COMBINATORIAL FORMULAS CONNECTED TO DIAGONAL HARMONICS AND MACDONALD POLYNOMIALS

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A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2009

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Acknowledgments

My greatest and eternal thanks go to my academic advisor Jim Haglund who showed consistent encouragement and boundless support. His mathematical instinct was truly inspiring and this work was only possible with his guidance. My gratitude is everlasting.

I am grateful to Jerry Kazdan and Jason Bandlow for kindly agreeing to serve on my dissertation committee, and Jennifer Morse, Sami Assaf and Antonella Grassi for graciously agreeing to write recommendation letters for me. I would also like to thank the faculty of the mathematics department for providing such a stimulating environment to study. I owe many thanks to the department secretaries, Janet, Monica, Paula and Robin. I was only able to finish the program with their professional help and cordial caring.

I would like to thank my precious friends Elena and Enka for being on my side and encouraging me whenever I had hard times. I owe thanks to Daeun for being a friend whom I can share my emotions with, Tong for being a person whom I can share my little worries with, Tomoko, Sarah and John for being such comfortable friends whom I shared cups of tea with, and Min, Sieun and all other my college classmates for showing such cheers for me. My thanks also go to my adorable friends Soong and Jinhee for keeping unswerving friendship with me.

My most sincere thanks go to my family, Hyun Suk Lee, Young Yoo, Mi Jeong Yoo and Cheon Gum Yoo, for their unconditional support and constant encouragement. I could not have made all the way through without their love and faith in me.

ABSTRACT

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We study bigraded S_n-modules introduced by Garsia and Haiman as an approach to prove the Macdonald positivity conjecture. We construct a combinatorial formula for the Hilbert series of Garsia-Haiman modules as a sum over standard Young tableaux, and provide a bijection between a group of fillings and the corresponding standard Young tableau in the hook shape case. This result extends the known property of Hall-Littlewood polynomials by Garsia and Procesi to Macdonald polynomials.

We also study the integral form of Macdonald polynomials and construct a combinatorial formula for the coefficients in the Schur expansion in the one-row case and the hook shape case.

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Chapter 1

Introduction and Basic Definitions

The theory of symmetric functions arises in various areas of mathematics such as algebraic combinatorics, representation theory, Lie algebras, algebraic geometry, and special function theory. In 1988, Macdonald introduced a unique family of symmetric functions with two parameters characterized by certain triangularity and orthogonality conditions which generalizes many well-known classical bases.

Macdonald polynomials have been in the core of intensive research since their introduction due to their applications in many other areas such as algebraic geometry and commutative algebra. However, given such an indirect definition of these polynomials satisfying certain conditions, the proof of existence does not give an explicit way of construction. Nonetheless, Macdonald conjectured that the integral form $J_\mu[X;q,t]$ of Macdonald polynomials, obtained by multiplying a certain polynomial to the Macdonald polynomials, can be extended in terms of modified Schur functions $s_\lambda[X(1-t)]$ with coefficients in $\mathbb{N}[q,t]$, i.e.,

$$J_\mu[X;q,t] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q,t) s_\lambda[X(1-t)], \quad K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t],$$

where the coefficients $K_{\lambda\mu}(q,t)$ are called *q,t-Kostka functions*. This is the famous Macdonald positivity conjecture. This conjecture was supported by the following known property of Hall-

Littlewood polynomials $P_\lambda(X;t) = P_\lambda(X;0,t)$ which was proved combinatorially by Lascoux and Schützenberger [LS78]

$$K_{\lambda\mu}(0,t) = K_{\lambda\mu}(t) = \sum_T t^{ch(T)},$$

summed over all SSYT of shape λ and weight μ, where $ch(T)$ is the *charge* statistic. In 1992, Stembridge [Ste94] found a combinatorial interpretation of $K_{\lambda\mu}(q,t)$ when μ has a hook shape, and Fishel [Fis95] found statistics for the two-column case which also gives a combinatorial formula for the two-row case.

To prove the Macdonald positivity conjecture, Garsia and Haiman [GH93] introduced certain bigraded S_n-modules and conjectured that the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ could be realized as the bigraded characters of those modules. This is the well-known $n!$ conjecture. Haiman proved this conjecture in 2001 [Hai01] by showing that it is intimately connected with the Hilbert scheme of n points in the plane and with the variety of commuting matrices, and this result proved the positivity conjecture immediately.

In 2004, Haglund [Hag04] conjectured and Haglund, Haiman and Loehr [HHL05] proved a combinatorial formula for the monomial expansion of the modified Macdonald polynomials. This celebrated combinatorial formula brought a breakthrough in Macdonald polynomial theory. Unfortunately, it does not give any combinatorial description of $K_{\lambda\mu}(q,t)$, but it provides a shortcut to prove the positivity conjecture. In 2007, Assaf [Ass07] proved the positivity conjecture purely combinatorially by showing Schur positivity of dual equivalence graphs and connecting them to the modified Macdonald polynomials.

In this thesis, we study the bigraded S_n-modules introduced by Garsia and Haiman [GH93] in their approach to the Macdonald positivity conjecture and construct a combinatorial formula for the Hilbert series of the Garsia-Haiman modules in the hook shape case. The monomial expansion formula of Haglund, Haiman and Loehr for the modified Macdonald polynomials gives a way of calculating the Hilbert series as a sum over all possible fillings from $n!$ permutations of n elements, but we introduce a combinatorial formula which calculates the same Hilbert series as a sum over standard Young tableaux (SYT, from now on) of the given hook shape. Noting that there are only $\frac{n!}{\prod_{c\in\lambda} h(c)}$, where $h(c) = a(c) + l(c) + 1$ (see Definition 1.1.6 for the descriptions of $a(c)$ and $l(c)$), many SYTs of shape λ, this combinatorial formula gives a way of calculating the Hilbert series much faster and easier than the monomial expansion formula.

The construction was motivated by a similar formula for the two-column case which was conjectured by Haglund and proved by Garsia and Haglund. To prove, we apply the similar strategy of deriving a recursion formula satisfied by both of the constructed combinatorial formulas and the Hilbert series that Garsia and Haglund used to prove the two-column case formula. In addition, we provide two independent proofs.

Also, we consider the integral form Macdonald polynomials, $J_\mu[X; q, t]$, and introduce a combinatorial formula for the Schur coefficients of $J_\mu[X; q, t]$ in the one-row case and the hook shape case. As we mentioned in the beginning, Macdonald originally considered $J_\mu[X; q, t]$ in terms of the modified Schur functions $s_\lambda[X(1 - t)]$ and the coefficients $K_{\lambda\mu}(q, t)$ have been studied a lot due to the positivity conjecture. But no research has been done concerning the Schur expansion of $J_\mu[X; q, t]$ so far. Along the way, Haglund noticed that the scalar product of $J_\mu[X; q, q^k]/(1 - q)^n$ and $s_\lambda(x)$, for any nonnegative k, is a polynomial in q with positive coefficients. Based on this observation, he conjectured that $J_\mu[X; q, t]$ has the following Schur expansion

$$J_\mu[X; q, t] = \sum_{\lambda \vdash n} \left[\sum_{T \in \mathrm{SSYT}(\lambda', \mu')} \prod_{c \in \mu} (1 - t^{l(c)+1} q^{q\mathrm{stat}(c,T)}) q^{ch(T)} \right] s_\lambda,$$

for certain unkown integers $q\mathrm{stat}(c, T)$. We define the $q\mathrm{stat}(c, T)$ for the one-row case and the hook shape case and construct the explicit combinatorial formula for the Schur coefficients in those two cases.

The thesis is organized as follows : in Chapter 1, we give definitions of basic combinatorial objects. In Section 1.2, we describe familiar bases of the space of symmetric functions including the famous Schur functions, and in Section 1.3, we define the Macdonald polynomials and introduce the monomial expansion formula for the Macdonald polynomials of Haglund, Haiman and Loehr.

In Section 1.4, we introduce the Garsia-Haiman modules and define their Hilbert series.

Chapter 2 is devoted to constructing and proving the combinatorial formula for the Hilbert series of Garsia-Haiman modules as a sum over standard Young tableaux. In Section 2.1, we review Garsia's proof for the two-column shape case, and in Section 2.2, we prove the combinatorial formula for the hook shape case. We provide three different proofs. The first one is by direct calculation using the monomial expansion formula of Macdonald polynomials, the second one is by deriving the recursive formula of Macdonald polynomials which is known by Garsia and Haiman, and the third one is by applying the science fiction conjecture. In addition to the combinatorial construction over SYTs, we provide a way of associating a group of fillings to one SYT, and prove that this association is a bijection.

In Chapter 3, we construct the combinatorial formula for the coefficients in the Schur expansion of the integral form of Macdonald polynomials. In Section 3.2, we construct and prove the combinatorial formula in one-row case and the hook case.

1.1 Basic Combinatorial Objects

Definition 1.1.1. A *partition* λ of a nonnegative integer n is a non increasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ satisfying

$$\lambda_1 \geq \cdots \geq \lambda_k \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = n.$$

We write $\lambda \vdash n$ to say λ is a partition of n. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ a partition of n, we say the *length* of λ is k (written $l(\lambda) = k$) and the *size* of λ is n (written $|\lambda| = n$). The numbers λ_i are referred to as the *parts* of λ. We may also write

$$\lambda = (1^{m_1}, 2^{m_2}, \dots)$$

where m_i is the number of times i occurs as a part of λ.

Definition 1.1.2. The *Young diagram* (also called a *Ferrers diagram*) of a partition λ is a collection of boxes (or *cells*), left justified and with λ_i cells in the i^{th} row from the bottom. The cells are indexed by pairs (i, j), with i being the row index (the bottom row is row 1), and j being the column index (the leftmost column is column 1). Abusing notation, we will write λ for both the partition and its diagram.

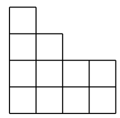


Figure 1.1: The Young diagram for $\lambda = (4, 4, 2, 1)$.

Definition 1.1.3. The *conjugate* λ' of a partition λ is defined by

$$\lambda'_j = \sum_{i \geq j} m_i.$$

The diagram of λ' can be obtained by reflecting the diagram of λ across the main diagonal. In particular, $\lambda'_1 = l(\lambda)$ and $\lambda_1 = l(\lambda')$. Obviously $\lambda'' = \lambda$.

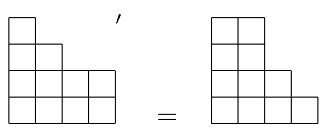


Figure 1.2: The Young diagram for $(4, 4, 2, 1)' = (4, 3, 2, 2)$.

Definition 1.1.4. For partitions λ, μ, we say μ is contained in λ and write $\mu \subset \lambda$, when the Young diagram of μ is contained within the diagram of λ, i.e., $\mu_i \leq \lambda_i$ for all i. In this case, we define the *skew diagram* λ/μ by removing the cells in the diagram of μ from the diagram of λ.

A skew diagram $\theta = \lambda/\mu$ is a *horizontal m-strip* (resp. a *vertical m-strip*) if $|\theta| = m$ and $\theta'_i \leqslant 1$ (resp. $\theta_i \leqslant 1$) for each $i \geqslant 1$. In other words, a horizontal (resp. vertical) strip has at most one square in each column (resp. row). Note that if $\theta = \lambda/\mu$, then a necessary and sufficient

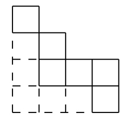


Figure 1.3: The Young diagram for $(4, 4, 2, 1)/(3, 1, 1)$.

condition for θ to be a horizontal strip is that the sequences λ and μ are interlaced, in the sense that $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots$.

A skew diagram λ/μ is a *border strip* (also called a *skew hook*, or a *ribbon*) if λ/μ is connected and contains no 2×2 block of squares, so that successive rows (or columns) of λ/μ overlap by exactly one square.

Definition 1.1.5. The *dominance order* is a partial ordering, denoted by $\leq$, defined on the set of partitions of n and this is given by defining $\lambda \leq \mu$ for $|\lambda| = |\mu| = n$ if for all positive integers k,

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i.$$

Definition 1.1.6. Given a square $c \in \lambda$, define the *leg* (respectively *coleg*) of c, denoted $l(c)$ (resp. $l'(c)$), to be the number of squares in λ that are strictly above (resp. below) and in the same column as c, and the *arm* (resp. *coarm*) of c, denoted $a(c)$ (resp. $a'(c)$), to be the number of squares in λ strictly to the right (resp. left) and in the same row as c. Also, if c has coordinates (i, j), we let south(c) denote the square with coordinates $(i - 1, j)$.

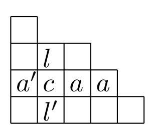


Figure 1.4: The arm a, leg l, coarm a' and coleg l'.

For each partition λ we define

$$n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$$

so that each $n(\lambda)$ is the sum of the numbers obtained by attaching a zero to each node in the first (bottom) row of the diagram of λ, a 1 to each node in the second row, and so on. Adding up the numbers in each column, we see that

$$n(\lambda) = \sum_{i \geq 1} \binom{\lambda_i'}{2}.$$

Definition 1.1.7. Let λ be a partition. A *tableau* of *shape* $\lambda \vdash n$ is a function T from the cells of the Young diagram of λ to the positive integers. The *size* of a tableau is its number of entries. If T is of shape λ then we write $\lambda = \mathrm{sh}(T)$. Hence the size of T is just $|\mathrm{sh}(T)|$. A *semistandard Young tableau* (SSYT) of *shape* λ is a tableau which is weakly increasing from the left to the right in every row and strictly increasing from the bottom to the top in every column. We may also think of an SSYT of shape λ as the Young diagram of λ whose boxes have been filled with positive integers (satisfying certain conditions). A semistandard Young tableau is *standard* (SYT) if it is a bijection from λ to $[n]$ where $[n] = \{1, 2, \ldots, n\}$.

6	9	9			
5	5	7			
2	4	4	5	5	
1	1	1	3	4	4

Figure 1.5: A SSYT of shape $(6, 5, 3, 3)$.

For a partition λ of n and a composition μ of n, we define

$$\begin{aligned}
\mathrm{SSYT}(\lambda) &= \{\text{semi-standard Young tableau } T : \lambda \to \mathbb{N}\}, \\
\mathrm{SSYT}(\lambda, \mu) &= \{\mathrm{SSYT}\ T : \lambda \to \mathbb{N} \text{ with entries } 1^{\mu_1}, 2^{\mu_2}, \ldots\}, \\
\mathrm{SYT}(\lambda) &= \{\mathrm{SSYT}\ T : \lambda \xrightarrow{\sim} [n]\} = \mathrm{SSYT}(\lambda, 1^n).
\end{aligned}$$

For $T \in \mathrm{SSYT}(\lambda, \mu)$, we say T is a SSYT of shape λ and weight μ. Note that if $T \in \mathrm{SSYT}(\lambda, \mu)$ for partitions λ and μ, then $\lambda \geqslant \mu$.

Definition 1.1.8. For a partition λ, we define the *hook polynomial* to be a polynomial in q by

$$H_\lambda(q) = \prod_{c \in \lambda} (1 - q^{a(c)+l(c)+1})$$

where $a(c)$ is the arm of c and $l(c)$ is the leg of c.

We say that T has *type* $\alpha = (\alpha_1, \alpha_2, \dots)$, denoted $\alpha = \mathrm{type}(T)$, if T has $\alpha_i = \alpha_i(T)$ parts equal to i. For any T of type α, write

$$x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots .$$

Note that $\alpha_i = |T^{-1}(i)|$.

1.2 Symmetric Functions

Consider the ring $\mathbb{Z}[x_1, \dots, x_n]$ of polynomials in n independent variables $x_1, \dots, x_n$ with rational integer coefficients. The symmetric group S_n acts on this ring by permuting the variables, and a polynomial is *symmetric* if it is invariant under this action. The symmetric polynomials form a subring

$$\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{S_n}.$$

Λ_n is a graded ring : we have

$$\Lambda_n = \oplus_{k \geq 0} \Lambda_n^k$$

where Λ_n^k consists of the homogeneous symmetric polynomials of degree k, together with the zero polynomial. If we add x_{n+1}, we can form $\Lambda_{n+1} = \mathbb{Z}[x_1, \dots, x_{n+1}]^{S_{n+1}}$, and there is a natural surjection $\Lambda_{n+1} \to \Lambda_n$ defined by setting $x_{n+1} = 0$. Note that the mapping $\Lambda_{n+1}^k \to \Lambda_n^k$ is a surjection for all $k \geq 0$, and a bijection if and only if $k \leq n$. If we define Λ^k as its inverse limit, i.e.,

$$\Lambda^k = \varprojlim_n \Lambda_n^k$$

for each $k \geq 0$, and let

$$\Lambda = \oplus_{k \geq 0} \Lambda^k$$

then this graded ring is called a *ring of symmetric functions*. For any commutative ring R, we write

$$\Lambda_R = \Lambda \otimes_{\mathbb{Z}} R, \quad \Lambda_{n,R} = \Lambda_n \otimes_{\mathbb{Z}} R$$

for the ring of symmetric functions (symmetric polynomials in n indeterminates, respectively) with coefficients in R.

1.2.1 Bases of Λ

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates, and let $n \in \mathbb{N}$. Let λ be any partition of size n. The polynomial

$$m_\lambda = \sum x^\alpha$$

summed over all *distinct* permutations α of $\lambda = (\lambda_1, \lambda_2, \ldots)$, is clearly symmetric, and the m_λ (as λ runs through all partitions of $\lambda \vdash n$) form a $\mathbb{Z}$-basis of Λ^n. Moreover, the set $\{m_\lambda\}$ is a basis for Λ. They are called *monomial symmetric functions*.

For each integer $n \geq 0$, the n^{th} *elementary symmetric function* e_n is the sum of all products of n distinct variables x_i, so that $e_0 = 1$ and

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = m_{(1^n)}$$

for $n \geq 1$. The generating function for the e_n is

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 + x_i t)$$

(t being another variable), as one sees by multiplying out the product on the right. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots .$$

For each $n \geq 0$, the n^{th} *complete symmetric function* h_n is the sum of all monomials of total degree n in the variables $x_1, x_2, \ldots$, so that

$$h_n = \sum_{|\lambda|=n} m_\lambda.$$

In particular, $h_0 = 1$ and $h_1 = e_1$. The generating function for the h_n is

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1} (1 - x_i t)^{-1}.$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots .$$

For each $n \geq 1$ the n^{th} *power sum symmetric function* is

$$p_n = \sum x_i^n = m_{(n)}.$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots .$$

The generating function for the p_n is

$$P(t) = \sum_{n\geq 1} p_n t^{n-1} = \sum_{i\geq 1} \sum_{n\geq 1} x_i^n t^{n-1} = \sum_{i\geq 1} \frac{x_i}{1 - x_i t} = \sum_{i\geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}.$$

Note that

$$P(t) = \frac{d}{dt} \log \prod_{i\geq 1} (1 - x_i t)^{-1} = \frac{d}{dt} \log H(t) = H'(t)/H(t),$$

and likewise

$$P(-t) = \frac{d}{dt} \log E(t) = E'(t)/E(t).$$

Proposition 1.2.1. *We have*

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$$

$$e_n = \sum_{\lambda \vdash n} \epsilon_\lambda z_\lambda^{-1} p_\lambda,$$

where $\epsilon_\lambda = (-1)^{|\lambda| - l(\lambda)}$, $z_\lambda = \prod_{i\geqslant 1} i^{m_i} \cdot m_i!$, m_i *is the number of parts of* λ *equal to* i.

Proof. See [Mac98, I, (2.14)] or [Sta99, 7.7.6]. ☐

Proposition 1.2.2. *The* m_λ, e_λ *and* h_λ *form* $\mathbb{Z}$*-bases of* Λ*, and the* p_λ *form a* $\mathbb{Q}$*-basis for* $\Lambda_\mathbb{Q}$*, where* $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$.

Proof. See [Mac98] or [Sta99]. ☐

Schur Functions

Definition 1.2.3. We define a scalar product on Λ so that the bases (h_λ) and (m_λ) become dual to each other :

$$< h_\lambda, m_\mu >= \delta_{\lambda\mu}.$$

This is called the *Hall inner product.* Note that the power sum symmetric functions are orthogonal with respect to the Hall inner product :

$$< p_\lambda, p_\mu >= z_\lambda \delta_{\lambda\mu}$$

where $z_\lambda = \prod_i i^{n_i} n_i!$ and $n_i = n_i(\lambda)$ is the number of i occurring as a part of λ.

Definition 1.2.4. Let λ be a partition. The *Schur function* s_λ of *shape* λ in the variables $x = (x_1, x_2, \dots)$ is the formal power series

$$s_\lambda(x) = \sum_T x^T$$

summed over all SSYTs T of shape λ.

Example 1.2.5. The SSYTs T of shape $(2, 1)$ with largest part at most three are given by

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

Hence

$$\begin{aligned} s_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3 \\ &= m_{21}(x_1, x_2, x_3) + 2m_{111}(x_1, x_2, x_3). \end{aligned}$$

We also define the *skew Schur functions*, $s_{\lambda/\mu}$, by taking the sum over semistandard Young tableaux of shape λ/μ. Note that the Schur functions are orthonormal with respect to the Hall inner product :

$$< s_\lambda, s_\mu >= \delta_{\lambda\mu}$$

so the s_λ form an *orthonormal* basis of Λ, and the s_λ such that $|\lambda| = n$ form an orthonormal basis of Λ^n. In particular, the Schur functions $\{s_\lambda\}$ could be defined as the unique family of symmetric functions with the following two properties :

(i) $s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu$,

(ii) $< s_\lambda, s_\mu >= \delta_{\lambda\mu}$.

The coefficient $K_{\lambda\mu}$ is known as the *Kostka number* and it is equal to the number of SSYT of shape λ and weight μ. The importance of Schur functions arises from their connections with many branches of mathematics such as representation theory of symmetric functions and algebraic geometry.

Proposition 1.2.6. *We have*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$

Proof. By the definition of the Hall inner product, $< h_\mu, m_\nu >= \delta_{\mu\nu}$ and by the property (i) of the above two conditions of Schur functions, $s_\lambda = \sum_\lambda K_{\lambda\mu} m_\mu$. So,

$$< h_\mu, s_\lambda >=< h_\mu, \sum_\lambda K_{\lambda\mu} m_\mu >= K_{\lambda\mu}.$$

□

The Littlewood-Richardson Rule

The integer $< s_\lambda, s_\mu s_\nu > = < s_{\lambda/\nu}, s_\mu > = < s_{\lambda/\mu}, s_\nu >$ is denoted $c^\lambda_{\mu\nu}$ and is called a *Littlewood-Richardson coefficient.* Thus

$$\begin{aligned} s_\mu s_\nu &= \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \\ s_{\lambda/\nu} &= \sum_\mu c^\lambda_{\mu\nu} s_\mu \\ s_{\lambda/\mu} &= \sum_\nu c^\lambda_{\mu\nu} s_\nu . \end{aligned}$$

We have a nice combinatorial interpretation of $c^\lambda_{\mu\nu}$ and to introduce it, we make several definitions first.

We consider a *word* w as a sequence of positive integers. Let T be a SSYT. Then we can derive a word w from T by reading the elements in T from right to left, starting from bottom to top.

Definition 1.2.7. A word $w = a_1 a_2 \dots a_N$ is said to be a *lattice permutation* if for $1 \leqslant r \leqslant N$ and $1 \leqslant i \leqslant n-1$, the number of occurrences of i in $a_1 a_2 \dots a_r$ is not less than the number of occurrences of $i+1$.

Now we are ready to state the Littlewood-Richardson rule.

Theorem 1.2.8. *Let λ, μ, ν be partitions. Then $c^\lambda_{\mu\nu}$ is equal to the number of SSYT T of shape λ/μ and weight ν such that $w(T)$, the word obtained from T, is a lattice permutation.*

Proof. See [Mac98, I. 9]. □

Example 1.2.9. Let $\lambda = (4,4,2,1), \mu = (2,1)$ and $\nu = (4,3,1)$. Then there are two SSYT's satisfying the Littlewood-Richardson rule :

$$\begin{array}{|c|c|c|c|} \hline 2 & & & \\ 1 & 3 & & \\ & 1 & 2 & 2 \\ & & 1 & 1 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|} \hline 3 & & & \\ 1 & 2 & & \\ & 1 & 2 & 2 \\ & & 1 & 1 \\ \hline \end{array}$$

Indeed, the words 11221312 and 11221213 are lattice permutations. Thus, $c^\lambda_{\mu\nu} = 2$.

Classical Definition of Schur Functions

We consider a finite number of variables, say $x_1, x_2, \ldots, x_n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\omega \in S_n$. As usual, we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and define

$$\omega(x^\alpha) = x_1^{\alpha_{\omega(1)}} \cdots x_n^{\alpha_{\omega(n)}}.$$

Now we define the polynomial a_α obtained by antisymmetrizing x^α, namely,

$$a_\alpha = a_\alpha(x_1, \ldots, x_n) = \sum_{\omega \in S_n} \epsilon(\omega)\omega(x^\alpha), \tag{1.2.1}$$

where $\epsilon(\omega)$ is the *sign* ($\pm$) of the permutation ω. Note that the right hand side of (1.2.1) is the expansion of a determinant, that is to say,

$$a_\alpha = \det(x_i^{\alpha_j})_{i,j=1}^n.$$

The polynomial a_α is skew-symmetric, i.e., we have

$$\omega(a_\alpha) = \epsilon(\omega)a_\alpha$$

for any $\omega \in S_n$, so $a_\alpha = 0$ unless all the α_i's are distint. Hence we may assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$, and therefore we may write $\alpha = \lambda + \delta$ where λ is a partition with $l(n) \leq n$ and $\delta = (n-1, n-2, \ldots, 1, 0)$. Since $\alpha_j = \lambda_j + n - j$, we have

$$a_\alpha = a_{\lambda+\delta} = \det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n. \tag{1.2.2}$$

This determinant is divisible in $\mathbb{Z}[x_1, \ldots, x_n]$ by each of the differences $x_i - x_j (1 \leq i < j \leq n)$, and thus by their product, which is the *Vandermonde determinant*

$$a_\delta = \det(x_i^{n-j}) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Moreover, since a_α and a_δ are skew-symmetric, the quotient is symmetric with homogeneous degree $|\lambda| = |\alpha| - |\delta|$, i.e., $a_\alpha / a_\delta \in \Lambda_n^{|\lambda|}$.

Theorem 1.2.10. *We have*

$$a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_n).$$

Proof. See [Mac98, I.3] or [Sta99, 7.15.1]. □

Proposition 1.2.11. *For any partition λ, we have*

$$s_\lambda(1, q, q^2, \dots) = \frac{q^{n(\lambda)}}{H_\lambda(q)}$$

where $H_\lambda(q) = \prod_{c \in \lambda}(1 - q^{a(c)+l(c)+1})$ is the hook polynomial.

Proof. By Theorem 1.2.10, we have

$$\begin{aligned} s_\lambda(1, q, q^2, \dots, q^{n-1}) &= \frac{\det\left(q^{(i-1)(\lambda_j+n-j)}\right)_{i,j=1}^n}{\det\left(q^{(i-1)(n-j)}\right)_{i,j=1}^n} \\ &= (-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} \frac{q^{\lambda_i+n-i} - q^{\lambda_j+n-j}}{q^{i-1} - q^{j-1}} \end{aligned}$$

Let $\mu_i = \lambda_i + n - i$, and note the q-integers $[k]_q = 1 - q^k$ and $[k]_q! = [1]_q[2]_q \cdots [k]_q$. Then (using the fact $\prod_{1 \le i < j \le n}[j-i]_q = \prod_{i=1}^n [n-i]_q!$)

$$\begin{aligned} s_\lambda(1, q, q^2, \dots, q^{n-1}) &= \frac{q^{\sum_{i<j}\mu_j} \prod_{i<j}[\mu_i - \mu_j]_q \cdot \prod_{i \ge 1}[\mu_i]_q!}{q^{\sum_{i<j}(i-1)} \prod_{i<j}[j-i]_q \cdot \prod_{i \ge 1}[\mu_i]_q!} \\ &= q^{n(\lambda)} \prod_{u \in \lambda} \frac{[n + c(u)]_q}{[h(u)]_q}, \end{aligned}$$

where $c(u) = l'(u) - a'(u)$, $h(u) = a(u) + l(u) + 1$, noting that

$$\prod_{u \in \lambda}[h(u)] = \frac{\prod_{i \ge 1}[\mu_i]_q!}{\prod_{1 \le i < j \le n}[\mu_i - \mu_j]_q}, \quad \prod_{u \in \lambda}[n + c(u)]_q = \prod_{i=1}^n \frac{[\mu_i]_q!}{[n-i]_q!}.$$

If we now let $n \to \infty$, then the numerator $\prod_{u \in \lambda}(1 - q^{n+c(u)})$ goes to 1, so we get

$$s_\lambda(1, q, q^2, \dots) = \frac{q^{n(\lambda)}}{H_\lambda(q)}.$$

□

Zonal Symmetric Functions

These are symmetric functions Z_λ characterized by the following two properties :

(i) $Z_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu$, for suitable coefficients $c_{\lambda\mu}$,

(ii) $< Z_\lambda, Z_\mu >_2 = 0$, if $\lambda \neq \mu$,

where the scalar product $< \quad , \quad >_2$ on Λ is defined by

$$< p_\lambda, p_\mu >_2 = \delta_{\lambda\mu} 2^{l(\lambda)} z_\lambda,$$

$l(\lambda)$ being the length of the partition λ and z_λ defined as before.

Jack's Symmetric Functions

The Jack's symmetric functions $P_\lambda^{(\alpha)} = P_\lambda^{(\alpha)}(X;\alpha)$ are a generalization of the Schur functions and the zonal symmetric functions and they are characterized by the following two properties :

(i) $P_\lambda^{(\alpha)} = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu$, for suitable coefficients $c_{\lambda\mu}$,

(ii) $< P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} >_\alpha = 0$, if $\lambda \neq \mu$,

where the scalar product $< \quad , \quad >_\alpha$ on Λ is defined by

$$< p_\lambda, p_\mu >_2 = \delta_{\lambda\mu} \alpha^{l(\lambda)} z_\lambda.$$

Remark 1.2.12. We have the following specializations :

(a) $\alpha = 1$: $P_\lambda^{(\alpha)} = s_\lambda$

(b) $\alpha = 2$: $P_\lambda^{(\alpha)} = Z_\lambda$

(c) $P_\lambda^{(\alpha)}(X;\alpha) \to e_{\lambda'}$ as $\alpha \to 0$

(d) $P_\lambda^{(\alpha)}(X;\alpha) \to m_\lambda$ as $\alpha \to \infty$

Hall-Littlewood Symmetric Functions

The Hall-Littlewood symmetric functions $P_\lambda(X;t)$ are characterized by the following two properties :

(i) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu$, for suitable coefficients $c_{\lambda\mu}$,

(ii) $< P_\lambda, P_\mu >_t = 0$, if $\lambda \neq \mu$,

where the scalar product $< \quad , \quad >_t$ on Λ is defined by

$$< p_\lambda, p_\mu >_t = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} (1 - t^{\lambda_i})^{-1}.$$

Note that when $t = 0$, the P_λ reduce to the Schur functions s_λ, and when $t = 1$ to the monomial symmetric functions m_λ.

Definition 1.2.13. The Kostka numbers $K_{\lambda\mu}$ defined by

$$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu$$

generalizes to the *Kostka-Foulkes polynomials* $K_{\lambda\mu}(t)$ defined as follows :

$$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu}(t) P_\mu(x; t)$$

where the $P_\mu(x; t)$ are the Hall-Littlewood functions.

Since $P_\lambda(x; 1) = m_\lambda$, we have $K_{\lambda\mu}(1) = K_{\lambda\mu}$. Foulkes conjectured, and Hotta and Springer [HS77] proved that $K_{\lambda\mu}(t) \in \mathbb{N}[t]$ using a cohomological interpretation. Later Lascoux and Schützenberger [LS78] proved combinatorially that

$$K_{\lambda\mu}(t) = \sum_T t^{ch(T)}$$

summed over all SSYT of shape λ and weight μ, where $ch(T)$ is the *charge* statistic.

Charge Statistic

We consider *words* (or sequences) $w = a_1 \cdots a_n$ in which each a_i is a positive integer. The *weight* of w is the sequence $\mu = (\mu_1, \mu_2, \dots)$, where μ_i counts the number of times i occurring in w. Assume that μ is a partition, i.e., $\mu_1 \geq \mu_2 \geq \cdots$. If $\mu = (1^n)$ so that w is a derangement of $12 \dots n$, then we call w a *standard* word.

To define a charge of word w, we assgin an *index* to each element of w in the following way.

(i) If w is a standard word, then we assign the index 0 to the number 1, and if r has index i, then $r+1$ has index i if it lies to the right of r and it has index $i+1$ if it lies to the left of r.

(ii) If w is any word with repeated numbers, then we extract standard subwords from w as follows. Reading from the left, choose the first 1 that occurs in w, then the first 2 to the right of the chosen 1, and so on. If at any stage there is no $s+1$ to the right of the chosen s, go back to the beginning. This procedure extracts a standard subword, say w_1, of w. We erase w_1 from w and repeat the same procedure to obtain a standard subword w_2, and so on. For each standard word w_j, we assign the index as we did for the standard word.

(iii) If a SSYT T is given, we can extract a word $w(T)$ from T by reading the elements from the right to the left, starting from the bottom to the top.

Then we define the *charge* $ch(w)$ of w to be the sum of the indices. If w has many subwords, then

$$ch(w) = \sum_j ch(w_j).$$

Example 1.2.14. Let $w = 21613244153$. Then we extract the first standard word $w_1 = 162453$ from $w : 2\ \underline{1}\ \underline{6}\ 1\ 3\ \underline{2}\ \underline{4}\ 4\ 1\ \underline{5}\ \underline{3}$. If we erase w_1 from w, we are left with 21341, and extract $w_2 = 2134$ from $\underline{2}\ \underline{1}\ \underline{3}\ \underline{4}\ 1$, and finally the last subword $w_3 = 1$. The indices (attached as subscripts) of w_1 are $1_0 6_2 2_0 4_1 5_1 3_0$, so $ch(w_1) = 2+1+1 = 4$. The indices of w_2 are $2_1 1_0 3_1 4_1$, so $ch(w_2) = 1+1+1 = 3$, and finally the index of w_3 is 1_0 and $ch(w_3) = 0$. Thus,

$$ch(w) = ch(w_1) + ch(w_2) + ch(w_3) = 4 + 3 + 0 = 7.$$

Then Lascoux and Schützenberger proves the following theorem in [LS78].

Theorem 1.2.15. *(i) We have*

$$K_{\lambda\mu}(t) = \sum_T t^{ch(T)}$$

summed over all SSYT T of shape λ and weight μ.

(ii) If $\lambda \geq \mu$, $K_{\lambda\mu}(t)$ is monic of degree $n(\mu) - n(\lambda)$. (Note that $K_{\lambda\mu}(t) = 0$ if $\lambda \ngeq \mu$).

Proof. See [LS78]. □

1.2.2 Quasisymmetric Functions

Definition 1.2.16. A formal power series $f = f(x) \in \mathbb{Q}[[x_1, x_2, \dots]]$ is *quasisymmetric* if for any composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, we have

$$f|_{x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}} = f|_{x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}}$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$.

Clearly every symmetric function is quasisymmetric, and sums and products of quasisymmetric functions are also quasisymmetric. Let $\mathcal{Q}^n$ denote the set of all homogeneous quasisymmetric functions of degree n, and let $\mathrm{Comp}(n)$ denote the set of compositions of n.

Definition 1.2.17. Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathrm{Comp}(n)$, define the *monomial quasisymmetric function* M_α by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$

Then it is clear that the set $\{M_\alpha : \alpha \in \mathrm{Comp}(n)\}$ forms a basis for $\mathcal{Q}^n$. One can show that if $f \in \mathcal{Q}^m$ and $g \in \mathcal{Q}^n$, then $fg \in \mathcal{Q}^{m+n}$, thus if $\mathcal{Q} = \mathcal{Q}^0 \oplus \mathcal{Q}^1 \oplus \cdots$, then $\mathcal{Q}$ is a $\mathbb{Q}$-algebra, and it is called the *algebra of quasisymmetric functions* (over $\mathbb{Q}$).

Note that there exists a natural one-to-one correspondence between compositions α of n and subsets S of $[n-1] = \{1, 2, \dots, n-1\}$, namely, we associate the set $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$ with the composition α, and the composition $\mathrm{co}(S) = (s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_{k-1})$ with the set $S = \{s_1, s_2, \dots, s_{k-1}\}_<$. Then it is clear that $\mathrm{co}(S_\alpha) = \alpha$ and $S_{\mathrm{co}(S)} = S$. Using these relations, we define another basis for the quasisymmetric functions.

Definition 1.2.18. Given $\alpha \in \mathrm{Comp}(n)$, we define the *fundamental quasisymmetric function* Q_α

$$Q_\alpha = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Proposition 1.2.19. *For $\alpha \in Comp(n)$ we have*

$$Q_\alpha = \sum_{S_\alpha \subseteq T \subseteq [n-1]} M_{co(T)}.$$

Hence the set $\{Q_\alpha : \alpha \in Comp(n)\}$ is a basis for $\mathcal{Q}^n$.

Proof. See [Sta99, 7.19.1]. □

The main purpose of introducing the quasisymmetric functions is because of the quasisymmetric expansion of the Schur functions. We define a *descent* of an SYT T to be an integer i such that $i+1$ appears in a upper row of T than i, and define the *descent set* $D(T)$ to be the set of all descents of T.

Theorem 1.2.20. *We have*

$$s_{\lambda/\mu} = \sum_T Q_{co(D(T))},$$

where T ranges over all SYTs of shape λ/μ.

Proof. See [Sta99, 7.19.7]. □

Example 1.2.21. For $n = 3$,

$$s_3 = \underset{Q_{\mathrm{co}(\varnothing)}}{\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}}$$

$$s_{21} = \underset{Q_{\mathrm{co}(1)}}{\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1&3\\ \hline\end{array}} + \underset{Q_{\mathrm{co}(2)}}{\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1&2\\ \hline\end{array}}$$

$$s_{111} = \underset{Q_{\mathrm{co}(1,2)}}{\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array}}$$

For $n = 4$,

$$s_4 = \underset{Q_{\mathrm{co}(\varnothing)}}{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline\end{array}}$$

$$s_{31} = \underset{Q_{\mathrm{co}(1)}}{\begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1&3&4\\ \hline\end{array}} + \underset{Q_{\mathrm{co}(2)}}{\begin{array}{|c|c|c|}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 1&2&4\\ \hline\end{array}} + \underset{Q_{\mathrm{co}(3)}}{\begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \hline 1&2&3\\ \hline\end{array}}$$

$$s_{22} = \underset{Q_{\mathrm{co}(2)}}{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}} + \underset{Q_{\mathrm{co}(1,3)}}{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}}$$

$$s_{211} = \underset{Q_{\mathrm{co}(1,2)}}{\begin{array}{|c|c}\cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array}} + \underset{Q_{\mathrm{co}(1,3)}}{\begin{array}{|c|c}\cline{1-1} 4 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}} + \underset{Q_{\mathrm{co}(2,3)}}{\begin{array}{|c|c}\cline{1-1} 4 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}}$$

$$s_{1111} = \underset{Q_{\mathrm{co}(1,2,3)}}{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}}$$

1.3 Macdonald Polynomials

1.3.1 Plethysm

In Proposition 1.2.2, we showed that the power sum functions p_λ form a basis of the ring of symmetric functions. This implies that the ring of symmetric functions can be realized as the ring of polynomials in the power sums $p_1, p_2, \ldots$. Under this consideration, we introduce an operation called *plethysm* which simplifies the notation for compositions of power sum functions and symmetric functions.

Definition 1.3.1. Let $E = E(t_1, t_2, \ldots)$ be a formal Laurent series with rational coefficients in $t_1, t_2, \ldots$. We define the *plethystic substitution* $p_k[E]$ by replacing each t_i in E by t_i^k, i.e.,

$$p_k[E] := E(t_1^k, t_2^k, \ldots).$$

For any arbitrary symmetric function f, the plethystic substitution of E into f, denoted by $f[E]$, is obtained by extending the specialization $p_k \mapsto p_k[E]$ to f.

Note that if $X = x_1 + x_2 + \cdots$, then for $f \in \Lambda$,

$$f[X] = f(x_1, x_2, \ldots).$$

For this reason, we consider this operation as a kind of substitution. In plethystic expression, X stands for $x_1 + x_2 + \cdots$ so that $f[X]$ is the same as $f(X)$. See [Hai99] for a fuller account.

Example 1.3.2. For a symmetric function f of degree d,

(a) $f[tX] = t^d f[X]$.

(b) $f[-X] = (-1)^d \omega f[X]$.

(c) $p_k[X+Y] = p_k[X] + p_k[Y]$.

(d) $p_k[-X] = -p_k[X]$.

Remark 1.3.3. Note that in plethystic notation, the indeterminates are not numeric variables, but they need to be considered as formal symbols.

1.3.2 Macdonald Polynomials

Macdonald introduced a family of symmetric polynomials which becomes a basis of the space of symmetric functions in infinitely many indeterminates $x_1, x_2, \ldots$, with coefficients in the field $\mathbb{Q}(q,t)$, $\Lambda_{\mathbb{Q}(q,t)}$. We first introduce a q,t-analog of the Hall inner product and define

$$< p_\lambda, p_\mu >_{q,t} = \delta_{\lambda\mu} z_\lambda(q,t)$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

In [Mac88], Macdonald proved the existence of the unique family of symmetric functions indexed by partitions $\{P_\lambda[X;q,t]\}$, with coefficients in $\mathbb{Q}(q,t)$, having triangularity with respect to the Schur functions and orthogonality with respect to the q,t-analog of the Hall inner product.

Theorem 1.3.4. *There exists a unique family of symmetric polynomials indexed by partitions,* $\{P_\lambda[X;q,t]\}$*, such that*

1. $P_\lambda = s_\lambda + \sum_{\mu<\lambda} \xi_{\mu,\lambda}(q,t) s_\mu$

2. $< P_\lambda, P_\mu >_{q,t} = 0$ *if* $\lambda \neq \mu$

where $\xi_{\mu,\lambda}(q,t) \in \mathbb{Q}(q,t)$.

Proof. See [Mac98]. □

Macdonald polynomials specialize to Schur functions, complete homogeneous, elementary and monomial symmetric functions and Hall-Littlewood functions.

Proposition 1.3.5.

$$P_\lambda[X;t,t] = s_\lambda[X], \qquad P_\lambda[X;q,1] = m_\lambda[X]$$

$$P_\lambda[X;1,t] = e_{\lambda'}[X], \qquad P_{(1^n)}[X;q,t] = e_n[X]$$

Proof. See [Mac88] or [Mac98]. □

Proposition 1.3.6.

$$P_\lambda[X;q,t] = P_\lambda[X;q^{-1},t^{-1}].$$

Proof. Note that

$$< f,g >_{q^{-1},t^{-1}} = (q^{-1}t)^n < f,g >_{q,t} .$$

Because of this property, $P_\lambda[X;q^{-1},t^{-1}]$ also satisfies two characteristic properties of $P_\lambda[X;q,t]$ in Theorem 1.3.4. By the uniqueness of such polynomials, we get the identity

$$P_\lambda[X;q,t] = P_\lambda[X;q^{-1},t^{-1}].$$

□

Integral Forms

In order to simplify the notations, we use the following common abbreviations.

$$h_\lambda(q,t) = \prod_{c\in\lambda}(1 - q^{a(c)}t^{l(c)+1}), \quad h'_\lambda(q,t) = \prod_{c\in\lambda}(1 - t^{l(c)}q^{a(c)+1}), \quad d_\lambda(q,t) = \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}.$$

We now define the *integral form* of Macdonald polynomials :

$$J_\mu[X;q,t] = h_\mu(q,t)P_\mu[X;q,t] = h'_\mu(q,t)Q_\mu[X;q,t]$$

where $Q_\lambda[X;q,t] = \frac{P_\lambda[X;q,t]}{d_\lambda(q,t)}$. Macdonald showed that the integral form of the Macdonald polynomials J_λ has the following expansion in terms of $\{s_\mu[X(1-t)]\}$:

$$J_\mu[X;q,t] := \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q,t)s_\lambda[X(1-t)],$$

where $K_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$ which satisfies $K_{\lambda\mu}(1,1) = K_{\lambda\mu}$. These functions are called the q,t-*Kostka* functions. Macdonald introduced the $J_\lambda[X;q,t]$ and the q, t-Kostka functions in [Mac88] and he conjectured that the q, t-Kostka functions were polynomials in $\mathbb{N}[q,t]$. This is the famous *Macdonald positivity conjecture*. This was proved by Mark Haiman in 2001 by showing that it is intimately connected with the Hilbert scheme of points in the plane and with the variety of commuting matrices, and Sami Assaf proved combinatorially by introducing the dual equivalence graphs in [Ass07], 2007.

Now we introduce a q,t-analog of the ω involution.

Definition 1.3.7. We define the homomorphism $\omega_{q,t}$ on $\Lambda_{\mathbb{Q}(q,t)}$ by

$$\omega_{q,t}(p_r) = (-1)^{r-1}\frac{1-q^r}{1-t^r}p_r$$

for all $r \geq 1$, and so

$$\omega_{q,t}(p_\lambda) = \epsilon_\lambda p_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Proposition 1.3.8. *We have*

$$\begin{aligned} \omega_{q,t}P_\lambda(X;q,t) &= Q_{\lambda'}(X;t,q) \\ \omega_{q,t}Q_\lambda(X;q,t) &= P_{\lambda'}(X;t,q). \end{aligned}$$

Note that since $\omega_{t,q} = \omega_{q,t}^{-1}$, these two assertions are equivalent.

Proof. See [Mac98]. □

We introduce two important properties of q, t-Kostka polynomials.

Proposition 1.3.9.

$$K_{\lambda\mu}(q,t) = q^{n(\mu')} t^{n(\mu)} K_{\lambda'\mu}(q^{-1}, t^{-1}). \tag{1.3.1}$$

Proof. Note that

$$\begin{aligned} h_\lambda(q^{-1}, t^{-1}) &= \prod_{c\in\lambda} (1 - q^{-a(c)} t^{-l(c)-1}) \\ &= (-1)^{|\lambda|} q^{-n(\lambda')} t^{-n(\lambda)-|\lambda|} h_\lambda(q,t), \end{aligned}$$

since $\sum_{c\in\lambda} a(c) = n(\lambda')$ and $\sum_{c\in\lambda} l(c) = n(\lambda)$. Hence,

$$\begin{aligned} J_\mu[X; q^{-1}, t^{-1}] &= h_\mu(q^{-1}, t^{-1}) P_\mu[X; q^{-1}, t^{-1}] = h_\mu(q^{-1}, t^{-1}) P_\mu[X; q, t] \\ &= (-1)^{|\mu|} q^{-n(\mu')} t^{-n(\mu)-|\mu|} h_\mu(q,t) P_\mu[X; q, t] \\ &= (-1)^{|\mu|} q^{-n(\mu')} t^{-n(\mu)-|\mu|} J_\mu[X; q, t]. \end{aligned}$$

Also, we note that

$$s_\lambda[X(1-t^{-1})] = (-t)^{|\lambda|} s_{\lambda'}[X(1-t)].$$

Then, on one hand

$$\begin{aligned} J_\mu[X; q^{-1}, t^{-1}] &= \sum_\lambda K_{\lambda\mu}(q^{-1}, t^{-1}) s_\lambda[X(1-t)] \\ &= \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q^{-1}, t^{-1}) (-t)^{|\lambda|} s_{\lambda'}[X(1-t)] \\ &= \sum_{\lambda'\vdash|\mu|} K_{\lambda'\mu}(q^{-1}, t^{-1}) (-t)^{|\lambda'|} s_\lambda[X(1-t)]. \end{aligned}$$

On the other hand,

$$\begin{aligned} J_\mu[X; q^{-1}, t^{-1}] &= (-1)^{|\mu|} q^{-n(\mu')} t^{-n(\mu)-|\mu|} J_\mu[X; q, t] \\ &= (-1)^{|\mu|} q^{-n(\mu')} t^{-n(\mu)-|\mu|} \sum_{\lambda\vdash|\mu|} K_{\lambda\mu} s_\lambda[X(1-t)]. \end{aligned}$$

By comparing two coefficients of $s_\lambda[X(1-t)]$, we get the desired identity

$$K_{\lambda\mu}(q,t) = q^{n(\mu')} t^{n(\mu)} K_{\lambda'\mu}(q^{-1}, t^{-1}).$$

□

Proposition 1.3.10.

$$K_{\lambda\mu}(q,t) = K_{\lambda'\mu'}(t,q). \tag{1.3.2}$$

Proof. Note that

$$\begin{aligned}\omega_{q,t}J_\mu[X;q,t] &= h_\mu(q,t)\omega_{q,t}P_\mu[X;q,t] \\ &= h_\mu(q,t)Q_{\mu'}[X;t,q]\end{aligned}$$

by Proposition 1.3.8. Since $h'_{\mu'}(t,q) = h_\mu(q,t)$, we have

$$\begin{aligned}\omega_{q,t}J_\mu[X;q,t] &= h'_{\mu'}(t,q)\omega_{q,t}Q_{\mu'}[X;t,q] \\ &= J_{\mu'}[X;t,q].\end{aligned}$$

Also, note that

$$\omega_{q,t}s_\lambda[X(1-t)] = s_{\lambda'}[X(1-q)].$$

So,

$$\begin{aligned}\omega_{q,t}J_\mu[X;q,t] &= \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q,t)\omega_{q,t}s_\lambda[X(1-t)] \\ &= \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q,t)s_{\lambda'}[X(1-q)].\end{aligned}$$

And

$$\begin{aligned}J_{\mu'}[X;t,q] &= \sum_{\lambda\vdash|\mu|} K_{\lambda\mu'}(t,q)s_\lambda[X(1-q)] \\ &= \sum_{\lambda'\vdash|\mu|} K_{\lambda'\mu'}(t,q)s_{\lambda'}[X(1-q)].\end{aligned}$$

Since $\omega_{q,t}J_\mu[X;q,t] = J_{\mu'}[X;t,q]$, by comparing the coefficients of $s_{\lambda'}[X(1-q)]$, we get

$$K_{\lambda\mu}(q,t) = K_{\lambda'\mu'}(t,q).$$

□

We can give a definite formula for $K_{\lambda\mu}(q,t)$ when μ has only one row or one column. To do that, we need several definitions.

If a is an indeterminate, we define

$$(a;q)_r = (1-a)(1-aq)\cdots(1-aq^{r-1})$$

and we define the infinite product, denoted by $(a;q)_\infty$,

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

regarded as a formal power series in a and q. For two sequences of independent indeterminates $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$, define

$$\prod(x, y; q, t) = \prod_{i,j} \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty}.$$

Note that then we have

$$\prod(x, y; q, t) = \sum_{\lambda} z_\lambda(q, t)^{-1} p_\lambda(x) p_\lambda(y).$$

Now let $g_n(x; q, t)$ denote the coefficient of y^n in the power-series expansion of the infinite product

$$\prod_{i\geq 1} \frac{(tx_iy; q)_\infty}{(x_iy; q)_\infty} = \sum_{n\geq 0} g_n(x; q, t) y^n,$$

and for any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ define

$$g_\lambda(x; q, t) = \prod_{i\geq 1} g_{\lambda_i}(x; q, t).$$

Then we have

$$g_n(x; q, t) = \sum_{\lambda\vdash n} z_\lambda(q, t)^{-1} p_\lambda(x),$$

and hence

$$\begin{aligned}\prod(x, y; q, t) &= \prod_j \left(\sum_{n\geq 0} g_n(x; q, t) y_j^n\right) \\ &= \sum_\lambda g_\lambda(x; q, t) m_\lambda(y).\end{aligned}$$

We note the following proposition.

Proposition 1.3.11. *Let $\{u_\lambda\}, \{v_\lambda\}$ be two $\mathbb{Q}(q,t)$-bases of $\Lambda^n_{\mathbb{Q}(q,t)}$, indexed by the partitions of n. For each $n \geq 0$, the following conditions are equivalent :*

(a) $< u_\lambda, v_\mu >= \delta_{\lambda\mu}$ *for all* λ, μ,

(b) $\sum_\lambda(x)v_\lambda(y) = \prod(x, y; q, t)$.

Proof. See [Mac98]. □

Then by Proposition 1.3.11,

$$< g_\lambda(X; q, t), m_\mu(x) >= \delta_{\lambda\mu} \tag{1.3.3}$$

so that the g_λ form a basis of $\Lambda_{\mathbb{Q}(q,t)}$ dual to the basis $\{m_\lambda\}$.

Now we consider P_λ when $\lambda = (n)$, i.e., λ has only one row with size n. By (1.3.3), g_n is orthogonal to m_μ for all partitions $\mu \neq (n)$, hence to all P_μ except for $\mu = (n)$. So g_n must be a scalar multiple of $P_{(n)}$, and actually $P_{(n)}$ is

$$P_{(n)} = \frac{(q;q)_n}{(t;q)_n} g_n.$$

And by the way of defining J_λ,

$$J_{(n)}(X; q, t) = (t; q)_n P_{(n)} = (q; q)_n g_n(X; q, t). \tag{1.3.4}$$

Proposition 1.3.12.

$$K_{\lambda,(n)}(q, t) = \frac{q^{n(\lambda)}(q;q)_n}{H_\lambda(q)}$$

and so by duality,

$$K_{\lambda,(1^n)}(q, t) = \frac{t^{n(\lambda')}(t;t)_n}{H_\lambda(t)},$$

where $H_\lambda(q)$ *is the hook polynomial defined in Definition 1.1.8.*

Proof. Note that we have

$$\begin{aligned}
\sum_{n\geq 0} g_n(X; q, t) &= \prod_{i,j} \frac{1 - t x_i q^{j-1}}{1 - x_i q^{j-1}} \\
&= \sum_\lambda s_\lambda(1, q, q^2, \dots) s_\lambda[X(1-t)] \\
&= \sum_\lambda q^{n(\lambda)} \frac{s_\lambda[X(1-t)]}{H_\lambda(q)}
\end{aligned}$$

by Theorem 1.2.11. So, in (1.3.4),

$$J_{(n)}(X;q,t) = (q;q)_n g_n(X;q,t) = \sum_{\lambda\vdash n} \frac{q^{n(\lambda)}(q;q)_n}{H_\lambda(q)} s_\lambda[X(1-t)]$$

and therefore

$$K_{\lambda,(n)}(q,t) = \frac{q^{n(\lambda)}(q;q)_n}{H_\lambda(q)}.$$

Note that $K_{\lambda\mu}(q,t)$ has a duality property (1.3.2)

$$K_{\lambda\mu}(q,t) = K_{\lambda'\mu'}(t,q)$$

and this property gives $K_{\lambda,(1^n)}(q,t) = t^{n(\lambda')}(t;t)_n/H_\lambda(t)$. □

Modified Macdonald Polynomials

In many cases, it is convenient to work with the modified Macdonald polynomials. We define

$$H_\mu[X;q,t] := J_\mu\left[\frac{X}{1-t};q,t\right] = \sum_{\lambda\vdash|\mu|} K_{\lambda,\mu}(q,t)s_\lambda[X].$$

We make one final modification to make the *modified Macdonald polynomials*

$$\tilde{H}_\mu[X;q,t] := t^{n(\mu)}H_\mu[X;q,1/t] = \sum_{\lambda\vdash|\mu|} \tilde{K}_{\lambda,\mu}(q,t)s_\lambda[X].$$

$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)}K_{\lambda,\mu}(q,t^{-1})$ are called the *modified q, t-Kostka functions.* Macdonald defined the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ in such a way that on setting $q=0$ they yield the famous (modified) Kostka-Foulkes polynomials $\tilde{K}_{\lambda\mu}(t) = \tilde{K}_{\lambda\mu}(0,t)$.

Proposition 1.3.13.

$$\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t].$$

Proof. See [Hai01] or [Ass07]. □

$\tilde{H}_\mu[X;q,t]$ can be characterized independently of the $P_\mu[X;q,t]$.

Proposition 1.3.14. *The functions $\tilde{H}_\mu[X;q,t]$ are the unique functions in $\Lambda_{\mathbb{Q}(q,t)}$ satisfying the following triangularity and orthogonality conditions :*

(1) $\tilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda\geq\mu} a_{\lambda\mu}(q,t)s_\lambda$,

(2) $\tilde{H}_\mu[X(1-t);q,t] = \sum_{\lambda\geq\mu'} b_{\lambda\mu}(q,t)s_\lambda$,

(3) $<\tilde{H}_\mu, s_{(n)}>= 1$,

for suitable coefficients $a_{\lambda\mu}, b_{\lambda\mu} \in \mathbb{Q}(q,t)$.

Proof. See [Hai99], [Hai01]. □

Corollary 1.3.15. *For all* μ*, we have*

$$\omega\tilde{H}_\mu[X;q,t] = t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu[X;q^{-1},t^{-1}]$$

and, consequently, $\tilde{K}_{\lambda'\mu}(q,t) = t^{n(\mu)}q^{n(\mu')}\tilde{K}_{\lambda\mu}(q^{-1},t^{-1})$.

Proof. One can show that $\omega t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu[X;q^{-1},t^{-1}]$ satisfies (1) and (2) of Proposition 1.3.14, and so it is a scalar multiple of $\tilde{H}_\mu$. (3) of Proposition 1.3.14 requires that $\tilde{K}_{(1^n),\mu} = t^{n(\mu)}q^{n(\mu')}$ which is equivalent to $K_{(1^n),\mu} = q^{n(\mu')}$ and this is known in [Mac98]. □

Proposition 1.3.16. *For all* μ*, we have*

$$\tilde{H}_\mu[X;q,t] = \tilde{H}_{\mu'}[X;t,q]$$

and consequently, $\tilde{K}_{\lambda\mu}(q,t) = \tilde{K}_{\lambda\mu'}(t,q)$.

Proof. The left hand side is

$$\begin{aligned}\tilde{H}_\mu[X;q,t] &= \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda[X] \\ &= \sum_\lambda t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})s_\lambda[X]. \end{aligned} \tag{1.3.5}$$

And the right hand side is

$$\begin{aligned}\tilde{H}_{\mu'}[X;t,q] &= \sum_\lambda \tilde{K}_{\lambda\mu'}(t,q)s_\lambda[X] \\ &= \sum_\lambda q^{n(\mu')}K_{\lambda\mu'}(t,q^{-1})s_\lambda[X]. \end{aligned} \tag{1.3.6}$$

Comparing (1.3.5) and (1.3.6), we need to show that

$$t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}) = q^{n(\mu')}K_{\lambda\mu'}(t,q^{-1}).$$

We note two properties of q,t-Kostka functions (1.3.1) and (1.3.2). Using those properties, $t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})$ becomes

$$\begin{aligned} t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}) &= t^{n(\mu)}q^{n(\mu')}t^{-n(\mu)}K_{\lambda\mu'}(t,q^{-1}) \\ &= q^{n(\mu')}K_{\lambda\mu'}(t,q^{-1}) \end{aligned}$$

and this finishes the proof. □

1.3.3 A Combinatorial Formula for Macdonald Polynomials

In 2004, Haglund [Hag04] conjectured a combinatorial formula for the monomial expansion of the modified Macdonald polynomials $\tilde{H}_\mu[x;q,t]$, and this was proved by Haglund, Haiman and Loehr [HHL05] in 2005. This celebrated combinatorial formula accelerated the research of symmetric functions theory concerning Macdonald polynomials. Beefore we give a detailed description of the formula, we introduce some definitions.

Definition 1.3.17. A *word* of length n is a function from $\{1,2,\dots,n\}$ to the positive integers. The *weight* of a word is the vector

$$wt(w) = \{|w^{-1}(1)|, |w^{-1}(2)|, \dots\}.$$

We will think of words as vectors

$$w = (w(1), w(2), \dots) = (w_1, w_2, \dots)$$

and we write the word $w = (w_1, w_2, \dots)$ as simply $w_1 w_2 \dots w_n$. A word with weight $(1,1,\dots,1)$ is called a *permutation*.

Definition 1.3.18. A *filling* of a diagram L of size n with a word σ of length n (written (σ, L)) is a function from the cells of the diagram to $\mathbb{Z}_+$ given by labeling the cells from top to bottom and left to right within rows by 1 to n in order, then applying σ. We simply use σ to denote a filled diagram.

Definition 1.3.19. The *reading order* is the total ordering on the cells of μ given by reading them row by row, from top to bottom, and from left to right within each row. More formally, $(i,j) < (i',j')$ in the reading order if $(-i,j)$ is lexicographically less than $(-i',j')$.

Definition 1.3.20. A *descent* of σ is a pair of entries $\sigma(u)$ and $\sigma(v)$, where the cell v is the south of u, that is $v = (i,j), u = (i+1,j)$, and the elements of u and v satisfy $\sigma(u) > \sigma(v)$. Define

$$\mathrm{Des}(\sigma) = \{u \in \mu : \sigma(u) > \sigma(v) \text{ is a descent }\},$$

$$\mathrm{maj}(\sigma) = \sum_{u \in \mathrm{Des}(\sigma)} (\mathrm{leg}(u) + 1).$$

Example 1.3.21. The example below has two descents, as shown.

$$\sigma = \begin{array}{|c|c|cc}\hline 6 & 2 & \multicolumn{2}{c}{} \\ \hline 2 & 5 & \multicolumn{1}{c|}{8} & \\ \hline 4 & 4 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \quad \mathrm{Des}(\sigma) = \begin{array}{|c|c|cc}\hline 6 & & \multicolumn{2}{c}{} \\ \hline & 5 & \multicolumn{1}{c|}{8} & \\ \hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \end{array}$$

The maj(σ)$= (0+1) + (1+1) + (0+1) = 4$.

Definition 1.3.22. Three cells $u, v, w \in \mu$ are said to form a *triple* if they are situated as shown below,

$$\begin{array}{|c|}\hline u \\ \hline v \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline w \\ \hline \end{array}$$

namely, v is directly below u, and w is in the same row as u, to its right. Let σ be a filling. When we order the entries $\sigma(u), \sigma(v), \sigma(w)$ from the smallest to the largest, if they make a counter-clockwise rotation, then the triple is called an *inversion triple.* If the two cells u and w are in the first row (i.e., in the bottom row), then they contribute an inversion triple if $\sigma(u) > \sigma(w)$. Define

$$\mathrm{inv}(\sigma) = \text{the number of inversion triples in } \sigma.$$

We also define a *coinversion triple* if the orientation from the smallest entry to the largest one is clockwise, and coinv(σ) is the number of coinversion triples in σ.

Now the combinatorial formula of Haglund, Haiman and Loehr is as follows.

Theorem 1.3.23.

$$\tilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{inv(\sigma)} t^{maj(\sigma)} X^\sigma. \tag{1.3.7}$$

Proof. See [HHL05]. □

1.4 Hilbert Series of S_n-Modules

To prove the positivity conjecture of Macdonald polynomials, Garsia and Haiman introduced certain bigraded S_n modules M_μ [GH93]. We give several important definitions and results of their research here.

Let μ be a partition. We shall identify μ with its Ferrers' diagram. Let $(p_1, q_1), \ldots, (p_n, q_n)$ denote the pairs (l', a') of the cells of the diagram of μ arranged in lexicographic order and set

$$\triangle_\mu(X,Y) = \triangle_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n) = \det \| \; x_i^{p_j} y_i^{q_j} \; \|_{i,j=1,\ldots,n} \; .$$

Example 1.4.1. For $\mu = (3,1)$, $\{(p_j,q_j)\} = \{(0,0),(0,1),(0,2),(1,0)\}$, and

$$\triangle_\mu = \det \begin{pmatrix} 1 & y_1 & y_1^2 & x_1 \\ 1 & y_2 & y_2^2 & x_2 \\ 1 & y_3 & y_3^2 & x_3 \\ 1 & y_4 & y_4^2 & x_4 \end{pmatrix}$$

This given, we let $M_\mu[X,Y]$ be the space spanned by all the partial derivatives of $\triangle_\mu(x,y)$. In symbols

$$M_\mu[X,Y] = \mathcal{L}[\partial_x^p \partial_y^q \triangle\mu(x,y)]$$

where $\partial_x^p = \partial_{x_1}^{p_1} \cdots \partial_{x_n}^{p_n}, \partial_y^p = \partial_{y_1}^{p_1} \cdots \partial_{y_n}^{p_n}$. The natural action of a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ on a polynomial $P(x_1,\ldots,x_n;y_1,\ldots,y_n)$ is the so called *diagonal action* which is defined by setting

$$\sigma P(x_1,\ldots,x_n;y_1,\ldots,y_n) := P(x_{\sigma(1)},\ldots,x_{\sigma(n)};y_{\sigma(1)},\ldots,y_{\sigma(n)}).$$

Since $\sigma\triangle_\mu = \pm\triangle_\mu$ according to the sign of σ, the space M_μ necessarily remains invariant under this action.

Note that, since $\triangle_\mu$ is bihomogeneous of degree $n(\mu)$ in x and $n(\mu')$ in y, we have the direct sum decomposition

$$M_\mu = \oplus_{h=0}^{n(\mu)} \oplus_{k=0}^{n(\mu')} \mathcal{H}_{h,k}(M_\mu),$$

where $\mathcal{H}_{h,k}(M_\mu)$ denotes the subspace of M_μ spanned by its bihomogeneous elements of degree h in x and degree k in y. Since the diagonal action clearly preserves bidegree, each of the subspaces $\mathcal{H}_{h,k}(M_\mu)$ is also S_n-invariant. Thus we see that M_μ has the structure of a bigraded module. We can write a *bivariate Hilbert series* such as

$$F_\mu(q,t) = \sum_{h=0}^{n(\mu)} \sum_{k=0}^{n(\mu')} t^h q^k \dim(\mathcal{H}_{h,k}(M_\mu)). \tag{1.4.1}$$

In dealing with graded S_n-modules, we will generally want to record not only the dimensions of homogeneous components but their characters. The generating function of the characters of its bihomogeneous components, which we shall refer to as the *bigraded character* of M_μ, may be written in the form

$$\chi^\mu(q,t) = \sum_{h=0}^{n(\mu)} \sum_{k=0}^{n(\mu')} t^h q^k \mathrm{char}\mathcal{H}_{h,k}(M_\mu).$$

We also have an associated *bigraded Frobenius characteristic* $\mathcal{F}(M_\mu)$ which is simply the image of $\chi^\mu(q,t)$ under the Frobenius map. Note that the *Frobenius map* from S_n-characters to symmetric functions homogeneous of degree n is defined by

$$\Phi(\chi) = \frac{1}{n!} \sum_{\omega \in S_n} \chi(\omega) p_{\tau(\omega)}(X),$$

where $\tau(\omega)$ is the partition whose parts are the lengths of the cycles of the permutation ω. Since the Schur function $s_\lambda(X)$ is the Frobenius image of the irreducible S_n-character χ^λ, we have, i.e., $\Phi(\chi^\lambda) = s_\lambda(X)$, for any character,

$$\Phi(\chi) = \sum_\lambda \mathrm{mult}(\chi^\lambda, \chi) s_\lambda.$$

Then we can define the *Frobenius series* of a doubly graded S_n module M_μ to be

$$\mathcal{F}_{M_\mu} = C_{M_\mu}(X;q,t) = \sum_{\lambda \vdash n} s_\lambda(X) C_{\lambda\mu}(q,t)$$

where $C_{\lambda\mu}(q,t)$ is the bivariate generating function of the multiplicity of χ^λ in the various bihomogeneous components of M_μ. M. Haiman proved [Hai02a] that the bigraded Frobenius series $\mathcal{F}_{M_\mu}$ is equal to the modified Macdonald polynomials $\tilde{H}[X;q,t]$.

Theorem 1.4.2.

$$C_{M_\mu}(X;q,t) = \tilde{H}_\mu(X;q,t)$$

which forces the equality

$$C_{\lambda\mu}(q,t) = \tilde{K}_{\lambda\mu}(q,t).$$

In particular, we have $\tilde{K}_{\lambda\mu} \in \mathbb{N}[q,t]$.

Proof. See [Hai02a, Hai02b]. □

Theorem 1.4.2 proves the Macdonald's positivity conjecture and in particular, it implies that the dimension of Garsia-Haiman module M_μ is $n!$ which was known as the "$n!$ conjecture".

Corollary 1.4.3. *The dimension of the space* M_μ *is* $n!$.

For a symmetric function f, we write

$$\partial_{p_1} f$$

to denote the result of differentiating f with respect to p_1 after expanding f in terms of the power sum symmetric function basis. Then it is known that for any Schur function s_λ, we have

$$\partial_{p_1} s_\lambda = \sum_{\nu \to \lambda} s_\nu$$

where "$\nu \to \lambda$" is to mean that the sum is carried out over partitions ν that are obtained from λ by removing one of its corners. When λ is a partition, the well-known *branching-rule*

$$\chi^\lambda \downarrow^{S_n}_{S_{n-1}} = \sum_{\nu \to \lambda} \chi^\nu$$

implies

$$\partial_{p_1}\tilde{H}_\mu(X;q,t) = \sum_{h=0}^{n(\mu)}\sum_{k=0}^{n(\mu')} t^h q^k \Phi(\mathrm{char}\mathcal{H}_{h,k}(M_\mu)\downarrow_{S_{n-1}}^{S_n}).$$

Namely, $\partial_{p_1}\tilde{H}_\mu(X;q,t)$ gives the bigraded Frobenius characteristic of M_μ restricted to S_{n-1}. In particular, we must have

$$F_\mu(q,t) = \partial_{p_1}^n \tilde{H}_\mu(X;q,t),$$

where $F_\mu(q,t)$ is the bivariate Hilbert series defined in (1.4.1). Noting the fact that the operator ∂_{p_1} is the Hall scalar product adjoint to multiplication by the elementary symmetric function e_1, we can transform one of the Pieri rules given by Macdonald in [Mac88] into the expansion of $\partial_{p_1}\tilde{H}_\mu(X;q,t)$ in terms of the polynomials $\tilde{H}_\nu(x;q,t)$ whose index ν immediately precedes μ in the Young partial order (which is defined simply by containment of Young diagrams). To explain this more precisely, we introduce some notations first.

Let μ be a partition of n and let $\{\nu^{(1)},\nu^{(2)},\ldots,\nu^{(d)}\}$ be the collection of partitions obtained by removing one of the corners of μ. And for a cell $c\in\mu$, we define *weight* to be the monomial $w(c) = t^{l'(c)}q^{a'(c)}$, where $a'(c)$ denotes the coarm of c and $l'(c)$ denotes the coleg of c as defined in Section 1.1. We call the weights of the corners

$$\mu/\nu^{(1)},\mu/\nu^{(2)},\ldots,\mu/\nu^{(d)}$$

respectively

$$x_1,x_2,\ldots,x_d.$$

Moreover, if $x_i = t^{l'_i}q^{a'_i}$, then we also let

$$u_i = t^{l'_{i+1}}q^{a'_i} \quad (\text{for } i=1,2,\ldots,d-1)$$

be the weights of what we might refer to as the *inner corners* of μ. Finally, we set

$$u_0 = t^{l'_1}/q, \quad u_d = q^{a'_d}/t \quad \text{and} \quad x_0 = 1/tq.$$

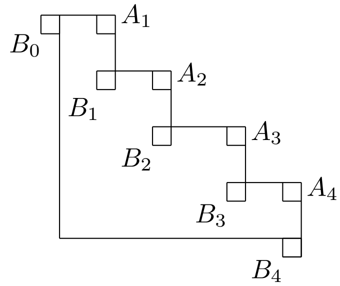


Figure 1.6: An example of 4-corner case with corner cells A_1, A_2, A_3, A_4 and inner corner cells B_0, B_1, B_2, B_3, B_4.

Proposition 1.4.4.

$$\partial_{p_1} \tilde{H}_\mu(X;q,t) = \sum_{i=1}^{d} c_{\mu\nu^{(i)}}(q,t)\tilde{H}_{\nu^{(i)}}(X;q,t) \tag{1.4.2}$$

where

$$c_{\mu\nu^{(i)}}(q,t) = \frac{1}{(1-1/t)(1-1/q)}\frac{1}{x_i}\frac{\prod_{j=0}^{d}(x_i-u_j)}{\prod_{j=1;j\neq i}^{d}(x_i-x_j)}.$$

Proof. See [BBG$^+$99]. □

Science Fiction

While studying the modules M_μ, Garsia and Haiman made a huge collection of conjectures based on representation theoretical heuristics and computer experiments. This collection of conjectures is called "*science fiction*" and most of them are still open. In particular, they conjectured the existence of a family of polynomials $G_D(X;q,t)$, indexed by arbitrary lattice square diagrams D, which the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ can be imbedded in. To be more precise, we give some definitions first.

We say that two lattice square diagrams D_1 and D_2 are *equivalent* and write $D_1 \approx D_2$ if and only if they differ by a sequence of row and column rearrangements.

The *conjugate* of D, denoted by D', is the diagram obtained by reflecting the diagram D about the diagonal $x = y$. We say that a diagram D is *decomposable* if D can be decomposed into the union of two diagrams D_1 and D_2 in such a way that no rook placed on a cell in D_1 attacks

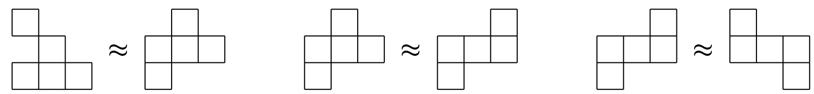


Figure 1.7: Examples of equivalent pairs.

any cell in D_2. If D is decomposable into D_1 and D_2, then we write $D = D_1 \times D_2$. This given, Garsia and Haiman conjectured the existence of a family of polynomials $\{G_D(X;q,t)\}_D$, indexed by diagrams D equivalent to skew Young diagrams, with the following properties :

(1) $G_D(X;q,t) = \tilde{H}_\mu(X;q,t)$ if D is the Young diagram of μ.

(2) $G_{D_1}(X;q,t) = G_{D_2}(X;q,t)$ if $D_1 \approx D_2$.

(3) $G_D(X;q,t) = G_{D'}(X;t,q)$.

(4) $G_D(X;q,t) = G_{D_1}(X;q,t)G_{D_2}(X;q,t)$ if $D = D_2 \times D_2$.

(5) The polynomials G_D satisfy the following equation :

$$\partial_{p_1} G_D(X;q,t) = \sum_{c \in D} q^{a'(c)} t^{l(c)} G_{D \backslash c}(X;q,t).$$

Proposition 1.4.5. *Properties* (3) *and* (5) *imply that for any lattice square diagram D we have*

$$\partial_{p_1} G_D(X;q,t) = \sum_{c \in D} q^{a(c)} t^{l'(c)} G_{D \backslash c}(X;q,t). \tag{1.4.3}$$

Proof. The property (5) gives

$$\partial_{p_1} G_{D'}(X;t,q) = \sum_{c \in D'} t^{a'(c)} q^{l(c)} G_{D' \backslash c}(X;t,q).$$

Applying the property (3) to the left hand side of the equation of (5) gives

$$\partial_{p_1} G_D(X;q,t) = \partial_{p_1} G_{D'}(X;t,q) = \sum_{c \in D'} t^{a'(c)} q^{l(c)} G_{D' \backslash c}(X;t,q).$$

Note that $l(c)$ for $c \in D'$ is equal to $a'(c)$ for $c \in D$ and $a'(c)$ for $c \in D'$ is equal to $l(c)$ for $c \in D$. Hence, by applying the property (3) to the right hand side, we get

$$\partial_{p_1} G_D(X;q,t) = \sum_{c \in D} q^{a(c)} t^{l'(c)} G_{D \backslash c}(X;q,t).$$

□

By Proposition 1.3.16, we can at least see that the condition (1) and (3) are consistent. It is possible to use these properties to explicitly determine the polynomials $G_D(X;q,t)$ in special cases. In particular, we consider the details for the hook case which will give useful information for later sections.

Hook Case

For convenience, we set the notation for diagram of hooks and broken hooks as

$$[l,b,a] = \begin{cases} (a+1,1^l) & \text{if } b=1, \\ (1^l)\times(a) & \text{if } b=0. \end{cases}$$

Namely, $[l,1,a]$ represents a hook with leg l and arm a, and $[l,0,a]$ represents the product of a column of length l by a row of length a. Then in the hook case, the property (5) and (1.4.3) yield the recursions

$$\partial_{p_1} G_{[l,1,a]} = [l]_t G_{[l-1,1,a]} + t^l G_{[l,0,a]} + q[a]_q G_{[l,1,a-1]}, \tag{1.4.4}$$

$$\partial_{p_1} G_{[l,1,a]} = t[l]_t G_{[l-1,1,a]} + q^a G_{[l,0,a]} + [a]_q G_{[l,1,a-1]}, \tag{1.4.5}$$

where $[l]_t = (1-t^l)/(1-t)$ and $[a]_q = (1-q^a)/(1-q)$. Subtracting (1.4.5) from (1.4.4) and simplifying gives the following two recursions

$$G_{[l,0,a]} = \frac{t^l-1}{t^l-q^a} G_{[l-1,1,a]} + \frac{1-q^a}{t^l-q^a} G_{[l,1,a-1]}, \tag{1.4.6}$$

$$G_{[l,1,a-1]} = \frac{1-t^l}{1-q^a} G_{[l-1,1,a]} + \frac{t^l-q^a}{1-q^a} G_{[l,0,a]}. \tag{1.4.7}$$

Transforming the Pieri rule ([Mac88]) gives the identity

$$\tilde{H}_{(1^l)}(X;q,t)\tilde{H}_{(a)}(X;q,t) = \frac{t^l-1}{t^l-q^a}\tilde{H}_{(a+1,1^{l-1})}(X;q,t) + \frac{1-q^a}{t^l-q^a}\tilde{H}_{(a,1^l)}(X;q,t)$$

which by comparing with (1.4.6) implies $G_{[l,0,a]} = \tilde{H}_{(1^l)}(X;q,t)\tilde{H}_{(a)}(X;q,t)$ with the initial conditions $G_{[l,1,a]} = \tilde{H}_{(a+1,1^l)}(X;q,t)$. For $\mu = (a+1,1^l)$, the formula in Proposition 1.4.4 gives

$$\partial_{p_1}\tilde{H}_{(a+1,1^l)} = [l]_t \frac{t^{l+1}-q^a}{t^l-q^a}\tilde{H}_{(a+1,1^{l-1})} + [a]_q \frac{t^l-q^{a+1}}{t^l-q^a}\tilde{H}_{(a,1^l)}. \tag{1.4.8}$$

On the other hand, by using (1.4.6) in (1.4.4), we can eliminate the broken hook term and get the following equation

$$\partial_{p_1} G_{[l,1,a]} = [l]_t \frac{t^{l+1} - q^a}{t^l - q^a} G_{[l-1,0,a]} + [a]_q \frac{t^l - q^{a+1}}{t^l - q^a} G_{[l,1,a-1]} \tag{1.4.9}$$

which is exactly consistent with (1.4.8). We set the Hilbert series for the hook shape diagram

$$F_{[l,b,a]} = n! G_{[l,b,a]}|_{p_1^n},$$

then (1.4.4) yields the *Hilbert series recursion*

$$F_{[l,1,a]} = [l]_t F_{[l-1,1,a]} + t^l F_{[l,0,a]} + q[a]_q F_{[l,1,a-1]}.$$

Note that we find in [Mac88] that

$$\tilde{H}_{(1^l)}(X;q,t) = (t)_l h\left[\frac{X}{1-t}\right] \quad \text{and} \quad \tilde{H}_{(a)}(X;q,t) = (q)_l h\left[\frac{X}{1-q}\right],$$

where $q_m = (1-q)(1-q^2)\cdots(1-q^{m-1})$. Combining this with $G_{[l,0,a]} = \tilde{H}_{(1^l)}(X;q,t)\tilde{H}_{(a)}(X;q,t)$ gives

$$F_{[l,0,a]} = \binom{l+a}{a} [l]_t! [a]_q! \tag{1.4.10}$$

where $[l]_t!$ and $[a]_q!$ are the t and q-analogues of the factorials.

We finish this section by giving a formula for the Macdonald hook polynomials which we can get by applying (1.4.5) recursively.

Theorem 1.4.6.

$$\tilde{H}_{(a+1,1^l)} = \sum_{i=0}^{l} \frac{(t)_l}{(t)_{l-i}} \frac{(q)_a}{(q)_{a+i}} \frac{t^{l-i} - q^{a+i+1}}{1 - q^{a+i+1}} \tilde{H}_{(1^{l-i})} \tilde{H}_{(a+i+1)}.$$

Proof. See [GH95]. □

Chapter 2

Combinatorial Formula for the Hilbert Series

In this chapter, we construct a combinatorial formula for the Hilbert series F_μ in (1.4.1) as a sum over SYT of shape μ. We provide three different proofs for this result, and in Section 2.2.4, we introduce a way of associating the fillings to the corresponding standard Young tableaux. We begin by recalling definitions of q-analogs :

$$[n]_q = 1 + q + \cdots + q^{n-1},$$

$$[n]_q! = [1]_q \cdots [n]_q.$$

2.1 Two column shape case

Let $\mu = (2^a, 1^b)$ (so $\mu' = (a+b, b)$). We use $\tilde{F}_{\mu'}(q,t)$ to denote the Hilbert series of the modified Macdonald polynomials $t^{n(\mu)}\tilde{H}_\mu(X; 1/t, q)$. Note that from the combinatorial formula for Macdonald polynomials in Theorem 1.3.7, the coefficient of $x_1 x_2 \cdots x_n$ in $\widetilde{H}_\mu\left(x; \frac{1}{t}, q\right) t^{n(\mu)}$ is given by

$$\tilde{F}_{\mu'}(q,t) := \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma,\mu')} t^{\mathrm{coinv}(\sigma,\mu')}, \tag{2.1.1}$$

where $\mathrm{maj}(\sigma,\mu')$ and $\mathrm{coinv}(\sigma,\mu')$ are as defined in Section 1.3.3. Now we Define

$$\tilde{G}_{\mu'}(q,t) := \sum_{T \in \mathrm{SYT}(\mu')} \prod_{i=1}^{n} [a_i(T)]_t \cdot \prod_{j=1}^{\mu'_2} \left(1 + qt^{b_j(T)}\right) \tag{2.1.2}$$

where $a_i(T)$ and $b_j(T)$ are determined in the following way : starting with the cell containing 1, add cells containing $2, 3, \ldots, i$, one at a time. After adding the cell containing i, $a_i(T)$ counts the number of columns which have the same height with the column containing the square just added with i. And $b_j(T)$ counts the number of cells in the first to the strictly right of the cell $(2,j)$ which contain bigger element than the element in $(2,j)$. Then we have the following theorem :

Theorem 2.1.1. *We have*

$$\tilde{F}_{\mu'}(q,t) = \tilde{G}_{\mu'}(q,t).$$

Remark 2.1.2. The motivation of the conjectured formula is from the equation (5.11) in [Mac98], p.229. The equation (5.11) is the first t-factor of our conjectured formula, i.e., when $q = 0$. Hence our formula extends the formula for Hall-Littlewood polynomials to Macdonald polynomials.

Example 2.1.3. Let $\mu = (2,1)$. We calculate $\tilde{F}_{\mu}(q,t) = \sum_{\sigma \in S_3} q^{\mathrm{maj}(\sigma,\mu')} t^{\mathrm{coinv}(\sigma,\mu')}$ first. All the possible fillings of shape $(2,1)$ are the followings.

$$\begin{array}{|c|c}\hline 1 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$$

From the above tableaux, reading from the left, we get

$$\tilde{F}_{(2,1)}(q,t) = t + 1 + qt + 1 + qt + q = 2 + q + t + 2qt. \tag{2.1.3}$$

Now we consider $\tilde{G}_{(2,1)}(q,t)$ over the two standard tableaux

$$T_1 = \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\ , \qquad T_2 = \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\ .$$

For the first SYT T_1, if we add 1, there is only one column with height 1, so we have $a_1(T_1) = 1$. And then if we add 2, since it is going on the top of the square with 1, it makes a column with height 2. There is only one column with height 2 which gives $a_2(T_1) = 1$ again. Adding the square with 3 gives the factor 1 as well, since the column containing the square with 3 is height 1 and there is only one column with height 1. Hence for the SYT T_1, the first factor is 1, i.e.,

$$a_i(T_1) : \underset{[1]_t}{\boxed{1}} \rightarrow \underset{[1]_t}{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}} \rightarrow \underset{[1]_t}{\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}} \quad \Rightarrow \quad \prod_{i=1}^{3} [a_i(T_1)]_t = 1.$$

To decide $b_j(T_1)$, we compare the element in the first row to the right of the square in the second row. In T_1, we only have one cell in the second row which has the element 2. 3 is bigger than 2 and the cell containing 3 is in the first row to the right of the first column. So we get $b_1(T_1) = 1$, i.e.,

$$\prod_{j=1}^{\mu_2'} \left(1 + qt^{b_j(T_1)}\right) = 1 + qt.$$

Hence, the first standard Young tableau T_1 gives $1 \cdot (1 + qt)$.

We repeat the same procedure for T_2 to get $a_i(T_2)$ and $b_j(T_2)$. If we add the second box containing 2 to the right of the box with 1, then it makes two columns with height 1, so we get $a_2(T_2) = 2$ and that gives the factor $[2]_t = (1 + t)$. Adding the last square gives the factor 1, so the first factor is $(1 + t)$.

$$a_i(T_2) : \underset{[1]_t}{\boxed{1}} \rightarrow \underset{[2]_t}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}} \rightarrow \underset{[1]_t}{\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}} \quad \Rightarrow \quad \prod_{i=1}^{3} [a_i(T_2)]_t = (1 + t).$$

Now we consider $b_j(T_2)$. Since 3 is the biggest element in this case and it is in the second row, $b_1(T_2) = 0$ and that makes the second factor $(1 + q)$. Hence from the SYT T_2, we get

$$\prod_{i=1}^{3} [a_i(T_2)]_t \cdot \prod_{j=1}^{\mu_2'=1} \left(1 + qt^{b_j(T_2)}\right) = (1 + t)(1 + q).$$

We add two polynomials from T_1 and T_2 to get $\tilde{G}_{(2,1)}(q, t)$ and the resulting polynomial is

$$\tilde{G}_{(2,1)}(q, t) = 1 \cdot (1 + qt) + (1 + t)(1 + q) = 2 + q + t + 2qt. \tag{2.1.4}$$

We compare (2.1.3) and (2.1.4) and confirm that

$$\tilde{F}_{(2,1)}(q, t) = \tilde{G}_{(2,1)}(q, t).$$

2.1.1 The Case When $\mu' = (n-1,1)$

We give a proof of Theorem 2.1.1 when $\mu = (2, 1^{n-2})$, or $\mu' = (n-1, 1)$.

Proposition 2.1.4. *We have*

$$\tilde{F}_{(n-1,1)}(q,t) = \tilde{G}_{(n-1,1)}(q,t).$$

Proof. First of all, we consider when $q = 1$. Then, then

$$\tilde{F}_{\mu'}(1,t) = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma,\mu')}\ t^{\mathrm{coinv}(\sigma,\mu')}\Big|_{q=1} = \sum_{\sigma \in S_n} t^{\mathrm{coinv}(\sigma,\mu')}$$

and for any element in the cell of the second row, say σ_1, $t^{\mathrm{coinv}(\sigma_2\sigma_3\cdots\sigma_n)}$ is always $[n-1]_t!$. So we have

$$\tilde{F}_{(n-1,1)}(1,t) = \sum_{\sigma \in S_n} t^{\mathrm{coinv}(\sigma,\mu')} = n \cdot [n-1]_t!. \tag{2.1.5}$$

On the other hand, to calculate $\tilde{G}_{(n-1,1)}(1,t)$, we consider a standard Young tableau with j in the second row. Then by the property of the standard Young tableaux, the first row has elements $1, 2, \cdots, j-1, j+1, \cdots, n$, from the left.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline j \\ \cline{1-1} 1 & \cdots & j-1 & j+1 & \cdots & n \\ \hline \end{array}$$

Then the first factor $\prod_{i=1}^{n} [a_i(T)]_t$ becomes

$$[j-1]_t! \cdot [j-1]_t \cdots [n-2]_t = [n-2]_t! \cdot [j-1]_t$$

and since there are $n-j$ many numbers which are bigger than j in the second row to the right of the first column, $b_j(T)$ is $n-j$. Thus we have

$$\tilde{G}_{(n-1,1)}(1,t) = \sum_{j=2}^{n} [n-2]_t! \cdot [j-1]_t \cdot \left(1 + t^{n-j}\right).$$

If we expand the sum and simplify,

$$\begin{aligned}
\tilde{G}_{(n-1,1)}(1,t) &= \sum_{j=2}^{n}[n-2]_t! \cdot [j-1]_t \cdot \left(1+t^{n-j}\right) \\
&= [n-2]_t! \left(\sum_{j=2}^{n}[j-1]_t \cdot (1+t^{n-j})\right) \\
&= \frac{[n-2]_t!}{t-1}\left(\sum_{j=2}^{n}(t^{j-1}-1)(1+t^{n-j})\right) \\
&= \frac{[n-2]_t!}{t-1}\left((t+\cdots+t^{n-1})+(n-1)(t^{n-1}-1)-(1+\cdots+t^{n-2})\right) \\
&= \frac{[n-2]_t!}{t-1}\cdot(t^{n-1}-1)\cdot n \\
&= [n-2]_t! \cdot [n-1]_t \cdot n \\
&= n \cdot [n-1]_t!. \qquad (2.1.6)
\end{aligned}$$

By comparing (2.1.5) and (2.1.6), we conclude that $\tilde{F}_{(n-1,1)}(1,t) = \tilde{G}_{(n-1,1)}(1,t)$.

In general, to compute $\tilde{G}_{(n-1,1)}(q,t)$, we only need to put q in front of t^{n-j} and simply we get

$$\tilde{G}_{(n-1,1)}(q,t) = \sum_{j=2}^{n}[n-2]_t![j-1]_t(1+qt^{n-j}). \qquad (2.1.7)$$

Now we compute $\tilde{F}_{(n-1,1)}(q,t)$. If 1 is the element of the cell in the second row, then there is no descent which makes maj zero, so as we calculated before, $t^{\mathrm{coinv}(\sigma,\mu')} = [n-1]_t!$. When 2 is in the second row, there are two different cases when 1 is right below 2 and when 3 or bigger numbers up to n comes below 2. In the first case, since the square containing 2 contributes a descent, it gives a q factor. So the whole factor becomes $q(t^{n-2} \cdot [n-2]_t!)$. In the second case, since there are no descents, the factor has only t's, and it gives us $\sum_{j=3}^{n} t^{n-j} \cdot [n-2]_t!$ in the end. In general, let's say the element i comes in the second row. Then we consider two cases when the element below i is smaller than i and when it is bigger than i. And these two different cases give us

$$q\left(t^{n-2} \cdot [n-2]_t! + \cdots + t^{n-i-1} \cdot [n-2]_t!\right) + \sum_{j=i+1}^{n} t^{n-j} \cdot [n-2]_t!.$$

The first term with q is from the first case, and the second with no q is from the second case. Lastly, when n comes in the second row, then the square in the second row always makes a descent and it gives $q([n-1]_t!)$. If we add all the possible terms, then we get

$$
\begin{aligned}
\tilde{F}_{(n-1,1)}(q,t) &= \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma,\mu')} t^{\mathrm{coinv}(\sigma,\mu')} \\
&= [n-1]_t! + \sum_{k=3}^{n} \left[q[n-2]_t! \left(\sum_{l=0}^{k-3} t^{n-2-l} \right) + \sum_{j=k}^{n} t^{n-j}[n-2]_t! \right] + q([n-1]_t!) \\
&= [n-2]_t! \left(\sum_{j=2}^{n} q t^{n-j} [j-1]_t + \sum_{j=2}^{n} [j-1]_t \right) \\
&= [n-2]_t! \sum_{j=2}^{n} [j-1]_t (1 + q t^{n-j}). \qquad (2.1.8)
\end{aligned}
$$

By comparing (2.1.8) to (2.1.7), we confirm that

$$\tilde{F}_{(n-1,1)}(q,t) = \tilde{G}_{(n-1,1)}(q,t).$$

□

2.1.2 General Case with Two Columns

Recursion

For the standard Young tableaux with two columns, i.e., of shape $\mu' = (n-s, s)$, there are two possible cases : the first case is when the cell with n is in the second row, and the second case is

Figure 2.1: (i) The first case (ii) The second case

when n comes in the end of the first row. Based on this observation, we can derive a recursion for $\tilde{G}_{(n-s,s)}(q,t)$: starting from the shape $(n-s, s-1)$, if we put the cell with n in the right most position in the second row, it gives $[s]_t$ factor since adding a cell in the second row increases the number of columns of height 2, from $s-1$ to s. And for b_j statistics, we get $b_{\mu'_2} = 0$ which contributes $(1+q)$ factor, because there is no elements in the first row which is bigger than n. So the first case contributes

$$[s]_t(1+q)\tilde{G}_{(n-s,s-1)}(q,t).$$

If the cell with n comes in the end of the first row, adding the cell with n in the end of the first row increases the number of height 1 columns, from $(n-2s-1)$ to $(n-2s)$, so it gives $a_n = [n-2s]_t$ factor. And having n in the right most position of the first row increases b_j's by 1. This can be expressed by changing q to qt, so in this case, we get

$$[n-2s]_t\tilde{G}_{(n-s-1,s)}(qt,t).$$

Therefore, the recursion formula for the tableaux with two rows is

$$\tilde{G}_{(n-s,s)}(q,t) = [s]_t(1+q)\tilde{G}_{(n-s,s-1)}(q,t) + [n-2s]_t\tilde{G}_{(n-s-1,s)}(qt,t).$$

Hence, we can prove Theorem 2.1.1 by showing that $\tilde{F}_{(n-s,s)}(q,t)$ satisfies the same recursion. To compare it to the Hilbert series $F_\mu(q,t)$ in (1.4.1), we do the transformations $\tilde{G}_{\mu'}(q,t) = G_\mu\left(\frac{1}{t};q\right)t^{n(\mu)}$, and get the recursion for $G_\mu(q,t)$

$$G_{(2^s,1^{n-2s})}(q,t) = [s]_t(1+q)G_{(2^{s-1},1^{n-2s+1})}(q,t) + [n-2s]_t t^s G_{(2^s,1^{n-2s-1})}(q/t,t). \tag{2.1.9}$$

Proof by recursion

We want to show that $F_{(2^a,q^{a-b})}(q,t)$, for $a=n-s, b=s$, satisfies the same recursion to (2.1.9), i,e.,

$$F_{(2^b,1^{a-b})}(q,t) = [b]_t(1+q)F_{(2^{b-1},1^{a-b+1})}(q,t) + [a-b]_t t^b F_{(2^b,1^{a-b-1})}(q/t,t). \tag{2.1.10}$$

Note that if ϕ is the Frobenius image of an S_n character χ, then the partial derivative $\partial_{p_1}\phi$[1] yields the Frobenius image of the restriction of χ to S_{n-1}. In particular, $\partial_{p_1}^n\tilde{H}_\mu$ must give the bigraded Hilbert series of μ, i.e., $F_\mu = \partial_{p_1}^n\tilde{H}_\mu$. Hence, (2.1.10) can be generalized as $\partial_{p_1}^{n-1}$ derivatives of the

[1]Here p_1 denotes the first power sum symmetric polynomial and $\partial_{p_1}\phi$ means the partial derivative of ϕ as a polynomial in the power symmetric function.

"Frobenius characteristic" recursion

$$\partial_{p_1}\tilde{H}_{(2^b,1^{a-b})}(q,t) = [b]_t(1+q)\tilde{H}_{(2^{b-1},1^{a-b+1})}(q,t) + [a-b]_t t^b \tilde{H}_{(2^b,1^{a-b-1})}(q/t,t). \tag{2.1.11}$$

A. Garsia proved (2.1.11) when he was visiting Penn. We provide the outline of his proof here. The proof will be divided into two separated parts. In the first part, we transform (2.1.11) into an equivalent simpler identity by eliminating common terms occurring in both sides. In the second part, we prove the simpler identity. The second part is based on the fact that the q,t-Kostka polynomials $K_{\lambda\mu}(q,t)$ for μ a two-column partition may be given as a completely explicit expression for all λ which is proved by Stembridge in [Ste94].

Proposition 2.1.5.

$$\partial_{p_1}\tilde{H}_{(2^b,1^{a-b})}(q,t) = [b]_t(1+q)\tilde{H}_{(2^{b-1},1^{a-b+1})}(q,t) + [a-b]_t t^b \tilde{H}_{(2^b,1^{a-b-1})}(q/t,t).$$

Proof. We can simplify the identity (2.1.11) by using Proposition 1.4.4 saying that we have the following identity

$$\partial_{p_1}\tilde{H}_\mu = \sum_{i=1}^{m} c_{\mu\nu^{(i)}}(q,t)\tilde{H}_{\nu^{(i)}}$$

where, by [GT96]

$$c_{\mu\nu^{(i)}}(q,t) = \frac{1}{(1-1/t)(1-1/q)}\frac{1}{x_i}\frac{\prod_{j=0}^{m}(x_i-u_j)}{\prod_{j=1,j\neq i}^{m}(x_i-x_j)}.$$

For the detailed description of x_i's and u_j's, see section 1.4. In the case with $\mu=(2^b,1^{a-b})$, the

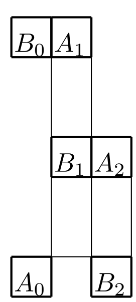


Figure 2.2: Diagram for $\mu=(2^b,1^{a-b})$.

diagram degenerates to Figure 2.2 with $A_0 = (-1,-1), A_1 = (1,a), A_2 = (2,b), B_0 = (-1,a), B_1 = (1,b), B_2 = (2,-1)$ and with weights

$$x_0 = 1/tq, \quad x_1 = t^{a-1}, \quad x_2 = t^{b-1}q$$

$$u_0 = t^{a-1}/q, \quad u_1 = t^{b-1}, \quad u_2 = q/t.$$

For our convenience, we set

$$\tilde{H}_{2^b1^{a-b}}(x;q,t) = \tilde{H}_{a,b}, \quad \tilde{H}_{2^b1^{a-b-1}}(x;q,t) = \tilde{H}_{a-1,b}, \quad \tilde{H}_{2^{b-1}1^{a-b+1}}(x;q,t) = \tilde{H}_{a,b-1},$$

$$c_{\mu\nu^{(1)}} = c_a, \quad c_{\mu\nu^{(2)}} = c_b.$$

Then by substituting the weights and c_a, c_b's and simplifying, we finally get

$$\partial_{p_1}\tilde{H}_{a,b} = \frac{(t^{a-b}-1)(t^a-q)}{(t-1)(t^{a-b}-q)}\tilde{H}_{a-1,b} + \frac{(t^b-1)(t^{a-b}-q^2)}{(t-1)(t^{a-b}-q)}\tilde{H}_{a,b-1}. \tag{2.1.12}$$

It turns out that the rational functions appearing in the right hand side can be unraveled by means of the substitutions

$$\tilde{H}_{a-1,b} = \phi_{a,b} + T_{a-1,b}\psi_{a,b}, \quad \tilde{H}_{a,b-1} = \phi_{a,b} + T_{a,b-1}\psi_{a,b}$$

with

$$T_{a-1,b} = t^{\binom{a-1}{2}+\binom{b}{2}}q^b, \quad T_{a,b-1} = t^{\binom{a}{2}+\binom{b-1}{2}}q^{b-1}.$$

By solving the above system of equations, we derive

$$\phi_{a,b} = \frac{T_{a,b-1}\tilde{H}_{a-1,b} - T_{a-1,b}\tilde{H}_{a,b-1}}{T_{a,b-1} - T_{a-1,b}}, \quad \psi_{a,b} = \frac{\tilde{H}_{a,b-1} - \tilde{H}_{a-1,b}}{T_{a,b-1} - T_{a-1,b}}$$

Using these expressions and simplifying expresses the recursion (2.1.12) in the following way :

$$\partial_{p_1}\tilde{H}_{a,b} = (1+q)[b]_t\tilde{H}_{a,b-1} + [a-b]_t(t^b\phi_{a,b} + T_{a-1,b}\psi_{a,b}).$$

Comparing this recursion to (2.1.11) reduces the problem to showing the following identity :

$$t^b\phi_{a,b} + T_{a-1,b}\psi_{a,b} = t^b\tilde{H}_{a-1,b}(q/t,t).$$

To make it more convenient, we shift the parameters a, b to $a+1$, b and prove the identity

$$t^b\phi_{a+1,b}+T_{a,b}\psi_{a+1,b}=t^b\tilde{H}_{a,b}(q/t,t). \tag{2.1.13}$$

To prove (2.1.13), we note the Macdonald specialization

$$\tilde{H}_\mu[1-x;q,t]=\prod_{i=1}^{l(\mu)}\prod_{j=1}^{\mu_i}(1-xt^{i-1}q^{j-1}) \qquad (\text{ where } l(\mu)=\mu_1') \tag{2.1.14}$$

which comes from the generalization of the Koornwinder-Macdonald reciprocity formula when $\lambda=0$. See [Hag08], [GHT99] for the explicit formula and the proof. For $\mu=(2^b,1^{a-b})$, (2.1.14) gives

$$\tilde{H}_{a,b}[1-x;q,t]=\prod_{i=1}^{a}(1-xt^{i-1})\times\prod_{i=1}^{n}(1-qxt^{i-1})=(x;t)_a(qx;t)_b. \tag{2.1.15}$$

For our convenience, we set $\tilde{H}_{a,b}(X;t)=\tilde{H}_{(2^b,1^{a-b})}(X;0,t)$. Then (2.1.15) reduces to

$$\tilde{H}_{a,b}(1-x;t)=(x;t)_a.$$

Since the Macdonald polynomials are triangularly related to the Hall-Littlewood polynomials, we have $\theta_s^{a,b}(q,t)$ satisfying

$$\tilde{H}_{a,b}(X;q,t)=\sum_{s=0}^{b}\tilde{H}_{a+s,b-s}(X;t)\theta_s^{a,b}(q,t). \tag{2.1.16}$$

By specializing the alphabet $X=1-x$, this relation becomes

$$(x;t)_a(qx;t)_b=\sum_{s=0}^{b}(x;t)_{a+s}\theta_s^{a,b}(q,t)$$

and (2.1.16) becomes

$$\tilde{H}_{a,b}(X;q,t)=\sum_{s=0}^{b}\tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_{a+s}}. \tag{2.1.17}$$

Making the shift in variables $a\to a+1$ and $b\to b-1$ in (2.1.17) gives

$$\tilde{H}_{a+1,b-1}(X;q,t)=\sum_{s=1}^{b}\tilde{H}_{a+s,b-s}(X;t)(x;t)_{a+1}(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}.$$

Using this and (2.1.17), we get

$$(t^{a-b+1}-q)\phi_{a+1,b}=t^{a-b+1}\tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b|_{(x;t)_a} \tag{2.1.18}$$
$$+\sum_{s=1}^{b}\tilde{H}_{a+s,b-s}(X;t)(t^{a-b+1}(x;t)_a(qx;t)_b-q(x;t)_{a+1}(qx;t)_{b-1})\Big|_{(x;t)_{a+s}}.$$

Now we have

$$\begin{aligned} & t^{a-b+1}(x;t)_a(qx;t)_b - q(x;t)_{a+1}(qx;t)_{b-1} \\ = \;& (x;t)_a(qx;t)_{b-1}(t^{a-b+1}(1-xqt^{b-1}) - q(1-xt^a)) \\ = \;& (t^{a-b+1}-q)(x;t)_a(qx;t)_{b-1} \end{aligned}$$

and (2.1.18) becomes

$$\begin{aligned} \phi_{a+1,b} = \frac{t^{a-b+1}}{t^{a-b+1}-q}\ \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a} \\ + \sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}. \end{aligned}$$

To get $T_{a,b}\psi_{a+1,b}$, we note that

$$\begin{aligned} \tilde{H}_{a,b}(X;q,t) - \tilde{H}_{a+1,b-1}(X;q,t) = \sum_{s=0}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_{a+s}} \\ - \sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_{a+1}(qx;t)_{b-1}\Big|_{(x;t)_{a+s}} \\ = \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a} + \sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}(-qxt^{b-1}+xt^a) \\ = \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a} + xt^{b-1}\sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}(-q+t^{a-b+1}) \end{aligned}$$

and then

$$\begin{aligned} T_{a,b}\psi_{a+1,b} = \frac{-q}{t^{a-b+1}-q}\ \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a} \\ -qxt^{b-1}\sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}. \end{aligned}$$

Note that (2.1.17) gives

$$t^b\tilde{H}_{a,b}(X;q/t,t) = (t^b - t^{b-1}qx)\sum_{s=0}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}. \tag{2.1.19}$$

To prove (2.1.13), we show that the right hand side of (2.1.19) can also be obtained by adding the right hand sides of the following two equalities.

$$t^b\phi_{a,b} = \frac{t^{a+1}}{t^{a-b+1}-q}\ \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a}$$

$$+t^b \sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}.$$

$$\psi_{a,b} = \frac{-q}{t^{a-b+1}-q}\ \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a}$$

$$-qxt^{b-1} \sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}.$$

Adding up the two above equalities gives

$$t^b\phi_{a,b} + \psi_{a,b} = \frac{t^{a+1}-q}{t^{a-b+1}-q}\ \tilde{H}_{a,b}(X;t)(x;t)_a(qx;t)_b\Big|_{(x;t)_a}$$

$$+(t^b - qxt^{b-1}) \sum_{s=1}^{b} \tilde{H}_{a+s,b-s}(X;t)(x;t)_a(qx;t)_{b-1}\Big|_{(x;t)_{a+s}}.$$

Comparing with the right hand side of (2.1.19) reduces us to proving the equality

$$\frac{t^{a+1}-q}{t^{a-b+1}-q}\ (x;t)_a(qx;t)_b|_{(x;t)_a} = (t^b - qxt^{b-1}) \sum_{s=1}^{b} (x;t)_a(qx;t)_{b-1}|_{(x;t)_a}$$

or, equivalently

$$(t^{a+1}-q)(x;t)_a(qx;t)_b - (t^{a-b+1}-q)(t^b-qxt^{b-1})(x;t)_a(qx;t)_{b-1}\big|_{(x;t)_a} = 0.$$

We can rewrite this in the form

$$((t^{a+1}-q)(1-qxt^{b-1}) - (t^{a-b+1}-q)(t^b-qxt^{b-1}))(x;t)_a(qx;t)_{b-1}\big|_{(x;t)_a} = 0. \qquad (2.1.20)$$

And a simple calculation gives

$$(t^{a+1}-q)(1-qxt^{b-1}) - (t^{a-b+1}-q)(t^b-qxt^{b-1}) = q(t^b-1)(1-xt^a)$$

and (2.1.20) becomes

$$q(t^b-1)(1-xt^a)(x;t)_a(qx;t)_{b-1}\big|_{(x;t)_a} = 0$$

or

$$q(t^b-1)(x;t)_{a+1}(qx;t)_{b-1}\big|_{(x;t)_a} = 0$$

which is clearly true. This completes the proof of (2.1.13) and thus the recursion (2.1.11) is established. □

Theorem 2.1.6.

$$F_{(2^a,1^{a-b})}(q,t) = G_{(2^a,1^{a-b})}(q,t).$$

Proof. The Frobenius characteristic recursion implies the recursion of the Hilbert series (2.1.10) and by comparing it to the recursion of $G_{(2^a,1^{a-b})}(q,t)$, (2.1.9), we can confirm that $F_{(2^a,1^{a-b})}(q,t)$ satisfies the same recursion. Based on the fact that $F_{(2,1)}(q,t) = G_{(2,1)}(q,t)$, having the same recursion implies that the two polynomials are the same. □

2.2 Hook Shape Case

In this section, we construct a combinatorial formula for the Hilbert series as a weighted sum over the standard Young tableaux of shape μ when μ has a hook shape and present proofs in three different ways.

Let $\mu' = (n-s, 1^s)$ and define

$$\tilde{G}_{(n-s,1^s)}(q,t) = \sum_{T\in \mathrm{SYT}(\mu')} \prod_{i=1}^{n} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)}\right)$$

where $a_i(T)$ is calculated in the same way as before, and $\alpha_j(T)$ is the number of cells in the first row with column height 1 (i.e., strictly to the right of the first column) which contain bigger element than the element in the $(\mu_1 - j + 1, 1)$ cell. Then for $\tilde{F}_{(n-s,1^s)}(q,t) = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma,\mu')} t^{\mathrm{coinv}(\sigma,\mu')}$, we have the following theorem :

Theorem 2.2.1.

$$\tilde{F}_{(n-s,1^s)}(q,t) = \tilde{G}_{(n-s,1^s)}(q,t).$$

Before we go into the details, we note a little fact :

Proposition 2.2.2. *In one column tableaux, of shape $\mu = (1^{n+1})$, all the permutations in the column fixing the number, say k, in the bottom $(1,1)$ cell gives*

$$\sum_{\substack{\sigma\in S_n,\\ k\ in\ (1,1)}} q^{maj(\sigma)} = q^{n+1-k}[n]_q!$$

where $n+1-k$ means the number of elements in the column above k which are bigger than k.

Proof. We prove by induction. When $n=2$, there are three possibilities for k : if $k=1$, then putting $1,2,3$ from the bottom gives q^3 and $1,3,2$ gives q^2 so adding them up gives

$$\sum_{\sigma\in S_2, 1 \text{ in } (1,1)} q^{\mathrm{maj}(\sigma)} = q^2(1+q) = q^2[2]_q.$$

If $k=2$, then $2,3,1$ from the bottom gives q^2 and $2,1,3$ gives q, so

$$\sum_{\sigma\in S_2, 2 \text{ in } (1,1)} q^{\mathrm{maj}(\sigma)} = q(1+q) = q[2]_q.$$

Similarly, when $k=3$, we have

$$\sum_{\sigma\in S_2, 3 \text{ in } (1,1)} q^{\mathrm{maj}(\sigma)} = 1+q = [2]_q.$$

Now we assume that for $\mu=(1^n)$, $\sum_{\sigma\in S_{n-1}, k \text{ in } (1,1)} q^{\mathrm{maj}(\sigma)} = q^{n-k}[n-1]_q!$ is true. First of all, if 1 is in the $(1,1)$ cell, then no matter what element comes above 1 it makes a descent, so it always contributes n to the major index maj. Then we consider the permutations in the column above 1. If 2 comes above 1, then the permutations of $n-1$ numbers not including 2 give $q^{n-1}[n-1]_q!$ by the induction hypothesis. If 3 comes above 1, then the sum of q^{maj} over the permutations of $n-1$ numbers is $q^{n-2}[n-1]_q!$. As the number above 1 gets larger by 1, the exponent of q in front of $[n-1]_q!$ decreases by 1, and finally when $n+1$ comes in the $(2,1)$ cell, we get $[n-1]_q!$. Adding them all gives

$$\sum_{\substack{\sigma\in S_n,\\ 1 \text{ in } (1,1)}} q^{\mathrm{maj}(\sigma)} = q^n(1+q+\cdots+q^{n-1})[n-1]_q! = q^n[n]_q!.$$

Secondly, we consider when $n+1$ is in the $(1,1)$ cell. Then it doesn't make descent no matter what element comes above it since $n+1$ is the biggest possible number, and the permutations of the column above $(1,1)$ cell is the same. Hence, this case gives

$$\sum_{\substack{\sigma\in S_n,\\ n+1 \text{ in } (1,1)}} q^{\mathrm{maj}(\sigma)} = (1+q+\cdots+q^{n-1})[n-1]_q! = [n]_q!.$$

In general, let's say k, $1 < k < n+1$, is in the $(1,1)$ cell. Then, if one of the numbers in $\{1, 2, \ldots, k-1\}$ comes above k, it doesn't make a descent. So the permutations of the upper $n-1$ numbers will give

$$q^{n-1}[n-1]_q!, \quad q^{n-2}[n-1]_q!, \quad \ldots, \quad q^{n-k+1}[n-1]_q!$$

as the elements above k changes from 1 to $k-1$. But if one of the numbers in $\{k+1, \ldots, n+1\}$ comes above k, then it makes a descent and it contributes n to maj. So the permutations of $n-1$ numbers in the column higher than row 2 give

$$q^{2n-k}[n-1]_q!, \quad q^{2n-k-1}[n-1]_q!, \quad \ldots, \quad q^{n}[n-1]_q!$$

as the number above k changes from $k+1$ to $n+1$. Then, adding them all up gives

$$\begin{aligned} \sum_{\sigma \in S_n, k \text{ in } (1,1)} q^{\mathrm{maj}(\sigma)} &= (q^{n-1} + \cdots q^{n-k+1} + q^{2n-k} + \cdots + q^{n})[n-1]_q! \\ &= q^{n-k+1}(1 + q + \cdots + q^{n-1})[n-1]_q! \\ &= q^{n+1-k}[n]_q! \end{aligned}$$

which proves the proposition. □

2.2.1 Proof of Theorem 2.2.1

When $s = 2$:

We start from a simple case when $s = 2$ first. Consider standard Young tableaux having k and j in the first column from the top, and calculate $\tilde{G}_{(n-s,1^2)}(q,t)$:

k								
j								
1	$\cdots$	$j-1$	$j+1$	$\cdots$	$k-1$	$k+1$	$\cdots$	n

$$\tilde{G}_{(n-2,1^2)}(q,t) = (1+q)\sum_{j=2}^{n-1}\sum_{k=j+1}^{n}[n-3]_t![j-1]_t(1 + qt^{n-k} + q^2t^{n-j-1}).$$

Note that this is equal to

$$\tilde{G}_{(n-2,1^2)}(q,t) = [n-3]_t![2]_q \sum_{j=2}^{n-1} [j-1]_t \left((n-j)(1+q^2t^{n-j-1}) + q[n-j]_t\right). \tag{2.2.1}$$

On the other hand, we calculate $\tilde{F}_{(n-2,1^2)}(q,t)$ by calculating $q^{\mathrm{maj}}t^{\mathrm{coinv}}$ and summing over all $n!$ permutations. Note that permutation of the numbers in the first row, from the second column to the last, gives $[n-3]_t!$ and we don't get any q factor from that part. Similarly, by Proposition 2.2.2, permutation of the top two cells in the first column gives $(1+q)$ factor, and it doesn't change any t factor. So we can factor out $[n-3]_t!(1+q)$ term, and think about the changes in q factor as we permute numbers in the first column. First, if we have 1 in the cell with $(1,1)$ coordinate, then no matter what number we have in $(1,2)$ coordinate cell, it makes a descent which contributes 1 to maj. Since there are $\binom{n-1}{2}$ many possible choices for the numbers in the top two cells of the first column, and 1 itself has $(n-3)$ coinversions, we get

$$\binom{n-1}{2} t^{n-3} q^2$$

in this case. If 2 comes in $(1,1)$ coordinate cell, then we could have 1 either in the first column upper than 2, or in the first row to the right of $(1,1)$ cell. If 1 is in the first column, we only have 1 maj in the first column, so it gives q factor. And there are $(n-2)$ possible choices for the other number in the first column, and 2 contributes $(n-3)$ coinversions. So this case gives $(n-2)t^{n-3}q$. If we have 1 in the first row, then both of two numbers in the first column are bigger than 2 and this gives q^2 factor. There are $\binom{n-2}{2}$ choices for them and 2 has $(n-4)$ inversions, so this case gives $\binom{n-2}{2}t^{n-4}q^2$. Adding them up, in the case when we have 2 in $(1,1)$ cell, we get

$$(n-2)t^{n-3}q + \binom{n-2}{2} t^{n-4} q^2.$$

If we have 3 in the $(1,1)$ coordinate cell, then we have three different cases :

(i) we have both of 1 and 2 in the first column,

(ii) we have either 1 or 2 in the first column and another bigger than 3,

(iii) we have two numbers bigger than 3 in the first column.

In case (i), we don't get any q factors, and 3 has $(n-4)$ coinversions, so we just get t^{n-3}. In case (ii), one number which is bigger than 3 gives q factor, and there are $2(n-3)$ possible choices for the numbers in the top two cells in the first column. And t has $(n-4)$ coinversions, so this case gives us $2(n-3)t^{n-4}q$. In case (iii), since we have two bigger numbers than 3 in the top two cells above 3, it gives q^2 factor, and this case 3 has $(n-5)$ many coinversions, so this case gives $\binom{n-3}{2}t^{n-5}q^2$. By summing up all the possible cases when 3 is in the $(1,1)$ cell, we get

$$t^{n-3} + 2(n-3)t^{n-4}q + \binom{n-3}{2}t^{n-5}q^2.$$

In general, when we have i in $(1,1)$ coordinate cell, for $3 \le i \le n-2$, considering all three different cases, we get

$$\binom{i-1}{2}t^{n-i} + (i-1)(n-i)t^{n-i-1}q + \binom{n-i}{2}t^{n-i-2}q^2.$$

If $(n-1)$ comes in the $(1,1)$ cell, then we don't have the third case, so we only get

$$\binom{n-2}{2}t + (n-2)q$$

and finally if we have n in $(1,1)$ cell, then all the numbers are smaller than n, so we only have $\binom{n-1}{2}$. Now we sum up all the possible terms, and get

$$\tilde{F}_{(n-2,1^2)}(q,t) = [n-3]_t![2]_q$$
$$\times \left(\sum_{j=2}^{n-1} \binom{j}{2} t^{n-j-1} + \sum_{j=2}^{n-1} (j-1)(n-j)t^{n-j-1}q + \sum_{j=2}^{n-1} \binom{n-j+1}{2} t^{n-j-1}q^2 \right).$$

Note that

$$\begin{aligned} \sum_{j=2}^{n-1} \binom{j}{2} t^{n-j-1} &= \sum_{j=2}^{n-1} \left(\sum_{k=1}^{j-1} k \right) t^{n-j-1} = \sum_{j=2}^{n-1} (n-j) \left(\sum_{k=1}^{j-1} t^{k-1} \right) \\ &= \sum_{j=2}^{n-1} (n-j)[j-1]_t, \end{aligned} \tag{2.2.2}$$

$$\sum_{j=2}^{n-1}[j-1]_t[n-j]_t$$
$$= (n-2)\cdot 1+(n-3)\cdot 2t+\cdots+(j-1)(n-j)t^{n-j-1}+\cdots+(n-2)t^{n-3}$$
$$= \sum_{j=2}^{n-1}(j-1)(n-j)t^{n-j-1}, \tag{2.2.3}$$

and

$$\sum_{j=2}^{n-1}\binom{n-j+1}{2}t^{n-j-1} = \sum_{j=2}^{n-1}\left(\sum_{k=1}^{n-j}k\right)t^{n-j-1}$$
$$= (1+2+\cdots+n-2)t^{n-3}+\cdots+(1+2+3)t^2+(1+2)t+1$$
$$= (n-2)t^{n-3}+(n-3)(1+t)t^{n-4}+\cdots+1\cdot[n-2]_t$$
$$= \sum_{j=2}^{n-1}[j-1]_t(n-j)t^{n-j-1}. \tag{2.2.4}$$

By using the above three identities (2.2.2), (2.2.3) and (2.2.4), $\tilde{F}_{(n-2,1^2)}(q,t)$ is equal to

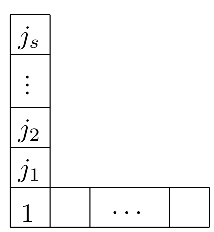
$$\tilde{F}_{(n-2,1^2)}(q,t) = [n-3]_t![2]_q\sum_{j=2}^{n-1}[j-1]_t\left((n-j)(1+q^2t^{n-j-1})+q[n-j]_t\right). \tag{2.2.5}$$

We compare this result to $\tilde{G}_{(n-2,1^2)}(q,t)$ in (2.2.1) and confirm that (2.2.1) and (2.2.5) are the same, i.e.,

$$\tilde{F}_{(n-2,1^2)}(q,t) = \tilde{G}_{(n-2,1^2)}(q,t).$$

For general $s \geq 2$:

Now we consider when $\mu' = (n-s, 1^s)$ with $s \geq 2$.



For standard tableaux with $j_s, \ldots, j_1$ in the first column from the top, we calculate $\tilde{G}_{(n-s,1^s)}(q,t)$:

$$\tilde{G}_{(n-s,1^s)}(q,t) = [n-s-1]_t![s]_q!\sum_{j_1=2}^{n-s+1}\sum_{j_2=j_1+1}^{n-s+2}\cdots$$

$$\cdots \sum_{j_s=j_{s-1}+1}^{n} [j_1-1]_t \left(1+qt^{n-j_s}+q^2t^{n-j_{s-1}-1}+\cdots+q^st^{n-j_1-(s-1)}\right). \tag{2.2.6}$$

On the other hand, to calculate $\tilde{F}_{(n-s,1^s)}(q,t)$, we vary the element in $(1,1)$ cell and calculate all the possible monomials, as we did in the previous case with $s=2$. By Proposition (2.2.2), we know that the permutations in the first row not including the very first cell give $[n-s-1]_t!$ factor, and from the permutations in the first column not including $(1,1)$ cell, we get $[s]_q!$ factor. To see the general case, let's say we have i in $(1,1)$ coordinate cell. If there are j many elements in the first column which are smaller than i, then there are $(s-j)$ many numbers in the first column which are bigger than i, so it gives q^{s-j} factor, and there are $n-i-(s-j)$ many numbers bigger than i in the first row, so it gives $t^{n-i-(s-j)}$ factor. For the elements in the first column excluding the $(1,1)$ cell, we have $\binom{i-1}{j}\binom{n-i}{s-j}$ many choices. Thus this case gives

$$[n-s-1]_t![s]_q!\binom{i-1}{j}\binom{n-i}{s-j}t^{n-i-(s-j)}q^{s-j},$$

where j changes from 0 to s, and i changes from 1 to n. So by considering all the permutations of n numbers, we get

$$\tilde{F}_{(n-s,1^s)}(q,t) = [n-s-1]_t![s]_q!\sum_{i=1}^{n}\left(\sum_{j=0}^{s}\binom{i-1}{j}\binom{n-i}{s-j}t^{n-i-(s-j)}q^{s-j}\right).$$

To show that this is the same to $\tilde{G}_{(n-s,1^s)}(q,t)$ in (2.2.6), we only need to show that the coefficients of q^k match, for $0\le k\le s$. In other words, we need to prove the following identity

$$\sum_{i=1}^{n}\binom{i-1}{j}\binom{n-i}{s-j}t^{n-i-(s-j)} = \sum_{j_1=2}^{n-s+1}\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{j_s=j_{s-1}+1}^{n}[j_1-1]_t t^{n-j_{s-j+1}-j+1} \tag{2.2.7}$$

for all $0\le j\le s$.

We first consider when $j=0$. Then what we want to show is the following :

$$\begin{aligned}\sum_{i=s+1}^{n}\binom{i-1}{s}t^{n-i} &= \sum_{j_1=2}^{n-s+1}\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{j_s=j_{s-1}+1}^{n}[j_1-1]_t \\ &= \sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{j_s=j_{s-1}+1}^{n}1. \end{aligned} \tag{2.2.8}$$

Lemma 2.2.3.

$$\sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} 1 = \binom{n-j_1}{s-1}.$$

Proof.

$$\begin{aligned}
\sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{\substack{j_{s-2}=\\ j_{s-3}+1}}^{n-2} \sum_{\substack{j_{s-1}=\\ j_{s-2}+1}}^{n-1} \sum_{\substack{j_s=\\ j_{s-1}+1}}^{n} 1 &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{\substack{j_{s-2}=\\ j_{s-3}+1}}^{n-2} \sum_{\substack{j_{s-1}=\\ j_{s-2}+1}}^{n-1} (n-j_{s-1}) \\
&= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-2}=j_{s-3}+1}^{n-2} \binom{n-j_{s-2}}{2}.
\end{aligned}$$

Note the following identity

$$\sum_{r=m}^{n} \binom{r}{k} = \binom{n+1}{k+1} - \binom{m}{k+1}. \tag{2.2.9}$$

By using the above identity, the inner most summation becomes

$$\sum_{j_{s-2}=j_{s-3}+1}^{n-2} \binom{n-j_{s-2}}{2} = \sum_{j_{s-2}=2}^{n-j_{s-3}-1} \binom{j_{s-3}}{2} = \binom{n-j_{s-3}}{3}.$$

We apply the above identity (2.2.9) repeatedly until we get

$$\sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} 1 = \sum_{j_2=j_1+1}^{n-s+2} \binom{n-j_2}{s-2} = \sum_{j_2=s-2}^{n-j_1-1} \binom{j_2}{s-2} = \binom{n-j_1}{s-1}$$

which proves the lemma. ☐

By applying Lemma (2.2.3) , the right hand side of (2.2.8) becomes

$$\sum_{j_1=2}^{n-s+1} [j_1-1]_t \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} 1 = \sum_{j_1=2}^{n-s+1} [j_1-1]_t \binom{n-j_1}{s-1}$$

and

$$\begin{aligned}
&\sum_{j_1=2}^{n-s+1} [j_1-1]_t \binom{n-j_1}{s-1} \\
= \;& \binom{n-2}{s-1} + (1+t)\binom{n-3}{s-1} + \cdots + (1+t+\cdots+t^{n-s-1}) \\
= \;& \sum_{j=s-1}^{n-2} \binom{j}{s-1} + \sum_{j=s-1}^{n-3} \binom{j}{s-1} t + \cdots + \binom{s-1}{s-1} t^{n-s-1} \\
= \;& \sum_{k=0}^{n-s-1} \sum_{j=s-1}^{n-2-k} \binom{j}{s-1} t^k = \sum_{k=0}^{n-s-1} \binom{n-1-k}{s} t^k \\
= \;& \sum_{i=s+1}^{n} \binom{i-1}{s} t^{n-i}.
\end{aligned}$$

This is the left hand side of (2.2.8), hence it shows that the coefficients of q^0 are equal.

Secondly we consider when $j = 1$. In this case, what we want to show is the following identity :

$$\begin{aligned} \sum_{i=s}^{n} \binom{i-1}{s-1} (n-i) t^{n-i-1} &= \sum_{j_1=2}^{n-s+1} \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} [j_1 - 1]_t t^{n-j_s} \\ &= \sum_{j_1=2}^{n-s+1} [j_1 - 1]_t \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} t^{n-j_s}. \end{aligned} \tag{2.2.10}$$

Lemma 2.2.4.

$$\sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} t^{n-j_s} = \sum_{k=0}^{n-j_1-s+1} \binom{n-j_1-1-k}{s-2} t^k.$$

Proof.

$$\begin{aligned} \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} t^{n-j_s} &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-1}=j_{s-2}+1}^{n-1} [n - j_{s-1}]_t \\ &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-2}=j_{s-3}+1}^{n-2} \sum_{k=0}^{n-j_{s-2}-2} (n - j_{s-2} - 1 - k) t^k \\ &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-3}=j_{s-4}+1}^{n-3} \sum_{k=0}^{n-j_{s-3}-3} \left(\sum_{r=1}^{n-j_{s-3}-2-k} r \right) t^k \\ &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-3}=j_{s-4}+1}^{n-3} \sum_{k=0}^{n-j_{s-3}-3} \binom{n-j_{s-3}-1-k}{2} t^k \\ &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-4}=j_{s-5}+1}^{n-4} \sum_{k=0}^{n-j_{s-4}-4} \left(\sum_{r=2}^{n-j_{s-4}-2-k} \binom{r}{2} \right) t^k \\ &= \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_{s-4}=j_{s-5}+1}^{n-4} \sum_{k=0}^{n-j_{s-4}-4} \binom{n-j_{s-4}-1-k}{3} t^k. \end{aligned}$$

Note that we are applying the previous identity (2.2.9) to get the binomial coefficients in each step. Keep applying (2.2.9) until we get

$$\begin{aligned} \sum_{j_2=j_1+1}^{n-s+2} \cdots \sum_{j_s=j_{s-1}+1}^{n} t^{n-j_s} &= \sum_{j_2=j_1+1}^{n-s+2} \sum_{k=0}^{n-j_2-(s-2)} \binom{n-j_2-1-k}{s-3} t^k \\ &= \sum_{k=0}^{n-j_1-(s-1)} \binom{n-j_1-1-k}{s-2} t^k. \end{aligned}$$

This finishes the proof. □

By applying Lemma 4.2, the right hand side of (2.2.10) becomes

$$\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{k=0}^{n-j_1-(s-1)}\binom{n-j_1-1-k}{s-2}t^k=\sum_{k=0}^{n-s-1}(k+1)\left(\sum_{r=s-2}^{n-3-k}\binom{r}{s-2}\right)t^k$$

$$=\sum_{k=0}^{n-s-1}(k+1)\binom{n-2-k}{s-1}t^k=\sum_{i=s}^{n-1}(n-i)\binom{i-1}{s-1}t^{n-i-1},$$

which is the left hand side of (2.2.10) and so this proves the case when $j=1$.

Now we compare the coefficients of q^j for general $0<j\leq s$, and we want to show

$$\sum_{i=s-j+1}^{n-j}\binom{i-1}{s-j}\binom{n-i}{j}t^{n-i-j}=\sum_{j_1=2}^{n-s+1}\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{j_s=j_{s-1}+1}^{n}[j_1-1]_t t^{n-j_{s-j+1}-j+1}. \qquad (2.2.11)$$

Note that the right hand side of (2.2.11) is equal to

$$\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{j_{s-j+1}=j_{s-j}+1}^{n-j+1}t^{n-j_{s-j+1}-j+1}\sum_{j_{s-j+2}=j_{s-j+1}+1}^{n-j+2}\cdots\sum_{j_s=j_{s-1}+1}^{n}1.$$

By applying the lemma (2.2.3),

$$\sum_{j_{s-j+2}=j_{s-j+1}+1}^{n-j+2}\cdots\sum_{j_s=j_{s-1}+1}^{n}1=\binom{n-j_{s-j+1}}{j-1}$$

so we can simplify the right hand side of (2.2.11) to

$$\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{\substack{j_{s-j+1}\\=j_{s-j}+1}}^{n-j+1}\binom{n-j_{s-j+1}}{j-1}t^{n-j_{s-j+1}-j+1}$$

$$=\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{\substack{j_{s-j}\\=j_{s-j-1}+1}}^{n-j}\left(1+\cdots+\binom{n-j_{s-j}-1}{j-1}t^{n-j_{s-j}-j}\right)$$

$$=\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{\substack{j_{s-j-1}\\=j_{s-j-2}+1}}^{n-j-1}\sum_{k=0}^{\substack{n-j_{s-j-1}\\-1-j}}\binom{j-1+k}{j-1}(n-j_{s-j-1}-j-k)t^k$$

$$=\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{\substack{j_{s-j-2}\\=j_{s-j-3}+1}}^{n-j-2}\sum_{k=0}^{\substack{n-j_{s-j-2}\\-2-j}}\binom{j-1+k}{j-1}\binom{n-j_{s-j-2}-j-k}{2}t^k$$

$\cdots$

$$=\sum_{j_1=2}^{n-s+1}[j_1-1]_t\sum_{k=0}^{n-j_1+1-s}\binom{j-1+k}{j-1}\binom{n-j_1-j-k}{s-j-1}t^k$$

$$= \sum_{k=0}^{n-s-1} \left(\sum_{r=s-j-1}^{n-j-2-k} \binom{r}{s-j-1} \right) \left(\sum_{r=j-1}^{j+k-1} \binom{r}{j-1} \right) t^k$$

$$= \sum_{k=0}^{n-s-1} \binom{n-j-1-k}{s-j} \binom{j+k}{j} t^k = \sum_{i=s-j+1}^{n-j} \binom{i-1}{s-j} \binom{n-i}{j} t^{n-i-j}.$$

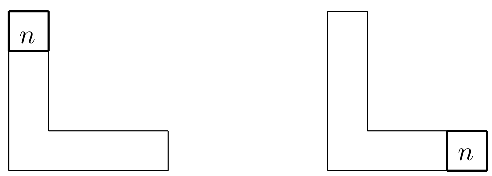
The last line of the equation is equal to the left hand side of (2.2.11), hence this proves that the coefficients of q^j, $0 \le j \le s$, are equal. Therefore, we proved that

$$\tilde{F}_{(n-s,1^s)}(q,t) = \tilde{G}_{(n-s,1^s)}(q,t)$$

for any s, and this finishes the proof of Theorem (2.2.1).

2.2.2 Proof by Recursion

We can derive a recursion formula in the hook shape case by fixing the position of the cell containing the largest number n.



Let's first start from a tableau of the shape $(n-s, 1^{s-1})$ with

$$\tilde{G}_{(n-s,1^{s-1})}(q,t) = \sum_{T \in \mathrm{SYT}((n-s,1^{s-1}))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s-1]_q! \left(1 + \sum_{j=1}^{s-1} q^j t^{\alpha_t(T)} \right)$$

and put the cell with n on the top of the first column. Then, since there is no other column with height $s+1$, adding the cell with n on the top of the first column gives $a_n(T) = 1$, hence it doesn't change the t-factor part of the above formula. Now as for the q part, we will have an additional factor of $[s]_q$, and all the q powers in the last parenthesis will be increased by 1 and it will have additional q from the top cell of the first column. The exponent of t with that q is 0 since n is the largest possible number. Hence, for the first case tableaux, the formula we get becomes

$$\sum_{T \in \mathrm{SYT}((n-s,1^{s-1}))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left[1 + q \left(1 + \sum_{j=1}^{s-1} q^j t^{\alpha_j(T)} \right) \right]$$

$$= \left(\sum_{T \in \mathrm{SYT}((n-s,1^{s-1}))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \right) + q[s]_q \tilde{G}_{(n-s,1^{s-1})}(q,t)$$

and in terms of $\tilde{G}_{(n-s,1^{s-1})}(q,t)$, this is equal to

$$[s]_q! \tilde{G}_{(n-s,1^{s-1})}(0,t) + q[s]_q \tilde{G}_{(n-s,1^{s-1})}(q,t). \tag{2.2.12}$$

For the second case, we start from a tableau of the shape $(n-s-1, 1^s)$ and add the cell with n in the end of the first row. This increases the number of columns with height 1 from $(n-s-2)$ to $(n-s-1)$, so it contributes the t factor $a_n = [n-s-1]_t$. Since it doesn't affect the first column, we don't get any extra q factor, but having the largest number n in the first row increases all α_j's by 1. In other words, if we let the formula for the shape $(n-s-1, 1^s)$ as

$$\tilde{G}_{(n-s-1,1^s)}(q,t) = \sum_{T \in \mathrm{SYT}((n-s-1,1^s))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)} \right),$$

then by adding the cell with n in the end of the first row, it becomes to

$$\sum_{T \in \mathrm{SYT}((n-s-1,1^s))} [n-s-1]_t \cdot \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)+1} \right)$$

$$= \sum_{T \in \mathrm{SYT}((n-s-1,1^s))} [n-s-1]_t \cdot \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left[t \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)} \right) + (1-t) \right].$$

Thus, in terms of $\tilde{G}_{(n-s-1,1^s)}$, this can be expressed as

$$t[n-s-1]_t \tilde{G}_{(n-s-1,1^s)}(q,t) + (1-t)[n-s-1]_t [s]_q! \tilde{G}_{(n-s-1,1^s)}(0,t). \tag{2.2.13}$$

Thus, we get the recursion for $\tilde{G}_{(n-s,1^s)}(q,t)$ by adding (2.2.12) and (2.2.13) :

$$\tilde{G}_{(n-s,1^s)}(q,t) = q[s]_q \tilde{G}_{(n-s,1^{s-1})}(q,t) + t[n-s-1]_t \tilde{G}_{(n-s-1,1^s)}(q,t)$$

$$+ [s]_q! (\tilde{G}_{(n-s,1^{s-1})}(0,t) + (1-t^{n-s-1}) \tilde{G}_{(n-s-1,1^s)}(0,t)).$$

To compare it to the recursion of the Hilbert series $F_\mu(q,t)$, we do the transformations $\tilde{G}_{\mu'}(q,t) = G_\mu \left(\frac{1}{t}, q \right) t^{n(\mu)}$, and get the final recursion of $G_\mu(q,t)$

$$G_{(s+1,1^{n-s-1})}(q,t) = q[s]_q G_{(s,1^{n-s-1})}(q,t) + [n-s-1]_t G_{(s+1,1^{n-s-2})}(q,t) \tag{2.2.14}$$

$$+[s]_q!(G_{(s,1^{n-s-1})}(0,t)+(t^{n-s-1}-1)G_{(s+1,1^{n-s-2})}(0,t)).$$

Simple calculation gives the following proposition.

Proposition 2.2.5.

$$[s]_q!(G_{(s,1^{n-s-1})}(0,t)+(t^{n-s-1}-1)G_{(s+1,1^{n-s-2})}(0,t)) = \binom{n-1}{s}t^{n-s-1}[n-s-1]_t![s]_q!. \tag{2.2.15}$$

Proof. By simple calculation, we find

$$G_{(s,1^{n-s-1})}(q,t) = t^{n-s-1}[n-s-1]_t![s-1]_q!\sum_{j_1=2}^{n-s+1}\cdots \cdots\sum_{j_{s-1}=j_{s-2}+1}^{n-1}\frac{[j_1-1]_t}{t^{j_1-2}}(1+qt^{-a_1}+\cdots+q^{s-1}t^{-a_{s-1}})$$

for appropriate a_i's. So

$$\begin{aligned}
G_{(s,1^{n-s-1})}(0,t) &= t^{n-s-1}[n-s-1]_t!\sum_{j_1=2}^{n-s+1}\cdots\sum_{j_{s-1}=j_{s-2}+1}^{n-1}\frac{[j_1-1]_t}{t^{j_1-2}} \\
&= t^{n-s-1}[n-s-1]_t!\sum_{j_1=2}^{n-s+1}\frac{[j_1-1]_t}{t^{j_1-2}}\left(\sum_{j_2=j_1+1}^{n-s+2}\cdots\sum_{j_{s-1}=j_{s-2}+1}^{n-1}1\right) \\
&= t^{n-s-1}[n-s-1]_t!\sum_{j_1=2}^{n-s+1}\frac{[j_1-1]_t}{t^{j_1-2}}\binom{n-1-j_1}{s-2} \\
&= t^{n-s-1}[n-s-1]_t!\sum_{i=s}^{n-1}\binom{i-1}{s-1}t^{-(n-1-i)} \\
&= [n-s-1]_t!\sum_{i=s}^{n-1}\binom{i-1}{s-1}t^{i-s}. \qquad (2.2.16)
\end{aligned}$$

Similarly, we have

$$G_{(s+1,1^{n-s-2})}(q,t) = t^{n-s-2}[n-s-2]_t![s]_q!\sum_{j_1=2}^{n-s}\cdots \cdots\sum_{j_s=j_{s-1}+1}^{n-1}\frac{[j_1-1]_t}{t^{j_1-2}}(1+qt^{-b_1}+\cdots+q^st^{-b_s})$$

for appropriate b_i's, and by plugging in $q=0$, we get

$$G_{(s+1,1^{n-s-2})}(0,t) = t^{n-s-2}[n-s-2]_t!\sum_{j_1=2}^{n-s}\frac{[j_1-1]_t}{t^{j_1-2}}\left(\sum_{j_2=j_1+1}^{n-s+1}\cdots\sum_{j_s=j_{s-1}+1}^{n-1}1\right)$$

$$= t^{n-s-2}[n-s-2]_t! \sum_{j_1=2}^{n-s} \frac{[j_1-1]_t}{t^{j_1-2}} \binom{n-1-j_1}{s-1}$$

$$= \quad t^{n-s-2}[n-s-2]_t! \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{-(n-1-i)}$$

$$= \quad [n-s-2]_t! \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{i-s-1}. \tag{2.2.17}$$

Using (2.2.16) and (2.2.17) in the left hand side of (2.2.15)

$$[s]_q![n-s-1]_t! \left(\sum_{i=s}^{n-1} \binom{i-1}{s-1} t^{i-s} + (t-1) \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{i-s-1} \right).$$

Note that the inside of the parenthesis simplifies as follows :

$$\begin{aligned}
&\sum_{i=s}^{n-1} \binom{i-1}{s-1} t^{i-s} + (t-1) \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{i-s-1} \\
= \quad & \sum_{i=s}^{n-1} \binom{i-1}{s-1} t^{i-s} + \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{i-s} - \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{i-s-1} \\
= \quad & \sum_{i=s+1}^{n-2} \left[\binom{i-1}{s-1} + \binom{i-1}{s} - \binom{i}{s} \right] t^{i-s} + \left[\binom{n-2}{s-1} + \binom{n-2}{s} \right] t^{n-s-1} \\
= \; & \binom{n-1}{s} t^{n-s-1}.
\end{aligned}$$

Therefore, we get that the left hand side of (2.2.15) is equal to

$$[s]_q![n-s-1]_t! \binom{n-1}{s} t^{n-s-1}$$

which is the right hand side of (2.2.15). This finishes the proof. □

Thus the recursion formula for $G_{(s+1,1^{n-s-1})}(q,t)$ in (2.2.14) becomes

$$G_\mu(q,t) = [n-s-1]_t G_{(s+1,1^{n-s-2})} + \binom{n-1}{s} t^{n-s-1}[n-s-1]_t![s]_q! + q[s]_q G_{(s,1^{n-s-1})}. \tag{2.2.18}$$

Garsia-Haiman Recursion

Now we note the Garsia-Haiman recursion introduced in [GH96]. To begin with, we need to imbed the collection of $\tilde{H}_\mu$'s into a larger family which includes $\tilde{H}$'s indexed by *broken hooks*.[2] More

[2] Diagrams which consist of the disjoint union of a row and a column.

precisely, we set

$$\begin{cases} a) \quad \tilde{H}_{[l,0,a]} = \tilde{H}_{1^l}(x;q,t)\tilde{H}_a(x;q,t) \\ b) \quad \tilde{H}_{[l,1,a]} = \tilde{H}_{b+1,1^a}(x;q,t) \end{cases}$$

Since $[l,1,a]$ represents a hook with leg l and arm a, we may visualize the symbol $[l,0,a]$ as the diagram obtained by removing the corner cell from $[l,1,a]$. Note that in (1.4.6), (1.4.7) and (1.4.8), we derived the following identities :

$$(a) \quad \tilde{H}_{[l,0,a]} = \frac{t^l-1}{t^l-q^a}\tilde{H}_{[l-1,1,a]} + \frac{1-q^a}{t^l-q^a}\tilde{H}_{[l,1,a-1]}. \tag{2.2.19}$$

$$(b) \quad \partial_{p_1}\tilde{H}_{[l,1,a]} = [l]_t\frac{t^{l+1}-q^a}{t^l-q^a}\tilde{H}_{[l-1,1,a]} + [a]_q\frac{t^l-q^{a+1}}{t^l-q^a}\tilde{H}_{[l,q,a-1]}. \tag{2.2.20}$$

Simple manipulations on identities (2.2.19) and (2.2.20) give the following expressions of $\partial_{p_1}\tilde{H}_{[l,1,a]}$.

Proposition 2.2.6.

$$(a) \quad \partial_{p_1}\tilde{H}_{[l,1,a]} = [l]_t\tilde{H}_{[l-1,1,a]} + t^l\tilde{H}_{[l,0,a]} + q[a]_q\tilde{H}_{[l,1,a-1]}. \tag{2.2.21}$$

$$(b) \quad \partial_{p_1}\tilde{H}_{[l,1,a]} = t[l]_t\tilde{H}_{[l-1,1,a]} + q^a\tilde{H}_{[l,0,a]} + [a]_q\tilde{H}_{[l,1,a-1]}. \tag{2.2.22}$$

Proof. Starting from (2.2.20),

$$\begin{aligned} \partial_{p_1}\tilde{H}_{[l,1,a]} &= [l]_t\frac{t^{l+1}-q^a}{t^l-q^a}\tilde{H}_{[l-1,1,a]} + [a]_q\frac{t^l-q^{a+1}}{t^l-q^a}\tilde{H}_{[l,q,a-1]} \\ &= [l]_t\frac{t^l-q^a+t^l(t-1)}{t^l-q^a}\tilde{H}_{[l-1,1,a]} + [a]_q\frac{q(t^l-q^a)+(1-q)t^l}{t^l-q^a}\tilde{H}_{[l,1,a-1]} \\ &= [l]_t\tilde{H}_{[l-1,1,a]} + t^l\left(\frac{t^l-1}{t^l-q^a}\tilde{H}_{[l-1,1,a]} + \frac{1-q^a}{t^l-q^a}\tilde{H}_{[l,1,a-1]}\right) + q[a]_q\tilde{H}_{[l,1,a-1]} \\ &= [l]_t\tilde{H}_{[l-1,1,a]} + t^l\tilde{H}_{[l,0,a]} + q[a]_q\tilde{H}_{[l,1,a-1]}. \end{aligned}$$

Similar manipulation on (2.2.20) gives (2.2.22). □

The Hilbert series of our module $M_{1^l,a+1}$ should be given by the expression [3]

$$F_{(1^l,a+1)}(q,t) = \partial_{p_1}^n\tilde{H}_{[l,1,a]}.$$

[3]Note that if ϕ is the Frobenius image of an S_n character χ, then the partial derivative $\partial_{p_1}\phi$ yields the Frobenius image of the restriction of χ to S_{n-1}. In particular, $\partial_{p_1}^n\tilde{H}_\mu$ must give the bigraded Hilbert series of μ, i.e., $F_\mu = \partial_{p_1}^n\tilde{H}_\mu$.

To this end $F_{(1^l,a+1)}(q,t)$ satisfies either of the following two recursions.

Proposition 2.2.7.

$$(a) \quad F_{[l,1,a]} = [l]_t F_{[l-1,1,a]} + \binom{l+a}{a} t^l [l]_t! [a]_q! + q[a]_q F_{[l,1,a-1]}. \tag{2.2.23}$$

$$(b) \quad F_{[l,1,a]} = t[l]_t F_{[l-1,1,a]} + \binom{l+a}{a} q^a [l]_t! [a]_q! + [a]_q F_{[l,1,a-1]}. \tag{2.2.24}$$

Proof. We derived that

$$F_{[l,0,a]} = \binom{l+a}{a} [l]_t! [a]_q!$$

in (1.4.10). Then (a) and (b) follow from Proposition 2.2.6. □

Note that for $\mu = (s+1, 1^{n-s-1})$, the recursion of the Hilbert series would be

$$F_\mu(q,t) = [n-s-1]_t F_{(s+1,1^{n-s-2})} + \binom{n-1}{s} t^{n-s-1} [n-s-1]_t! [s]_q! + q[s]_q F_{(s,1^{n-s-1})}. \tag{2.2.25}$$

Proof of Theorem 2.2.1

We compare the recursion of $G_{(s+1,1^{n-s-1})}(q,t)$ (2.2.18) to (2.2.25) and prove Theorem 2.2.1 by verifying the equality of them. By comparing (2.2.18) to the recursion of $F_\mu(q,t)$ in (2.2.25), we can confirm that $G_\mu(q,t)$ and $F_\mu(q,t)$ satisfy the same recursion formula for the hook shape partition $\mu = (s+1, 1^{n-s-1})$. Note that we already saw that $G_{(2,1)}(q,t) = F_{(2,1)}(q,t)$. Therefore, we have

$$G_{(s+1,1^{n-s-2})}(q,t) = F_{(s+1,1^{n-s-2})}(q,t)$$

and since $\tilde{F}_{\mu'}(q,t) = t^{n(\mu)} F_\mu\left(\frac{1}{t}, q\right)$, $\tilde{G}_{\mu'}(q,t) = t^{n(\mu)} G_\mu\left(\frac{1}{t}, q\right)$, finally we get

$$\tilde{G}_{(n-s,1^s)}(q,t) = \tilde{F}_{(n-s,1^s)}(q,t)$$

and this finishes the proof of Theorem 2.2.1.

Remark 2.2.8. We have the way of expressing the Hilbert series $F_\mu(q,t)$ as a sum over the hook shape standard Young tableaux as follows :

$$F_\mu(q,t) = \sum_{T \in \mathrm{SYT}(\mu)} \prod_{i=1}^{n} [a_i(T)]_t [\mu_1 - 1]_q! \left(\sum_{j=1}^{\mu_1 - 1} q^{j-1} t^{b_j(T)} + q^{\mu_1 - 1} \right)$$

where $a_i(T)$ counts the number of rows having the same width with the row containing i as adding the cell i, from 1 to n, and $b_j(T)$ counts the number of cells in the first column in rows strictly higher than row 1 containing bigger numbers than the element in the cell $(1, j+1)$. This can be verified by the facts $\tilde{G}_{\mu'}(q,t) = \tilde{F}_{\mu'}(q,t)$ and $\tilde{F}_{\mu'}(q,t) = t^{n(\mu)} F_\mu\left(\frac{1}{t}, q\right)$.

2.2.3 Proof by Science Fiction

There is another way of deriving the recursion formula (2.2.25) for $F_\mu(q,t)$, $\mu = (s+1, q^{n-s-1})$ by using the theory of modules occurring in [BG99]. We consider the classical identity from Proposition 1.4.4

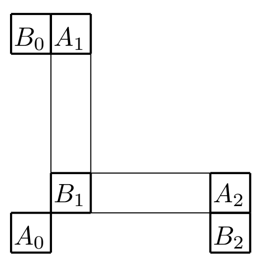
$$\partial_{p_1} \tilde{H}_\mu = \sum_{i=1}^{m} c_{\mu\nu^{(i)}}(q,t) \tilde{H}_{\nu^{(i)}}$$

where

$$c_{\mu\nu^{(i)}}(q,t) = \frac{1}{(1-1/t)(1-1/q)} \frac{1}{x_i} \frac{\prod_{j=0}^{m}(x_i - u_j)}{\prod_{j=1, j\neq i}^{m}(x_i - x_j)}.$$

In the hook shape case, we have

$$m = 2, \quad \mu = (s+1, 1^{n-s-1}), \quad \nu^{(1)} = (s+1, 1^{n-s-2}), \quad \nu^{(2)} = (s, 1^{n-s-1}).$$



Considering the diagram of a hook, the outer and inner corner cells are $A_0 = (-1,-1), A_1 = (1, n-s), A_2 = (s+1, 1)$ and $B_0 = (-1, n-s), B_1 = (1,1), B_2 = (s+1, -1)$, and the weights are

$$x_0 = 1/tq, \quad x_1 = t^{n-s-1}, \quad x_2 = q^s$$

and

$$u_0 = t^{n-s-1}/q, \quad u_1 = 1, \quad u_2 = q^s/t.$$

By plugging in these weights to $c_{\mu\nu^{(i)}}(q,t)$ and simplifying, we get

$$c_{\mu\nu^{(1)}}(q,t) = \frac{(t^{n-s-1}-1)(t^{n-s}-q^s)}{(t-1)(t^{n-s-1}-q^s)}, \quad c_{\mu\nu^{(2)}}(q,t) = \frac{(q^{s+1}-t^{n-s-1})(q^s-1)}{(q-1)(q^s-t^{n-s-1})}.$$

So the $\partial_{p_1}^{n-1}$ derivative of the Frobenius characteristic recursion gives

$$\begin{aligned}\partial_{p_1}\tilde{H}_{(s+1,1^{n-s-1})} = \frac{(t^{n-s-1}-1)(t^{n-s}-q^s)}{(t-1)(t^{n-s-1}-q^s)}\tilde{H}_{(s+1,1^{n-s-2})} \\ +\frac{(q^{s+1}-t^{n-s-1})(q^s-1)}{(q-1)(q^s-t^{n-s-1})}\tilde{H}_{(s,1^{n-s-1})}.\end{aligned}$$

The right hand side of the above recursion can be unraveled by means of the substitutions

$$\tilde{H}_{(s+1,1^{n-s-2})} = \phi_{(s+1,1^{n-s-1})} + T_{(s+1,1^{n-s-2})}\psi_{(s+1,1^{n-s-1})},$$

$$\tilde{H}_{(s,1^{n-s-1})} = \phi_{(s+1,1^{n-s-1})} + T_{(s,1^{n-s-1})}\psi_{(s+1,1^{n-s-1})}$$

where

$$T_{(s+1,1^{n-s-2})} = t^{\binom{n-s-1}{2}}q^{\binom{s+1}{2}}, \quad T_{(s,1^{n-s-1})} = t^{\binom{n-s}{2}}q^{\binom{s}{2}}.$$

By solving the system of equations we derive

$$\begin{aligned}\phi_{(s+1,1^{n-s-1})} &= \frac{T_{(s,1^{n-s-1})}\tilde{H}_{(s+1,1^{n-s-2})}(q,t) - T_{(s+1,1^{n-s-2})}\tilde{H}_{(s,1^{n-s-1})}}{T_{(s,1^{n-s-1})} - T_{(s+1,1^{n-s-2})}}, \\ \psi_{(s+1,1^{n-s-1})} &= \frac{\tilde{H}_{(s,1^{n-s-1})} - \tilde{H}_{(s+1,1^{n-s-2})}}{T_{(s,1^{n-s-1})} - T_{(s+1,1^{n-s-2})}}.\end{aligned}$$

By definition of T, we get

$$\frac{T_{(s,1^{n-s-1})}}{T_{(s+1,1^{n-s-2})}} = \frac{t^{\binom{n-s}{2}}q^{\binom{s}{2}}}{t^{\binom{n-s-1}{2}}q^{\binom{s+1}{2}}} = \frac{t^{n-s-1}}{q^s}$$

and using this expression simplifies ϕ and ψ to

$$\begin{aligned}\phi_{(s+1,1^{n-s-1})} &= \frac{t^{n-s-1}\tilde{H}_{(s+1,1^{n-s-2})} - q^s\tilde{H}_{(s,1^{n-s-1})}}{t^{n-s-1}-q^s}, \\ T_{(s,1^{n-s-1})}\psi_{(s+1,1^{n-s-1})} &= t^{n-s-1}\cdot\frac{\tilde{H}_{(s,1^{n-s-1})} - \tilde{H}_{(s+1,1^{n-s-2})}}{t^{n-s-1}-q^s}.\end{aligned}$$

Proposition 2.2.9.

$$\begin{aligned}\partial_{p_1}\tilde{H}_{(s+1,1^{n-s-1})} &= [n-s-1]_t\tilde{H}_{(s+1,1^{n-s-2})} + q[s]_q\tilde{H}_{(s,1^{n-s-1})} \\ &\quad +t^{n-s-1}\phi_{(s+1,1^{n-s-1})} + T_{(s,1^{n-s-1})}\psi_{(s+1,1^{n-s-1})}.\end{aligned}$$

Proof. Let's denote

$$\mu = (s+1, 1^{n-s-1}), \quad \nu^{(1)} = (s+1, 1^{n-s-2}), \quad \nu^{(2)} = (s, 1^{n-s-1}).$$

From above, we have the Frobenius characteristic recursion

$$\begin{aligned}
\partial_{p_1} \tilde{H}_\mu &= \frac{(t^{n-s-1}-1)(t^{n-s}-q^s)}{(t-1)(t^{n-s-1}-q^s)} \tilde{H}_{\nu^{(1)}} + \frac{(q^{s+1}-t^{n-s-1})(q^s-1)}{(q-1)(q^s-t^{n-s-1})} \tilde{H}_{\nu^{(2)}} \\
&= [n-s-1]_t \left(\frac{(t^{n-s-1}-q^s)+(t-1)t^{n-s-1}}{t^{n-s-1}-q^s} \right) \tilde{H}_{\nu^{(1)}} \\
&\qquad + [s]_q \left(\frac{q(q^s-t^{n-s-1})+t^{n-s-1}(q-1)}{q^s-t^{n-s-1}} \right) \tilde{H}_{\nu^{(2)}} \\
&= [n-s-1]_t \tilde{H}_{\nu^{(1)}} + q[s]_q \tilde{H}_{\nu^{(2)}} \\
&\qquad + \frac{t^{n-s-1}}{t^{n-s-1}-q^s} \left((t^{n-s-1}-1)\tilde{H}_{\nu^{(1)}} + (1-q^s)\tilde{H}_{\nu^{(2)}} \right) \\
&= [n-s-1]_t \tilde{H}_{\nu^{(1)}} + q[s]_q \tilde{H}_{\nu^{(2)}} \\
&\quad + t^{n-s-1} \left(\frac{t^{n-s-1}\tilde{H}_{\nu^{(1)}} - q^s \tilde{H}_{\nu^{(2)}}}{t^{n-s-1}-q^s} \right) + t^{n-s-1} \left(\frac{\tilde{H}_{\nu^{(1)}} - \tilde{H}_{\nu^{(2)}}}{t^{n-s-1}-q^s} \right) \\
&= [n-s-1]_t \tilde{H}_{\nu^{(1)}} + q[s]_q \tilde{H}_{\nu^{(2)}} + t^{n-s-1}\phi_\mu + T_{\nu^{(2)}}\psi_\mu.
\end{aligned}$$

∎

The recursion in Proposition 2.2.9 gives the recursion for the Hilbert series :

$$\begin{aligned}
F_{(s+1,1^{n-s-1})}(q,t) &= [n-s-1]_t F_{(s+1,1^{n-s-2})}(q,t) + q[s]_q F_{(s,1^{n-s-1})}(q,t) \\
&\quad + \frac{t^{n-s-1}}{t^{n-s-1}-q^s} \left((t^{n-s-1}-1) F_{(s+1,1^{n-s-2})} + (1-q^s) F_{(s,1^{n-s-1})} \right).
\end{aligned} \tag{2.2.26}$$

Proposition 2.2.10.

$$\begin{aligned}
&\frac{t^{n-s-1}}{t^{n-s-1}-q^s} \left((t^{n-s-1}-1) F_{(s+1,1^{n-s-2})} + (1-q^s) F_{(s,1^{n-s-1})} \right) \\
&\qquad = \binom{n-1}{s} t^{n-s-1} [n-s-1]_t! [s]_q!.
\end{aligned} \tag{2.2.27}$$

Proof. We calculate the Hilbert series by using the combinatorial formula for Macdonald polynomials in Theorem 1.3.7

$$F_\mu(q,t) = \sum_{S:\mu \to \mathbb{N}} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)}.$$

We fix an element in the $(1,1)$ cell, and think about all the possible different monomials. From the permutation of the first column not including the $(1,1)$ cell, we get $[n-s-1]_t!$ factor, and the permutation of the first row excluding the left most cell gives $[s-1]_q!$ factor. Having common factors $[n-s-1]_t![s-1]_q!$, assume that we have an element i in the $(1,1)$ coordinate cell. If there are j many elements in the first row which are smaller than i, then it gives q^j factor, and there are $n-i-(s-j)$ many elements bigger than i in the first column which gives $t^{n-i-s+j}$ factor. For the elements in the first row excluding $(1,1)$ cell, we have $\binom{i-1}{j}\binom{n-i}{s-j}$ choices. Therefore, adding up all the possible monomials gives

$$F_{(s+1,1^{n-s-1})}(q,t) = [n-s-1]_t![s]_q!\sum_{i=1}^{n}\sum_{j=0}^{s}\binom{i-1}{j}\binom{n-i}{s-j}t^{n-i-s+j}q^j$$

and thus

$$F_{(s+1,1^{n-s-2})}(q,t) = [n-s-2]_t![s]_q!\sum_{i=1}^{n-1}\sum_{j=0}^{s}\binom{i-1}{j}\binom{n-i-1}{s-j}t^{n-i-s+j-1}q^j,$$

$$F_{(s,1^{n-s-1})}(q,t) = [n-s-1]_t![s-1]_q!\sum_{i=1}^{n-1}\sum_{j=0}^{s-1}\binom{i-1}{j}\binom{n-i-1}{s-j-1}t^{n-i-s+j}q^j.$$

By using these formulas, the left hand side of (2.2.27) becomes

$$\begin{aligned}
&\frac{t^{n-s-1}}{t^{n-s-1}-q^s}\left((t^{n-s-1}-1)F_{(s+1,1^{n-s-2})}(q,t)+(1-q^s)F_{(s,1^{n-s-1})}(q,t)\right)\\
&= [n-s-1]_t![s]_q!\frac{t^{n-s-1}}{t^{n-s-1}-q^s}\Bigg((t-1)\sum_{i=1}^{n-1}\sum_{j=0}^{s}\binom{i-1}{j}\binom{n-i-1}{s-j}t^{n-i-s+j-1}q^j\\
&\qquad +(1-q)\sum_{i=1}^{n-1}\sum_{j=0}^{s-1}\binom{i-1}{j}\binom{n-i-1}{s-j-1}t^{n-i-s+j}q^j\Bigg).
\end{aligned}$$

Let's consider the inside of the parenthesis in the above equation and calculate each coefficient of q^k for $0 \le k \le s$. First of all, the coefficient of $q^0 = 1$ is

$$\begin{aligned}
&(t-1)\sum_{i=1}^{n-1}\binom{n-i-1}{s}t^{n-i-s-1}+\sum_{i=1}^{n-1}\binom{n-i-1}{s-1}t^{n-i-s}\\
&=\sum_{i=1}^{n-1}\binom{n-i-1}{s}t^{n-i-s}-\sum_{i=2}^{n}\binom{n-i}{s}t^{n-i-s}+\sum_{i=1}^{n-1}\binom{n-i-1}{s-1}t^{n-i-s}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\binom{n-2}{s} + \binom{n-2}{s-1}\right] t^{n-s-1} \\
&\qquad + \sum_{i=2}^{n-1} \left[\binom{n-i-1}{s} + \binom{n-i-1}{s-1} - \binom{n-i}{s}\right] t^{n-i-s} \\
&= \binom{n-1}{s} t^{n-s-1}.
\end{aligned}
$$

Similarly, the coefficient of q^s is

$$
\begin{aligned}
& (t-1)\sum_{i=1}^{n-1} \binom{i-1}{s} t^{n-i-1} - \sum_{i=1}^{n-1} \binom{i-1}{s-1} t^{n-i-1} \\
&= \sum_{i=s}^{n-2} \binom{i}{s} t^{n-i-1} - \sum_{i=s+1}^{n-1} \binom{i-1}{s} t^{n-i-1} - \sum_{i=s}^{n-1} \binom{i-1}{s-1} t^{n-i-1} \\
&= \sum_{i=s+1}^{n-2} \left[\binom{i}{s} - \binom{i-1}{s} - \binom{i-1}{s-1}\right] t^{n-i-1} - \left[\binom{n-2}{s} + \binom{n-2}{s-1}\right] \\
&= -\binom{n-1}{s}.
\end{aligned}
$$

Finally, for any $k \neq 0, s$, the coefficient of q^k is

$$
\begin{aligned}
& (t-1)\sum_{i=1}^{n-1} \binom{i-1}{k}\binom{n-i-1}{s-k} t^{n-i-s+k-1} + \sum_{i=1}^{n-1} \binom{i-1}{k}\binom{n-i-1}{s-k-1} t^{n-i-s+k} \\
&\qquad - \sum_{i=1}^{n-1} \binom{i-1}{k-1}\binom{n-i-1}{s-k} t^{n-i-s+k-1} \\
&= \sum_{i=1}^{n-1} \binom{i-1}{k}\binom{n-i-1}{s-k} t^{n-i-s+k} + \sum_{i=1}^{n-1} \binom{i-1}{k}\binom{n-i-1}{s-k-1} t^{n-i-s+k} \\
&\qquad - \sum_{i=1}^{n-1} \binom{i-1}{k}\binom{n-i-1}{s-k} t^{n-i-s+k-1} - \sum_{i=1}^{n-1} \binom{i-1}{k-1}\binom{n-i-1}{s-k} t^{n-i-s+k-1} \\
&= \sum_{i=1}^{n-1} \binom{i-1}{k}\binom{n-i}{s-k} t^{n-i-s+k} - \sum_{i=1}^{n-1} \binom{i}{k}\binom{n-i-1}{s-k} t^{n-i-s+k-1} \\
&= \sum_{i=1}^{n-2} \binom{i}{k}\binom{n-i-1}{s-k} t^{n-i-s+k-1} - \sum_{i=1}^{n-2} \binom{i}{k}\binom{n-i-1}{s-k} t^{n-i-s+k-1} \\
&= 0.
\end{aligned}
$$

Hence, the left hand side of (2.2.27) becomes

$$\begin{aligned} & [n-s-1]_t![s]_q!\frac{t^{n-s-1}}{t^{n-s-1}-q^s}\left(\binom{n-1}{s}t^{n-s-1}-\binom{n-1}{s}q^s\right) \\ = \quad & [n-s-1]_t![s]_q!t^{n-s-1}\binom{n-1}{s}\frac{t^{n-s-1}-q^s}{t^{n-s-1}-q^s} \\ = \quad & [n-s-1]_t![s]_q!t^{n-s-1}\binom{n-1}{s}, \end{aligned}$$

which is exactly the right hand side of (2.2.27). □

Using Proposition 2.2.10 in (2.2.26) gives the recursion for the Hilbert series (2.2.25), and we already showed that $G_\mu(q,t)$ satisfies the same recursion. Hence, this completes the proof of Theorem 2.2.1.

2.2.4 Association with Fillings

For the Hilbert series $F_\mu(q,t)$ of the Garsia-Haiman modules M_μ, we have shown that the sum of $n!$ many monomials can be calculated as a sum of polynomials over standard Young tableaux of shape μ. Noting that the number of SYT's of shape μ is $n!/\prod_{c\in\mu} h(c)$ where $h(c) = a(c) + l(c) + 1$, obviously the combinatorial formula for $G_\mu(q,t)$ reduces the number of tableaux that we need to consider. Since we showed that $F_\mu(q,t) = G_\mu(q,t)$, now we are interested in finding the correspondence between a group of fillings and a standard Young tableaux giving the same polynomial for the Hilbert series. In this section, we construct a grouping table which gives a way of grouping the sets of fillings corresponding SYT's, and introduce a modified Garsia-Procesi tree which makes complete bijection of the grouping table correspondence.

$\mu = (2, 1^{n-2})$ **case**

Recall the way of calculating the Hilbert series $F_\mu(q,t)$ over the hook shape standard Young tableaux :

$$F_\mu(q,t) = \sum_{T\in\mathrm{SYT}(\mu)} \prod_{i=1}^{n} [a_i(T)]_t [\mu_1 - 1]_q! \left(\sum_{j=1}^{\mu_1 - 1} q^{j-1} t^{b_j(T)} + q^{\mu_1 - 1} \right) \tag{2.2.28}$$

where $a_i(T)$ counts the number of rows having the same width with the row containing i as adding the cell i, from 1 to n, and $b_j(T)$ counts the number of cells in the first column in rows strictly higher than row 1 containing bigger numbers than the element in the cell $(1, j+1)$. From now on, we are going to use this formula to calculate the Hilbert series $F_\mu(q,t)$ of the hook shape tableaux. Since we are now considering the case when $\mu_1 = 2$, the formula simplifies to

$$F_{(2,1^{n-2})}(q,t) = \sum_{T \in \mathrm{SYT}((2,1^{n-2}))} \prod_{i=1}^{n} [a_i(T)]_t (t^{b_1(T)} + q)$$

and we can express it even more explicitly

$$F_{(2,1^{n-2})}(q,t) = [n-2]_t! \sum_{k=2}^{n} [k-1]_t (t^{n-k} + q)$$

where k denotes the element in the $(1,2)$ cell. Note that the permutation on the first column not including $(1,1)$ cell gives $[n-2]_t!$ factor, and the permutation of the first row determines whether it has q factor or not. So we are going to focus on the pair of elements in the first row, i.e., $(1,1)$ and $(1,2)$ cell. Keeping it mind that we have the $[n-2]_t!$ common factor, if we consider the change of t powers and q power, as we change the elements in the first row, we get Table 2.1. We interpret the Table 2.1 as follows : the monomial in (i,j) coordinate gives the factor in front of $[n-2]_t!$ when we have i in the $(1,1)$ cell and j in the $(1,2)$ cell, and add the monomials from all the possible fillings by permuting the first column above the first row. Now, we go back to the standard tableaux and consider one with $k, k \geq 2$, in $(1,2)$ cell. According to the above formula, this standard tableaux gives

$$[n-2]_t![k-1]_t(t^{n-k} + q)$$

which can be expressed as

$$[n-2]_t!((t^{n-2} + t^{n-3} + \cdots + t^{n-k}) + q(1 + t + t^2 + \cdots + t^{k-2})). \tag{2.2.29}$$

We look at the table again and notice that the terms in the k^{th} column above the diagonal give the first part of the above polynomial (2.2.29) with no q factor, and the terms in the $(n-(k-1))^{\text{th}}$ column below the diagonal give the rest part combined with q. In other words, by combining the

$\overset{(1,1)}{\text{element}} \setminus \overset{(1,2)}{\text{element}}$	1	2	3	$\cdots$	k	$\cdots$	$n-(k-1)$	$\cdots$	$n-1$	n
1	$\diagdown$	t^{n-2}	t^{n-2}	$\cdots$	t^{n-2}	$\cdots$	t^{n-2}	$\cdots$	t^{n-2}	t^{n-2}
2	qt^{n-2}	$\diagdown$	t^{n-3}	$\cdots$	t^{n-3}	$\cdots$	t^{n-3}	$\cdots$	t^{n-3}	t^{n-3}
3	qt^{n-3}	qt^{n-3}	$\diagdown$	$\cdots$	t^{n-4}	$\cdots$	t^{n-4}	$\cdots$	t^{n-4}	t^{n-4}
$\cdots$	$\cdots$		$\cdots$		$\cdots$		$\cdots$		$\cdots$	$\cdots$
$k-1$	$qt^{n-(k-1)}$	$\cdots$	$qt^{n-(k-1)}$	$\diagdown$	t^{n-k}	$\cdots$	t^{n-k}	$\cdots$	t^{n-k}	t^{n-k}
$\cdots$	$\cdots$		$\cdots$		$\diagdown$		$\cdots$		$\cdots$	$\cdots$
$\cdots$			$\cdots$	$\cdots$		$\cdots$	$\diagdown$		$\cdots$	
$n-(k-2)$	qt^{k-2}		$\cdots$	$\cdots$			qt^{k-2}	$\diagdown$		$\cdots$
$\cdots$	$\cdots$			$\cdots$			$\cdots$	$\diagdown$	t	t
$n-1$	qt	qt	$\cdots$			$\cdots$	qt	qt	$\diagdown$	1
n	q	q	$\cdots$			$\cdots$	q	q	q	$\diagdown$

Table 2.1: Association table for $\mu = (2, 1^{n-2})$.

terms in the k^{th} column above the diagonal and the terms in the $(n-(k-1))^{\text{th}}$ column below the diagonal, we get the entire polynomial coming from the standard tableaux with k in the $(1,2)$ cell. Hence, this table gives the association of the fillings with the standard Young tableaux, i.e., the fillings with pairs $[1,k]$, $[2,k]$, $\ldots$, $[k-1,k]$ and $[n-(k-2), n-(k-1)]$, $[n-(k-2)+1, n-(k-1)]$, $\ldots$, $[n-1, n-(k-1)]$, $[n, n-(k-1)]$ in the first row in this order ([element of $(1,1)$, elements of $(1,2)$]) correspond to the standard tableaux with k in the $(1,2)$ cell.

$\mu = (n-1,1)$ **case**

Note that in the case when $\mu = (n-1,1)$, the Hilbert series is

$$F_{(n-1,1)}(q,t) = (1+t)[n-1]_q! + \sum_{k=3}^{n} [n-2]_q!(t + qt + \cdots + q^{k-3}t + q^{k-2} + \cdots + q^{n-2}). \quad (2.2.30)$$

The first summand comes from the standard Young tableaux with 2 in the second row, and in the second summation, k denotes the element in the second row. Notice that the common factor $[n-2]_q!$ comes from the permutation of the first row not including the $(1,1)$ cell. So in this case, we are going to fix the pair of elements in the first column and see how the factor changes. Let's consider the case when we have k, $1 < k < n$, in the second row. If 1 comes right below k, then it makes a descent and since 1 is the smallest number, it doesn't make any inversion triples. So all the possible fillings with k in the second row and 1 in the $(1,1)$ cell give

$$t[n-2]_q!.$$

Now, if 2 comes in the $(1,1)$ cell, and if k is bigger than 2, then we still have a descent, and by having 1 in the first row to the right of 2, it will create one inversion no matter where 1 goes. So this fillings with k and 2 in the first column give

$$qt[n-2]_q!.$$

As the element in $(1,1)$ cell gets bigger, the power of q increases by 1. But once we get the bigger number than k in the $(1,1)$ cell, we lose the t factor, and so, summation of all possible monomials with k in the second row gives

$$[n-2]_q!(t+qt+\cdots+q^{k-2}t+q^{k-1}+\cdots q^{n-2}).$$

Notice that $q^{k-2}t$ comes from the fillings with $k-1$ in the $(1,1)$ cell, and q^{k-1} comes from the ones with $k+1$ in the $(1,1)$ cell. Comparing this polynomial to the Hilbert series (2.2.30), we can know that this is exactly the same polynomial coming from the standard Young tableaux with $(k+1)$ in the second row. In this way, we can find all possible fillings corresponding to the polynomials in the summation in (2.2.30). The only thing left is the polynomial from the standard tableaux with 2 on the second row, and this polynomial can be expressed as

$$[n-1]_q!+t[n-1]_q!.$$

Notice that the first term comes from the fillings with 1 in the second row, since it has no t factor, meaning there is no descents, and the second term comes from the fillings with n in the second row, which always have one descent no matter what element comes below n. We can make a table with the pair of elements in the first column as Table 2.2. In this table, the (i,j) factor gives the factor in front of $[n-2]_q!$ from the fillings with i in the $(1,1)$ cell and j in the $(2,1)$ cell. According to the rule that we found above, the k^{th} column, for $1<k<n$, will give the fillings corresponding to the standard tableaux with $k+1$ in the second row. And the sum of all monomials in the first and the last column will give the polynomial corresponding to the standard Young tableaux with 2 in the second row.

$\mu=(s,1^{n-s})$ case

Before we go into the details, we first introduce the Garsia-Procesi tree introduced in [GP92]. Note that in [GP92], they proved that this tree gives the Hilbert series when $q=0$, i.e., $F_\mu(0,t)$. Figure 2.3 is an example of the tree for the partition $\mu=(2,1,1)$. We see that the paths from the leaves can be identified with standard tableaux by filling the cells from 1 to $n=4$ following the tree from the bottom to top. The contribution to the Hilbert series from each of these leaves can then be

$(1,1)\setminus(2,1)$	1	2	3	$\cdots$	k	$\cdots$	$n-1$	n
1	$\diagdown$	t	t	$\cdots$	t	$\cdots$	t	t
2	1	$\diagdown$	qt	$\cdots$	qt	$\cdots$	qt	qt
3	q	q	$\diagdown$	$\cdots$	q^2t	$\cdots$	q^2t	q^2t
$\cdots$					$\cdots$	$\cdots$	$\cdots$	$\cdots$
$k-1$				$\diagdown$	$q^{k-2}t$	$\cdots$	$q^{k-2}t$	$q^{k-2}t$
k		$\cdots$			$\diagdown$			
$k+1$	q^{k-1}				q^{k-1}	$\diagdown$	$\cdots$	
$\cdots$	$\cdots$				$\cdots$			$\cdots$
$n-1$	q^{n-3}		$\cdots$		q^{n-3}	q^{n-3}	$\diagdown$	$q^{n-2}t$
n	q^{n-2}	q^{n-2}	$\cdots$		q^{n-2}	q^{n-2}	q^{n-2}	$\diagdown$

Table 2.2: Association table for $\mu = (n-1, 1)$.

obtained by multiplying the contributions corresponding to each of the multiple edges encountered along the path. For instance, the contribution corresponding to the double edge joining shape $(2,1,1)$ to shape $(2,1)$ is $(t+t^2)$; that corresponding to the triple edge joining $(1,1,1)$ to $(1,1)$ is $(1+t+t^2)$. Hence, from the left most leaves, we trace the tree and get a standard Young tableau having the filling 3214.

3	
2	
1	4

The Hilbert series corresponding to this SYT is $(1+t)(1+t+t^2)$ which is exactly the same thing that we can get from our combinatorial construction. Similarly, from the middle leaves and the right most ones, we get the following standard tableaux.

4	
2	
1	3

4	
3	
1	2

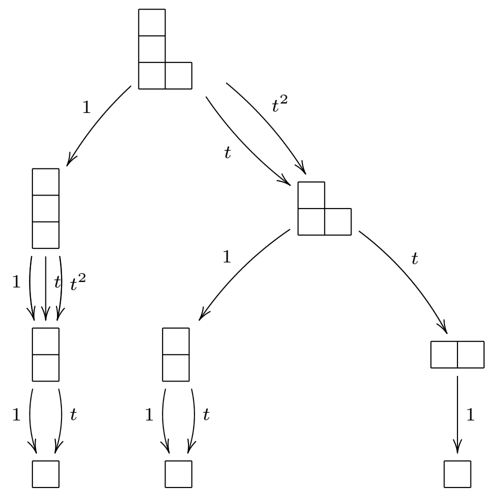


Figure 2.3: Garsia-Procesi tree for a partition $\mu = (2, 1, 1)$.

The corresponding Hilbert series would be $(1+t)(t+t^2)$ and $t(t+t^2)$. Again, we can confirm that they are the same things that we get from our method when $q = 0$.

We are going to modify the Garsia-Procesi tree to make it to give the complete polynomial corresponding to the standard Young tableaux. In the beginning of the tree, we put 0's on the top of the cells in the first row to the right of the $(1, 1)$ corner cell. And whenever losing a cell from the first column not in the first row, instead of labeling edges with t and higher power of t's, we start labeling from 1 and increase the number above the cells in the first column by 1. As soon as the cell in the first row is removed, the number above it will be fixed, and in the end, these numbers over the first row will give the b_i statistic in (2.2.28). For instance, according to our modification, the above example tree for $\mu = (2, 1, 1)$ changes to Figure 2.4. The underbar of the numbers on the top of the first row cells means that the corresponding number has been fixed. So by the above tree, we can read the complete polynomial involving both of q and t corresponding to the standard Young tableau. For instance, the left most leaves give

$$(1+t)(1+t+t^2)(1+q),$$

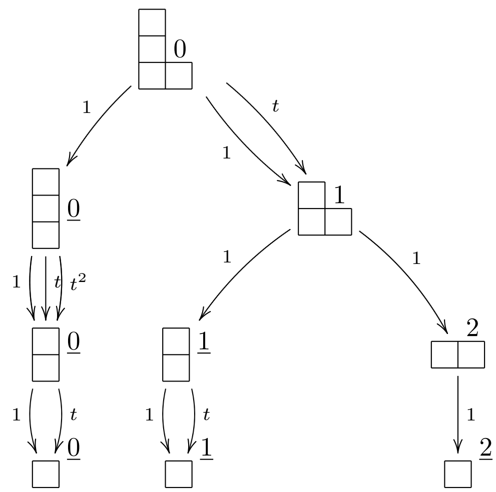


Figure 2.4: Modified Garsia-Procesi tree for a partition $\mu = (2, 1, 1)$.

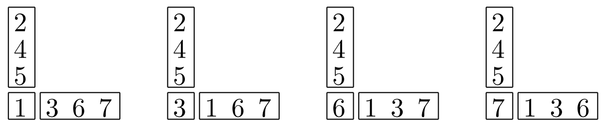
the middle and the last ones give

$$(1+t)^2(t+q), \quad (1+t)(t^2+q)$$

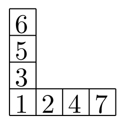
which are exactly the same polynomials that we get from the combinatorial construction over the standard Young tableau.

Now, for the association with fillings, we are going to introduce a *grouping table*. For the general hook of shape $\mu = (s, 1^{n-s})$, the way that we construct the table is the following : first we choose s many numbers including 1 and n, in all possible ways. Note that for this we have $\binom{n-2}{s-2}$ many choices. The unchosen $n-s$ many numbers will be put in the first column above the $(1,1)$ cell, in all possible ways, and the chosen s many numbers will come in the first row, in all possible ways. Then, this set of fillings will correspond to one standard Young tableau. We read out the polynomial corresponding the tableau in the following way : keeping in mind that we are calculating $q^{\text{inv}}t^{\text{maj}}$, since the permutations in the first column without including the $(1,1)$ cell give $[n-s]_t!$ factor and the permutations in the first row without the $(1,1)$ cell give $[s-1]_q!$ factor, we just consider s many different cases as we change the element coming in the $(1,1)$ cell by the chosen ones and each will give $q^a t^b [n-s]_t! [s-1]_q!$ where a is the number of elements in the first row to the right of $(1,1)$ cell which are smaller than the element in the $(1,1)$ cell, and b is the number of elements in the first column above $(1,1)$ cell which are bigger than the one in the $(1,1)$ cell. For instance, for $\mu = (4,1,1,1)$, if we choose $1,3,6$ and 7, then $2,4$ and 5 will be placed in the first column not including $(1,1)$ cell in all possible ways, and the chosen $1,3,6,7$ in the first row, and we read out the monomials t^3, qt, q^2 and q^3, multiplied by $[3]_t![3]_q!$, from the left, and summing up them all gives

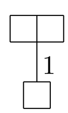
$$[3]_t![3]_q!(t^3 + qt^2 + q^2 + q^3).$$



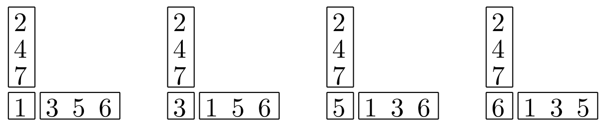
We note that these fillings will correspond to the following standard Young tableau.



We can check that this standard Young tableau gives the same polynomial that we've got from the fillings. We find the corresponding standard Young tableaux from the modified Garsia-Procesi tree, by comparing the power of t's and the numbers above the first column in the bottom of the tree. For example, in the above example, the powers of t are $3,2,0$, so we find the same numbers above three cells in the first row in the tree, and trace back to the top of the tree to complete the standard tableau. This way will give us all the standard tableaux which have the following looking tail part in the tree :

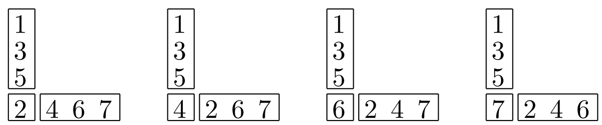


Secondly, we choose s many numbers within 1 to $n-1$ including 1 and $n-1$. Say we chose 1, $\alpha_1, \alpha_2, \ldots, \alpha_{s-2}$ and $n-1$. Then the rest of the procedure will be the same. But in this case, $2, \alpha_1+1, \alpha_2+1, \ldots, \alpha_{s-2}+1$ and n will give another set of fillings and all of them will be in the same group of fillings corresponding to one SYT. Note that by the way of choosing the set of numbers filling the first row, the monomial factors from the first set have 1 more power of t than the ones from the second set, so we can factor out $(1+t)$ so that it is enough to consider the monomials from the second set only. For these numbers, we repeat the same procedure as well. In the above running example for $\mu = (4,1,1,1)$, say we have chosen $1,3,5$ and 6.

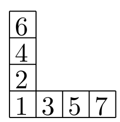


This will give the polynomial

$$t(t^2 + qt + q^2 + q^3)[3]_t![3]_q!.$$



And the numbers $2,4,6$ and 7 give $[3]_t![3]_q!(t^2+qt+q^2+q^3)$ and adding them all gives $[3]_t![3]_q!(1+t)(t^2+qt+q^2+q^3)$. This set of fillings will correspond to the following standard Young tableau.



We can find this standard Young tableau from the modified Garsia-Procesi tree by finding the powers of t from the numbers above the three cells in the first row. We repeat the similar procedure as reducing the range of numbers by 1 until we get to choose $1,2,\ldots,s$ in the same group. Note that if the range of numbers is reduced by $(k-1)$, i.e., if we get to choose s many numbers within 1 and $n-(k-1)$ including 1 and $n-(k-1)$, then we will have k many different sets of s many numbers, increased by 1 until we get to choose s many numbers between k to n. Lastly when we choose $1,2,\ldots,s$, we also choose $2,3,\ldots,s+1$, and $2,3,\ldots,s+2$ up to $n-s+1,\ldots,n$, as well. Then we will have $n-s+1$ different sets of s numbers. We can see this grouping process easily by making a table.

For example, for $\mu = (4,1,1)$, the grouping table is the following. In Table 2.3, the $\times$'s denote the chosen numbers filling the first row, and $\circ$'s are the ones placed in the first column above the

1	2	3	4	5	6
$\times$	$\times$	$\times$	$\circ$	$\circ$	$\times$
$\times$	$\times$	$\circ$	$\times$	$\circ$	$\times$
$\times$	$\times$	$\circ$	$\circ$	$\times$	$\times$
$\times$	$\circ$	$\times$	$\times$	$\circ$	$\times$
$\times$	$\circ$	$\times$	$\circ$	$\times$	$\times$
$\times$	$\circ$	$\circ$	$\times$	$\times$	$\times$
$\times$	$\times$	$\times$	$\circ$	$\times$	$\circ$
$\circ$	$\times$	$\times$	$\times$	$\circ$	$\times$
$\times$	$\times$	$\circ$	$\times$	$\times$	$\circ$
$\circ$	$\times$	$\times$	$\circ$	$\times$	$\times$
$\times$	$\circ$	$\times$	$\times$	$\times$	$\circ$
$\circ$	$\times$	$\circ$	$\times$	$\times$	$\times$
$\times$	$\times$	$\times$	$\times$	$\circ$	$\circ$
$\circ$	$\times$	$\times$	$\times$	$\times$	$\circ$
$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$

Table 2.3: The grouping table for $\mu = (4, 1, 1)$.

$(1, 1)$ cell. Here is the way of reading the table. The fillings between the lines will give fillings corresponding to one standard Young tableaux and the monomial coefficient factors $q^a t^b$ multiplied by $[n-s]_t![s-1]_q!$ will come by the following way : say k is placed in the $(1, 1)$ cell, then a is the number of chosen elements (i.e., $\times$ marked in the table) strictly smaller than k, and b is the number of unchosen elements (i.e., $\circ$ marked in the table) strictly bigger than k. For instance, in the first case, we choose $1, 2, 3$ and 6 for the elements in the first row, and $4, 5$ will be put in the first column above the $(1,1)$ cell.

$$\begin{array}{l}\boxed{\begin{matrix}4\\5\end{matrix}}\\ \boxed{1}\boxed{2\ 3\ 6}\end{array}\qquad\begin{array}{l}\boxed{\begin{matrix}4\\5\end{matrix}}\\ \boxed{2}\boxed{1\ 3\ 6}\end{array}\qquad\begin{array}{l}\boxed{\begin{matrix}4\\5\end{matrix}}\\ \boxed{3}\boxed{1\ 2\ 6}\end{array}\qquad\begin{array}{l}\boxed{\begin{matrix}4\\5\end{matrix}}\\ \boxed{6}\boxed{1\ 2\ 3}\end{array}$$

By permuting the first column without $(1,1)$ cell, we get $[2]_t$, and by permuting the first row not including $(1,1)$ cell, we get $[3]_q!$. And the coefficient monomials are t^2, qt^2, q^2t^2 and q^3, from the left, and summing up them all gives

$$[2]_t[3]_q!(t^2 + qt^2 + q^2t^2 + q^3)$$

which corresponds to the following standard Young tableau.

$$\begin{array}{|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-1} 5 \\ \hline 1 & 2 & 3 & 4 \\ \hline\end{array}$$

Example 2.2.11. We go over an example for $\mu = (3,1,1)$. The *grouping table* for $\mu = (3,1,1)$ would be:

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\ \hline
\times & \times & \circ & \circ & \times \\ \hline
\times & \circ & \times & \circ & \times \\ \hline
\times & \circ & \circ & \times & \times \\ \hline
\times & \times & \circ & \times & \circ \\
\circ & \times & \times & \circ & \times \\ \hline
\times & \circ & \times & \times & \circ \\
\circ & \times & \circ & \times & \times \\ \hline
\times & \times & \times & \circ & \circ \\
\circ & \times & \times & \times & \circ \\
\circ & \circ & \times & \times & \times \\ \hline
\end{array}$$

The rows between dividing lines will be grouped in the same set and the entries marked by $\times$ will come in the first row and permute in all possible ways. The entries marked by $\circ$ will come in the first column above the $(1,1)$ cell, and permute in all possible ways. All the fillings obtained by these permutations of row and column will correspond to one standard Young tableau. For instance, from the first grouping $\times \quad \times \quad \circ \quad \circ \quad \times$, we get 12 different fillings.

$$\begin{array}{lll}4&&\\3&&\\1&2&5\end{array}\quad\begin{array}{lll}4&&\\3&&\\1&5&2\end{array}\quad\begin{array}{lll}3&&\\4&&\\1&2&5\end{array}\quad\begin{array}{lll}3&&\\4&&\\1&5&2\end{array}\quad\begin{array}{lll}4&&\\3&&\\2&1&5\end{array}\quad\begin{array}{lll}4&&\\3&&\\2&5&1\end{array}$$

$$\begin{array}{lll}3&&\\4&&\\2&1&5\end{array}\quad\begin{array}{lll}3&&\\4&&\\2&5&1\end{array}\quad\begin{array}{lll}4&&\\3&&\\5&1&2\end{array}\quad\begin{array}{lll}4&&\\3&&\\5&1&3\end{array}\quad\begin{array}{lll}3&&\\4&&\\5&1&2\end{array}\quad\begin{array}{lll}3&&\\4&&\\5&2&1\end{array}$$

If we calculate $t^{\mathrm{maj}} q^{\mathrm{inv}}$ from the left of the top line, we get

$$t^3 + qt^3 + t^2 + qt^2 + qt^3 + q^2t^3 + qt^2 + q^2t^2 + q^2t + q^3t + q^2 + q^3$$

$$= (1+t)(1+q)(t^2 + qt^2 + q^2).$$

On the other hand, according to the way that we read the grouping table, we put 3 and 4 in the first column above $(1,1)$ cell, and put $1, 2$, and 5 changing the element in $(1,1)$ cell :

$$\begin{array}{ll}3&\\4&\\1&2\ 3\end{array}\qquad\begin{array}{ll}3&\\4&\\2&1\ 5\end{array}\qquad\begin{array}{ll}3&\\4&\\5&1\ 2\end{array}$$

From the permutations of 3 and 4, and permutations of two numbers in the first row not including $(1,1)$ cell, we get the common factors $[2]_t!$ and $[2]_q!$. And we read out the monomials t^2, qt^2 and q^2, from the left, hence we get

$$(1+t)(1+q)(t^2 + qt^2 + q^2).$$

This set of fillings will correspond to the following standard tableau.

$$\begin{array}{lll}5&&\\4&&\\1&2&3\end{array}$$

Noting that the permutations of either column or row not including the $(1,1)$ cell will give $(1+t)$ from the permutations of the column, and $(1+q)$ from the permutations of the row, the monomial coefficient factors $q^a t^b$ multiplied by $(1+t)(1+q)$ will be calculated by the following way : say k

is placed in the $(1,1)$ cell, then a is the number of chosen elements (i.e., $\times$ marked in the table) strictly smaller than k (or number of $\times$'s strictly to the left of k), and b is the number of unchosen elements (i.e., $\circ$ marked in the table) strictly bigger than k (or number of $\circ$'s strictly to the right of k). Following this method, we read out the following polynomials from the grouping table, from the second line

$$(1+t)(1+q)(t^2+qt+q^2),$$

$$(1+t)(1+q)(t^2+q+q^2),$$

$$(1+t)^2(1+q)(t+qt+q^2),$$

$$(1+t)^2(1+q)(t+q+q^2),$$

$$(1+t)(1+t+t^2)(1+q)(1+q+q^2).$$

And the corresponding standard tableaux are from the top

$$\begin{array}{|c|cc} \hline 5 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 4 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 5 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 4 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \; .$$

We use the modified Garsia-Procesi tree [GP92] to find the corresponding standard Young tableau given the polynomial from the set of fillings. For the running example for $\mu = (3,1,1)$, the modified Garsia-Procesi tree is given in Figure 2.5.

The way of finding the standard tableau from the modified Garsia-Procesi tree is the following : given the polynomial from the fillings, either of the form $(1+t)(1+q)(t^a+qt^b+q^2)$ or $(1+t)^2(1+q)(t^a+qt^b+q^2)$, compare (a,b) with the numbers on the right-top of bottom leaves in the tree. Finding the same numbers in the tree, trace back the tree from the bottom to top filling the cells with numbers from 1 to n (here $n=5$) as the tree adds the cells. Then on the top of the tree, we get the corresponding standard tableau giving exactly the same polynomial as we calculated from the table. We can check that all the possible 6 standard tableaux are covered in the grouping table and each set of fillings in the grouping table gives the polynomial corresponding to one standard Young tableau.

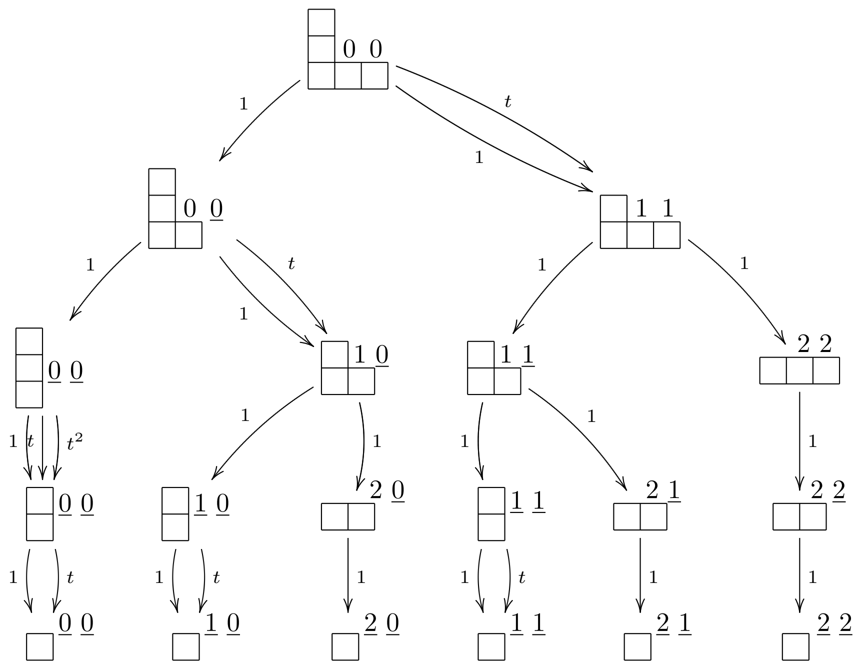


Figure 2.5: Modified Garsia-Procesi tree for $\mu = (3,1,1)$.

Proposition 2.2.12. *The grouping table gives the complete Hilbert series $F_\mu(q,t)$.*

Proof. By the way of constructing the grouping table that we don't count the same filling multiple times, we only need to check that the number of fillings that are counted in the grouping table is $n!$. From the permutations on the first column and the first row not including $(1,1)$ we count $(s-1)!(n-s)!$. In the grouping table, the set with k lines will be $\binom{n-(k-1)}{s-2}$ many and each line represents s different fillings. By adding them all, the number of fillings that the grouping table counts is

$$s!(n-s)!\left(\binom{n-2}{s-1}+2\binom{n-3}{s-2}+\cdots+(n-s)\binom{s-1}{s-2}+(n-s+1)\binom{s-2}{s-2}\right).$$

Therefore, we want to show the following identity

$$\binom{n}{s}=\binom{n-2}{s-1}+2\binom{n-3}{s-2}+\cdots+(n-s)\binom{s-1}{s-2}+(n-s+1)\binom{s-2}{s-2}. \tag{2.2.31}$$

Note the identity

$$\sum_{j=k}^{n-1}\binom{j}{k}=\binom{n}{k}. \tag{2.2.32}$$

Then, by applying (2.2.32) twice, the right hand side of (2.2.31) is

$$\begin{aligned}
&\binom{n-2}{s-1} + 2\binom{n-3}{s-2} + \cdots + (n-s)\binom{s-1}{s-2} + (n-s+1)\binom{s-2}{s-2} \\
= &\left(\binom{n-2}{s-1} + \binom{n-3}{s-2} + \cdots + \binom{s-1}{s-2} + \binom{s-2}{s-2}\right) \\
&+ \left(\binom{n-3}{s-2} + \cdots + \binom{s-1}{s-2} + \binom{s-2}{s-2}\right) + \cdots + \binom{s-2}{s-2} \\
= &\binom{n-1}{s-1} + \binom{n-2}{s-1} + \cdots + \binom{s}{s-1} + \binom{s-1}{s-1} \\
= &\binom{n}{s}.
\end{aligned}$$

This shows that we considered all $n!$ possible fillings, hence the grouping table gives the complete Hilbert series. ☐

Proposition 2.2.13. *The grouping table gives the association with the fillings corresponding to the standard Young tableaux.*

Proof. Remind that for the hook of shape $\mu = (s, 1^{n-s})$, the Hilbert series will be expressed as the following.

$$F_\mu(q,t) = [n-s]_t![s-1]_q! \sum_{j_1=2}^{n-s+1} \cdots \sum_{\substack{j_{s-1}=\\ j_{s-2}+1}}^{n} [j_1-1]_t(t^{b_1} + qt^{b_2} + \cdots + q^{s-2}t^{b_{s-1}} + q^{s-1}) \quad (2.2.33)$$

where b_i is the number of elements in the first column above the $(1,1)$ cell which are bigger than j_i. We start from the case where we have 1-lined set in the grouping table. Note that by knowing the tail part of the standard tableaux in the Garsia-Procesi tree, we know that in this case, the standard tableaux look like Figure 2.6. Then there are $\binom{n-2}{s-2}$ possibilities for the choice of the rest

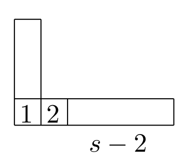


Figure 2.6: SYT corresponding to 1-lined set in grouping table

of the $(s-2)$ elements in the first row. Let's say $1, a_1, \ldots, a_{s-2}, n$, $a_i < a_j$ for $i < j$, are chosen in the grouping table. If 1 comes in the $(1,1)$ cell, then all the elements in the first row to the right and all the elements in the first column above 1 are bigger than 1, so the monomial factor will be t^{n-s}. And if a_1 comes in the $(1,1)$ cell, then we gain one power of q since 1 will be to the right of 2 in the first row, and the power of t will depend on a_1. Similarly, as a_i comes in the $(1,1)$ cell, as i gets larger by 1, the power of q will be increased by 1, and finally when n comes in the $(1,1)$ cell, since there are no bigger elements than n, it doesn't have any t powers and the power of q will be $s-1$, since all the rest of the chosen numbers are smaller than n. So this case gives the following form of polynomial

$$[n-s]_t![s-1]_q!(t^{n-s} + qt^{b_1} + \cdots + q^{s-2}t^{b_{s-2}} + q^{s-1})$$

where b_i is the number of elements in the unchosen ones which are bigger than a_i. Note that the fact that we don't have any repeated lines in the grouping table guarantees that we don't get the same polynomials multiple times, since the power of t, b_i, is the number of unchosen ones to the right of a_i in the grouping table. Secondly, consider the two-lined sets in the grouping table. Again, by the tail looking of the standard tableaux in the tree, we know that this case takes care

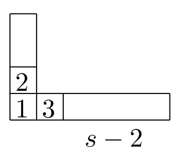


Figure 2.7: SYT corresponding to 2-lined set in grouping table

of the standard tableaux of the kind in Figure 2.7. Then there are $\binom{n-3}{s-2}$ many choices for the $s-2$ elements in the rest of the first row. Let's say we have chosen $1, a_1, \ldots, a_{s-2}, n-1$ in the first line. Then, by the construction, $2, a_1+1, \ldots, a_{s-2}+1, n$ will be chosen in the second line. Notice that in the grouping table, the lost of one $\circ$ under n in the second line means that all the monomial factors from the first line have 1 more power of t than the ones from the second line, hence we get $(1+t)$ factor after we sum them up all. So the polynomial that we get from this

1	2	a_1	$\dots$	a_{s-2}	$n-1$	n
$\times$	[			]	$\times$	$\circ$
$\circ$	$\times$	[			]	$\times$

Table 2.4: 2-lined set of the grouping table.

case is the following

$$[n-s]_t![s-1]_q!(1+t)(t^{n-s-1}+qt^{b_1}+\cdots+q^{s-2}t^{b_{s-s}}+q^{s-1})$$

where b_i is the number of circles in the grouping table to the right of a_i+1. Now, we consider a general case when we have k-lined set in the grouping table. This case takes care of the form of standard tableaux in Figure 2.8. Again, there are $\binom{n-(k+1)}{s-2}$ different possibilities for the different

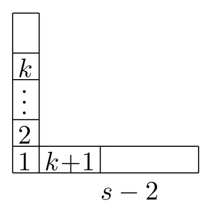


Figure 2.8: SYT corresponding to k-lined set in grouping table

choice of numbers coming in the rest of the first column. Let's say we choose $1, a_1, \dots, a_{s-2}, n-k+1$ in the first line, then $2, a_1+1, \dots, a_{s-2}+1, n-k+2$ will be chosen in the second line, and finally in the k^{th} line, $k, a'_1, \dots, a'_{s-2}, n$ will be chosen where $a'_i = a_i + k - 1$. Keeping in mind that the right most consecutive circles in the same line will give the common t powers and having the same number of circles and the same pattern means that the k lines have the common factor which comes from the k^{th} line, this case gives the polynomials as follows

$$[n-s]_t![s-1]_q![k]_t(t^{n-s-(k-1)}+qt^{b_1}+\cdots+q^{s-2}t^{b_{s-2}}+q^{s-1})$$

where b_i is the number of circles to the right of a'_i. In the last set of the grouping table, we choose $1, 2, \dots, s$ in the first line and $n-s+1, \dots, n$ in the last (which is $(n-s+1)^{\text{th}}$) line. This whole set of fillings will correspond to the standard tableaux of the kind in Figure 2.9. The fact that

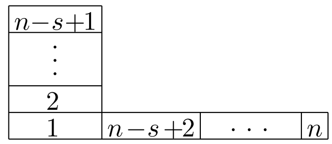


Figure 2.9: SYT corresponding to the last line in grouping table

there is no $\circ$ between $\times$ marks means there is no t powers combined with q and the consecutive circles to the right of the chosen elements will give the common t factors. Adding up them all will give

$$[n-s]_t![s-1]_q![n-s+1]_t(1+q+q^2+\cdots+q^{s-1}).$$

By comparing the polynomials coming from the grouping table and the polynomials added in the Hilbert series, we can confirm that the sets in the grouping table give the polynomials corresponding to standard Young tableaux. Since we know that the grouping table gives the complete Hilbert series by Proposition 2.2.12, we conclude that the grouping table gives the association with fillings to the standard Young tableaux. □

Remark 2.2.14. We note that the grouping table doesn't give the information about what the right corresponding standard Young tableau is. But if the modified Garsia-Procesi tree is given, by using the powers of t combined with q's, we can trace back the tree to construct the corresponding standard Young tableau, as we did in previous examples.

Remark 2.2.15. The case when $\mu = (n-1, 1)$, the table method introduced in the beginning is consistent with the grouping table, in other words, they both give the same association with fillings to the standard tableaux. But when $\mu = (2, 1^{n-2})$, the grouping table's grouping is different from the one that the Table 2.1 (introduced previously) gives. Note that the association with the fillings from the grouping table can be seen in the Table 2.1 for this case in the following way : for the fillings corresponding to the standard Young tableaux with 2 in the $(1, 2)$ cell, combine q in the left most column in the bottom and t^{n-2} in the right most column in the top. And for the standard tableaux with 3 in the $(1, 2)$ cell, combine qt in the left most column above the previous q and q in the bottom of the second column, and t^{n-2} in the top of the $(n-1)^{\text{th}}$ column and t^{n-3} below t^{n-3} in the right most column. In the similar way, we group the monomials diagonally so that we get $k-1$ monomials from the lower triangular part and $k-1$ terms from the upper triangular part, for the fillings corresponding to the standard tableaux with k in $(1, 2)$ cell. Finally, combining the inner most diagonal $(n-1)$ terms from both of upper and lower triangular part will give the fillings corresponding to the standard tableaux with n in the $(1, 2)$ cell.

Chapter 3

Schur Expansion of J_μ

In this chapter, we construct a combinatorial formula for the Schur coefficients of the integral form of the Macdonald polynomials $J_\mu[X; q, t]$ when μ is a hook. We recall the definition of the integral form of the Macdonald polynomials.

3.1 Integral Form of the Macdonald Polynomials

For each partition λ, we define

$$h_\lambda(q,t) = \prod_{c\in\lambda}(1 - q^{a(c)}t^{l(c)+1}), \quad h'_\lambda(q,t) = \prod_{c\in\lambda}(1 - t^{l(c)}q^{a(c)+1}), \quad d_\lambda(q,t) = \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}.$$

Then the *integral form* of the Macdonald polynomials is defined by

$$J_\mu[X;q,t] = h_\mu(q,t)P_\mu[X;q,t] = h'_\mu(q,t)Q_\mu[X;q,t]$$

where $Q_\lambda[X;q,t] = P_\lambda[X;q,t]/d_\lambda(q,t)$.

Remark 3.1.1. If $q = t$,

$$h_\lambda(q,q) = \prod_{c\in\lambda}(1 - q^{a(c)+l(c)+1}) = H_\lambda(q),$$

where $H_\lambda(q)$ is the hook polynomial of λ, defined in Definition 1.1.8. Since we know that

$P_\lambda[X;q,q] = s_\lambda,$

$$J_\lambda[X;q,q] = H_\lambda(q) s_\lambda.$$

In [Mac98, (8.11)], Macdonald showed that $J_\mu[X;q,t]$ can be expressed in terms of the modified Schur functions $s_\lambda[X(1-t)]$

$$J_\mu[X;q,t] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q,t) s_\lambda[X(1-t)],$$

where $K_{\lambda\mu}(q,t)$ is the q,t-Kostka polynomials. Using the relationship between $J_\mu[X;q,t]$ and $\tilde{H}_\mu[X;q,t]$ ([Mac98]) given by

$$\begin{aligned} J_\mu[X;q,t] &= t^{n(\mu)} \tilde{H}_\mu[X(1-t);q,t^{-1}] \\ &= t^{n(\mu)+n} \tilde{H}_\mu[X(t^{-1}-1);q,t^{-1}] \\ &= t^{n(\mu)+n} \tilde{H}_{\mu'}[X(t^{-1}-1);t^{-1},q], \end{aligned}$$

the combinatorial formula for monomial expansion of the Macdonald polynomials in Theorem 1.3.7 gives the following formula for $J_\mu[X;q,t]$.

Proposition 3.1.2. *For any partition μ,*

$$\begin{aligned} J_\mu[X;q,t] = &\sum_{\substack{\sigma:\mu' \to \mathbb{Z}_+ \\ non\text{-}attacking}} x^\sigma q^{maj(\sigma,\mu')} t^{coinv(\sigma,\mu')} \\ &\times \prod_{\substack{u \in \mu' \\ \sigma(u)=\sigma(south(u))}} (1 - q^{leg(u)+1} t^{arm(u)+1}) \prod_{\substack{u \in \mu' \\ \sigma(u)\neq\sigma(south(u))}} (1-t), \end{aligned}$$

where each square in the bottom row is included in the last product, and σ is non-attacking if $|\sigma(u)| \neq |\sigma(v)|$ for $u = (i,j)$ and $v = (i,k)$ (i.e., they are in the same row) or $u = (i+1,k)$ and $v = (i,k), j < k$ (i.e., they are in the consecutive rows, with u in the upper row strictly to the right of v).

Proof. See [HHL05, Prop. 8.1] or [Hag08, A.11.1]. □

We introduce another combinatorial formula for J_μ as an expansion of quasisymmetric functions $\{Q_\alpha\}$ defined in Definition 1.2.18. For the formula, we need several definitions.

Definition 3.1.3. For a positive filling σ and $s \in \mu$, let

$$\mathrm{maj}_s(\sigma,\mu) = \begin{cases} \mathrm{leg}(s) & \text{if North}(s) \in \mathrm{Des}(\sigma,\mu) \\ 0 & \text{else} \end{cases}$$

$$\mathrm{nondes}_s(\sigma,\mu) = \begin{cases} \mathrm{leg}(s)+1 & \text{if } s \notin \mathrm{Des}(\sigma,\mu) \text{ and South}(s) \in \mu \\ 0 & \text{else} \end{cases}$$

Definition 3.1.4. Given a triple u, v, w of μ, we define the *middle square* of u, v, w with respect to σ to be the square containing the number which is neither the largest nor the smallest of the set $\{\sigma(u), \sigma(v), \sigma(w)\}$. And define $\mathrm{coinv}_s(\sigma,\mu)$ to be the number of coinversion triples for which s is the middle square, and $\mathrm{inv}_s(\sigma,\mu)$ to be the number of inversion triples for which s is the middle square.

Definition 3.1.5. A filling of μ is called *primary* if all 1's occur in the first row of μ, all 2's occur in the first two rows of μ, and in general, all k's occur in the first k rows of μ.

Corollary 3.1.6.

$$J_\mu[X;q,t] = \sum_{\substack{\beta \in S_n \\ primary}} Q_{Des(\beta^{-1})}(X) \prod_{s \in \mu} (q^{inv_s(\beta,\mu)} t^{nondes_s(\beta,\mu)} - q^{coinv_s(\beta,\mu)} t^{1+maj_s(\beta,\mu)}).$$

Proof. See [Hag08]. □

Let u be a new indeterminate, and define a homomorphism

$$\epsilon_{u,t} : \Lambda_{\mathbb{Q}(q,t)} \to \mathbb{Q}(q,t)[u]$$

by

$$\epsilon_{u,t}(p_r) = \frac{1-u^r}{1-t^r} \tag{3.1.1}$$

for each $r \geqslant 1$. Then for $u = t^n$, n a positive integer, we have

$$\begin{aligned} \epsilon_{t^n,t}(p_r) &= \frac{1-t^{nr}}{1-t^r} = 1 + t^r + \cdots + t^{(n-1)r} \\ &= p_r(1, t, \ldots, t^{n-1}) \end{aligned}$$

and so for any symmetric function f, we have

$$\epsilon_{t^n,t}(f) = f(1, t, \dots, t^{n-1}). \tag{3.1.2}$$

Proposition 3.1.7. *We have*

$$\epsilon_{u,t}(P_\lambda(q,t)) = \prod_{c\in\lambda} \frac{q^{a'(c)}u - t^{l'(c)}}{q^{a(c)}t^{l(c)+1} - 1}.$$

Proof. See [Mac88, (5.3)]. □

Using this result, we can prove a recursive formula of J_μ when μ is a hook.

Proposition 3.1.8.

$$J_r \cdot J_{1^s} = \frac{1-t^s}{1-q^r t^s} J_{(r+1,1^{s-1})} + \frac{1-q^r}{1-q^r t^s} J_{(r,1^s)}.$$

Proof. By [Mac88, (4.8)], it is known that the J_λ expansion of $J_r \cdot J_\mu$ only involves those partitions λ such that λ/μ is a horizontal r-strip. Hence, we can find $c_1, c_2 \in \mathbb{Q}(q,t)$ such that

$$J_{(r)} \cdot J_{(1^s)} = c_1 J_{(r+1,1^{s-1})} + c_2 J_{(r,1^s)}. \tag{3.1.3}$$

Note that by Proposition 3.1.7,

$$\epsilon_{a,t} J_\mu[X;q,t] = \prod_{c\in\lambda} (t^{l'(c)} - q^{a'(c)}a).$$

We apply $\epsilon_{a,t}$ on both sides of (3.1.3), and get

$$(1-a) = c_1(1-q^r a) + c_2(t^s - a),$$

after cancelling the common factors. If we let $a = t^s$ and $a = q^{-r}$, we get

$$c_1 = \frac{1-t^s}{1-q^r t^s}, \quad c_2 = \frac{1-q^r}{1-q^r t^s}.$$

□

3.2 Schur Expansion of J_μ

Upon observing several examples of calculations, Haglund noticed that for any nonnegative integer $k \in \mathbb{N}$,

$$\left\langle \frac{J_\mu[X;q,q^k]}{(1-q)^n}, s_\lambda \right\rangle \in \mathbb{N}[q].$$

Based on this, Haglund conjectured that the Schur expansion of the integral form of Macdonald polynomials would have the following form :

$$J_\mu[X;q,t] = \sum_{\lambda \vdash n} \left[\sum_{T \in \mathrm{SSYT}(\lambda',\mu')} \prod_{c\in\mu} (1 - t^{l(c)+1} q^{q\mathrm{stat}(c,T)}) q^{ch(T)} \right] s_\lambda,$$

for certain appropriate q-statistics $q\mathrm{stat}(c,T)$. Motivated by this conjecture and based on explicit calculations of several examples, we could find the right q-statistic in one-row case and in the hook shape case.

3.2.1 Combinatorial Formula for the Schur coefficients of $J_{(r)}$

In this section, we construct a combinatorial formula for the Schur coefficient of J_μ when μ has only one row.

Theorem 3.2.1. *When $\mu = (r)$, J_μ has the following Schur expansion*

$$\begin{aligned} J_{(r)}[X;q,t] &= \sum_{\lambda\vdash r} \left[\prod_{c\in\lambda} (1 - q^{a'(c)-l'(c)} t) \right] \left(\sum_{T \in SYT(\lambda')} q^{ch(T)} \right) s_\lambda[X] \\ &= \sum_{\lambda\vdash r} \left[\prod_{c\in\lambda} (1 - q^{a'(c)-l'(c)} t) \right] K_{\lambda',1^r}(q) s_\lambda[X], \end{aligned}$$

where $K_{\lambda\mu}(q)$ is the Kostka-Foulkes polynomial.

Proof. In (1.3.4), we showed that

$$J_{(r)}[X;q,t] = (t;q)_r P_{(r)}[X;q,t] = (q;q)_r g_r(X;q,t)$$

where

$$g_r(X;q,t) = \sum_{\mu\vdash r} \frac{(t;q)_\mu}{(q;q)_\mu} m_\mu(X)$$

by [Mac98, Example 1, p.314]. So,

$$J_{(r)}[X;q,t] = (q;q)_r \sum_{\mu \vdash r} \frac{(t;q)_\mu}{(q;q)_\mu} m_\mu(x) \tag{3.2.1}$$

On the other hand, noting that $s_\lambda = \sum_{\mu \vdash r} K_{\lambda\mu} m_\mu$ and $K_{\lambda\mu} = 0$ if $\lambda < \mu$,

$$\begin{aligned}
&\sum_{\lambda \vdash r} \left[\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right] K_{\lambda',1^r}(q) s_\lambda[X] \\
&= \sum_{\lambda \vdash r} \left[\left(\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right) K_{\lambda',1^r}(q) \left(\sum_{\mu \leqslant \lambda} K_{\lambda\mu} m_\mu \right) \right] \\
&= \sum_{\mu \vdash r} \left[\sum_{\lambda \geqslant \mu} K_{\lambda\mu} \left(\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right) K_{\lambda',1^r}(q) \right] m_\mu.
\end{aligned}$$

In Proposition 1.3.12, we showed that

$$K_{\lambda',(1^r)}(q) = \frac{q^{n(\lambda)} (q;q)_r}{H_\lambda(q)},$$

and so the above formula becomes

$$\sum_{\mu \vdash r} \left[\sum_{\lambda \geqslant \mu} K_{\lambda\mu} \left(\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right) \frac{q^{n(\lambda)} (q;q)_r}{H_\lambda(q)} \right] m_\mu. \tag{3.2.2}$$

By comparing the coefficients of m_μ of (3.2.1) and (3.2.2), we need to show the following identity

$$\frac{(t;q)_\mu}{(q;q)_\mu} = \sum_{\lambda \geqslant \mu} K_{\lambda\mu} \left[\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right] \frac{q^{n(\lambda)}}{H_\lambda(q)} \tag{3.2.3}$$

for each $\mu \vdash r$. For $t = q^k$, for any nonnegative integer k, the left hand side of (3.2.3) is

$$\left. \frac{(t;q)_\mu}{(q;q)_\mu} \right|_{t=q^k} = \frac{(q^k;q)_\mu}{(q;q)_\mu} = h_\mu(1, q, \ldots, q^{k-1}),$$

by [Sta99, 7.8.3].

On the other hand, by Proposition 3.1.7,

$$\left[\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right] \frac{q^{n(\lambda)}}{H_\lambda(q)} = \prod_{c \in \lambda} \frac{q^{l'(c)} - q^{a'(c)} t}{1 - q^{a(c)+l(c)+1}} = \epsilon_{t,q}(P_\lambda[X;q;q]).$$

Hence, the right hand side of (3.2.3) becomes

$$\sum_{\lambda \geqslant \mu} K_{\lambda\mu} \left[\prod_{c \in \lambda} (1 - q^{a'(c)-l'(c)} t) \right] \frac{q^{n(\lambda)}}{H_\lambda(q)} = \sum_{\lambda \geqslant \mu} K_{\lambda\mu} \cdot \epsilon_{t,q}(P_\lambda[X;q;q])$$

$$
\begin{aligned}
&= \sum_{\lambda \geqslant \mu} K_{\lambda\mu}\epsilon_{t,q}(s_\lambda) = \epsilon_{t,q}\left(\sum_{\lambda \geqslant \mu} K_{\lambda\mu}s_\lambda\right) \\
&= \epsilon_{t,q}(h_\mu),
\end{aligned}
$$

since $h_\mu = \sum_\lambda K_{\lambda\mu}s_\lambda$ by Proposition 1.2.6. Then for $t = q^k$,

$$\epsilon_{q^k,q}(h_\mu) = h_\mu(1, q, \ldots, q^{k-1})$$

by (3.1.2). So, for any $t = q^k$, k a positive integer, (3.2.3) is true, and thus for any t. This finishes the proof. □

3.2.2 Schur Expansion of $J_{(r,1^s)}$

Recall that in Proposition 3.1.8, we proved the following recursive formula of J_μ when μ is a hook.

$$J_{(r)} \cdot J_{(1^s)} = \frac{1-t^s}{1-q^r t^s} J_{(r+1,1^{s-1})} + \frac{1-q^r}{1-q^r t^s} J_{(r,1^s)}. \tag{3.2.4}$$

Note that by Corollary 3.1.6, we get

$$J_{(1^s)}[X; q, t] = (t;t)_s s_{(1^s)}. \tag{3.2.5}$$

Having this 1-column formula and 1-row formula from Theorem 3.2.1, by plugging in various values of r to (3.2.4) starting from $r = 1$, we can derive $J_{(r+1,1^{s-1})}$ formula, when $n = r + s$.

Example 3.2.2. If we let $r = 1, s = n-1$, (3.2.4) gives

$$J_{(1)} \cdot J_{(1^{n-1})} = \frac{1-t^{n-1}}{1-qt^{n-1}} J_{(2,1^{n-2})} + \frac{1-q}{1-qt^{n-1}} J_{(1^n)}.$$

Knowing that $J_{(1^n)} = (t;t)_n s_{(1^n)}$ and especially $J_{(1)} = (t;t)_1 s_{(1)}$, we can derive the Schur expansion for $J_{(2,1^{n-2})}$:

$$
\begin{aligned}
J_{(2,1^{n-2})} &= \frac{1}{1-t^{n-1}}\big((1-qt^{n-1})J_{(1)} \cdot J_{(1^{n-1})} - (1-q)J_{(1^n)}\big) \\
&= \frac{1}{1-t^{n-1}}\big((1-qt^{n-1})(t;t)_1 s_{(1)} \cdot (t;t)_{n-1} s_{(1^{n-1})} - (1-q)(t;t)_n J_{(1^n)}\big) \\
&= \frac{(1-t)^n[n-1]_t!}{1-t^{n-1}}((1-qt^{n-1})s_{(1)} \cdot s_{(1^{n-1})} - (1-q)[n]_t s_{(1^n)})
\end{aligned}
$$

$$= (1-t)^{n-1}[n-2]_t!((1-qt^{n-1})(s_{(2,1^{n-2})}+s_{(1^n)}) - (1-q)[n]_t s_{(1^n)})$$

$$= (t;t)_{n-2}\{(1-t)(1-qt^{n-1})s_{(2,1^{n-2})} + (q-t)(1-t^{n-1})s_{(1^n)}\}.$$

Similarly, by letting $r = 2, s = n-2$ and using this formula for $J_{(2,1^{n-2})}$ in (3.2.4), we can get the formula for $J_{(3,1^{n-3})}$:

$$J_{(3,1^{n-3})} = (t;t)_{n-3}\{(1-t)(1-qt)(1-q^2t^{n-2})s_{(3,1^{n-3})} + (1-t)(q-t)(1-q^2t^{n-2})s_{(221^{n-4})}$$

$$+(1-t)(q-t)(1-qt^{n-2})(1+q)s_{(2,1^{n-2})} + (q-t)(q^2-t)(1-t^{n-2})s_{(1^n)}\}.$$

Note that since we know the Schur expansion of $J_{(r)}$, by recursively using (3.2.4), we can derive the formula for $J_{(r+1,1^{s-1})}$ from $J_{(r,1^s)}$.

Based on examples for small r values, we construct the following combinatorial formula for the Schur expansion of J_μ when $\mu = (r, 1^s), n = r + s$:

Theorem 3.2.3. [1] *For $\mu = (r, 1^s), n = r+s$, we have*

$$J_{(r,1^s)} = (t;t)_s \sum_{\substack{\lambda\vdash n\\ \lambda\leqslant\mu}} \left[\prod_{c\in 1^{l(\lambda)}/1^{s+1}} (1-q^{-l'(c)-1}t) \cdot \prod_{c\in\lambda/1^{l(\lambda)}} (1-q^{a'(c)-l'(c)}t) \right]$$

$$\times \quad (1-q^{n-l(\lambda)}t^{s+1}) \left(\sum_{T\in SSYT(\lambda',\mu')} q^{ch(T)} \right) s_\lambda.$$

Proof. We prove by using this formula in (3.2.4) and showing that the recursion formula is still true. Using Theorem 3.2.1, the left hand side of (3.2.4) becomes

$$\begin{aligned} J_{(r)} \cdot J_{(1^s)} &= (t;t)_s \left(\sum_{\nu\vdash r} \left[\prod_{c\in\nu} (1-q^{a'(c)-l'(c)}t) \right] K_{\nu',1^r}(q) s_\nu \right) \cdot s_{(1^s)} \\ &= (t;t)_s \sum_{\nu\vdash r} \left\{ \left[\prod_{c\in\nu} (1-q^{a'(c)-l'(c)}t) \right] K_{\nu',1^r}(q) \left(\sum_{\lambda} c^{\lambda}_{\nu,1^s} s_\lambda \right) \right\}, \end{aligned}$$

where $c^\lambda_{\nu,1^s}$ is the Littlewood-Richardson coefficient (see Theorem 1.2.8). It is known that $c^\lambda_{\nu,1^s} = 1$ ([Sta99, 7.15.7]) where λ changes over all partitions λ of n containing ν and 1^s such that λ/ν is a

[1] The combinatorial formula in Theorem 3.2.3 holds when s is large enough so that the coefficients of all the possible s_λ's, $\lambda \vdash n$, $\lambda \leqslant (r, 1^s)$, occur in the expansion.

vertical strip of size s. So,

$$J_{(r)} \cdot J_{(1^s)} = (t;t)_s \sum_{\lambda} \left\{ \sum_{\nu \vdash r} \left[\prod_{c \in \nu} (1 - q^{a'(c) - l'(c)} t) \right] K_{\nu', 1^r}(q) \right\} s_\lambda \tag{3.2.6}$$

summed over all partitions $\lambda \vdash n$ for which λ/ν is a vertical strip of size s.

Now, we consider the right hand side of (3.2.4). First of all,

$$\begin{aligned}
& (1-t^s) J_{(r+1,1^{s-1})} + (1-q^r) J_{(r,1^s)} \\
= \; & (t;t)_s \sum_{\substack{\lambda \vdash n \\ \lambda \leqslant (r+1,1^{s-1})}} \left[\prod_{c \in 1^{l(\lambda)}/1^s} (1 - q^{-l'(c)-1} t) \cdot \prod_{c \in \lambda/1^{l(\lambda)}} (1 - q^{a'(c)-l'(c)} t) \right] \\
& \qquad \times \; (1 - q^{n-l(\lambda)} t^s) K_{\lambda', (s,1^r)}(q) s_\lambda \\
+ \; & (1-q^r)(t;t)_s \sum_{\substack{\lambda \vdash n \\ \lambda \leqslant (r,1^s)}} \left[\prod_{c \in 1^{l(\lambda)}/1^{s+1}} (1 - q^{-l'(c)-1} t) \cdot \prod_{c \in \lambda/1^{l(\lambda)}} (1 - q^{a'(c)-l'(c)} t) \right] \\
& \qquad \times \; (1 - q^{n-l(\lambda)} t^{s+1}) K_{\lambda', (s+1,1^{r-1})}(q) s_\lambda.
\end{aligned} \tag{3.2.7}$$

For every possible λ, we compare the coefficient of s_λ of $(1-q^r t^s) J_{(r)} \cdot J_{(1^s)}$ from (3.2.6) and the one from (3.2.7). Note that up to the order of λ (according to the dominance order), $(1 - t^s) J_{(r+1,1^{s-1})} + (1-q^r) J_{(r,1^s)}$ can be decomposed in the following way :

$$\begin{aligned}
& (1-t^s) J_{(r+1,1^{s-1})} + (1-q^r) J_{(r,1^s)} \\
= \; & (t;t)_s (1 - q^r t^s)(t;q)_r s_{(r+1,1^{s-1})} & (3.2.8) \\
+ \; & (t;t)_s \sum_{\substack{\lambda \vdash n, \lambda \leqslant (r,1^s) \\ l(\lambda) = s}} \left[\prod_{c \in \lambda/1^s} (1 - q^{a'(c)-l'(c)} t) \right] (1 - q^r t^s) K_{\lambda', (s,1^r)}(q) s_\lambda & (3.2.9) \\
+ \; & (t;t)_s \sum_{\substack{\lambda \vdash n, \lambda \leqslant (r,1^s) \\ l(\lambda) \geqslant s+1}} \left[\prod_{c \in 1^{l(\lambda)}/1^s} (1 - q^{-l'(c)-1} t) \cdot \prod_{c \in \lambda/1^{l(\lambda)}} (1 - q^{a'(c)-l'(c)} t) \right] & (3.2.10) \\
& \times \left[(1 - q^{n-l(\lambda)} t^s) K_{\lambda', (s,1^r)}(q) + \frac{(1-q^r)(1 - q^{n-l(\lambda)} t^{s+1})}{(1 - q^{s-l(\lambda)} t)} K_{\lambda', (s+1,1^{r-1})}(q) \right] s_\lambda.
\end{aligned}$$

The coefficient of $s_{(r+1,1^{s-1})}$ in $(1-q^r t^s) J_{(r)} \cdot J_{(1^s)}$ is

$$(t;t)_s (1 - q^r t^s) \left[\prod_{c \in \nu} (1 - q^{a'(c)-l'(c)} t) \right]$$

where $\nu = (r)$, since $\nu = (r)$ is the only partition for which λ/ν becomes a vertical s-strip for $\lambda = (r+1, 1^{s-1})$, and for $\nu = (r)$, $K_{\nu', (1^r)}(q) = 1$. For each $c \in \nu$, $l'(c) = 0$ and $a'(c)$ changes from 0 to $r-1$, so

$$\prod_{c\in\nu}(1-q^{a'(c)-l'(c)}t)=(t;q)_r.$$

Hence,

$$(t;t)_s(1-q^rt^s)\left[\prod_{c\in\nu}(1-q^{a'(c)-l'(c)}t)\right]=(t;t)_s(1-q^rt^s)(t;q)_r$$

which is $s_{(r+1,1^{s-1})}$ coefficient of $(1-t^s)J_{(r+1,1^{s-1})}+(1-q^r)J_{(r,1^s)}$ in (3.2.8). Secondly, we compare the coefficients of s_λ when $\lambda\leqslant(r,1^s)$ and $l(\lambda)=s$. Note that because of these two conditions of λ, for each λ, there is only one ν such that λ/ν becomes a vertical s-strip, namely, $\nu=\lambda/(1^s)$. And note that

$$K_{\lambda',(s,1^r)}(q)=K_{\nu',1^r}(q)$$

since $\lambda_1'=s$, all the s many 1's fill the first row when we make the filling for the SSYT of shape λ' and weight $(s,1^r)$ and those 1's contribute nothing to the charge statistic, and so, we can just ignore the first row with 1's. Hence, in (3.2.9), the coefficient of s_λ becomes

$$(1-q^rt^s)(t;t)_s\left[\prod_{c\in\nu}(1-q^{a'(c)-l'(c)}t)\right]K_{\nu',(1^r)}(q)$$

and if we divide it by $(1-q^rt^s)$, the result is exactly the coefficient of s_λ in $J_{(r)}\cdot J_{(1^s)}$ of (3.2.6).

Lastly, we consider λ's such that $\lambda\leqslant(r,1^s),l(\lambda)\geqslant s+1$. We first simplify the terms in the second square bracket of (3.2.10).

$$\begin{aligned}
&(1-q^{n-l(\lambda)}t^s)K_{\lambda',(s,1^r)}(q)+\frac{(1-q^r)(1-q^{n-l(\lambda)}t^{s+1})}{(1-q^{s-l(\lambda)}t)}K_{\lambda',(s+1,1^{r-1})}(q)\\
&=(1-q^rt^s+q^rt^s(1-q^{s-l(\lambda)}))K_{\lambda',(s,1^r)}(q)\\
&\qquad\qquad+\quad(1-q^r)\left(q^rt^s+\frac{1-q^rt^s}{1-q^{s-l(\lambda)}t}\right)K_{\lambda',(s+1,1^{r-1})}(q)\\
&=(1-q^rt^s)\left[K_{\lambda',(s,1^r)}(q)+\frac{1-q^r}{1-q^{s-l(\lambda)}t}K_{\lambda',(s+1,1^{r-1})}(q)\right]\\
&\qquad+\quad q^rt^s\left[(1-q^{s-l(\lambda)})K_{\lambda',(s,1^r)}(q)+(1-q^r)K_{\lambda',(s+1,1^{r-1})}(q)\right].
\end{aligned}\tag{3.2.11}$$

Later, in Proposition 3.2.6, we prove that

$$(1-q^{s-l(\lambda)})K_{\lambda',(s,1^r)}(q)+(1-q^r)K_{\lambda',(s+1,1^{r-1})}(q)=0.$$

Using this identity, we can even more simplify (3.2.11) :

$$K_{\lambda',(s,1^r)}(q) + \frac{1-q^r}{1-q^{s-l(\lambda)}t} K_{\lambda',(s+1,1^{r-1})}(q) = \frac{q^{s-l(\lambda)}(1-t)}{1-q^{s-l(\lambda)}t} K_{\lambda',(s,1^r)}(q).$$

So, the coefficient of s_λ in (3.2.10) becomes

$$(t;t)_s(1-q^r t^s)\left[\prod_{c\in 1^{l(\lambda)}/1^s}(1-q^{-l'(c)}t)\cdot \prod_{c\in\lambda/1^{l(\lambda)}}(1-q^{a'(c)-l'(c)}t)\right] q^{s-l(\lambda)} K_{\lambda',(s,1^r)}(q).$$

In Proposition 3.2.5, we prove that

$$q^{s-l(\lambda)} K_{\lambda',(s,1^r)}(q) = \sum_\nu K_{\nu',1^r}(q),$$

where ν ranges over partitions of r such that λ/ν is a vertical strip of size s. Note that in Proposition 1.3.12, we showed that

$$K_{\nu',1^r}(q) = K_{\nu',1^r}(0,q) = \frac{q^{n(\nu)}(q;q)_r}{H_\nu(q)}$$

and by Proposition 1.2.11, $q^{n(\nu)}/H_\nu(q) = s_\nu(1,q,q^2,\dots)$, hence

$$\begin{aligned}\sum_\nu K_{\nu',1^r}(q) &= (q;q)_r \sum_\nu \frac{q^{n(\nu)}}{H_\nu(q)}\\ &= (q;q)_r \sum_\nu s_\nu(1,q,q^2,\dots)\\ &= (q;q)_r s_{\lambda/1^s}(1,q,q^2,\dots).\end{aligned}$$

The last line is by [Sta99, 7.15.7]. Finally dividing by $(1-q^r t^s)$ gives the s_λ-coefficient of $\frac{1-t^s}{1-q^r t^s} J_{(r+1,1^{s-1})} + \frac{1-q^r}{1-q^r t^s} J_{(r,1^s)}$ which is

$$(t;t)_s(q;q)_r\left[\prod_{c\in 1^{l(\lambda)}/1^s}(1-q^{-l'(c)}t)\cdot \prod_{c\in\lambda/1^{l(\lambda)}}(1-q^{a'(c)-l'(c)}t)\right] s_{\lambda/1^s}(1,q,q^2,\dots)$$

$$= (t;t)_s(q;q)_r\left[\prod_{c\in\lambda/1^s}(1-q^{a'(c)-l'(c)}t)\right] s_{\lambda/1^s}(1,q,q^2,\dots) \tag{3.2.12}$$

assuming that s is large enough so that $s \geqslant \lambda'_2$. By Proposition 3.1.7 (or [Mac88, p.161]),

$$\prod_{c\in\lambda/1^s}(1-q^{a'(c)-l'(c)}t) = \frac{\epsilon_{t,q} J_{\lambda/1^s}[X;q,q]}{q^{n(\lambda/1^s)}} = \frac{H_{\lambda/1^s}(q)}{q^{n(\lambda/1^s)}}\cdot \epsilon_{t,q}(s_{\lambda/1^s}[X]),$$

since $J_{\lambda/1^s}[X;q,q] = H_{\lambda/1^s}(q)s_{\lambda/1^s}[X]$. Again, by Proposition 1.2.11,

$$q^{n(\lambda/1^s)}/H_{\lambda/1^s}(q) = s_{\lambda/1^s}(1,q,q^2,\dots),$$

thus (3.2.12) becomes

$$(t;t)_s(q;q)_r \cdot \frac{\epsilon_{t,q}(s_{\lambda/1^s}[X])}{s_{\lambda/1^s}(1,q,q^2,\dots)} \cdot s_{\lambda/1^s}(1,q,q^2,\dots)$$

$$= (t;t)_s(q;q)_r\epsilon_{t,q}(s_{\lambda/1^s}[X]). \tag{3.2.13}$$

Now we consider the s_λ-coefficient of $J_{(r)} \cdot J_{(1^s)}$, which is

$$(t;t)_s \sum_{\nu\vdash r}\left[\prod_{c\in\nu}(1-q^{a'(c)-l'(c)}t)\right]K_{\nu',1^r}(q) \tag{3.2.14}$$

summed over all partitions $\nu \vdash r$ for which λ/ν is a vertical strip of size s. Note that

$$K_{\nu',1^r}(q) = K_{\nu',1^r}(0,q) = \frac{q^{n(\nu)}(q;q)_r}{H_\nu(q)}$$

and so

$$\sum_{\nu\vdash r}\left[\prod_{c\in\nu}(1-q^{a'(c)-l'(c)}t)\right]K_{\nu',1^r}(q) = (q;q)_r\sum_{\nu\vdash r}\left[\prod_{c\in\nu}\frac{q^{l'(c)}-q^{a'(c)}t}{1-q^{a(c)+l(c)+1}}\right].$$

By Proposition 3.1.7,

$$\prod_{c\in\nu}\frac{q^{l'(c)}-q^{a'(c)}t}{1-q^{a(c)+l(c)+1}} = \epsilon_{t,q}(P_\nu[X;q,q]) = \epsilon_{t,q}(s_\nu[X])$$

and if we sum $\epsilon_{t,q}(s_\nu)$ over all partitions ν of r for which λ/ν is a vertical strip of size s, then

$$\sum_{\nu\vdash r}\epsilon_{t,q}(s_\nu) = \epsilon_{t,q}\sum_{\nu\vdash r}s_\nu = \epsilon_{t,q}(s_{\lambda/1^s}),$$

where the last equality comes from [Sta99, 7.15.7]. Using this, (3.2.14) becomes

$$(t;t)_s(q;q)_r\epsilon_{t,q}(s_{\lambda/1^s}[X]).$$

This is the s_λ coefficient of $J_{(r)} \cdot J_{(1^s)}$ and it is the same to (3.2.13) which is the s_λ coefficient of $\frac{1-t^s}{1-q^rt^s}J_{(r+1,1^{s-1})} + \frac{1-q^r}{1-q^rt^s}J_{(r,1^s)}$. Hence, the J_μ recursion (3.2.4) is satisfied and this finishes the proof. □

We provide the proofs of the following propositions that we used in the proof of Theorem 3.2.3. Before we prove, we introduce a maj statistic.

Definition 3.2.4. Let λ be a partition. For $T \in$ SYT(λ), let $D(T)$ denote the corresponding descent set ; i.e., the subset of $[n-1]$ with $i \in D(T)$ if and only if $i+1$ appears in a row strictly higher than i in T. For $T \in$ SYT(λ), we define the *major index* maj(T) by

$$\mathrm{maj}(T) = \sum_{k \in D(T)} k.$$

Proposition 3.2.5. *For a given partition λ of n, we have*

$$q^{s-l(\lambda)} K_{\lambda',(s,1^r)}(q) = \sum_{\nu} K_{\nu',1^r}(q)$$

where ν ranges over partitions of r such that λ/ν is a vertical strip of size s.

Proof. Note that Morris' Lemma in [Mor63, p.114] of the recursive formula of Kostka polynomials $K_{\lambda\mu}(q)$ reduces to

$$K_{(m,\tau)(\lambda_0,\lambda)}(q) = \sum_{\substack{\mu \\ \mu/\tau \text{ horizontal } (m-\lambda_0)\text{-strip}}} q^{m-\lambda_0} K_{\mu\lambda}(q)$$

by Butler [But94, 2.2.6]. For the detailed proof, see [But94, 2.5.1]. To apply this recursion, we let

$$m = l(\lambda), \quad \tau = \lambda'/(l(\lambda)), \quad \lambda_0 = s, \quad \lambda = 1^r.$$

Then we get the recursion for $K_{\lambda',(s,1^r)}(q)$:

$$K_{\lambda',(s,1^r)}(q) = \sum_{\substack{\mu \\ \mu/\tau \text{ horizontal } (l(\lambda)-s)\text{-strip}}} q^{l(\lambda)-s} K_{\mu,1^r}(q).$$

Then

$$q^{s-l(\lambda)} K_{\lambda',(s,1^r)}(q) = \sum_{\substack{\mu \\ \mu/\tau \text{ horizontal } (l(\lambda)-s)\text{-strip}}} K_{\mu,1^r}(q).$$

So, if we can make a one-to-one correspondence between μ' such that μ/τ is a horizontal $(l(\lambda)-s)$-strip (or, $\mu'/(\lambda/(1^{l(\lambda)}))$ is a vertical $(l(\lambda)-s)$-strip) and ν such that λ/ν is a vertical strip of size s, then that finishes the proof. But this correspondence is obvious since $\nu/(\lambda/(1^{l(\lambda)}))$ is a vertical $(l(\lambda)-s)$-strip by the construction of ν. ◻

Proposition 3.2.6. *For a given partition λ of n, we have*

$$(1-q^{s-l(\lambda)})K_{\lambda',(s,1^r)}(q)+(1-q^r)K_{\lambda',(s+1,1^{r-1})}(q)=0.$$

Proof. The identity is equivalent to

$$K_{\lambda',(s,1^r)}(q)+(1-q^r)K_{\lambda',(s+1,1^{r-1})}(q)=q^{s-l(\lambda)}K_{\lambda',(s,1^r)}(q),$$

and by Proposition 3.2.5, the right hand side is

$$\sum_{\nu} K_{\nu',1^r}(q)$$

where ν ranges over partitions of r such that λ/ν is a vertical strip of size s, which we showed in the middle of the proof of Theorem 3.2.3 that is equal to

$$\sum_{\nu}(q;q)_r s_\nu(1,q,q^2,\dots).$$

So we can show that the following identity holds :

$$K_{\lambda',(s,1^r)}(q)+(1-q^r)K_{\lambda',(s+1,1^{r-1})}(q)=\sum_{\nu}(q;q)_r s_\nu(1,q,q^2,\dots), \tag{3.2.15}$$

where ν satisfies the specified conditions. It is known by [Sta99, 7.19.11] that

$$s_\nu(1,q,q^2,\dots)=\frac{\sum_{T\in\ \mathrm{SYT}(\nu)} q^{\mathrm{maj}(T)}}{(q;q)_r}.$$

So the right hand side of (3.2.15) becomes

$$\sum_{\nu}\sum_{T\in\ \mathrm{SYT}(\nu)} q^{\mathrm{maj}(T)}, \tag{3.2.16}$$

where ν ranges over partitions of r such that λ/ν is a vertical strip of size s.

Now we consider the left hand side of (3.2.15). Note that

$$K_{\lambda\mu}(q)=K_{\lambda\mu}(0,q)=K_{\lambda'\mu'}(q,0),$$

where the last equality comes from the duality property of the q,t-Kostka polynomials. So the left hand side of (3.2.15) becomes

$$\begin{aligned} & K_{\lambda',(s,1^r)}(q) + (1-q^r)K_{\lambda',(s+1,1^{r-1})}(q) \\ = \; & K_{\lambda',(s,1^r)}(0,q) + (1-q^r)K_{\lambda',(s+1,1^{r-1})}(0,q) \\ = \; & K_{\lambda,(r+1,1^{s-1})}(q,0) + (1-q^r)K_{\lambda,(r,1^s)}(q,0). \end{aligned} \tag{3.2.17}$$

In [Ste94], Stembridge showed that

$$\sum_{T\in\ \mathrm{SYT}(\lambda)} q^{\alpha_r(T)}t^{\beta_{r+1}(T)} = \frac{1-t^s}{1-q^rt^s}K_{\lambda,(r+1,1^{s-1})}(q,t) + \frac{1-q^r}{1-q^rt^s}K_{\lambda,(r,1^s)}(q,t),$$

where $\alpha_r(T)$ and $\beta_r(T)$ are defined by

$$\alpha_r(T) = \sum_{\substack{1\leqslant k<r \\ k\in D(T)}} k, \qquad \beta_r(T) = \sum_{\substack{r\leqslant k<n \\ k\notin D(T)}} n-k.$$

(3.2.17) comes from the above formula by plugging in $t=0$, and we get

$$K_{\lambda,(r+1,1^{s-1})}(q,0) + (1-q^r)K_{\lambda,(r,1^s)}(q,0) = \sum_{T\in\ \mathrm{SYT}(\lambda)} q^{\alpha_r(T)}$$

where $\beta_{r+1}(T)=0$. If $\beta_{r+1}(T)=0$, then the numbers from $r+1$ to n should be in the vertical s-border strip in λ. Hence, for those fillings $T\in\mathrm{SYT}(\lambda)$,

$$\sum_{T\in\ \mathrm{SYT}(\lambda)} q^{\alpha_r(T)} = \sum_{\nu}\sum_{T\in\ \mathrm{SYT}(\nu)} q^{\mathrm{maj}(T)}$$

where ν ranges over partitions of r such that λ/ν is a vertical s-strip, and this is exactly the right hand side (3.2.16) of (3.2.15). This finishes the proof. □

Chapter 4

Further Research

4.1 Combinatorial formula of the Hilbert series of M_μ for μ with three or more columns

In this thesis, we constructed a combinatorial formula for the Hilbert series of S_n-modules M_μ for the hook case, and Garsia and Haglund found a similar formula for the two-column case. We would like to find a combinatorial formula for the general shapes of μ, but after doing several computational experiments, we don't believe that we can extend the same (or similar) combinatorial construction for the Hilbert series when μ has three or more columns. However, we do believe that there is a way that we can group the monomials corresponding to standard Young tableaux and express the Hilbert series as a sum of polynomials with certain statistics in q and t over standard Young tableaux. Based on the fact that the modified Garsia-Procesi tree gives a correspondence of fillings to standard Young tableaux, by studying Garsia-Procesi tree or modifying it in a certain way, we might be able to find a way of associating fillings to the standard Young tableaux for μ in general shape.

4.2 Finding the basis for the Hilbert series of S_n-modules

In 2001, Haiman [Hai01] proved the $n!$ conjecture, which implies that the dimension of M_μ is $n!$, by using techniques in algebraic geometry, but the proof doesn't give an explicit way of constructing the basis set of M_μ. Garsia expected that the combinatorial formula for the Hilbert series of M_μ would give some clues to find the basis of M_μ, so we hope that the recursion formula that we used to prove the combinatorial formula in the hook shape case to give a way of constructing the basis for the Garsia-Haiman modules of the hook shape case. We should mention that there are unproven conjectures [BG99] made in the process of constructing the Garsia-Haiman modules. Constructing the basis of the Garsia-Haiman modules would be the key to prove those conjectures.

4.3 Constructing the coefficients in the Schur expansion of $J_\mu[X;q,t]$ for general μ

We can continue to construct the combinatorial formula for the Schur coefficients of $J_\mu[X;q,t]$ for the general μ's. We expect that

$$J_\mu[X;q,t] = \sum_{\lambda\vdash n}\left[\sum_{T\in \mathrm{SSYT}(\lambda',\mu')}\prod_{c\in\mu}(1-t^{l(c)+1}q^{q\mathrm{stat}(c,T)})q^{ch(T)}\right]s_\lambda$$

holds in general. Thus, we hope to find the appropriate q-statistics $q\mathrm{stat}(c,T)$ for the general μ's.

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