EQUIVARIANT TENSORS ON POLAR MANIFOLDS

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A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2011

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Acknowledgments

I could not have arrived at this dissertation alone. Let me try to list everyone who has helped me in the work leading to this dissertation, and apologize in advance if I forget someone. Thank you!

Wolfgang for all the advice; generous donation of his time in frequent meetings; the secret seminar; the opportunity to go to Rio for a semester; and general support.

The rest of Wolfgang's group, my academic "brothers", of different generations: Martin Kerin, Chenxu He, Jason DeVito, John Olsen, Lee Kennard and Marco Radeschi.

Prof. Kirillov, for a couple of meetings where he gave me good advice about Coxeter groups.

A few combinatorialists who helped me with questions regarding generating functions: Prof. Herb Wilf, Paul Levande and Prof. Christian Krattenthaler.

A few geometers with whom I discussed my thesis projects: Prof. Thorbergsson, Claudio Gorodski, Marcos Alexandrino, Alexander Lytchak.

The Math Dept community at Penn for providing a nice environment: other graduate students , of different generations; faculty and staff, including the secretaries for taking care of so many helpless and clueless graduate students like me!

The community at IMPA, in Rio de Janeiro, where I spent one semester and got an important part of my thesis work done.

The audiences at Unicamp, in Campinas, Brazil, of talks I gave about my work, on two different occasions.

My home desktop computer, for doing computations that deal with exceptional Weyl groups, with code written in GAP ([23] and [57]) using the package CHEVIE ([24]), as well as the developers of these programs.

All the friends, teachers and mentors from Unicamp, where I studied mathematics from 2001 to 2006. At the beginning of my PhD I found myself quite well-prepared, and have them to thank for.

Finally my parents, Carmen and Manoel, for the support they have always showed, and still do! I might not have been able to pursue a career in Academia if it weren't for them.

ABSTRACT

EQUIVARIANT TENSORS ON POLAR MANIFOLDS

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This PhD dissertation has two parts, both dealing with extension questions for equivariant tensors on a polar G-manifold M with section $\Sigma \subset M$.

Chapter 3 contains the first part, regarding the so-called smoothness conditions: If a tensor defined only along Σ is equivariant under the generalized Weyl group $W(\Sigma)$, then it exends to a G-equivariant tensor on M if and only if it satisfies the smoothness conditions. The main result is stated and proved in the first section, and an algorithm is also provided that calculates smoothness conditions.

Chapter 4 contains the second part, which consists of a proof that every equivariant symmetric 2-tensor defined on the section of a polar manifold extends to a symmetric 2-tensor defined on the whole manifold. This is stated in detail in the first section, with proof. The main technical result used, called the Hessian Theorem, concerns the Invariant Theory of reflection groups, and is possibly of independent interest.

Contents

Chapter 1

Introduction

1.1 Symmetry and Reduction of variables

Let M be a Riemnnian manifold. The concept of symmetry of M is encoded in the action of a group G by isometries. For simplicity we will assume G to be a compact Lie group.

In dealing with a geometric problem in the presence of symmetries, one can restrict one's attention to the objects which are invariant, thus reducing the number of free variables in the problem to the so-called *cohomogeneity* of the action, which is defined to be the dimension of the orbit space M/G.

For example the Einstein equation is a PDE on the metric g:

$$\mathrm{Ric}(g) = \lambda g$$

where $\mathrm{Ric}(g)$ is the Ricci curvature tensor, and λ is a constant. But if there is a a group acting with cohomogeneity 1, and we consider only invariant metrics, the equation becomes an ODE. If the action is transitive, that is, cohomogeneity 0, then it reduces to an algebraic equation. More can be said in these highly symmetric cases than in general.

See [9], chapter 7, and [70] for more about the homogeneous case, and [18] for more about the cohomogeneity one case.

Here are more examples of the use of reduction of variables to study homogeneous and cohomogeneity one metrics satisfying some property:

- Positive Ricci curvature. See [6] for the homogeneous case and [32] for cohomogeneity one. In both cases admitting positive Ricci curvature is equivalent to the fundamental group being finite.

- Positive sectional curvature. See [2], [5], [7] and [69] for the classification of all simply-connected compact homogeneous spaces admitting positive sectional curvature. See [31] and [30] for cohomogeneity one.

- Ricci flow. See [44] (page 726) for a discussion of Ricci flow on homogeneous manifolds. It is an ODE, and it is useful to see the Ricci flow as a flow of Lie bracket structures with fixed metric instead of a flow of metrics with a fixed Lie bracket structure.

1.2 Polar actions

Cohomogeneity one actions have a nice property: if Gp is a principal orbit, then a geodesic γ which intersects Gp orthogonally will intersect every orbit, and always orthogonally. Thus we can view reduction of variables very concretely as restriction to γ.

From this point of view *polar* actions are a natural generalization of cohomogeneity one actions: By definition they are required to admit a submanifold Σ, called the section, which meets all orbits, and always orthogonally. Such a section is automatically totally geodesic and has dimension equal to the cohomogeneity of the action. Examples:

1. Polar coordinates (cohomogeneity one). Here M is the real plane $\mathbb{R}^2$, G is the group SO(2) of rotations. The orbits are the origin and all circles centered at the origin, and any line through the origin is a section. See Figure 1 below.

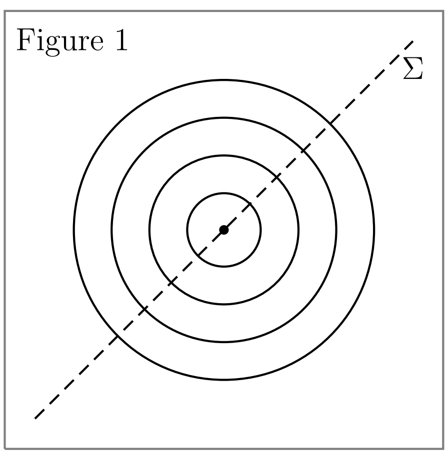

Figure 1

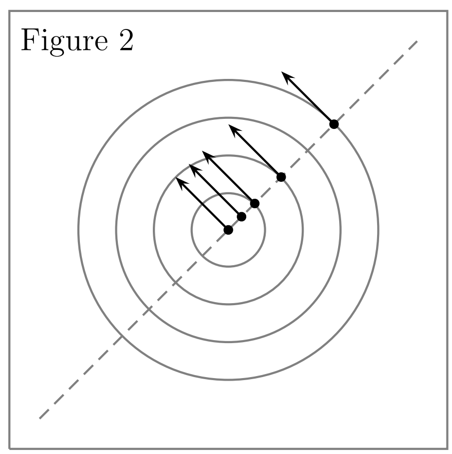
Figure 2

2. Lie groups (cohomogeneity equals the rank). Here M is a compact Lie group G with a bi-invariant metric, and G is the same group, acting on itself by conjugation. Any maximal torus is a section.

See chapter 2 for a longer review. Some references: [67] for a recent survey, [55] for the paper where polar actions on manifolds were introduced, and [15] where Dadok introduced and classified polar *representations*, that is, linear polar actions.

Note that topologically polar manifolds of cohomogeneity at least 2 behave differently from the cohomogeneity 0 and 1. Indeed, the latter are always rational elliptic, while polar manifolds of coh. ≥ 2 may be hyperbolic. See Example 2.1.1 for a sequence of cohomogeneity 2 polar manifolds with sum of Betti numbers going to infinity. By Corollary 2.9 in [21], every rational elliptic space of dimension n has sum of Betti numbers at most 2^n , and hence all but a finite number of elements in such a sequence must be rational hyperbolic.

Every section Σ is acted upon by a finite group called its *generalized Weyl group*, which we shall denote by $W(\Sigma)$. Its orbits coincide with the intersection of G-orbits with Σ, and the quotient spaces M/G and $\Sigma/W(\Sigma)$ are canonically isometric. In the first example above $W(\Sigma) = \pm 1$ and in the second one $W(\Sigma)$ is the usual Weyl group.

1.3 Smoothness conditions

The restriction map for invariant functions is an isomorphism:

$$C^\infty(M)^G \to C^\infty(\Sigma)^{W(\Sigma)}$$

This is called the Chevalley Restriction Theorem (see Theorem 2.3.6).

The corresponding statement for equivariant tensors other than functions is in general false. A W-equivariant tensor defined along Σ only extends to a G-equivariant tensor defined on all of M if it satisfies an extra set of conditions, called the *smoothness conditions*.

Thus computation of these smoothness condition allows one to define a G-equivariant tensor τ by specifying its values only along the section Σ. If one also translates a geometric property that one wants τ to have into a property of the restriction $\tau|_\Sigma$, one achieves reduction of variables for the problem at hand.

In the example of polar coordinates above, one can consider metrics defined along Σ of the form

$$g = dr^2 + (f(r))^2 d\theta^2$$

where f is an odd function. These are all smooth and $W(\Sigma)$-invariant tensors along Σ, but such a tensor extends smoothly to the whole plane if and only if f satisfies the smoothness condition $f'(0) = 1$. See figure 2 above for a pictorial illustration of this smoothness condition. There we see that a unitary radial vector based at the origin is the limit of vectors tangent to orbits, and whose lenghts converge to $f'(0)$.

My original thesis problem was to study such smoothness conditions for polar actions, and find formulas for them. For context, I found in the literature such formulas for specific cohomogeneity one actions, and for specific tensor bundles. See for example the calculation of the smoothness conditions for equivariant Sym^2-tensors on the Kervaire spheres starting on page 211 in [4]. In [18] there are some more general results in the cohomogeneity one case.

My results for this first part of the dissertation are contained in chapter 3. The statements are in the first section. The proofs are also, but refer to lemmas from the remainder of the chapter.

I did not find general formulas, except for the simplest examples of cohomogeneity one actions, see section 3.5. But there was enough general structure in these smoothness conditions that there is an algorithm to compute them in any given concrete example, see section 3.3.3. Such general structure also yields a proof that in the cohomogeneity-one case there is only a finite number of conditions at each singular point. Recently I saw a paper ([37]) in Representation Theory which seems to give formulas for smoothness conditions for the special but important class of adjoint actions of compact Lie groups, although not in the same language as this dissertation.

1.4 Symmetric 2-tensors

My second thesis problem was closely related to the smoothness conditions for the bundle $\mathrm{Sym}^2 T^*M$. It consisted of showing that all $W(\Sigma)$-equivariant symmetric 2-tensors on Σ extend to a G-equivariant symmetric 2-tensor on M (see theorem 4.1.1). The difference between this and the above is that at each $p \in \Sigma$ only the component in $\mathrm{Sym}^2 T_p\Sigma^*$ is specified, not the whole $\mathrm{Sym}^2 T_pM$. Because of this such extensions are not unique anymore.

Moreover, if the symmetric 2-tensor on Σ one starts with is a *metric*, then the extension to M may be chosen to be a metric as well, with respect to which the action of G is polar with the same sections.

This last part has some interesting applications to the study of polar actions, as it allows one to change the metric to one whose restriction to Σ has desirable properties, for instance with constant sectional curvature in the cohomogeneity two case. See example 2.1.1 for a construction using connected sums which produces in any dimension ≥ 4 polar manifolds having any closed orientable surface as section.

The context is a result due to Michor, called the Basic Forms Theorem (see theorem 4.1.2), which is an analogous statement for exterior differential forms. The proof relies on the Chevalley Restriction Theorem and a theorem about reflection groups due to Solomon, theorem 4.1.3. My proof is analogous to Michor's.

My work on this second problem is contained in chapter 4, including a detailed statement and the proof in section 4.1. The main algebraic lemma, corresponding to Solomon's theorem in the analogy with exterior forms, I called the Hessian Theorem (theorem 4.2.1), and may be of independent interest.

The proof of the Hessian Theorem ultimately relies on case-by-case arguments for the different types of Weyl groups. The exceptional groups of type E and F required computations which were performed by my home desktop computer using the program GAP and the package CHEVIE. See sections 4.3.7 and 4.3.6 for details. The code can be found at
http://dl.dropbox.com/u/374152/GAPcode.zip

1.5 More about polar actions

In the above I have emphasized extension questions for polar actions because there is where my work lies. But people have studied other aspects, and this section contains examples, with no attempt to be thorough. See [67] for a survey of polar actions.

- Relation with Isoparametric submanifolds: Palais and Terng's [56] , Terng's [64], Thorbergsson's [66].
- Copolarity. A generalization of polar actions by Gorodski, Olmos and Tojeiro: [28].
- Singular riemannian foliations with sections: Alexandrino and Gorodski's [1].
- Polar representations: In addition to Dadok's [15] which introduced and classified them see also Eschenburg and Heintze's [17]. See Dadok and Kac's [16] for the complex version.
- Equivariant theories: Hurder and Töben have studied the LS category of a polar manifold in [34].
- Classification of polar actions for special classes of total manifold M: Symmetric Spaces (Kollross [38],[39], [40]), compact Euclidean hypersurfaces (Moutinho, Tojeiro [50]).
- Generalizing to polar actions results about the adjoint action of a Lie group: Weyl-type integration formula by Magata [46], Chevalley Restriction Theorem by Terng using theory of isoparametric submanifolds (see theorem 2.3.6)

Chapter 2

Review of polar actions

2.1 Polar Actions

In this section I define polar actions and some of the objects associated to them, as well as the basic facts about them. A good reference is [55].

2.1.1 Definition and examples

Definition 2.1.1. Let G be a compact Lie group and M a Riemannian manifold on which G acts by isometries. We say that the action is *polar* if there is a connected, closed, embedded submanifold $\Sigma \subseteq M$, called a *section*, which meets all the orbits orthogonally. That is, $G \cdot \Sigma = M$ and, for every $x \in \Sigma$, $T_x\Sigma$ is contained in the slice $T_x(Gx)^{\perp}$.

If in addition Σ is flat (with the induced metric), we call the action hyperpolar.

Definition 2.1.2. Given a polar G-manifold M with section Σ, one defines:

- the *normalizer* by $N(\Sigma) = \{g \in G \mid g \cdot \Sigma = \Sigma\}$;
- the *centralizer* by $Z(\Sigma) = \{g \in G \mid gx = x \quad \forall x \in \Sigma\}$; and
- the *generalized Weyl group* by $W_\Sigma = \mathrm{N}(\Sigma)/\mathrm{Z}(\Sigma)$.

Note that $N(\Sigma)$ acts on Σ with innefective kernel $Z(\Sigma)$, so that W_Σ acts effectively on Σ.

Here are some examples:

- $G = S^1$ acting by rotations on $M = \mathbb{R}^2$. The sections are the straight lines through the origin. They have trivial centralizer and both the normalizer and the generalized Weyl group are $\{\pm 1\}$.

- G any compact Lie group, M is G with a bi-invariant metric, and the action is by conjugation. Any maximal torus is a section, with centralizer equal to itself, and the generalized Weyl group is the usual Weyl group.

- Linearization of the above: M is the Lie algebra of G, action is given by Ad. The sections are the Cartan subalgebras, and the generalized Weyl group is again the usual Weyl group.

- $G = O(n)$ acting on the space M of real symmetric $n \times n$ matrices by conjugation. The metric is given by $< A, B >= \mathrm{tr(AB)}$. A section is the set Σ of all diagonal matrices. Its generalized Weyl group is the symmetric group S_n, and it acts on Σ by permuting the diagonal entries.

  Note that this representation is the isotropy representation of the symmetric space $GL(n)/O(n)$. We will see later that every isotropy representation of a symmetric space is polar.

- Any finite group action on a connected manifold is polar, the section being the whole manifold.

- Hermann actions. Let G be a compact Lie group, and H, K symmetric subgroups of G. This means that there are involutions θ_1 and θ_2 of G such that
  $$(F_1)_0 \subseteq H \subseteq F_1 \quad \text{and} \quad (F_2)_0 \subseteq K \subseteq F_2$$
  where $F_i = \mathrm{Fix}(\theta_i)$ are the fixed-point subgroups associated to the θ_i. Then the action of H on the symmetric space G/K is hyperpolar. See for example [26] for more about Hermann actions.

2.1.2 Elementary facts

Let M be a Riemannian G-manifold, and M_0 the union of principal orbits (i.e., the regular part). Since the action of G on M_0 has only one isotropy type, the quotient M_0/G is a Riemmanian manifold and the projection map $M_0 \to M_0/G$ is a Riemannian submersion. Denote by $\mathcal{V}$ the distribution tangent to the orbits, called the *vertical* distribution, and by $\mathcal{H}$ the distribution perpendicular to the orbits, called the *horizontal* distribution.

If the action is polar, the horizontal distribution is integrable, the leaves being the intersections of sections with M_0. This implies:

Proposition 2.1.1. *Let M be a polar G-manifold. Then the sections are totally geodesic submanifolds of M. In particular there is exactly one section passing through any regular point $x \in M$, namely $exp(T_x(G \cdot x)^{\perp})$.*

Corollary 2.1.1. *G acts transitively on the set of sections. The generalized Weyl groups associated to any two sections are isomorphic.*

Proof. Let Σ_1 and Σ_2 be two sections. Choose a principal orbit O, points $x_i \in O \cap \Sigma_i$ and $g \in G$ such that $x_2 = gx_1$. Since $g\Sigma_1$ is a section passing through x_2, the proposition above implies that $\Sigma_2 = g\Sigma_1$.

To prove the second statement note that $N(g\Sigma) = gN(\Sigma)g^{-1}$ and $Z(g\Sigma) = gZ(\Sigma)g^{-1}$. □

Corollary 2.1.2. *Let G be a compact Lie group acting by isometries on the Riemannian manifold M, and Σ a submanifold of M. Denote by G_0 the connected component of G containing the identity element. Assume M is connected. Then Σ is a section for the G-action if and only if it is a section for the induced G_0-action. In particular the action of G is polar if and only if the action of G_0 is polar.*

Proof. It is clear that Σ meets G-orbits orthogonally if and only if it meets G_0-orbits orthogonally, so it's enough to show that $G\Sigma = M$ if and only if $G_0\Sigma = M$.

The other direction being obvious, let's assume that $G\Sigma = M$ and show that $G_0\Sigma = M$. Since Σ is a section for the G-action, choosing an $x \in \Sigma$ we get that $\Sigma = \exp(\mathrm{T_x(G \cdot x)}^\perp)$ by the proposition above.

Let $y \in M$. Since M is connected, there is a minimal geodesic from y to the orbit $G_0 \cdot x$. Composing with some $g \in G_0$ we get a geodesic from $G_0 \cdot y$ to x which meets $G_0 \cdot x$ orthogonally. Therefore Σ meets all G_0-orbits. □

Proposition 2.1.2. *Let M be a polar G-manifold with section Σ, and x be a point in Σ. Denote by K the isotropy subgroup G_x, and by $V = T_x(Gx)^\perp$ the slice at x. Then the slice representation of K on V is polar, and $T_x\Sigma$ is a section. Moreover, the generalized Weyl group of $T_x\Sigma$ is just the isotropy $(W_\Sigma)_x$.*

Proof. (See Theorem 4.6 in [55].)

First we prove that $T_x\Sigma$ meets orbits orthogonally. Let $v, w \in T_x\Sigma$, and $X \in \mathfrak{k}$. We want to show that Xv is orthogonal to w in the flat metric. On a small ball around the origin we also have the pull-back metric by $\exp_x$, and they coincide at the origin. The vectors Xv and w at tv are orthogonal in the pull-back metric for every $t > 0$, because $\frac{1}{t}X \cdot \exp_x(tv)$ is tangent to the orbit $G \exp_x(tv)$ at $\exp_x(tv) \in \Sigma$ and Σ is a section of M. Since the pull-back metric at tv converges to the flat metric as $t \to 0$, we are done with the first part.

Now we show that $T_x\Sigma$ meets all orbits. Let $v \in V$ be a regular vector of the slice representation. Since Σ is a section for M, there is $g \in G$ such that $y = g \cdot \exp_x(v)$ belongs to Σ.

But $y = \exp_{g_1x}(w)$ for some $g_1 \in G$ and $w \in T_{g_1x}(Gx)^\perp$. Since $Gy = G\exp_x(v)$ is a principal orbit, $\Sigma = \exp_y(T_y(Gy)^\perp)$, and in particular $g_1x \in \Sigma$. This implies that there is $g_2 \in N(\Sigma)$ such that $g_2g_1x = x$, that is, $g_2g_1 \in K$. Therefore $(g_2g_1)v \in T_x\Sigma$.

We have proved that $T_x\Sigma$ meets every principal K-orbit in V, or, in other words, that $V_0 \subseteq K \cdot T_x\Sigma$, where V_0 denotes the set of regular vectors. To finish the proof we observe that since K is compact, $K \cdot T_x\Sigma$ is closed, and that V_0 is dense in V, so that $K \cdot T_x\Sigma = V$. □

This proposition together with the slice theorem in some sense reduces the local study of polar manifolds to the study of polar representations.

Here are some more elementary facts. Proofs can be found in [55].

Proposition 2.1.3. *Let G be a compact Lie group with a polar action on M, with section Σ. Then:*

- *The generalized Weyl group W_Σ is finite;*
- *Let $x \in \Sigma$. Then $(G \cdot x) \cap \Sigma = W_\Sigma \cdot x$;*
- *The inclusion $\Sigma \subseteq M$ induces an isometry between the quotients $\Sigma/W_\Sigma = M/G$;*

2.1.3 Connected Sums

In this section we describe the operation of connected sum of two polar G-manifolds at fixed points with equivalent isotropy representations, which I learned from [29]. We need a lemma:

Lemma 2.1.1. *Let (M, σ_0) be a polar G-manifold with section Σ. Assume it has a fixed point p. Then for small enough $\epsilon > 0$, there is a metric σ on $M - p$ such that:*

a) $\sigma = \sigma_0$ on $M - B_p(2\epsilon/3)$;

b) $\sigma|_{B_p(\epsilon/3)-p}$ is the product metric on $S(T_pM) \times (0, \epsilon/3)$; and

c) $\Sigma - p$ is σ-orthogonal to the G-orbits.

Proof. Let $\epsilon > 0$ be such that the exponential map

$$\exp_p : B_0^{T_pM}(\epsilon) \to B_p(\epsilon)$$

is a (G-equivariant) diffeomorphism, and let

$$\psi : S(T_pM) \times (0, \epsilon) \to B_0^{T_pM}(\epsilon) - 0$$

be the (G-equivariant) diffeomorphism given by $(\theta, r) \mapsto r\theta$, where $S = S(T_pM)$ denotes the unit sphere in T_pM. From now on we are going to identify $B_p^M(\epsilon) - p$ with $S \times (0, \epsilon)$ through $\exp_p \circ \psi$.

Let $\rho : (0, \epsilon) \to [0, 1]$ be a smooth function which equals 1 on $(0, \epsilon/3)$ and 0 on $(2\epsilon/3, \epsilon)$, and $\tilde{\rho} : S \times (0, \epsilon) \to [0, 1]$ be given by $\tilde{\rho}(\theta, r) = \rho(r)$.

Denoting by σ_1 the product metric on $S \times (0, \epsilon)$, define the new metric σ by

$$\sigma = (1 - \tilde{\rho})\sigma_0 + \tilde{\rho}\sigma_1$$

Since σ_0, σ_1 and $\tilde{\rho}$ are G-invariant, so is σ. Since $(\Sigma - 0) \cap B_p(\epsilon)$ is both σ_0- and σ_1-orthogonal to orbits, it is also σ-orthogonal to orbits. ∎

Now to the construction:

Let M_1, M_2 be two polar G-manifolds with sections Σ_1, Σ_2. Assume they have fixed points p_1, p_2 whose isotropy representations are equivalent. Use the lemma above to find metrics σ_1 and σ_2 on $M - p_1$ and $M - p_2$. Then, using a G-equivariant isometry $T_{p_1}M_1 \to T_{p_2}M_2$ we find a G-equivariant isometry

$$\phi : (B_{p_1}(\epsilon/3) - p_1,\ \sigma_1) \to (B_{p_2}(\epsilon/3) - p_2,\ \sigma_2)$$

By replacing Σ_2 with another section if necessary, we may assume that ϕ sends $\Sigma_1 \cap B_{p_1}(\epsilon/3) - p_1$ to $\Sigma_2 \cap B_{p_2}(\epsilon/3) - p_2$.

Define the connected sum by

$$M_1 \# M_2 = (M_1 - p_1) \cup_\phi (M_2 - p_2)$$

Since Σ_1 and Σ_2 match under ϕ, we get a submanifold

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 - p_1) \cup_\phi (\Sigma_2 - p_2) \subseteq M_1 \# M_2$$

Proposition 2.1.4. *With the notations above, $M_1 \# M_2$ is a polar G-manifold, and the submanifold $\Sigma_1 \# \Sigma_2$ meets all the orbits orthogonally, and therefore is a section if it is connected. In general any of its connected components is a section. (Of course, $\Sigma_1 \# \Sigma_2$ can be disconnected only in the cohomogeneity one case)*

Moreover $N(\Sigma_1 \# \Sigma_2) = N(\Sigma_1) = N(T_{p_1}\Sigma_1) = N(T_{p_2}\Sigma_2) = N(\Sigma_2)$, and similarly for the centralizers $Z(\cdot)$. Thus if $\Sigma_1 \# \Sigma_2$ is connected, its generalized Weyl group is isomorphic to the ones associated to Σ_1 and Σ_2.

Here are two examples:

Example 2.1.1. Let S^1 act on S^2 by rotating around an axis, with fixed points n and s, and let $G = S^1 \times S^1$ be the product action on $M = S^2 \times S^2$. Denote by $S^1 \subseteq S^2$ any great circle passing through n and s, that is, any section for the S^1-action. Then $\Sigma = S^1 \times S^1 \subseteq M$ is a section for the G action, and its generalized Weyl group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. G acts with four fixed points: (n, n), (n, s), (s, n) and (s, s). They all have equivalent isotropy representations.

Thus we get a polar G-action on $M^{\#g}$ for any $g = 0, 1, 2, 3, \ldots$, with section a genus g closed orientable surface, and generalized Weyl group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $n, m \geq 2$. Replacing one of the copies of S^1 acting on S^2 with $SO(n)$ acting on S^n and the other with $SO(m)$ acting on S^m, we get a polar $SO(n) \times SO(m)$-manifold $(S^n \times S^m)^{\#g}$, with section the surface of genus g and generalized Weyl group $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Note that by theorem 4.1.1 there is a metric on these spaces, also equivariant and polar with the same sections, which restricted to Σ has constant sectional curvature.

Example 2.1.2. $G = S^1$ acting on $M = \mathbb{R}^2$ by rotations, $\Sigma = \mathbb{R}$. Then $M \# M$ is diffeomorphic to a cylinder, and $\Sigma \# \Sigma$ is the disjoint union of two lines. These sections have trivial generalized Weyl group.

There are other surgeries one can make on polar manifolds, for example cutting along an orbit which is bigger than just one point, or even a whole stratum. It would be interesting to work out the exact conditions under which one can do such surgeries and the effect it has on the sections and their generalized Weyl groups.

It would also be interesting to find out what is the relation between these surgery procedures and the process of desingularization of a polar manifold (see for example [68] or [10])

2.1.4 Equivariant maps from a polar manifold

Let's now consider (smooth) G-equivariant maps from M to a G-manifold N. Such a map is completely determined by its restriction to a section Σ, because Σ meets all orbits. The restriction is automatically invariant under the normalizer $N(\Sigma)$. In other words, the restriction map is injective:

$$|_\Sigma : C^\infty(M, N)^G \to C^\infty(\Sigma, N)^{N(\Sigma)}$$

The same is true if we replace smooth with some other regularity condition, or no regularity condition at all, for that matter. I choose smooth maps because these are the ones I am personally most interested in. But as we will see later, in the case where M and N are vector spaces with linear actions, considering *polynomial* maps turns out to give important information about the smooth ones.

Restriction to Σ is generally not surjective. Here is a simple example: $G = U(1)$, $M = \mathbb{C}$, $N = \mathbb{C}$. $z \in U(1)$ acts by sending $m \in M$ to zm and $n \in N$ to $z^2 n$. A section for M is $\Sigma = \mathbb{R}$, and the normalizer is $N(\Sigma) = \pm 1$. Thus $N(\Sigma)$ acts trivially on N, and any even map $\mathbb{R} \to N$ is $N(\Sigma)$-equivariant. On the other hand, if $F : M \to N$ is G-equivariant, then $F(0)$ must be zero.

But sometimes it is surjective! For example, when $N = \mathbb{R}$ with trivial G-action, that is, for the case of real-valued *invariant* functions. This is called the Chevalley Restriction Theorem (see section 2.3.6 below).

The conditions under which an $N(\Sigma)$-equivariant map from Σ to N extends to a G-equivariant map from M to N are called the *smoothness conditions*. My thesis problem consisted of finding such conditions in the particular case where N is the total space of a G-equivariant bundle over M, and the maps considered are equivariant bundle sections.

Let's finish this section with a simple lemma relating smoothness conditions for the polar actions of G and G_0 on M:

Lemma 2.1.2. *Let M be a polar G-manifold with section Σ. Denote by G_0 the identity component of G, and by $N(\Sigma)_0$ the normalizer of Σ in G_0, that is, $G_0 \cap N(\Sigma)$. Then:*

a) $G_0 \cdot N(\Sigma) = G$

b) Let N be a G-manifold. Then

$$|_\Sigma(C^\infty(M,N)^{G_0}) \cap C^\infty(\Sigma,N)^{N(\Sigma)} = |_\Sigma(C^\infty(M,N)^G)$$

In words, the set of smoothness conditions that governs whether an $N(\Sigma)$-equivariant function on Σ extends to a G-equivariant function on M is the same as the corresponding set of conditions associated to G_0.

Proof. a) Let $g \in G$. Since $g\Sigma$ is a section for the action of G_0, and G_0 acts transitively on the set of sections, there is $g_0 \in G_0$ such that $g_0 g\Sigma = \Sigma$. Thus $g_0 g \in N(\Sigma)$, and $g \in G_0 N(\Sigma)$.

b) Let $f \in C^\infty(M,N)^{G_0}$ such that $f|_\Sigma$ is $N(\Sigma)$-equivariant. We need to show that f is G-equivariant.

Let $g \in G$, and use part a) above to write $g = g_0 g_1$, with $g_0 \in G_0$ and $g_1 \in N(\Sigma)$. Then $g_1 \cdot f = f$ because $(g_1 \cdot f)|_\Sigma = g_1 \cdot (f|_\Sigma) = f|_\Sigma$. Therefore $g \cdot f = g_0 \cdot (g_1 \cdot f) = f$.

□

2.2 Polar Representations

Polar representation were introduced and essentially classified in [15]. See also [16], and [17].

Definition 2.2.1. We say that a polar action of the compact Lie group K on an Euclidean vector space V is a polar representation if the action is linear.

Note that a section Σ is always a vector subspace. Indeed, it is totally geodesic and must pass through the orbit $\{0\}$.

Definition 2.2.2. Let K be a compact, connected Lie group with Lie algebra $\mathfrak{k}$, and $\pi : K \to SO(V)$ a representation. We say that π is a *symmetric space representation* if there is a real semisimple Lie algebra $\mathfrak{h}$ with a Cartan decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$, a Lie algebra isomorophism $A : \mathfrak{k} \to \mathfrak{g}$ and a vector space isomorphism $L : V \to \mathfrak{p}$ such that

$$L(\pi(X)(y)) = [A(X), y] \qquad \text{for all } X \in \mathfrak{k},\ y \in \mathfrak{p}$$

Theorem 2.2.1. *Symmetric space representations are polar. The sections are the maximal flats, that is, the maximal abelian subalgebras of $\mathfrak{k}$ contained in $\mathfrak{p}$.*

Proof. (taken from [8] - see theorem 3.2.13) Let $v \in \mathfrak{p}$ be a regular vector, and let $\Sigma = \nu_v(Gv)$ be the normal space at v to the principal orbit through v.

i) Σ meets all the orbits. Indeed, let $w \in \mathfrak{p}$. We may assume that $d(v,w) = d(Gv, Gw)$. Then $v - w$ is orthogonal to $T_v(Gv)$, which implies $w \in \Sigma$.

ii) Σ is an abelian subalgebra. First note that $\Sigma = \{u \in \mathfrak{p} \mid [v,u]=0\}$, because $< u, [\mathfrak{g}, v] >=< \mathfrak{g}, [u,v] >$. Now let $u, w \in \Sigma$. We claim that $[u,w]=0$. Indeed, by the Jacobi identity we get

$$[[u,w],v] = [[u,v],w] + [u,[w,v]] = 0$$

which implies that $[u,w] \in \mathfrak{g}_v$. Now, since v is regular, and w is in the slice through v, $\mathfrak{g}_v \subseteq \mathfrak{g}_w$, so that $[[u,w],w]=0$. Therefore

$$< [u,w],[u,w] >= - < [[u,w],w],u >= 0$$

and so $[u,w]=0$, as we wanted.

iii) Σ meets orbits orthogonally. Indeed, if $u, w \in \Sigma$, then

$$< [\mathfrak{g}, u], w >=< \mathfrak{g}, [u,w] >= 0$$

iv) Σ is a *maximal* abelian subalgebra. This follows from $\Sigma = \{u \in \mathfrak{p} \mid [v,u]=0\}$.

□

In [15] Dadok studied polar representations, and found a classification of *irreducible* polar representations of *connected* compact Lie groups, which implies:

Theorem 2.2.2. *Let V be a (not necessarily irreducible) polar representation of the connected compact Lie group K. Then there is a connected compact Lie group $\tilde{K}$ with a symmetric space representation on V which is orbit equivalent to the K-representation.*

As a consequence we get:

Proposition 2.2.1. *Let K be a connected compact Lie group, and V a polar K-representation with section Σ. Then the image of the generalized Weyl group W_Σ in $O(\Sigma)$ is generated by reflections across hyperplanes, and is crystallographic, that is, it is a Weyl group.*

Proof. By the theorem above, $K \to SO(V)$ is orbit equivalent to a symmetric space representation. Since orbits of the generalized Weyl group parametrize the intersection of K-orbits with Σ, the actions of the two generalized Weyl groups on Σ are also orbit-equivalent. Now, for finite group representations orbit equivalence implies equivalence. Therefore W_Σ is generated by reflections because that is the case for symmetric space representations. □

2.3 Isoparametric Submanifolds and the Chevalley Restriction Theorem

There is an important interplay between representations and the geometry of their orbits. Principal orbits of polar representations have particularly simple (submanifold geometric) invariants: they are *isoparametric submanifolds*. See Thorbergsson's survey article [66] for the rich history of this subject. There is also a good exposition of the theory in [8].

Definition 2.3.1. Let N be an embedded, complete, connected submanifold of the Euclidean vector space V. We say N is isoparametric if the normal bundle is globabally flat and the principal curvatures along any parallel normal field are constant.

Proposition 2.3.1. *Principal orbits of polar representations of a* connected *compact Lie group are isoparametric.*

Proof. See Theorem 5.7.1 in [56], or [8]. □

Thorbergsson has proved a partial converse to the above proposition:

Theorem 2.3.1 (Thorbergsson)**.** *Let $\mathcal{F}$ be an isoparametric foliation of $\mathbb{R}^{n+r}$ whose isoparametric submanifolds are compact, irreducible and full with codimension $r \geq 3$. Then there is a symmetric space $X = G/K$ and an isometry $A : R^{n+r} \to T_{(K)}X$ that carries the leaves of $\mathcal{F}$ onto the orbits of the isotropy representation of X.*

Proof. See [65]. □

The full converse is false. There are non-homogeneous isoparametric submanifolds of codimension 2: See [19] or [53] and [54]. See [51] and [52] for the theory of compact rank 2 isoparametric submanifolds, or, equivalently, isoparametric hypersurfaces *in spheres.*

Terng has discovered that to an isoparametric submanifold N of V there is an associated finite reflection subgroup W^q acting on the normal space $\nu_q N$ at any point $q \in N$.

Theorem 2.3.2. *Given a homogeneous degree k W^q-invariant polynomial on $\nu_q N$, it extends to a unique homogeneous degree k polynomial on V which is constant on N.*

Proof. See Theorem C in [64]. □

When N is a principal orbit of a polar representation, the reflection group W^q coincides with the generalized Weyl group, and the theorem above then implies:

Theorem 2.3.3 (Chevalley Restriction Theorem for polynomials)**.** *Let V be a polar K-representation, where K is a compact Lie group, Σ be a section, and W_Σ its generalized Weyl group. Then restriction to Σ gives an isomorphism*

$$|_\Sigma : \mathbb{R}[V]^K \to \mathbb{R}[\Sigma]^{W_\Sigma}$$

where $\mathbb{R}[V]^K$ denotes the set of K-invariant polynomial functions on V and similarly for $\mathbb{R}[\Sigma]^{W_\Sigma}$.

Proof. (Sketch) Let K_0 be the connected component of $\{e\}$ in K, W_Σ^0 the corresponding generalized Weyl group, and N a principal K_0-orbit. Then N is an isoparametric submanifold. If $f \in \mathbb{R}[\Sigma]^{W_\Sigma}$, then by the theorem above there is a unique $F \in \mathbb{R}[V]$ which restricts to f and is constant on N. By examining the proof of the theorem above one can see that F is actually constant on all nearby principal orbits, and therefore $F \in \mathbb{R}[V]^{K_0}$. Finally we apply Lemma 2.1.2 to conclude that $F \in \mathbb{R}[V]^K$. □

It's a fact due to Hilbert that $\mathbb{R}[V]^K$ is finitely generated (this is true for any linear action of a compact group, not only polar representations). Let $\rho_1, \ldots, \rho_n$ be generators. Then one has the following theorem:

Theorem 2.3.4 (Schwarz)**.** *Let $\rho : V \to \mathbb{R}^n$ be the map whose coordinates are $\rho_1, \ldots, \rho_n$. Then composition with ρ gives a surjection*

$$\rho^* : C^\infty(\mathbb{R}^n) \to C^\infty(V)^K$$

Proof. See [58]. □

Using this one can prove the smooth version of the Chevalley Restriction Theorem for polar representations:

Theorem 2.3.5. *Let V be a polar representation of the compact Lie group K, with section Σ and generalized Weyl group W_Σ. Then restriction to Σ is an isomorphism:*

$$|_\Sigma : C^\infty(V)^K \to C^\infty(\Sigma)^{W_\Sigma}$$

Proof. By the Chevalley Restriction Theorem for polynomials, $\rho_1|_\Sigma, \ldots, \rho_n|_\Sigma$ generate $\mathbb{R}[\Sigma]^{W_\Sigma}$. So, given $f \in C^\infty(\Sigma)^{W_\Sigma}$, we can apply Schwarz's theorem to the action of W_Σ on Σ to get a $F \in C^\infty(\mathbb{R}^n)$ such that $f = F \circ \rho|_\Sigma$. Therefore f extends to the K-invariant smooth function $F \circ \rho$. □

Combining this with the Slice Theorem one gets the full-blown version of the Restriction Theorem:

Theorem 2.3.6 (Chevalley Restriction Theorem)**.** *Let M be a polar G-manifold with section Σ. Then restriction to Σ is an isomorphism:*

$$|_\Sigma : C^\infty(M)^G \to C^\infty(\Sigma)^{W_\Sigma}$$

Proof. Let $f \in C^\infty(\Sigma)^{W_\Sigma}$. Then f can be extended uniquely to a G-invariant function $F : M \to \mathbb{R}$, and we need to prove that F is smooth.

Let $x \in M$, $K = G_x$, $V = \nu_x(G \cdot x)$, and $\epsilon > 0$ such that

$$\exp^\perp : G \times_K V^\epsilon \to \mathcal{U}$$

is a G-equivariant diffeomorphism onto the G-invariant open set $\mathcal{U}$ in M.

With this identification in mind, $f|_{T_x(\Sigma)^\epsilon}$ is a $W_{T_x(\Sigma)}$-invariant smooth function, and thus, by the Restriction Theorem for representations, $F|V^\epsilon$ is smooth. But since K acts freely on $G \times V^\epsilon$, restriction to the slice is an isomorphism

$$|_{V^\epsilon} : C^\infty(G \times_K V^\epsilon)^G \to C^\infty(V^\epsilon)^K$$

and therefore $F|_\mathcal{U}$ is smooth, as we wanted to show. $\square$

Chapter 3

Equivariant tensors - Smoothness Conditions

3.1 Introduction and Main Theorem

Let M be a polar G-manifold with section Σ and consider a G-equivariant vector bundle $\pi : E \to M$. Examples of E include the tangent and cotangent bundles TM and T^*M, and other tensor bundles. Let $\sigma : M \to E$ be a smooth G-equivariant cross-section.

As in the case of functions, σ is completely determined by its restriction to Σ. In other words the map

$$|_\Sigma : C^\infty(E)^G \to C^\infty(E|_\Sigma), \qquad \sigma \mapsto \sigma|_\Sigma$$

is injective. The goal of this chapter is to describe the image of this map.

First note that $E|_\Sigma$ is a $N(\Sigma)$-equivariant vector bundle, where $N(\Sigma)$ is the normalizer of Σ in G, $\{g \in G \mid g\Sigma = \Sigma\}$. So the centralizer $Z(\Sigma) = \{g \in G \mid gp = p \ \forall p \in \Sigma\}$ also acts on $E|_\Sigma$, and the union of $E_p^{Z(\Sigma)}$ for $p \in \Sigma$ turns out to form a vector subbundle $(E|_\Sigma)^{Z(\Sigma)}$ of $E|_\Sigma$. Since $Z(\Sigma)$ acts trivially on this subbundle, it is a $W(\Sigma) = N(\Sigma)/Z(\Sigma)$-equivariant vector bundle over Σ.

Theorem 3.1.1 (Main Theorem)**.** *Let M be a polar G-manifold with section Σ, and let $E \to M$ be a G-equivariant vector bundle over M. Then*

1. *Let $\tau \in C^\infty(E|_\Sigma)$ be a cross-section of E defined along Σ. It extends to a $\tilde{\tau} \in C^\infty(E)^G$ if and only if*
   - *τ has image in the subbundle $(E|_\Sigma)^{Z(\Sigma)}$;*
   - *τ is $W(\Sigma)$-equivariant; and*
   - *for every* singular *point $p \in \Sigma$, τ satisfies a certain set of "smoothness conditions" at p, which are linear conditions on the Taylor series of τ at p.*

2. *For every singular point $p \in \Sigma$, there is an algorithm to compute the "smoothness conditions".*

3. *If dim $\Sigma = 1$, at every singular point $p \in \Sigma$ there is only a* finite *number of smoothness conditions.*

See section 3.5 for formulas that give the smoothness conditions in the two simplest cases, namely at points whose slice representation is either $S^1 \subset O(\mathbb{R}^2)$ or $SO(3) \subset \mathbb{R}^3$. They work for any equivariant bundle.

The main technical algebraic result we need is:

Theorem 3.1.2. *Let $V = (T_p(G \cdot p))^\perp$ be the slice at p and $K = G_p$ the isotropy. Denote by $\mathbb{R}[V, E_p]^K$ the space of polynomial maps from V to E_p which are K-equivariant, and similarly for $\mathbb{R}[T_p\Sigma, E_p^{Z(\Sigma)}]^{(W_\Sigma)_p}$.*

These are modules over the ring of invariant polynomials $\mathbb{R}[V]^K = \mathbb{R}[T_p\Sigma]^{(W_\Sigma)_p}$ which are free of the same rank, namely $l = \dim(E_p)^{Z(\Sigma)}$.

The proof of this theorem involves complexifying the group K and the K-representations V and E_p, then using results from Representation Theory by G. Schwarz, V. Kac, J. Dadok, B. Kostant and others (for example [59], [16], [42]), and pulling these results back to the real case. For more details see section 3.3 below.

Proof of Theorem 3.1.1. We break the proof into steps:

Let $\tau \in C^\infty(E|_\Sigma)$.

1. It is clear that if it is the restriction of a $\tilde{\tau} \in C^\infty(E)^G$, then at every $p \in \Sigma$, $\tau(p) \in E_p$ is fixed by $Z(\Sigma)$.

2. This is a local problem: τ extends to $\tilde{\tau} \in C^\infty(E)^G$ if and only $\tau|_{\mathcal{U}\cap\Sigma}$ extends, where $\mathcal{U}$ runs through a covering of M by G-stable open sets.

3. Let $p \in \Sigma$ be any point, and we will look at a small tubular neighbourhood $\mathcal{U}$ around the orbit through p.

   Consider $V = (T_pGp)^\perp$, the slice at p. It is a polar $K = G_p$-representation with section $T_p\Sigma$ and generalized Weyl group $W(\Sigma)_p$ (see Proposition 2.1.2).

   Using the Slice Theorem, there is a tubular neighbourhood $\mathcal{U}$ of Gp which is G-equivariantly diffeomorphic to $G \times_K V$. Identify V and $T_p\Sigma$ with the images of $\{e\} \times V$ and $\{e\} \times T_p\Sigma$ in $\mathcal{U}$. By Proposition 3.2.1 the restriction mapping
$$|_V : C^\infty(E|_\mathcal{U})^G \to C^\infty(E|_V)^K$$
   is an isomorphism. Similarly, the restriction map
$$|_\Sigma : C^\infty(E_{\Sigma\cap\mathcal{U}})^{N(\Sigma)} \to C^\infty(E|_{T_p\Sigma})^{N(\Sigma)_p}$$

is also an isomorphism. Therefore $\tau_{\Sigma\cap\mathcal{U}}$ extends to $C^\infty(\mathcal{U})^G$ if and only if $\tau_{T_p\Sigma}$ extends to $C^\infty(E|_V)^K$.

In other words, we have reduced to the linear case. Also note that if p is regular, then $V = T_p\Sigma$, and so there are no smoothness conditions.

4. Since $T_p\Sigma$ and V are representations, the equivariant bundles $E|_{T_pM}$ and $E|_V$ are trivial, that is, equivalent to the bundles $T_p\Sigma \times E_p$ and $V \times E_p$ with bundle projection equal to the projection onto the first factor. See Proposition 3.2.3. In this frame a cross-section of $E|_V$ is simply a function $V \to E_p$, and similarly for $T_p\Sigma$. In particular this proves that $(E|_{T_p\Sigma})^{Z(\Sigma)}$, which here becomes $T_p\Sigma \times E_p^{Z(\Sigma)}$, is indeed a sub-bundle.

5. K is not necessarily connected. Let K_0 be the connected component containing the identity, and $W_0 = W_p \cap K_0$. By lemma 2.1.2 $\tau|_{T_p\Sigma} \in C^\infty(T_p\Sigma, E_p^{Z(\Sigma)})^{W_p}$ extends to $C^\infty(V, E_p)^K$ if and only if it extends to $C^\infty(V, E_p)^{K_0}$. So we may as well assume K connected from now on.

6. Since these are vector spaces, we may consider equivariant *polynomial* maps between them:
$$\mathbb{R}[V, E_p]^K \qquad \mathbb{R}[T_p\Sigma, E_p^Z]^{W_p}$$
By Theorem 3.1.2 these are free modules of the same rank l over the same ring $\mathbb{R}[V]^K = \mathbb{R}[T_p\Sigma]^{W_p}$. Let $\sigma_1, \ldots \sigma_l$ be a basis for the first module, and $\tau_1, \ldots \tau_l$ a basis for the second. See section 3.3.3 for an algorithm that finds such bases.

7. An application of the Malgrange Division Theorem shows that $\sigma_1, \ldots \sigma_l$ also generate $C^\infty(V, E_p)^K$ over $C^\infty(V)^K$ (see proposition 3.4.1), and similarly for $T_p\Sigma$. This means that τ extends if and only if for all $v \in T_p\Sigma$, the Taylor series of τ at v belongs to the module generated by the Taylor series at v of $\sigma_1|_{T_p\Sigma}, \ldots, \sigma_l|_{T_p\Sigma}$ over the ring of formal power series $\mathbb{R}[[T_p\Sigma]]$. We may ignore the conditions at $v \neq 0$, because these are already covered by conditions coming from a different p.

8. Since τ is W_p-equivariant, its Taylor series at $v = 0$ already belongs to the ideal generated by the τ_i, so many of the conditions above are superfluous. Discarding these the space of "smoothness conditions" is isomorphic to
$$\{\text{smoothness conditions}\} = \frac{\mathbb{R}[T_p\Sigma, E_p^Z]^{W_p}}{\mathbb{R}[V, E_p]^K}$$

9. Since the base ring A is a polynomial ring in n variables, where n is the cohomogeneity of the action, a quotient of two free modules of the same rank is always finite-dimensional over $\mathbb{R}$ for $n = 1$, and is either zero or infinite-dimensional over $\mathbb{R}$ for $n \geq 2$. See proposition 3.4.2 for a proof.

□

3.2 Equivariant Vector Bundles

3.2.1 Set-up and restricting to the slice

Let M be a polar G-manifold, and $\Sigma \subset M$ be a section. Let $\pi : E \to M$ be a smooth G-equivariant vector bundle.

Since we wish to study smoothness of bundle sections of E, which is a local condition (in M), the first step is to apply the Slice Theorem (see Bredon [12]): For $x \in M$, we call the vector space $V = (T_x(G \cdot x))^\perp \subset T_x M$ the slice at x. It is a K-representation, where $K = G_x$ is the isotropy at x . The Slice Theorem then says that there is a G-equivariant diffeomorphism

$$\mathcal{U} \to G \times_K V$$

where $\mathcal{U}$ is an open G-neighbourhood of x. From now on we will identify them through this isomorphism.

Denote by $C^\infty(G \times_K V)^G$ the $\mathbb{R}$-algebra of G-invariant smooth functions on $G \times_K V$ and by $C^\infty(V)^K$ the $\mathbb{R}$-algebra of K-invariant smooth functions on V. Moreover, denote by $C^\infty(E|_{\mathcal{U}})^G$ the $C^\infty(G \times_K V)^G$-module of smooth G-equivariant bundle sections of $E|_{\mathcal{U}}$, and by $C^\infty(E|_V)^K$ the $C^\infty(V)^K$-module of smooth K-equivariant bundle sections of $E|_V$. Implicit here is the identification of V with the subspace $\{[(e, v)] \mid v \in V\}$ of $G \times_K V$.

Lemma 3.2.1. *Let N be any smooth manifold, and $f : G \times_K V \to N$ a function. Then f is smooth if and only if $\tilde{f} = f \circ p$ is smooth, where*

$$p : G \times V \to G \times_K V, \qquad (g, v) \mapsto [(g, v)]$$

is the natural projection.

Proof. This follows from the fact that the action of K on $G \times V$ is free, and from the definition of smooth structure on $G \times_K V$. □

Proposition 3.2.1. *There are no smoothness conditions when restricting either functions or bundle sections to the slice. More precisely:*

(i) The restriction map

$$|_V : C^\infty(G \times_K V)^G \to C^\infty(V)^K$$

is an isomorphism of $\mathbb{R}$-algebras;

(ii) *The restriction map*

$$|_V : C^\infty(E|_{\mathcal{U}})^G \to C^\infty(E|_V)^K$$

is an isomorphism of $C^\infty(V)^K$-modules, where we view $C^\infty(E)^G$ as a module over $C^\infty(V)^K$ through the isomorphism in (i).

Proof. (i) $|_V$ is clearly a homomorphism of $\mathbb{R}$-algebras, and it is injective because $V \subset G \times_K V$ meets all the G-orbits. Now we show surjectivity:

Let $h \in C^\infty(V)^K$ and define the smooth function $\tilde{f} : G \times V \to \mathbb{R}$ by $\tilde{f}(g, v) = g \cdot h(v)$.

Since h is K-invariant, $\tilde{f}$ descends to a G-invariant $f : G \times_K V \to \mathbb{R}$. f is smooth by the lemma above, and $f|_V = h$ by construction.

(ii) Analogous to the proof of (i).

□

Since the slice V is a polar representation (with section $T_x\Sigma$, see proposition 2.1.2, or Theorem 4.6 in [55] or Theorem 5.6.21 in [56]), Proposition 3.2.1 above reduces the local question of extension of equivariant tensors in polar manifolds to the same question for polar representations.

3.2.2 Trivial equivariant bundles

Given a K-equivariant vector bundle $\pi : F \to V$ over a K-representation V, the fiber $W = \pi^{-1}(0)$ over zero is again a K-representation, because $0 \in V$ is fixed by the action of K. Then it turns out that F is trivial in the sense that it is equivalent, as a smooth K-equivariant bundle, to $p_1 : V \times W \to V$, where p_1 is projection onto the first factor, and where K acts on $V \times W$ in the obvious way. This is a well-known result, but since I was not able to find a proof in the literature (in the smooth category), I write one here.

We start by recalling some standard facts.

The action of K on a manifold M makes TM into a K-equivariant vector bundle, and so we get an action on $C^\infty(TM)$:

$$(g_*X)(v) = g \cdot X(g^{-1}v)$$

Similarly, K acts on the algebra of smooth functions by $g_*f = f \circ g^{-1}$.

For any K-equivariant vector bundle F over M, the actions of K on $C^\infty(M)$ and on $C^\infty(F)$ interact with the module structure in the following way:

$$g_*(f \cdot Y) = (g_*f) \cdot g_*Y \tag{3.2.1}$$

Now, $C^\infty(TM)$ also acts on $C^\infty(M)$ by taking directional derivatives, and this interacts with the K-actions by:

$$g_*(X \cdot f) = (g_*X) \cdot (g_*f) \tag{3.2.2}$$

Lemma 3.2.2. *There is a K-equivariant connection on F, that is, a connection ∇ satisfying*

$$\nabla_{g_* X}(g_* Y) = g_*(\nabla_X Y)$$

for all group elements $g \in K$, vector fields $X \in C^\infty(TM)$ and bundle sections $Y \in C^\infty(F)$.

Here g_* denotes the usual actions of K on bundle sections of TM and F.

Proof. K acts on

$$L = \mathrm{Hom}(C^\infty(TM) \times C^\infty(F), C^\infty(F)) \quad \text{by}$$

$$(g \star \nabla)_X Y = g_*(\nabla_{g_*^{-1} X} g_*^{-1} Y)$$

The space of connections on F is the affine subspace of L defined by the equations

$$\nabla_{fX} Y = f \nabla_X Y$$

$$\nabla_X fY = (X \cdot f)Y + f \nabla_X Y$$

Let $\tilde{\nabla}$ be any connection on F. (Every vector bundle admits a connection, because the frame bundle always admits a horizontal distribution: put any metric on its total space, and define the horizontal distribution to be the orthogonal complement to the vertical distribution.)

First we prove that $g \star \tilde{\nabla}$ is a connection for all $g \in K$.

$$\begin{aligned}
(g \star \tilde{\nabla})_{fX} Y &= g_*\left(\tilde{\nabla}_{g_*^{-1}(fX)} g_*^{-1} Y\right) && \text{by definition} \\
&= g_*\left(\tilde{\nabla}_{(g_*^{-1} f)(g_*^{-1} X)} g_*^{-1} Y\right) && \text{by (3.2.1)} \\
&= g_*\left((g_*^{-1} f) \tilde{\nabla}_{(g_*^{-1} X)} g_*^{-1} Y\right) && \text{because } \tilde{\nabla} \text{ is a connection} \\
&= f (g \star \tilde{\nabla})_X Y && \text{by (3.2.1) again}
\end{aligned}$$

$$\begin{aligned}
(g \star \tilde{\nabla})_X (fY) &= \\
&= g_*\left(\tilde{\nabla}_{g_*^{-1} X} g_*^{-1}(fY)\right) && \text{by definition} \\
&= g_*\left(\tilde{\nabla}_{g_*^{-1} X}((g_*^{-1} f)(g_*^{-1} Y))\right) && \text{by (3.2.1)} \\
&= g_*\left(((g_*^{-1} X) \cdot (g_*^{-1} f)) g_*^{-1} Y + (g_*^{-1} f) \tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Y\right) && \text{because } \tilde{\nabla} \text{ is a connection} \\
&= g_*\left(((g_*^{-1}(X \cdot f)) g_*^{-1} Y + (g_*^{-1} f) \tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Y\right) && \text{by (3.2.2)} \\
&= (X \cdot f) Y + f((g \star \tilde{\nabla})_X Y) && \text{by (3.2.1) twice}
\end{aligned}$$

Now let $\int_K$ be a Haar integral on K with total volume 1, and define

$$\nabla = \int_{g \in K} (g \star \tilde{\nabla}) dg$$

This is a connection because the space of connections is an affine subspace of L: convex linear combinations (even infinite ones) of vectors in an affine subspace stay in that affine subspace. And it is K-equivariant by construction.

□

Remark 3.2.1. Of course when E is a tensor bundle, we can simply choose the Levi-Civita connection, which is equivariant because the group acts by isometries.

Proposition 3.2.2. *Let N be a K-manifold and $\phi : N \times [0,1] \to M$ be a smooth homotopy through K-equivariant maps between $\phi_0 = \phi(\cdot, 0)$ and $\phi_1 = \phi(\cdot, 1)$.*

Let $\pi : F \to M$ be a K-equivariant vector bundle, and ∇ a K-equivariant connection on F (which always exists by the lemma above). Then parallel translation relative to ∇ gives a K-equivariant isomorphism between the K-equivariant bundles $\phi_0^* F$ and $\phi_1^* F$.

Proof. Consider the K-equivariant bundle over N:

$$L = \mathrm{Hom}(\phi_0^* F, \phi_1^* F)$$

Define the bundle section $\psi : N \to L$ by $\psi(n) = P_\gamma$, where P_γ denotes parallel translation relative to ∇ along the curve $\gamma(t) = \phi(n,t)$, $t \in [0,1]$. Then define a bundle map $\eta : \phi_0^* F \to \phi_1^* F$ by $\eta(v) = \psi(\pi(v))v$.

Note that if we start with $\phi_-(n,t) = \phi(n, 1-t)$ in the construction above we get the η_- which is the inverse of η. Therefore we have two things left to prove: (i) that ψ is smooth, and (ii) that ψ is K-equivariant.

(i) ψ was defined in terms of the solution of the parallel transport ODE:

$$\psi(n)(v) = \sigma(1)$$

where $\sigma(t)$ is the bundle section of F along $\gamma(t) = \phi(n,t)$ satisfying

$$\frac{D}{dt}\sigma(t) = 0 \quad \text{and} \quad \sigma(0) = v$$

The coefficients of this ODE change smoothly with the parameter $n \in N$. Now the standard ODE result about smooth dependence on parameters shows that ψ is smooth. (see Arnold [3], chapter 2, section 8.5)

(ii) Apply $g \in K$ to the situation above. We get a section $g_*\sigma(t) = g\sigma(t)$ of F along the curve

$$\bar{\gamma}(t) = \phi(gn, t) = g\phi(n, t) = g\gamma(t)$$

It is parallel because ∇ is K-equivariant:

$$\nabla_{\bar{\gamma}'(t)}(g_*\sigma) = \nabla_{g\gamma'(t)}(g_*\sigma) = g\nabla_{\gamma'(t)}\sigma = 0$$

Since $(g_*\sigma)(0) = gv$, we get equivariance of ψ:

$$\psi(gn)(gv) = (g_*\sigma)(1) = g(\psi(n)(v))$$

□

Proposition 3.2.3. *Let V be a (real) K-representation, where K is a compact Lie group, and $\pi : F \to V$ a smooth K-equivariant vector bundle. Let W be the K-representation $W = \pi^{-1}(0)$. Then F is trivial in the sense that it is equivalent, as a smooth K-equivariant vector bundle, to $p_1 : V \times W \to V$, where K acts on $V \times W$ by $g(v, w) = (gv, gw)$ and p_1 is the projection onto the first factor.*

Proof. The map $\phi : V \times [0, 1] \to V$ given by $\phi(v, t) = tv$ is a smooth homotopy through K-equivariant maps from $\phi_0 = 0$ to $\phi_1 = \mathrm{id}_V$, and so, by the Proposition above, the bundles $F = \phi_1^* F$ and $(p_1 : V \times W \to V) = \phi_0^* F$ are equivalent as K-equivariant vector bundles over V. □

Since the space of equivariant bundle sections of the trivial bundle $V \times W$ is equal to the space $C^\infty(V, W)^K$ of equivariant maps from V to W, we have reduced the extension problem for tensors on a polar representation to the extension problem for maps to the representation W.

3.3 Polynomial coefficients

Let V be a polar K-representation with section Σ, and W be any K-representation. In this section we study invariants and equivariants with polynomial coefficients.

Notations: $A = \mathbb{R}[V]^K = \mathbb{R}[\Sigma]^{W(\Sigma)}$ is the ring of K-invariant polynomials on V, which is isomorphic through the restriction map to $\mathbb{R}[\Sigma]^{W(\Sigma)}$ by the polynomial Chevalley Restriction Theorem, see proposition 2.3.3. $\mathbb{R}[V, W]^K$ and $\mathbb{R}[\Sigma, W^{Z(\Sigma)}]^{W(\Sigma)}$ are the equivariant polynomial maps between the indicated vector spaces, and both are modules over A.

3.3.1 Cofreeness of polar representations

In this subsection we investigate when the polar K-representation V is cofree, that is, when $\mathbb{R}[V]$ is a free A-module, where $A = \mathbb{R}[V]^K$ as above. This will eventually imply that the A-modules we are most interested in, $\mathbb{R}[V, W]^K$ and $\mathbb{R}[\Sigma, W]^{N(\Sigma)}$, are free of the same rank, namely $\dim W^{Z(\Sigma)}$ (see Corollaries 3.3.2 and 3.3.3 below). The main result of this subsection is Theorem 3.3.3 below, which lists several conditions equivalent to cofreeness.

We start by looking at the representation of the finite group W_Σ on Σ. Recall that a subgroup of $\mathrm{GL}(U)$ (where U is a real vector space) is called a reflection group when it is generated by reflections in hyperplanes in U. See Humphreys' book [33] for more information.

Theorem 3.3.1 (Chevalley)**.** *Suppose W_Σ is a reflection group. Then $A = \mathbb{R}[\Sigma]^{W_\Sigma}$ is a free polynomial ring in a number of variables equal to the dimension of Σ, and the set of all polynomials on Σ, $\mathbb{R}[\Sigma]$, is a free A-module of rank equal to the cardinality of the Weyl Group, $|W_\Sigma|$.*

Proof. See Humphreys [33], Part I, Theorems 3.5 and Proposition 3.6. □

The converse to the theorem above is:

Theorem 3.3.2 (Shephard-Todd)**.** *Let W be a finite subgroup of $GL(\Sigma)$. If $A = \mathbb{R}[\Sigma]^W$ is a free polynomial ring, then W is a reflection group.*

Proof. See Theorem 3.11 in [33], or the original paper [61]. □

For some of the implications in Theorem 3.3.3 we will need to complexify the representation V, then use the theory of such representations developed by Schwarz, Kac, Dadok, Vinberg, Popov etc, and finally see what that implies for the original real representation V of K. For the definition of complexification of groups and its basic properties see Chapter 3 of [13]. For a summary of the theory of complex transformation groups (much of which was developed by Luna) in the case of representations, especially complexifications of real representations, see section 5 of [60].

Let $K_\mathbb{C}$ be the complexification of K. $K_\mathbb{C}$ is a reductive, linear algebraic complex group. Let $V_\mathbb{C} = \mathbb{C} \otimes V$ be the regular representation of $K_\mathbb{C}$ extending V and $\Sigma_\mathbb{C} = \mathbb{C} \otimes \Sigma$.

Note that $\mathbb{C}[V_\mathbb{C}] = \mathbb{C} \otimes \mathbb{R}[V]$, and $\mathbb{C}[V_\mathbb{C}]^{K_\mathbb{C}} = \mathbb{C}[V_\mathbb{C}]^K = \mathbb{C} \otimes \mathbb{R}[V]^K$, and similarly for polynomials on Σ and $\Sigma_\mathbb{C}$.

Proposition 3.3.1. *Suppose that W_Σ is a finite reflection group. Then $V_\mathbb{C}$ is cofree, that is, $\mathbb{C}[V_\mathbb{C}]$ is a free $\mathbb{C}[V_\mathbb{C}]^{K_\mathbb{C}}$-module.*

Proof. Complexifying Chevalley's Restriction Theorem (see Theorem 2.3.6), we get that the restriction to $\Sigma_{\mathbb{C}}$ is an isomorphism

$$\mathbb{C}[V_{\mathbb{C}}]^{K_{\mathbb{C}}} \to \mathbb{C}[\Sigma_{\mathbb{C}}]^{N(\Sigma)_{\mathbb{C}}}$$

Moreover, by the complexification of Chevalley's Theorem 3.3.1, $\mathbb{C}[\Sigma_{\mathbb{C}}]$ is a free $\mathbb{C}[\Sigma_{\mathbb{C}}]^{N(\Sigma)_{\mathbb{C}}}$-module, and so we can apply Lemma 2.5 in [59] to conclude that $V_{\mathbb{C}}$ is cofree. ☐

Remark 3.3.1. In [16] Dadok and Kac have defined and studied a complex analogue of polar representations of connected groups, and proved (see theorem 2.10 in their paper) that such representations are always cofree. So an alternative way of proving the Proposition above, in the special case where K is connected (see Theorem 3.3.3), is to show that $V_{\mathbb{C}}$ is polar in their sense, which can be done as follows:

Let $v \in \Sigma$ be a regular vector. Then v is also regular in $V_{\mathbb{C}}$: see Lemma 3.3.1 below. Let $c_v = \{x \in V_{\mathbb{C}} \mid \mathfrak{g}x \subseteq \mathfrak{g}v\}$. Then clearly $\Sigma_{\mathbb{C}} \subseteq c_v$, and so $c_v \oplus \mathfrak{g}v = V_{\mathbb{C}}$. Now apply Corollary 2.5 in [16] to conclude that $V_{\mathbb{C}}$ is polar.

Let $\mathbb{R}[V]^K_+$ be the space of polynomials in $\mathbb{R}[V]^K$ with zero constant term, and $I = \mathbb{R}[V]^K_+ \cdot \mathbb{R}[V]$ be the ideal of $\mathbb{R}[V]$ generated by $\mathbb{R}[V]^K_+$.

Denote by $I_{\mathbb{C}}$ the complexification $\mathbb{C}\otimes I$ of I, which is also equal to $\mathbb{C}[V_{\mathbb{C}}]^{K_{\mathbb{C}}}_+ \cdot \mathbb{C}[V_{\mathbb{C}}]$

Proposition 3.3.2. *Suppose $V_{\mathbb{C}}$ is cofree. Consider a K-stable graded vector subspace $\mathcal{H} \subseteq \mathbb{R}[V]$ such that we have a direct sum*

$$\mathbb{R}[V] = \mathcal{H} \oplus I$$

(i) Such an $\mathcal{H}$ exists;

(ii) Let $\mathcal{H}_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{H}$. It is a $K_{\mathbb{C}}$-invariant graded subspace of $\mathbb{C}[V_{\mathbb{C}}]$ such that

$$\mathbb{C}[V_{\mathbb{C}}] = \mathcal{H}_{\mathbb{C}} \oplus I_{\mathbb{C}}$$

and multiplication gives an isomorphism

$$\phi_{\mathbb{C}} : \mathbb{C}[V_{\mathbb{C}}]^{K_{\mathbb{C}}} \otimes \mathcal{H}_{\mathbb{C}} \to \mathbb{C}[V_{\mathbb{C}}]$$

which respects the grading, the structure of $\mathbb{C}[V_{\mathbb{C}}]^{K_{\mathbb{C}}}$-modules and that of $K_{\mathbb{C}}$-modules;

(iii) Multiplication induces an isomorphism

$$\phi : \mathbb{R}[V]^K \otimes \mathcal{H} \to \mathbb{R}[V]$$

which respects the grading, the structure of $\mathbb{R}[V]^K$-modules and that of K-modules. In particular, the K-representation V is cofree.

Proof. (i) $\mathcal{H}$ can be built degree by degree: let $\mathcal{H}_j$ be a K-invariant complement of I_j in $\mathbb{R}[V]_j$, and define $\mathcal{H} = \bigoplus_{j=0}^{\infty} \mathcal{H}_j$

(ii) The first statement is obvious. Kostant has proved that in this situation, cofreeness of $V_{\mathbb{C}}$ (which we have) is equivalent to $\phi_{\mathbb{C}}$ being an isomorphism: see Lemma 1 (and also Proposition 1) in [42].

(iii) Since $\phi_{\mathbb{C}}$ is the complexification of ϕ and $\phi_{\mathbb{C}}$ is an isomorphism, so is ϕ.

□

Remark 3.3.2. An $\mathcal{H}_{\mathbb{C}}$ satisfying the properties of (ii) above can also be described as $K_{\mathbb{C}}$-harmonic polynomials. Here harmonicity is defined as follows: (see [42], pages 335, 340)

Let $\mathrm{Sym}(V)$ be the symmetric algebra on V. Choosing linear coordinates $\{z_1, \ldots, z_n\}$ on V, $\mathrm{Sym}(V)$ can be identified with the set of constant coefficient differential operators

$$\partial = \sum a_{i_1,\ldots,i_n} \left(\frac{\partial}{\partial z_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{i_n}$$

$\mathrm{Sym}(V)$ is a graded $K_{\mathbb{C}}$-module. Let J be the set of elements in $\mathrm{Sym}(V)$ which have zero constant term and are $K_{\mathbb{C}}$-invariant. Then a polynomial f is called $K_{\mathbb{C}}$-harmonic if $(\partial f) = 0$ for all $\partial \in J$.

If $K = \mathrm{SO}(n)$, $V = \mathbb{R}^n$, then one gets the usual harmonic functions.

Theorem 3.3.3. *Let K be a compact Lie group, and V a polar K-representation, with section $\Sigma \subseteq V$ and generalized Weyl group $W_\Sigma = N(\Sigma)/Z(\Sigma)$. Then the following are equivalent:*

(i) W_Σ is a (finite) reflection subgroup of $GL(\Sigma)$, i.e., it is generated by reflections in hyperplanes of Σ;

(ii) The base ring $A = \mathbb{R}[V]^K = \mathbb{R}[\Sigma]^{W_\Sigma}$ is a free polynomial ring in dim(Σ) generators;

(iii) V is a cofree K-representation, i.e., $\mathbb{R}[V]$ is a free $\mathbb{R}[V]^K$-module;

(iv) Σ is a cofree W_Σ-representation;

Moreover, if K is connected, all the equivalent conditions above are satisfied.

Proof.

- (i)⇒(ii) See Theorem 3.3.1;
- (ii)⇒(i) See Theorem 3.3.2;

- (i)⇒(iii) See Propositions 3.3.1 and 3.3.2;
- (iii)⇒(ii) Since V is cofree, so is $V_{\mathbb{C}}$. This implies that $V_{\mathbb{C}}$ is coregular: see Proposition 17.29 in [60]. Then V is coregular.
- (ii)⇔(iv) Cofreeness and coregularity are equivalent for finite group representations, and equivalent to the group being a reflection group, see chapter 18 in [35].

Finally, when K is connected, we can use Dadok's classification to conclude that W_{Σ} is a reflection group. Actually, it is even a Weyl group: see proposition 2.2.1. □

Remark 3.3.3. According to Theorem 1 in [25], the equivalent conditions above are also equivalent to property SP. This means that the convex hull of the orbits form a set closed under Minkowski addition $A + B = \{a + b \mid a \in A,\ b \in B\}$.

3.3.2 Structure of the modules of covariants

Assume V and Σ are cofree representation (see Theorem 3.3.3 above). We want to describe the structure of $\mathbb{R}[V]$ and $\mathbb{R}[\Sigma]$ as graded A-modules and as K- (respectively W_{Σ}-) modules.

We start with $\mathbb{R}[V]$. Let $\mathcal{H}$ be as in Proposition 3.3.2. It is clearly enough to describe the graded and K-module structures of $\mathcal{H}$.

As a K-representation, $\mathcal{H}$ decomposes as a direct sum of irreducible representations, and the number of times each one appears in this decomposition is called its multiplicity.

We need a lemma:

Lemma 3.3.1. *Let $v \in \Sigma$ be a regular vector of V. Then, viewed as a vector in $V_{\mathbb{C}}$, it is also regular, that is: $K_{\mathbb{C}} \cdot v$ is closed and has maximal dimension among closed orbits. The isotropy is*

$$(K_{\mathbb{C}})_v = (K_v)_{\mathbb{C}} = Z(\Sigma)_{\mathbb{C}}$$

where $Z(\Sigma) = \{g \in K \mid gv = v\ \forall v \in \Sigma\}$ is the centralizer of Σ in K.

Proof. We first show that v is semi-simple, that is, that $K_{\mathbb{C}} \cdot v$ is closed. Choose any K-invariant inner product on V, and extend it to a Hermitian inner product $< \cdot, \cdot >$ on $V_{\mathbb{C}}$. Since $< \mathfrak{k}v, v >= 0$, we get $< \mathfrak{k}_{\mathbb{C}}v, v >= 0$, that is, the function

$$g \in K_{\mathbb{C}} \mapsto < gv, gv > \in \mathbb{R}$$

has a critical point at $e \in K_{\mathbb{C}}$. By a theorem of Kempf and Ness (see theorem 1.1 in [16], or theorems 0.1 and 0.2 in [36]), v is semi-simple.

The dimension of $K_{\mathbb{C}} \cdot v$ is maximal among closed orbits because the isotropy types of semi-simple vectors in $V_{\mathbb{C}}$ are the complexifications of the isotropy types of vectors in V (see Proposition 5.8(2) in [60]).

To show that $(K_{\mathbb{C}})_v = (K_v)_{\mathbb{C}}$, we note that v being real, $(K_{\mathbb{C}})_v$ is preserved by conjugation, and so is equal to the complexification of $K \cap (K_{\mathbb{C}})_v$, which is exactly K_v.

Since V is a polar K-representation and $v \in \Sigma$ is regular, Σ is the slice at v. But the slice representation at a regular vector is trivial, and so, any $g \in K_v$ also fixes Σ pointwise. In other words, $K_v = Z(\Sigma)$.

□

Proposition 3.3.3. *Let $\mathcal{H}$ and $\mathcal{H}_{\mathbb{C}}$ be as in Proposition 3.3.2, and U be any irreducible $K_{\mathbb{C}}$-representation. The multiplicity of U in $\mathcal{H}_{\mathbb{C}}$ is equal to* $\dim(U^*)^{Z(\Sigma)_{\mathbb{C}}}$, *where U^* is the dual representation to U.*

Proof. Since $V_{\mathbb{C}}$ is the complexification of a real representation of a compact group, it has generically closed orbits, that is, the union of the closed orbits is Zariski dense in $V_{\mathbb{C}}$: see Corollary 5.9 in [60].

Then, by Theorem 4.6 in [59], the multiplicity of U in $\mathcal{H}_{\mathbb{C}}$ is $\dim(U^*)^H$, for any principal isotropy subgroup $H \subseteq K_{\mathbb{C}}$. By the lemma above, we can take $H = Z(\Sigma)_{\mathbb{C}}$. □

Corollary 3.3.1. *Let W be an irreducible real representation of K, and $\mathcal{H}$ as above. Then the multiplicity of W in $\mathcal{H}$ is equal to:*

- $\dim(W^*)^{Z(\Sigma)}$ *if $\mathbb{C} \otimes W$ is irreducible;*
- $\frac{1}{2}\dim(W^*)^{Z(\Sigma)}$ *if $\mathbb{C} \otimes W \simeq U \oplus \bar{U}$ where U is an irreducible complex representation of complex type, i.e., such that $U \not\simeq \bar{U}$;*
- $\frac{1}{4}\dim(W^*)^{Z(\Sigma)}$ *if $\mathbb{C} \otimes W \simeq U \oplus U$ where U is an irreducible complex representation of quaternionic type;*

Proof. The multiplicity of W in $\mathcal{H}$ is equal to the number m of times $\mathbb{C}\otimes W$ appears in $\mathcal{H}_{\mathbb{C}}$.

- If $\mathbb{C} \otimes W$ is irreducible, then the proposition above says that

$$m = \dim_{\mathbb{C}}((\mathbb{C} \otimes W)^*)^{Z(\Sigma)_{\mathbb{C}}} = \dim_{\mathbb{R}}(W^*)^{Z(\Sigma)}$$

- if $\mathbb{C} \otimes W \simeq U \oplus \bar{U}$ with $U \not\simeq \bar{U}$, then m equals the multiplicity of U in $\mathcal{H}_{\mathbb{C}}$, which by the proposition above is $\dim_{\mathbb{C}}(U^*)^{Z(\Sigma)_{\mathbb{C}}}$. Since $\dim_{\mathbb{C}}(U^*)^{Z(\Sigma)_{\mathbb{C}}} = \dim_{\mathbb{C}}(\bar{U}^*)^{Z(\Sigma)_{\mathbb{C}}}$ we have

$$m = \dim_{\mathbb{C}}(U^*)^{Z(\Sigma)_{\mathbb{C}}} = \frac{1}{2}\dim_{\mathbb{C}}((\mathbb{C} \otimes W)^*)^{Z(\Sigma)_{\mathbb{C}}} = \frac{1}{2}\dim_{\mathbb{R}}(W^*)^{Z(\Sigma)}$$

- if $\mathbb{C} \otimes W \simeq U \oplus U$, m is equal to half the multiplicity of U in $\mathcal{H}_{\mathbb{C}}$, and so

$$m = \frac{1}{4} \dim(W^*)^{Z(\Sigma)}$$

□

Remark 3.3.4. Using Frobenius Reciprocity, Proposition 3.3.3 and Corollary 3.3.1 above can be restated in a more natural way: $\mathcal{H}$ (respectively $\mathcal{H}_{\mathbb{C}}$) is equivalent to (the localy finite part of) the representation of K (resp. $K_{\mathbb{C}}$) induced from the trivial one-dimensional representation of the subgroup $Z(\Sigma)$ (resp. $Z(\Sigma)_{\mathbb{C}}$). (See chapter 3 of [13] for the definition of induced representation). Compare with page 328 in [42] and pages 758, 759 in [41].

Now we need a version of Schur's lemma for real representations:

Lemma 3.3.2. *Let W be a real irreducible K-representation. Then* $\mathrm{End}_{\mathbb{R}}(W)^K$ *is isomorphic, as an $\mathbb{R}$-algebra, to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, according to whether $\mathbb{C} \otimes W$ is irreducible, isomorphic to $U \oplus \bar{U}$ with $U \not\simeq \bar{U}$, or to $U \oplus U$.*

Corollary 3.3.2. *Let W be any real representation of K. Then $\mathbb{R}[V, W]^K$ is a free $\mathbb{R}[V]^K$-module of rank equal to* $\dim W^{Z(\Sigma)}$.

Proof. First note that both numbers are additive in W, so we may as well asuume W irreducible.

Let $\mathcal{H}$ be as in Proposition 3.3.2. Then, since $\mathbb{R}[V] = \mathbb{R}[V]^K \otimes \mathcal{H}$, we get:

$$\mathbb{R}[V, W]^K = (\mathbb{R}[V] \otimes W)^K = \mathbb{R}[V]^K \otimes (\mathcal{H} \otimes W)^K$$

So $\mathbb{R}[V, W]^K$ is a free $\mathbb{R}[V]^K$-module of rank equal to $\dim(\mathcal{H} \otimes W)^K$, which by Schur's lemma is equal to 1, 2 or 4 times the multiplicity of W^* in $\mathcal{H}$. In all cases, using the Corollary above we get that the rank is equal to $\dim W^{Z(\Sigma)}$. □

Remark 3.3.5. It would be nice to have a description of $\mathcal{H}$ as a *graded* K-module, for this would give us the degrees of the elements in an A-basis for $\mathbb{R}[V, W]^K$, which would help in actually finding such bases in concrete situations.

Kostant and Rallis (see [41], pages 759, 760) have done this starting from $K_{\mathbb{C}}$ as the full fixed point set of an involution of a complex reductive linear algebraic group G, and $V_{\mathbb{C}}$ the isotropy representation of the associated symmetric space. It would be interesting to see whether their method also works in the current context. (another account of the Kostant-Rallis theorem can be found in chapter 12 of [27])

Molien's formula (see theorem 4.2.2) also works for compact groups instead of finite groups, with a sum replaced with an integral over the group. This integral in principle also calculates the degrees of the elements in an A-basis for $\mathbb{R}[V, W]^K$.

Now we turn to $\mathbb{R}[\Sigma]$ and $\mathbb{R}[\Sigma, W^{Z(\Sigma)}]^{W_\Sigma}$. Recall that W_Σ is the generalized Weyl group $N(\Sigma)/Z(\Sigma)$, which are assuming to be a finite reflection group.

Theorem 3.3.4. *Σ is a cofree W_Σ-representation and there is a W_Σ-invariant graded subspace $\mathcal{I} \subseteq \mathbb{R}[\Sigma]$ such that multiplication induces an isomorphism*

$$\mathcal{I} \otimes \mathbb{R}[\Sigma]^{W_\Sigma} \to \mathbb{R}[\Sigma]$$

Moreover, the W_Σ-module structure of $\mathcal{I}$ is that of the regular representation, that is, the representation with basis $\{e_g \mid g \in W_\Sigma\}$ and action $h \cdot e_g = e_{hg}$.

Proof. See [14] □

Corollary 3.3.3. *Let W be any K-representation. Then $\mathbb{R}[\Sigma, W^{Z(\Sigma)}]^{W_\Sigma}$ is a free $\mathbb{R}[\Sigma]^{W_\Sigma}$-module of rank equal to $\dim W^{Z(\Sigma)}$.*

Proof. Using the theorem above, we see that

$$\mathbb{R}[\Sigma, W^{Z(\Sigma)}]^{W_\Sigma} = \mathbb{R}[\Sigma]^{W_\Sigma} \otimes (\mathcal{I} \otimes W^{Z(\Sigma)})^{W_\Sigma}$$

is a free $\mathbb{R}[\Sigma]^{W_\Sigma}$-module of rank equal to the dimension of $(\mathcal{I} \otimes W^{Z(\Sigma)})^{W_\Sigma}$.

But elements of $\mathcal{I} \otimes W^{Z(\Sigma)}$ can be uniquely written as

$$\sigma_f = \sum_{g \in W_\Sigma} e_g \otimes f(g)$$

where f is a function $W_\Sigma \to W^{Z(\Sigma)}$, and σ_f is fixed by W_Σ if and only if f is equivariant (with the understanding that W_Σ acts on itself by left multiplication).

Therefore the map $\sigma_f \mapsto f(e)$ gives an isomorphism between $(\mathcal{I} \otimes W^{Z(\Sigma)})^{W_\Sigma}$ and $W^{Z(\Sigma)}$, completing the proof. □

Remark 3.3.6. One can describe $\mathcal{I}$ as a *graded* W_Σ-module using Molien's formula (see theorem 4.2.2).

3.3.3 An algorithm

Recall the notations we have been using: K is a connected compact Lie group, V is a polar K-representation with section Σ, and W is an arbitrary K-representation. Denote by A the ring of invariant polynomials $A = \mathbb{R}[V]^K = \mathbb{R}[\Sigma]^{W_\Sigma}$, and by M and N the A-modules of equivariant maps to W:

$$M = \mathbb{R}[V, W]^K \qquad N = \mathbb{R}[\Sigma, W^{Z(\Sigma)}]^{W_\Sigma}$$

They are all graded: $A = \bigoplus_{i=0}^\infty A_i$, $M = \bigoplus_{i=0}^\infty M_i$, $N = \bigoplus_{i=0}^\infty N_i$. The grading respects the module structures in the sense that $A_i M_j \subseteq M_{i+j}$ and similarly for N, and the inclusion $M \to N$ (given by restriction to the section) also respects the grading. Denote by A_+ the ideal $\bigoplus_{i=1}^\infty A_i$, and by $\bar{M}$, $\bar{N}$ the quotients

$$\bar{M} = \frac{M}{A_+ M} \qquad \bar{N} = \frac{N}{A_+ N}$$

They are graded modules over $A/A_+ = A_0 = \mathbb{R}$.

We have seen above that M and N are free A-modules of rank $\dim W^{Z(\Sigma)}$. Then we have:

Proposition 3.3.4. *Let $\{m_i\}$ be a subset of M, and $\{\bar{m}_i\}$ their images in $\bar{M}$. Then $\{m_i\}$ is an A-basis for M if and only if $\{\bar{m}_i\}$ is an $\mathbb{R}$-basis for $\bar{M}$. (and similarly for N)*

Proof. See section II.4 in [43]. □

Now, for each i one can find bases for A_i, M_i:

$$A_i = (\mathrm{Sym}^i V^*)^K \qquad M_i = ((\mathrm{Sym}^i V^*) \otimes W)^K$$

and if we have bases for A_1 up to A_i and M_0 up to M_i, we can also find a basis for $(A_+M)_i = A_1 M_{i-1} + A_2 M_{i-2} + \cdots A_i M_0$, and hence also for $\bar{M}_i = M_i/(A_+M)_i$.

Since we have an a priori knowledge of the dimension of $\bar{M}$, we get the following algorithm for finding an A-basis S for M:

Step 0: Find a subset $S_0 \subseteq M_0$ such that $\bar{S}_0$ is a basis for $\bar{M}_0$.

Step $i+1$: Suppose we have subsets $S_0, S_1, \ldots, S_i$ of $M_0, \ldots M_i$ which map to bases of $\bar{M}_0, \ldots, \bar{M}_i$. If $|S_0 \cup \cdots \cup S_i| = \dim W^{Z(\Sigma)}$, halt and return $S = S_0 \cup \cdots \cup S_i$. Otherwise find $S_{i+1} \subseteq M_{i+1}$ mapping to a basis for $\bar{M}_{i+1}$ and continue to the next step.

3.4 Smooth coefficients

Having obtained information about the structure of $\mathbb{R}[V,W]^K$ as a module over $R[V]^K$ in the preceding section, we would like to relate it to the structure of $C^\infty(V,W)^K$ over $C^\infty(V)^K$. (And similarly for Σ). This is a job for the Malgrange Division Theorem, and ultimately relies on the density of polynomials in smooth functions. We start with a motivating special case of the MDT:

Consider smooth real-valued functions on $\mathbb{R}$ (coordinate: t). Such a function f can be divided by t, that is, can be written as $f(t) = t \cdot g(t)$ for $g \in C^\infty(\mathbb{R})$ if and only if $f(0) = 0$. Here is a proof: The other implication being obvious, let's assume that $f(0) = 0$. Then we can define $g(t) = \int_0^1 f'(ts)ds$.

Theorem 3.4.1 (Malgrange Division Theorem)**.** *Let $a_1, \ldots, a_s$ be analytic functions on $\mathbb{R}^n$ with values in $\mathbb{R}^m$, and I the $C^\infty(\mathbb{R}^n)$-submodule of $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ generated by the a_i. Then*

$$I = \{F \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \quad | \quad \forall x \in \mathbb{R}^n, \quad T_x F \in \sum_{i=1}^{s} \mathbb{R}[[\mathbb{R}^n]] \cdot T_x a_i\}$$

where $\mathbb{R}[[\mathbb{R}^n]]$ denotes the ring of formal power series on $\mathbb{R}^n$, and T_xF, T_xa_i denote the Taylor series of F, a_i at x.

Proof. See [47], chapter VI. ☐

Now let V, W be representations of the compact Lie group K, and assume V is polar.

Then the set $\mathbb{R}[V,W]^K$ of equivariant polynomial maps from V to W is finitely generated over the ring of invariant polynomial functions $\mathbb{R}[V]^K$ (see preceding section). Let $a_1,\ldots,a_s$ be generators. Then one has:

Proposition 3.4.1. *With the notations above, $a_1,\ldots,a_s$ generate $C^\infty(V,W)^K$ over $C^\infty(V)^K$.*

Proof. (taken from [20], see lemma 3.1) Let

$$\mathcal{M} = \sum_{i=1}^{s} C^\infty(V)\cdot a_i \qquad \mathcal{M}^K = \sum_{i=1}^{s} C^\infty(V)^K \cdot a_i$$

We are trying to show that $\mathcal{M}^K = C^\infty(V,W)^K$. First note that $\mathcal{M}^K = \mathcal{M}\cap C^\infty(V,W)^K$. Indeed, if $\lambda_i \in C^\infty(V)$ are such that $\sum_i \lambda_i a_i$ is K-equivariant, then

$$\sum_i \lambda_i a_i = \int_K \left(\sum_i \lambda_i a_i\right) = \sum_i \left(\int_K \lambda_i\right) a_i \in \mathcal{M}^K$$

Since the a_i are polynomials, they are analytic, and so we can apply the Malgrange Division Theorem. It implies that $\mathcal{M}$ is a closed subspace of $C^\infty(V,W)$ with respect to the C^∞ topology. Therefore $\mathcal{M}^K$ is closed in $C^\infty(V,W)^K$.

On the other hand, since $a_1,\ldots,a_s$ generate $\mathbb{R}[V,W]^K$ over $\mathbb{R}[V]^K$, $\mathcal{M}^K$ contains $\mathbb{R}[V,W]^K$, and hence is dense in $C^\infty(V,W)^K$. Therefore $\mathcal{M}^K = C^\infty(V,W)^K$. ☐

Let $\Sigma \subseteq V$ be a section. Then $a_i|_\Sigma$ generate $|_\Sigma(C^\infty(V,W)^K)$ over $C^\infty(\Sigma)^K$, and applying the Malgrange Division Theorem together with the same averaging argument used in the proof above, we get:

$$|_\Sigma(C^\infty(V,W)^K) = \left\{\sigma \in C^\infty(\Sigma, W^{Z(\Sigma)})^{W(\Sigma)} \quad | \quad \forall x\in\Sigma,\ T_x\sigma \in \sum_{i=1}^{s}\mathbb{R}[[\Sigma]]\cdot T_x(a_i)\right\}$$

The conditions on the Taylor series of σ listed on the right-hand side are the *smoothness conditions*. The space of smoothness conditions at 0 is isomorphic to a complement of $\mathbb{R}[V,W]^K$ in $\mathbb{R}[[\Sigma]]\otimes W$. But since $\sigma \in C^\infty(\Sigma, W^{Z(\Sigma)})^{W(\Sigma)}$, it already satisfies many of these conditions. Indeed, $T_0\sigma$ satisfies a set of conditions isomorphic to a complement of $\mathbb{R}[\Sigma, W^Z]^W$, and therefore the effective set of smoothness conditions on σ is isomorphic to the quotient

$$\frac{\mathbb{R}[\Sigma, W^Z]^W}{\mathbb{R}[V,W]^K}$$

We saw in the preceding section that the numerator and denominator are free of the same rank l.

Proposition 3.4.2. *Let A be a polynomial ring in n variables, and M a module which is the quotient of two graded free modules of the same rank l. Then for $n = 1$ M is finite-dimensional over $\mathbb{R}$, and for $n \geq 2$ it is either zero or infinite-dimensional over $\mathbb{R}$.*

Proof. If $n = 1$, A is a PID, and since M is torsion, by the classification of finitely generated modules over a PID it must be finite-dimensional over $\mathbb{R}$.

If $n \geq 2$, the Poincaré series of M of the form

$$\left(\sum_{i=1}^{l} (t^{a_i} - t^{b_i}) \right) P_t(A)$$

where $P_t(A) = (1 - t^{d_1})^{-1} \cdots (1 - t^{d_n})^{-1}$. Note that $P_t(A)$ has a pole at 1 of order $n \geq 2$. Since $t^{a_i} - t^{b_i}$ can have a zero of order at most 1 at $t = 1$, unless $a_i = b_i$, the Poincaré series can only be a polynomial if it is zero. □

In section 3.5 there are examples that show that the number of smoothness conditions in cohomogeneity one can be arbitrarily large.

3.5 Two Formulas

In this section we find formulas for the two smallest chomogeneity one representations.

These are formulas for smoothness conditions for any equivariant bundle when the slice representation is of the given type K, V. That is, given a general representation W of K it gives a basis for $\mathbb{R}[\Sigma, W^Z]^W$ and a basis for $\mathbb{R}[V, W]^K$ as combinations of the first basis. Here 'basis' always means 'basis over $A = \mathbb{R}[V]^K$'.

It is enough to give formulas for the irreducible W only.

3.5.1 $K = S^1$ on $V = \mathbb{R}^2$

This is cohomogeneity one, so $\mathbb{R}[V]^K = \mathbb{R}[t^2]$.

Let W be a real irreducible K-representation. Then it is either trivial 1-dimensional or 2-dimensional depending on an integral parameter $m > 0$. If one identifies $W = \mathbb{R}^2$ with $\mathbb{C}$, and $K = S^1$ with $U(1)$, the action of $z \in K$ takes w to $z^m w$.

Proposition 3.5.1. *With W depending on m as above, $\mathbb{R}[\Sigma, W]^{W(\Sigma)}$ is*

$$\mathbb{R}[t^2] \otimes W \qquad m \text{ even}$$

$$\mathbb{R}[t^2] t \otimes W \qquad m \text{ odd}$$

and the image of $\mathbb{R}[V, W]^K$ in $\mathbb{R}[\Sigma, W]^{W(\Sigma)}$ is

$$\mathbb{R}[t^2] t^m \otimes W$$

Proof. V has weights $\pm i$, and W has weights $\pm mi$. Using the formulas for weights of symmetric powers and tensor products one computes the Poincaré series

$$P_t(\mathbb{R}[V,W]^K) = 2t^m + 2t^{m+2} + \dots$$

$-1 \in W(\Sigma)$ acts on W by ± 1 according to the parity of m, and we get:

$$P_t(\mathbb{R}[\Sigma,W]^{W(\Sigma)}) = 2t^{(m \mod 2)}(1 + t^2 + \dots)$$

Therefore $\mathbb{R}[V,W]^K$ and $\mathbb{R}[\Sigma,W]^{W(\Sigma)}$ are as stated. □

3.5.2 $K = S^3$ on $V = \mathbb{R}^3$

Again $\mathbb{R}[V]^K = \mathbb{R}[t^2]$.

Let F be the standard representation of K on $\mathbb{C}^2$. The complex irreducible representations of K are the $\mathrm{Sym}^k F$, for $k = 0, 1, 2, \dots$. The real irreducible representations are the $\mathrm{Sym}^k F$ for k even, and $\mathrm{Sym}^k F \oplus \mathrm{Sym}^k F$ for k odd.

Consider $W = \mathrm{Sym}^k F$, for k even. Note that this representation descends to $SO(3)$. A basis for W is

$$\{x^k, x^{k-1}y, \dots y^k\}$$

Our domain representation V is $\mathrm{Sym}^2 F$, and we choose $\Sigma = \mathbb{R}xy$. Then

$$W^{Z(\Sigma)} = \mathbb{R}x^{k/2}y^{k/2}$$

Proposition 3.5.2. *In the notation above,*

a) If $W = \mathrm{Sym}^k F \oplus \mathrm{Sym}^k F$ for k odd, then both $\mathbb{R}[\Sigma, W^Z]^W$ and $\mathbb{R}[V,W]^K$ are zero.

b) If $W = \mathrm{Sym}^k F$, for k even, then $\mathbb{R}[\Sigma, W^Z]^{W(\Sigma)}$ is

$$\mathbb{R}[t^2] \otimes x^{k/2}y^{k/2} \qquad k/2 \text{ even}$$

$$\mathbb{R}[t^2]t \otimes x^{k/2}y^{k/2} \qquad k/2 \text{ odd}$$

and the image of $\mathbb{R}[V,W]^K$ in $\mathbb{R}[\Sigma, W^Z]^{W(\Sigma)}$ is

$$\mathbb{R}[t^2]t^{k/2} \otimes x^{k/2}y^{k/2}$$

Proof. Let W_n denote $\mathrm{Sym}^n F$, so that $V = W_2$. We have the following decomposition formulas (see Exercises 11.14 and 11.11 in [22])

$$\mathrm{Sym}^n(V) = \bigoplus_{\alpha=0}^{\lfloor n/2 \rfloor} W_{2n-4\alpha}$$

$$W_a \otimes W_b = W_{a+b} \oplus W_{a+b-2} \oplus \cdots W_{|a-b|}$$

From these we get that the Poincaré series are zero when $W = W_k \oplus W_k$ for k odd.

And when $W = W_k$, for k even, the Poincaré series are

$$P_t(\mathbb{R}[V, W]^K) = t^{k/2}(1 + t^2 + \dots)$$

$$P_t(\mathbb{R}[\Sigma, W]^{W(\Sigma)}) = t^{(k/2 \bmod 2)}(1 + t^2 + \dots)$$

Therefore $\mathbb{R}[V, W]^K$ and $\mathbb{R}[\Sigma, W]^{W(\Sigma)}$ are as stated. □

Remark 3.5.1. In the same way one could also get formulas for the polar representation of $K = S^3$ on $V = \mathbb{R}^4 = \mathbb{C}^2$. In the notation above $V = W_1 \oplus W_1$.

Chapter 4

Extending symmetric 2-tensors

4.1 Introduction and Main Theorem

The second part of this thesis consists of the following result:

Theorem 4.1.1 (Main Theorem)**.** *Let M be a polar G-manifold with section Σ, and $W(\Sigma)$ the generalized Weyl group associated to Σ.*

Let $\sigma \in C^\infty(\mathrm{Sym}^2(T^*\Sigma))^{W(\Sigma)}$ be a $W(\Sigma)$-equivariant symmetric two-tensor on the section Σ. Then there exists a G-equivariant symmetric two-tensor $\tilde{\sigma}$ on M which restricts to σ.

Moreover, if σ is a metric, that is, positive definite at every point, then we can choose $\tilde{\sigma}$ to be a metric with respect to which the action of G remains polar with the same sections.

This is analogous to Michor's Basic Forms Theorem (see [48] and [49]):

Theorem 4.1.2 (Michor)**.** *Let M be a polar G-manifold with section Σ, and $W(\Sigma)$ the generalized Weyl group associated to Σ.*

Then the restriction map

$$|_\Sigma : C^\infty(\Lambda^p(T^*M))^G_{\mathrm{hor}} \to C^\infty(\Lambda^p(T^*\Sigma))^{W(\Sigma)}$$

is an isomorphism. Here $C^\infty(\Lambda^p(T^*M))^G_{\mathrm{hor}}$ denotes the space of G-equivariant exterior p-forms which are horizontal, that is, that vanish when contracted with any vector tangent to an orbit.

The main part of Michor's proof consists of combining Chevalley's Restriction Theorem with the following result about reflection groups:

Theorem 4.1.3 (Solomon)**.** *Let $W \subseteq O(V)$ be a finite reflection group, where V is a finite-dimensional Euclidean vector space. Let $\rho_1, \ldots \rho_n$ be basic invariants. Then the space of W-equivariant exterior p-forms, $\mathbb{R}[V, \Lambda^p V^*]^W$, is a free $\mathbb{R}[V]^W$-module with basis*

$$\{d\rho_{i_1} \wedge \ldots \wedge d\rho_{i_p} \quad | \quad 1 \leq i_1 < \ldots < i_p \leq n\}$$

Proof. See the original paper [63], or [35] section 22. ∎

Our proof follows the same strategy, with Solomon's theorem replaced with a new result about reflection groups, the Hessian theorem (see section 4.2).

Most of the work goes into proving a linear version:

Proposition 4.1.1. *Let V be a polar K-representation, where K is a compact Lie group, not necessarily connected, with section $\Sigma \subset V$ and generalized Weyl group W. Denote by $i : \Sigma \to V$ the inclusion.*

Let $\sigma \in C^\infty(\Sigma, \mathrm{Sym}^2\Sigma^*)^W$. Then there exists $\tilde{\sigma} \in C^\infty(V, \mathrm{Sym}^2 V^*)^K$ such that $i^*\tilde{\sigma} = \sigma$ and $\tilde{\sigma}(X, Y) = 0$ for X horizontal and Y vertical. (Recall: vertical means tangent to the orbit, and horizontal means normal to the orbit.)

Moreover, if σ is positive definite at 0, then so is $\tilde{\sigma}$.

Proof. Recall that

$$N(\Sigma) = \{g \in K \mid g\Sigma = \Sigma\} \qquad Z(\Sigma) = \{g \in K \mid gp = p \ \forall p \in \Sigma\}$$

are called the normalizer and centralizer of Σ in K, and that the generalized Weyl group was defined by

$$W(\Sigma) = \frac{N(\Sigma)}{Z(\Sigma)}$$

Let K_0 denote the connected component of K containing the identity, and by $N(\Sigma)_0$ and $Z(\Sigma)_0$ the normalizer and centralizer of Σ in K_0, which equal $N(\Sigma)_0 = N(\Sigma) \cap K_0$ and $Z(\Sigma)_0 = Z(\Sigma) \cap K_0$.

From Dadok's classification it follows that $W(\Sigma)_0 = N(\Sigma)_0/Z(\Sigma)_0$ is a Weyl group, that is, a crystallographic reflection group – see proposition 2.2.1 on page 13. Therefore, by the Hessian theorem, there are homogeneous polynomial W_0-invariants $Q_1, \ldots Q_l$ whose Hessians generate the module $\mathbb{R}[\Sigma, \mathrm{Sym}^2\Sigma^*]^{W_0}$ over the algebra of incariants $\mathbb{R}[V]^{W_0}$.

By Proposition 3.4.1 on page 33, $\mathrm{Hess}(Q_1), \ldots, \mathrm{Hess}(Q_l)$ also generate $C^\infty(\Sigma, \mathrm{Sym}^2\Sigma^*)^{W_0}$ over $C^\infty(V)^{W_0}$.

Since σ is $W(\Sigma)$-equivariant, it is also $W(\Sigma)_0$-equivariant, and so there are smooth W_0-invariants a_i such that

$$\sigma = \sum_i a_i \cdot \mathrm{Hess}(Q_i)$$

By the Chevalley Restriction Theorem (see theorem 2.3.6 on page 15), there are unique extensions of a_i and Q_i to

$$\tilde{a}_i \in C^\infty(V)^{K_0} \qquad \tilde{Q}_i \in \mathbb{R}[V]^{K_0}$$

Define $\tilde{\sigma}_0$ by

$$\tilde{\sigma}_0 = \sum_i \tilde{a}_i \cdot \mathrm{Hess}(\tilde{Q}_i) \quad \in C^\infty(V, \mathrm{Sym}^2 V^*)^{K_0}$$

and $\tilde{\sigma}$ by

$$\tilde{\sigma} = \int_{g \in K} g \cdot \tilde{\sigma}_0 \quad \in C^\infty(V, \mathrm{Sym}^2 V^*)^K$$

where the integral sign denotes Haar integration of total volume 1.

We claim that $i^*\tilde{\sigma} = \sigma$. The idea is to show that $i^*(g \cdot \tilde{\sigma}_0) = \sigma$ for all $g \in K$.

Indeed, given a $g \in K$, $g^{-1}\Sigma$ is another section, and since the action of K_0 is polar with the same sections as K, it must act transitively on the sections, and thus there is an $h \in K_0$ such that $g^{-1}\Sigma = h^{-1}\Sigma$, that is, $gh^{-1} \in N(\Sigma)$. (see section 2.1.2)

Then

$$g \cdot \tilde{\sigma}_0 = gh^{-1}h \cdot \tilde{\sigma}_0 = gh^{-1} \cdot \tilde{\sigma}_0$$

because $\tilde{\sigma}_0$ is K_0-equivariant. Applying i^* to both sides gives

$$i^*(g \cdot \tilde{\sigma}_0) = [gh^{-1}] \cdot i^*(\tilde{\sigma}_0) = [gh^{-1}] \cdot \sigma = \sigma$$

where $[gh^{-1}]$ denotes the class of gh^{-1} in the quotient $W = N(\Sigma)/Z(\Sigma)$. This finishes the proof that $i^*\tilde{\sigma} = \sigma$.

Now we turn to the second statement. Let $X, Y \in T_pV$ with X vertical and Y horizontal, that is, tangent and normal to the orbit through p. Extend them to parallel (in the Euclidean metric) vector fields, also called X and Y.

For each $\tilde{Q}_i$, let $f = \frac{\partial \tilde{Q}_i}{\partial X}$. Since Σ is a vector subspace, X is orthogonal to Σ at every point of Σ. So at every regular $q \in \Sigma$, $X(q)$ is tangent to the orbit. Since $\tilde{Q}_i$ is constant on orbits, $f = 0$ at every regular $q \in \Sigma$, and hence on all of Σ.

In particular $\mathrm{Hess}(\tilde{Q}_i)_p(X, Y) = \frac{\partial}{\partial Y}(f) = 0$. Since $\tilde{\sigma}_0$ is a linear combination of such Hessians and $\tilde{\sigma} = \int_g g \cdot \tilde{\sigma}_0$, we get $\tilde{\sigma}(X, Y) = 0$ as well.

Finally, assume σ is positive-definite. We are going to show that $\tilde{\sigma}$ is positive definite at $0 \in V$.

It suffices to do so for

$$\tilde{\sigma}_0 = \sum_{i=1}^{l} \tilde{a}_i \cdot \mathrm{Hess}(\tilde{Q}_i)$$

Since we are evaluating at 0, only the terms with $\deg(\tilde{Q}_i) = 2$ survive. There is one such term, $\tilde{Q}_{i_j}$ for each irreducible factor V_j of the K-representation $V = \bigoplus_j V_j$, and it is, up to a non-zero constant, equal to $\tilde{Q}_{i_j}(v) = |P_{V_j}v|^2$, where P_{V_j} is the orthogonal projection to V_j.

Moreover, the W-representation Σ has one irreducible factor $\Sigma_j \subseteq V_j$, for each V_j (see [15]), and so $Q_{i_j}(v) = |P_{\Sigma_j}v|^2$ for $v \in \Sigma$.

Thus

$$\sigma(0) = 2\sum_j a_{i_j}(0)\mathrm{Id}_{\Sigma_j}$$

Since this is positive definite, each $a_{i_j}(0) = \tilde{a}_{i_j}(0)$ is positive, which means that

$$\tilde{\sigma}_0(0) = 2\sum_j a_{i_j}(0)\mathrm{Id}_{V_j}$$

is positive-definite as well. □

Now we are ready for the general case:

proof of Main Theorem, 4.1.1. Let $p \in \Sigma$, and $\mathcal{U}$ be a small tubular neighbourhood sround the orbit Gp. Let $K = G_p$ be the isotropy, and V be the slice at p, which we identify with a submanifold of $\mathcal{U}$ passing through p.

V is a polar K-representation with section $T_p\Sigma$. By the linear case above, we get a $\tilde{\sigma} \in C^\infty(\mathrm{Sym}^2 V^*)^K$. If σ is a metric, we may assume that $\tilde{\sigma}$ is a metric also (perhaps by shrinking the tubular neighbourhood), with respect to which $T_p\Sigma$ meets the K-orbits orthogonally.

Choose any K-equivariant $\tau : V \to \mathrm{Sym}^2((T_pGp)^*)$. In the metric case τ should be positive-definite. Combining with $\tilde{\sigma}$ we get something in $C^\infty(\mathrm{Sym}^2 TM^*|_V)^K$ which is a metric, and with respect to which $T_p\Sigma$ meets G-orbits orthogonally.

By Proposition 3.2.1 this extends uniquely to $C^\infty(\mathrm{Sym}^2 TM^*|_{\mathcal{U}})^G$.

With these and a G-invariant partition of unity one then defines a $\tilde{\sigma}$ globally. □

4.2 Hessian Theorem for Weyl groups

Theorem 4.2.1 (Hessian Theorem)**.** *Let $W \subseteq O(V)$ be a Weyl group, that is, a crystallographic finite reflection group, where V is an Euclidean vector space of dimension n.*

Then there are homogeneous invariants $Q_1, \ldots Q_l \in \mathbb{R}[V]^W$ whose Hessians form a basis for the space of W-equivariant symmetric 2-tensors, $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$, where $l = \dim(\mathrm{Sym}^2 V^*) = (n^2 + n)/2$.

We start with subsections of preliminaries.

4.2.1 The Poincaré series and Molien's formula

Definition 4.2.1. A real algebra A is *graded* if it decomposes as a direct sum of vector subspaces $A = A_0 \oplus A_1 \oplus \cdots$ such that $A_i \cdot A_j \subseteq A_{i+j}$ and A_0 is the set of scalar multiples of $1 \in A$. Similarly, if M is an A-module, it is *graded* if $M = M_0 \oplus M_1 \oplus \cdots$ such that $A_i \cdot M_j \subseteq M_{i+j}$.

In our situation, $\mathbb{R}[V]$ and $\mathbb{R}[V]^W$ are both graded algebras, where A_i = homogeneous polynomials of degree i, and similarly for the modules $\mathbb{R}[V, \mathrm{Sym}^2 V^*]$ and $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$.

Definition 4.2.2. Let A be a graded real algebra. The *Poincaré series* of A is the formal power series defined by:

$$P_t(A) = \sum_{i=0}^{\infty} (\dim A_i) t^i$$

And similarly for modules.

Examples:

$$P_t(\mathbb{R}[V]) = \frac{1}{(1-t)^n}$$

$$P_t(\mathbb{R}[V]^W) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}}$$

$$P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]) = \frac{n^2+n}{2} \frac{1}{(1-t)^n}$$

where $n = \dim V$, and d_i are the degrees of the Weyl group W.

To compute Poincaré series we use:

Theorem 4.2.2 (Molien's formula)**.** *Let G be any finite group, $\rho : G \to \mathrm{GL}(V)$ be a representation, and U be another representation, with character χ. Then*

$$P_t(\mathbb{R}[V, U]^G) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi(g)}{\det(1 - t\rho(g))}$$

Proof. See [35] (section 24, page 249) □

4.2.2 Linear Independence

Let A be $\mathbb{R}[V]$ or $\mathbb{R}[V]^W$, where $W \subseteq O(V)$ is a finite reflection group, and V is an Euclidean vector space. Let M be a free graded module over A, with (homogeneous) basis $e_1, \ldots e_l$, and let $f_1, \ldots f_l$ another set of l elements of M. Then $f_1, \ldots f_l$ are linearly independent over A if and only d is non-zero, where d is the determinant of the $l \times l$ matrix with entries in A obtained by writing the f_i in the basis e_j. The way we will usually prove that such a d is non-zero is by evaluating at a vector $v \in V$.

Let U be an arbitrary W-representation, with basis $e_1, \ldots e_l$.

Then $e_1, \ldots e_l$ is also an $\mathbb{R}[V]$-basis of $\mathbb{R}[V, U]$.

Now put $M = \mathbb{R}[V, U]^W$ and $A = \mathbb{R}[V]^W$. To show that f_i, $i = 1, \ldots l \in M$ are linearly independent over $\mathbb{R}[V]^W$ it is sufficient to show that the determinant of the matrix of coefficients of f_i in terms of e_j is non-zero. That is because linear independence over $\mathbb{R}[V]$ implies linear independence over $\mathbb{R}[V]^W$. (Remark: this determinant is not necessarily invariant)

4.2.3 Hessians

Let $\rho_1, \dots \rho_n \in \mathbb{R}[V]^W$ be basic invariants. Define the map $\rho : V \to \mathbb{R}^n$ by $\rho(v) = (\rho_1(v), \dots, \rho_n(v))$. Then we have:

Lemma 4.2.1 (Chain Rule)**.** *Let $f \in \mathbb{R}[Y_1, \dots Y_n]$ be a polynomial map on $\mathbb{R}^n$. Then*

$$\mathrm{Hess}(\rho^* f) = J^t(\rho^*(\mathrm{Hess} f))J + \sum_{k=1}^{n} \left(\rho^* \left(\frac{\partial f}{\partial Y_k} \right) \right) \mathrm{Hess}(\rho_k)$$

where J is the Jacobian matrix $\left(\frac{\partial \rho_i}{\partial X_j}\right)_{i,j}$

Corollary 4.2.1. *Let $M \subseteq \mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ be the $\mathbb{R}[V]^W$-submodule generated by the Hessians of all invariants. Then M is generated by the Hessians of*

$$\rho_1, \ \dots \ \rho_n, \ \rho_1^2, \ \rho_1\rho_2, \ \rho_1\rho_3, \ \dots \ \rho_n^2$$

4.2.4 The proof of the Hessian Theorem

We first reduce the proof to the irreducible case:

Proposition 4.2.1. *Let $W_i \subseteq O(V_i)$, $i = 1, 2$ be two finite reflection groups in the Euclidean vector spaces V_i of dimensions n_i. Let $\rho_j \in \mathbb{R}[V_1]^{W_1}$, $j = 1, \dots n_1$ and $\psi_j \in \mathbb{R}[V_2]^{W_2}$, $j = 1, \dots n_2$ be basic invariants on V_1 and V_2 respectively, and $Q_j \in \mathbb{R}[V_1]^{W_1}$, for $j = 1, \dots (n_1^2 + n_1)/2$, $R_j \in \mathbb{R}[V_2]^{W_2}$, for $j = 1, \dots (n_2^2 + n_2)/2$ be homogeneous invariants whose Hessians form a basis for the corresponding spaces of equivariant symmetric 2-tensors.*

Then the Hessians of the following set of $W = W_1 \times W_2$-invariant polynomials on $V = V_1 \times V_2$ form a basis for the space of equivariant symmetric 2-tensors on V:

$$\{Q_j\} \cup \{R_j\} \cup \{\rho_i \psi_j, \quad i = 1 \dots n_1, j = 1 \dots n_2\}$$

In particular, if the Hessian theorem is true for $W_i \subseteq O(V_i)$, $i = 1, 2$, then it is also true for the action of $W = W_1 \times W_2$ on $V = V_1 \times V_2$.

Proof. Let's use Molien's formula (Theorem 4.2.2 on page 41). Note that for $g = (g_1, g_2) \in W$, we have

$$\det(1 - tg) = \det(1 - tg_1) \cdot \det(1 - tg_2)$$

Then note that

$$\mathrm{Sym}^2(V_1^* \oplus V_2^*) = \mathrm{Sym}^2 V_1^* \oplus \mathrm{Sym}^2 V_2^* \oplus (V_1^* \otimes V_2^*)$$

as representations of W. Thus $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ breaks down as a direct sum of three $\mathbb{R}[V]^W$-modules accordingly.

Consider the first piece, namely $\mathbb{R}[V, \mathrm{Sym}^2 V_1^*]^W$. We claim that it is a free $\mathbb{R}[V]^W$-module with basis $\{\mathrm{Hess} Q_j\}$. Indeed, using Molien's formula its Poincaré series is

$$\frac{1}{|W|} \sum_{g_1 \in W_1, g_2 \in W_2} \frac{\mathrm{tr}(\mathrm{Sym}^2 g_1)}{\det(1 - tg_1) \cdot \det(1 - tg_2)} = P_t(\mathbb{R}[V_1, \mathrm{Sym}^2 V_1^*]^{W_1}) \cdot P_t(\mathbb{R}[V_2]^{W_2})$$

and $\{\mathrm{Hess} Q_j\}$, being linearly independent over $\mathbb{R}[V_1]^{W_1}$, are also l.i. over $\mathbb{R}[V]^W$.

The same argument shows that $\{\mathrm{Hess} R_j\}$ form a basis for the free $\mathbb{R}[V]^W$-module $\mathbb{R}[V, \mathrm{Sym}^2 V_2^*]^W$ (second piece).

Finally consider the third piece, $\mathbb{R}[V, V_1^* \otimes V_2^*]^W$. Since $\mathrm{tr}(g_1 \otimes g_2) = (\mathrm{tr} g_1)(\mathrm{tr} g_2)$, Molien's formula gives

$$P_t(\mathbb{R}[V, V_1^* \otimes V_2^*]^W) = P_t(\mathbb{R}[V_1, V_1^*]^{W_1}) \cdot P_t(\mathbb{R}[V_2, V_2^*]^{W_2})$$

By Solomon's theorem (Theorem 4.1.3 on page 37) we know that $\{d\rho_i\}$ is a basis for $\mathbb{R}[V_1, V_1^*]^{W_1}$ and similarly for $\{d\psi_i\}$ and $\mathbb{R}[V_2, V_2^*]^{W_2}$. Then $\{d\rho_i \otimes d\psi_j\}$ are linearly independent over $\mathbb{R}[V]^W$, and therefore form a basis for $\mathbb{R}[V, V_1^* \otimes V_2^*]^W$ over $\mathbb{R}[V]^W$.

To conclude the proof we use the product rule:

$$\mathrm{Hess}(\rho_i \psi_j) = d\rho_i \otimes d\psi_j + \rho_i \mathrm{Hess} \psi_j + \psi_j \mathrm{Hess} \rho_i$$

□

For completeness, let's prove the converse of the above proposition:

Proposition 4.2.2. *Let $W \subseteq O(V)$ be a finite reflection group of the Euclidean vector space V, and let $V' \subseteq V$ be a W-stable subspace.*

Suppose the Hessian theorem holds for the action of W on V, so that there are $Q_j \in \mathbb{R}[V]^W$ whose Hessians generate $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ as an $\mathbb{R}[V]^W$-module. Then their restrictions to V', $Q_j|V'$, generate $\mathbb{R}[V', \mathrm{Sym}^2 (V')^*]^W$ as an $\mathbb{R}[V']^W$-module, so in particular the Hessian theorem also holds for the action of W on V'.

Proof. Let V'' be the orthogonal complement to V' in V, and $P : V \to V'$ the orthogonal projection onto V'.

$\mathrm{Sym}^2 V^*$ decomposes as

$$\mathrm{Sym}^2 V^* = \mathrm{Sym}^2 (V')^* \oplus \mathrm{Sym}^2 (V'')^* \oplus ((V')^* \otimes (V'')^*)$$

Let $i : \mathrm{Sym}^2 (V')^* \to \mathrm{Sym}^2 V^*$ be the natural inclusion.

Given $\sigma \in \mathbb{R}[V', \mathrm{Sym}^2 (V')^*]^W$ define $\tilde{\sigma} \in \mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ by $\tilde{\sigma} = i \circ \sigma \circ P$. It is equivariant because both i and P are, and it satisfies $\tilde{\sigma}|_{V'} = \sigma$.

By hypothesis there are $a_j \in \mathbb{R}[V]^W$ such that $\tilde{\sigma} = \sum a_j \mathrm{Hess}(Q_j)$. Therefore $\sigma = \sum a_j|_{V'} \mathrm{Hess}(Q_j|_{V'})$, as wanted. □

There is a classification of irreducible finite reflection groups. Here is the list of the ones which are crystallographic, that is, Weyl groups:

- A_n
- B_n (also C_n)
- D_n
- E_6, E_7, E_8
- F_4

In the next section we will prove the Hessian theorem for each type, by finding specific bases $\mathrm{Hess}(Q_j)$. For the exceptional groups of type E and F the proof relies on calculations performed in a computer by GAP. The GAP scripts can be found at
http://dl.dropbox.com/u/374152/GAPcode.zip

As a corollary of the specific computations for each type, we get:

Theorem 4.2.3. *Let $W \subseteq O(V)$ be an irreducible Weyl group, and ρ_j, for $j = 1, \ldots n$ be basic invariants. Let $S = \{\rho_i \rho_j, \quad 1 \leq i \leq j \leq n\}$ be the set of the $(n^2+n)/2$ binomials in the basic invariants. Then there is a subset $T \subset S$ with n elements such that*

$$\{\mathrm{Hess}(Q), \quad Q \in S - T\} \cup \{\mathrm{Hess}(\rho_i), \quad i = 1, \ldots n\}$$

is a basis for $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ as a free module over $\mathbb{R}[V]^W$.

Here is the general strategy of the proof:

- We know from general theory (see Corollary 3.3.2 on page 30) that $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ is a free $\mathbb{R}[V]^W$-module of rank equal to the dimension of $\mathrm{Sym}^2 V^*$. Using Molien's formula (Theorem 4.2.2 on page 41) we compute the Poincaré polynomial of $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ over $\mathbb{R}[V]^W$ and extract from it the degrees of a basis. Then we choose an appropriate subset $T \subset S = \{\rho_i \rho_j, \quad 1 \leq i \leq j \leq n\}$ such that the degrees of $\{\mathrm{Hess}(Q), Q \in S - T\} \cup \{\mathrm{Hess}(\rho_i)\}$ are the same as those of a basis.

- At this point it is enough to show that the chosen Hessians are linearly independent over $\mathbb{R}[V]^W$, because then the submodule generated by them will have the same Poincaré series as $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$, forcing them to be equal for dimension reasons in each degree. And to show that such a set of tensors is linearly independent we show that their values at a particular vector $v \in V$ are linearly independent over $\mathbb{R}$. Here v can be any regular vector.

4.3 The proofs by type

4.3.1 Summary

For the dihedral groups the Poincaré series is computed directly, and seen to match the degrees of the Hessians of $\{\rho_1, \rho_2, \rho_1^2\}$. Listing the uppertriangular entries of these Hessians we get a 3×3 matrix, whose determinant is computed directly and seen to be non-zero.

For type A we use a combinatorial identity called the cycle index formula of S_n (see 4.3.2 or [71] chapter 4.7 page 141). Generating functions-type manipulations with this identity together with Molien's formula (see Theorem 4.2.2 on page 41) yield the Poincaré series of the Sym^2 tensors.

Making a particular choice of basic invariants, namely the power sum polynomials, and of a regular vector $v = (1, \xi, \xi^2, \ \ldots \ \xi^{n-1})$, where $\xi = \exp(2\pi i/n)$, the Hessians of the n basic invariants evaluated at v have particularly simple forms. Looking at these n matrices it is then easy to see which sets of Hessians are linearly independent when evaluated at v.

Types B and D: The cycle index formulas corresponding to these groups can be easily proved using the formula for type A, and from then on everything works very similarly to type A.

For the exceptional types E and F we use a computer. The Poincaré series can be computed from the character table, which comes already computed in the package CHEVIE. We find explicit basic invariants very similar to the power sum polynomials, associated to a finite W-stable subset of V^*, the so-called *orbit Chern classes* (see [62]). A particular regular vector $v \in V$ is chosen and for each set of Hessians with degrees matching the Poincaré polynomial the program lists all the entries of these Hessians into a square matrix, computes its determinant evaluated at v, and checks that it is non-zero.

4.3.2 Dihedral groups

Let $V = \mathbb{R}^2$ and $W = D_n$, the dihedral group with $2n$ elements. It is generated by a, b, where a is counterclockwise rotation by $2\pi/n$ and b is the reflection across the x-axis. Thus

$$D_n = \{1, a, \ldots, a^{n-1}, b, ab, \ldots a^{n-1}b\}$$

It is convenient to introduce complex notation: identify (x, y) with $z = x + iy$. Then a becomes complex multiplication with $\xi = e^{2\pi i/n}$, and b becomes complex conjugation.

It is well-known that we can take the basic invariants to be

$$\rho_1(x, y) = z\bar{z} = x^2 + y^2 \qquad \rho_2(x, y) = \mathrm{Re}(z^n)$$

Thus the degrees of W are $d_1 = 2$, $d_2 = n$.

In the following subsections we will prove

Theorem 4.3.1. *A basis for the free $\mathbb{R}[V]^W$-module $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ is*

$$\{\mathrm{Hess}(\rho_1), \mathrm{Hess}(\rho_1^2), \mathrm{Hess}(\rho_2)\}$$

Poincaré series

Since the degrees are 2 and n, the Poincaré series of the ring of invariants is

$$P_t(\mathbb{R}[V]^W) = \frac{1}{(1-t^2)(1-t^n)}$$

Proposition 4.3.1. *The Poincaré series for the space of equivariant symmetric 2-tensors is*

$$P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W) = (1 + t^2 + t^{n-2}) P_t(\mathbb{R}[V]^W)$$

Proof. Since the eigenvalues of a^j are ξ^j and ξ^{-j}, we see that

$$\det(1 - ta^j) = (t - \xi^j)(t - \xi^{-j}) \qquad \mathrm{tr}(\mathrm{Sym}^2 a^j) = 1 + \xi^{2j} + \xi^{-2j}$$

The $a^j b$ are reflections, so that

$$\det(1 - ta^j b) = 1 - t^2 \qquad \mathrm{tr}(\mathrm{Sym}^2 a^j b) = 0$$

Applying Molien's formula (Thoerem 4.2.2 on page 41) gives

$$\frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)} = \frac{(t^2-1)(t^n-1)}{2n} \sum_{j=0}^{n-1} \frac{1 + \xi^{2j} + \xi^{-2j}}{(t-\xi^j)(t-\xi^{-j})}$$

Let's call this polynomial $f(t)$. We are trying to show it is equal to $1 + t^2 + t^{n-2}$. Since they have degrees less than or equal to n, it is enough to check that they have the same values at n distinct points, which we take to be $t = 1, \xi, \xi^2, \ldots \xi^{n-1}$.

- $t = 1$. Since $(t^2 - 1)(t^n - 1)$ has a zero of order 2 at $t = 1$, only one term of the sum survives:

  $$f(1) = \left.\frac{(1+t)(1+t+\ldots+t^{n-1})}{2n}\right|_{t=1} \cdot 3 = 3$$

  which equals $(1 + t^2 + t^{n-2})|_{t=1}$, as wanted.

- n is even and $t = \xi^{n/2} = -1$. Again only one term survives:

$$f(-1) = \left.\frac{(t-1)(t-1)(t-\xi)\cdots(\widehat{t-\xi^{n/2}})\cdots(t-\xi^{n-1})}{2n}\right|_{t=-1} \cdot 3$$

$$= \frac{-2}{2n}\left.\left(\frac{d}{dt}t^n\right)\right|_{t=-1} \cdot 3 = 3$$

  which equals $(1 + t^2 + t^{n-2})|_{t=-1}$, as wanted.

- $t = \xi^k$ with $k \neq 0, n/2$. The terms $j = k, n-k$ survive, and they are equal:

$$\begin{aligned}
f(\xi^k) &= \frac{(\xi^{2k}-1)(1+\xi^{2k}+\xi^{-2k})}{n} \prod_{0\leq j\leq n-1 \quad j\neq k,n-k} (\xi^k - \xi^j) \\
&= \frac{(\xi^{2k}-1)(1+\xi^{2k}+\xi^{-2k})\xi^k}{n(\xi^k - \xi^{n-k})\xi^k} \prod_{0\leq j\leq n-1 \quad j\neq k} (\xi^k - \xi^j) \\
&= \frac{(1+\xi^{2k}+\xi^{-2k})\xi^k}{n} \left(\frac{d}{dt}t^n\right)\bigg|_{t=\xi^k} \\
&= 1 + \xi^{2k} + \xi^{-2k}
\end{aligned}$$

which equals $(1+t^2+t^{n-2})|_{t=\xi^k}$

□

Independence of generators

Proposition 4.3.2. *The following are linearly idependent over $\mathbb{R}[V]$, and in particular also over $\mathbb{R}[V]^W$:*

$$\{\mathrm{Hess}(\rho_1), \mathrm{Hess}(\rho_1^2), \mathrm{Hess}(\rho_2)\}$$

Proof. $\mathbb{R}[V, \mathrm{Sym}^2 V^*]$ is a free $\mathbb{R}[V]$-module with basis

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

We are going to write $\{\mathrm{Hess}(\rho_1), \mathrm{Hess}(\rho_1^2), \mathrm{Hess}(\rho_2)\}$ in terms of this basis and show that the determinant of the 3×3 matrix thus obtained is non-zero.

It's more convenient and enough to do the computation of the Hessians in the basis $z, \bar{z}$ instead of x, y:

$$\mathrm{Hess}(z\bar{z}) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \mathrm{Hess}(z^2\bar{z}^2) = \begin{bmatrix}2\bar{z}^2 & 4z\bar{z}\\ 4z\bar{z} & 2z^2\end{bmatrix}$$

$$\mathrm{Hess}\left(\frac{z^n + \bar{z}^n}{2}\right) = \frac{n(n-1)}{2}\begin{bmatrix}z^{n-2} & 0\\ 0 & \bar{z}^{n-2}\end{bmatrix}$$

Therefore

$$\begin{bmatrix}\mathrm{Hess}(\rho_1)\\ \mathrm{Hess}(\rho_1^2)\\ \mathrm{Hess}(\rho_2)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 2\bar{z}^2 & 4z\bar{z} & 2z^2\\ \frac{n(n-1)}{2}z^{n-2} & 0 & \frac{n(n-1)}{2}\bar{z}^2\end{bmatrix} \cdot \begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\\ \\ \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\ \\ \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\end{bmatrix}$$

The determinant of the 3×3 matrix above is

$$-2n(n-1)i \ \mathrm{Im}(z^n) \neq 0$$

$\square$

4.3.3 Type A

Let $W = S_n$, the permutation group in n letters, and $V = \mathbb{R}^n$, with standard basis $e_1, \dots e_n$. W acts on V by permuting the basis elements.

Let V' be the W-stable subspace $V' = (1, \dots 1)^\perp = \{(x_1, \dots, x_n) \mid x_1 + \cdots + x_n = 0\}$. The Weyl group of type A_{n-1} is this action of W on V'.

We will prove the Hessian theorem for the non-irreducible action of W on V because it is more convenient, and it is sufficient by proposition 4.2.2 on page 43.

The set of basic invariants we are going to use are the renormalized power sum polynomials:

$$\rho_j(x_1, \dots x_n) = \frac{1}{j} \sum_{i=1}^{n} (x_i)^j$$

Thus the degrees are $d_i = i$, $i = 1, \dots n$.

Poincaré series

Given $g \in S_n$, write it as a product of disjoint cycles, and denote by

$$k_i(g) = \text{number of cycles of length } i$$

Then, seeing g as a permutation matrix, we have

$$\mathrm{tr}(g) = k_1$$
$$\mathrm{tr}(\mathrm{Sym}^2 g) = \frac{\mathrm{tr}(g^2) + (\mathrm{tr}(g))^2}{2} = \frac{k_1^2}{2} + \frac{k_1}{2} + k_2$$
$$\det(1 - tg) = (1-t)^{k_1}(1-t^2)^{k_2} \cdots (1-t^n)^{k_n}$$

Therefore from Molien's formula (Theorem 4.2.2 on page 41) we get:

$$\begin{aligned} P_t(\mathbb{R}[V]^W) &= \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \\ &= \frac{1}{n!} \sum_{g \in S_n} \frac{1}{(1-t)^{k_1}(1-t^2)^{k_2}\cdots(1-t^n)^{k_n}} \end{aligned}$$

Now to compute

$$P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W) = \frac{1}{n!} \sum_{g \in S_n} \frac{\frac{k_1^2}{2} + \frac{k_1}{2} + k_2}{(1-t)^{k_1}(1-t^2)^{k_2}\cdots(1-t^n)^{k_n}}$$

we will need to use the:

Theorem 4.3.2 (Cycle index formula for S_n)**.**

$$\exp\left(\sum_{j=1}^{\infty} \frac{z^j x_j}{j}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{g \in S_n} \mathbf{x}^{\mathbf{k}}$$

where $\mathbf{x}^{\mathbf{k}}$ *denotes* $x_1^{k_1(g)} x_2^{k_2(g)} \cdots x_n^{k_n(g)}$.

Proof. See [71] chapter 4.7 page 141. □

Lemma 4.3.1. *Replacing* x_j *with* $(1-t^j)^{-1}$ *in the cycle index formula gives*

$$\exp\left(\sum_{j=1}^{\infty} \frac{z^j}{j(1-t^j)}\right) = \sum_{n=1}^{\infty} \frac{z^n}{(1-t)(1-t^2)\cdots(1-t^n)}$$

Proof. Follows immediately from the formula for $P_t(\mathbb{R}[y_1 \dots y_n]^{S_n})$ and Molien's formula. Alternatively it is possible to prove this directly with a few manipulations of generating functions. □

Now we are ready to attack the Poincaré series of the space of equivariant symmetric 2-tensors:

Proposition 4.3.3 (Poincaré series for Sym^2)**.**

$$\frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)} = \frac{1-t^n}{1-t} + \frac{(1-t^{n-1})(1-t^n)}{(1-t)(1-t^2)}$$

Proof. (this argument involving the differential operator D was pointed out by Christian Krattenthaler)

Define the differential operator

$$D = \frac{x_1^2}{2} \frac{\partial^2}{\partial x_1^2} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

Apply D to both sides of the Cycle Index formula:

$$\left(\frac{x_1^2}{2} z^2 + x_1 z + x_2 \frac{z^2}{2}\right) \exp\left(\sum_{j=1}^{\infty} \frac{z^j}{j} x_j\right) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{g \in S_n} \left(\frac{k_1^2}{2} + \frac{k_1}{2} + k_2\right) \mathbf{x}^{\mathbf{k}}$$

Now replace x_j with $(1-t^j)^{-1}$ and use the lemma:

$$\left(\frac{z}{1-t} + \frac{z^2}{2}\left(\frac{1}{(1-t)^2} + \frac{1}{1-t^2}\right)\right) \sum_{n=1}^{\infty} \frac{z^n}{(1-t)\cdots(1-t^n)} =$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{g \in S_n} \frac{\frac{k_1^2}{2} + \frac{k_1}{2} + k_2}{(1-t)^{k_1} \cdots (1-t^n)^{k_n}}$$

Since

$$\frac{1}{2}\left(\frac{1}{(1-t)^2}+\frac{1}{1-t^2}\right)=\frac{1}{(1-t)(1-t^2)}$$

when we take the coefficient of z^n on both sides we get exactly what we wanted to prove.

□

Remark 4.3.1. Of course the same method can be applied to compute the Poincaré series of tensors of any kind. One just writes the character as a polynomial in the numbers k_i, and defines a differential operator D by replacing each k_i in this polynomial with $x_i \frac{\partial}{\partial x_i}$, and multiplication with composition of operators.

Independence of generators

With our choice of basic invariants $\rho_j = \frac{1}{j}\sum_{i=1}^{n}(x_i)^j$, the Jacobian matrix J is the Vandermonde matrix:

$$J = \left(\frac{\partial \rho_i}{\partial x_j}\right)_{i,j} = (x_j^{i-1})_{i,j}$$

whose determinant is $\det J = \prod_{i>j}(x_i - x_j)$

The Hessians of the basic invariants are

$$\mathrm{Hess}(\rho_i) = (i-1)\begin{bmatrix} x_1^{i-2} & & & \\ & x_2^{i-2} & & \\ & & \ddots & \\ & & & x_n^{i-2} \end{bmatrix}$$

Let $v = (1, \xi, \xi^2, \ldots, \xi^{n-1})$, where $\xi = \exp(2\pi i/n)$. It is a regular vector.

Lemma 4.3.2. *Let B_k denote the $k \times k$ matrix with ones in the antidiagonal and zeros everywhere else.*

$$B_k = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

Then at the regular vector v we have:

$$(J^T)^{-1}\mathrm{Hess}(\rho_i)J^{-1} = \frac{i-1}{n}\begin{bmatrix} B_{i-1} & 0 \\ 0 & B_{n-i+1} \end{bmatrix}$$

That is, the (a, b) entry is $(i-1)/n$ if $a + b - i = 0$ or n, and zero otherwise.

Proof. It's equivalent to show that

$$J^T \begin{bmatrix} B_{i-1} & 0 \\ 0 & B_{n-i+1} \end{bmatrix} J = n \begin{bmatrix} 1 & & & \\ & \xi^{i-2} & & \\ & & \ddots & \\ & & & \xi^{(i-2)(n-1)} \end{bmatrix}$$

This is a computation: The $(l.m)$ entry of the LHS is

$$\sum_{a,b=1}^{n} J_{l,a} M_{a,b} J_{b,m}$$

where $M_{a,b}$ equals 1 if $a+b=i$ or $i+n$, and zero otherwise; $J_{l,a} = \xi^{(l-1)(a-1)}$ and similarly for $J_{b,m}$. Thus the sum becomes

$$= \sum_{a+b=i,i+n} \xi^{(l-1)(a-1)+(b-1)(m-1)} \tag{4.3.1}$$

$$= \sum_{a=1}^{i-1} \xi^{(l-1)(a-1)+(i-a-1)(m-1)} + \sum_{a=i}^{n} \xi^{(l-1)(a-1)+(n+i-a-1)(m-1)} \tag{4.3.2}$$

$$= \sum_{a=1}^{n} \xi^{(l-1)(a-1)+(i-a-1)(m-1)} \tag{4.3.3}$$

$$= \xi^{(i-1)(m-1)-(l-1)} \sum_{a=1}^{n} \xi^{(l-m)a} \tag{4.3.4}$$

We have replaced $(n+i-a-1)$ with $(i-a-1)$ because $\xi^n = 1$.

The last sum $\sum_{a=1}^{n} \xi^{(l-m)a}$ equals 0 or n according to whether ξ^{l-m} equals 1, that is, $l=m$. In the latter case $(i-1)(m-1)-(l-1) = (i-2)(m-1)$. ☐

Theorem 4.3.3. *Let $S = \{\rho_i\rho_j, \quad 1 \le i \le j \le n\}$ and $T \subseteq S$ be a subset with $n-1$ elements which has one element $\rho_i\rho_j$ with $i+j=k$ for each $k = n+2, \dots 2n$. Then*

$$\{\mathrm{Hess}(Q), \quad Q \in S - T\} \cup \{\mathrm{Hess}(\rho_i), \quad i = 2, \dots n\}$$

is a basis for $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ as a free module over $\mathbb{R}[V]^W$.

Proof. We first show that the Hessians are linearly independent over $\mathbb{R}[V]$, by showing that their values at the regular vector v are linearly independent over $\mathbb{R}$.

By the chain rule, for $\rho_i\rho_j \in S - T$ we set $f = y_i y_j$ and get

$$\mathrm{Hess}(\rho_i\rho_j) = J^t(\rho^*(\mathrm{Hess} f))J + \rho_i \mathrm{Hess}(\rho_j) + \rho_j \mathrm{Hess}(\rho_i)$$

where J is the Jacobian matrix $\left(\frac{\partial \rho_i}{\partial x_j}\right)_{i,j}$, and $\rho^*(\mathrm{Hess} f)$ is the matrix $E_{i,j}$ with (i,j) and (j,i) entries equal to one, and the others to zero (except when $i=j$, in which case the (i,i) entry is 2).

Since the Hessians of the basic invariants are in the set we are trying to show is linearly independent, we may ignore the second and third terms above.

Now we apply the invertible linear transformation $M \mapsto (J^T)^{-1}MJ^{-1}$ and are left to show that the set of $E_{i,j}$ corresponding to $S - T$, together with $(J^T)^{-1}\mathrm{Hess}(\rho_i)J^{-1}$ (at v) for $i = 2, \ldots n$, are linearly independent.

This is clear by the shape of $(J^T)^{-1}\mathrm{Hess}(\rho_i)J^{-1}$, described in the lemma above, thus finishing the proof of linear independence.

To show that these Hessians generate $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ it is enough to show that the Poincaré series of their span is equal to that of $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$. The proposition above says that

$$\frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)} = \frac{1 - t^n}{1 - t} + \frac{(1 - t^{n-1})(1 - t^n)}{(1 - t)(1 - t^2)}$$

The RHS equals

$$t^{-2}\left(t(t + t^2 + \ldots + t^n) + (t^2 + t^3 + \ldots + t^n) + \sum_{1 \le i \le j \le n-1} t^{i+j}\right)$$

Recall that the Hessian decreases degree by two. Since the degree of ρ_i is i it is easy to see that $S - T$ matches the formula above. Indeed,

- $P_t(\mathrm{span}\{\mathrm{Hess}(\rho_1^2), \mathrm{Hess}(\rho_1\rho_2), \ldots, \mathrm{Hess}(\rho_1\rho_n)\}) = t^{-2}t(t + \ldots t^n)$
- $P_t(\mathrm{span}\{\mathrm{Hess}(\rho_2), \mathrm{Hess}(\rho_3), \ldots, \mathrm{Hess}(\rho_n)\}) = t^{-2}(t^2 + \ldots t^n)$
- $P_t(\text{span of the remaining Hessians of } S - T) = t^{-2}\sum_{1 \le i \le j \le n-1} t^{i+j}$

□

4.3.4 Types B and C

Let $V = \mathbb{R}^n$, and $W = S_n \ltimes \{\pm 1\}^n \subseteq O(V)$ be the subgroup of signed permutation matrices, that is, matrices with entries in $\{0, -1, +1\}$ with exactly one non-zero entry in each row and column. These are all of the form $\tau \cdot \mathrm{diag}(\epsilon)$, where τ is a permutation matrix and $\epsilon = (\epsilon_1, \ldots \epsilon_n) \in \{\pm 1\}^n$.

The degrees are $2, 4, \ldots 2n$.

Poincaré series

Given $g = \tau \cdot \mathrm{diag}(\epsilon) \in W$, write τ as a product of disjoint cycles, and call each cycle $(i_1, \ldots i_l)$ positive or negative according to the sign of the product $\epsilon_{i_1} \cdots \epsilon_{i_l}$. Denote by k_j^+ the number of positive cycles of length j, by k_j^- the negative ones, and by $k_j = k_j^+ + k_j^-$ the total number of j-cycles.

Proposition 4.3.4. *With the notations above, for $g \in W$ we have:*

$$\det(1 - tg) = \prod_{i=1}^{n} (1 - t^i)^{k_i^+} \prod_{i=1}^{n} (1 + t^i)^{k_i^-}$$

$$\mathrm{tr}(\mathrm{Sym}^2(g)) = k_1^+ + k_1^- + \binom{k_1^+}{2} + \binom{k_1^-}{2} + k_2^+ - k_1^+ k_1^- - k_2^-$$

Proof. The first formula is pretty straightforward. The second can be proved using $\mathrm{tr}(\mathrm{Sym}^2(g)) = \frac{(\mathrm{tr}(g))^2 + \mathrm{tr}(g^2)}{2}$ and the observation that the trace of a signed permutation matrix equals the number of positive 1-cycles minus the number of negative ones. □

We need to compute

$$\frac{1}{n!2^n} \sum_{\tau \in S_n} \sum_{\epsilon \in \{\pm 1\}^n} \frac{\mathrm{tr}(\mathrm{Sym}^2(\tau \cdot \mathrm{diag}(\epsilon)))}{\det(1 - t\tau \cdot \mathrm{diag}(\epsilon))}$$

The strategy is to compute the cycle index formula for $S_n \ltimes \{\pm 1\}^n$, and then proceed in the same way as in type A.

Proposition 4.3.5. *The cycle index formula for $S_n \ltimes \{\pm 1\}^n$ is:*

$$\exp\left(\sum_{j=1}^{\infty} \frac{z^j (x_j^+ + x_j^-)}{2j} \right) = \sum_{n=1}^{\infty} \frac{z^n}{n!2^n} \sum_{g \in S_n \ltimes (\pm 1)^n} \mathbf{x}^{\mathbf{k}}$$

where $\mathbf{x}^{\mathbf{k}}$ denotes $\prod_{j=1}^{n} (x_j^+)^{k_j^+(g)} (x_j^-)^{k_j^-(g)}$.

Proof. Fix a $\tau \in S_n$ and consider the inner sum

$$\sum_{\epsilon \in (\pm 1)^n} \mathbf{x}^{\mathbf{k}}$$

The sign of each cycle $(i_1 \ldots i_l)$ in τ depends only on the values of $\epsilon_1 \ldots \epsilon_l$, and is actually positive for 2^{l-1} such values and negative for the remaining 2^{l-1}. Thus this inner sum becomes

$$\prod_{\text{cycles } \sigma} (2^{l(\sigma)-1} x_{l(\sigma)}^- + 2^{l(\sigma)-1} x_{l(\sigma)}^+) = 2^n \prod_{j=1,\ldots n} \left(\frac{x_k^+ + x_j^-}{2} \right)^{k_j}$$

Now the result follows from the cycle index formula for S_n. □

Lemma 4.3.3. *Replacing x_j^+ with $(1 - t^j)^{-1}$ and x_j^- with $(1 + t^j)^{-1}$ in the RHS of the cycle index formula for W_n gives*

$$\sum_{n=1}^{\infty} \frac{z^n}{(1 - t^2) \cdots (1 - t^{2n})}$$

Proof. Follows from Molien's formula, and the fact that the degrees are $2, 4, \dots 2n$. ☐

Proposition 4.3.6 (Poincaré series for Sym^2)**.**

$$\frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)} = \frac{1-t^{2n}}{1-t^2} + \frac{(1-t^{2n-2})(1-t^{2n})t^2}{(1-t^2)(1-t^4)}$$

Proof. Start with the formula

$$\mathrm{tr}(\mathrm{Sym}^2(g)) = k_1^+ + k_1^- + \binom{k_1^+}{2} + \binom{k_1^-}{2} + k_2^+ - k_1^+ k_1^- - k_2^-$$

and use it to define a differential operator $D =$

$$x_1^+ \frac{\partial}{\partial x_1^+} + x_1^- \frac{\partial}{\partial x_1^-} + \frac{(x_1^+)^2}{2} \frac{\partial^2}{\partial (x_1^+)^2} + \frac{(x_1^-)^2}{2} \frac{\partial^2}{\partial (x_1^-)^2} + x_2^+ \frac{\partial}{\partial x_2^+} - x_1^+ x_1^- \frac{\partial^2}{\partial x_1^+ \partial x_1^-} - x_2^- \frac{\partial}{\partial x_2^-}$$

so that

$$D\mathbf{x}^{\mathbf{k}} = \mathrm{tr}(\mathrm{Sym}^2(g))\mathbf{x}^{\mathbf{k}}$$

Apply D to both sides of the cycle index formula for $S_n \ltimes (\pm 1)^n$:

$$D\left(\exp\left(\sum_{j=1}^{\infty} \frac{z^j (x_j^+ + x_j^-)}{2j}\right)\right) = D\left(\sum_{n=1}^{\infty} \frac{z^n}{n! 2^n} \sum_{g \in S_n \ltimes (\pm 1)^n} \mathbf{x}^{\mathbf{k}}\right)$$

and then replace x_j^+ with $(1-t^j)^{-1}$ and x_j^- with $(1+t^j)^{-1}$. Let's look at both the LHS and the RHS of the equation we get.

On the RHS, the coefficient of z^n is the Poincaré series we are looking for.

The exponential on the LHS gets multiplied with

$$x_1^+ \frac{z}{2} + x_1^- \frac{z}{2} + \frac{(x_1^+)^2}{2} \frac{z^2}{4} + \frac{(x_1^-)^2}{2} \frac{z^2}{4} + x_2^+ \frac{z^2}{4} - x_1^+ x_1^- \frac{z^2}{4} - x_2^- \frac{z^2}{4}$$

Replace x_j^+ with $(1-t^j)^{-1}$ and x_j^- with $(1+t^j)^{-1}$, and then simplify to get

$$\frac{z}{1-t^2} + \frac{z^2 t^2}{(1-t^2)(1-t^4)}$$

Therefore the coefficient of z^n in the LHS becomes

$$\left(\frac{1-t^{2n}}{1-t^2} + \frac{(1-t^{2n-2})(1-t^{2n})t^2}{(1-t^2)(1-t^4)}\right) P_t(\mathbb{R}[V]^W)$$

as wanted. ☐

Independence of generators

We take the basic invariants to be

$$\rho_i = \frac{1}{2i}\sum_{j=1}^{n}(x_j)^{2i} \qquad i = 1, \dots n$$

The Jacobian matrix J is:

$$J = \left(\frac{\partial \rho_i}{\partial x_j}\right)_{i,j} = (x_j^{2i-1})_{i,j}$$

whose determinant is $\det J = x_1 \cdots x_n \prod_{i>j}(x_i^2 - x_j^2)$

The Hessians of the basic invariants are

$$\mathrm{Hess}(\rho_i) = (2i-1)\begin{bmatrix} x_1^{2i-2} & & & \\ & x_2^{2i-2} & & \\ & & \ddots & \\ & & & x_n^{2i-2} \end{bmatrix}$$

Let $v = (1, \xi, \xi^2, \dots, \xi^{n-1})$, where $\xi = \exp(2\pi i/(2n))$. It is a regular vector.

Lemma 4.3.4. *Let B_k denote the $k \times k$ matrix with ones in the antidiagonal and zeros everywhere else. Then at the regular vector v we have:*

$$(J^T)^{-1}\mathrm{Hess}(\rho_i)J^{-1} = \frac{2i-1}{n}\begin{bmatrix} B_{i-1} & 0 \\ 0 & B_{n-i+1} \end{bmatrix}$$

That is, the (a,b) entry is $(2i-1)/n$ if $a+b-i = 0$ or n, and zero otherwise.

Proof. This proof is analogous to the proof of lemma 4.3.2 from type A. □

Putting this lemma together with the formula for the Poincaré series gives:

Theorem 4.3.4. *Let $S = \{\rho_i\rho_j, \quad 1 \le i \le j \le n\}$ and $T \subseteq S$ be a subset with n elements which has one element $\rho_i\rho_j$ with $i+j = k$ for each $k = n+1, \dots 2n$. Then*

$$\{\mathrm{Hess}(Q), \quad Q \in S - T\} \cup \{\mathrm{Hess}(\rho_i), \quad i = 1, \dots n\}$$

is a basis for $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ as a free module over $\mathbb{R}[V]^W$.

Proof. Analogous to the type A case. One needs to use that

$$\frac{1-t^{2n}}{1-t^2} + \frac{(1-t^{2n-2})(1-t^{2n})t^2}{(1-t^2)(1-t^4)} = \sum_{i=0}^{n-1} t^{2i} + \sum_{1\le i\le j\le n-1} t^{2(i+j-1)}$$

□

4.3.5 Type D

Let $V = \mathbb{R}^n$ and $W = S_n \ltimes H_n \subset O(V)$ be the subgroup of matrices of the form $\tau \cdot \operatorname{diag}(\epsilon)$, where $\tau \in S_n$ is a permutation matrix and $\epsilon = (\epsilon_1, \dots \epsilon_n) \in H_n$ with $\epsilon_i = \pm 1$ and $\epsilon_1 \cdots \epsilon_n = +1$. This a subgroup of index 2 in the reflection group of type B_n (or C_n).

Poincaré series

Since W is a subgroup of the group of type B, the formulas are the same as in that case:

$$\det(1 - tg) = \prod_{i=1}^{n} (1 - t^i)^{k_i^+} \prod_{i=1}^{n} (1 + t^i)^{k_i^-}$$

$$\operatorname{tr}(\operatorname{Sym}^2(g)) = k_1^+ + k_1^- + \binom{k_1^+}{2} + \binom{k_1^-}{2} + k_2^+ - k_1^+ k_1^- - k_2^-$$

The strategy is the same as in types A and B, namely find a cycle index formula, use the formula for $\operatorname{tr}(\operatorname{Sym}^2(g))$ to define an appropriate differential operator D, and apply D to both sides of the cycle index formula.

Proposition 4.3.7. *The cycle index formula for the Weyl group of type D_n is*

$$\sum_{n=1}^{\infty} \frac{z^n}{n! 2^{n-1}} \sum_{g \in S_n \ltimes H_n} \mathbf{x}^{\mathbf{k}} = \exp\left(\sum_{j=1}^{\infty} \frac{z^j (x_j^+ + x_j^-)}{2j} \right) + \exp\left(\sum_{j=1}^{\infty} \frac{z^j (x_j^+ - x_j^-)}{2j} \right)$$

where $\mathbf{x}^{\mathbf{k}}$ denotes $\prod_{j=1}^{n} (x_j^+)^{k_j^+(g)} (x_j^-)^{k_j^-(g)}$, and $H_n = \{\epsilon \in \{\pm 1\}^n \mid \epsilon_1 \cdots \epsilon_n = +1\}$

Proof. Let A denote the quantity

$$A = \frac{1}{n! 2^{n-1}} \sum_{g \in S_n \ltimes H_n} \mathbf{x}^{\mathbf{k}}$$

which we are trying to compute, and B the complement sum

$$B = \frac{1}{n! 2^{n-1}} \sum_{g \in (S_n \ltimes \{\pm 1\}^n - S_n \ltimes H_n)} \mathbf{x}^{\mathbf{k}}$$

Using the cycle index formula for type B_n one sees that

$$A + B = 2 \exp\left(\sum_{j=1}^{\infty} \frac{z^j (x_j^+ + x_j^-)}{2j} \right)$$

$$A - B = 2 \exp\left(\sum_{j=1}^{\infty} \frac{z^j (x_j^+ - x_j^-)}{2j} \right)$$

and thus $A = (A + B + A - B)/2$ is as stated. $\square$

Lemma 4.3.5. *Replacing x_j^+ with $(1-t^j)^{-1}$ and x_j^- with $(1+t^j)^{-1}$ in the exponentials on the right-hand side of the cycle index formula gives:*

$$\exp\left(\sum_{j=1}^{\infty} \frac{z^j(x_j^+ + x_j^-)}{2j}\right) \longmapsto \sum_{n=1}^{\infty} \frac{z^n}{(1-t^2)\cdots(1-t^{2n})}$$

$$\exp\left(\sum_{j=1}^{\infty} \frac{z^j(x_j^+ - x_j^-)}{2j}\right) \longmapsto \sum_{n=1}^{\infty} \frac{z^n t^n}{(1-t^2)\cdots(1-t^{2n})}$$

Since the character of Sym^2 is the same as in type B, the differential operator is also the same $D =$

$$x_1^+\frac{\partial}{\partial x_1^+} + x_1^-\frac{\partial}{\partial x_1^-} + \frac{(x_1^+)^2}{2}\frac{\partial^2}{\partial (x_1^+)^2} + \frac{(x_1^-)^2}{2}\frac{\partial^2}{\partial (x_1^-)^2} + x_2^+\frac{\partial}{\partial x_2^+} - x_1^+x_1^-\frac{\partial^2}{\partial x_1^+\partial x_1^-} - x_2^-\frac{\partial}{\partial x_2^-}$$

D has the property (it is also determined by this property)

$$D\mathbf{x}^{\mathbf{k}} = \mathrm{tr}(\mathrm{Sym}^2(g))\mathbf{x}^{\mathbf{k}}$$

Recall that the degrees of W are $2, 4, 6, \ldots 2n-2, n$, so that

$$P_t(\mathbb{R}[V]^W) = \frac{1}{(1-t^2)(1-t^4)\cdots(1-t^{2n-2})(1-t^n)}$$

Proposition 4.3.8. *The Poincaré series for symmetric 2-tensors is:*

$$P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W) = \left(\frac{1-t^{2n}}{1-t^2} + \frac{(t^2+t^{n-2})(1-t^{2n-2})(1-t^n)}{(1-t^2)(1-t^4)}\right) P_t(\mathbb{R}[V]^W)$$

Proof. Apply the operator D defined above to both sides of the cycle index formula, and then replace x_j^+ with $(1-t^j)^{-1}$ and x_j^- with $(1+t^j)^{-1}$. The coefficient of z^n on the LHS is exactly the Poincaré series we are trying to compute.

As for the RHS, after using the lemma above and some simplification it becomes:

$$\left(\frac{z}{1-t^2} + \frac{z^2t^2}{(1-t^2)(1-t^4)}\right)\sum_{n=1}^{\infty}\frac{z^n}{(1-t^2)\cdots(1-t^{2n})} +$$

$$+\left(\frac{zt}{1-t^2} + \frac{1}{(1-t^2)(1-t^4)}\right)\sum_{n=1}^{\infty}\frac{z^n t^n}{(1-t^2)\cdots(1-t^{2n})}$$

Taking the coefficient of z^n gives the formula stated. ∎

Let's rewrite the Poincaré series above in a way that is useful in the next subsection:

$$\frac{P_t(\mathbb{R}[V,\mathrm{Sym}^2V^*]^W)}{P_t(\mathbb{R}[V]^W)} = P_t(B_{n-1}) + t^{2n-2} + t^{n-2}(1+t^2+\ldots+t^{2n-4})$$

where $P_t(B_{n-1})$ denotes the corresponding quotient of Poincaré series for the Weyl group of type B_{n-1}, namely

$$P_t(B_{n-1}) = \frac{1-t^{2n}}{1-t^2} + \frac{(1-t^{2n-2})(1-t^{2n})t^2}{(1-t^2)(1-t^4)} = \sum_{i=0}^{n-1} t^{2i} + \sum_{1\le i\le j\le n-1} t^{2(i+j-1)}$$

Independence of generators

We take the basic invariants to be

$$\rho_i(x_1,\ldots x_n) = \frac{1}{2i}\sum_{j=1}^{n} x_j^{2i} \qquad i=1,\ldots n-1$$

$$\rho_n(x_1,\ldots x_n) = x_1\cdots x_n$$

Let $v=(1,\xi,\xi^2,\ldots\xi^{n-2},0)\in\mathbb{R}^n$, where $\xi=\exp(2\pi i/(2n-2))$. It is a regular vector.

Recall the notations: $J=(\partial\rho_i/\partial x_j)_{i,j}$ is the Jacobian matrix, and B_k is the $k\times k$ matrix with ones in the main antidiagonal and zeros everywhere else.

Lemma 4.3.6. *At the regular vector v we have:*

$$(J^t)^{-1}\mathrm{Hess}(\rho_i)J^{-1} = (2i-1)\begin{bmatrix} \frac{1}{n-1}B_{i-1} & 0 & 0 \\ 0 & \frac{1}{n-1}B_{n-i} & 0 \\ 0 & 0 & (-1)^n \end{bmatrix}$$

for $1\le i\le n-1$ and

$$(J^t)^{-1}\mathrm{Hess}(\rho_n)J^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}$$

Theorem 4.3.5. *Let $S=\{\rho_i\rho_j, \quad 1\le i\le j\le n-1\}$ and $T\subseteq S$ be a subset with $n-1$ elements which has one element $\rho_i\rho_j$ with $i+j=k$ for each $k=n,\ldots 2n-2$. Then*

$$\{\mathrm{Hess}(Q)\mid Q\in S-T\}\cup\{\mathrm{Hess}(\rho_i\rho_n)\mid i\ne n-1\}\cup\{\mathrm{Hess}(\rho_i)\mid i=1,\ldots n\}$$

is a basis for $\mathbb{R}[V,\mathrm{Sym}^2V^*]^W$ as a free module over $\mathbb{R}[V]^W$.

4.3.6 F_4 - computer assisted

A few of the proofs below required computations that were performed by a computer. The code was written for GAP, uses the package CHEVIE and can be found in:
http://dl.dropbox.com/u/374152/GAPcode.zip

Let $V = \mathbb{R}^4$ and $W \subseteq O(V)$ be the Weyl group of type F_4. It is generated by the reflections in the hyperplanes orthogonal to the 48 vectors in the following root system:

$$\{\pm e_i,\ \pm e_i \pm e_j,\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$$

where e_1, e_2, e_3, e_4 is the standard basis for $\mathbb{R}^4$.

Let $v = (1, 2, 3, 5)$. It is a regular vector.

The degrees are $(d_1, d_2, d_3, d_4) = (2, 6, 8, 12)$.

Poincaré series

Since the degrees are $2, 6, 8, 12$,

$$P_t(\mathbb{R}[V]^W) = \frac{1}{(1-t^2)(1-t^6)(1-t^8)(1-t^{12})}$$

Proposition 4.3.9.

$$\frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)} = 1 + t^2 + t^4 + 2t^6 + t^8 + 2t^{10} + t^{12} + t^{14}$$

To describe how the program computes this polynomial, we need to recall a few facts (see Theorem 3.3.4 above). Let I be the ideal in $\mathbb{R}[V]$ generated by the homogeneous invariants of positive degree. The quotient $\mathbb{R}[V]/I$ is known to be isomorphic, as a W-representation, to the regular representation, but it is also a graded vector space. Fixing an irreducible representation/character ξ, the Poincaré polynomial $\mathrm{FD}_\xi(t)$ of the subspace with components isomorphic to ξ is called the *fake degree* of ξ. Moreover $\mathbb{R}[V]$ is isomorphic to $(\mathbb{R}[V]/I) \otimes \mathbb{R}[V]^W$. Thus the Poincaré series of the vector subspace in $\mathbb{R}[V]$ given by the direct sum of all irreducible subspaces isomorphic to ξ equals $\mathrm{FD}_\xi(t) P_t(\mathbb{R}[V]^W)$.

The way the program computes $P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)$ is as follows:

It first computes the character χ of $\mathrm{Sym}^2 V^*$, and decomposes it into a sum of irreducible characters, using character tables that come with CHEVIE.

$$\chi = \sum_{\xi \text{ irreducible}} c_\xi \xi$$

It then uses a command in CHEVIE that returns the fake degrees of the irreducible characters ξ, and computes

$$\sum_{\xi} c_\xi \mathrm{FD}_\xi(t)$$

Using Schur's lemma one sees that this equals $\frac{P_t(\mathbb{R}[V,\mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)}$.

Independence of generators

We start by constructing basic invariants, following [45]. Let $\{x_i\}$ be the basis dual to $\{e_i\}$, and

$$\mathcal{O} = \{\pm x_i \pm x_j \mid 1 \leq i \leq j \leq 4\}$$

Proposition 4.3.10. *$\mathcal{O}$ is W-stable.*

Since W permutes the linear polynomials in $\mathcal{O}$, for each m we get a W-invariant polynomial of degree m

$$\psi_m = \sum_{\lambda \in \mathcal{O}} \lambda^m$$

Proposition 4.3.11. *The polynomials $\rho_i = \psi_{d_i}$, $i = 1, 2, 3, 4$, form a set of basic invariants.*

Proof. The program computes the Jacobian determinant of the ψ, evaluates it at the vector v, and checks that the value is non-zero. This proves both that the ψ are algebraically independent and hence a set of basic invariants because they have the right degree; and that v is indeed a regular vector. □

Theorem 4.3.6. *Let*

$$S = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_1^2, \rho_2^2, \rho_3^2, \rho_1\rho_2, \rho_1\rho_3\}$$

Then both sets below form a basis for $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ as a free module over $\mathbb{R}[V]^W$:

$$\{\mathrm{Hess}(Q) \mid Q \in S \cup \{\rho_1\rho_4\}\}$$

$$\{\mathrm{Hess}(Q) \mid Q \in S \cup \{\rho_2\rho_3\}\}$$

Proof. For each set of 10 Hessians the program lists the 10 upper triangular entries of these Hessians and computes the determinant of the matrix thus obtained, evaluated at the regular vector v, and checks that it is non-zero. This shows they are linearly independent.

Since their degrees match the Poincaré polynomial computed above, they must form a basis. □

Remark 4.3.2. The invariants

$$\psi_m = \sum_{\lambda \in \mathcal{O}} \lambda^m$$

constructed from a union of orbits of a finite group in the dual space V^*, are called orbit Chern classes. For more on this and the relations with algebraic topology, see [62].

4.3.7 Type E - computer-assisted

For the root systems, see tables in [11].

The degrees are:

- E_6: $(d_1 \dots d_6) = (2, 5, 6, 8, 9, 12)$
- E_7: $(d_1 \dots d_7) = (2, 6, 8, 10, 12, 14, 18)$
- E_8: $(d_1 \dots d_8) = (2, 8, 12, 14, 18, 20, 24, 30)$

Poincaré series

Recall that

$$P_t(\mathbb{R}[V]^W) = \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}}$$

Proposition 4.3.12. *The Poincaré series of symmetric 2-tensors for Weyl groups of type E are:*

$$\frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)} =$$

- E_6*:*

$$t^{16} + t^{15} + t^{14} + t^{13} + 2t^{12} + t^{11} + 2t^{10} + 2t^9 + 2t^8 + t^7 + 2t^6 + t^5 + t^4 + t^3 + t^2 + 1$$

- E_7*:*

$$t^{26} + t^{24} + 2t^{22} + 2t^{20} + 3t^{18} + 3t^{16} + 3t^{14}$$
$$+3t^{12} + 3t^{10} + 2t^8 + 2t^6 + t^4 + t^2 + 1$$

- E_8*:*

$$t^{46} + t^{42} + t^{40} + t^{38} + 2t^{36} + 2t^{34} + t^{32} + 3t^{30} + 2t^{28} + 2t^{26} + 3t^{24}$$
$$+2t^{22} + 2t^{20} + 3t^{18} + t^{16} + 2t^{14} + 2t^{12} + t^{10} + t^8 + t^6 + t^2 + 1$$

The way the program computes these polynomials is the same as in type F_4.

Independence of generators

We construct the basic invariants as in type F_4, by finding a finite W-stable subset $\mathcal{O} \subseteq V^*$ and taking orbit Chern classes associated to it.

Define $\mathcal{O}$ by (following notation in [45])

- E_6: Denote the 6 variables by $y_1, y_2, y_3, y_4, y_5, y$, and define a linear functional $y_6 = -y_1 - \dots y_5$.

  $$\mathcal{O} = \{y_i + y \mid i = 1, \dots 6\} \cup \{y_i - y \mid i = 1, \dots 6\} \cup \{-y_i - y_j \mid 1 \leq i < j \leq 6\}$$

- E_7: Call the variables $y_1, \dots y_7$, and define a linear functional $y_8 = -y_1 - \dots y_7$.

  $$\mathcal{O} = \{y_i + y_j \mid 1 \leq i < j \leq 8\}$$

- E_8: Call the variables $r_1, \dots r_8$.

  $$\mathcal{O} = \{\pm r_i \pm r_j \mid 1 \leq i < j \leq 8\} \cup \{\epsilon_1 r_1 + \cdots \epsilon_8 r_8 \mid \epsilon_i = \pm 1,\ \epsilon_1 \cdots \epsilon_8 = -1\}$$

Regular vector v chosen:

- E_6: $(2, 3, 5, 7, 11, 13)$
- E_7: $(2, 3, 5, 7, 9, 11, 12)$
- E_8: $(2, 3, 5, 7, 9, 11, 12, 13)$

Theorem 4.3.7. *Let W be a Weyl group of type E_n, for $n = 6, 7, 8$, and $\rho_1, \dots \rho_n$ a set of basic invariants.*

Let $S = \{\rho_i \rho_j, \quad 1 \leq i \leq j \leq n\}$ and $T \subseteq S$ be a subset with n elements such that

$$\sum_{Q \in (S-T) \cup \{\rho_1, \dots \rho_n\}} t^{\deg(Q)-2} = \frac{P_t(\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W)}{P_t(\mathbb{R}[V]^W)}$$

The number of choices of such a T is:

- *9 for E_6*
- *48 for E_7*
- *96 for E_8*

For all of them,

$$\{\mathrm{Hess}(Q) \mid Q \in S - T\} \cup \{\mathrm{Hess}(\rho_i) \mid i = 1 \dots n\}$$

is a basis for $\mathbb{R}[V, \mathrm{Sym}^2 V^*]^W$ as a free module over $\mathbb{R}[V]^W$.

Proof. Same as in type F_4. □

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