpha \in \mathbb{C}$, and which satisfy

Hypotheses 1.

There exists $N > 0$ *such that for all* $\boldsymbol{\sigma} \in \mathbb{R}^n_{>N}$*,*

$$\sum_{\alpha \in \mathcal{A}} \frac{|c_\alpha|}{\mathbf{m}_\alpha^{\boldsymbol{\sigma}}} < \infty. \tag{3.0.1}$$

Under these hypotheses,

$$K(\mathbf{s}) \;=\; K(\mathbf{m}, c; \mathbf{s}) \quad := \quad \sum_{\alpha \in \mathcal{A}} \frac{c_\alpha}{\mathbf{m}_\alpha^{\mathbf{s}}} \tag{3.0.2}$$

defines an analytic function of $\mathbf{s}$ in the region $\mathbb{C}^n_{>N}$. We will write $N_K = N$. From now on, when we write $K(\mathbf{m}, c; \mathbf{s})$, we will implicitly assume that hypotheses 1 hold.

If $(\varphi, u) \in \mathscr{Z}(\mathbb{R}^n_{\geq 0})$, define

$$\zeta_{\varphi,u}(K; s) = \sum_{\alpha \in \mathcal{A}} c_\alpha \frac{\varphi(\mathbf{m}_\alpha)}{u(\mathbf{m}_\alpha)^s}, \quad s \in \mathbb{C}_{>nN_K}. \tag{3.0.3}$$

Since u is a distance function,

$$u(\mathbf{x}) \gg \|\mathbf{x}\| \gg \mathbf{x}^{n^{-1}\mathbf{1}} \tag{3.0.4}$$

for $\mathbf{x} \in \mathbb{R}^n_{>0}$ (the second estimate follows from the quadratic-geometric inequality). Thus, for $s \in \mathbb{C}_{>nN_K}$,

$$|c_\alpha \varphi(\mathbf{m}_\alpha) u(\mathbf{m}_\alpha)^{-s}| = O\left(|c_\alpha| \mathbf{m}_\alpha^{-N'\mathbf{1}}\right), \qquad N' = \Re(s)/n > N_K,$$

and hence (3.0.1) implies that the series in (3.0.3) converges absolutely, and defines an analytic function on $\mathbb{C}_{>nN_K}$. Following Essouabri [8], we call this a *mixed zeta function*[1].

In proposition 3.2.1, we will show that when $(\varphi, u) \in \mathscr{Z}^\infty(\mathbb{R}^n_{\geq 0})$, $\zeta_{\varphi,u}(K; s)$ has an integral representation which involves a 'beta' function associated to (φ, u). If K satisfies additional hypotheses, we will show that $\zeta_{\varphi,u}(K; s)$ extends to a meromorphic function with a larger domain (theorems 3.4.1 and 3.4.2).

---

[1] The definition in [8] is more restrictive than ours. The Dirichlet series considered by Essouabri can be obtained from ours by taking $\varphi(\mathbf{x}) = 1$, $u(\mathbf{x}) = \sum_{j=1}^n b_j x_j$, where $b_1, \dots, b_n > 0$, $\mathcal{A} = \mathbb{Z}^n_{>0}$, and $\mathbf{m}_\mathbf{a} = \mathbf{a}^M$ for a matrix M of non-negative integers, none of whose rows are zero, and such that the sum of the entries in each column is the same. In other words, $u(\mathbf{m}_\mathbf{a})$ is a homogeneous polynomial with non-negative coefficients which depends effectively on all variables.

3.1 The beta function of φ and u

Suppose $(\varphi, u) \in \mathcal{Z}(\mathbb{R}^n_{>0})$. Fix $z_1, \ldots, z_n \in \mathbb{C}_{>0}$. Then the differential form

$$\varphi(t_1, \ldots, t_n)u(t_1, \ldots, t_n)^{-z_1-\ldots-z_n}|t_1|^{z_1-1} \ldots |t_n|^{z_n-1}dt_1 \ldots dt_n$$

on $\mathbb{R}^n_{>0}$ is invariant with respect to the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n_{>0}$, so defines a differential form $\omega_{z_1,\ldots,z_n}$ on $\mathbb{R}^n_{>0}/\mathbb{R}_{>0}$. Set

$$\mathrm{B}_{\varphi,u}(z_1, \ldots, z_n) \quad := \quad \int_{\mathbb{R}^n_{>0}/\mathbb{R}_{>0}} \omega_{z_1,\ldots,z_n}. \tag{3.1.1}$$

To see that this is well-defined, pick $i \in \{1, \ldots, n\}$, and define the chart

$$\gamma_i : \mathbb{R}^n_{>0}/\mathbb{R}_{>0} \to \mathbb{R}^{n-1}_{>0} : [t_1 : \ldots : t_n] \mapsto \frac{1}{t_i}\left(t_1, \ldots, \hat{t}_i, \ldots, t_n\right),$$

(where the hat means omit). Then

$$\begin{aligned} &\mathrm{B}_{\varphi,u}(z_1, \ldots, z_n) \\ &\quad = \int_{\mathbb{R}^{n-1}_{>0}} \varphi\ u^{-z_1-\ldots-z_n}(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) \prod_{j\neq i} t_j^{z_j-1} dt_1 \ldots \widehat{dt_i} \ldots dt_n, \end{aligned}$$

and the integral converges, since, for $c_1 = \sup\{|\varphi(\mathbf{t})| \mid \mathbf{t} \in \mathbb{R}^n_{\geq 0}\}$ and $c_2 = \min\{u(\mathbf{t}) \mid \mathbf{t} \in \mathbb{R}^n_{\geq 0},\ \sum_{j=1}^n t_j = 1\}$,

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}_{>0}} \left| \varphi\ u^{-z_1-\ldots-z_n}(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) \prod_{j\neq i} t_j^{z_j-1} \right| dt_1 \ldots \widehat{dt_i} \ldots dt_n \\ &\quad \leq \ c_1 c_2^{-\Re(z_1+\ldots+z_n)} \int_{\mathbb{R}^{n-1}_{\geq 0}} (t_1 + \ldots + t_{i-1} + 1 + t_{i+1} + \ldots + t_n)^{-\Re(z_1+\ldots+z_n)} \\ &\qquad\qquad \times \prod_{j\neq i} t_j^{\Re(z_j)-1} dt_1 \ldots \widehat{dt_i} \ldots dt_n \\ &\quad = \ c_1 c_2^{-\Re(z_1+\ldots+z_n)} \frac{\prod_{j=1}^n \Gamma(\Re(z_j))}{\Gamma(\sum_{j=1}^n \Re(z_j))}. \end{aligned}$$

Remark 3.1.1. If $\varphi(\mathbf{t}) = 1$ and $u(\mathbf{t}) = |\mathbf{t}|$, then $\mathrm{B}_{\varphi,u}(z_1, \ldots, z_n) = \frac{\Gamma(z_1)\cdots\Gamma(z_n)}{\Gamma(z_1+\ldots+z_n)}$; in particular, if $n = 2$, $\mathrm{B}_{\varphi,u}(z_1, z_2) = \mathrm{B}(z_1, z_2)$, which is why we call this the beta function of φ and u.

It will be useful to give the following alternative expression for $\mathrm{B}_{\varphi,u}$. For $\mathbf{z} \in \mathbb{C}^n_{>0}$, $k \in \mathbb{Z}_{\geq 0}$, put

$$G^k_{\varphi,u}(\mathbf{z}) := \int_{\mathbb{R}^n_{\geq 0}} \varphi(\mathbf{y}) u(\mathbf{y})^k e^{-u(\mathbf{y})} \mathbf{y}^{\mathbf{z}-\mathbf{1}} d\mathbf{y}. \tag{3.1.2}$$

Then

$$\begin{aligned}
&\Gamma(|\mathbf{z}| + k)\mathrm{B}_{\varphi,u}(\mathbf{z}) \\
&= \int_0^\infty e^{-t} t^{|\mathbf{z}|+k-1} dt \int_{\mathbb{R}^{n-1}_{\geq 0}} \varphi u^{-|\mathbf{z}|}(\mathbf{x}{:}1)(\mathbf{x}{:}1)^{\mathbf{z}-\mathbf{1}}\ d\mathbf{x} \\
&= \int_{\mathbb{R}^{n-1}_{\geq 0}} \int_0^\infty e^{-y_n u(\mathbf{x}:1)} \varphi u^k(\mathbf{x}{:}1)(\mathbf{x}{:}1)^{\mathbf{z}-\mathbf{1}} y_n^{|\mathbf{z}|+k-1}\ dy_n\ d\mathbf{x} \qquad (t = y_n u(\mathbf{x}{:}1)) \\
&= \int_{\mathbb{R}^n_{\geq 0}} e^{-u(\mathbf{y})} \varphi u^k(\mathbf{y}) \mathbf{y}^{\mathbf{z}-\mathbf{1}}\ d\mathbf{y} \qquad (x_i = y_i/y_n \text{ for } i = 1, \ldots, n-1) \\
&= G^k_{\varphi,u}(\mathbf{z}),
\end{aligned}$$

so

$$\mathrm{B}_{\varphi,u}(z_1, \ldots, z_n) = \frac{G^k_{\varphi,u}(z_1, \ldots, z_n)}{\Gamma(z_1 + \ldots + z_n + k)}. \tag{3.1.3}$$

Remark 3.1.2. For $\boldsymbol{\lambda} \in \mathbb{R}^n_{>0}$, define $\delta_{\boldsymbol{\lambda}} : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{x} \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n)$. Then (3.1.2) and (3.1.3) imply

$$\mathrm{B}_{\varphi\circ\delta_{\boldsymbol{\lambda}}, u\circ\delta_{\boldsymbol{\lambda}}}(\mathbf{z}) = \boldsymbol{\lambda}^{-\mathbf{z}} \mathrm{B}_{\varphi,u}(\mathbf{z}). \tag{3.1.4}$$

We write $G_{\varphi,u} = G^0_{\varphi,u}$, and from (3.1.3), it follows that

$$G^k_{\varphi,u}(z_1, \ldots, z_n) = (z_1 + \ldots + z_n)^+_k G_{\varphi,u}(z_1, \ldots, z_n).$$

We are now ready to prove a functional equation between different beta functions.

Proposition 3.1.3. *If* $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{\geq 0})$*, then for each* $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$*, there exist* $\psi_{\mathbf{k},j} \in \mathcal{H}^\infty_0(\mathbb{R}^n_{\geq 0})$*,* $(j = 0, \ldots, |\mathbf{k}|)$*, which satisfy*

$$(\mathbf{z})^+_{\mathbf{k}} \mathrm{B}_{\varphi,u}(\mathbf{z}) \;=\; \sum_{j=0}^{|\mathbf{k}|} |\mathbf{z}|^j \mathrm{B}_{\psi_{\mathbf{k},j},u}(\mathbf{z}+\mathbf{k}), \quad \mathbf{z} \in \mathbb{C}^n_{>0}. \tag{3.1.5}$$

.

Proof. We prove this by induction on $\mathbf{k}$. If $\mathbf{k} = \mathbf{0}$, then (3.1.5) holds with $\psi_{\mathbf{0},0} = \varphi$. If $\mathbf{z} \in \mathbb{C}^n_{>0}$, then by integration by parts,

$$\begin{aligned}
z_j G^1_{\varphi,u}(\mathbf{z}) &= \int_{\mathbb{R}^n_{>0}} \varphi(\mathbf{x}) u(\mathbf{x}) e^{-u(\mathbf{x})} z_j \mathbf{x}^{\mathbf{z}-\mathbf{1}} d\mathbf{x} \\
&= -\int_{\mathbb{R}^n_{>0}} \partial_j \left(\varphi(\mathbf{x}) u(\mathbf{x}) e^{-u(\mathbf{x})}\right) \mathbf{x}^{\mathbf{z}+\mathbf{e}_j-\mathbf{1}} d\mathbf{x} \\
&= -\int_{\mathbb{R}^n_{>0}} \left(\partial_j(\varphi u) - \varphi u \partial_j u\right)(\mathbf{x}) e^{-u(\mathbf{x})} \mathbf{x}^{\mathbf{z}+\mathbf{e}_j-\mathbf{1}} d\mathbf{x} \\
&= -G^0_{\partial_j(\varphi u),u}(\mathbf{z}+\mathbf{e}_j) + G^1_{\varphi\partial_j u,u}(\mathbf{z}+\mathbf{e}_j).
\end{aligned}$$

After dividing by $\Gamma(|\mathbf{z}|+1)$, we obtain, from (3.1.3),

$$\begin{aligned}
z_j \mathrm{B}_{\varphi,u}(\mathbf{z}) &= -\mathrm{B}_{\partial_j(\varphi u),u}(\mathbf{z}+\mathbf{e}_j) + (|\mathbf{z}|+1)\mathrm{B}_{\varphi\partial_j u,u}(\mathbf{z}+\mathbf{e}_j) \\
&= |\mathbf{z}|\mathrm{B}_{\varphi\partial_j u,u}(\mathbf{z}+\mathbf{e}_j) - \mathrm{B}_{u\partial_j\varphi,u}(\mathbf{z}+\mathbf{e}_j). \qquad (3.1.6)
\end{aligned}$$

If (3.1.5) holds for $\mathbf{k}$, then for $\mathbf{z} \in \mathbb{C}^n_{>0}$,

$$\begin{aligned}(\mathbf{z})^+_{\mathbf{k}+\mathbf{e}_j} \mathrm{B}_{\varphi,u}(\mathbf{z}) &= \sum_{\ell=0}^{|\mathbf{k}|} |\mathbf{z}|^\ell (s_j + k_j) \mathrm{B}_{\psi_{\mathbf{k},\ell},u}(\mathbf{z}+\mathbf{k}) \\ &= \sum_{\ell=0}^{|\mathbf{k}|} |\mathbf{z}|^\ell \left(|\mathbf{z}+\mathbf{k}| \mathrm{B}_{\psi_{\mathbf{k},\ell}\partial_j u,u}(\mathbf{z}+\mathbf{k}+\mathbf{e}_j) - \mathrm{B}_{u\partial_j \psi_{\mathbf{k},\ell},u}(\mathbf{z}+\mathbf{k}+\mathbf{e}_j) \right) \\ &= \sum_{\ell=0}^{|\mathbf{k}+\mathbf{e}_j|} |\mathbf{z}|^\ell \mathrm{B}_{\psi_{\mathbf{k}+\mathbf{e}_j,\ell},u}(\mathbf{z}+\mathbf{k}+\mathbf{e}_j),\end{aligned}$$

where

$$\psi_{\mathbf{k}+\mathbf{e}_j,\ell} \quad := \quad (\psi_{\mathbf{k},\ell-1} + |\mathbf{k}|\psi_{\mathbf{k},\ell})\partial_j u - u\partial_j \psi_{\mathbf{k},\ell} \tag{3.1.7}$$

(and we define $\psi_{\mathbf{k},-1} = \psi_{\mathbf{k},|\mathbf{k}|+1} = 0$). □

Remark 3.1.4. A simple inductive proof shows that the functions $\psi_{\mathbf{k},\ell}$ are characterised by

$$\sum_{\ell=0}^{|\mathbf{k}|} s^\ell \psi_{\mathbf{k},\ell} \quad = \quad (-1)^{|\mathbf{k}|} u^{|\mathbf{k}|+s} \partial^{\mathbf{k}}(\varphi u^{-s}). \tag{3.1.8}$$

3.2 The integral representation

We will denote the pointwise product of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ by $\mathbf{x} \odot \mathbf{y} = (x_1 y_1, \ldots, x_n y_n)$. For $\boldsymbol{\tau} \in \mathbb{R}^n_{>0}$, let

$$\theta(\boldsymbol{\tau}) = \theta_{\varphi,u}(K; \boldsymbol{\tau}) := \sum_{\alpha \in \mathcal{A}} c_\alpha \varphi(\boldsymbol{\tau} \odot \mathbf{m}_\alpha) e^{-u(\boldsymbol{\tau} \odot \mathbf{m}_\alpha)},$$

which converges absolutely since

$$e^{-u(\boldsymbol{\tau} \odot \mathbf{m}_\alpha)} \leq e^{-C|\boldsymbol{\tau} \odot \mathbf{m}_\alpha|} = O_{\boldsymbol{\tau}}(\mathbf{m}_\alpha^{-2N_K \mathbf{1}}).$$

(Here we use the fact that for any $\lambda > 0$, $e^{-cm} = O_{c,\lambda}(m^{-\lambda})$ for $m > 0$). Then

$$\Theta_{\varphi,u}(K;t) := \theta_{\varphi,u}(K;t,\ldots,t) = \sum_{\alpha\in\mathcal{A}} c_\alpha \varphi(\mathbf{m}_\alpha) e^{-tu(\mathbf{m}_\alpha)}, \quad t > 0$$

is the exponential series corresponding to $\zeta_{\varphi,u}(K;s)$, so that

$$\Gamma(s)\zeta_{\varphi,u}(K;s) = \int_0^\infty \Theta_{\varphi,u}(K;t)t^{s-1}dt.$$

Proposition 3.2.1. *Suppose K is as above, and $(\varphi,u) \in \mathscr{Z}^\infty(\mathbb{R}^n_{\geq 0})$. If $c > N_K$, then for $\Re(s) > nc$,*

$$\begin{aligned} \zeta_{\varphi,u}(K;s) \;\; &= \;\; \int_{(c)} \cdots \int_{(c)} K(z_1,\ldots,z_{n-1}, s - z_1 - \ldots - z_{n-1}) \\ &\qquad \times \mathrm{B}_{\varphi,u}(z_1,\ldots,z_{n-1}, s - z_1 - \ldots - z_{n-1}) dz_1 \ldots dz_{n-1}. \end{aligned} \tag{3.2.1}$$

Proof. Suppose $\mathbf{z} \in \mathbb{C}^n_{>N_K}$. We take the n-fold Mellin transform of $\theta(\tau_1,\ldots,\tau_n)$ with respect to $\tau_1,\ldots,\tau_n$, switch the order of integration and summation, and then use the change of variable $\mathbf{x} = \boldsymbol{\tau} \odot \mathbf{m}_\alpha$:

$$\begin{aligned} \int_{\mathbb{R}^n_{\geq 0}} \theta(\boldsymbol{\tau})\boldsymbol{\tau}^{\mathbf{z}-1} d\boldsymbol{\tau} \;\; &= \;\; \sum_{\alpha\in\mathcal{A}} c_\alpha \int_{\mathbb{R}^n_{\geq 0}} \varphi(\boldsymbol{\tau} \odot \mathbf{m}_\alpha) e^{-u(\boldsymbol{\tau}\odot\mathbf{m}_\alpha)} \boldsymbol{\tau}^{\mathbf{z}-1} d\boldsymbol{\tau} \\ &= \;\; \sum_{\alpha\in\mathcal{A}} c_\alpha \mathbf{m}_\alpha^{-\mathbf{z}} \int_{\mathbb{R}^n_{\geq 0}} \varphi(\mathbf{x}) e^{-u(\mathbf{x})} \mathbf{x}^{\mathbf{z}-1} d\mathbf{x} \\ &= \;\; K(\mathbf{z}) G_{\varphi,u}(\mathbf{z}) \;\; = \;\; K(\mathbf{z})\Gamma(|\mathbf{z}|)\mathrm{B}_{\varphi,u}(\mathbf{z}). \end{aligned} \tag{3.2.2}$$

We now take the n-fold inverse Mellin transform of (3.2.2), and set $\tau_1 = \ldots = \tau_n =$

$t > 0$. If $c > N_K$,

$$\begin{aligned}
\Theta_{\varphi,u}(t) &= \theta(t,\ldots,t) \\
&= \int_{(c)}\ldots\int_{(c)} t^{-z_1}\ldots t^{-z_n} K(z_1,\ldots,z_n) \\
&\qquad \times\Gamma(z_1+\ldots+z_n)\mathrm{B}_{\varphi,u}(z_1,\ldots,z_n)dz_1\ldots dz_n \\
&= \int_{(nc)} t^{-z}\Gamma(z)\int_{(c)}\ldots\int_{(c)} K(z_1,\ldots,z_{n-1},z-z_1-\ldots-z_{n-1}) \\
&\qquad \times\mathrm{B}_{\varphi,u}(z_1,\ldots,z_{n-1},z-z_1-\ldots-z_{n-1})dz_1\ldots dz_{n-1}\ dz.
\end{aligned}$$

Therefore, for $\Re(s) > nc$,

$$\begin{aligned}
\Gamma(s)\zeta_{\varphi,u}(K;s) &= \int_0^\infty \Theta_{\varphi,u}(t)t^{s-1}dt \\
&= \Gamma(s)\int_{(c)}\ldots\int_{(c)} K(z_1,\ldots,z_{n-1},s-z_1-\ldots-z_{n-1}) \\
&\qquad \times\mathrm{B}_{\varphi,u}(z_1,\ldots,z_{n-1},s-z_1-\ldots-z_{n-1})dz_1\ldots dz_{n-1},
\end{aligned}$$

which proves the proposition. □

Remark 3.2.2. This implies

$$\begin{aligned}
\zeta_{\varphi\circ\delta_{\mathbf{x}:1},u\circ\delta_{\mathbf{x}:1}}(K;s) &= \int_{(\mathbf{c})} K(\mathbf{z},s-\mathbf{z})\mathrm{B}_{\varphi\circ\delta_{\mathbf{x}:1},u\circ\delta_{\mathbf{x}:1}}(\mathbf{z},s-\mathbf{z})d\mathbf{z} \\
&= \int_{(\mathbf{c})} K(\mathbf{z},s-\mathbf{z})\mathrm{B}_{\varphi,u}(\mathbf{z},s-\mathbf{z})(\mathbf{x}\colon 1)^{-\mathbf{z}}d\mathbf{z} \qquad \text{(by (3.1.4))}.
\end{aligned}$$

By applying the $n-1$ fold Mellin transform, we see that

$$K(\mathbf{z},s-\mathbf{z})\mathrm{B}_{\varphi,u}(\mathbf{z},s-\mathbf{z}) = \int_{\mathbb{R}_{\geq 0}^{n-1}} \zeta_{\varphi\circ\delta_{\mathbf{x}:1},u\circ\delta_{\mathbf{x}:1}}(K;s)(\mathbf{x}\colon 1)^{\mathbf{z}-\mathbf{1}}d\mathbf{x},$$

so for $\mathbf{z} \in \mathbb{C}^n_{>N_K}$,

$$K(\mathbf{z})\mathrm{B}_{\varphi,u}(\mathbf{z}) = \int_{\mathbb{R}_{\geq 0}^{n-1}} \zeta_{\varphi\circ\delta_{\mathbf{x}:1},u\circ\delta_{\mathbf{x}:1}}(K;|\mathbf{z}|)(\mathbf{x}\colon 1)^{\mathbf{z}-\mathbf{1}}d\mathbf{x}.$$

In section 3.4, we will show that if $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{\geq 0})$, then under certain conditions on K, $\zeta_{\varphi,u}(K;s)$ extends to a meromorphic function on a larger region (theorem 3.4.1). If K satisfies stronger conditions, $\zeta_{\varphi,u}(K;s)$ extends to a meromorphic function on $\mathbb{C}$ (theorem 3.4.2). We will see later that the hypotheses on K and u can be weakened.

Given what we know about $\mathrm{B}_{\varphi,u}$, the proofs of theorems 3.4.1 and 3.4.2 will be fairly immediate consequences of some general lemmas (3.3.3 and 3.3.5), the proofs of which are a bit technical.

3.3 Some general lemmas

We start with a preliminary lemma.

Lemma 3.3.1. *Let $I_1, \ldots, I_n \subset \mathbb{R}$ be compact intervals. If $I_1 = [a_1, b_1]$, let $I_1(\delta) = [a_1 + \delta, b_1 - \delta]$, for $\delta > 0$. Suppose $J(s_1, \ldots, s_n)$ is an analytic function on a neighbourhood of $\mathcal{E}_{I_1,\ldots,I_n} := (\prod_{j=1}^n I_j)_\mathbb{C}$ which satisfies the growth condition*

$$J(s_1, \ldots, s_n) = O_\lambda\left((1 + |\Im(s_n)|)^{f(\lambda)} \prod_{i=1}^{n-1}(1 + |\Im(s_i)|)^{-\lambda}\right), \quad \forall \lambda > 0,$$

on $\mathcal{E}_{I_1,\ldots,I_n}$ for some function $f : \mathbb{R}_{>0} \to \mathbb{R}$. Then for $k \in \mathbb{Z}_{>0}$ and $\delta, \lambda > 0$,

$$\partial^k_{s_1} J(s_1, \ldots, s_n) \quad = \quad O_{\delta,\lambda}\left((1 + |\Im(s_n)|)^{f(\lambda)} \prod_{i=2}^{n-1}(1 + |\Im(s_i)|)^{-\lambda}\right), \qquad (3.3.1)$$

on $\mathcal{E}_{I_1(\delta),I_2,\ldots,I_n}$.

Proof. We use Cauchy's formula to estimate the derivative. For $T > 0$, let C_T be the rectangular contour which bounds the region $\Re(s) \in I_1$, $|\Im(s)| \leq T$. If

$\Re(s_1) \in I_1(\delta)$ and $\Re(\xi) \in \partial I_1$, then

$$|\xi - s_1| \geq \frac{1}{\sqrt{2}}(|\Re(\xi - s_1)| + |\Im(\xi - s_1)|) \geq \frac{1}{\sqrt{2}}(\delta + |\Im(\xi) - \Im(s_1)|).$$

If $(s_1, \ldots, s_n) \in \mathcal{E}_{I_1(\delta), I_2, \ldots, I_n}$, then for $T > |\Im(s_1)| + 1$, we can use Cauchy's derivative formula with the contour C_T to estimate

$$\begin{aligned}
|\partial_{s_1}^k J(s_1, \ldots, s_n)| &= \left| \frac{(-1)^k k!}{2\pi i} \int_{C_T} \frac{J(\xi, s_2, \ldots, s_n)}{(\xi - s_1)^{k+1}} d\xi \right| \\
&\ll_\lambda (1 + |\Im(s_n)|)^{f(\lambda)} \prod_{i=2}^{n-1} (1 + |\Im(s_i)|)^{-\lambda} \\
&\quad \times \left[\int_{-T}^{T} \frac{(1+|t|)^{-\lambda}}{(\delta + |t - \Im(s_1)|)^{k+1}} dt + \int_{a_1}^{b_1} (1+T)^{-\lambda} dt \right].
\end{aligned}$$

In the limit $T \to \infty$, the term in square parentheses is equal to

$$\int_{-\infty}^{\infty} \frac{(1+|t|)^{-\lambda}}{(\delta + |t - \tau|)^{k+1}} dt = \int_{-\infty}^{\infty} \frac{(1+|t+\tau|)^{-\lambda}}{(\delta + |t|)^{k+1}} dt \leq \int_{-\infty}^{\infty} \frac{1}{(\delta + |t|)^{k+1}} dt = \frac{2\delta^{-k}}{k}.$$

□

If $L(\mathbf{s})$ is a product of degree 1 real polynomials:

$$L(\mathbf{s}) = \prod_{i=1}^{m} L_i(\mathbf{s})^{n_i}, \quad L_i(\mathbf{s}) = \sum_{j=1}^{n} a_{i,j} s_j - b_i, \tag{3.3.2}$$

let $\mathcal{I}_L$ be the set of all affine subspaces of $\mathbb{R}^n$ obtained by intersecting a non-empty subset of the real hyperplanes $H_i = \{\mathbf{x} \in \mathbb{R}^n \mid L_i(\mathbf{x}) = 0\}$.

Let $J(\mathbf{s})$ be a meromorphic function defined on $\Omega_{\mathbb{C}} \subseteq \mathbb{C}^n$, where $\Omega \subseteq \mathbb{R}^n$ is open and convex. We call $\Omega_{\mathbb{C}}$ a *tube domain*. Suppose J satisfies the following hypotheses:

Hypotheses 2.

- *There exists a polynomial* $L(\mathbf{z})$ *as in (3.3.2), such that*

  (a) $\tilde{J}(\mathbf{z}) := L(\mathbf{z})J(\mathbf{z})$ *is analytic on* $\Omega_{\mathbb{C}}$,

  (b) *For every* $\mathbf{x} \in \Omega$, *there exists a compact neighbourhood,* $D \subset \Omega$, *and a function* $f : \mathbb{R}_{>0} \to \mathbb{R}$ *such that for all* $\lambda > 0$,

$$\tilde{J}(\mathbf{z}) \;=\; O_{D,\lambda}\left((1 + |\Im(z_n)|)^{f(\lambda)} \prod_{i=1}^{n-1}(1 + |\Im(z_i)|)^{-\lambda}\right), \quad \mathbf{z} \in D_{\mathbb{C}}. \quad (3.3.3)$$

Remark 3.3.2.

(i) If **(a)** holds, then up to multiplication by elements of $\mathbb{R}^{\times}$, there exists a unique minimal[2] real polynomial, L, such that $L(\mathbf{s})J(\mathbf{z})$ is analytic on $\Omega_{\mathbb{C}}$. We call this the *denominator* of J.

(ii) Condition **(b)** is equivalent to being able to cover Ω with open sets D such that (3.3.3) holds for some $f = f_D$.

(iii) The truth of **(b)** does not depend on the choice of L in **(a)**.

(iv) We may replace $\Im(z)$ by z in (3.3.3), since if $\Re(z)$ is bounded, $1 + |z| \ll 1 + |\Im(z)| \ll 1 + |z|$.

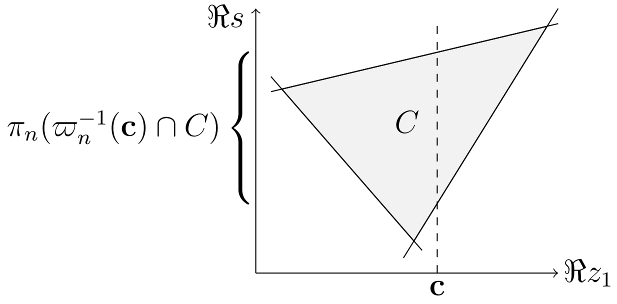
For $i \in [n]$, let $\pi_i : \Omega \to \mathbb{R}$ be the projection onto the i-th coordinate, and $\varpi_i : \Omega \to \mathbb{R}^{n-1}$ the projection onto the $n-1$ coordinates other than i. Let $\mathcal{C}_L$ be the set of connected components of $\{\mathbf{x} \in \Omega \mid L(\mathbf{x}) \neq 0\}$. For $C \in \mathcal{C}_L$ and $\mathbf{c} \in \varpi_n(C)$,

---

[2] Ordered by divisibility.

we may define an analytic function $\mathcal{F}_{\mathbf{c}}$ on $\pi_n(\varpi_n^{-1}(\mathbf{c}) \cap C)_{\mathbb{C}} \subset \mathbb{C}$ by

$$\mathcal{F}_{\mathbf{c}}(s) \;=\; \int_{(c_1)} \cdots \int_{(c_{n-1})} J(z_1, \ldots, z_{n-1}, s) dz_{n-1} \ldots dz_1. \tag{3.3.4}$$



The residue theorem implies that for $\mathbf{c}, \mathbf{c}' \in \varpi_n(C)$, the functions $\mathcal{F}_{\mathbf{c}}$ and $\mathcal{F}_{\mathbf{c}'}$ agree on the intersection of their domains, so this defines an analytic function on $\pi_n(C)_{\mathbb{C}}$, which we call $\mathcal{F}_C$.

We will call an affine subspace in $\mathbb{R}^n$ *horizontal* if A is perpendicular to $\mathbf{e}_n$. If $F \subset \mathbb{R}^n$ is a polyhedron, we say F is horizontal if the smallest affine subspace containing F is horizontal.

Let $\lambda_{\inf} = \inf \pi_n(C) \in \mathbb{R} \cup \{-\infty\}$, and define $C_{\inf} = \{\mathbf{x} \in \bar{C} \mid x_n = \lambda_{\inf}\}$. Define $\lambda_{\sup}$ and $C_{\sup}$ analogously, and note that the domain of $\mathcal{F}_C$ is

$$\{s \in \mathbb{C} \mid \lambda_{\inf} < \Re(s) < \lambda_{\sup}\}.$$

Lemma 3.3.3. *Suppose J satisfies hypotheses 2. With the notation above, for any $C \in \mathcal{C}_L$ with dist$(\mathbf{d}, \mathbb{R}^n \smallsetminus \Omega) > 0$ for some $\mathbf{d} \in C_{\inf}$, there exists $\epsilon > 0$ such that $\mathcal{F}_C$*

extends to a meromorphic function $\widehat{\mathcal{F}}_C$ *on*

$$\{s \in \mathbb{C} \mid \lambda_{\text{inf}} - \epsilon < \Re(s) < \lambda_{\text{sup}}\}.$$

Let $I \subset (\lambda_{\text{inf}} - \epsilon, \lambda_{\text{sup}})$ *be a compact interval. Then there exists* $\mu > 0$ *such that*

$$\widehat{\mathcal{F}}_C(\sigma + it) \;=\; O(|t|^{\mu}) \text{ for } \sigma \in I,\ |t| \geq 1. \tag{3.3.5}$$

Remark 3.3.4. By decreasing ϵ if necessary, we may assume that the meromorphic function has at most one pole, at $s = \lambda_{\text{inf}}$. By symmetry, we can extend the domain of $\mathcal{F}_C$ beyond $\Re(s) = \lambda_{\text{sup}}$ if there exists $\mathbf{d} \in C_{\text{sup}}$ with $\text{dist}(\mathbf{d}, \mathbb{R}^n \smallsetminus \Omega) > 0$.

Proof. We need to show that we can define a meromorphic function on a tube neighbourhood of λ_{inf} which coincides with $\mathcal{F}_C$ on the intersection of their domains. The proof will be by induction on the dimension, n. For $n = 1$, the conclusion holds by assumption. Suppose the lemma holds for $n - 1$. If $C_{\text{inf}} = \varnothing$ there is nothing to prove, so suppose $C_{\text{inf}} \neq \varnothing$. We may also assume that no factor $L_k(\mathbf{s})$ of $L(\mathbf{s})$ is of the form $a_{k,n}s + b_k$, else this can be taken outside the integral in (3.3.4). In other words, we may assume none of the hyperplanes H_k are horizontal.

Choose a point $\mathbf{d} \in C_{\text{inf}}$ such that $\text{dist}(\mathbf{d}, \mathbb{R}^n \smallsetminus \Omega) > 0$, and let I be the set of indices of those hyperplanes H_k which intersect $\mathbf{d}$. Shrink Ω so that $\mathbf{d} \in \Omega_{\mathbb{R}}$ and $H_k \cap \Omega_{\mathbb{R}} = \varnothing$ iff $k \notin I$. This may shrink the domain of $\mathcal{F}_C$, but it still has $\Re(s) = \lambda_{\text{inf}}$ as its lower boundary. Pick a point

$$\mathbf{c} \in \varpi_n(C) \smallsetminus \bigcup_{A \in \mathcal{I}_L, \operatorname{codim} A \geq 2} \varpi_n(A) \tag{3.3.6}$$

such that $\lambda_{\inf} \in \pi_n(\varpi_n^{-1}(\mathbf{c}))$. Then for some $t_0 \in \mathbb{R}$, $(\mathbf{c}, t_0) \in C$. Consider the point $(\mathbf{c}, t)$ as t varies within the interval $\pi_n(\varpi_n^{-1}(\mathbf{c}))$. As t decreases, starting from t_0, label the connected components the point travels through with increasing indices, $0, 1, \ldots$, so that $t' < t$ for all $(\mathbf{c}, t) \in C_k$, $(\mathbf{c}, t') \in C_{k+1}$. Let r be the smallest index for which $\lambda_{\inf} \in \pi_n(C_r)$.

The condition (3.3.6) ensures that for $k = 0, \ldots, r-1$ the polytopes $\bar{C}_k$ and $\bar{C}_{k+1}$ have a common facet, F_k, contained in a hyperplane, which we may take to be H_k (after relabelling, if necessary). The domains of $\mathcal{F}_{C_k}$ and $\mathcal{F}_{C_{k+1}}$ have intersection equal to $\pi_n(C_k) \cap \pi_n(C_{k+1}) = \operatorname{int} \pi_n(F_k)$, which is non-empty, since H_k is not horizontal. Recall that $H_k = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{j=1}^n a_{k,j} x_j = b_k\}$, where we may assume $a_{k,1} \neq 0$. Thus, if $\Re(s) \in \operatorname{int} \pi_n(F_k)$, the residue theorem gives

$$\mathcal{F}_{C_k}(s) - \mathcal{F}_{C_{k+1}}(s) \;=\; \int_{(c_{n-1})} \cdots \int_{(c_2)} J_k(z_2, \ldots, z_{n-1}, s) dz_2 \ldots dz_{n-1}, \qquad (3.3.7)$$

where $(c_1, \ldots, c_{n-1}) \in \varpi_n(\pi_n^{-1}(\Re(s)) \cap \operatorname{relint} F_k)$, and

$$J_k(z_2, \ldots, z_{n-1}, s) := \operatorname{Res}_{z_1 = a_{k,n-1}^{-1}(b_k - \sum_{j=2}^{n-1} a_{k,j} z_j - a_{k,n} s)} J(z_1, \ldots, z_{n-1}, s),$$

defined on $\varpi_1(\Omega \cap H_k)$. We claim that J_k satisfies hypotheses 2.

If this is true, then by induction, the right hand side of (3.3.7) extends to a meromorphic function ϕ_k on

$$\{s \in \mathbb{C} \mid \lambda_{\inf} - \epsilon_k < \Re(s) < \sup \pi_n(F_k)\}$$

for some $\epsilon_k > 0$. Thus $\mathcal{F}_C = \mathcal{F}_{C_0} = \sum_{k=0}^{r-1} \phi_k + \mathcal{F}_{C_r}$ on the intersection of their domains, and since the right hand side is defined on a tube neighbourhood of $\lambda_{\inf}$, the lemma will follow.

To see that J_k satisfies hypotheses 2, let

$$\theta_k = \theta_k(\mathbf{z}, s) = a_{k,1}^{-1}(b_k - \sum_{j=2}^{n-1} a_{k,j} z_j - a_{k,n} s),$$

and write $L(\mathbf{z}, s) = L_k(\mathbf{z}, s)^{n_k} \not{L}_k(\mathbf{z}, s) = a_{k,1}^{n_k}(z_1 - \theta_k)^{n_k} \not{L}_k(\mathbf{z}, s)$, so that

$$\begin{aligned} J_k(z_2, \ldots, z_{n-1}, s) &= \mathrm{Res}_{z_1=\theta_k} J(\mathbf{z}, s) \\ &= a_{k,1}^{-n_k} \mathrm{Res}_{z_1=\theta_k} \frac{\not{L}_k(\mathbf{z}, s)^{-1} \tilde{J}(\mathbf{z}, s)}{(z_1 - \theta_k)^{n_k}} \\ &= \frac{a_{k,1}^{-n_k}}{(n_k-1)!} \partial_{z_1}^{n_k-1} \left[\not{L}_k(\mathbf{z}, s)^{-1} \tilde{J}(\mathbf{z}, s) \right] \Big|_{z_1=\theta_k} \\ &= \frac{a_{k,1}^{-n_k}}{(n_k-1)! \not{L}_k(\mathbf{z}, s)^{n_k}} \sum_{j=0}^{n_k-1} P_{k,j}(\mathbf{z}, s) \partial_{z_1}^{j} \tilde{J}(\mathbf{z}, s) \Bigg|_{z_1=\theta_k} \end{aligned}$$

for some polynomials $P_{k,j}$. Note that condition **(b)** of hypotheses 2 implies that $\tilde{J}$ satisfies the hypotheses of lemma 3.3.1, so for all $\lambda > 0$, $j = 0, \ldots, n_k - 1$, we have $\partial_{z_1}^{j} \tilde{J}(\mathbf{z}, s) = O_{\lambda,j}((1 + |\Im(s)|)^{f(\lambda)} \prod_{i=2}^{n-1}(1 + |\Im(z_i)|)^{-\lambda})$ when $z_1, \ldots, z_{n-1}, s$ have real parts restricted to compact intervals.

In this region, $P_{k,j}|_{z_1=\theta_k}(\mathbf{z}, s) \ll (1 + |\Im(s)|)^{\deg P_{k,j}} \prod_{i=2}^{n-1}(1 + |\Im(z_i)|)^{\deg P_{k,j}}$, so if $d = \max \deg P_{k,j}$, then

$$\begin{aligned} &\not{L}_k(\mathbf{z}, s)^{n_k}|_{z_1=\theta_k} J_k(z_2, \ldots, z_{n-1}, s) \\ &\quad \ll \quad (1 + |\Im(s)|)^d \prod_{i=2}^{n-1}(1 + |\Im(z_i)|)^d \times (1 + |\Im(s)|)^{f(\lambda)} \prod_{i=2}^{n-1}(1 + |\Im(z_i)|)^{-\lambda} \\ &\quad \ll \quad (1 + |\Im(s)|)^{d+f(\lambda)} \prod_{i=2}^{n-1}(1 + |\Im(z_i)|)^{d-\lambda}, \end{aligned}$$

which shows that J_k satisfies hypotheses 2.

It remains to prove (3.3.5). By (3.3.7) and induction, it is enough to show that (3.3.5) holds for $I \subset \pi_n(C)$ equal to a compact neighbourhood of each point $x \in \pi_n(C)$. Pick $\mathbf{c} \in \varpi_n(\pi_n^{-1}(x))$, and let I be a compact neighbourhood of x contained in $\pi_n(\varpi_n^{-1}(\mathbf{c}))$. Then $D = \{\mathbf{c}\} \times I \subset C$ is compact, so if $\Re(s) \in I$, then for any $\lambda > 1$,

$$\begin{aligned}\int_{(\mathbf{c})} |J(\mathbf{z}, s)| d\mathbf{z} &\ll \int_{(\mathbf{c})} |\tilde{J}(\mathbf{z}, s)| d\mathbf{z} \\ &\ll_{D,\lambda} (1 + |\Im(s)|)^{f(\lambda)} \int_{(\mathbf{c})} \prod_{i=1}^{n-1} (1 + |\Im(z_i)|)^{-\lambda} d\mathbf{z} \ll_{\lambda} (1 + |\Im(s)|)^{f(\lambda)}.\end{aligned}$$

□

The previous lemma was concerned with showing that the domain of a function can be made strictly larger, with no other conditions on the size of the new domain. The next lemma deals with the opposite extreme.

Lemma 3.3.5. *Let J be a meromorphic function on $\mathbb{C}^n$, and suppose there exists a sequence of open tube domains $(\Omega_1)_{\mathbb{C}} \subset (\Omega_2)_{\mathbb{C}} \subset \ldots \subseteq \mathbb{C}^n$ whose union is $\mathbb{C}^n$, such that $J|_{(\Omega_j)_{\mathbb{C}}}$ satisfies hypotheses 2 for each $j = 1, 2, \ldots$.*

Let L be the denominator of $J|_{(\Omega_1)_{\mathbb{C}}}$, and for $C \in \mathcal{C}_L$, define $\mathcal{F}_C$ as before. Then $\mathcal{F}_C$ extends to a meromorphic function on $\mathbb{C}$, with poles contained in $\bigcup_{A \in \mathcal{H}_L} \pi_n(A)$, where $\mathcal{H}_L$ is the set of horizontal affine subspaces in $\mathcal{I}_L$.

If $I \subset \mathbb{R}$ is a compact interval, then there exists $\mu > 0$ such that

$$\widehat{\mathcal{F}}_C(\sigma + it) = O(1 + |t|^{\mu}) \text{ for } \sigma \in I,\ |t| \geq 1. \tag{3.3.8}$$

Proof. The proof is mostly the same as that of lemma 3.3.3. For j fixed, pick

$$\mathbf{c} \in \varpi_n(C) \setminus \bigcup_{A \in \mathcal{I}_{L_j}, \operatorname{codim} A \geq 2} \varpi_n(A),$$

where L_j is the denominator of $J|_{(\Omega_j)_{\mathbb{C}}}$. Thus we can find a sequence $C_0 = C, C_1, \ldots$ $\ldots, C_r$ as before, but where now $\inf \pi_n(C_r) = \inf \pi_n(\Omega_j \cap \varpi_n^{-1}(\mathbf{c}))$. By induction, we may assume that $\mathcal{F}_{C_i} - \mathcal{F}_{C_{i+1}} = \phi_i$ has a meromorphic extension to $\mathbb{C}$ with the required properties, so the function

$$\widehat{\mathcal{F}_C}(s) = \sum_{i=0}^{k-1} \phi_i(s) + \mathcal{F}_{C_k}(s), \quad \Re(s) \in \pi_n(C_k)$$

is a well-defined meromorphic function on $\pi_n(\cup_{k=0}^{r} C_k)$ with the required properties, and which agrees with $\mathcal{F}_C$ on $\pi_n(C)$.

Since we can do this for each j, we obtain the desired meromorphic continuation of $\mathcal{F}_C$. □

3.4 Meromorphic continuation of $\zeta_{\varphi,u}(K;s)$

Now consider the following hypotheses on a meromorphic function, K, defined on a tube domain $\Omega_{\mathbb{C}} \subseteq \mathbb{C}^n$:

Hypotheses 3.

- *There exists a non-zero polynomial L as in (3.3.2) such that*

  (a) $\tilde{K}(\mathbf{s}) := L(\mathbf{s})K(\mathbf{s})$ *is analytic on* $\Omega_{\mathbb{C}}$*,*

(b) *Ω can be covered by compact subsets D, for which there exist $\boldsymbol{\lambda} = \boldsymbol{\lambda}_D \in \mathbb{R}^n_{>0}$ such that*

$$\tilde{K}(\mathbf{s}) = O_D\left(\prod_{i=1}^{n}(1+|\Im(s_i)|)^{\lambda_i}\right) \quad \textit{for } \mathbf{s} \in D_{\mathbb{C}}.$$

If $K(\mathbf{s}) = K(\mathbf{m}, c, \mathbf{s})$ extends to a meromorphic function on $\Omega_{\mathbb{C}}$ which satisfies hypotheses 3, with denominator L, let Σ_K be the connected component of $(N_K+1)\mathbf{1}$ in $\{\mathbf{x} \in \Omega \cap \mathbb{R}^n_{>0} \mid L(\mathbf{x}) \neq 0\}$, and let $\rho = \inf\{|\mathbf{x}| \mid \mathbf{x} \in \Sigma_K\}$.

Theorem 3.4.1. *Suppose $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{\geq 0})$. With the hypotheses on K above, $\zeta_{\varphi,u}(K;s)$ extends to an analytic function on $\Re(s) > \rho$. If, in addition, there exists a point $\mathbf{x} \in \Omega$ in the boundary of Σ_K with $|\mathbf{x}| = \rho$, then $\zeta_{\varphi,u}(K;s)$ extends to a meromorphic function on $\Re(s) > \rho - \epsilon$, for some $\epsilon > 0$.*

In either case, for each compact interval I in the extended domain of $\zeta_{\varphi,u}(K;s)$, there exists $\mu > 0$ such that

$$\zeta_{\varphi,u}(K;s) \ll |\Im(s)|^\mu \quad \textit{for } \Re(s) \in I \textit{ and } \Im(s) \geq 1.$$

Proof. The theorem will follow from the integral representation 3.2.1 and lemma 3.3.3 once we show that $J(\mathbf{z}, s) := \mathrm{B}_{\varphi,u}(\mathbf{z}, s - |\mathbf{z}|)$ satisfies hypotheses 2 with $\hat{\Omega} = \{(\mathbf{z}, s) \in \mathbb{C}^n \mid (\mathbf{z}, s - |\mathbf{z}|) \in \Omega\}$. Recall that we showed in (3.1.5) that for $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$,

$$(\mathbf{z})^+_{\mathbf{k}}\mathrm{B}_{\varphi,u}(\mathbf{z}) \;=\; \sum_{j=0}^{|\mathbf{k}|} |\mathbf{z}|^j \mathrm{B}_{\psi_{\mathbf{k},j},u}(\mathbf{z}+\mathbf{k}), \quad \mathbf{z} \in \mathbb{C}^n_{>0}.$$

By analytic continuation, this holds for $\mathbf{z} \in \prod_{j=1}^n \mathbb{C}_{>-k_j}$.

Note that we may restrict Ω so that $\Omega \subseteq \mathbb{R}^n_{>-p+1}$ for some $p \in \mathbb{Z}_{\geq 0}$. We claim that hypotheses 2 are satisfied with $L(\mathbf{z}, s) = (\mathbf{z}, s - |\mathbf{z}|)^+_{p\mathbf{1}}$. Indeed, for $\mathbf{z} \in \Omega_{\mathbb{C}}$,

$(\mathbf{z})^+_{p\mathbf{1}}\mathrm{B}_{\varphi,u}(\mathbf{z})$ is analytic, and for $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{z} \in D_{\mathbb{C}}$ with D compact,

$$\begin{aligned}(\mathbf{z})^+_{p\mathbf{1}}\mathrm{B}_{\varphi,u}(\mathbf{z}) &= \frac{1}{(\mathbf{z}+p\mathbf{1})^+_{\mathbf{k}}} \sum_{j=0}^{|p\mathbf{1}+\mathbf{k}|} |\mathbf{z}|^j \mathrm{B}_{\psi_{p\mathbf{1}+\mathbf{k},j},u}(\mathbf{z}+p\mathbf{1}+\mathbf{k}) \\ &\ll_{D,\mathbf{k}} (1+|\mathbf{z}|)^{|p\mathbf{1}+\mathbf{k}|} \prod_{j=1}^{n} (1+|\Im(z_j)|)^{-k_j}.\end{aligned}$$

Therefore

$$L(\mathbf{z},s)J(\mathbf{z},s) \ll_{D,\mathbf{k}} (1+|s|)^{|p\mathbf{1}+\mathbf{k}|} \prod_{j=1}^{n-1} (1+|\Im(z_j)|)^{-k_j}$$

□

With stronger hypotheses, we can ensure that $\zeta_{\varphi,u}(K;s)$ extends to a meromorphic function on $\mathbb{C}$. Let K be a meromorphic function on $\mathbb{C}^n$.

Hypotheses 4. *There exists a sequence of tube domains* $(\Omega_1)_{\mathbb{C}} \subset (\Omega_2)_{\mathbb{C}} \subset \ldots \subseteq \mathbb{C}^n$ *whose union is* $\mathbb{C}^n$*, and such that for each* $j \in \mathbb{Z}_{>0}$*,* $K|_{\Omega_j}$ *satisfies hypotheses 3.*

Theorem 3.4.2. *Let* $K(\mathbf{s}) = K(\mathbf{m}, c, \mathbf{s})$*, and suppose* $K(\mathbf{s})$ *extends to a meromorphic function on* $\mathbb{C}^n$ *which satisfies hypotheses 4. If* $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{>0})$*, then* $\zeta_{\varphi,u}(K;s)$ *extends to a meromorphic function on* $\mathbb{C}$ *with real poles and polynomial growth in vertical strips.*

Proof. This follows from lemma 3.3.5 and the proof of theorem 3.4.1, since the proof shows $\mathrm{B}_{\varphi,u}(\mathbf{z}, s-|\mathbf{z}|)$ satisfies hypotheses 2. □

Examples

Theorems 3.4.1 and 3.4.2 would not be much use if hypotheses 3 and 4 were never satisfied, so we give several ways of constructing $(\mathbf{m}, c)$ such that $K(\mathbf{m}, c; s)$ satisfies the hypotheses. We will only refer to hypotheses 4 below, but everything remains true for hypotheses 3.

1. Suppose $K_1(\mathbf{s}) = K_1(\mathbf{m}^{(1)}, c^{(1)}; \mathbf{s})$ and $K_2(\mathbf{s}) = K_2(\mathbf{m}^{(2)}, c^{(2)}; \mathbf{s})$ satisfy hypotheses 4. If $\mathbf{m}_{\alpha_1,\alpha_2} = \mathbf{m}^{(1)}_{\alpha_1} : \mathbf{m}^{(2)}_{\alpha_2}$ and $c_{\alpha_1,\alpha_2} = c^{(1)}_{\alpha_1} c^{(2)}_{\alpha_2}$, then

   $$K(\mathbf{m}, c; \mathbf{s} : \mathbf{s}') = K_1(\mathbf{s}) K_2(\mathbf{s}')$$

   also satisfies the hypotheses.

   In particular, suppose we have n complex sequences $(c_{j,a})_{a=0}^{\infty}$, and n sequences of positive reals $(\lambda_{j,a})_{a=0}^{\infty}$ $(j = 1, \ldots, n)$ such that each Dirichlet series $\sum_{a=0}^{\infty} \frac{c_{j,a}}{\lambda_{j,a}^{s_j}}$ converges absolutely for $\Re(s)$ sufficiently large, and extends to a meromorphic function on $\mathbb{C}$ with real poles and polynomial growth in each vertical strip of finite width. For $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, set $\mathbf{m}_{\mathbf{a}} = (\lambda_{1,a_1}, \ldots, \lambda_{n,a_n})$ and $c_{\mathbf{a}} = \prod_{j=1}^{n} c_{j,a_j}$. Then

   $$K(s_1, \ldots, s_n) = \prod_{j=1}^{n} \sum_{a \in \mathbb{Z}_{\geq 0}} \frac{c_{j,a}}{\lambda_{j,a}^{s_j}},$$

   and so K satisfies hypotheses 4[3].

2. Suppose $K(\mathbf{m}, c; \mathbf{s})$ satisfies hypotheses 4. If M is an $n \times n'$ matrix with non-negative entries, none of whose rows are zero, then

   $$K(\mathbf{m}^M, c; \mathbf{s}') = K(\mathbf{m}, c; M\mathbf{s}')$$

   also satisfies the hypotheses, where $(\mathbf{m}^M)_\alpha = \mathbf{m}^M_\alpha$.

---

[3] Various authors have considered Dirichlet series similar to $\zeta_u(K; s)$ that arise for such K, where u is replaced by a polynomial with coefficients in $\mathbb{C}_{>0}$, and where the growth condition is weakened. See [27], [28], [30] and [34], for example.

3. Suppose $K(\mathbf{m}, c; \mathbf{s})$ satisfies hypotheses 4. Fix $\mathbf{z} \in \mathbb{C}^n$, and define $\tilde{c}_\alpha = c_\alpha \mathbf{m}_\alpha^{\mathbf{z}}$. Then

   $$K(\mathbf{m}, \tilde{c}; \mathbf{s}) = K(\mathbf{m}, c; \mathbf{s} - \mathbf{z})$$

   also satisfies the hypotheses.

4. In chapter 6 we will see that if $K(\mathbf{m}, c; \mathbf{s})$ satisfies hypotheses 4 and $\varphi \in \mathcal{H}_0^\infty(\mathbb{R}_{\geq 0}^n)$, $u_j \in \mathcal{D}^\infty(\mathbb{R}_{\geq 0}^n)$ for $j = 1, \ldots, r$, then the hypotheses are satisfied with $\widehat{c}_\alpha = c_\alpha \varphi(\mathbf{m}_\alpha)$ and $\widehat{\mathbf{m}}_\alpha = (u_1(\mathbf{m}_\alpha), \ldots, u_r(\mathbf{m}_\alpha))$.

   Note that if $\widehat{K} = K(\widehat{\mathbf{m}}, \widehat{c};\ )$, then for $(\widehat{\varphi}, \widehat{u}) \in \mathcal{Z}^\infty(\mathbb{R}_{\geq 0}^r)$,

   $$\zeta_{\widehat{\varphi}, \widehat{u}}(\widehat{K}; s) = \zeta_{\check{\varphi}, \check{u}}(K; s),$$

   where $\check{\varphi}(\mathbf{x}) = \varphi(\mathbf{x})\widehat{\varphi}(u_1(\mathbf{x}), \ldots, u_r(\mathbf{x}))$ and $\check{u}(\mathbf{x}) = \widehat{u}(u_1(\mathbf{x}), \ldots, u_r(\mathbf{x}))$, so applying this construction directly does not produce new mixed zeta functions.

   However, if we apply the transformations in 2 and 3 above, we do get new mixed zeta functions.

3.4.1 An example: The two-dimensional Epstein zeta function

Let $Q = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ be positive definite, so that $\Delta := 4ac - b^2 > 0$. Write $u_\pm(\mathbf{x}) = \sqrt{ax_1^2 \pm bx_1x_2 + cx_2^2}$. The (two-dimensional) Epstein zeta function is defined by

$$\begin{aligned} Z_Q(s) &= \sum_{\mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} (\mathbf{n}^T Q \mathbf{n})^s \\ &= 2\left[(a^{-s} + c^{-s})\zeta(2s) + \zeta_{1,u_+}(K; 2s) + \zeta_{1,u_-}(K; 2s)\right], \end{aligned} \tag{3.4.1}$$

where $K(s_1, s_2) = \zeta(s_1)\zeta(s_2)$.

Set $\mathrm{B}_u := \mathrm{B}_{1,u_+} + \mathrm{B}_{1,u_-}$, so that

$$\frac{1}{2}Z_Q(s/2) = (a^{-s/2} + c^{-s/2})\zeta(s) + \int_{(c_0)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_u(z_1, s - z_1)dz_1 \tag{3.4.2}$$

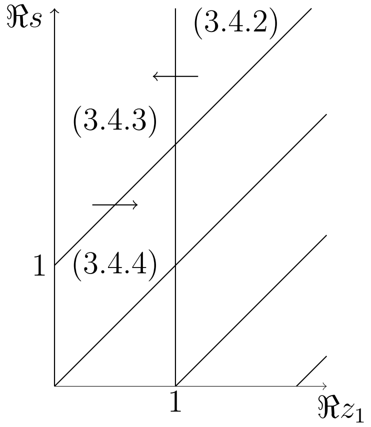
for $c_0 > 1$, $\Re(s) > c_0 + 1$. By the residue theorem,

$$\begin{aligned}\frac{1}{2}Z_Q(s/2) &= (a^{-s/2} + c^{-s/2})\zeta(s) + \zeta(s-1)\mathrm{B}_u(1, s-1) \\ &\quad + \int_{(c_1)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_u(z_1, s - z_1)dz_1,\end{aligned} \tag{3.4.3}$$

for $0 < c_1 < 1$ and $\Re(s) > c_1 + 1$. If we fix $1 < \Re(s) < 2$, then the residue theorem implies

$$\begin{aligned}\frac{1}{2}Z_Q(s/2) &= (a^{-s/2} + c^{-s/2})\zeta(s) + \zeta(s-1)[\mathrm{B}_u(1, s-1) + \mathrm{B}_u(s-1, 1)] \\ &\quad + \int_{(c_2)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_u(z_1, s - z_1)dz_1,\end{aligned} \tag{3.4.4}$$

for $\Re(s) - 1 < c_2 < 1$. By analytic continuation, (3.4.4) holds for $c_2 < \Re(s) < c_2 + 1$. The following diagram illustrates the transition from (3.4.2) to (3.4.3) to (3.4.4).



Next, we express $\mathrm{B}_u(z_1, z_2)$ in terms of a hypergeometric function. First consider the case $a = c = 1$, $b = \lambda \in (-2, 2)$. Since $(t+1)^2 \geq 4t > \lambda^2 t$, it follows from the binomial expansion that

$$(t + \lambda t^{1/2} + 1)^\alpha + (t - \lambda t^{1/2} + 1)^\alpha \quad = \quad 2\sum_{n=0}^{\infty} \binom{\alpha}{2n} \lambda^{2n} t^n (t+1)^{\alpha - 2n}.$$

Thus, for $z_1, z_2 \in \mathbb{C}_{>0}$ and $z = z_1 + z_2$,

$$\begin{aligned}
\mathrm{B}_u(z_1, z_2) &= \int_0^\infty \left[u(t,1)^{-z} + u(-t,1)^{-z}\right] t^{z_1 - 1} dt \\
&= \frac{1}{2}\int_0^\infty \left[(t + \lambda t^{1/2} + 1)^{-z/2} + (t - \lambda t^{1/2} + 1)^{-z/2}\right] t^{z_1/2 - 1} dt \\
&= \int_0^\infty \sum_{n=0}^{\infty} \binom{-z/2}{2n} \lambda^{2n} (t+1)^{-z/2 - 2n} t^{n + z_1/2 - 1} dt \\
&= \sum_{n=0}^{\infty} \binom{-z/2}{2n} \lambda^{2n} \mathrm{B}\left(\frac{z_1}{2} + n, \frac{z_2}{2} + n\right) \\
&= \sum_{n=0}^{\infty} \binom{-z/2}{2n} \lambda^{2n} \frac{(z_1/2)_n^+ (z_2/2)_n^+}{(z/2)_{2n}^+} \mathrm{B}\left(\frac{z_1}{2}, \frac{z_2}{2}\right) \\
&= \mathrm{B}\left(\frac{z_1}{2}, \frac{z_2}{2}\right) \sum_{n=0}^{\infty} \frac{(z_1/2)_n^+ (z_2/2)_n^+}{(1/2)_n^+ n! 4^n} \lambda^{2n} \\
&= \mathrm{B}\left(\frac{z_1}{2}, \frac{z_2}{2}\right) F\left(\frac{z_1}{2}, \frac{z_2}{2}; \frac{1}{2}; \frac{\lambda^2}{4}\right), \qquad (3.4.5)
\end{aligned}$$

where $F = {}_2F_1$ is Gauss' hypergeometric function. For the general case, we can write

$$\sqrt{ax^2 + bxy + cy^2} = \sqrt{x^2 + \lambda xy + y^2} \circ \delta_{(\sqrt{a}, \sqrt{c})}, \qquad \lambda = \frac{b}{\sqrt{ac}},$$

so by (3.1.4) and (3.4.5),

$$\mathrm{B}_u(z_1, z_2) \quad = \quad a^{-z_1/2} c^{-z_2/2} \mathrm{B}\left(\frac{z_1}{2}, \frac{z_2}{2}\right) F\left(\frac{z_1}{2}, \frac{z_2}{2}; \frac{1}{2}; \frac{b^2}{4ac}\right). \qquad (3.4.6)$$

Since ${}_2F_1(a, b; a; z) = {}_1F_0(b; \ ; z) = (1 - z)^{-b}$,

$$\begin{aligned}
\mathrm{B}_u(1, s-1) &= a^{-1/2}c^{-(s-1)/2}\mathrm{B}\left(\frac{1}{2}, \frac{s-1}{2}\right) F\left(\frac{1}{2}, \frac{s-1}{2}; \frac{1}{2}; \frac{b^2}{4ac}\right) \\
&= a^{-1/2}c^{-(s-1)/2}\frac{\Gamma(1/2)\Gamma((s-1)/2)}{\Gamma(s/2)}\left(1 - \frac{b^2}{4ac}\right)^{(1-s)/2} \\
&= 2^(s-1)a^{s/2-1}\sqrt{\pi}\frac{\Gamma((s-1)/2)}{\Gamma(s/2)}\Delta^{(1-s)/2}. \qquad (3.4.7)
\end{aligned}$$

Now we demonstrate how the function equation of the Epstein zeta function can be derived using Euler's transformation

$$F(a, b; c; z) \ = \ (1 - z)^{c-a-b}F(c - a, c - b; c; z) \qquad (3.4.8)$$

and the functional equation for the Riemann zeta function. For simplicity, we will only consider the case $a = c = 1$, $b = \lambda \in (-2, 2)$. Note that Euler's transformation implies

$$F\left(\frac{1-z_1}{2}, \frac{1-z_2}{2}; \frac{1}{2}; \frac{\lambda^2}{4}\right) \ = \ \left(\frac{\Delta}{4}\right)^{(z_1+z_2-1)/2} F\left(\frac{z_1}{2}, \frac{z_2}{2}; \frac{1}{2}; \frac{\lambda^2}{4}\right). \qquad (3.4.9)$$

Since $u_{Q^{-1}}(x, y) = \frac{2}{\sqrt{\Delta}}\sqrt{x^2 - \lambda xy + y^2}$, $\xi(s; Q^{-1}) = \left(\frac{2}{\sqrt{\Delta}}\right)^{-s}\xi(s; Q)$. We wish to show that if $\xi(s; Q) = \pi^{-s/2}\Gamma(s/2)Z_Q(s/2)$, then

$$\xi(2 - s, Q) = (\det Q)^{-1/2}\xi(s; Q^{-1}) = \left(\frac{2}{\sqrt{\Delta}}\right)^{1-s}\xi(s; Q).$$

It is enough to show this for s with $1/2 < \Re(s) < 3/2$, so we may use (3.4.4). Set $\rho(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, so that $\rho(s) = \rho(1 - s)$. Then (3.4.4) becomes

$$\frac{1}{2}\xi(s; Q) \ = \ T_1(s; Q) + T_2(s; Q) + T_3(s; Q),$$

where

$$\begin{aligned}
T_1(s;Q) &= \pi^{-s/2}\Gamma(s/2)2\zeta(s) = 2\rho(s), \\
T_2(s;Q) &= 2\pi^{(1-s)/2}\zeta(s-1)\Gamma((s-1)/2)\left(\frac{\sqrt{\Delta}}{2}\right)^{1-s} \qquad \text{(using (3.4.7))}, \\
T_3(s;Q) &= \pi^{-s/2}\Gamma(s/2)\int_{(1/2)}\zeta(z_1)\zeta(s-z_1)\mathrm{B}_{u_Q}(z_1,s-z_1)\ dz_1 \\
&= \int_{(1/2)}\rho(z_1)\rho(s-z_1)F\left(\frac{z_1}{2},\frac{s-z_1}{2};\frac{1}{2};\frac{\lambda^2}{4}\right)\ dz_1.
\end{aligned}$$

Then

$$\begin{aligned}
T_1(2-s;Q) \ &= \ 2\rho(2-s) \ = \ 2\rho(s-1) \\
&= \ 2\pi^{-(s-1)/2}\Gamma((s-1)/2)\zeta(s-1) \ = \ \left(\frac{2}{\sqrt{\Delta}}\right)^{1-s} T_2(s;Q),
\end{aligned}$$

so $T_2(2-s;Q) = \left(\frac{2}{\sqrt{\Delta}}\right)^{1-s} T_1(s;Q)$. Finally,

$$\begin{aligned}
T_3(2-s;Q) &= \int_{(1/2)}\rho(z_1)\rho(2-s-z_1)F\left(\frac{z_1}{2},\frac{2-s-z_1}{2};\frac{1}{2};\frac{\lambda^2}{4}\right)dz_1 \\
&= \int_{(1/2)}\rho(1-z_1)\rho(1+z_1-s)F\left(\frac{1-z_1}{2},\frac{1+z_1-s}{2};\frac{1}{2};\frac{\lambda^2}{4}\right)dz_1 \\
&= \int_{(1/2)}\rho(z_1)\rho(s-z_1)\left(\frac{\Delta}{4}\right)^{(s-1)/2}F\left(\frac{z_1}{2},\frac{s-z_1}{2};\frac{1}{2};\frac{\lambda^2}{4}\right)dz_1 \\
&= \left(\frac{2}{\sqrt{\Delta}}\right)^{1-s} T_3(s;Q).
\end{aligned}$$

In the second line, we have used the change of variable $z_1 \mapsto 1 - z_1$, and in the third line, we have used (3.4.9) and the functional equation of the Riemann zeta function.

3.4.2 Examples of meromorphic continuation where hypotheses 2 do not apply

We give two examples which show that the hypotheses on K are not the most general under which we can deduce the meromorphic continuation of $\zeta_{\varphi,u}(K;s)$. In both examples, the function K has infinitely many zeros on vertical strips of the form $\Re(s) = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

Example 1

Fix $\theta > 1$, and set $\mathcal{A} = \mathbb{Z}_{\geq 0}$, $\mathbf{m}_a = (\theta^a, 1)$, $c_a = 1$ and $u(x,y) = x + y$. (A more general example of this form is considered by Peter in [30]). Then

$$\zeta_u(K;s) = \sum_{a=1}^{\infty} \frac{1}{(\theta^a + 1)^s}, \qquad K(s_1, s_2) = \sum_{a=1}^{\infty} \theta^{-as_1} = \frac{1}{\theta^{s_1} - 1}.$$

If we write $\beta = 2\pi i/\log\theta$, then $K(s_1, s_2)$ has poles in the set $s_1 = \beta n$, $n \in \mathbb{Z}$, with residue $1/\log\theta$. Fix s for the moment. In the region $\Re(z) < \Re(s)$, $z \mapsto \frac{1}{\theta^z - 1}\mathrm{B}(z, s-z)$ has simple poles at $z = \beta n$, $n \in \mathbb{Z}\smallsetminus\{0\}$ and at $z \in \mathbb{Z}_{<0}$, and a double pole at $z = 0$. The residue of the double pole is $\frac{1}{2} - \frac{\gamma + \psi(s)}{\log\theta}$, where $\psi(s) = \Gamma'(s)/\Gamma(s)$.

For $s \in \mathbb{C}_{>0}$,

$$\begin{aligned}
\zeta_u(K;s) &= \int_{(1/2)} \frac{1}{\theta^z - 1}\mathrm{B}(z, s-z)dz \\
&= \sum_{n \in \mathbb{Z}\smallsetminus\{0\}} \frac{\mathrm{B}(\beta n, s - \beta n)}{\log\theta} + \frac{1}{2} - \frac{\gamma + \psi(s)}{\log\theta} + \int_{(-1/2)} \frac{1}{\theta^z - 1}\mathrm{B}(z, s-z)dz \\
&= \sum_{n \in \mathbb{Z}\smallsetminus\{0\}} \frac{\mathrm{B}(\beta n, s - \beta n)}{\log\theta} + \frac{1}{2} - \frac{\gamma + \psi(s)}{\log\theta} + \sum_{j=1}^{k} \frac{(-1)^j (s)_j^+}{(\theta^j - 1)j!} \\
&\quad + \int_{(-k-1/2)} \frac{1}{\theta^z - 1}\mathrm{B}(z, s-z)dz
\end{aligned}$$

The last line gives the meromorphic extension to $\Re(s) > -k - 1/2$, where $k \in \mathbb{Z}_{\geq 0}$. Since $\frac{(-1)^j (s)_j^+}{(\theta^j - 1)j!} = \binom{-s}{j}(\theta^j - 1)^{-1} = O(\theta^{-j})$ for s restricted to a compact set, we can take the limit as $k \to \infty$:

$$\zeta_u(K;s) = \sum_{n \in \mathbb{Z}\smallsetminus\{0\}} \frac{\mathrm{B}(\beta n, s - \beta n)}{\log\theta} + \frac{1}{2} - \frac{\gamma + \psi(s)}{\log\theta} + \sum_{j=1}^{\infty} \frac{(-1)^j (s)_j^+}{(\theta^j - 1)j!}.$$

Of course, we have the much simpler representation

$$\zeta_u(K;s) = \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{\theta^{s+k} - 1},$$

but this does not give the bounds on the growth along vertical strips.

Example 2

Let $\mathcal{A} = \mathbb{Z}_{>0}^2$, $\mathbf{m}_{(n,m)} = (n,m)$, and let $c_{(n,m)}$ be the characteristic function of $\{(n,m) \in \mathbb{Z}_{>0}^2 \mid n > \theta m\}$, where $\theta > 0$ is a quadratic irrational. The Euler-

Maclaurin formula gives

$$\begin{aligned}\sum_{n>\theta m} n^{-s_1}m^{-s_2} &= \sum_{m=1}^{\infty} m^{-s_2}\left[\int_{\theta m}^{\infty} x^{-s_1}dx + \sum_{k=0}^{N-1}(s_1)_k^+ B_{k+1}(\theta m)(\theta m)^{-s_1-k}\right.\\ &\qquad \left. -(s_1)_N^+\int_{\theta m}^{\infty} B_N(x)x^{-s_1-N}dx\right]\\ &= \frac{\theta^{1-s_1}}{s_1-1}\zeta(s_1+s_2-1) + \sum_{k=0}^{N-1}(s_1)_k^+\theta^{-s_1-k}\sum_{m=1}^{\infty} B_{k+1}(\theta m)m^{-s_1-s_2-k}\\ &\quad -(s_1)_N^+\mathcal{R}_N(s_1,s_2),\end{aligned}$$

where $\mathcal{R}_N(s_1,s_2) = \sum_{m=1}^{\infty} O(m^{-s_1-s_2-N})$ is an analytic function on $\Re(s_1+s_2) > 1-N$, which is bounded by a function of $\Re(s_1+s_2)$. Therefore

$$K(z,s-z) = \frac{\theta^{1-z}}{z-1}\zeta(s-1) + \sum_{k=0}^{N-1}(z)_k^+\theta^{-z-k}Z_{k+1}(\theta,s+k) - (z)_N^+\mathcal{R}_N(z,s-z),$$

where $Z_k(\theta,s) := \sum_{m=1}^{\infty} B_k(\theta m)m^{-s}$. Thus it suffices to prove that for $k \in \mathbb{Z}_{>0}$, $Z_k(\theta,s)$ extends to a meromorphic function with polynomial growth in vertical strips. This was conjectured by Hardy and Littlewood [11], and follows from a result of Mahler [26], as pointed out by G. Tenenbaum (see [7]). This also follows by adapting an argument of G. Lowther given in [24].

As with the previous example, one can use (3.2.1) to show that $\zeta_{\varphi,u}(K;s)$ extends to a meromorphic function on $\mathbb{C}$ when $(\varphi,u) \in \mathscr{Z}^{\infty}(\mathbb{R}^n_{\geq 0})$. If $F = \mathbb{Q}(\theta)$, then the poles are contained in $2\mathbb{Z}_{\leq 0} + \frac{2\pi i}{\log \eta}$, where η is the unique fundamental unit of the number ring $\mathcal{O}_F$ which is greater than 1. In the special case where u and φ come from homogenizing an elliptic polynomial (see chapter 4) and a ratio of elliptic polynomials respectively, we recover Mahler's [26].

3.5 A theorem of Hlawka.

As mentioned in section 2.1, Hlawka [18] considers distance functions u which are smooth away from $\mathbf{0}$, and such that $\mathcal{B}(u)$ is convex and the product of the principal curvatures of the unit sphere $\partial\mathcal{B}(u)$ is positive everywhere. Among other things, he concludes that $\zeta_u(s) := \zeta_{1,u}(s)$ vanishes at negative even integers. However, we will show that this cannot be true.

For $\lambda > 0$, define $u_\lambda(x,y) = \sqrt[4]{x^4 + \lambda x^2 y^2 + y^4}$. One can check that the curvature condition is equivalent to having $0 < \lambda < 6$. We will show that if $u_{\lambda,\theta}(x,y) := u_\lambda(x, \theta y)$, then there exists $\theta > 0$ and $\lambda \in (0,2)$ for which $\zeta_{u_{\lambda,\theta}}(-2) \neq 0$.

We can calculate B_{u_λ} as follows. If $0 < \lambda < 2$, then $\left|\frac{\lambda t^{1/2}}{t+1}\right| < 1$ for all $t \geq 0$, so we can apply the binomial expansion to $(t + \lambda t^{1/2} + 1)^{-z/4} = (t+1)^{-z/4}\left(1 + \frac{\lambda t^{1/2}}{t+1}\right)^{-z/4}$ to compute

$$\begin{aligned}
\mathrm{B}_{u_\lambda}(z_1, z_2) &= \int_0^\infty (t^4 + \lambda t^2 + 1)^{-z/4} t^{z_1 - 1} dt \qquad (z := z_1 + z_2) \\
&= \frac{1}{4}\int_0^\infty (t + \lambda t^{1/2} + 1)^{-z/4} t^{z_1/4 - 1} dt \\
&= \frac{1}{4}\int_0^\infty \sum_{n=0}^\infty \binom{-z/4}{n} (t+1)^{-z/4-n} t^{n/2 + z_1/4 - 1} \lambda^n dt \\
&= \frac{1}{4}\sum_{n=0}^\infty \binom{-z/4}{n} \mathrm{B}\left(\frac{n}{2} + \frac{z_1}{4}, \frac{n}{2} + \frac{z_2}{4}\right)\lambda^n. \qquad (3.5.1)
\end{aligned}$$

Changing the order of integration and summation above can be justified using Stirling's approximation, and this also shows that the series in (3.5.1) converges to a meromorphic function on $\mathbb{C}^2$, with poles at $z_i = -2n$, $n \in \mathbb{Z}_{\geq 0}$, $i = 1, 2$. Therefore, for fixed s with $\Re(s) > 5/2$, the only pole of $\zeta(z_1)\zeta(s - z_1)\mathrm{B}_{u_\lambda}(z_1, s - z_1)$

with $\Re(z_1) < 1$ is at $z_1 = 0$, where the residue is $\zeta(0)\zeta(s) = -\frac{1}{2}\zeta(s)$.

Starting from (3.5.1), we can show, using several hypergeometric identities, that

$$\mathrm{B}_{u_\lambda}(z_1, z_2) = \frac{1}{2}\mathrm{B}\left(\frac{z_1}{2}, \frac{z_2}{2}\right) F\left(\frac{z_1}{2}, \frac{z_2}{2}; \frac{z}{4} + \frac{1}{2}; \frac{2-\lambda}{4}\right).$$

However, for our purposes it is enough to note that $\mathrm{B}_{u_\lambda}(1, -3)$ is non-zero for some $\lambda \in (0, 2)$, since the power series defining $\mathrm{B}_{u_\lambda}(1, -3)$ has a non-zero first term.

For $\Re(s) > 5/2$,

$$\begin{aligned}
\zeta_{u_{\lambda,\alpha}}(s) &= 4\int_{(3/2)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_{u_{\lambda,\alpha}}(z_1, s - z_1)dz_1 + 2[1 + \alpha^{-s}]\zeta(s) \\
&= 4\int_{(3/2)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_{u_\lambda}(z_1, s - z_1)\alpha^{z_1 - s}dz_1 + 2[1 + \alpha^{-s}]\zeta(s) \\
&= 4\int_{(1/2)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_{u_\lambda}(z_1, s - z_1)\alpha^{z_1 - s}dz_1 \\
&\quad +4\zeta(s-1)\mathrm{B}_{u_\lambda}(1, s-1)\alpha^{1-s} + 2[1 + \alpha^{-s}]\zeta(s) \\
&= 4\int_{(-7/2)} \zeta(z_1)\zeta(s - z_1)\mathrm{B}_{u_\lambda}(z_1, s - z_1)\alpha^{z_1 - s}dz_1 \\
&\quad +4\zeta(s-1)\mathrm{B}_{u_\lambda}(1, s-1)\alpha^{1-s} + 2\zeta(s).
\end{aligned}$$

The last line gives the meromorphic extension of $\zeta_{u_{\lambda,\alpha}}(s)$ to $\Re(s) > -5/2$, so

$$\begin{aligned}
\zeta_{u_{\lambda,\alpha}}(-2) &= 4\int_{(-7/2)} \zeta(z_1)\zeta(-2 - z_1)\mathrm{B}_{u_\lambda}(z_1, -2 - z_1)\alpha^{z_1 + 2}dz_1 \\
&\quad +4\zeta(-3)\mathrm{B}_{u_\lambda}(1, -3)\alpha^3.
\end{aligned}$$

Choose λ such that $\mathrm{B}_{u_\lambda}(1, -3) \neq 0$. If $\zeta_{u_{\lambda,\alpha}}(-2)$ were equal to zero, we would have

$$\int_{(-7/2)} \zeta(z_1)\zeta(-2 - z_1)\mathrm{B}_{u_\lambda}(z_1, -2 - z_1)\alpha^{z_1 + 2}dz_1 \quad = \quad -\zeta(-3)\mathrm{B}_{u_\lambda}(1, -3)\alpha^3$$

for all $\alpha > 0$. However, the left-hand side is $O(\alpha^{-3/2})$, so this is impossible. In fact, the left hand side decays faster than any polynomial in α, since we are free to move the line of integration arbitrarily far to the left.

It appears that the error in Hlawka's argument arises in ([17], §2, Satz 2), where Satz 1 of §1 is applied to the distance function $(\mathbf{x}:\mathbf{y}) \mapsto \sqrt{f(\mathbf{x})^2 + \|\mathbf{y}\|^2}$, which is not necessarily smooth at points where $\mathbf{x} = \mathbf{0}$.

Chapter 4

Dirichlet series associated to polynomials

4.1 Elliptic polynomials

A polynomial is *elliptic* on $\mathbb{R}^n_{\geq 0}$ if it is positive on $\mathbb{R}^n_{\geq 0}$ and the highest degree homogeneous part is positive on $\mathbb{R}^n_{\geq 0} \smallsetminus \{\mathbf{0}\}$. In [25], Mahler showed that if $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$ are polynomials with P non-constant and elliptic on $\mathbb{R}^n_{\geq 0}$, then the Dirichlet series $\sum_{\mathbf{m} \in \mathbb{Z}^n_{>0}} \frac{Q(\mathbf{m})}{P(\mathbf{m})^s}$, which converges for $\Re(s) > (n + \deg Q)/\deg P$, has a meromorphic continuation to $\mathbb{C}$. We can recover this result from theorem 3.4.2, as follows.

If P is a degree d polynomial which is elliptic on $\mathbb{R}^n_{\geq 0}$, the homogenized polynomial $\tilde{P}(\mathbf{x} : x_{n+1}) := x^d_{n+1} P(\mathbf{x}/x_{n+1})$ determines a distance function $u_P = \tilde{P}^{1/d}$

on $\mathbb{R}^{n+1}_{\geq 0}$. Set $\mathcal{A} = \mathbb{Z}^n_{>0}$, $c_{\mathbf{a}} = Q(\mathbf{a})$ and $\mathbf{m}_{\mathbf{a}} = (\mathbf{a}, 1) \in \mathbb{Z}^{n+1}_{>0}$ for $\mathbf{a} \in \mathbb{Z}^n_{>0}$. If $Q(\mathbf{x}) = \sum_{\mathbf{k}} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{R}[x_1, \ldots, x_n]$, then

$$K(\mathbf{s} \colon s_{n+1}) = \sum_{\mathbf{a} \in \mathbb{Z}^n_{>0}} \frac{Q(\mathbf{a})}{\mathbf{a}^{\mathbf{s}}} = \sum_{\mathbf{k}} b_{\mathbf{k}} \prod_{j=1}^{n} \zeta(s_j - k_j)$$

satisfies hypotheses 4, and

$$\sum_{\mathbf{a} \in \mathbb{Z}^n_{>0}} Q(\mathbf{a}) P(\mathbf{a})^{-s} \quad = \quad \zeta_{u_P}(K; sd). \tag{4.1.1}$$

4.2 More general polynomials

A number of authors have extended Mahler's result to larger classes of polynomials, which we now describe. We note that the theorems of Sargos, Lichtin and Essouabri which we will refer to below, are more refined than the versions we will state here.

If $P = \sum_{\mathbf{k} \in \mathbb{Z}^n_{\geq 0}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial, its *support* is

$$\operatorname{supp}(P) = \{\mathbf{k} \in \mathbb{Z}^n_{\geq 0} \mid a_{\mathbf{k}} \neq 0\},$$

and its *Newton polyhedron* $\Delta(P) = \operatorname{conv}(\operatorname{supp}(P))$ is the convex hull of its support. We say that $\Delta(P)$ is *of full dimension* if $\dim \operatorname{span} \Delta(P) = n$. The *Newton polyhedron at infinity* is

$$\Gamma_\infty(P) = \operatorname{conv}(\operatorname{supp}(P) - \mathbb{R}^n_{\geq 0}).$$

Note that if $P^{[\mathbf{a}]}(\mathbf{x}) := P(\mathbf{a}+\mathbf{x})$ is the shift of P by $\mathbf{a} \in \mathbb{R}^n$, then $\Gamma_\infty(P^{[\mathbf{a}]}) = \Gamma_\infty(P)$, and for generic[1] $\mathbf{a}$,

$$\Delta(P^{[\mathbf{a}]}) = \Gamma_\infty(P) \cap \mathbb{R}^n_{\geq 0}.$$

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[1] specifically, for $\mathbf{a}$ such that $\partial^{\mathbf{k}} P(\mathbf{a}) \neq 0$ for all $\mathbf{k}$ which are vertices of $\Gamma_\infty(P) \cap \mathbb{R}^n_{\geq 0}$.

We write $P^* = \sum_{\mathbf{k}\in\mathcal{V}(P)} \mathbf{x}^{\mathbf{k}}$, where $\mathcal{V}(P)$ is the set of vertices of $\Gamma_\infty(P)$.

4.2.1 Nondegenerate polynomials

If $X \subseteq \mathbb{R}^n$, a polynomial P is said to be *nondegenerate with respect to its Newton polygon at infinity on* X (or just *nondegenerate on* X) if $P^* = O(P)$ on X. If we do not specify X, it should be understood to be J^n, where $J = [1, \infty)$. As shown in [3], nondegeneracy on J^n is equivalent to having $\frac{\partial^{\mathbf{k}} P}{P}(\mathbf{x}) = O(\mathbf{x}^{-\mathbf{k}})$ on J^n for all $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$.

Every elliptic polynomial is nondegenerate, but not conversely. For example, $x^2 + y$ is nondegenerate, but is not elliptic. Thus the following extends Mahler's result.

Theorem 4.2.1. *(Sargos, [31])*

If $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$, *where* P *is nondegenerate and* $P(\mathbf{x}) \to \infty$ *as* $|\mathbf{x}| \to \infty$ *on* J^n, *the series* $\sum_{\mathbf{m}\in\mathbb{Z}^n_{>0}} \frac{Q(\mathbf{m})}{P(\mathbf{m})^s}$ *defines an analytic function on* $\mathbb{C}_{>\eta}$ *for some* $\eta > 0$, *and extends to a meromorphic function on* $\mathbb{C}$ *with rational poles and polynomial growth in vertical strips.*

4.2.2 Hypoelliptic polynomials

A polynomial P is *hypoelliptic* if there exists $b \in (0, 1)$ such that

(i) $P(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, $\mathbf{x} \in [b, \infty)^n$

(ii) For all $\mathbf{k} \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\}$, $\frac{P^{(\mathbf{k})}}{P}(\mathbf{x}) \to 0$ as $\|\mathbf{x}\| \to \infty$ on $[b, \infty)^n$.

Every elliptic polynomial in hypoelliptic, but not conversely. For example, $(x-y)^2+x$ is hypoelliptic, but is degenerate, hence not elliptic. There is no inclusion relation between the classes of nondegenerate and hypoelliptic polynomials, since xy is nondegenerate and not hypoelliptic. Lichtin [22] showed that for $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$ with P hypoelliptic, the series $\sum_{\mathbf{m} \in \mathbb{Z}^n_{>0}} \frac{Q(\mathbf{m})}{P(\mathbf{m})^s}$ defines an analytic function on $\mathbb{C}_{>\eta}$ for some $\eta > 0$, and extends to a meromorphic function on $\mathbb{C}$ with rational poles and polynomial growth in vertical strips.

4.2.3 The class $H_0 S$

Essouabri [5] defined the class $H_0 S$ to be those polynomials P for which there exists $b \in (0,1)$ such that

(i) $P(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, $\mathbf{x} \in [b, \infty)^n$

and such that one of the following equivalent conditions is satisfied:

(ii) The distance between $[b, \infty)^n$ and the set of complex zeros of P is positive.

(ii)$'$ There exists $\epsilon > 0$ such that for $\mathbf{x} \in [b, \infty)^n$ and $\mathbf{y} \in B(\mathbf{0}, \epsilon)$, $P(\mathbf{x} + i\mathbf{y}) \neq 0$.

(ii)$''$ For all $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$, $\frac{P^{(\mathbf{k})}}{P}(\mathbf{x}) = O(1)$ as $\|\mathbf{x}\| \to \infty$ on $[b, \infty)^n$.

In [5], Essouabri shows that for $P \in H_0 S$, $\sum_{\mathbf{m} \in \mathbb{Z}^n_{>0}} P(\mathbf{m})^{-s}$ defines an analytic function on $\mathbb{C}_{>\eta}$ for some $\eta > 0$, and extends to a meromorphic function on $\mathbb{C}$, with rational poles and polynomial growth in vertical strips.

4.3 Completely nonvanishing polynomials

We now introduce a class of polynomials which are closely related to nondegenerate polynomials.

For $P \in \mathbb{R}[x_1, \ldots, x_n]$ and Γ a face of $\Delta(P)$, we let P_Γ denote the truncated polynomial $\sum_{\mathbf{k}\in\mathbb{Z}^n_{\geq 0}\cap\Gamma} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$. Following [29], we say that P is *completely nonvanishing* on a set $X \subset \mathbb{R}^n$, if P has no zeros in X, and if, for all faces Γ of $\Delta(P)$, the truncated polynomial P_Γ has no zeros in X.

The following theorem is theorem 2.2 of [31] chapter III in the case where P is real.

Theorem 4.3.1. *Suppose $P \in \mathbb{R}[x_1, \ldots, x_n]$ is positive on J^n. Then the following properties are equivalent:*

(i) *P is nondegenerate on J^n.*

(ii) *For every facet[2] F of $\Gamma_\infty(P)$, P_F is nondegenerate on J^n.*

(iii) *For every face F of $\Gamma_\infty(P)$, P_F is positive on J^n.*

Remark 4.3.2. By replacing P by $P \circ \delta_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathbb{R}^n_{>0}$, we see that theorem 4.3.1 remains true with J^n replaced by $\prod_{j=1}^n [\lambda_j, \infty)$. Thus, if P is positive and nondegenerate on $\mathbb{R}^n_{>0}$, P_F is positive on $\mathbb{R}^n_{>0}$ for every face F of $\Gamma_\infty(P)$.

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[2] A warning to English-speaking readers of [31] (and French papers concerning polyhedra in general): Sargos uses the terminology of Bourbaki, where a facet is a *face* in French, and a face is a *facette*.

Theorem 4.3.1 implies that every polynomial which is completely nonvanishing on J^n is nondegenerate. While the converse is not true, we will show that we can still relate nondegenerate polynomials to completely nonvanishing polynomials on $\mathbb{R}^n_{>0}$. This will be done in section 4.3.1. The reason for considering completely nonvanishing polynomials is that their homogenizations determine well-behaved beta functions, provided their Newton polyhedra are of full dimension.

Let $\Delta_\infty(P) \subset \mathbb{R}^{n+1}_{\geq 0}$ be the convex cone generated by $\operatorname{supp}(\tilde{P})$, where $\tilde{P}$ is the homogenization of P. The following theorem is a restatement of theorem 2.2 of [1], in the special case $m = 1$:

Theorem 4.3.3. *Suppose $P \in \mathbb{R}[x_1, \ldots, x_n]$ is completely nonvanishing on $\mathbb{R}^n_{>0}$, and that its Newton polyhedron is of full dimension. Let $d = \deg P$. Then*

$$\mathrm{B}_P(\mathbf{z}) := \int_{\mathbb{R}^n_{\geq 0}} P(\mathbf{t})^{-|\mathbf{z}|/d} (\mathbf{t} : 1)^{\mathbf{z}-\mathbf{1}} d\mathbf{t}$$

converges to an analytic function in the tube domain int$(\Delta_\infty(P))_{\mathbb{C}}$.

Let $\Gamma_1, \ldots, \Gamma_N$ be the facets of $\Delta_\infty(P)$. We can write

$$\Delta_\infty(P) \;=\; \bigcap_{i=1}^{N} \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \langle \mathbf{x}, \boldsymbol{\mu}_i \rangle \geq 0\}, \tag{4.3.1}$$

where $\boldsymbol{\mu}_i \in \mathbb{Z}^{n+1}$ is an inward-pointing normal vector to Γ_i, and where $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_{n+1} y_{n+1}$ is the diagonal bilinear form.

The next proposition can be derived from ([1], theorem 2.4) and its proof, but we give a more direct proof, using the ideas in [1].

Proposition 4.3.4. *Suppose the same hypotheses as in theorem 4.3.3 hold. For each* $\mathbf{k} \in \mathbb{Z}_{\geq 0}^N$*, there exists a finite set* $S_{\mathbf{k}} \subset \mathbb{Z}^{n+1}$ *such that* $\langle \mathbf{h}, \boldsymbol{\mu}_i \rangle \geq k_i$ *for* $\mathbf{h} \in S_{\mathbf{k}}$*,* $i = 1, \ldots, N$*, and there exist polynomials* $Q_{\mathbf{k},\mathbf{h}}$ *for each* $\mathbf{h} \in S_{\mathbf{k}}$*, of degree at most* $|\mathbf{k}|$*, such that*

$$\prod_{i=1}^{N} (\langle \mathbf{z}, \boldsymbol{\mu}_i \rangle)_{k_i}^{+} \mathrm{B}_P(\mathbf{z}) \;=\; \sum_{\mathbf{h} \in S_{\mathbf{k}}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|) \mathrm{B}_P(\mathbf{z} + \mathbf{h}) \tag{4.3.2}$$

for $\mathbf{z} \in \Delta_\infty(P)$*.*

Proof. Let $\widetilde{P}(\mathbf{x}) = \sum_{\mathbf{h}} \alpha_{\mathbf{h}} \mathbf{x}^{\mathbf{h}}$. Then for $i \in [n]$,

$$\begin{aligned} \Gamma(|\mathbf{z}|/d)\mathrm{B}_P(\mathbf{z}) &= \int_{\mathbb{R}_{\geq 0}^{n+1}} e^{-\widetilde{P}(\mathbf{y})} \mathbf{y}^{\mathbf{z}-\mathbf{1}} d\mathbf{y} \\ &= z_i^{-1} \int_{\mathbb{R}_{\geq 0}^{n+1}} e^{-\widetilde{P}(\mathbf{y})} \widetilde{P}^{(\mathbf{e}_i)}(\mathbf{y}) \mathbf{y}^{\mathbf{z}+\mathbf{e}_i-\mathbf{1}} d\mathbf{y} \\ &= z_i^{-1} \sum_{\mathbf{h} \in \mathrm{supp}(\widetilde{P})} \alpha_{\mathbf{h}} h_i \Gamma(|\mathbf{z}+\mathbf{h}|/d) \mathrm{B}_P(\mathbf{z}+\mathbf{h}), \end{aligned}$$

where we have used integration by parts in the second line. Note that $|\mathbf{h}| = d$ for $\mathbf{h} \in \mathrm{supp}(\widetilde{P})$, so $\Gamma(|\mathbf{z}+\mathbf{h}|/d) = \Gamma(|\mathbf{z}|/d)|\mathbf{z}|/d$, hence

$$z_i \mathrm{B}_P(\mathbf{z}) \;=\; \frac{|\mathbf{z}|}{d} \sum_{\mathbf{h} \in \mathrm{supp}(\widetilde{P})} \alpha_{\mathbf{h}}\, h_i\, \mathrm{B}_P(\mathbf{z}+\mathbf{h}).$$

Therefore, for $j = 1, \ldots, N$,

$$\langle \mathbf{z}, \boldsymbol{\mu}_j \rangle\, \mathrm{B}_P(\mathbf{z}) \;=\; \frac{|\mathbf{z}|}{d} \sum_{\mathbf{h} \in \mathrm{supp}(\widetilde{P}) \smallsetminus \Gamma_j} \alpha_{\mathbf{h}}\, \langle \mathbf{h}, \boldsymbol{\mu}_j \rangle\, \mathrm{B}_P(\mathbf{z}+\mathbf{h}), \tag{4.3.3}$$

We can now prove (4.3.2) by induction on $|\mathbf{k}|$. For $|\mathbf{k}| = 0$ it is trivially true. Suppose it holds for $|\mathbf{k}|$. Fix $j \in [N]$, and set $\mathbf{k}^* = \mathbf{k} + \mathbf{e}_j$. Write $S_{\mathbf{k}}$ as the disjoint union $S_{\mathbf{k},j} \cup S'_{\mathbf{k},j}$, where $\mathbf{h} \in S_{\mathbf{k},j}$ iff $\langle \mathbf{h}, \boldsymbol{\mu}_j \rangle > k_j$ (so $\mathbf{h} \in S'_{\mathbf{k},j}$ iff $k_j - \langle \mathbf{h}, \boldsymbol{\mu}_j \rangle = 0$). Then

$$\begin{aligned}
\prod_{i=1}^{N} (\langle \mathbf{z}, \boldsymbol{\mu}_i \rangle)^{+}_{k_i^*} \mathrm{B}_P(\mathbf{z}) &= \sum_{\mathbf{h} \in S_{\mathbf{k}}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|)(\langle \mathbf{z}, \boldsymbol{\mu}_j \rangle + k_j) \mathrm{B}_P(\mathbf{z} + \mathbf{h}) \\
&= \sum_{\mathbf{h} \in S_{\mathbf{k}}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|)(\langle \mathbf{z} + \mathbf{h}, \boldsymbol{\mu}_j \rangle + k_j - \langle \mathbf{h}, \boldsymbol{\mu}_j \rangle) \mathrm{B}_P(\mathbf{z} + \mathbf{h}) \\
&= \sum_{\mathbf{h} \in S_{\mathbf{k}}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|) \langle \mathbf{z} + \mathbf{h}, \boldsymbol{\mu}_j \rangle \mathrm{B}_P(\mathbf{z} + \mathbf{h}) \\
&\quad + \sum_{\mathbf{h} \in S_{\mathbf{k},j}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|)(k_j - \langle \mathbf{h}, \boldsymbol{\mu}_j \rangle) \mathrm{B}_P(\mathbf{z} + \mathbf{h}) \\
&= \sum_{\mathbf{h} \in S_{\mathbf{k}}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|) \frac{|\mathbf{z}|}{d} \sum_{\mathbf{h}' \in \mathrm{supp}(\widetilde{P}) \smallsetminus \Gamma_j} \alpha_{\mathbf{h}'} \langle \mathbf{h}', \boldsymbol{\mu}_j \rangle \mathrm{B}_P(\mathbf{z} + \mathbf{h} + \mathbf{h}') \\
&\quad + \sum_{\mathbf{h} \in S_{\mathbf{k},j}} Q_{\mathbf{k},\mathbf{h}}(|\mathbf{z}|)(k_j - \langle \mathbf{h}, \boldsymbol{\mu}_j \rangle) \mathrm{B}_P(\mathbf{z} + \mathbf{h}).
\end{aligned}$$

If we set $S_{\mathbf{k}^*} = (S_{\mathbf{k}} + (\mathrm{supp}(\widetilde{P}) \smallsetminus \Gamma_j)) \cup S_{\mathbf{k},j}$, then it is clear that we can find polynomials $Q_{\mathbf{k}^*,\mathbf{h}}$ for $\mathbf{h} \in S_{\mathbf{k}^*}$, of degree at most $|\mathbf{k}| + 1 = |\mathbf{k}^*|$, such that (4.3.2) holds.

It remains to show that for $\mathbf{h} \in S_{\mathbf{k}^*}$, $i = 1, \ldots, N$, we have $\langle \mathbf{h}, \boldsymbol{\mu}_i \rangle \geq k_i^* = k_i + \delta_{i,j}$. If $\mathbf{h} \in S_{\mathbf{k}}$ and $\mathbf{h}' \in \mathrm{supp}(\widetilde{P}) \smallsetminus \Gamma_j$, then

$$\langle \mathbf{h} + \mathbf{h}', \boldsymbol{\mu}_i \rangle = \langle \mathbf{h}, \boldsymbol{\mu}_i \rangle + \langle \mathbf{h}', \boldsymbol{\mu}_i \rangle \geq k_i + \delta_{i,j},$$

while if $\mathbf{h} \in S_{\mathbf{k},j}$, then by definition, $\langle \mathbf{h}, \boldsymbol{\mu}_i \rangle \geq k_i + \delta_{i,j}$. □

The right-hand side of (4.3.2) defines an analytic function for $\mathbf{z} \in \mathbb{C}^{n+1}$ with

$$\begin{aligned}
\Re(\mathbf{z}) &\in \bigcap_{\mathbf{h} \in S_{\mathbf{k}}} (\Delta_\infty(P) - \mathbf{h}) \\
&= \bigcap_{\mathbf{h} \in S_{\mathbf{k}}} \bigcap_{i=1}^{N} \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \langle \mathbf{x} + \mathbf{h}, \boldsymbol{\mu}_i \rangle \geq 0\} \\
&= \bigcap_{i=1}^{N} \bigcap_{\mathbf{h} \in S_{\mathbf{k}}} \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \langle \mathbf{x}, \boldsymbol{\mu}_i \rangle \geq -\langle \mathbf{h}, \boldsymbol{\mu}_i \rangle\} \qquad (4.3.4)
\end{aligned}$$

Since $\langle \mathbf{h}, \boldsymbol{\mu}_i \rangle \geq k_i$ for all $\mathbf{h} \in S_{\mathbf{k}}$, $i = 1, \ldots, N$, (4.3.4) will contain the region

$$\bigcap_{i=1}^{N} \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \langle \mathbf{x}, \boldsymbol{\mu}_i \rangle \geq -k_i\}.$$

We thus obtain the meromorphic continuation of $\mathrm{B}_P(\mathbf{z})$.

The following proposition is a modified version of proposition 3.2.1; likewise for its proof.

Proposition 4.3.5. *Suppose* $K = K(\mathbf{m}, c;\ )$, *and* $P \in \mathbb{R}[x_1, \ldots, x_n]$ *is completely nonvanishing on* $\mathbb{R}^n_{>0}$ *and of full dimension. Let* C_K *be the connected component of* $\mathbb{R}^n \smallsetminus \cup\{$*polar divisors of* $K\}$ *which contains* $(N_K + 1)\mathbf{1}$, *and let* $\Sigma_{K,P} = C_K \cap \Delta_\infty(P)$. *If* $\mathbf{c} \in \varpi_n(int(\Sigma_{K,P}))$, *then for* $s \in \mathbb{C}$ *with* $(\mathbf{c}, \Re(s) - |\mathbf{c}|) \in int(\Sigma_{K,P})$,

$$\zeta_{\varphi,u}(K; s) = \int_{(\mathbf{c})} K(\mathbf{z}, s - |\mathbf{z}|) \mathrm{B}_{\varphi,u}(\mathbf{z}, s - |\mathbf{z}|) d\mathbf{z}. \qquad (4.3.5)$$

As before, we can apply lemma 3.3.3 to conclude that

Theorem 4.3.6. *Suppose* $K(\mathbf{m}, c;\ )$ *satisfies hypotheses 4. If* $\rho_0 = \min\{|\mathbf{x}| \mid \mathbf{x} \in \Sigma_{K,P}\}$, *then* $\zeta_{\varphi,u}(K; s)$ *has a meromorphic continuation to* $\mathbb{C}$ *with real poles and polynomial growth in vertical strips, and is analytic in* $\mathbb{C}_{>\rho_0}$.

4.3.1 Sargos' theorem

With some more work, we can use theorem 4.3.6 to give a new proof of theorem 4.2.1. For $A \in \mathrm{GL}_n(\mathbb{R})$, define $\omega_A : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0} : \mathbf{x} \mapsto \mathbf{x}^A$. Note that for $\boldsymbol{\lambda} \in \mathbb{R}^n_{>0}$, $\omega_A \circ \delta_{\boldsymbol{\lambda}} = \delta_{\boldsymbol{\lambda}^A} \circ \omega_A$.

The following theorem is theorem 2.1. of Sargos [32] in the case $r = 1$:

Theorem 4.3.7. *Let Δ be a bounded integral polyhedron in $\mathbb{R}^n_{\geq 0}$. Then there exists a finite subset $\mathcal{M} \subset GL_n(\mathbb{Q}) \cap M_{n\times n}(\mathbb{Z}_{\geq 0})$ such that the following two properties are satisfied:*

- ***(i)*** *The family $(\omega_A(J^n))_{A\in\mathcal{M}}$ is, up to a set of measure zero, a partition of J^n.*
- ***(ii)*** *For each $A \in \mathcal{M}$, the polyhedron $A\Delta$ has a largest*[3] *vertex.*

Lemma 4.3.8. *If $P \in \mathbb{R}[x_1, \ldots, x_n]$ is nondegenerate on J^n, it is nondegenerate on $[\eta, \infty)^n$ for some $\eta \in (0, 1)$.*

The proof below uses ideas from the proof of theorem 2.2 of [31], chapter III.

Proof. We first show that this is true when P has a largest monomial, say $\mathbf{x}^{\mathbf{d}}$. By assumption, there exists $\lambda > 0$ such that $\mathbf{x}^{\mathbf{d}} \leq \lambda P(\mathbf{x})$ on J^n. Then $\widehat{P}(\mathbf{x}) := P(\mathbf{x}^{-\mathbf{1}})\mathbf{x}^{\mathbf{d}}$ is bounded below by $1/\lambda$ on $(0, 1]^n$. Since $\widehat{P}$ is a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$, it is continuous, so there exists $\epsilon > 0$ such that $\widehat{P}(\mathbf{x}) \geq 1/(2\lambda)$ on $[-\epsilon, 1 + \epsilon]^n$, and so the lemma follows with $\eta = 1/(1 + \epsilon)$.

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[3] with respect to the product partial order.

For the general case, we use the fact that there exists a finite collection $\mathcal{M} \subset \mathrm{GL}_n(\mathbb{Q}) \cap M_{n\times n}(\mathbb{Z}_{\geq 0})$ such that (i) and (ii) of theorem 4.3.7 hold for $\Delta = \Delta(P)$. For each $A \in \mathcal{M}$, $\Delta(P \circ \omega_A) = A\Delta(P)$ has a largest monomial, so $P \circ \omega_A$ has a largest monomial. Thus, for some $\eta_A \in (0,1)$, $(P \circ \omega_A)^* \ll P \circ \omega_A$ on $[\eta_A, \infty)^n$. But $P^* \circ \omega_A \ll (P^* \circ \omega_A)^* = (P \circ \omega_A)^*$ on $[\eta_A, \infty)^n$, so $P^* \ll P$ on $\omega_A([\eta_A, \infty)^n)$. Choose $\eta < 1$ such that for each $A \in \mathcal{M}$, $\boldsymbol{\eta} := (\eta\mathbf{1})^{A^{-1}} > \eta_A\mathbf{1}$. Then

$$[\eta, \infty)^n = \delta_{\eta\mathbf{1}}(J^n) = \bigcup_{A\in\mathcal{M}} \delta_{\eta\mathbf{1}}(\omega_A(J^n)) = \bigcup_{A\in\mathcal{M}} \omega_A(\delta_{\boldsymbol{\eta}}(J^n)) \subset \bigcup_{A\in\mathcal{M}} \omega_A([\eta_A, \infty)^n),$$

so $P^* \ll P$ on $[\eta, \infty)^n$. $\square$

Proposition 4.3.9. *Suppose P is positive on J^n and $P(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$ on J^n. If P is nondegenerate on J^n, then for some $\boldsymbol{\eta} \in [0,1)^n$, $P^{[\boldsymbol{\eta}]}$ is completely nonvanishing on $\mathbb{R}^n_{>0}$ and $\Delta(P^{[\boldsymbol{\eta}]})$ is of full dimension.*

Proof. If P is positive and nondegenerate on J^n and $P(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$ on J^n, then P is positive and nondegenerate on $[\eta_0, \infty)^n$ for some $\eta_0 \in (0,1)$, by the previous lemma, and we may choose $\boldsymbol{\eta} \in [\eta_0, 1)^n$ such that $\Delta(P^{[\boldsymbol{\eta}]}) = \Gamma_\infty(P) \cap \mathbb{R}^n_{\geq 0}$. The assumption $P(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$ on J^n implies that P depends effectively on all variables, so $\Delta(P^{[\boldsymbol{\eta}]})$ is of full dimension.

Therefore, on $\mathbb{R}^n_{\geq 0}$, $P^{[\boldsymbol{\eta}]} \gg (P^*)^{[\boldsymbol{\eta}]} > P^* = (P^{[\boldsymbol{\eta}]})^*$, where we use the fact that P^* has positive coefficients and $\boldsymbol{\eta} \in \mathbb{R}^n_{>0}$ to conclude $(P^*)^{[\boldsymbol{\eta}]} > P^*$. Thus $P^{[\boldsymbol{\eta}]}$ is positive and nondegenerate on $\mathbb{R}^n_{\geq 0}$, so by remark 4.3.2, $(P^{[\boldsymbol{\eta}]})_F$ is positive on $\mathbb{R}^n_{>0}$ for all faces F of $\Gamma_\infty(P^{[\boldsymbol{\eta}]})$. To conclude that $P^{[\boldsymbol{\eta}]}$ is completely nonvanishing on

$\mathbb{R}^n_{>0}$, we need to show that $(P^{[\boldsymbol{\eta}]})_F$ is positive on $\mathbb{R}^n_{>0}$ for all faces F of $\Delta(P^{[\boldsymbol{\eta}]})$ which lie in one of the coordinate hyperplanes.

Let H_i be the coordinate hyperplane $x_i = 0$. We may assume that $F \subset H_i$ iff $i > m$. If we write $\boldsymbol{\eta} = (\boldsymbol{\eta}', \boldsymbol{\eta}'')$ and $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, then

$$(P^{[\boldsymbol{\eta}]})_{\cap_{i=m+1}^n H_i} = P^{[\boldsymbol{\eta}]}|_{\mathbf{x}''=\mathbf{0}} = (P|_{\mathbf{x}''=\boldsymbol{\eta}''})^{[\boldsymbol{\eta}']}.$$

Let $Q = P|_{\mathbf{x}''=\boldsymbol{\eta}''}$. If we write $F = F' \times \{\mathbf{0}_{n-m}\}$ for a face F' of $\Delta(P^{[\boldsymbol{\eta}]}|_{\mathbf{x}''=\mathbf{0}}) \subset \mathbb{R}^{n-m}$, then $(P^{[\boldsymbol{\eta}]})_F = ((P^{[\boldsymbol{\eta}]})_{\cap_{i=m+1}^n H_i})_{F'} = (Q^{[\boldsymbol{\eta}']})_{F'}$. If we can show that Q is nondegenerate on $\prod_{j=1}^m [\eta_j, \infty)$, then by lemma 4.3.1, $(P^{[\boldsymbol{\eta}]})_F$, regarded as a function of $\mathbf{x}'$, will be positive on $\mathbb{R}^m_{>0}$, and so, as a function of $\mathbf{x}$, will be positive on $\mathbb{R}^n_{>0}$.

Write $P = \sum_{\mathbf{k}} R_{\mathbf{k}}(\mathbf{x}'')\mathbf{x}'^{\mathbf{k}}$. Then if $\mathcal{V}_m(P)$ is the set of vertices of $\Gamma_\infty(P) \cap \mathbb{R}^m$, $Q^* = \sum_{\mathbf{k}\in\mathcal{V}_m(P)} \mathbf{x}'^{\mathbf{k}}$, while $P^*|_{\mathbf{x}''=\boldsymbol{\eta}''} = \sum_{\mathbf{k}\in\mathcal{V}_m(P)} R^*_{\mathbf{k}}(\boldsymbol{\eta}'')\mathbf{x}'^{\mathbf{k}}$. Therefore

$$Q^* \ll P^*|_{\mathbf{x}''=\boldsymbol{\eta}''} \ll P|_{\mathbf{x}''=\boldsymbol{\eta}''} = Q$$

on $\prod_{j=1}^m [\eta_j, \infty)$. $\square$

Therefore, if $P \in \mathbb{R}[x_1, \ldots, x_n]$ is positive and nondegenerate on J^n and $P(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$ on J^n, we may write

$$\sum_{\mathbf{a}\in\mathbb{Z}^n_{>0}} P(\mathbf{a})^{-s} = \sum_{\mathbf{a}\in\mathbb{Z}^n_{>0}-\boldsymbol{\eta}} P^{[\boldsymbol{\eta}]}(\mathbf{a})^{-s} = \zeta_{P^{[\boldsymbol{\eta}]}}(K; s),$$

where $\boldsymbol{\eta}$ is as in proposition 4.3.9, and

$$K(\mathbf{z}) = \sum_{\mathbf{a}\in\mathbb{Z}^n_{>0}-\boldsymbol{\eta}} (\mathbf{a} : 1)^{-\mathbf{z}} = \prod_{i=1}^n \zeta(z_i, 1-\eta_i)$$

is a product of Hurwitz zeta functions.

If $Q \in \mathbb{C}[x_1, \ldots, x_n]$ and P is as above, we can also express $\sum_{\mathbf{a} \in \mathbb{Z}^n_{>0}} Q(\mathbf{a}) P(\mathbf{a})^{-s}$ as a mixed zeta function, as in (4.1.1).

Chapter 5

Counting problems

We will show that if $(\varphi, u) \in \mathbb{Z}^{\infty}(\mathbb{R}^n_{\geq 0})$ and $K = K(\mathbf{m}, c;\ )$ satisfies hypotheses 3, then one can give estimates for the growth of the weighted counting function

$$\mathcal{N}_{\varphi,u}(K;t) := \sum_{\alpha \in \mathcal{A}, u(\mathbf{m}_\alpha) < t} c_\alpha \varphi(\mathbf{m}_\alpha).$$

If $\mathcal{A} = \mathbb{Z}^n_{>0}$ and $\mathbf{m_a} = \mathbf{a}$, then this amounts to finding an estimate for the growth of weighted sums over the integer lattice points inside $t\mathcal{B}(u) \cap \mathbb{R}^n_{>0}$.

5.1 Rates of growth in vertical strips

The following lemma a minor modification of a lemma of Sargos ([32], lemme 6.1).

Lemma 5.1.1. *Let $f(s)$ be a function which is holomorphic in $\mathbb{C}_{>\kappa}$, for some $\kappa \in \mathbb{R}$. Suppose there exist $\sigma_a > \kappa$ and $A > 0$, such that*

(i) *$f(\sigma + it) = O(1)$ for $\sigma > \sigma_a, |t| > 1$,*

(ii) *$f(\sigma + it) = O_\sigma(|t|^A)$ for $\sigma > \kappa, |t| > 1$.*

Then, for all $\epsilon > 0$, we have

$$f(\sigma + it) = O_\epsilon(1 + |t|^{B(\sigma_a - \sigma)+\epsilon}) \quad (\sigma > \kappa, |t| > 1),$$

where $B = A/(\sigma_a - \kappa)$.

5.2 Estimates for counting functions

A version of the following lemma is stated in ([6], prop 3.1), and can be proved by modifying the proof of ([23], theorem B-4), which in turn is based on the proof of a Tauberian theorem due to Landau ([20]).

Lemma 5.2.1. *Let $(a_k)_k$ be a sequence of complex numbers, and $0 < \lambda_1 < \lambda_2 < \dots$ a sequence of reals, such that*

$$Z(s) = \sum_{k=1}^{\infty} a_k \lambda_k^{-s}$$

satisfies

(i) *$Z(s)$ converges absolutely in a half-plane of the form $\mathbb{C}_{>\alpha}$. We let σ_a denote the abscissa of absolute convergence.*

(ii) *There exists $\delta > 0$ such that $Z(s)$ extends to a meromorphic function on $\{\Re(s) > \sigma_a - \delta\}$, with a finite number of poles, which are all real. We denote the poles of $s^{-1}Z(s)$ in this half-plane by $\sigma_0 > \dots > \sigma_r$.*

(iii) *There exists $A > 0$ such that for all $\epsilon > 0$,*

$$Z(\sigma + i\tau) = O(1 + |\tau|^{A(\sigma_a - \sigma)}) \quad \textit{for } \sigma > \sigma_a - \delta \textit{ and } |\tau| \geq 1.$$

For $k = 0, \ldots, r$, define $Q_k(x) = e^{-\sigma_k x} Res_{s=\sigma_k}(s^{-1}Z(s)e^{sx})$, and set $\mu = \sup\{1/\delta, A\}$. Then for every $\epsilon > 0$,

$$\sum_{\lambda_n < t} c_n = \sum_{k=0}^{r} t^{\sigma_k} Q_k(\log t) + O_\epsilon(t^{\sigma_0 - \lfloor \mu\delta \rfloor/(1+\lfloor \mu\delta \rfloor)\mu + \epsilon})$$

as $t \to \infty$.

Corollary 5.2.2. *Suppose $K(\mathbf{m}, c, \mathbf{z})$ satisfies hypotheses 3 and $(\varphi, u) \in \mathcal{Z}^\infty(\mathbb{R}^n_{\geq 0})$. Let Σ_K be the connected component of $(N_K + 1)\mathbf{1}$ in $\{\mathbf{x} \in \Omega \cap \mathbb{R}^n_{>0} \mid L(\mathbf{x}) \neq 0\}$, and let $\rho = \inf\{|\mathbf{x}| \mid \mathbf{x} \in \Sigma_K\}$. If there exists a point $\mathbf{x} \in \Omega$ in the boundary of Σ_K with $|\mathbf{x}| = \rho$, then there exists a polynomial Q_0 and $\theta > 0$ such that $\mathcal{N}_{\varphi,u}(K; s) = t^\rho Q_0(\log t) + O(t^{\rho - \theta})$.*

We do not address the question of how to describe Q_0 explicitly, but in certain cases, this can be done (see [8]).

Chapter 6

Multivariable mixed zeta functions

For $a \in \mathbb{R}$, $\mathbf{d} \in \mathbb{R}^r_{>0}$, let $\mathcal{Z}^{\infty}_{a,\mathbf{d}}(\mathbb{R}^n_{\geq 0})$ be the set of pairs $(\varphi, \mathbf{u})$, with $\mathbf{u} = (u_1, \ldots, u_r)$, where $\varphi \in \mathcal{H}_a(\mathbb{R}^n_{\geq 0})$ and $u_1^{1/d_1}, \ldots, u_r^{1/d_r} \in \mathcal{D}(\mathbb{R}^n_{\geq 0})$ are all smooth on $\mathbb{R}^n_{\geq 0} \smallsetminus \{\mathbf{0}\}$.

Suppose $K(\mathbf{s}) = K(\mathbf{m}, c; \mathbf{s})$ and $(\varphi, \mathbf{u}) \in \mathcal{Z}^{\infty}_{0,\mathbf{1}}(\mathbb{R}^n_{\geq 0})$. Since u_j is a continuous distance function on $\mathbb{R}^n_{\geq 0}$, $\|\mathbf{x}\| \ll u_j(\mathbf{x}) \ll \|\mathbf{x}\|$ for $j = 1, \ldots, r$. In particular, $u_j(\mathbf{m}_\alpha)^\lambda \ll_\lambda \|\mathbf{m}_\alpha\|^\lambda$ for any real λ and $\alpha \in \mathcal{A}$. Therefore, if $\mathbf{s} \in \mathbb{C}^r$ with $|\mathbf{s}| > nN_K$,

$$\sum_{\alpha \in \mathcal{A}} |c_\alpha| \prod_{j=1}^{r} u_j(\mathbf{m}_\alpha)^{-\Re(s_j)} \ll_{\mathbf{s}} \sum_{\alpha \in \mathcal{A}} |c_\alpha| \, \|\mathbf{m}_\alpha\|^{-\Re(|\mathbf{s}|)} \ll \sum_{\alpha \in \mathcal{A}} |c_\alpha| \mathbf{m}_\alpha^{-n^{-1}\Re(|\mathbf{s}|)\mathbf{1}} < \infty,$$

by (3.0.4). Thus the series defined by

$$\zeta_{\varphi,\mathbf{u}}(K; \mathbf{s}) := \sum_{\alpha \in \mathcal{A}} c_\alpha \varphi(\mathbf{m}_\alpha) \prod_{j=1}^{r} u_j(\mathbf{m}_\alpha)^{-s_j}$$

converges to an analytic function in the region $\{\mathbf{s} \in \mathbb{C}^r \mid \Re(|\mathbf{s}|) > nN_K\}$.

6.1 Meromorphic continuation

Following the method of proof of 3.2.1, we can prove

Proposition 6.1.1. *Let* $(\varphi, \mathbf{u}) \in \mathcal{Z}^{\infty}_{0,\mathbf{1}}(\mathbb{R}^n_{\geq 0})$*, and suppose* $K(\mathbf{z}) = K(\mathbf{m}, c; \mathbf{z})$ *satisfies hypotheses 4. Choose* $\varphi_1, \ldots, \varphi_r \in \mathcal{H}_0(\mathbb{R}^n_{\geq 0})$ *which are smooth on* $\mathbb{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$*, and such that* $\prod_{i=1}^r \varphi_i = \varphi$*. (For example, one could take* $\varphi_1 = \varphi$*, and* $\varphi_2 = \ldots = \varphi_r = 1$*).*

If $\mathbf{c} > \frac{N_K}{r}\mathbf{1}_{[n-1]}$*, then for* $\Re(\mathbf{s}) > n|\mathbf{c}|\mathbf{1}_{[r]}$*,*

$$\begin{aligned}\zeta_{\varphi,\mathbf{u}}(K;\mathbf{s}) \;=\; & \int_{(\mathbf{c})} \cdots \int_{(\mathbf{c})} K(\mathbf{y}_1 + \ldots + \mathbf{y}_r, s_1 + \ldots + s_r - |\mathbf{y}_1 + \ldots + \mathbf{y}_r|) \\ & \times \prod_{j=1}^r \mathrm{B}_{\varphi_j,u_j}(\mathbf{y}_j, s_j - |\mathbf{y}_j|) d\mathbf{y}_1 \ldots d\mathbf{y}_r. \end{aligned} \tag{6.1.1}$$

Proof. If $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r \in \mathbb{R}^n_{>0}$, define

$$J(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r) = \sum_{\alpha \in \mathcal{A}} c_\alpha \prod_{j=1}^r \left[\varphi_j(\boldsymbol{\tau}_j \odot \mathbf{m}_\alpha) e^{-u_j(\boldsymbol{\tau}_j \odot \mathbf{m}_\alpha)}\right].$$

Suppose $\mathbf{z}_1, \ldots, \mathbf{z}_r \in \mathbb{C}^n_{>N}$. We take the nr-fold Mellin transform of $J(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r)$ with respect to $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r$, switch the order of integration and summation, and then use the change of variables $\mathbf{x}_j = \boldsymbol{\tau}_j \odot \mathbf{m}_\alpha$:

$$\begin{aligned} & \int_{\mathbb{R}^n_{\geq 0}} \cdots \int_{\mathbb{R}^n_{\geq 0}} J(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r) \boldsymbol{\tau}_1^{\mathbf{z}_1 - \mathbf{1}} \ldots \boldsymbol{\tau}_r^{\mathbf{z}_r - \mathbf{1}} d\boldsymbol{\tau}_1 \ldots d\boldsymbol{\tau}_r \\ = \; & \sum_{\alpha \in \mathcal{A}} c_\alpha \int_{\mathbb{R}^n_{\geq 0}} \cdots \int_{\mathbb{R}^n_{\geq 0}} \prod_{j=1}^r \left[\varphi_j(\boldsymbol{\tau}_j \odot \mathbf{m}_\alpha) e^{-u_j(\boldsymbol{\tau}_j \odot \mathbf{m}_\alpha)} \boldsymbol{\tau}_j^{\mathbf{z}_j - \mathbf{1}}\right] d\boldsymbol{\tau}_1 \ldots d\boldsymbol{\tau}_r \\ = \; & \sum_{\alpha \in \mathcal{A}} c_\alpha \mathbf{m}_\alpha^{-\mathbf{z}_1 - \ldots - \mathbf{z}_r} \prod_{j=1}^r \int_{\mathbb{R}^n_{\geq 0}} \varphi_j(\mathbf{x}_j) e^{-u_j(\mathbf{x}_j)} \mathbf{x}_j^{\mathbf{z}_j - \mathbf{1}} d\mathbf{x}_j \\ = \; & K(\mathbf{z}_1 + \ldots + \mathbf{z}_r) \prod_{j=1}^r \Gamma(|\mathbf{z}_j|) \ldots \Gamma(|\mathbf{z}_r|) \mathrm{B}_{\varphi_j,u_j}(\mathbf{z}_j). \end{aligned} \tag{6.1.2}$$

As in the proof of theorem 3.2.1, we deduce (6.1.1) by taking the nr-fold inverse Mellin transform of (6.1.2), and setting $\boldsymbol{\tau}_j = t_j \mathbf{1}$, $t_j > 0$. $\square$

The proof of proposition 3.3.5 extends to show that:

Theorem 6.1.2. *If K satisfies hypotheses 4, then $\zeta_{\varphi,\mathbf{u}}(K;\mathbf{s})$ extends to a meromorphic function on $\mathbb{C}^r$ with polynomial growth in vertical strips. The polar divisor consists of real hyperplanes.*

In the next two sections, we will show that under certain assumptions on K, we obtain generalizations of two theorems concerning relations between values of Dirichlet series at zero, as well as relations between the first derivatives of Dirichlet series at $s = 0$.

The assumptions are given by the following hypotheses:

Hypotheses 5.

- ***(i)*** *$K(\mathbf{z}) = K(\mathbf{m}, c; \mathbf{z})$ satisfies hypotheses 4.*
- ***(ii)*** *The poles of $K(\mathbf{z})$ are at most simple, and occur along $z_i = \kappa_i$, for some positive constants κ_i ($i = 1, \ldots, n$).*

For example, such a function can be constructed from n Dirichlet series in one variable, $s \mapsto \sum_k c_{j,k} \lambda_{j,k}^{-s}$, $j = 1, \ldots, n$, as in example 1 on pg 35.

In the proof of proposition 6.1.4 below, we will need the following lemma, which is a multivariable version of the partial fraction decomposition:

Lemma 6.1.3. *Let $\mathbb{F}$ be a field, and let L be a product of degree 1 polynomials in $\mathbb{F}[x_1, \dots, x_n]$. Then in $\mathbb{F}(x_1, \dots, x_n)$,*

$$\frac{1}{L} = \sum_{j=1}^{\ell} \frac{\alpha_j}{L_j},$$

where for each $j = 1, \dots, \ell$, $\alpha_j \in \mathbb{F}$ and $L_j \in \mathbb{F}[x_1, \dots, x_n]$ is a product of degree 1 polynomials such that $\cap_{L_j} := \bigcap_{T|L_j} \{\mathbf{x} \in \mathbb{F}^n \mid T(\mathbf{x}) = 0\}$ is non-empty.

Proof. If $\cap_L \neq \varnothing$, there is nothing to prove, so suppose $\cap_L = \varnothing$. Write $L(\mathbf{s}) = \prod_{j=1}^{d}(\langle \mathbf{s}, \boldsymbol{\mu}_j \rangle - \nu_j)$, and let M be the matrix whose j-th row is $\boldsymbol{\mu}_j$. Then the matrix equation $M\mathbf{x}^T = \boldsymbol{\nu}^T$ has no solutions, so a linear combination of the rows of the augmented matrix $M : \boldsymbol{\nu}^T$ is equal to $(\mathbf{0}, -1)$. In other words, there exist constants $\alpha_j \in \mathbb{F}$ such that $\sum_j \alpha_j(\langle \mathbf{s}, \boldsymbol{\mu}_j \rangle - \nu_j) = \langle \mathbf{s}, \mathbf{0} \rangle - (-1) = 1$.

Therefore

$$\frac{1}{L} = \sum_{j=1}^{d} \frac{\alpha_j}{\prod_{i \neq j}(\langle \mathbf{s}, \boldsymbol{\mu}_i \rangle - \nu_i)}.$$

By induction on $d = \deg L$, each term $\frac{1}{\prod_{i \neq j}(\langle \mathbf{s}, \boldsymbol{\mu}_i \rangle - \nu_i)}$ can be written in the form we desire. □

Proposition 6.1.4. *If K satisfies hypotheses 5 and $(\varphi, \mathbf{u}) \in \mathcal{Z}^{\infty}_{0,\mathbf{1}}(\mathbb{R}^n_{\geq 0})$, then each irreducible component of the polar divisor of $\zeta_{\varphi,\mathbf{u}}(K; \mathbf{s})$ is of the form $|\mathbf{s}| = \lambda$, for $\lambda \in \mathbb{R}$, and has multiplicity one.*

Proof. We first show that all poles have multiplicity one. Define the following hyperplanes: For $k \in \mathbb{Z}_{\geq 0}$, $i = 1, \ldots, r$,

$$A_{i,j}(k) \quad : \quad \begin{cases} x_{i,j} = -k, & j = 1, \ldots, n-1, \\ x_{i,n} - \sum_{\ell=1}^{n-1} x_{i,\ell} = -k, & j = n, \end{cases}$$

$$B_j \quad : \quad \begin{cases} x_{1,j} + \ldots + x_{r,j} = \kappa_j, & j = 1, \ldots, n-1, \\ x_{1,n} + \ldots + x_{r,n} - \sum_{\ell=1}^{n-1}(x_{1,\ell} + \ldots + x_{r,\ell}) = \kappa_n, & j = n. \end{cases}$$

If $s_i = x_{i,n}$ and $x_{i,j} = z_{i,j}$ ($i \in [r]$, $j \in [n-1]$), then the hyperplanes above give the (potential) poles of the integrand in (6.1.1). We will abuse terminology by referring to a degree 1 polynomial P as the hyperplane $P(\mathbf{x}) = 0$ (here P is only defined up to multiplication by elements in $\mathbb{C}^\times$). By lemma 6.1.3, we may assume that the denominator of the integrand in (6.1.1) is a product of hyperplanes with non-empty intersection. Thus for $i \in [r]$, $j \in [n]$, the set S of factors of the denominator contains at most one hyperplane of the form $A_{i,j}(k)$, and for each $j \in [n]$, at most r hyperplanes from the set $S_j = \{A_{1,j}(k_1), \ldots, A_{r,j}(k_r), B_j\}$.

To each hyperplane, we can associate a vector $(\boldsymbol{\mu}, -\nu)$, such that the hyperplane is $\langle \boldsymbol{\mu}, \mathbf{x} \rangle = \nu$, where the entries of $\boldsymbol{\mu}$ and $\mathbf{x}$ are indexed by $(i,j) \in [r] \times [n]$ (again, this is only defined up to multiplication by elements in $\mathbb{C}^\times$). If a pole occurs with multiplicity greater than 1, the set of vectors associated to S is linearly dependent. For $j \in [n]$, consider the set of vectors associated to S_j. Since each vector is in the span of the others, we may assume that B_j is not in S. In other words, we may assume S is a subset of $\{A_{i,j}(k_{i,j}) \mid i \in [r], j \in [n]\}$. But the corresponding set of rows is linearly independent, so there can be no poles with multiplicity greater than 1.

Finally, note that any hyperplane not of the form $|\mathbf{s}| = \text{const}$ will intersect $\{\mathbf{s} \in \mathbb{C}^r \mid \Re(|\mathbf{s}|) > nN_K\}$, but we showed that $\zeta_{\varphi,\mathbf{u}}(K; \mathbf{s})$ is analytic in this region.

$\square$

6.2 Relations between Laurent coefficients of Dirichlet series at $s = 0$

6.2.1 Values at $s = 0$

Friedman and Pereira prove the following theorem:

Theorem 6.2.1. *([9], thm 1.1) Let Q and P_j $(1 \leq j \leq r)$ be real polynomials in n variables, where each P_j is elliptic on $\mathbb{R}^n_{\geq 0}$, of degree d_j. Then the Dirichlet series*

$$Z(Q, P_j; s) := \sum_{\mathbf{n} \in \mathbb{Z}^n_{\geq 0}} Q(\mathbf{n}) P_j(\mathbf{n})^{-s}, \tag{6.2.1}$$

defined for $\Re(s) > (n + \deg Q)/\deg P$, can be analytically continued to $s = 0$, and the following product rule at $s = 0$ holds:

$$\left(\sum_{j=1}^{r} d_j\right) Z\left(Q, \prod_{j=1}^{r} P_j; 0\right) = \sum_{j=1}^{r} d_j \cdot Z(Q, P_j; 0). \tag{6.2.2}$$

Remark 6.2.2. The theorem remains true if we sum over $\mathbf{n} \in \mathbb{Z}^n_{>0}$ instead of $\mathbf{n} \in \mathbb{Z}^n_{\geq 0}$.

To see this, let

$$Z_+(Q, P_j; s) := \sum_{\mathbf{n}\in\mathbb{Z}^n_{>0}} Q(\mathbf{n})P_j(\mathbf{n})^{-s}, \tag{6.2.3}$$

and note that $Z_+(Q, P_j; s) = Z(Q^{[\mathbf{1}]}, P_j^{[\mathbf{1}]}; s)$.

Conversely, suppose one can show that (6.2.2) holds with Z replaced by Z_+, for all Q, P_j as in the statement of the theorem. Then since

$$Z(Q, P_j; s) = \sum_{I\subseteq[n]} \sum_{\mathbf{n}\in\mathbb{Z}^I_{>0}} Q(\mathbf{n}\!:\!\mathbf{0})P_j(\mathbf{n}\!:\!\mathbf{0})^{-s} = \sum_{J\subseteq[n]} Z_+(Q|_{\mathbf{x}_J=\mathbf{0}_J}, P_j|_{\mathbf{x}_J=\mathbf{0}_J}; s),$$

(6.2.2) follows.

6.2.2 The discrepancy of zeta regularized products

If $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is a sequence of positive numbers such that $Z(\mathbf{a}; s) := \sum_{n=1}^{\infty} a_n^{-s}$ converges absolutely for $\Re(s)$ sufficiently large, and extends to an analytic function in a neighbourhood of 0, then we define the zeta-regularized product

$$\widehat{\prod}\mathbf{a} := \exp(-Z'(\mathbf{a}; 0)).$$

(see [19]). In general, this construction does *not* commute with taking finite products: If $\mathbf{a}_j = (a_{j,n})_n$, $j = 1, \ldots, r$ are r sequences such that $\widehat{\prod}\mathbf{a}_j$ exists for $j = 1, \ldots, r$, and if $\widehat{\prod}(\prod_j \mathbf{a}_j)$ exists (where $\prod_j \mathbf{a}_j = (\prod_j a_{j,n})_n$ is the pointwise product), then $\prod_j \widehat{\prod}\mathbf{a}_j$ and $\widehat{\prod}(\prod_j \mathbf{a}_j)$ may be different.

The *discrepancy* $F_r(\mathbf{a}_1, \ldots, \mathbf{a}_r)$ of the zeta-regularized products is defined by

$$F_r = Z'\left(\textstyle\prod_j \mathbf{a}_j, 0\right) - \sum_j Z'(\mathbf{a}_j, 0),$$

so that

$$\exp(F_r) = \frac{\widehat{\prod}(\prod_j \mathbf{a}_j)}{\prod_j \widehat{\prod}\mathbf{a}_j}$$

measures the extent to which taking regularized products fails to commute with taking finite products.

In [2], Castillo-Garate and Friedman show that when the sequences $\mathbf{a}_j$ come from elliptic polynomials $P_1, \ldots, P_r$ in several variables evaluated at points in $\mathbb{Z}^n_{\geq 0}$, then the discrepancy can be expressed in terms of the discrepancies associated to pairs of distinct polynomials P_i. To be explicit,

Theorem 6.2.3. *([2] thm 1.1) Let $P_1, \ldots, P_r \in \mathbb{R}[x_1, \ldots, x_n]$ be elliptic polynomials, and let $d_j = \deg P_j$. Write $F_r(P_1, \ldots, P_r)$ for the discrepancy associated to the sets $\{P_j(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}_{\geq 0}}$ for $j = 1, \ldots, r$. Then*

$$\left(\sum_{j=1}^{r} d_j\right) F_r(P_1, \ldots, P_r) \;=\; \sum_{1\leq i<j\leq r} (d_i + d_j) F_2(P_i, P_j). \tag{6.2.4}$$

In terms of derivatives of Dirichlet series, this is

$$\left(\sum_{j=1}^{r} d_j\right) Z'(P_1 \cdots P_r; 0) \;=\; \sum_{1\leq i<j\leq r} (d_i + d_j) Z'(P_i P_j; 0) - (r-2) \sum_{j=1}^{r} d_j Z'(P_j; 0). \tag{6.2.5}$$

where $Z(P; s)$ is the meromorphic continuation of the series $\sum_{\mathbf{n}\in\mathbb{Z}^n_{\geq 0}} P(\mathbf{n})^{-s}$.

Note that remark 6.2.2 applies to theorem 6.2.3 too.

6.2.3 A general relation

We will prove a general theorem that implies the identities (6.2.2) and (6.2.5) when all functions are analytic at $s = 0$. In fact, (6.2.2) and (6.2.5) are true in general, provided we replace the zeta function by its 'regularization at $s = 0$':

$$\widehat{\zeta}_{\varphi,u}(K;s) := \zeta_{\varphi,u}(K;s) - s^{-1}\mathrm{Res}_{z=0}\zeta_{\varphi,u}(K;z), \quad s \neq 0,$$

which extends to a regular function at $s = 0$, assuming $\zeta_{\varphi,u}(K;z)$ has a pole of order at most 1 at $z = 0$.

If $d_I := \sum_{i\in I} d_i$ and $v_I := \prod_{i\in I} v_i$, set

$$C_r^k(i) \quad := \sum_{I\subseteq[r],\#I=i} d_I \cdot \widehat{\zeta}_{\varphi,v_I}{}^{(k-1)}(K;0) \tag{6.2.6}$$

for $k, r, i \geq 1$. We can use this to rewrite (6.2.2) and (6.2.5) when the sum defining $Z(P,s)$ is over $\mathbb{Z}^n_{>0}$. By expressing $Z(P;s)$ as a mixed zeta function, as we did in (4.1.1), we find that for $v_i = \widetilde{P}_i$, (6.2.2) becomes

$$C_r^1(r) \quad = \quad C_r^1(1), \tag{6.2.7}$$

and (6.2.5) becomes

$$C_r^2(r) \quad = \quad C_r^2(2) - (r-2)C_r^2(1) \tag{6.2.8}$$

These two identities are the cases $k = 1$ and $k = 2$ in the following theorem.

Theorem 6.2.4. *Suppose K satisfies hypotheses 5, and $(\varphi, \mathbf{v}) \in \mathcal{Z}^{\infty}_{0,\mathbf{d}}(\mathbb{R}^n_{\geq 0})$ for $\mathbf{d} \in \mathbb{R}^r_{>0}$. Then $d_j Res_{s=0}\zeta_{\varphi,v_j}(K;s)$ is independent of j, and for positive integers k*

and r *with* $r \geq k+1$,

$$C_r^k(r) \;=\; \sum_{i=1}^{k}(-1)^{k-i}\binom{r-1-i}{k-i}C_r^k(i). \tag{6.2.9}$$

Proof. Around $\mathbf{s} = \mathbf{0}$,

$$\zeta_{\varphi,v_1,\ldots,v_r}(K;\mathbf{s}) \;=\; \zeta_{\varphi,v_1^{1/d_1},\ldots,v_r^{1/d_r}}(K;d_1s_1,\ldots,d_rs_r) \;=\; \frac{\sum_{i=0}^{k}H_i(\mathbf{s})+O(\|\mathbf{s}\|^{k+1})}{d_1s_1+\ldots+d_rs_r},$$

where $H_i(\mathbf{s})$ is a homogeneous, degree i polynomial in $\mathbf{s}$. Therefore, for $I \subseteq [r]$,

$$\begin{aligned}\zeta_{\varphi,v_I}(K;s) \;&=\; \zeta_{\varphi,v_1,\ldots,v_r}(K;s\mathbf{1}_I\!:\!\mathbf{0})\\ &=\; \frac{\sum_{i=0}^{k}H_i(s\mathbf{1}_I\!:\!\mathbf{0})+O(s^{k+1})}{d_Is} = d_I^{-1}\left(\sum_{i=0}^{k}H_i(\mathbf{1}_I\!:\!\mathbf{0})s^{i-1}+O(s^k)\right)\end{aligned}$$

as $s \to 0$. We write

$$H_i(\mathbf{s}) = \sum_{\mathbf{j}\in[r]^i}\alpha_{\mathbf{j}}\prod_{\ell=1}^{i}s_{j_\ell},$$

where $\alpha_{\mathbf{j}} \in \mathbb{C}$ are constants which are invariant under permutations of the entries of $\mathbf{j}$. Thus $d_j\mathrm{Res}_{s=0}\zeta_{\varphi,v_j}(K;s) = H_0(\mathbf{1}_I\!:\!\mathbf{0}) = \alpha_\varnothing$ is independent of j, and for $k \in \mathbb{Z}_{>0}$,

$$d_I\widehat{\zeta}_{\varphi,v_I}{}^{(k-1)}(K;0) \;=\; (k-1)!H_k(\mathbf{1}_I\!:\!\mathbf{0}) \;=\; (k-1)!\sum_{\mathbf{j}\in I^k}\alpha_{\mathbf{j}}. \tag{6.2.10}$$

Let $s(\mathbf{j})$ be the set of entries in $\mathbf{j}$. Equations (6.2.6) and (6.2.10) imply

$$\frac{C_r^k(i)}{(k-1)!} \;=\; \sum_{I\subseteq[r],\#I=i}\;\sum_{\mathbf{j}\in I^k}\alpha_{\mathbf{j}} \;=\; \sum_{\mathbf{j}\in[r]^k}\;\sum_{s(\mathbf{j})\subseteq I\subseteq[r],\#I=i}\alpha_{\mathbf{j}} \;=\; \sum_{\mathbf{j}\in[r]^k}\binom{r-\#s(\mathbf{j})}{i-\#s(\mathbf{j})}\alpha_{\mathbf{j}}.$$

Therefore, if we put $j = \#s(\mathbf{j})$, the coefficient of $\alpha_{\mathbf{j}}$ in the right-hand side of (6.2.9) is $(k-1)!$ times

$$\begin{aligned}&\sum_{i=j}^{k}(-1)^{k-i}\binom{r-1-i}{k-i}\binom{r-j}{i-j}\\ &\quad=\; \sum_{i=j}^{k}\binom{k-r}{k-i}\binom{r-j}{i-j} = \sum_{\ell=0}^{k-j}\binom{k-r}{k-j-\ell}\binom{r-j}{\ell} = \binom{k-j}{k-j} = 1,\end{aligned}$$

by the Chu-Vandermonde identity. □

Remark 6.2.5. One can prove other relations in this manner. For example, under the hypotheses of the theorem,

$$\sum_{i=1}^{r} (-1)^i C_r^k(i) \quad = \quad 0. \tag{6.2.11}$$

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