BIQUOTIENTS WITH ALMOST POSITIVE CURVATURE

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ABSTRACT

BIQUOTIENTS WITH ALMOST POSITIVE CURVATURE

Martin Kerin

Wolfgang Ziller, Advisor

When does a manifold admit a metric with positive sectional curvature? This is one of the most fundamental and difficult problems in differential geometry. One attempt at understanding this problem is to begin with a non-negatively curved manifold and examine how large is the set of points with positive curvature. More precisely, given a manifold, does it admit a non-negatively curved metric for which there is an open set of points with positive curvature (quasi-positive curvature), or an open dense set of such points (almost positive curvature)? We construct new examples of biquotients which admit such metrics.

Contents

List of Tables

Chapter 1

Introduction

When does a manifold admit a metric with positive sectional curvature? This is a fundamental and difficult problem in differential geometry. There are many examples of manifolds with non-negative curvature. For example, all homogeneous spaces G/H and all biquotients $G/\!\!/U$ inherit non-negative curvature from the bi-invariant metric on G. It was also shown in [GZ1] that all cohomogeneity-one manifolds, namely manifolds admitting an isometric group action with one-dimensional orbit space, with singular orbits of codimension ≤ 2 admit metrics with non-negative curvature.

On the other hand, the known examples with positive curvature are very sparse. Other than the rank-one symmetric spaces there are isolated examples in dimensions $6, 7, 12, 13$ and 24 due to Wallach [Wa] and Berger [Ber2], and two infinite families, one in dimension 7 (Eschenburg spaces; see [AW], [E1], [E2]) and the other in dimension 13 (Bazaikin spaces; see [Ba1]).

In the simply connected case there are no known obstructions to admitting positive curvature that are not already obstructions to admitting a metric of non-negative curvature. Some of the standard theorems relating topology and positive curvature are:

Bonnet-Myers Let M^n be a complete Riemannian manifold and suppose that the Ricci curvature satisfies $\mathrm{Ric} \geq \delta > 0$. Then M is compact and $\pi_1(M)$ is finite.

Synge Let M^n be a compact manifold with positive sectional curvature, $\sec > 0$.

(i) If M is orientable and n is even, then M is simply connected;

(ii) If n is odd, then M is orientable.

Sphere Theorem Let M^n be a compact, simply connected, Riemannian manifold.

(i) If $1 < \sec \leq 4$ then M^n is diffeomorphic to S^n;

(ii) If $1 \leq \sec \leq 4$ then M^n is either diffeomorphic to S^n or isometric to a CROSS.

The Sphere Theorem was established up to homeomorphism by Berger [Ber1] and Klingenberg [K], and up to diffeomorphism by Brendle and Schoen [BS1], [BS2]. In fact, Brendle and Schoen proved more general rigidity results which imply the Sphere Theorem as a special case. Soon after the announcement of the proof by Brendle and Schoen, Ni and Wolfson [NW] announced an alternate proof in the special case of the differential Sphere Theorem.

We are interested in the study of manifolds which lie "between" those with non-negative and those with positive curvature.

Definition. *A Riemannian manifold* $(M, \langle\ ,\ \rangle)$ *has* quasi-positive curvature *(resp.* almost positive curvature*) if* $(M, \langle\ ,\ \rangle)$ *has non-negative sectional curvature and there is a point (resp. an open dense set of points) at which all* 2*-planes have positive sectional curvature.*

It should be noted that in the definition of quasi-positive curvature we could replace "point" with "an open set of points".

One of the major motivations for studying manifolds with quasi-positive curvature is the well-known Deformation Conjecture which we rewrite in our language.

Conjecture. *Suppose* $(M, \langle\ ,\ \rangle)$ *is a complete Riemannian manifold with quasi-positive curvature. Then* M *admits a metric with positive curvature.*

There is some evidence in support of this conjecture. Aubin [Au] and Ehrlich [Eh] proved the analogous statements for scalar and Ricci curvature. Perelman's proof of the soul conjecture [Pe] shows that a non-compact manifold with quasi-positive curvature is diffeomorphic to $\mathbb{R}^n$ and hence admits a metric with positive curvature. Moreover, Hamilton [Ha] has shown that the Deformation Conjecture is true in dimension three.

Petersen and Wilhelm [PW] provided the first examples of manifolds with almost positive curvature when they showed that the unit tangent bundle of S^4 and a real cohomology $\mathbb{C}P^3$ admit such metrics.

Most of the known examples of manifolds with almost positive curvature appear in the work of Wilking [Wi]. In particular he proves that

Theorem (Wilking)**.** *Each of the following compact manifolds admits almost positive curvature.*

(i) The projective tangent bundles $P_{\mathbb{R}}T\mathbb{R}P^n$*,* $P_{\mathbb{C}}T\mathbb{C}P^n$*, and* $P_{\mathbb{H}}T\mathbb{H}P^n$ *of* $\mathbb{R}P^n$*,* $\mathbb{C}P^n$ *and* $\mathbb{H}P^n$ *respectively.*

(ii) The homogeneous space $M^{4n-1}_{k,\ell} = U(n+1)/H_{k,\ell}$*, with* $k, \ell \in \mathbb{Z}$*,* $k\ell < 0$*, and* $n \geq 2$*, where*

$$H_{k,\ell} = \{\mathrm{diag}(z^k, z^\ell, A) \mid z \in S^1, A \in U(n-1)\}.$$

Since the universal cover of $P_{\mathbb{R}}T\mathbb{R}P^n$ is T^1S^n, the unit tangent bundle of S^n, it is clear that T^1S^n also admits a metric with almost positive curvature. We also note that Wallach [Wa] has shown that the flag manifolds $P_{\mathbb{C}}T\mathbb{C}P^2$ and $P_{\mathbb{H}}T\mathbb{H}P^2$ admit homogeneous metrics with positive curvature.

The homogeneous spaces described by $M^{4n-1}_{k,\ell}$ should be thought of as generalisations of the 7-dimensional positively curved Aloff-Wallach spaces, $W^7_{k,\ell}$. However it is clear that the metric on $M^{4n-1}_{k,\ell}$ cannot be homogeneous, since otherwise we could left-translate a zero-curvature plane to every point of $M^{4n-1}_{k,\ell}$. Furthermore, an examination of the Gysin sequence for the fibration $S^1 \longrightarrow M^{4n-1}_{k,\ell} \longrightarrow P_{\mathbb{C}}T\mathbb{C}P^n$ shows that there are infinitely many homotopy types of simply connected manifolds of a fixed dimension $4n-1$ which admit almost positive curvature. Recall that the Aloff-Wallach spaces are homogeneous spaces $SU(3)/S^1_{k,\ell}$, where $S^1_{k,\ell} \subset SU(3)$ via $z \longmapsto \mathrm{diag}(z^k, z^\ell, \bar{z}^{k+\ell})$. Of course $W^7_{k,\ell}$ may be rewritten in the form $M^{4n-1}_{k,\ell}$ with $n = 2$ as in Wilking's theorem. In [AW] the authors show that $W^7_{k,\ell}$ admits positive curvature if and only if $k\ell(k+\ell) \neq 0$. There is thus a unique Aloff-Wallach space, namely $W^7_{-1,1}$, which is not known to admit positive curvature. Since left-translation is an isometry, it is clear that $W^7_{-1,1}$ must have a zero-curvature plane at every point with respect to the homogeneous metric. Therefore we see that Wilking's result deforms the homogeneous metric on $W^7_{-1,1}$ to a non-homogeneous metric with almost positive curvature. The integral cohomology ring of $W^7_{-1,1}$ is the same as that of $S^2 \times S^7$ and it is an open problem to decide whether these manifolds are homotopy equivalent.

Notice that Wilking shows there are odd-dimensional, non-orientable manifolds, for example $\mathbb{R}P^3 \times \mathbb{R}P^2$ and $\mathbb{R}P^7 \times \mathbb{R}P^6$, which admit almost positive curvature. By Synge's Theorem such manifolds cannot admit positive curvature. Thus these manifolds are counter-examples to the Deformation Conjecture. However, all of these counter-examples have non-trivial fundamental group. Therefore it is still possible that the Deformation Conjecture holds for simply connected manifolds with quasi-positive curvature. Moreover, in [PW] the authors suggest that consideration should be given to the following modification of the Deformation Conjecture:

Question. *Does a Riemannian manifold with quasi-positive curvature admit a metric with almost positive curvature?*

From Wilking's counter-examples to the Deformation Conjecture we see that the class of manifolds admitting almost positive curvature is strictly larger than the class of manifolds admitting positive curvature. Moreover, the class of manifolds having quasi-positive curvature is strictly contained in the class of non-negatively curved manifolds, as a necessary condition for admitting quasi-positive curvature is the possession of a finite fundamental group. This follows in the non-compact case from Perelman's proof of the soul conjecture [Pe], and in the compact case from Bonnet-Myers together with the results of Aubin [Au] and Ehrlich [Eh] on deformations of metrics with non-negative Ricci curvature and positive Ricci curvature at a point. From the question above we see that it is unknown whether the class of almost positively curved manifolds is strictly contained in the class of those with quasi-positive curvature.

There is an even more profound reason for investigating the validity of the Deformation Conjecture for simply connected manifolds. The fact that $\mathbb{R}P^3 \times \mathbb{R}P^2$ and $\mathbb{R}P^7 \times \mathbb{R}P^6$ admit metrics with almost positive curvature implies that $S^3 \times S^2$ and $S^7 \times S^6$ admit such metrics. If it were possible to deform these metrics to have positive curvature then we would have counter-examples to the celebrated Hopf Conjecture, which asserts that a product of spheres cannot admit positive curvature.

The first example of a manifold with quasi-positive curvature was given in [GM, '74]. Here it was shown that the Gromoll-Meyer exotic 7-sphere $\Sigma^7 = Sp(2)/\!\!/Sp(1)$

inherits quasi-positive curvature from the bi-invariant metric on $Sp(2)$. In [W, '01] Wilhelm showed that the bi-invariant metric on $Sp(2)$ may be deformed in such a way as to induce almost positive curvature on Σ^7. The deformed metric on $Sp(2)$ is no longer left-invariant. In [EK, '07] it is shown that there exits a left-invariant metric on $Sp(2)$ which induces almost positive curvature on Σ^7. The set of points with zero-curvature planes is given by a finite union of subvarieties of codimension ≥ 1 and can be explicitly determined. If one could further deform the metric on Σ^7 to have positive curvature, then it would be the first example of an exotic sphere admitting a metric of this kind.

The only other previously known examples of manifolds with almost positive or quasi-positive curvature are given in [PW, '99], [Wi, '01] and [Ta1, '03]. In particular Tapp shows:

Theorem (Tapp)**.** *The following manifolds admit metrics with quasi-positive curvature:*

(i) *The unit tangent bundles* $T^1\mathbb{C}P^n$, $T^1\mathbb{H}P^n$ *and* $T^1Ca\mathbb{P}^2$*;*

(ii) *The homogeneous space* $M^{4n-1}_{k,\ell} = U(n+1)/H_{k,\ell}$, *with* $k, \ell \in \mathbb{Z}$, $(k,\ell) \neq (0,0)$, *and* $n \geq 2$, *where*

$$H_{k,\ell} = \{\mathrm{diag}(z^k, z^\ell, A) \mid z \in S^1, A \in U(n-1)\}.$$

Notice that Tapp proves that those generalised Aloff-Wallach spaces $M^{4n-1}_{k,\ell}$ for which $k\ell \geq 0$ admit quasi-positive curvature. These are exactly the $M^{4n-1}_{k,\ell}$ not included in Wilking's examples of almost positively curved manifolds.

We recall briefly that the Eschenburg spaces are defined by $E^7_{p,q} = SU(3)/\!\!/S^1_{p,q}$ where $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{Z}^3$, $\sum p_i = \sum q_i$, and $S^1_{p,q}$ acts on $SU(3)$ via

$$z \star A = \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) A\, \mathrm{diag}(\bar{z}^{q_1}, \bar{z}^{q_2}, \bar{z}^{q_3}), \quad z \in S^1, A \in SU(3).$$

Similarly the Bazaikin spaces are defined by $B^{13}_{q_1,\ldots,q_5} = SU(5)/\!\!/(Sp(2) \cdot S^1_{q_1,\ldots,q_5})$, where $Sp(2) \cdot S^1_{q_1,\ldots,q_5} = (Sp(2) \times S^1_{q_1,\ldots,q_5})/\mathbb{Z}_2$ acts on $SU(5)$ via

$$[A, z] \star B = \mathrm{diag}(z^{q_1}, \ldots, z^{q_5}) B\, \mathrm{diag}(A, \bar{z}^q),$$

$z \in S^1$, $A \in Sp(2) \subset SU(4)$, $B \in SU(5)$, and $q = \sum q_i$. We will discuss the Eschenburg and Bazaikin spaces in much more detail in subsequent chapters.

We are now in a position to state our main result in which we describe some new examples of manifolds admitting almost or quasi-positive curvature.

Theorem A.

(i) All Eschenburg spaces $E^7_{p,q} = SU(3)/\!\!/S^1_{p,q}$ admit metrics with quasi-positive curvature.

(ii) The Eschenburg space $E^7_{p,q}$, $p = (1, 1, 0)$, $q = (0, 0, 2)$, admits almost positive curvature.

(iii) All Bazaikin spaces $B^{13}_{q_1,\ldots,q_5} = SU(5)/\!\!/(Sp(2) \cdot S^1_{q_1,\ldots,q_5})$ such that four of the q_j share the same sign admit quasi-positive curvature.

(iv) The Bazaikin space $B^{13}_{-1,1,1,1,1}$ admits almost positive curvature.

(v) There is a free circle action on $S^7 \times S^7$ such that $M^{13} = S^1 \backslash (S^7 \times S^7)$ admits a metric with quasi-positive curvature. Furthermore, M^{13} is not homeomorphic to $\mathbb{C}P^3 \times S^7$.

(vi) There is a free S^3-action on $S^7 \times S^7$ such that $N^{11} = S^3 \backslash (S^7 \times S^7)$ admits a metric with quasi-positive curvature. Furthermore, N^{11} is not homeomorphic to $S^4 \times S^7$.

The topology of Eschenburg spaces has been studied extensively (see, for example, [E2], [CEZ], [K1], [K2], [Sh2]). In particular the cohomology groups of the Eschenburg spaces are $H^0 = H^2 = H^5 = H^7 = \mathbb{Z}$ and $H^4 = \mathbb{Z}_{|s|}$, where $s := p_1p_2 + p_1p_3 + p_2p_3 - q_1q_2 - q_1q_3 - q_2q_3$. Moreover s is always odd (see [K1], Remark 1.4). There is a special subfamily of Eschenburg spaces with $p = (1, 1, n), q = (0, 0, n + 2)$ ([Sh2], [GSZ]) which we denote by E^7_n. We note that E^7_n and $E^7_{-(n+1)}$ describe the same manifold, and E^7_n has positive curvature if $n \geq 1$ ([E1]). Since $H^4(E^7_n) = \mathbb{Z}_{2n+1}$, $n \geq 0$, it is clear that every cyclic group of odd order is achieved, and moreover that there are infinitely many positively curved Eschenburg spaces which are distinct even up to homotopy equivalence. We note that, as for the Aloff-Wallach space $W^7_{-1,1}$, the integral cohomology ring of E^7_0 agrees with that of $S^2 \times S^5$ and it is unknown whether these manifolds are homotopy equivalent. On the other hand, in [CEZ] it is shown that there are only finitely many positively curved Eschenburg spaces for a given cohomology ring. This should be viewed in the context of the Klingenberg-Sakai conjecture. It states that there are only finitely many positively curved manifolds in a given homotopy type, and the result in [CEZ] raises the question of whether the conjecture is true even for cohomology. In the present context, it is natural to ask the following question.

Question. *Are there infinitely many pairwise non-homotopy equivalent Eschenburg spaces which share the same cohomology ring?*

In the event of a positive answer to this question, only finitely many of these Eschenburg spaces can admit positive curvature by our previous remark. Theorem A(i) would provide the first examples of infinite families of simply connected, non-homotopy equivalent manifolds with quasi-positive curvature which share the same cohomology.

Notice that the resolution of this question again highlights the importance of the Deformation Conjecture for simply connected manifolds with quasi-positive curvature. If the Deformation Conjecture is true in this case, then the cohomology Klingenberg-Sakai Conjecture is false. Equivalently, if the cohomology Klingenberg-Sakai conjecture is true, then the Deformation Conjecture must clearly be false for simply connected manifolds.

It follows from the results in [FZ1] that, although the almost positively curved Bazaikin space $B^{13}_{-1,1,1,1,1}$ has the same integral cohomology ring as $\mathbb{C}P^2 \times S^9$, consideration of the respective Pontrjagin classes shows that these manifolds are not even homotopy equivalent. Using results of Taimanov [T], one easily notices that $B^{13}_{-1,1,1,1,1}$ contains both the exceptional Aloff-Wallach space $W^7_{-1,1}$ and the Eschenburg space E_0^7 as totally geodesic submanifolds (Proposition 4.2.3). Theorem A(ii) shows that E_0^7 also admits almost positive curvature, whereas $W^7_{-1,1}$ has a zero-curvature plane at every point ([AW]). As we discussed earlier, Wilking [Wi] has shown that one can deform the metric on $W^7_{-1,1}$ such that it admits almost positive curvature.

If we now relax the constraint that U acts freely on G by allowing U to act almost freely (i.e. all isotropy groups are finite), then we can find the following orbifold examples:

Theorem B.

(i) All of the Eschenburg orbifolds $\mathcal{E}^7_{p,q} = SU(3)/\!\!/S^1_{p,q}$ *with* $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{Z}^3$ *satisfying*

$$q_1 < q_2 = p_1 < p_2 \leq p_3 < q_3 \tag{†}$$

admit almost positive curvature.

(ii) There are infinitely many orbifolds of the form $(S^3 \times S^3)/\!\!/T^2$ *admitting almost positive curvature.*

Remark (a). There are no free $S^1_{p,q}$-actions on $SU(3)$ satisfying condition (†). Moreover, (†) is essential to the proof of almost positive curvature, i.e. we cannot use a similar proof to get almost positive curvature for free actions.

Remark (b). It is interesting to note that for the T^2-actions on $S^3 \times S^3$ which we consider the proof of almost positive curvature on $(S^3 \times S^3)/\!\!/T^2$ breaks down exactly when the action is required to be free, namely for the quotient manifolds $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

Remark (c). Among the orbifolds $(S^3 \times S^3) /\!\!/ T^2$, there are examples with one singular point with $\mathbb{Z}_3$-isotropy, and examples with two singular points, each with $\mathbb{Z}_2$-isotropy. These examples are described in Tables 6.2, 6.3, 6.4 and 6.5.

Chapter 2

Biquotient actions and metrics

In his Habilitation, [E1, '84], Eschenburg studied biquotients in great detail. In particular he provided a classification of maximal rank torus actions on simple Lie groups. The following sections borrow heavily from the material in [E1] and establish the basic language, notation and results which will be used throughout the subsequent chapters.

2.1 Biquotients

Let G be a compact Lie group, $U \subset G \times G$ a closed subgroup, and let U act on G via

$$(u_1, u_2) \star g = u_1 g u_2^{-1}, \quad g \in G, (u_1, u_2) \in U.$$

The action is free if and only if, for all non-trivial $(u_1, u_2) \in U$, u_1 is never conjugate in G to u_2. The resulting manifold is called a *biquotient*.

Recall that every element of a compact Lie group is conjugate to an element of 'the' maximal torus. Thus in order to check that the action of $U \subset G \times G$ on G is free it is enough to show that t_1 and t_2 are never conjugate in G, where (t_1, t_2) is a non-trivial element of $U \cap (T \times T)$, T a maximal torus of G. Therefore checking for freeness is enormously simplified and often reduces to comparing the eigenvalues of matrices acting on the left and right of G.

Let $K \subset G$ be a closed subgroup, $\langle \ , \ \rangle$ be a left-invariant, right K-invariant metric on G, and $U \subset G \times K \subset G \times G$ act freely on G as above. Let $g \in G$. Define

$$\begin{aligned} U_L^g &:= \{(gu_1g^{-1}, u_2) \mid (u_1, u_2) \in U\}, \\ U_R^g &:= \{(u_1, gu_2g^{-1}) \mid (u_1, u_2) \in U\}, \quad \text{and} \\ \widehat{U} &:= \{(u_2, u_1) \mid (u_1, u_2) \in U\}. \end{aligned}$$

Then U_L^g, U_R^g and $\widehat{U}$ act freely on G, and $G/\!\!/U$ is isometric to $G/\!\!/U_L^g$, diffeomorphic to $G/\!\!/U_R^g$ (isometric if $g \in K$), and diffeomorphic to $G/\!\!/\widehat{U}$ (isometric if $U \subset K \times K$).

In the case of U_L^g this follows from the fact that left-translation $L_g : G \longrightarrow G$ is an isometry which satisfies $gu_1g^{-1}(L_g g')u_2^{-1} = L_g(u_1 g' u_2^{-1})$. Therefore L_g induces an isometry of the orbit spaces $G/\!\!/U$ and $G/\!\!/U_L^g$. Similarly we find that $R_{g^{-1}}$ induces a diffeomorphism between $G/\!\!/U$ and $G/\!\!/U_R^g$, which is an isometry if $g \in K$.

Consider now $\widehat{U}$. The actions of U and $\widehat{U}$ are equivariant under the diffeomorphism $\tau : G \longrightarrow G$, $\tau(g) := g^{-1}$. That is, $u_1\tau(g)u_2^{-1} = \tau(u_2 g u_1^{-1})$. Notice that this is an isometry only if $U \subset K \times K$. In general $G/\!\!/U$ and $G/\!\!/\widehat{U}$ are therefore diffeomorphic but not isometric.

Homogeneous spaces, G/H, provide the most trivial examples of biquotients. We include some more interesting examples below.

Example 2.1.1. The Gromoll-Meyer sphere, $\Sigma^7 = Sp(2)/\!\!/Sp(1)$, where $Sp(1)$ is embedded in $Sp(2) \times Sp(2)$ via

$$q \longmapsto \left(\begin{pmatrix} q & \\ & 1 \end{pmatrix}, \begin{pmatrix} q & \\ & q \end{pmatrix} \right), \qquad q \in Sp(1).$$

Example 2.1.2. The Eschenburg spaces, $E^7_{p,q} := SU(3)/\!\!/S^1_{p,q}$, where $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3) \in \mathbb{Z}^3$, $\sum p_i = \sum q_i$, and $S^1_{p,q}$ acts on $SU(3)$ via

$$z \star A = \begin{pmatrix} z^{p_1} & & \\ & z^{p_2} & \\ & & z^{p_3} \end{pmatrix} A \begin{pmatrix} \bar{z}^{q_1} & & \\ & \bar{z}^{q_2} & \\ & & \bar{z}^{q_3} \end{pmatrix}, \quad A \in SU(3), z \in S^1.$$

The action is free if and only if $(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1$ for all permutations $\sigma \in S_3$.

Example 2.1.3. The Bazaikin spaces, $B^{13}_{q_1,\dots,q_5} = SU(5)/\!\!/(Sp(2) \cdot S^1_{q_1,\dots,q_5})$, where all $q_1, \dots, q_5 \in \mathbb{Z}$ are odd and the action of $Sp(2) \cdot S^1_{q_1,\dots,q_5} = (Sp(2) \times S^1_{q_1,\dots,q_5})/\mathbb{Z}_2$ is given by

$$[z, A] \star B = \begin{pmatrix} z^{q_1} & & \\ & \ddots & \\ & & z^{q_5} \end{pmatrix} B \begin{pmatrix} A & \\ & \bar{z}^{q} \end{pmatrix},$$

with $z \in S^1$, $A \in Sp(2) \subset SU(4)$, $B \in SU(5)$, and $q = \sum q_i$. It is easy to check that such an action is free if and only if all q_i are odd and $(q_{\sigma(1)} + q_{\sigma(2)}, q_{\sigma(3)} + q_{\sigma(4)}) = 2$ for all $\sigma \in S_5$.

2.2 Submersions and metric deformations

Recall that a differentiable map $\pi : M^n \longrightarrow N^{n-k}$ is called a submersion if f is surjective, and for all $p \in M$, $d\pi_p : T_pM \longrightarrow T_{\pi(p)}N$ has rank $n-k$. The submersion

π is said to be Riemannian if, for all $p \in M$, $d\pi_p$ preserves the lengths of horizontal vectors at p. The O'Neill formula for a Riemannian submersion $\pi : M^n \longrightarrow N^{n-k}$ is

$$\sec_N(X,Y) = \sec_M(\widetilde{X},\widetilde{Y}) + \frac{3}{4}\left|\left|\left[\widetilde{X}',\widetilde{Y}'\right]^{\mathcal{V}}\right|\right|^2,$$

where $\widetilde{X}$ denotes the horizontal lift to T_pM of $X \in T_{\pi(p)}N$, $\widetilde{X}'$ denotes a local horizontal extension of $\widetilde{X}$, and $Z^{\mathcal{V}} \in T_pM$ is the component of $Z \in T_pM$ tangent to the fibre $\pi^{-1}(\pi(p))$.

Notice that π is curvature non-decreasing. Therefore if $\sec_M \geq 0$ then $\sec_N \geq 0$, and zero-curvature planes on N lift to horizontal zero-curvature planes on M. In general, because of the Lie bracket term in the O'Neill formula, the converse is not true, namely horizontal zero-curvature planes in M cannot be expected to project to zero-curvature planes on N. However, we will see at the end of this section that in many situations we have $\sec_N(X,Y) = 0$ if and only if $\sec_M(\widetilde{X},\widetilde{Y}) = 0$.

Let $K \subset G$ be Lie groups, $\mathfrak{k} \subset \mathfrak{g}$ the corresponding Lie algebras, and $\langle\ ,\ \rangle_0$ a bi-invariant metric on G. Note that $(G, \langle\ ,\ \rangle_0)$ has $\sec \geq 0$, and $\sigma = \mathrm{Span}\,\{X,Y\}$ has $\sec(\sigma) = 0$ if and only if $[X,Y] = 0$. We can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $\langle\ ,\ \rangle_0$. Given $X \in \mathfrak{g}$ we will always use $X_{\mathfrak{k}}$ and $X_{\mathfrak{p}}$ to denote the $\mathfrak{k}$ and $\mathfrak{p}$ components of X respectively.

Recall that

$$G \cong (G \times K)/\Delta K$$

via $(g,k) \longmapsto gk^{-1}$, where ΔK acts diagonally on the right of $G \times K$. Thus we may define a new left-invariant, right K-invariant metric $\langle\ ,\ \rangle_1$ (with $\sec \geq 0$) on G via the Riemannian submersion

$$\begin{aligned} (G \times K, \langle\ ,\ \rangle_0 \oplus t\langle\ ,\ \rangle_0|_{\mathfrak{k}}) &\longrightarrow (G, \langle\ ,\ \rangle_1) \\ (g,k) &\longmapsto gk^{-1}, \end{aligned}$$

where $t > 0$ and

$$\langle\ ,\ \rangle_1 = \langle\ ,\ \rangle_0|_{\mathfrak{p}} + \lambda\langle\ ,\ \rangle_0|_{\mathfrak{k}}, \quad \lambda = \frac{t}{t+1} \in (0,1). \tag{2.2.1}$$

Lemma 2.2.1. *The metric $\langle\ ,\ \rangle_0 \oplus t\langle\ ,\ \rangle_0|_{\mathfrak{k}}$ on $G \times K$ induces the metric $\langle\ ,\ \rangle_1$ on G.*

Proof. Let π be the submersion $G \times K \longrightarrow G$ arising from the diagonal action of K. Since $\langle\ ,\ \rangle_0 \oplus t\langle\ ,\ \rangle_0|_{\mathfrak{k}}$ is bi-invariant we will restrict our attention to the inner product on $\mathfrak{g} \oplus \mathfrak{k}$. Recall that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $\langle\ ,\ \rangle_0$.

The vertical subspace at (e,e) of the K-action is

$$\mathcal{V} = \{(W,W) \mid W \in \mathfrak{k}\}.$$

The horizontal subspace at (e,e) with respect to $\langle\ ,\ \rangle_0 \oplus t\langle\ ,\ \rangle_0|_{\mathfrak{k}}$ is therefore given by

$$\mathcal{H} = \left\{ \left(Z,\ -\frac{1}{t}Z_{\mathfrak{k}}\right) \;\middle|\; Z \in \mathfrak{g} \right\}.$$

Now, since

$$\begin{aligned} d\pi_{(e,e)} : \mathfrak{g} \oplus \mathfrak{k} &\longrightarrow \mathfrak{g} \\ (X,Y) &\longmapsto X - Y, \end{aligned}$$

it is clear that the horizontal lift of $X \in \mathfrak{g}$ is given by

$$\widetilde{X} = \left(X_{\mathfrak{p}} + \frac{t}{1+t} X_{\mathfrak{k}},\ -\frac{1}{1+t} X_{\mathfrak{k}} \right) \in \mathfrak{g} \oplus \mathfrak{k}.$$

Then

$$\begin{aligned}
\langle \widetilde{X}, \widetilde{Y} \rangle &= \left\langle X_{\mathfrak{p}} + \frac{t}{1+t} X_{\mathfrak{k}}, Y_{\mathfrak{p}} + \frac{t}{1+t} Y_{\mathfrak{k}} \right\rangle_0 + t \left\langle \frac{1}{1+t} X_{\mathfrak{k}}, \frac{1}{1+t} Y_{\mathfrak{k}} \right\rangle_0 \\
&= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle_0 + \left(\frac{t^2}{(1+t)^2} + \frac{t}{(1+t)^2} \right) \langle X_{\mathfrak{k}}, Y_{\mathfrak{k}} \rangle_0 \\
&= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle_0 + \frac{t}{1+t} \langle X_{\mathfrak{k}}, Y_{\mathfrak{k}} \rangle_0 \\
&= \langle X, Y \rangle_1
\end{aligned}$$

as desired. □

In particular notice that

$$\langle X, Y \rangle_1 = \langle X, \Phi(Y) \rangle_0, \quad \text{where } \Phi(Y) = Y_{\mathfrak{p}} + \lambda Y_{\mathfrak{k}},\ \lambda \in (0,1).$$

It is clear that the metric tensor Φ is invertible with inverse given by $\Phi^{-1}(Y) = Y_{\mathfrak{p}} + \frac{1}{\lambda} Y_{\mathfrak{k}}$.

Lemma 2.2.2 (Eschenburg)**.** *Let (G, K) be a symmetric pair. Then a plane $\sigma = \operatorname{Span}\{\Phi^{-1}(X), \Phi^{-1}(Y)\}$ has $\sec(\sigma) = 0$ with respect to $\langle\ ,\ \rangle_1$ if and only if*

$$0 = [X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}].$$

Recall that for a bi-invariant metric we get $\sec(X, Y) = 0$ if and only if $[X, Y] = 0$. For our left-invariant metric $\langle\ ,\ \rangle_1$ we have two extra conditions which must be satisfied for a plane to have zero-curvature, and hence we may have reduced the number of such planes.

Suppose we have a biquotient $G/\!\!/U$, where $U \subset G\times K \subset G\times G$ and G is equipped with a left-invariant, right K-invariant metric constructed as above. Then U acts by isometries on G and therefore the submersion $G \longrightarrow G/\!\!/U$ induces a metric on $G/\!\!/U$ from the metric on G. By our discussion of the O'Neill formula above we know that a zero-curvature plane on $G/\!\!/U$ with respect to the induced metric must lift to a horizontal zero-curvature plane in G.

In order to determine what it means for a plane to be horizontal we must first determine the vertical distribution on G. Note that this is independent of the choice of left-invariant metric on G. The fibre through a particular point $g \in G$ is

$$F_g := \{u_1 g u_2^{-1} \mid (u_1, u_2) \in U\}.$$

If $u(t) := \exp(tX)$, where $X = (X_1, X_2) \in \mathfrak{u}$ and $\mathfrak{u}$ is the Lie algebra of U, then $u_1(t)\, g\, u_2(t)^{-1}$ is a curve in F_g and

$$\frac{d}{dt}\, u_1(t)\, g\, u_2(t)^{-1}\, \Big|_{t=0} = (R_g)_* X_1 - (L_g)_* X_2 =: v_g(X)$$

is a typical vertical vector. The vector field $v(X)$ on G defined in such a way is the Killing vector field associated to X. Since G is equipped with a left-invariant metric we may shift the vertical space $V_g = v_g(\mathfrak{u})$ to the identity $e \in G$ by left-translation and get

$$\mathcal{V}_g := (L_{g^{-1}})_* V_g = \underline{v}_g(\mathfrak{u})$$

where

$$\underline{v}_g(X) := (L_{g^{-1}})_* v_g(X) = \mathrm{Ad}_{g^{-1}} X_1 - X_2.$$

We may therefore define the horizontal subspace at $g \in G$ by

$$\mathcal{H}_g := (L_{g^{-1}})_* H_g = \mathcal{V}_g^{\perp}.$$

It is important to remark that the horizontal subspace at g depends on the choice of left-invariant metric as it is defined by $\mathcal{V}_g^{\perp}$, where we are taking the orthogonal complement with respect to our metric.

The collection $\{\mathcal{V}_g \mid g \in G\}$ is a family of subspaces of $\mathfrak{g}$, none of which are naturally Lie algebras in general. Since left-translations are isometries, the transition from V_g and H_g to $\mathcal{V}_g$ and $\mathcal{H}_g$ will have no effect on our computations.

Suppose G is equipped with a bi-invariant metric. Eschenburg [E1] provides some sufficient conditions under which a horizontal zero-curvature in G projects to a zero-curvature plane in a biquotient $G/\!/U$. Wilking [Wi] has generalised this to show that, given any biquotient submersion $G \longrightarrow G/\!/U$, a horizontal zero-curvature plane in G must always project to a zero-curvature plane in $G/\!/U$. Tapp [Ta2] has recently generalised this result even further. We state his theorem below without proof.

Theorem 2.2.3 (Tapp, '07)**.** *Suppose G is a compact Lie group equipped with a bi-invariant metric and that $G \longrightarrow B$ is a Riemannian submersion. Then a horizontal zero-curvature plane in G projects to a horizontal zero-curvature plane in B.*

It follows immediately from the above theorem that if we have a pair of Riemannian submersions $G \longrightarrow M \longrightarrow B$, where G is equipped with a bi-invariant metric, then a horizontal zero-curvature plane in M must project to a zero-curvature plane in B.

Notice that in the metric construction on $G/\!\!/U$ described above we have Riemannian submersions $G \times K \longrightarrow G \longrightarrow G/\!\!/U$ where $G \times K$ is equipped with a bi-invariant metric. Therefore in order to find zero-curvature planes in $(G, \langle\ ,\ \rangle_1)/\!\!/U$ we may concentrate exclusively on the more tractable problem of finding horizontal zero-curvature planes in G.

Chapter 3

Eschenburg Spaces

3.1 Eschenburg's results

Recall that the Eschenburg spaces are defined as $E^7_{p,q} := SU(3)/\!\!/S^1_{p,q}$, where $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3) \in \mathbb{Z}^3$, $\sum p_i = \sum q_i$, and $S^1_{p,q}$ acts on $SU(3)$ via

$$z \star A = \begin{pmatrix} z^{p_1} & & \\ & z^{p_2} & \\ & & z^{p_3} \end{pmatrix} A \begin{pmatrix} \bar{z}^{q_1} & & \\ & \bar{z}^{q_2} & \\ & & \bar{z}^{q_3} \end{pmatrix}, \quad A \in SU(3), z \in S^1.$$

The action is free if and only if

$$(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1 \ \text{ for all } \ \sigma \in S_3. \tag{3.1.1}$$

Let $K = U(2) \hookrightarrow G = SU(3)$ via

$$A \in U(2) \longmapsto \begin{pmatrix} A & \\ & \alpha \end{pmatrix} \in SU(3), \quad \alpha = \overline{\det(A)}.$$

(G, K) is a rank one symmetric pair. Let $\langle\ ,\ \rangle_0$ be the bi-invariant metric on G given by $\langle X, Y\rangle_0 = -\operatorname{Re}\operatorname{tr}(XY)$. We can write $\mathfrak{su}(3) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to

$\langle\ ,\ \rangle_0$. We define a new left-invariant, right K-invariant metric $\langle\ ,\ \rangle_1$ (with $\sec \geq 0$) on G as in (2.2.1) and may therefore apply Lemma 2.2.2.

From §2.1 we know that, for the $S^1_{p,q}$-action, permuting the p_i's and permuting q_1, q_2 are isometries, while permuting the q_i's and swapping p, q are diffeomorphisms.

Let

$$Y_1 := i\begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad Y_3 := i\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \in \mathfrak{g} = \mathfrak{su}(3).$$

Using Lemma 2.2.2 Eschenburg [E1] showed that in this special case we can easily determine when a plane in $\mathfrak{g}$ has zero-curvature.

Lemma 3.1.1 (Eschenburg)**.** *$\sigma = \mathrm{Span}\,\{X, Y\} \subset \mathfrak{su}(3)$ has $\sec(\sigma) = 0$ with respect to $\langle\ ,\ \rangle_1$ if and only if either $Y_3 \in \sigma$, or $\mathrm{Ad}(k)Y_1 \in \sigma$ for some $k \in K$.*

We are now in a position to discuss when an Eschenburg space $E^7_{p,q}$ admits positive curvature.

Theorem 3.1.2 (Eschenburg '84)**.** *$E^7_{p,q} := (SU(3), \langle\ ,\ \rangle_1)/\!\!/S^1_{p,q}$ has positive curvature if and only if*

$$q_i \notin [\underline{p}, \overline{p}] \quad \text{for } i = 1, 2, 3, \tag{3.1.2}$$

where $\underline{p} := \min\{p_1, p_2, p_3\}$, $\overline{p} := \max\{p_1, p_2, p_3\}$.

Proof. We will first prove that the condition (3.1.2) gives positive curvature. By Lemma 3.1.1 we need only show that we may choose an ordering on the q_i's so that Y_3 and $\mathrm{Ad}(k)Y_1$ are never horizontal.

From our discussion of vertical spaces in §2.1 we find that the vertical subspace at $A = (a_{ij}) \in SU(3)$ is

$$\mathcal{V}_A = \left\{ \theta\, v_A \;\middle|\; \theta \in \mathbb{R}, v_A := \operatorname{Ad}_{A^*} P - Q, P = i \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & p_3 \end{pmatrix}, Q = i \begin{pmatrix} q_1 & & \\ & q_2 & \\ & & q_3 \end{pmatrix} \right\}.$$

Then

$$0 = \langle v_A, Y_3 \rangle_1 \quad \Longleftrightarrow \quad \sum_{j=1}^{3} |a_{j3}|^2 p_j = q_3 \tag{3.1.3}$$

$$0 = \langle v_A, \operatorname{Ad}_k Y_1 \rangle_1 \quad \Longleftrightarrow \quad \sum_{j=1}^{3} |(Ak)_{j1}|^2 p_j = |k_{11}|^2 q_1 + |k_{21}|^2 q_2. \tag{3.1.4}$$

In order to derive equation (3.1.3), notice that $Y_3 \in \mathfrak{k}$. Thus $0 = \langle v_A, Y_3 \rangle_1$ if and only if $0 = \langle v_A, Y_3 \rangle_0$. Now, since $\langle X, Y \rangle_0 = -\operatorname{Re} \operatorname{tr}(XY)$ and $\operatorname{Ad}_{A^*}(w_{ij}) = \left(\sum_{k,\ell=1}^{3} \bar{a}_{ki} a_{\ell j} w_{k\ell} \right)$, it follows that

$$\begin{aligned}
0 &= \langle v_A, Y_3 \rangle_0 \\
&= \langle \operatorname{Ad}_{A^*} P, Y_3 \rangle_0 - \langle Q, Y_3 \rangle_0 \\
&= \left(\sum_{\ell=1}^{3} (|a_{\ell 1}|^2 + |a_{\ell 2}|^2 - 2|a_{\ell 3}|^2) p_\ell \right) - (q_1 + q + 2 - 2q_3) \\
&= \left(\sum_{\ell=1}^{3} (1 - 3|a_{\ell 3}|^2) p_\ell \right) - \left(\sum_{\ell=1}^{3} q_\ell - 3q_3 \right) \qquad \text{since } A \text{ is unitary} \\
&= -3 \left(\sum_{\ell=1}^{3} |a_{\ell 3}|^2 p_\ell \right) + 3q_3 \qquad \text{since } \sum p_\ell = \sum q_\ell,
\end{aligned}$$

as desired. Equation (3.1.4) follows similarly.

Now, since $q_i \notin [\underline{p}, \overline{p}], i = 1, 2, 3$, and $\sum p_j = \sum q_j$, we know that two of the q_i's must lie on one side of $[\underline{p}, \overline{p}]$, and one on the other. We reorder and relabel the q_i's so that q_1, q_2 lie on the same side of $[\underline{p}, \overline{p}]$. Since A and k are both unitary we therefore have that there are no solutions to either (3.1.3) or (3.1.4). Hence $E^7_{p,q}$ has positive curvature.

For the converse suppose that $E^7_{p,q}$ has positive curvature. If $q_i \in [\underline{p}, \overline{p}]$ for some $i = 1, 2, 3$ then by continuity there exists a solution to either (3.1.3) or (3.1.4), and hence either Y_3 or $\operatorname{Ad}_k Y_1$ is horizontal. Since the orbits of $S^1_{p,q}$ are one-dimensional and by Lemma 3.1.1, we can always find another horizontal vector X which, together with either Y_3 or $\operatorname{Ad}_k Y_1$, will span a zero-curvature plane. Theorem 2.2.3 then implies that this horizontal zero-curvature plane must project to a zero-curvature plane in $E^7_{p,q}$ and so we have a contradiction.

□

3.2 New results

We will now discuss some new results on the curvature of general Eschenburg spaces.

Theorem 3.2.1. *All Eschenburg spaces admit a metric with quasi-positive curvature.*

Proof. We need to find a point in $SU(3)$ at which there are no horizontal zero-curvature planes, i.e. at which Y_3 and $\operatorname{Ad}(k)Y_1$ are not horizontal.

Consider $A = \left(\begin{smallmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{smallmatrix}\right) \in SU(3)$. Thus $|a_i| = 1, i = 1, 2, 3,$ and so equation

(3.1.4) becomes

$$\begin{aligned} & |k_{11}|^2 p_1 + |k_{21}|^2 p_2 = |k_{11}|^2 q_1 + |k_{21}|^2 q_2 \\ \Longleftrightarrow \quad & (p_1 - q_1)|k_{11}|^2 + (p_2 - q_2)|k_{21}|^2 = 0. \end{aligned}$$

Therefore, if

$$(p_1 - q_1)(p_2 - q_2) > 0 \tag{3.2.1}$$

there is no $k \in K$ satisfying (3.1.4), i.e. $\mathrm{Ad}_k Y_1$ is not horizontal at A.

Equation (3.1.3) becomes $p_3 = q_3$. However, (3.2.1), together with $\sum p_i = \sum q_i$, implies that $p_3 \neq q_3$, i.e. that Y_3 is not horizontal at A.

Thus, if (3.2.1) holds, then $E^7_{p,q}$ has sec > 0 at $[A]$, where $A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in SU(3)$.

Recall the freeness condition (3.1.1) and that permuting the p_i's and q_j's are diffeomorphisms. Therefore, as long as there is no $i \in \{1, 2, 3\}$ such that $p_i = q_j$ for all $j \in \{1, 2, 3\}$, we may always reorder and relabel the p_i's and q_j's such that (3.2.1) holds.

By (3.1.1), the only Eschenburg space satisfying the condition "*there is an* $i \in \{1, 2, 3\}$ *such that* $p_i = q_j$ *for all* $j \in \{1, 2, 3\}$" is the Aloff-Wallach space $W_{-1,1} := E^7_{p,q}, p = (-1, 1, 0), q = (0, 0, 0)$. However, Wilking [Wi] has shown that $W_{-1,1}$ admits a metric with almost positive curvature, and so we are done. □

The subfamily $E^7_n := E^7_{p,q}, p = (1, 1, n), q = (0, 0, n+2)$, admits a cohomogeneity-one action by $SU(2) \times SU(2)$. These cohomogeneity-one Eschenburg spaces are discussed in great detail in [GSZ]. We may assume that $n \geq 0$ since $E^7_n \cong E^7_{-(n+1)}$. This is a simple consequence of the facts that $\Delta S^1 = \{\mathrm{diag}(z,z,z) \mid z \in S^1\}$ commutes with $SU(3)$ and that taking the complex conjugate of elements in $S^1_{p,q}$ preserves the orbits of the $S^1_{p,q}$-action. Moreover, by Theorem 3.1.2, $n > 0$ implies that E^7_n admits a metric with positive curvature.

Theorem 3.2.2. *E^7_0 admits a metric with almost positive curvature.*

Proof. Given $p = (1,1,0)$ and $q = (0,0,2)$, equations (3.1.3) and (3.1.4) become

$$2 = |a_{13}|^2 + |a_{23}|^2 \tag{3.2.2}$$

and

$$\begin{aligned} & |(Ak)_{11}|^2 + |(Ak)_{21}|^2 = 0 \\ \iff \quad & (Ak)_{11} = (Ak)_{21} = 0 \\ \iff \quad & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} k_{11} \\ k_{21} \end{pmatrix} = 0 \end{aligned} \tag{3.2.3}$$

respectively. Since $A \in SU(3)$ it is clear that (3.2.2) cannot be satisfied. Since $k \in K = U(2)$, we are only interested in solutions $\left(\begin{smallmatrix} k_{11} \\ k_{21} \end{smallmatrix}\right) \neq 0$. This occurs if and only if

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0,$$

which defines a codimension two sub-variety $\Omega \subset SU(3)$ of points with horizontal zero-curvature planes. Moreover it is easy to check that Ω is a smooth sub-variety. Since the equation which defines Ω is preserved under the $S^1_{p,q}$-action, E^7_0 has almost positive curvature and points in E_0^7 with zero-curvature planes form a smooth codimension two submanifold. $\square$

We may fix a particular metric on $E_{p,q}^7$ by choosing $p_1 \leq p_2 \leq p_3$ and $q_1 \leq q_2 \leq q_3$. Therefore Eschenburg's positive curvature condition is

$$q_1 \leq q_2 < p_1 \leq p_2 \leq p_3 < q_3 \quad \text{or} \quad q_1 < p_1 \leq p_2 \leq p_3 < q_2 \leq q_3. \tag{3.2.4}$$

It is natural to ask what happens when $q_2 = p_1$ or $q_2 = p_3$, which we refer to as the "boundary" of the positive curvature condition.

Lemma 3.2.3. *The only free $S_{p,q}^1$-actions on $SU(3)$ satisfying $q_2 = p_1$ or $q_2 = p_3$ are, up to diffeomorphism,*

(i) $p = (0,0,0)$ *and* $q = (-1,0,1)$, *and*

(ii) $p = (0,1,1)$ *and* $q = (0,0,2)$.

Proof. We need only consider the case $q_2 = p_1$, since it is clear that $E_{p,q}^7$ is diffeomorphic to $E_{p',q'}^7$, where $p' = (-p_3,-p_2,-p_1)$, $q' = (-q_3,-q_2,-q_1)$. Since ΔS^1 commutes with $SU(3)$ we may write $p = (0,p_2,p_3)$ and $q = (q_1,0,q_3)$ without loss of generality. By considering the freeness condition (3.1.1) and the ordering of our integers we must have $p = (0,p_2,p_3)$ and $q = (p_2-1,0,p_2+1)$. Since $\sum p_i = \sum q_i$ we have $p = (0,p_2,p_2)$ and $q = (p_2-1,0,p_2+1)$. Hence, since we have assumed that our triples of integers are ordered, i.e. $0 \leq p_2$ and $p_2 - 1 \leq 0 \leq p_2 + 1$, either $p_2 = 0$ or $p_2 = 1$ as desired. $\square$

Notice that the resulting manifolds are diffeomorphic to the exceptional Aloff-Wallach space $W^7_{-1,1}$ and the exceptional cohomogeneity-one Eschenburg space E^7_0 for actions (i) and (ii) respectively. As previously discussed, both manifolds have been shown to admit metrics with almost positive curvature. Note also that action (i) is the action given by $q_1 < q_2 = p_1 = p_2 = p_3 < q_3$, and action (ii) is the action given by $q_1 = q_2 = p_1 < p_2 = p_3 < q_3$. Even though there are no other manifolds on the boundary of the positive curvature condition, we can prove the following:

Theorem 3.2.4. *If*

$$q_1 < q_2 = p_1 < p_2 \le p_3 < q_3 \quad \text{or} \quad q_1 < p_1 \le p_2 < p_3 = q_2 \le q_3, \tag{3.2.5}$$

then the singular space $E^7_{p,q}$ admits a metric with almost positive curvature. In particular, all orbifolds $E^7_{p,q}$ satisfying (3.2.5) have almost positive curvature.

Proof. As in the proof of Lemma 3.2.3, we need only consider

$$q_1 < q_2 = p_1 < p_2 \le p_3 < q_3, \tag{3.2.6}$$

since $E^7_{p,q}$ is diffeomorphic to $E^7_{p',q'}$, where as before $p' = (-p_3, -p_2, -p_1)$ and $q' = (-q_3, -q_2, -q_1)$.

Notice that (3.2.6) implies that (3.1.3) has no solutions, since $q_3 > p_i$ for all $i = 1, 2, 3$.

Consider for a moment the more general case of Eschenburg spaces $E^7_{p,q}$ given by $q_1 < p_1 \le q_2 < p_2 \le p_3 < q_3$, hence not admitting positive curvature. Suppose that there is a $k \in K$ such that $\mathrm{Ad}_k Y_1$ is horizontal at some $A \in SU(3)$. Then (3.1.4) implies that

$$p_1 \leq \sum_{j=1}^{3} |(Ak)_{j1}|^2 p_j = |k_{11}|^2 q_1 + |k_{21}|^2 q_2 \leq q_2.$$

Since $|k_{11}|^2 + |k_{21}|^2 = 1$ we thus have

$$p_1 \leq |k_{11}|^2(q_1 - q_2) + q_2 \leq q_2 \quad \text{and} \quad p_1 \leq q_1 + |k_{21}|^2(q_2 - q_1) \leq q_2,$$

which are equivalent to

$$0 \leq |k_{11}|^2 \leq \frac{q_2 - p_1}{q_2 - q_1} \quad \text{and} \quad \frac{p_1 - q_1}{q_2 - q_1} \leq |k_{21}|^2 \leq 1.$$

In particular, when the hypothesis of the theorem is satisfied, namely $p_1 = q_2$, we get $|k_{11}|^2 = 0$ and $|k_{21}|^2 = 1$, i.e.

$$k = \begin{pmatrix} 0 & k_{12} & 0 \\ k_{21} & 0 & 0 \\ 0 & 0 & -\overline{k_{12}k_{21}} \end{pmatrix} \in K = U(2).$$

Hence (3.1.4) becomes

$$\begin{aligned} & |a_{12}|^2 p_1 + |a_{22}|^2 p_2 + |a_{32}|^2 p_3 = q_2 = p_1 \\ \iff \quad & |a_{22}|^2 (p_2 - p_1) + |a_{32}|^2 (p_3 - p_1) = 0, \quad \text{since} \quad A \in SU(3) \\ \iff \quad & a_{22} = a_{32} = 0, \quad \text{since} \quad p_1 < p_2 \leq p_3 \\ \iff \quad & A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \in SU(3). \end{aligned}$$

The set of such $A \in SU(3)$ is preserved under the $S^1_{p,q}$-action, hence projects to a set of measure zero in $E^7_{p,q}$. Therefore $E^7_{p,q}$ has almost positive curvature. $\square$

We may also examine how large the set of zero-curvature planes is at each point of the set $\left\{ S^1_{p,q} \star A \;\middle|\; A = \left(\begin{smallmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{smallmatrix}\right) \right\} \subset E^7_{p,q}$, with $q_1 < q_2 = p_1 < p_2 \leq p_3 < q_3$. This is equivalent to determining how large the set of horizontal zero-curvature planes is at each $A = \left(\begin{smallmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{smallmatrix}\right) \in SU(3)$.

Proposition 3.2.5. *If $q_1 < q_2 = p_1 < p_2 \leq p_3 < q_3$ then there is a one-dimensional family of horizontal zero-curvature planes at each point $A = \left(\begin{smallmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{smallmatrix}\right) \in SU(3)$.*

Proof. Recall we have shown in the proof of Theorem 3.2.4 that Y_3 is never horizontal, and $\mathrm{Ad}_k Y_1$ being horizontal at A implies that $k = \left(\begin{smallmatrix} 0 & k_{12} & 0 \\ k_{21} & 0 & 0 \\ 0 & 0 & -\overline{k_{12}k_{21}} \end{smallmatrix}\right) \in K = U(2)$. Hence $\mathrm{Ad}_k Y_1 = Y_2 := i \left(\begin{smallmatrix} 1 & & \\ & -2 & \\ & & 1 \end{smallmatrix}\right)$.

Let $Y = \Phi^{-1}(Y_2)$, where $\langle X, Z\rangle_1 = \langle X, \Phi(Z)\rangle_0$. Let $X \in \mathcal{H}_A$ be such that $\mathrm{Span}\,\{X, Y\}$ is a horizontal zero-curvature plane. Then, by Lemma 2.2.2 and since $\Phi^{-1}(Y_2) = \frac{1}{\lambda} Y_2 \in \mathfrak{k}$, $[X, Y] = [X_{\mathfrak{k}}, Y] = 0$, which is equivalent to

$$\begin{aligned} & [X, Y_2] = [X_{\mathfrak{k}}, Y_2] = 0 \\ \iff \quad & [X, Y_2] = 0 \\ \iff \quad & X = \begin{pmatrix} is & 0 & x \\ 0 & it & 0 \\ -\bar{x} & 0 & -i(s+t) \end{pmatrix}, \end{aligned}$$

where $s, t \in \mathbb{R}$, $x \in \mathbb{C}$. We may assume without loss of generality that $\langle X, Y\rangle_1 = 0$.

Hence

$$X = \begin{pmatrix} is & 0 & x \\ 0 & 0 & 0 \\ -\bar{x} & 0 & -is \end{pmatrix}.$$

The set of such X is 3-dimensional. We also require that X is horizontal, i.e. $\langle X, \mathrm{Ad}_{A^*} P - Q\rangle_1 = 0$, and without loss of generality we may assume that $||X||^2 = 1$.

Thus, for each $A = \left(\begin{smallmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{smallmatrix}\right) \in SU(3)$ there is a one-dimensional family of horizontal zero-curvature planes Span $\{X, Y\}$. □

3.3 Lens spaces and closed geodesics

Recall that a lens space $L(p, q; d)$, $(p, d) = 1$, $(q, d) = 1$, is defined as the quotient

$$L(p, q; d) := S^3/\mathbb{Z}_d,$$

where $\mathbb{Z}_d = \{\xi \in S^1 \mid \xi^d = 1\}$ acts on $S^3 = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\}$ via

$$\xi \cdot (x.y) = (\xi^p x, \xi^q y).$$

Now there are nine copies of $U(2)$ embedded in $SU(3)$, namely

$$U(2)_{ij} := \left\{ \tau_i \begin{pmatrix} A & \\ & \overline{\det A} \end{pmatrix} \tau_j \;\middle|\; A \in U(2) \right\}, \quad 1 \leq i, j \leq 3,$$

where $\tau_\ell \in O(3)$ is the linear map that interchanges the ℓth vector of the canonical basis with the third one. Note that (i, j) denotes the position of the entry which has norm 1.

Similarly there are six 2-dimensional tori embedded in $SU(3)$, namely

$$T^2_\sigma := \left\{ \sigma \begin{pmatrix} z & & \\ & w & \\ & & \overline{zw} \end{pmatrix} \;\middle|\; z, w \in S^1 \right\},$$

where σ is an element of S_3 which permutes the columns of $SU(3)$.

We define $\mathcal{L}_{ij} := \pi(U(2)_{ij})$ and $\mathcal{C}_\sigma := \pi(T^2_\sigma)$, where $\pi : SU(3) \longrightarrow E^7_{p,q}$. It is clear that the $\mathcal{C}_\sigma$ are circles.

Proposition 3.3.1 (Florit, Ziller '06). *$\mathcal{L}_{ij}$ is a totally geodesic lens space in $E^7_{p,q}$.*

Proof. As the general case is analogous, we will prove only that $\mathcal{L}_{33}$ is a lens space, i.e. the case

$$U(2)_{33} = \left\{ \begin{pmatrix} A & \\ & \overline{\det A} \end{pmatrix} \;\middle|\; A \in U(2) \right\}.$$

Notice that every $B \in U(2)_{33}$ may be written uniquely in the form

$$B = \begin{pmatrix} 1 & & \\ & \lambda & \\ & & \bar{\lambda} \end{pmatrix} \begin{pmatrix} x & y & \\ -\bar{y} & \bar{x} & \\ & & 1 \end{pmatrix}, \quad \lambda \in S^1, \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \in SU(2),$$

since $U(2) \cong S^1 \ltimes SU(2)$. Hence, since $\sum p_i = \sum q_i$, elements of the $S^1_{p,q}$-orbit of B have the form

$$\begin{pmatrix} 1 & & \\ & \lambda \bar{z}^{(p_3-q_3)} & \\ & & \bar{\lambda} z^{(p_3-q_3)} \end{pmatrix} \begin{pmatrix} x z^{(p_1-q_1)} & y z^{(p_1-q_2)} & \\ -\bar{y} \bar{z}^{(p_1-q_2)} & \bar{x} \bar{z}^{(p_1-q_1)} & \\ & & 1 \end{pmatrix}.$$

By choosing z such that $z^{(p_3-q_3)} = \lambda$ we see that each orbit intersects $SU(2)$, and so we need only consider the image under $S^1_{p,q}$ of points in $SU(2)$. However, there are $p_3 - q_3$ choices for z, namely $\{\lambda, \lambda\xi, \ldots, \lambda\xi^{p_3-q_3-1} \mid \xi^{p_3-q_3} = 1\}$, i.e. each orbit intersects $SU(2)$ in $p_3 - q_3$ places.

Recall that $SU(2) \cong S^3$ via $\left(\begin{smallmatrix} x & y \\ -\bar{y} & \bar{x} \end{smallmatrix}\right) \longmapsto (x, y)$. Therefore the group

$$\mathbb{Z}_{p_3-q_3} = \{\xi \mid \xi^{p_3-q_3} = 1\}$$

acts on S^3 via $\xi \cdot (x, y) = (x\xi^{p_1-q_1}, y\xi^{p_1-q_2})$, and

$$(p_1 - q_1, p_3 - q_3) = 1, \quad (p_1 - q_2, p_3 - q_3) = 1$$

by the freeness of the $S^1_{p,q}$-action on $SU(3)$. Hence, by our definition of lens spaces, $\mathcal{L}_{33} = L(p_1 - q_1, p_1 - q_2; p_3 - q_3)$.

In order to prove that $\mathcal{L}_{ij}$ is totally geodesic in $E^7_{p,q}$ we first assume that $U(2)_{ij}$ is totally geodesic in $SU(3)$ (with respect to $\langle\ ,\ \rangle_1$). A geodesic γ in $\mathcal{L}_{ij}$ lifts to a horizontal geodesic $\tilde{\gamma}$ in $U(2)_{ij}$, since $S^1_{p,q} \star A \subset U(2)_{i,j}$ for all $A \in U(2)_{ij}$ and $\tilde{\gamma}'(0) \perp S^1_{p,q} \star \tilde{\gamma}(0)$.

Now, since we assumed that $U(2)_{ij}$ is totally geodesic, this implies that $\tilde{\gamma}$ is a horizontal geodesic in $SU(3)$. Hence $\tilde{\gamma}$ projects to a geodesic in $E^7_{p,q}$, which by uniqueness must be γ. Thus $\mathcal{L}_{ij}$ is totally geodesic in $E^7_{p,q}$.

It remains to show that $U(2)_{ij}$ is totally geodesic in $SU(3)$. Consider isometries of $SU(3)$ given by $A \longmapsto z^r \cdot A \cdot \bar{z}^{r_\sigma}$, where

$$z^r := \begin{pmatrix} z^{r_1} & & \\ & z^{r_2} & \\ & & z^{r_3} \end{pmatrix}$$

with two of r_1, r_2, r_3 equal, and

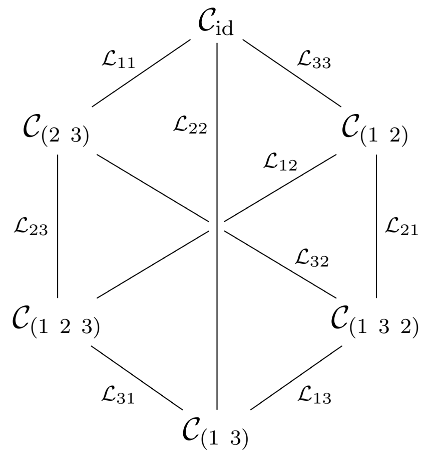
$$z^{r_\sigma} := \begin{pmatrix} z^{r_{\sigma(1)}} & & \\ & z^{r_{\sigma(2)}} & \\ & & z^{r_{\sigma(3)}} \end{pmatrix}, \quad \sigma \in S_3.$$

Each $U(2)_{ij}$ is the fixed point set of such an isometry and hence totally geodesic in $SU(3)$. □

Now notice that the circles $\mathcal{C}_\sigma$ are the intersections of two totally geodesic submanifolds. Hence we have immediately

Corollary 3.3.2. *$\mathcal{C}_\sigma$ is a closed geodesic in $E^7_{p,q}$.*

We can arrange the lens spaces and closed geodesics in the following schematic diagram



Suppose now that we fix a metric on an Eschenburg space $E^7_{p,q}$ by specifying $p_1 \leq p_2 \leq p_3$ and $q_1 \leq q_2 \leq q_3$. By again examining equations (3.1.3) and (3.1.4) we can find conditions under which all of the points in each of the nine lens spaces $\mathcal{L}_{ij}$ and each of the six closed geodesics $\mathcal{C}_\sigma$ admit positive curvature. These conditions are collected in Tables 3.1 and 3.2.

As an example, if we assume that $p_1 > q_1$ and $p_2 > q_2$, then we have positive curvature on $\mathcal{L}_{11}, \mathcal{L}_{22}$ and $\mathcal{L}_{32}$ (and hence on $\mathcal{C}_{\mathrm{id}}, \mathcal{C}_{(2\ 3)}, \mathcal{C}_{(1\ 3\ 2)}$ and $\mathcal{C}_{(1\ 3)}$). With the extra assumption that $q_2 < 0 < p_2$ we get positive curvature on $\mathcal{L}_{13}$. However we get positive curvature on the remaining lens spaces if and only if $q_2 < p_1$, i.e.

$$q_1 \leq q_2 < p_1 \leq p_2 \leq p_3 < q_3, \quad q_2 < 0 < p_2,$$

in which case the entire manifold $E^7_{p,q}$ (where we have assumed the action is free) has positive curvature, by Theorem 3.1.2.

Notice that, in particular, the "worst" lens space when $p_1 > q_1$ and $p_2 > q_2$, i.e. the lens space containing no points of positive curvature, is exactly the lens space

Table 3.1: Conditions for positive curvature on $\mathcal{L}_{ij}$

$\mathcal{L}_{ij}$	Conditions for sec > 0
$\mathcal{L}_{11}$	$(p_1 - q_1)(p_2 - q_2) > 0$ and $(p_1 - q_1)(p_3 - q_2) > 0$
$\mathcal{L}_{12}$	$(p_1 - q_2)(p_2 - q_1) > 0$ and $(p_1 - q_2)(p_3 - q_1) > 0$
$\mathcal{L}_{13}$	$p_2p_3 > 0$, $q_1q_2 > 0$, $p_2q_1 < 0$; or $p_2 = p_3 = 0$, $q_1q_2 > 0$; or $q_1 = q_2 = 0$, $p_2p_3 > 0$
$\mathcal{L}_{21}$	$(p_1 - q_2)(p_2 - q_1) > 0$ and $(p_2 - q_1)(p_3 - q_2) > 0$
$\mathcal{L}_{22}$	$(p_1 - q_1)(p_2 - q_2) > 0$ and $(p_3 - q_1)(p_2 - q_2) > 0$
$\mathcal{L}_{23}$	$p_1p_3 > 0$, $q_1q_2 > 0$, $p_1q_1 < 0$; or $p_1 = p_3 = 0$, $q_1q_2 > 0$; or $q_1 = q_2 = 0$, $p_1p_3 > 0$
$\mathcal{L}_{31}$	$(p_1 - q_2)(p_3 - q_1) > 0$ and $(p_2 - q_2)(p_3 - q_1) > 0$
$\mathcal{L}_{32}$	$(p_1 - q_1)(p_3 - q_2) > 0$ and $(p_2 - q_1)(p_3 - q_2) > 0$
$\mathcal{L}_{33}$	$p_1p_2 > 0$, $q_1q_2 > 0$, $p_1q_1 < 0$; or $p_1 = p_2 = 0$, $q_1q_2 > 0$; or $q_1 = q_2 = 0$, $p_1p_2 > 0$

Table 3.2: Conditions for positive curvature on $\mathcal{C}_\sigma$

$\mathcal{C}_\sigma$	Condition for sec > 0
$\mathcal{C}_{\text{id}}$	$(p_1 - q_1)(p_2 - q_2) > 0$
$\mathcal{C}_{(1\ 2)}$	$(p_1 - q_2)(p_2 - q_1) > 0$
$\mathcal{C}_{(1\ 3)}$	$(p_3 - q_1)(p_2 - q_2) > 0$
$\mathcal{C}_{(2\ 3)}$	$(p_1 - q_1)(p_3 - q_2) > 0$
$\mathcal{C}_{(1\ 2\ 3)}$	$(p_1 - q_2)(p_3 - q_1) > 0$
$\mathcal{C}_{(1\ 3\ 2)}$	$(p_2 - q_1)(p_3 - q_2) > 0$

$\mathcal{L}_{12}$ which arises as the set of points admitting zero-curvature planes in the case $q_1 < q_2 = p_1 < p_2 \leq p_3 < q_3$, of which there are only singular examples, and these examples admit almost positive curvature.

Chapter 4

Bazaikin Spaces

4.1 Positive curvature

The proof of positive curvature on an infinite subfamily of the Bazaikin spaces follows from essentially the same techniques as in the case of the Eschenburg spaces. We recall the proof as given in [Zi1], with a slight modification which will allow us to prove the results in Theorem A(iii) and (iv).

Recall that the Bazaikin spaces are defined as

$$B^{13}_{q_1,\dots,q_5} := SU(5)/\!\!/Sp(2)\cdot S^1_{q_1,\dots,q_5},$$

where $q_1,\dots,q_5 \in \mathbb{Z}$, and

$$Sp(2)\cdot S^1_{q_1,\dots,q_5} = (Sp(2)\times S^1_{q_1,\dots,q_5})/\mathbb{Z}_2, \quad \mathbb{Z}_2 = \{\pm(1,I)\},$$

acts effectively on $SU(5)$ via

$$[A, z] \star B = \begin{pmatrix} z^{q_1} & & \\ & \ddots & \\ & & z^{q_5} \end{pmatrix} B \begin{pmatrix} \hat{A} & \\ & \bar{z}^q \end{pmatrix},$$

with $z \in S^1$, $A \in Sp(2) \hookrightarrow SU(4)$, $B \in SU(5)$, and $q = \sum q_i$. We recall that

$$\begin{array}{rcl} Sp(2) & \hookrightarrow & SU(4) \\ A = S + Tj & \longmapsto & \hat{A} = \begin{pmatrix} S & T \\ -\bar{T} & \bar{S} \end{pmatrix}. \end{array}$$

It is not difficult to show that the action of $Sp(2) \cdot S^1_{q_1,\dots,q_5}$ is free if and only all $q_1, \dots, q_5$ are odd and $(q_{\sigma(1)} + q_{\sigma(2)}, q_{\sigma(3)} + q_{\sigma(4)}) = 2$ for all $\sigma \in S_5$.

Let $G = SU(5) \supset K = U(4)$, where $K \hookrightarrow G$ via

$$A \longmapsto \begin{pmatrix} A & \\ & \overline{\det A} \end{pmatrix}.$$

Then (G, K) is a rank one symmetric pair, with Lie algebras $(\mathfrak{g}, \mathfrak{k})$. With respect to the bi-invariant metric $\langle X, Y \rangle_0 = -\operatorname{Re} \operatorname{tr} XY$ we may write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Define a metric, $\langle\ ,\ \rangle_1$, on G as in 2.2.1 which is left-invariant and right K-invariant. In particular we have $\langle X, Y \rangle_1 = \langle X, \Phi(Y) \rangle_0$, where $\Phi(Y) = Y_{\mathfrak{p}} + \lambda Y_{\mathfrak{k}}$, $\lambda \in (0, 1)$. By Lemma 2.2.2 we know that a plane $\sigma = \operatorname{Span}\{\Phi^{-1}(X), \Phi^{-1}(Y)\} \subset \mathfrak{g}$ has zero-curvature with respect to $\langle\ ,\rangle_1$ if and only if

$$0 = [X, Y] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}].$$

It is clear that the action of $U := Sp(2) \cdot S^1_{q_1,\dots,q_5}$ is by isometries since U is contained in $SU(5) \times U(4) \subset SU(5) \times SU(5)$. Therefore we get an induced submersion metric on $B^{13}_{q_1,\dots,q_5} = G /\!\!/ U$.

Since $\langle \ , \ \rangle_1$ is left-invariant we may left-translate back to the Lie algebra $\mathfrak{g}$ without changing our computations. Therefore the vertical subspace at $A \in SU(5)$ with respect to the U-action may be written as

$$\mathcal{V}_A = \left\{ \theta \operatorname{Ad}_{A^*} Q - \begin{pmatrix} X & \\ & i\theta q \end{pmatrix} \;\middle|\; \theta \in \mathbb{R}, Q = i \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_5 \end{pmatrix}, X \in \mathfrak{sp}(2) \subset \mathfrak{su}(4) \right\}$$

where $A^* = \bar{A}^t$. Our aim is to determine when zero-curvature planes with respect to $\langle \ , \ \rangle_1$ are horizontal at $A \in SU(5)$. A vector $\Phi^{-1}(X)$ is orthogonal to $\mathcal{V}_A$ with respect to $\langle \ , \ \rangle_1$ if and only if

$$\left\langle X, \operatorname{Ad}_{A^*} Q - \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & iq \end{pmatrix} \right\rangle_0 = 0 \quad \text{and} \quad X \perp_0 \mathfrak{sp}(2) \subset \mathfrak{su}(4), \tag{4.1.1}$$

where $\perp_0$ denotes orthogonality with respect to $\langle \ , \ \rangle_0$.

Lemma 4.1.1. $\sigma = \operatorname{Span}\{\Phi^{-1}(X), \Phi^{-1}(Y)\} \subset \mathfrak{g}$ *is a horizontal zero-curvature plane with respect to* $\langle \ , \ \rangle_1$ *if and only if either*

$$W_1 := \operatorname{diag}(i, i, i, i, -4i) \quad \text{or} \quad W_2 := \operatorname{Ad}_k \operatorname{diag}(2i, -3i, 2i, -3i, 2i),$$

for some $k \in Sp(2)$*, is in* σ *and is horizontal.*

Proof. Suppose that $\sigma = \operatorname{Span}\{\Phi^{-1}(X), \Phi^{-1}(Y)\}$ has zero-curvature with respect to $\langle \ , \ \rangle_1$. Then, since $[X_\mathfrak{p}, Y_\mathfrak{p}] = 0$ by Lemma 2.2.2, we may assume without loss of generality that $Y_\mathfrak{p} = 0$, i.e $X = X_\mathfrak{p} + X_\mathfrak{k}$, $Y = Y_\mathfrak{k}$.

If we also have $X_\mathfrak{p} = 0$, then $X, Y \in \mathfrak{k}$. Notice that $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{sp}(2) \oplus \mathfrak{m}$, where $\mathfrak{z} \perp \mathfrak{su}(4)$ is the centre of $\mathfrak{k}$, generated by $\operatorname{diag}(i, i, i, i, -4i)$, and $\mathfrak{m} = \mathfrak{sp}(2)^\perp \subset \mathfrak{su}(4)$. But we have assumed that $X, Y \perp_0 \mathfrak{sp}(2)$. Thus $X, Y \in \mathfrak{z} \oplus \mathfrak{m}$, and $[X, Y] = 0$ if and only if $[X_{\mathfrak{m}}, Y_{\mathfrak{m}}] = 0$. Now $SU(4) = Spin(6)$, $Sp(2) = Spin(5)$ and $(SU(4), Sp(2))$ is a rank one symmetric pair. Therefore $X_{\mathfrak{m}}, Y_{\mathfrak{m}}$ must be linearly dependent and we may assume without loss of generality that $X = X_{\mathfrak{m}}$, $Y = Y_{\mathfrak{z}}$. Then $\mathfrak{z} \subset \sigma$, i.e. $W_1 = \mathrm{diag}(i, i, i, i, -4i) \in \sigma$.

We now note that W_1 being horizontal is not only a necessary condition for $\sigma \subset \mathfrak{k}$ to be a horizontal zero-curvature plane, but also sufficient for the existence of such a plane as, for dimension reasons, we may always find a vector $X \in \mathfrak{m}$ such that $\sigma = \mathrm{Span}\,\{\Phi^{-1}(X), \Phi^{-1}(W_1)\}$ is a horizontal zero-curvature plane.

On the other hand, suppose now that $X_{\mathfrak{p}} \neq 0$. Then the conditions for zero-curvature become $0 = [X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}]$. Suppose that

$$X_{\mathfrak{p}} = \begin{pmatrix} 0 & x \\ -\bar{x}^t & 0 \end{pmatrix}, \quad Y = Y_{\mathfrak{k}} = \begin{pmatrix} Z & \\ & -\operatorname{tr} Z \end{pmatrix},$$

where $x \in \mathbb{C}^4$ and $Z \in \mathfrak{u}(4) = \mathfrak{z} \oplus \mathfrak{su}(4)$. Then $0 = [X_{\mathfrak{p}}, Y_{\mathfrak{k}}]$ if and only if $Zx = -(\operatorname{tr} Z)x$. Let $Z = itI + Z' \in \mathfrak{z} \oplus \mathfrak{su}(4)$, $t \in \mathbb{R}$. Since it is required that $Y \perp \mathfrak{sp}(2)$ we have $Z' \perp \mathfrak{sp}(2) \subset \mathfrak{su}(4)$. Recall that $SU(4) = Spin(6)$, $Sp(2) = Spin(5)$. Therefore $SU(4)/Sp(2) = S^5$ and, since $Sp(2) = Spin(5)$ acts transitively on distance spheres in $\mathfrak{m} = \mathfrak{sp}(2)^{\perp} \subset \mathfrak{su}(4)$, we may write

$$Z' = k \begin{pmatrix} is & & & \\ & -is & & \\ & & is & \\ & & & -is \end{pmatrix} k^{-1}, \quad k \in Sp(2).$$

This in turn implies that Z may be written as

$$Z = k \begin{pmatrix} i(t+s) & & & \\ & i(t-s) & & \\ & & i(t+s) & \\ & & & i(t-s) \end{pmatrix} k^{-1}, \quad k \in Sp(2).$$

But we established above that $-\operatorname{tr} Z = -4it$ is an eigenvalue of Z. Therefore either $-4t = t + s$ or $-4t = t - s$, i.e. $s = -5t$ or $s = 5t$. Thus we have shown that Y must be conjugate by an element of $Sp(2)$ to either $\operatorname{diag}(-4it, 6it, -4it, 6it, -4it)$ or $\operatorname{diag}(6it, -4it, 6it, -4it, -4it)$, and so up to scaling we have

$$Y = k \begin{pmatrix} 2i & & & & \\ & -3i & & & \\ & & 2i & & \\ & & & -3i & \\ & & & & 2i \end{pmatrix} k^{-1}, \quad k \in Sp(2) \subset SU(4) \subset SU(5).$$

Notice that $\Phi^{-1}(Y)$ is a multiple of Y and so we have $Y \in \sigma$. Conversely, if such a vector Y is horizontal it is not difficult to find a complementary vector X such that $\sigma = \operatorname{Span}\{\Phi^{-1}(X), \Phi^{-1}(Y)\}$ is a horizontal zero-curvature plane. Set $X_{\mathfrak{k}} = 0$. X is therefore automatically orthogonal to $\mathfrak{sp}(2)$ and it remains to choose $X_{\mathfrak{p}}$ such that X satisfies the first condition of (4.1.1), namely that X is orthogonal to a one-dimensional subspace. A choice of appropriate $X_{\mathfrak{p}}$ is equivalent to choosing an eigenvector for Z above. The set of such eigenvectors has dimension > 1 and so we may thus choose $X_{\mathfrak{p}}$ such that X has the desired properties. $\square$

It is at this stage that we modify the proof of positive curvature on the Bazaikin spaces to suit our future purposes. In [Zi1] a clever lemma due to Eschenburg is applied here to avoid direct computations. However, at the expense of some elegance in the proof of positive curvature, in our case we need to perform these computations in order to derive some equations which we can exploit in our proof of Theorem A (iii) and (iv).

Lemma 4.1.2. *$W_1 = \operatorname{diag}(i,i,i,i,-4i)$ and $W_2 = \operatorname{Ad}_k \operatorname{diag}(2i,-3i,2i,-3i,2i)$, $k \in Sp(2)$, are horizontal with respect to $\langle\ ,\ \rangle_1$ at $A = (a_{ij}) \in SU(5)$ if and only if*

$$q = \sum_{\ell=1}^{5} |a_{\ell 5}|^2 q_\ell, \quad \text{and} \tag{4.1.2}$$

$$0 = \sum_{\ell=1}^{5} (|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2) q_\ell \tag{4.1.3}$$

respectively.

Proof. We first recall that both W_1 and W_2 lie in $\mathfrak{k} = \mathfrak{u}(4)$. Therefore W_1 and W_2 are horizontal with respect to $\langle\ ,\ \rangle_1$ if and only if they are horizontal with respect to $\langle\ ,\ \rangle_0$. Moreover, W_1 and W_2 are both orthogonal to $\mathfrak{sp}(2)$ with respect to the bi-invariant metric by our discussion above. Hence we need only obtain expressions for W_1 and W_2 being orthogonal with respect to $\langle\ ,\ \rangle_0$ to $v_A := \operatorname{Ad}_{A^*} Q - \operatorname{diag}(0,0,0,0,iq)$, where $Q = \operatorname{diag}(iq_1, \ldots, iq_5)$.

Recall that $\langle X, Y\rangle_0 = -\operatorname{Re}\operatorname{tr}(XY)$. Then W_1 is horizontal if and only if

$$\begin{aligned}
-4q &= \langle \operatorname{diag}(0,0,0,0,iq), W_1\rangle_0 \\
&= \langle \operatorname{Ad}_{A^*} Q, W_1\rangle_0 \\
&= \left\langle \left(i \sum_{\ell=1}^{5} \bar{a}_{\ell i} a_{\ell j} q_\ell \right), W_1 \right\rangle_0 \\
&= -\operatorname{Re}\operatorname{tr}\left(i \sum_{\ell=1}^{5} \bar{a}_{\ell i} a_{\ell j} q_\ell (W_1)_{jj} \right) \\
&= -\operatorname{Re}\left(i \sum_{\ell,j=1}^{5} |a_{\ell j}|^2 q_\ell (W_1)_{jj} \right) \\
&= \sum_{\ell=1}^{5} (|a_{\ell 1}|^2 + |a_{\ell 2}|^2 + |a_{\ell 3}|^2 + |a_{\ell 4}|^2 - 4|a_{\ell 5}|^2) q_\ell.
\end{aligned}$$

Now using the fact that A is unitary together with $q = \sum_{\ell=1}^{5} q_\ell$ yields

$$\begin{aligned} -4q &= \sum_{\ell=1}^{5}((1-|a_{\ell 5}|^2) - 4|a_{\ell 5}|^2)q_\ell \\ &= \sum_{\ell=1}^{5}(1-5|a_{\ell 5}|^2)q_\ell \\ &= q - 5\sum_{\ell=1}^{5}|a_{\ell 5}|^2 q_\ell \end{aligned}$$

as desired.

Consider now $W_2 = \mathrm{Ad}_k \widehat{W}$, where $\widehat{W} = \mathrm{diag}(2i, -3i, 2i, -3i, 2i)$. Then W_2 is horizontal if and only if

$$\begin{aligned} 2q &= \left\langle \mathrm{diag}(0,0,0,0,iq), \widehat{W} \right\rangle_0 \\ &= \left\langle \mathrm{Ad}_{k^*}\, \mathrm{diag}(0,0,0,0,iq), \widehat{W} \right\rangle_0 \quad \text{for } \; k \in Sp(2) \subset SU(4) \\ &= \langle \mathrm{diag}(0,0,0,0,iq), W_2 \rangle_0 \\ &= \langle \mathrm{Ad}_{A^*} Q, W_2 \rangle_0 \\ &= \left\langle \mathrm{Ad}_{(Ak)^*} Q, \widehat{W} \right\rangle_0 \\ &= \left\langle \left(i \sum_{\ell=1}^{5} \overline{(Ak)}_{\ell i} (Ak)_{\ell j} q_\ell \right), \widehat{W} \right\rangle_0 \\ &= -\,\mathrm{Re}\ \mathrm{tr} \left(i \sum_{\ell=1}^{5} \overline{(Ak)}_{\ell i} (Ak)_{\ell j} q_\ell (\widehat{W})_{jj} \right) \\ &= -\,\mathrm{Re} \left(i \sum_{\ell, j=1}^{5} |(Ak)_{\ell j}|^2 q_\ell (\widehat{W})_{jj} \right) \\ &= \sum_{\ell=1}^{5} \left(2|(Ak)_{\ell 1}|^2 - 3|(Ak)_{\ell 2}|^2 + 2|(Ak)_{\ell 3}|^2 - 3|(Ak)_{\ell 4}|^2 + 2|(Ak)_{\ell 5}|^2 \right) q_\ell \\ &= \sum_{\ell=1}^{5} \left(2 - 5\left(|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2 \right) \right) q_\ell, \quad \text{since } A \text{ is unitary.} \end{aligned}$$

Equation (4.1.3) now follows immediately from $q = \sum_{\ell=1}^{5} q_\ell$. $\square$

We may now state the conditions for a Bazaikin space to admit positive curvature.

Theorem 4.1.3. *The Bazaikin space* $B^{13}_{q_1,\dots,q_5} = (SU(5), \langle\ ,\ \rangle_1) /\!\!/ S^1_{q_1,\dots,q_5} \cdot Sp(2)$ *admits positive sectional curvature if and only if*

$$q_{\sigma(1)} + q_{\sigma(2)} > 0 \quad (\text{or } < 0) \quad \textit{for all permutations } \sigma \in S_5. \tag{4.1.4}$$

Proof. Suppose $q_{\sigma(1)} + q_{\sigma(2)} > 0$ for all permutations $\sigma \in S_5$. In particular notice that this implies that $q > 0$. Through Lemmas 4.1.1 and 4.1.2 we have established that we need only examine equations (4.1.2) and (4.1.3) to obtain the desired result. Consider equation (4.1.2) in the alternative form

$$\sum_{\ell=1}^{5} (1 - |a_{\ell 5}|^2) q_\ell = 0.$$

Since $A \in SU(5)$ is unitary we know that either $1 - |a_{\ell 5}|^2 \neq 0$ for all $\ell = 1, \dots, 5$, or $1 - |a_{\ell_0 5}|^2 = 0$ for exactly one $\ell_0 \in \{1, \dots, 5\}$. In the first case we have, for some $\ell_0 \in \{1, \dots, 5\}$,

$$\sum_{\ell=1}^{5} (1 - |a_{\ell 5}|^2) q_\ell \geq (1 - |a_{\ell_0 5}|^2) \sum_{\ell=1}^{5} q_\ell = (1 - |a_{\ell_0 5}|^2) q > 0,$$

and so there are no solutions to equation (4.1.2). In the second case equation (4.1.2) reduces to, without loss of generality, $q_1 = q$. Thus in order to have solutions we require $q_2 + q_3 + q_4 + q_5 = 0$, which is impossible by our hypothesis. Hence we have established that there can be no solutions to equation (4.1.2).

Consider now equation (4.1.3). Since A is unitary there must be at least two $\ell \in \{1, \ldots, 5\}$ such that $|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2 \neq 0$. Without loss of generality we may assume that $\ell = 1$ gives the minimal $|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2 \neq 0$. Then, defining $\hat{q}$ to be the difference $\sum_{\ell=1}^{5} q_\ell -$(the sum of those q_j for which $|(Ak)_{j2}|^2 + |(Ak)_{j4}|^2 = 0$), we have

$$\sum_{\ell=1}^{5} \left(|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2\right) q_\ell \geq \left(|(Ak)_{12}|^2 + |(Ak)_{14}|^2\right) \hat{q} > 0$$

since $\hat{q}$ is the sum of at least two q_ℓ and must therefore be positive by our hypothesis. Hence equation (4.1.3) has no solutions.

We have thus shown that $q_{\sigma(1)} + q_{\sigma(2)} > 0$ for all permutations $\sigma \in S_5$ implies positive curvature.

Say now that $B^{13}_{q_1,\ldots,q_5}$ admits positive curvature. Suppose without loss of generality that $q_1 + q_2 \leq 0$ and $q_2 + q_3 > 0$. If we have $q_1 + q_2 = 0$ then choosing $A \in SU(5)$ such that $|a_{15}|^2 = |a_{25}|^2 = \frac{1}{2}$ yields a solution of equation (4.1.2) and by Lemmas 4.1.1 and 4.1.2 there exists a horizontal zero-curvature plane at this A, and hence at the image point in $B^{13}_{q_1,\ldots,q_5}$, which is a contradiction. On the other hand, say $q_1 + q_2 < 0$ and $q_2 + q_3 > 0$. Since both $A \in SU(5)$ and $k \in Sp(2) \subset SU(4)$ are unitary we may choose an A and k such that $|(Ak)_{12}|^2 = 1$ and $|(Ak)_{24}|^2 = 1$, and similarly an A and k such that $|(Ak)_{22}|^2 = 1$ and $|(Ak)_{34}|^2 = 1$. Thus we have

$$q_1 + q_2 \leq \sum_{\ell=1}^{5} (|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2) q_\ell \leq q_2 + q_3$$

with equality achieved at both sides for some A and k. Therefore by varying $A \in$

$SU(5)$ and $k \in Sp(2)$ continuously we may obtain a solution to equation (4.1.3) and hence a zero-curvature plane by Lemmas 4.1.1 and 4.1.2. □

4.2 Almost and quasi-positive curvature

Consider a general Bazaikin space $B^{13}_{q_1,\ldots,q_5}$. Since each q_j is odd, it is clear that at least three of the q_j must have the same sign. Suppose that four of the q_j share the same sign. By the discussion in §2.1 we may assume without loss of generality that $q_1, \ldots, q_4$ are all positive. We remark that by Theorem 4.1.3 we get positive curvature if $q_j + q_5 > 0$ for all $j = 1, \ldots, 4$. We are now in a position to prove Theorem A (iii).

Theorem 4.2.1. *All $B^{13}_{q_1,\ldots,q_5}$ with $q_1, \ldots, q_4 > 0$ admit quasi-positive curvature.*

Proof. As we established in Lemmas 4.1.1 and 4.1.2, there is a horizontal zero-curvature plane at $A \in SU(5)$ if and only if we can solve either equation (4.1.2) or equation (4.1.3) at A. If we allow A to be diagonal then equations (4.1.2) and (4.1.3) become

$$q_5 = \sum_{\ell=1}^{5} q_\ell, \quad \text{and} \tag{4.2.1}$$

$$0 = \sum_{\ell=1}^{5} \left(|k_{\ell 2}|^2 + |k_{\ell 4}|^2\right) q_\ell \tag{4.2.2}$$

respectively. By hypothesis $q_1, \ldots, q_4 > 0$ and therefore equality in (4.2.1) is impossible. On the other hand, because of how we have embedded $Sp(2)$ in $SU(5)$, both k_{52} and k_{54} are zero. Now since k is unitary there are at least two non-zero coefficients $|k_{\ell 2}|^2 + |k_{\ell 4}|^2$, $\ell = 1, \ldots 4$. Therefore the right-hand side of equation (4.2.2) is positive and thus no solutions exist. We have shown there are no horizontal zero-curvature planes at diagonal $A \in SU(5)$, which in turn implies the desired result. □

In particular, Theorem 4.2.1 tells us that the Bazaikin spaces $B^{13}_p := B^{13}_{1,1,1,1,p}$, where p is odd, admit quasi-positive curvature. This is a one-parameter family of cohomogeneity-one spaces under the action of $S(U(4)U(1)) = U(4)$ on the left and describes all cohomogeneity-one Bazaikin spaces, see [Zi1] or [GSZ] for details. By Theorem 4.1.3 B^{13}_p admits positive curvature if and only if $p > 0$. In particular $p = 1$ gives the positively curved homogeneous Berger space. It is natural to ask whether we can make a stronger curvature statement than quasi-positive curvature in the "boundary" case $p = -1$. We use "boundary" here in relation to the positive curvature condition, namely that $q_i + q_j = 0$ for some i, j. We can now prove Theorem A (iv).

Theorem 4.2.2. *The cohomogeneity-one space B^{13}_{-1} admits almost positive curvature.*

Proof. Since $A \in SU(5)$ equation (4.1.2) reduces to $|a_{55}|^2 = -1$ which clearly has no solutions. Similarly equation (4.1.3) may be reduced to

$$|(Ak)_{52}|^2 + |(Ak)_{54}|^2 = 1.$$

Since Ak is unitary this implies that $|(Ak)_{51}|^2 = |(Ak)_{53}|^2 = |(Ak)_{55}|^2 = 0$. In particular we have $(Ak)_{55} = 0$. But $(Ak)_{55} = a_{55}k_{55}$ because of our embedding of $Sp(2)$ in $SU(5)$, and for the same reason $|k_{55}| = 1$. Hence $a_{55} = 0$, and so B^{13}_{-1} has almost positive curvature since this is clearly invariant under the action of $S^1_{q_1,\cdots,q_5} \cdot Sp(2)$. ☐

As it turns out, we may apply a result of Taimanov [T] to show that the Bazaikin space B^{13}_{-1} contains two interesting totally geodesic submanifolds.

Proposition 4.2.3. *B^{13}_{-1} contains the exceptional Aloff-Wallach space $W^7_{-1,1}$ and the exceptional cohomogeneity-one Eschenburg space E^7_0 as totally geodesic submanifolds.*

Proof. Let $\sigma = \mathrm{diag}(-1,-1,1,1,1) \in SU(5)$. σ acts on $SU(5)$ via conjugation, and hence induces an action on a general Bazaikin space $B^{13}_{q_1,\ldots,q_5}$. The induced action is given by $\sigma \star [A] = [\sigma \star A]$, for $A \in SU(5)$, $[A] \in B^{13}_{q_1,\ldots,q_5}$, since conjugation by σ is in the normalizer of $Sp(2) \subset SU(5)$. Clearly the σ-action on $B^{13}_{q_1,\ldots,q_5}$ is an isometry. Recall that each component of the fixed point set of an isometry is a totally geodesic submanifold. Taimanov shows in [T] that one component of $Fix(\sigma)$, the fixed point set of the σ-action, is given by

$$Fix(\sigma)^0 := S(U(2)U(3))/\!\!/(U(2) \cdot S^1_{q_1,\ldots,q_5}) \subset B^{13}_{q_1,\ldots,q_5},$$

where the $U(2)$ in $U(2) \cdot S^1_{q_1,\ldots,q_5}$ is the usual embedding of $U(2)$ into $Sp(2)$. In fact Taimanov shows that $Fix(\sigma)^0$ is isometric to the Eschenburg space $E^7_{a,b}$ with

$a = (q_3, q_4, q_5)$ and $b = (-q_1, -q_2, q)$, where we recall that $q = \sum q_i$.

Thus we immediately see that $B^{13}_{-1} = B^{13}_{1,1,1,1,-1}$ contains the Eschenburg space described by $a = (1, 1, -1)$, $b = (-1, -1, 3)$ as a totally geodesic submanifold. However this is none other than E^7_0 in disguise, where E^7_0 is the Eschenburg space described by $a = (1, 1, 0)$, $b = (0, 0, 2)$.

On the other hand, recall from §2.1 that we may conjugate the left-hand side of the action of $Sp(2) \cdot S^1_{q_1,\ldots,q_5}$ by some element of $SU(5)$ to get an isometric Bazaikin space. Let $g \in SU(5)$ be an element which permutes the entries on the diagonal of the left-hand side of the action of $Sp(2) \cdot S^1_{q_1,\ldots,q_5}$, namely $g\,\mathrm{diag}(z, z, z, z, \bar{z})g^{-1} = \mathrm{diag}(\bar{z}, z, z, z, z)$. Let $f_g : B^{13}_{1,1,1,1,-1} \longrightarrow B^{13}_{-1,1,1,1,1}$ be the resulting isometry. By the results of Taimanov $B^{13}_{-1,1,1,1,1}$ contains the Eschenburg space described by $a = (1, 1, 1)$, $b = (1, -1, 3)$ as a totally geodesic submanifold. However again we notice that this is none other than the Aloff-Wallach space $W^7_{-1,1}$ described by $a = (0, 0, 0)$, $b = (0, -1, 1)$. Now since f_g is an isometry we see that $f_g^{-1}(W^7_{-1,1}) \cong W^7_{-1,1}$ is a totally geodesic submanifold of B^{13}_{-1} as desired. □

As we mentioned in the introduction, $W^7_{-1,1}$ has a zero-curvature plane at every point, whereas we showed in Theorem 3.2.2 that E^7_0 has almost positive curvature.

Chapter 5

Quotients of $S^7 \times S^7$

5.1 The Cayley numbers, G_2 and its Lie algebra

We recall without proof some well known facts about Cayley numbers, the Lie group G_2 and its Lie algebra. More details may be found in [GWZ] and [M].

We may write the Cayley numbers as $Ca = \mathbb{H} + \mathbb{H}\ell$. Thus we have a natural orthonormal basis

$$\{e_0 = 1, e_1 = i, e_2 = j, e_3 = k, e_4 = \ell, e_5 = i\ell, e_6 = j\ell, e_7 = k\ell\}$$

for Ca. Note that this description of Ca differs slightly from that given in [M]. This will account for the difference in the descriptions of the Lie algebra $\mathfrak{g}_2$. Multiplication in Ca is non-associative and defined via

$$(a + b\ell)(c + d\ell) = (ac - \bar{d}b) + (da + b\bar{c})\ell, \quad a, b, c, d \in \mathbb{H}, \tag{5.1.1}$$

and hence we have the following multiplication table, where the order of multiplication is given by (row)∗(column):

Table 5.1: Multiplication table for Ca

	$e_1 = i$	$e_2 = j$	$e_3 = k$	$e_4 = \ell$	$e_5 = i\ell$	$e_6 = j\ell$	$e_7 = k\ell$
$e_1 = i$	-1	k	$-j$	$i\ell$	$-\ell$	$-k\ell$	$j\ell$
$e_2 = j$	$-k$	-1	i	$j\ell$	$k\ell$	$-\ell$	$-i\ell$
$e_3 = k$	j	$-i$	-1	$k\ell$	$-j\ell$	$i\ell$	$-\ell$
$e_4 = \ell$	$-i\ell$	$-j\ell$	$-k\ell$	-1	i	j	k
$e_5 = i\ell$	ℓ	$-k\ell$	$j\ell$	$-i$	-1	$-k$	j
$e_6 = j\ell$	$k\ell$	ℓ	$-i\ell$	$-j$	k	-1	$-i$
$e_7 = k\ell$	$-j\ell$	$i\ell$	ℓ	$-k$	$-j$	i	-1

Recall that the Lie group G_2 is the automorphism group of $Ca \cong \mathbb{R}^8$. In fact G_2 is a connected subgroup of $SO(7) \subset SO(8)$, where $SO(8)$ acts on $Ca \cong \mathbb{R}^8$ by orthogonal transformations and $SO(7)$ is that subgroup consisting of elements which leave $e_0 = 1$ fixed. $SO(8)$ also contains two copies of $Spin(7)$ which are not conjugate in $SO(8)$, and G_2 is the intersection of these two subgroups.

As our eventual goal is to prove Theorem A(v) and (vi), it is useful to recall the fact that G_2 appears in the descriptions of some interesting homogeneous spaces. The following results are well-known and we state them without proof. More details may be found in, for example, [M], [J] (page 93).

Theorem 5.1.1.

(i) $Spin(7)/G_2 = S^7$, *which inherits positive curvature from the bi-invariant metric on* $Spin(7)$*;*

(ii) $Spin(8)/G_2 = S^7 \times S^7$ *and* $SO(8)/G_2 = (S^7 \times S^7)/\mathbb{Z}_2$*, where* $\mathbb{Z}_2 = \{\pm \mathrm{id}\}$*;*

(iii) $G_2/SU(3) = S^6$*.*

These statements follow from applications of the triality principle for $SO(8)$. More details may be obtained in [M]. $SO(8)/G_2 = (S^7 \times S^7)/\mathbb{Z}_2$ has a geometric interpretation as the set of all possible Cayley multiplications on $\mathbb{R}^8$. Note that for $\mathbb{H}$ we get $SO(4)/SO(3) = S^3$ as the set of all possible quaternionic multiplications on $\mathbb{R}^4$.

We now turn our attention to the Lie algebra of G_2. The proof of the following theorem follows exactly as in [M] except that we use the basis and multiplication conventions for Ca as in Table 5.1. Recall that $\mathfrak{so}(n) = \{A \in M_n(\mathbb{R}) \mid A^t = -A\}$.

Theorem 5.1.2. *The Lie algebra of* G_2*, denoted by* $\mathfrak{g}_2$*, consists of matrices* $A = (a_{ij}) \in \mathfrak{so}(7)$ *which satisfy* $a_{ij} + a_{ji} = 0$ *and*

$$\begin{aligned} a_{23} + a_{45} + a_{76} &= 0 \\ a_{12} + a_{47} + a_{65} &= 0 \\ a_{13} + a_{64} + a_{75} &= 0 \\ a_{14} + a_{72} + a_{36} &= 0 \\ a_{15} + a_{26} + a_{37} &= 0 \\ a_{16} + a_{52} + a_{43} &= 0 \\ a_{17} + a_{24} + a_{53} &= 0. \end{aligned}$$

Hence $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ *is* 14*-dimensional and consists of matrices of the form*

$$\begin{pmatrix} 0 & x_1+x_2 & y_1+y_2 & x_3+x_4 & y_3+y_4 & x_5+x_6 & y_5+y_6 \\ -(x_1+x_2) & 0 & \alpha_1 & -y_5 & x_5 & -y_3 & x_3 \\ -(y_1+y_2) & -\alpha_1 & 0 & x_6 & y_6 & -x_4 & -y_4 \\ -(x_3+x_4) & y_5 & -x_6 & 0 & \alpha_2 & y_1 & -x_1 \\ -(y_3+y_4) & -x_5 & -y_6 & -\alpha_2 & 0 & x_2 & y_2 \\ -(x_5+x_6) & y_3 & x_4 & -y_1 & -x_2 & 0 & \alpha_1+\alpha_2 \\ -(y_5+y_6) & -x_3 & y_4 & x_1 & -y_2 & -(\alpha_1+\alpha_2) & 0 \end{pmatrix}. \tag{5.1.2}$$

Recall that G_2 is a rank 2 Lie group. Thus, by examining the elements (5.1.2) of $\mathfrak{g}_2$, we can exponentiate and see that the maximal torus of G_2 is given by

$$T^2 = \left\{ \begin{pmatrix} 1 & & & \\ & R(\theta) & & \\ & & R(\varphi) & \\ & & & R(\theta+\varphi) \end{pmatrix} \;\middle|\; R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}. \tag{5.1.3}$$

5.2 Free isometric actions on $SO(8)$

Consider the rank 1 symmetric pair $(G, K) = (SO(8), SO(7))$ where

$$\begin{aligned} SO(7) &\hookrightarrow SO(8) \\ A &\longmapsto \begin{pmatrix} 1 & \\ & A \end{pmatrix}, \end{aligned}$$

with Lie algebras $\mathfrak{g}, \mathfrak{k}$ respectively. Let $\langle\ ,\ \rangle_0$ be the bi-invariant metric on G. With respect to $\langle\ ,\ \rangle_0$ we thus have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. As in (2.2.1) we define a left-invariant, right K-invariant metric $\langle\ ,\ \rangle_1$ on G by

$$\langle X, Y\rangle_1 = \langle X, \Phi(Y)\rangle_0, \tag{5.2.1}$$

where $\Phi(Y) = Y_{\mathfrak{p}} + \lambda Y_{\mathfrak{k}}$, $\lambda \in (0,1)$. Recall that by Lemma 2.2.2 we know that a plane

$$\sigma = \operatorname{Span}\{\Phi^{-1}(X), \Phi^{-1}(Y)\} \subset \mathfrak{g}$$

has zero-curvature with respect to $\langle\ ,\ \rangle_1$ if and only if

$$0 = [X, Y] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}]. \tag{5.2.2}$$

We now equip G with a K-invariant metric $\langle\!\langle\ ,\ \rangle\!\rangle$ induced via the diffeomorphism

$$\begin{aligned} \Delta G\backslash(G \times G) &\longrightarrow G \\ [(g_1, g_2)] &\longmapsto g_1^{-1} g_2, \end{aligned}$$

where $G \times G$ is equipped with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$.

Consider the isometric action of $U_{p_1,p_2,p_3} := S^1_{p_1,p_2,p_3} \times G_2 \subset K \times K$ on $SO(8)$ defined by

$$A \longmapsto \tilde{R}_{p_1,p_2,p_3}(\theta) \cdot A \cdot g^{-1}, \tag{5.2.3}$$

where $A \in SO(8)$, $g \in G_2$, and

$$\tilde{R}_{p_1,p_2,p_3}(\theta) = \begin{pmatrix} I_{2\times 2} & & & \\ & R(p_1\theta) & & \\ & & R(p_2\theta) & \\ & & & R(p_3\theta) \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5.2.4}$$

Lemma 5.2.1. *U_{p_1,p_2,p_3} acts freely and isometrically on $(G, \langle\!\langle\ ,\ \rangle\!\rangle)$ if and only if (p_1, p_2, p_3) is equal to $(0, 0, 1)$ (up to sign and permutations of the p_i).*

Proof. Recall that conjugation of either factor of U_{p_1,p_2,p_3} by elements of G is a diffeomorphism, and that a biquotient action is free if and only if non-trivial elements in each factor are never conjugate to one another in G. Thus we need only show that non-trivial elements of $S^1_{p_1,p_2,p_3}$ and T^2 are never conjugate in G if and only if (p_1, p_2, p_3) has one of the values listed above, where T^2 is the maximal torus of G_2 described in (5.1.3). This amounts to investigating when the sets of 2×2 blocks on each side are equal up to conjugation by an element of the Weyl group of $SO(8)$. We recall that the Weyl group of $SO(2n)$ acts via permutations of the 2×2 blocks and changing an even number of signs, where by a change of sign we mean $R(\theta) \longmapsto R(-\theta)$.

We consider first the case where we allow only permutations of the 2×2 blocks, i.e.

$$\{I_{2\times2}, R(p_1\theta), R(p_2\theta), R(p_3\theta)\} = \{I_{2\times2}, R(s), R(t), R(s + t)\}.$$

This is equivalent to examining when the sets

$$\{1, z^{p_1}, z^{p_2}, z^{p_3}\}, \quad \text{some } z \in \mathbb{C}, |z| = 1,$$

and

$$\{1, w_1, w_2, w_1w_2\}, \quad \text{some } w_j \in \mathbb{C}, |w_j| = 1,$$

are equal.

Suppose 1 = 1, i.e. $\{z^{p_1}, z^{p_2}, z^{p_3}\} = \{w_1, w_2, w_1w_2\}$. Then $z^{p_{\sigma(1)}} = w_1, z^{p_{\sigma(2)}} = w_2$, and $z^{p_{\sigma(3)}} = w_1w_2$, some $\sigma \in S_3$. Hence $z^{p_{\sigma(1)}+p_{\sigma(2)}-p_{\sigma(3)}} = 1$. Since $z = w_1 = w_2 = 1$ is a necessary condition for U_{p_1,p_2,p_3} to act freely, we also require that

$$p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} = \pm 1, \quad \text{for all} \quad \sigma \in S_3. \tag{5.2.5}$$

Thus by (5.2.5) if we are to have a free action we require that

$$p_1 + p_2 - p_3 = \pm 1, \tag{5.2.6}$$

$$p_1 + p_3 - p_2 = \pm 1, \tag{5.2.7}$$

$$p_2 + p_3 - p_1 = \pm 1. \tag{5.2.8}$$

If (5.2.6), (5.2.7), (5.2.8) have the same sign, then $p_1 = p_2 = p_3 = \pm 1$. Otherwise, if one of (5.2.6), (5.2.7), (5.2.8) has a different sign, then $(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = (0, 0, \pm 1)$, for some $\sigma \in S_3$. Therefore, up to a permutation and change of sign, the only possible triples (p_1, p_2, p_3) resulting in free actions are $(p_1, p_2, p_3) = (0, 0, 1)$ and $(p_1, p_2, p_3) = (1, 1, 1)$.

Suppose first that $p_1 = p_2 = p_3 = 1$. If we now allow $z = z^{p_j} = 1$, some $j \in \{1, 2, 3\}$, then $z = 1$ and hence $w_1 = w_2 = 1$. Thus we have shown that the sets $\{I_{2\times 2}, R(p_1\theta), R(p_2\theta), R(p_3\theta)\}$ and $\{I_{2\times 2}, R(s), R(t), R(s+t)\}$ cannot be equal when $(p_1, p_2, p_3) = (1, 1, 1)$.

Suppose on the other hand that $(p_1, p_2, p_3) = (0, 0, 1)$. Since the case $z^1 = 1$ is trivial, we assume $z^{p_j} = 1$ with $p_j = 0$, i.e. 1 = 1. We have already shown that this implies $z = w_1 = w_2 = 1$ and the sets $\{I_{2\times 2}, R(p_1\theta), R(p_2\theta), R(p_3\theta)\}$ and

$\{I_{2\times 2}, R(s), R(t), R(s+t)\}$ cannot therefore be equal.

Let us now deal with the cases $(p_1, p_2, p_3) = (1, 1, 1)$ and $(p_1, p_2, p_3) = (0, 0, 1)$ independently. In both cases we have already dealt with the situation where there are no sign changes under conjugation by an element of the Weyl group. We now allow conjugation of the $SO(2)$ on the left by Weyl group elements. Note that, up to a change of coordinate, changing the sign of all four 2×2 blocks is equivalent to making no sign changes. We therefore need only consider the case where the sign of two of the 2×2 blocks are changed.

We begin with $(p_1, p_2, p_3) = (1, 1, 1)$. We will show that the action described by this triple is not free by showing it is possible to have

$$\{I_{2\times 2}, R(-\theta), R(-\theta), R(\theta)\} = \{I_{2\times 2}, R(s), R(t), R(s+t)\}$$

for non-trivial $R(\theta)$, $R(s)$, $R(t)$. This is equivalent to showing that there is a non-trivial equality between the sets

$$\{1, \bar{z}, \bar{z}, z\}, \quad \text{some } z \in \mathbb{C}, |z| = 1,$$

and

$$\{1, w_1, w_2, w_1 w_2\}, \quad \text{some } w_j \in \mathbb{C}, |w_j| = 1.$$

When $1 = 1$, $\bar{z} = w_1 = w_2$, $z = w_1 w_2$, we find that $z^3 = 1$ and hence we have a non-trivial equality. Therefore the action given by $(p_1, p_2, p_3) = (1, 1, 1)$ is not free as desired.

We now turn our attention to $(p_1, p_2, p_3) = (0, 0, 1)$. Notice that changing the sign of two of the 2×2 blocks on the left leaves the three $I_{2\times 2}$ blocks invariant. Therefore it is clear that we will always have at least two of $R(s)$, $R(t)$ and $R(s+t)$ equal to $I_{2\times 2}$, which in turn implies $R(\theta) = R(s) = R(t) = I_{2\times 2}$. Hence the action given by $(p_1, p_2, p_3) = (0, 0, 1)$ is free. $\square$

Note that there are many other free $S^1 \times G_2$ actions on G. For example, a similar analysis to above shows that we have a free S^1 action on the left of G/G_2 by matrices of the form

$$\begin{pmatrix} R(\theta) & & & \\ & R(\theta) & & \\ & & R(\theta) & \\ & & & R(k\theta) \end{pmatrix} \tag{5.2.9}$$

where $(k, 3) = 1$.

However, only the action in Lemma 5.2.1 is isometric with respect to the K-invariant metric $\langle\!\langle\ ,\rangle\!\rangle$ on G, and henceforth we will denote this action by U.

It follows immediately from the long exact homotopy sequence for fibrations that a biquotient $Spin(8)/\!\!/(S^1 \times G_2) = S^1 \backslash (S^7 \times S^7)$ must be simply connected. By the lifting criterion for covering spaces the action by U on $SO(8)$ described above lifts to some action by $S^1 \times G_2$ on $Spin(8)$. Therefore, together with Theorem 5.1.1, one might expect that the resulting simply connected biquotient $Spin(8)/\!\!/(S^1 \times G_2) = S^1 \backslash (S^7 \times S^7)$ is a non-trivial finite cover of $SO(8)/\!\!/(S^1 \times G_2)$. In fact the lemma below will demonstrate that this covering map is a diffeomorphism.

Lemma 5.2.2. *$M^{13} := SO(8)/\!\!/(S^1 \times G_2)$ is simply connected and hence a quotient of $S^7 \times S^7$ by an S^1 action.*

Proof. Consider a general embedding

$$\begin{array}{ccc} S^1_q & \hookrightarrow & SO(8) \\ R(\theta) & \longmapsto & \begin{pmatrix} R(q_1\theta) & & & \\ & R(q_2\theta) & & \\ & & R(q_3\theta) & \\ & & & R(q_4\theta) \end{pmatrix} \end{array}$$

where $q = (q_1, q_2, q_3, q_4) \in \mathbb{Z}^4$, where $R(u) \in SO(2)$. The long exact homotopy sequence for the fibration $S^1_q \times G_2 \longrightarrow SO(8) \longrightarrow SO(8)/\!\!/S^1_q \times G_2$ yields

$$\ldots \longrightarrow \pi_1(S^1_q \times G_2) = \mathbb{Z} \longrightarrow \pi_1(SO(8)) = \mathbb{Z}_2 \longrightarrow \pi_1(SO(8)/\!\!/S^1_q \times G_2) \longrightarrow 0.$$

Thus to obtain the desired result we need only show that the map $\mathbb{Z} \longrightarrow \mathbb{Z}_2$ is surjective.

Recall that the homomorphism $\iota_* : \pi_1(S^1_q) \longrightarrow \pi_1(SO(n))$ is determined by the weights $q = (q_1, \ldots, q_m)$, $m = \lfloor \frac{n}{2} \rfloor$, of the embedding, namely $\iota_*(1) = \sum q_i \mod 2$. Therefore ι_* is onto exactly when $\sum q_i$ is odd. In our case we have $q = (0, 0, 0, 1)$, and so ι_* is a surjection. □

Notice that the action of U on $SO(8)$ given in Lemma 5.2.1 may be enlarged to an isometric action by $SO(3) \times G_2$, and the resulting biquotient we call N^{11}. Now recall that for all n we have a 2-fold cover $Spin(n) \longrightarrow SO(n)$ with $\pi_1(Spin(n)) = 0$ and $\pi_1(SO(n)) = \mathbb{Z}_2$. Thus, by the lifting criterion for covering spaces, the inclusion $SO(3) \hookrightarrow SO(8)$ must lift to $Spin(3) = S^3 \hookrightarrow Spin(8)$. As in the case of $U = S^1 \times G_2$ above we show that $N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ is simply connected and hence diffeomorphic to $Spin(8)/\!\!/S^3 \times G_2 = S^3 \backslash (S^7 \times S^7)$.

Lemma 5.2.3. *$N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ is simply connected and hence a quotient of $S^7 \times S^7$ by an S^3 action.*

Proof. Consider the chain of embeddings $i \circ j : S^1 = SO(2) \hookrightarrow SO(3) \hookrightarrow SO(8)$ given by enlarging S^1 above to an $SO(3)$ in $SO(8)$. We thus have an induced homomorphism on fundamental groups $(i \circ j)_* = i_* \circ j_* : \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$. But i_* and $(i \circ j)_*$ are simply the homomorphism ι_* from Lemma 5.2.2. Hence $i_*(1) = 1 \mod 2$ and $(i \circ j)_*(1) = 1 \mod 2$. This implies $j_*(1) = 1 \mod 2$ and therefore j_* is a surjection. An examination of the long exact homotopy sequence of the fibration $SO(3) \times G_2 \longrightarrow SO(8) \longrightarrow N^{11}$ yields the result. □

5.3 Quasi-positive curvature

Given Lemma 5.2.2 we are now in a position to perform the curvature computations for the circle quotient of $S^7 \times S^7$ mentioned in Theorem A, namely

$$M^{13} = SO(8)/\!\!/(S^1 \times G_2) = G/\!\!/U,$$

where S^1 is the circle giving a free isometric action U as in Lemma 5.2.1.

Consider the inclusions $G = SO(8) \supset K = SO(7) \supset G_2$. With respect to the bi-invariant metric $\langle\ ,\ \rangle_0$ on G we have

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \quad \text{and} \quad \mathfrak{k} = \mathfrak{m} \oplus \mathfrak{g}_2,$$

where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & w \\ -w^t & 0 \end{pmatrix} \;\middle|\; w \in \mathbb{R}^7 \right\} \tag{5.3.1}$$

and, by (5.1.2),

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 0 & -v_1 & 0 & v_7 & v_6 & -v_5 & v_4 & -v_3 \\ 0 & -v_2 & -v_7 & 0 & -v_5 & -v_6 & v_3 & v_4 \\ 0 & -v_3 & -v_6 & v_5 & 0 & v_7 & -v_2 & v_1 \\ 0 & -v_4 & v_5 & v_6 & -v_7 & 0 & -v_1 & -v_2 \\ 0 & -v_5 & -v_4 & -v_3 & v_2 & v_1 & 0 & -v_7 \\ 0 & -v_6 & v_3 & -v_4 & -v_1 & v_2 & v_7 & 0 \end{pmatrix} \right\}. \tag{5.3.2}$$

Recall that the action U in Lemma 5.2.1 is contained in $K \times K$. Thus, equipping $G \times G$ with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$ as before, we may induce a metric on $G /\!\!/ U$ via the diffeomorphism

$$\Delta G \backslash G \times G / U \longrightarrow (G, \langle\!\langle\ ,\ \rangle\!\rangle) /\!\!/ U.$$

Every $(\Delta G \times U)$-orbit in $G \times G$ passes through a point of the form (A, I). The vertical subspace at (A, I) is given by

$$\mathcal{V}_A = \{(\mathrm{Ad}_{A^t} X - Y_L, X - Y_R) \mid X \in \mathfrak{g}, (Y_L, Y_R) \in Lie(U) = (Lie(S^1), \mathfrak{g}_2)\}.$$

Thus the horizontal subspace at (A, I) with respect to $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$ is given by

$$\mathcal{H}_A = \{(-\Phi^{-1}(\mathrm{Ad}_{A^t} W), \Phi^{-1}(W)) \mid W_{\mathfrak{g}_2} = 0, \langle W, \mathrm{Ad}_A \Theta \rangle_0 = 0, \forall\ \Theta \in Lie(S^1)\}.$$

Suppose that

$$\sigma = \mathrm{Span}\, \{(-\Phi^{-1}(\mathrm{Ad}_{A^t} X), \Phi^{-1}(X)), (-\Phi^{-1}(\mathrm{Ad}_{A^t} Y), \Phi^{-1}(Y))\}$$

is a horizontal zero-curvature plane in $\mathfrak{g} \oplus \mathfrak{g}$. Since we have equipped $G \times G$ with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$, σ must project to zero-curvature planes $\check{\sigma}_i$, $i = 1, 2$, on each factor.

Lemma 5.3.1. *If* $\sigma = \mathrm{Span}\,\{(-\Phi^{-1}(\mathrm{Ad}_{A^t} X), \Phi^{-1}(X)), (-\Phi^{-1}(\mathrm{Ad}_{A^t} Y), \Phi^{-1}(Y))\}$ *is a horizontal zero-curvature plane then it may be assumed without loss of generality that* $X \in \mathfrak{p}$ *and* $Y \in \mathfrak{m}$.

Proof. $\check{\sigma}_2$ has zero-curvature and hence X, Y must satisfy the conditions in (5.2.2). $[X_\mathfrak{p}, Y_\mathfrak{p}] = 0$ implies that, since (G, K) is a rank one symmetric pair, we may assume $Y_\mathfrak{p} = 0$ without loss of generality. Hence $X \in \mathfrak{p} \oplus \mathfrak{m}, Y \in \mathfrak{m}$, since $X_{\mathfrak{g}_2} = Y_{\mathfrak{g}_2} = 0$. Now $0 = [X_\mathfrak{k}, Y_\mathfrak{k}]$ if and only if $0 = [X_\mathfrak{m}, Y_\mathfrak{m}]$. But, by Theorem (i) and since the bi-invariant metric on $Spin(7)$ induces positive curvature on $Spin(7)/G_2 = S^7$, there are no independent commuting vectors in $\mathfrak{m}$. Then, without loss of generality, $X \in \mathfrak{p}, Y \in \mathfrak{m}$. □

Thus we have

$$X = \begin{pmatrix} 0 & w \\ -w^t & 0 \end{pmatrix} \in \mathfrak{p}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & (v_{ij}) \end{pmatrix} \in \mathfrak{m},$$

and $[X, Y] = 0$.

Now $\check{\sigma}_1$ has zero curvature if and only if $[(\mathrm{Ad}_{A^t} X)_\mathfrak{p}, (\mathrm{Ad}_{A^t} Y)_\mathfrak{p}] = 0$, since

$$[\mathrm{Ad}_{A^t} X, \mathrm{Ad}_{A^t} Y] = \mathrm{Ad}_{A^t}([X, Y]) = 0.$$

Again, (G, K) is a rank one symmetric pair, thus $\check{\sigma}_1$ has zero curvature if and only if $(\mathrm{Ad}_{A^t} X)_\mathfrak{p}, (\mathrm{Ad}_{A^t} Y)_\mathfrak{p}$ are linearly dependent. Since an element of $\mathfrak{p}$ is determined by its first row, for $A=(a_{ij})\in SO(8)$ we have

$$\begin{aligned}(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}} &\sim \sum_{k,\ell=1}^{8} a_{k1}x_{k\ell}a_{\ell j}=\sum_{\ell=2}^{8}(a_{11}a_{\ell j}-a_{\ell 1}a_{1j})w_\ell, \quad j=2,\ldots,8, \quad \text{and}\\ (\mathrm{Ad}_{A^t}Y)_{\mathfrak{p}} &\sim \sum_{k,\ell=1}^{8} a_{k1}y_{k\ell}a_{\ell j}=\sum_{k,\ell=2}^{8}a_{k1}v_{k\ell}a_{\ell j}, \quad j=2,\ldots,8.\end{aligned}$$

If we assume that

$$A=\begin{pmatrix} R(\theta) & \\ & I_{6\times 6}\end{pmatrix}\in SO(8), \quad \theta\neq \frac{n\pi}{2}, n\in\mathbb{Z},$$

then

$$\begin{aligned}(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}} &\sim (0,w_2,w_3\cos\theta,w_4\cos\theta,w_5\cos\theta,w_6\cos\theta,w_7\cos\theta,w_8\cos\theta) \quad \text{and}\\ (\mathrm{Ad}_{A^t}Y)_{\mathfrak{p}} &\sim (0,0,v_1\sin\theta,v_2\sin\theta,v_3\sin\theta,v_4\sin\theta,v_5\sin\theta,v_6\sin\theta).\end{aligned}$$

Now $(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}}, (\mathrm{Ad}_{A^t}Y)_{\mathfrak{p}}$ are linearly dependent if and only if either $(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}}=0$ or $(\mathrm{Ad}_{A^t}Y)_{\mathfrak{p}}=0$ or $(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}}=s(\mathrm{Ad}_{A^t}Y)_{\mathfrak{p}}$, some $s\in\mathbb{R}-\{0\}$.

Suppose $(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}}=0$. Then $w_2=0$ and $w_j\cos\theta=0$, $j=3,\ldots,8$. But $\theta\neq\frac{n\pi}{2}$, hence $w_j=0$ for all j and so $X=X_{\mathfrak{p}}=0$. Thus $(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}}=0$ is impossible.

Suppose now that $(\mathrm{Ad}_{A^t}X)_{\mathfrak{p}}=s(\mathrm{Ad}_{A^t}Y)_{\mathfrak{p}}$, some $s\in\mathbb{R}-\{0\}$. Then $w_2=0$ and $w_j=s\ v_{j-2}\tan\theta$, $3\leq j\leq 8$. However, $[X,Y]=0$ implies that

$$\sum_{j=3}^{8} w_j v_{j-2}=0$$

and so we have a contradiction, as $s\neq 0$, $\theta\neq\frac{n\pi}{2}$.

Finally consider $(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} = 0$. $\theta \neq \frac{n\pi}{2}$ implies that $v_j = 0$, $1 \leq j \leq 6$, and so

$$Y = Y_{\mathfrak{m}} = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & v_7 & & & & \\ & & -v_7 & 0 & & & & \\ & & & & 0 & v_7 & & \\ & & & & -v_7 & 0 & & \\ & & & & & & 0 & -v_7 \\ & & & & & & v_7 & 0 \end{pmatrix}.$$

Recall that $\langle Y, \mathrm{Ad}_A \Theta \rangle_0 = 0$ for all $\Theta \in Lie(S^1)$. But $\mathrm{Ad}_A \Theta = \Theta$ since A commutes with S^1. Hence we must have $v_7 = 0$, i.e. $Y = 0$, and so $(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} = 0$ is impossible.

We have shown that there are no horizontal zero-curvature planes at (A, I), and therefore have proved that the image of (A, I) in $G/\!\!/U$ is a point of positive curvature. We have proved the curvature part of Theorem A(v):

Theorem 5.3.2. *The manifold* $M^{13} = SO(8)/\!\!/(S^1 \times G_2) = G/\!\!/U$ *admits a metric with quasi-positive curvature.*

Since extending the U action to an action by $SO(3) \times G_2$ increases the number of conditions which must be satisfied in order for a zero-curvature plane to be horizontal, Theorem A(vi) follows immediately:

Corollary 5.3.3. *The manifold* $N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ *admits a metric with quasi-positive curvature.*

5.4 Topology

Two questions now arise. Are M^{13} and N^{11} known manifolds, and are they known to admit positive curvature? We begin with the following theorem.

Theorem 5.4.1. *The biquotient M^{13} has the same cohomology groups and ring structure as $\mathbb{C}P^3 \times S^7$, and N^{11} has the same cohomology as $S^4 \times S^7$.*
In particular M^{13} and N^{11} are not manifolds known to admit positive curvature.

Proof. Consider any circle bundle

$$S^1 \longrightarrow S^7 \times S^7 \xrightarrow{p} M^{13}.$$

Since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_j(S^1) = 0$ for $j > 1$, the long exact homotopy sequence for fibre bundles implies $\pi_1(M^{13}) = 0$, $\pi_2(M^{13}) = \mathbb{Z}$, $\pi_j(M^{13}) \cong \pi_j(S^7 \times S^7)$ for $j \geq 3$, and in particular yields

$$\pi_j(M^{13}) = \begin{cases} 0, & \text{if } j = 1,3,4,5,6,11,12; \\ \mathbb{Z}, & \text{if } j = 2; \\ \mathbb{Z} \times \mathbb{Z}, & \text{if } j = 7; \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } j = 8,9; \\ \mathbb{Z}_{24} \times \mathbb{Z}_{24}, & \text{if } j = 10. \end{cases}$$

Since $\pi_1(M^{13}) = 0$ there is a Gysin sequence for the bundle $S^1 \longrightarrow S^7 \times S^7 \longrightarrow M^{13}$,

$$\cdots \longrightarrow H^{i-2}(M;\mathbb{Z}) \xrightarrow{\smile e} H^i(M;\mathbb{Z}) \xrightarrow{p^*} H^i(S^7 \times S^7;\mathbb{Z}) \longrightarrow H^{i-1}(M;\mathbb{Z}) \longrightarrow \cdots$$

Recall that

$$H^j(S^7 \times S^7; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, 14; \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } j = 7; \\ 0, & \text{otherwise.} \end{cases}$$

Since we have a circle bundle $S^1 \longrightarrow S^7 \times S^7 \longrightarrow M^{13}$, there is an isomorphism $H^0(M; \mathbb{Z}) \cong H^0(S^7 \times S^7) = \mathbb{Z}$, and the Euler class $e \in H^2(M; \mathbb{Z})$.

The Gysin sequence gives groups $H^j(M; \mathbb{Z}) = \mathbb{Z}$, $j = 0, 2, 4, 6$, and $H^j(M; \mathbb{Z}) = 0$, $j = 1, 3, 5$. By Poincaré Duality and the Universal Coefficient Theorem we thus have

$$H^j(M^{13}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, 2, 4, 6, 7, 9, 11, 13; \\ 0, & \text{if } j = 1, 3, 5, 8, 10, 12. \end{cases}$$

Hence, looking at the Serre spectral sequence for a fibration $S^1 \longrightarrow S^7 \times S^7 \longrightarrow M^{13}$, we see that M^{13} has cohomology ring $H^*(M; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^4, \beta^2 \rangle$, where $\alpha \in H^2(M)$ and $\beta \in H^7(M)$. Therefore M^{13} has the same cohomology as $\mathbb{C}P^3 \times S^7$.

The analogous Gysin sequence computation for $S^3 \longrightarrow S^7 \times S^7 \longrightarrow N^{11}$ yields

$$H^j(N^{11}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, 4, 7, 11; \\ 0, & \text{if } j = 1, 2, 3, 5, 6, 8, 9, 10, \end{cases}$$

and the Euler class $e \in H^4(N^{11}; \mathbb{Z})$. Looking at the Serre spectral sequence for a fibration $S^3 \longrightarrow S^7 \times S^7 \longrightarrow N^{11}$ we find that N^{11} has cohomology ring structure $H^*(N^{11}; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^2, \beta^2 \rangle$, , where $\alpha \in H^4(N^{11})$ and $\beta \in H^7(N^{11})$, and so N^{11} has the same cohomology as $S^4 \times S^7$. □

Before we continue we prove an easy lemma which will prove useful in the topological computations to follow.

Lemma 5.4.2. *Consider a triple* (r_1, r_2, r_3) *such that* $\sum r_i = 0$*. Let* $\sigma_i(r)$ *and* $\sigma_i(r^2)$ *denote the* i^{th} *elementary symmetric polynomials in* r_1, r_2, r_3 *and* r_1^2, r_2^2, r_3^2 *respectively. Then* $\sigma_1(r^2) = -2\sigma_2(r)$ *and* $\sigma_2(r^2) = \sigma_2(r)^2$*.*

Proof. Since $\sigma_1(r) = \sum r_i = 0$ we have

$$\begin{aligned} 0 &= \sigma_1(r)^2 \\ &= (r_1^2 + r_2^2 + r_3^2) + 2(r_1r_2 + r_1r_3 + r_2r_3) \\ &= \sigma_1(r^2) + 2\sigma_2(r) \end{aligned}$$

as desired. On the other hand

$$\begin{aligned} \sigma_2(r)^2 - \sigma_2(r^2) &= (r_1r_2 + r_1r_3 + r_2r_3)^2 - (r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2) \\ &= 2(r_1^2r_2r_3 + r_1r_2^2r_3 + r_1r_2r_3^2) \\ &= 2r_1r_2r_3(r_1 + r_2 + r_3) \\ &= 0. \end{aligned}$$

□

In [E1] (pages *vii* and 139), Eschenburg provides a beautiful diagram which explicitly describes the embedding of the root system G_2 into B_3. Recall that B_3 is the root system corresponding to the Lie algebra $\mathfrak{so}(7)$ and is given by

$$B_3 = \{\pm t_i \mid 1 \leq i \leq 3\} \cup \{\pm(t_i \pm t_j) \mid 1 \leq i < j \leq 3\}.$$

The root system G_2 lies on a hypersurface in Span $\{B_3\}$ and is given by

$$G_2 = \{\pm s_i \mid 1 \le i \le 3\} \cup \{\pm(s_i - s_j) \mid 1 \le i < j \le 3\},$$

where $s_i = \frac{1}{3}(2t_i - t_j - t_k)$, $\{i, j, k\} = \{1, 2, 3\}$. Notice that $\sum s_i = 0$ and that $s_i - s_j = t_i - t_j \in B_3$. Furthermore, s_i is the projection of $t_i \in B_3$ and $-(t_j + t_k) \in B_3$ onto the hypersurface containing G_2.

Since the Lie group G_2 is simply connected and has no centre, we see that the inclusions

$$\exp^{-1}(I) = \text{integral lattice of } G_2 \subset \text{root lattice of } G_2 \subset \text{weight lattice of } G_2$$

are in fact equalities. Therefore, by our above discussion of the roots of G_2, the integral and weight lattices of G_2 are spanned by $\{s_i \mid 1 \le i \le 3\}$, $\sum s_i = 0$. Thus by an abuse of notation we may assume that $\{s_i \mid 1 \le i \le 3\}$, $\sum s_i = 0$, spans $H^1(T_{G_2}; \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z})$, where T_{G_2} is a maximal torus of G_2 and Γ is the integral lattice of G_2.

Recall that Lemma 5.2.2 showed that

$$M^{13} := SO(8) /\!\!/ (S^1 \times G_2)$$

is a quotient of $S^7 \times S^7$ by a particular S^1 action.

Theorem 5.4.3. *The first Pontrjagin class of M^{13} is*

$$p_1(M^{13}) \quad = \quad 2\alpha^2$$

where α is a generator of $H^2(M^{13}; \mathbb{Z}_p) = \mathbb{Z}_p$, p prime, $p \ge 3$.

Before we provide the proof we remark that, in terms of integral cohomology, the theorem tells us only that $p_1(M^{13})$ is not divisible by any primes $p \geq 3$. Thus $p_1(M^{13}) = \pm 2^k \in \mathbb{Z} = H^4(M^{13})$, for some $k \in \mathbb{Z}$, $k \geq 0$. Since $p(M \times N) = p(M) \otimes p(N)$, $p(S^n) = 1$, and $p(\mathbb{C}P^n) = (1 + \beta^2)^{n+1}$, where β is the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$, we find that $p_1(\mathbb{C}P^3 \times S^7) = 4(\beta \otimes 1)^2$. Therefore we are unable to distinguish M^{13} and $\mathbb{C}P^3 \times S^7$ using the theorem.

Proof of Theorem 5.4.3. We follow the techniques developed in [BH], [E2] and [Si] (see also [FZ1]). Our strategy is to compute the first Pontrjagin class of the quotient M^{13} using the descriptions given in Lemma 5.2.2 as biquotients of $SO(8)$ by $S^1 \times G_2$.

Let $G = SO(8)$. Define the inclusions

$$\begin{aligned} f_q : H := S^1_q &\hookrightarrow G \\ R(u) &\longmapsto \begin{pmatrix} R(q_1 u) & & & \\ & R(q_2 u) & & \\ & & R(q_3 u) & \\ & & & R(q_4 u) \end{pmatrix} \end{aligned}$$

where $q = (q_1, q_2, q_3, q_4) \in \mathbb{Z}^4$, and

$$g : K := G_2 \hookrightarrow G$$

given by the embedding of G_2 into $SO(7) \subset SO(8)$. For a general Lie group L, let E_L denote a contractible space on which L acts freely, and denote the classifying space E_L/L by B_L. For the sake of notation we denote a product of Lie groups $L_1 \times L_2$ by $L_1 L_2$.

Consider the following commutative diagram of fibrations

$$\begin{array}{ccc} G \times E_{GG} & \longrightarrow & G \times E_{GG} \\ \downarrow & & \downarrow \\ G \times_{HK} E_{GG} & \xrightarrow{\varphi_G} & G \times_{GG} E_{GG} = B_{\Delta G} \\ {\scriptstyle (\varphi_H, \varphi_K)} \downarrow & & \downarrow {\scriptstyle B_\Delta} \\ B_H \times B_K & \xrightarrow[(B_{f_q}, B_g)]{} & B_G \times B_G \end{array} \tag{5.4.1}$$

where φ_G, φ_K, and φ_H are the respective classifying maps, and $\Delta : G \longrightarrow GG$ denotes the diagonal embedding. Now, since projection onto the first factor in each case is a homotopy equivalence, we have $G \simeq G \times E_{GG}$ and $H \backslash G/K \simeq G \times_{HK} E_{GG}$. Thus, up to homotopy, we can consider the diagram as

$$\begin{array}{ccc} G & \longrightarrow & G \\ \downarrow & & \downarrow \\ H \backslash G/K & \longrightarrow & B_{\Delta G} \\ \downarrow & & \downarrow {\scriptstyle B_\Delta} \\ B_H \times B_K & \xrightarrow[(B_{f_q}, B_g)]{} & B_G \times B_G \end{array}$$

Recall that $SO(8)$ and G_2 have torsion in their cohomology for coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$ (see [MT]). Therefore, using $\mathbb{Z}_p$ coefficients with $p \geq 3$ and prime, we have

$$\begin{aligned} H^*(G; \mathbb{Z}_p) &= \Lambda(y_1, y_2, y_3, y_4), \quad y_1 \in H^3, y_2, y_4 \in H^7, y_3 \in H^{11}, \\ H^*(K; \mathbb{Z}_p) &= \Lambda(x_1, x_2), \quad x_1 \in H^3, x_2 \in H^{11}, \\ H^*(H; \mathbb{Z}_p) &= \Lambda(u), \quad u \in H^1. \end{aligned}$$

Hence

$$\begin{aligned} H^*(B_G;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4], \quad \bar{y}_1\in H^4, \bar{y}_2,\bar{y}_4\in H^8, \bar{y}_3\in H^{12} \\ H^*(B_K;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{x}_1,\bar{x}_2], \quad \bar{x}_1\in H^4, \bar{x}_2\in H^{12} \\ H^*(B_H;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{u}], \quad \bar{u}\in H^2, \end{aligned}$$

where $\bar{y}_i$, $\bar{x}_j$ and $\bar{u}$ denote the transgressions of y_i, x_j and u respectively.

Let T_G and T_K be the maximal tori of G and K, with coordinates (t_1,t_2,t_3,t_4) and (s_1,s_2,s_3), $\sum s_i = 0$, respectively. By an abuse of notation we will identify t_i and s_j with the elements $t_i\in H^1(T_G)$ and $s_j\in H^1(T_K)$. Hence $\bar{t}_i\in H^2(B_{T_G})$, $\bar{s}_j\in H^2(B_{T_K})$. Since G and K do not have any torsion in their cohomologies we have

$$\begin{aligned} H^*(B_G) &= H^*(B_{T_G})^{W_G} = \mathbb{Z}_p[\bar{t}_1,\dots,\bar{t}_4]^{W_G} \quad \text{and} \\ H^*(B_K) &= H^*(B_{T_K})^{W_K} = \mathbb{Z}_p[\bar{s}_1,\bar{s}_2,\bar{s}_3]^{W_K} \end{aligned}$$

where W_L denotes the Weyl group of L.

W_G acts on $H^*(B_{T_G})$ via permutations in $\bar{t}_i$ and an even number of sign changes. Therefore a basis for $H^*(B_{T_G})^{W_G}$ is given by elementary symmetric polynomials $\sigma_i(\bar{t}^2) := \sigma_i(\bar{t}_1^2,\dots,\bar{t}_4^2)$, $i=1,2,3$, and $\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$. Hence we choose $\bar{y}_i = \sigma_i(\bar{t}^2)$, $i=1,2,3$, and $\bar{y}_4 = \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$.

$W_{G_2} = W_K$ is the dihedral group of order twelve. Its action on the root system G_2 is by rotations of $\frac{\pi}{3}$ and by reflections through the horizontal axis. Therefore, given our description of the root system of G_2 above, W_K acts on $H^*(B_{T_K})$ via permutations in $\bar{s}_i$ and a simultaneous sign change of all $\bar{s}_i$. Thus we see that elements of $H^*(B_{T_K})$ which are invariant under W_K are given by sums and products of the elementary symmetric polynomials $\sigma_2(\bar{s}) := \sigma_2(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ and $\sigma_i(\bar{s}^2) := \sigma_i(\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2)$, $i = 1, 2, 3$. However, since $\sum s_i = 0$, Lemma 5.4.2 shows that a basis for $H^*(B_{T_K})^{W_K}$ is given by the symmetric polynomials $\sigma_i(\bar{s}^2)$, $i = 1, 3$. Thus we identify $\bar{x}_1 = \sigma_1(\bar{s}^2)$ and $\bar{x}_2 = \sigma_3(\bar{s}^2)$.

Therefore we have

$$H^*(B_G; \mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{t}^2), \sigma_2(\bar{t}^2), \sigma_3(\bar{t}^2), \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4], \tag{5.4.2}$$

$$H^*(B_K; \mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{s}^2), \sigma_3(\bar{s}^2)]. \tag{5.4.3}$$

Let $h : L_1 \longrightarrow L_2$ be a homomorphism of Lie groups. Then the commutative diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{h} & L_2 \\ \uparrow & & \uparrow \\ T_{L_1} & \xrightarrow[h]{} & T_{L_2} \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} B_{L_1} & \xrightarrow{B_h} & B_{L_2} \\ \uparrow & & \uparrow \\ B_{T_{L_1}} & \xrightarrow[B_h]{} & B_{T_{L_2}} \end{array} \tag{5.4.4}$$

which in turn induces the commutative diagram

$$\begin{array}{ccc} H^*(B_{L_1}) & \xleftarrow{(B_h)^*} & H^*(B_{L_2}) \\ \downarrow & & \downarrow \\ H^*(B_{T_{L_1}}) & \xleftarrow[(B_h)^*]{} & H^*(B_{T_{L_2}}) \end{array} \tag{5.4.5}$$

Recall that

$$\begin{aligned} H^*(B_{GG}) = H^*(B_G) \otimes H^*(B_G) &= H^*(B_{T_G})^{W_G} \otimes H^*(B_{T_G})^{W_G} \\ &= \mathbb{Z}_p[\bar{t}_1, \ldots, \bar{t}_4]^{W_G} \otimes \mathbb{Z}_p[\bar{t}_1, \ldots, \bar{t}_4]^{W_G} \\ &= \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4] \otimes \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4], \end{aligned}$$

where $\mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4] = \mathbb{Z}_p[\sigma_1(\bar{t}^2), \sigma_2(\bar{t}^2), \sigma_3(\bar{t}^2), \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4]$ as before. Consider the diagonal embedding $\Delta : G \hookrightarrow GG$. In coordinates $\Delta|_{T_G}$ is given by $t_i \longmapsto (t_i, t_i)$, $i = 1, \ldots, 4$. We have commutative diagrams as in (5.4.4) and (5.4.5). Now

$$\begin{aligned} \Delta^* : H^1(T_G) \otimes H^1(T_G) &\longrightarrow H^1(T_G) \\ t_i \otimes 1 &\longmapsto t_i \\ 1 \otimes t_i &\longmapsto t_i, \end{aligned}$$

which in turn implies

$$\begin{aligned} (B_\Delta)^* : H^2(B_{T_G}) \otimes H^2(B_{T_G}) &\longrightarrow H^2(B_{T_G}) \\ \bar{t}_i \otimes 1 &\longmapsto \bar{t}_i \\ 1 \otimes \bar{t}_i &\longmapsto \bar{t}_i. \end{aligned}$$

Therefore,

$$\begin{aligned} (B_\Delta)^* : H^*(B_{GG}) &\longrightarrow H^*(B_G) \\ \bar{y}_i \otimes 1 &\longmapsto \bar{y}_i \\ 1 \otimes \bar{y}_i &\longmapsto \bar{y}_i. \end{aligned}$$

Since the diagram (5.4.1) is commutative we see that

$$\begin{aligned}\varphi_G^*(\bar{y}_i) &= \varphi_G^*((B_\Delta)^*(\bar{y}_i \otimes 1)) = \varphi_H^*((B_{f_q})^*(\bar{y}_i)), \quad \text{and} \\ \varphi_G^*(\bar{y}_i) &= \varphi_G^*((B_\Delta)^*(1 \otimes \bar{y}_i)) = \varphi_K^*((B_g)^*(\bar{y}_i)).\end{aligned}$$

Consider now $f_q : H := S_q^1 \hookrightarrow G$ as above. In coordinates f_q is given by

$$u \longmapsto (q_1 u, \ldots, q_4 u).$$

We get commutative diagrams as in (5.4.4) and (5.4.5). Now

$$\begin{aligned}(f_q)^* : H^1(T_G) &\longrightarrow H^1(S_q^1) \\ t_i &\longmapsto q_i u,\end{aligned}$$

which implies that

$$\begin{aligned}(B_{f_q})^* : H^2(B_{T_G}) &\longrightarrow H^2(B_{S_q^1}) \\ \bar{t}_i &\longmapsto q_i \bar{u}.\end{aligned}$$

Therefore, letting $q^2 := (q_1^2, \ldots, q_4^2)$, we have

$$\begin{aligned}(B_{f_q})^* : H^*(B_G) &\longrightarrow H^*(B_{S_q^1}) \\ \bar{y}_i &\longmapsto \sigma_i(q^2)\bar{u}^{2i}, \quad i = 1, 2, 3, \\ \bar{y}_4 &\longmapsto \sigma_4(q)\bar{u}^4.\end{aligned} \tag{5.4.6}$$

On the other hand, now consider $g : K := G_2 \hookrightarrow G \subset SO(8)$ as above. In particular, $g|_{T_K} : T_K \longrightarrow T_G$, and examining T_K as in (5.1.3) we see that in coordinates $g|_{T_K}$

is given by $(s_1, s_2, s_3) \longmapsto (0, s_1, s_2, -s_3)$, $\sum s_i = 0$. Again we get commutative diagrams as in (5.4.4) and (5.4.5). Now

$$\begin{aligned} (g|_{T_K})^* : H^1(T_G) &\longrightarrow H^1(T_K) \\ t_1 &\longmapsto 0, \\ t_i &\longmapsto s_{i-1}, \quad i = 2, 3, \\ t_4 &\longmapsto -s_3 \end{aligned}$$

and hence

$$\begin{aligned} (B_{g|_{T_K}})^* : H^2(B_{T_G}) &\longrightarrow H^2(B_{T_K}) \\ \bar{t}_1 &\longmapsto 0, \\ \bar{t}_i &\longmapsto \bar{s}_{i-1}, \quad i = 2, 3, \\ \bar{t}_4 &\longmapsto -\bar{s}_3 \end{aligned}$$

Therefore we have

$$\begin{aligned} (B_g)^* : H^*(B_G) &\longrightarrow H^*(B_K) \\ \bar{y}_1 &\longmapsto \sigma_1(\bar{s}^2) = \bar{x}_1, \\ \bar{y}_2 &\longmapsto \sigma_2(\bar{s}^2), \\ \bar{y}_3 &\longmapsto \sigma_3(\bar{s}^2) = \bar{x}_2, \\ \bar{y}_4 &\longmapsto 0. \end{aligned} \tag{5.4.7}$$

Thus $(B_g)^*(\bar{y}_1) = \bar{x}_1$ and $(B_g)^*(\bar{y}_3) = \bar{x}_2$.

We are now in a position to compute the Pontrjagin class of $H\backslash G/K$, and in particular p_1. Let τ be the tangent bundle of $H\backslash G/K$. In [Si] the following vector bundles over $H\backslash G/K$ were introduced. Let $\alpha_H := (G/K) \times_H \mathfrak{h}$, where H acts on G/K on the left, and on $\mathfrak{h}$ via Ad_H. Let $\alpha_K := (H\backslash G) \times_K \mathfrak{k}$, where K acts on $H\backslash G$ on the right, and on $\mathfrak{k}$ via Ad_K. Finally, let $\alpha_G := ((H\backslash G) \times (G/K)) \times_G \mathfrak{g}$, where G acts on $(H\backslash G) \times (G/K)$ via $(Hg_1, g_2K) \star g = (Hg_1g, g^{-1}g_2K)$, and on $\mathfrak{g}$ via Ad_G. Since $H \times K$ acts freely on G we have

$$\tau \oplus \alpha_H \oplus \alpha_K = \alpha_G.$$

Recall from [BH] that the Pontrjagin class of a homogeneous vector bundle $\alpha_L = P \times_L V$ associated to the L-principal bundle $P \longrightarrow B := P/L$ is given by

$$p(\alpha_L) = 1 + p_1(\alpha_L) + p_2(\alpha_L) + \cdots = \varphi_L^*(a), \quad a := \prod_{\alpha_i \in \Delta_L^+} (1 + \bar{\alpha}_i^2),$$

where Δ_L^+ is the set of positive weights of the representation of L on V, and $\varphi_L : B \longrightarrow B_L$ is the classifying map of the L-principal bundle. We have identified $\alpha_i \in H^1(T_L) \cong H^2(B_{T_L})$, and hence $a \in H^*(B_{T_L})^{W_L} \cong H^*(B_L)$

In our case the vector bundles α_H, α_K, α_G are associated to a principal bundle and the weights are the roots of the corresponding Lie group.

Since $H = S^1$ we have $p(\alpha_H) = 1$, and since, if V, W are vector bundles over some manifold M, $p(V \oplus W) = p(V) \smile p(W)$, we have

$$p(\tau)p(\alpha_K) = p(\alpha_G).$$

By our discussion above and since inverses are well-defined in the polynomial algebra $H^*(B_K)$ it follows that

$$p(\tau) = \varphi_G^*(a)\varphi_K^*(b^{-1}),$$

where $a := \prod_{\alpha_i \in \Delta_G^+}(1 + \bar{\alpha}_i^2)$ and $b := \prod_{\beta_j \in \Delta_K^+}(1 + \bar{\beta}_j^2)$. In particular, note that

$$\begin{aligned} p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_K) \\ &= \varphi_G^*\left(\sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2\right) - \varphi_K^*\left(\sum_{\beta_j \in \Delta_K^+} \bar{\beta}_j^2\right). \end{aligned}$$

The positive roots of $G = SO(8)$ are $t_i \pm t_j$, $1 \le i < j \le 4$. Hence

$$\begin{aligned} \sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2 &= \sum_{1 \le i < j \le 4} \left((\bar{t}_i - \bar{t}_j)^2 + (\bar{t}_i + \bar{t}_j)^2\right) \\ &= 2\sum_{1 \le i < j \le 4} (\bar{t}_i^2 + \bar{t}_j^2) \\ &= 6\sum_{i=1}^{4} \bar{t}_i^2 \\ &= 6\bar{y}_1. \end{aligned}$$

Now $\varphi_G^*(\bar{y}_1) = \varphi_H^*((B_{f_q})^*(\bar{y}_1)) = \sigma_1(q^2)\varphi_H^*(\bar{u}^2)$. Hence $p_1(\alpha_G) = 6\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)$.

From our earlier description of the roots of G_2, the positive roots of $K = G_2$ are

$$s_1,\ s_2,\ -s_3,\ s_1 - s_3,\ s_2 - s_1,\ s_2 - s_3,$$

where $\sum s_i = 0$. Then

$$\begin{aligned}
\sum_{\beta_j \in \Delta_K^+} \bar{\beta}_j^2 &= \bar{s}_1^2 + \bar{s}_2^2 + \bar{s}_3^2 + (\bar{s}_1 - \bar{s}_3)^2 + (\bar{s}_2 - \bar{s}_1)^2 + (\bar{s}_2 - \bar{s}_3)^2 \\
&= \sigma_1(\bar{s}^2) + (\bar{s}_1^2 + \bar{s}_3^2 - 2\bar{s}_1\bar{s}_3) + (\bar{s}_1^2 + \bar{s}_2^2 - 2\bar{s}_1\bar{s}_2) + (\bar{s}_2^2 + \bar{s}_3^2 - 2\bar{s}_2\bar{s}_3) \\
&= 3\sigma_1(\bar{s}^2) - 2\sigma_2(\bar{s}) \\
&= 4\sigma_1(\bar{s}^2) \qquad \text{by Lemma 5.4.2} \\
&= 4\bar{x}_1
\end{aligned}$$

Thus, since

$$\varphi_K^*(\bar{x}_1) = \varphi_K^*((B_g)^*(\bar{y}_1)) = \varphi_H^*((B_{f_q})^*(\bar{y}_1)) = \sigma_1(q^2)\varphi_H^*(\bar{u}^2),$$

we have $p_1(\alpha_K) = 4\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)$. Hence

$$\begin{aligned}
p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_K) \\
&= 2\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)
\end{aligned}$$

In our case we have $q = (0, 0, 0, 1)$. Therefore $p_1(M^{13}) = p_1(\tau) = 2\varphi_H^*(\bar{u}^2)$.

Consider the Serre spectral sequence for the fibration $G \longrightarrow H\backslash G/K \longrightarrow B_{HK}$. Notice that $\bar{u} \in H^2(B_H) = H^2(B_{HK}) = E_2^{2,0}$ will survive until E_∞ since $H^*(G)$ contains no elements of degree 1. Recall that the classifying map φ_H^* is the edge homomorphism

$$\varphi_H^* : H^i(B_{HK}) = E_2^{i,0} \twoheadrightarrow E_\infty^{i,0} \hookrightarrow H^i(H\backslash G/K).$$

Therefore, given Theorem 5.4.1, $\varphi_H^*(\bar{u})$ is mapped to a non-zero element, i.e. a generator, of $H^2(H\backslash G/K; \mathbb{Z}_p) = \mathbb{Z}_p$ and hence $\varphi_H^*(\bar{u}^2) \neq 0$. $\square$

Whilst we are as yet unable to distinguish M^{13} and $\mathbb{C}P^3 \times S^7$, we have more luck when we consider N^{11}.

Theorem 5.4.4. *The manifold* $N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ *has first Pontrjagin class*

$$p_1(N^{11}) = \alpha,$$

where α *is a generator of* $H^4(N^{11}; \mathbb{Z}_p) = \mathbb{Z}_p$*,* p *prime,* $p \geq 3$.

Proof. We let G and K be as above, $H = SO(3)$, and retain the notation and techniques used in the proof of Theorem 5.4.3.

Again using $\mathbb{Z}_p$ coefficients with $p \geq 3$ and prime, we have

$$\begin{aligned} H^*(G;\mathbb{Z}_p) &= \Lambda(y_1, y_2, y_3, y_4), \quad y_1 \in H^3, y_2, y_4 \in H^7, y_3 \in H^{11}, \\ H^*(K;\mathbb{Z}_p) &= \Lambda(x_1, x_2), \quad x_1 \in H^3, x_2 \in H^{11}, \\ H^*(H;\mathbb{Z}_p) &= \Lambda(w), \quad w \in H^3. \end{aligned}$$

Hence

$$\begin{aligned} H^*(B_G;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4], \quad \bar{y}_1 \in H^4, \bar{y}_2, \bar{y}_4 \in H^8, \bar{y}_3 \in H^{12} \\ H^*(B_K;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{x}_1, \bar{x}_2], \quad \bar{x}_1 \in H^4, \bar{x}_2 \in H^{12} \\ H^*(B_H;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{w}], \quad \bar{w} \in H^2, \end{aligned}$$

where $\bar{y}_i$, $\bar{x}_j$ and $\bar{w}$ denote the transgressions of y_i, x_j and w respectively.

Let T_H be the maximal tori of H with coordinate u. By an abuse of notation we will identify u with the element $u \in H^1(T_H)$. Hence $\bar{u} \in H^2(B_{T_H})$. Since G, K

and H do not have any torsion in their cohomologies we have

$$\begin{aligned} H^*(B_G) &= H^*(B_{T_G})^{W_G} = \mathbb{Z}_p[\bar{t}_1, \ldots, \bar{t}_4]^{W_G}, \\ H^*(B_K) &= H^*(B_{T_K})^{W_K} = \mathbb{Z}_p[\bar{s}_1, \bar{s}_2, \bar{s}_3]^{W_K}, \quad \text{and} \\ H^*(B_H) &= H^*(B_{T_H})^{W_H} = \mathbb{Z}_p[\bar{u}]^{W_H}, \end{aligned}$$

where W_L denotes the Weyl group of L.

W_G acts on $H^*(B_{T_G})$ via permutations in $\bar{t}_i$ and an even number of sign changes. Therefore a basis for $H^*(B_{T_G})^{W_G}$ is given by elementary symmetric polynomials $\sigma_i(\bar{t}^2) := \sigma_i(\bar{t}_1^2, \ldots, \bar{t}_4^2)$, $i = 1, 2, 3$, and $\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$. Hence we choose $\bar{y}_i = \sigma_i(\bar{t}^2)$, $i = 1, 2, 3$, and $\bar{y}_4 = \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$.

Similarly, since W_K is the dihedral group of order twelve and acts on $H^*(B_{T_K})$ via permutations in $\bar{s}_i$ and a simultaneous sign change, a basis for $H^*(B_{T_G})^{W_K}$ is given by the symmetric polynomials $\sigma_i(\bar{s}^2) := \sigma_i(\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2)$, $i = 1, 3$. Thus we identify $\bar{x}_1 = \sigma_1(\bar{s}^2)$ and $\bar{x}_2 = \sigma_3(\bar{s}^2)$.

W_H acts on $H^*(B_{T_H})$ via a sign change. Hence a basis for $H^*(B_{T_H})^{W_H}$ is given by $\bar{u}^2$.

Therefore we have

$$H^*(B_G; \mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{t}^2), \sigma_2(\bar{t}^2), \sigma_3(\bar{t}^2), \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4], \tag{5.4.8}$$

$$H^*(B_K; \mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{s}^2), \sigma_3(\bar{s}^2)], \tag{5.4.9}$$

$$H^*(B_H; \mathbb{Z}_p) = \mathbb{Z}_p[\bar{u}^2]. \tag{5.4.10}$$

Recall from the proof of Theorem 5.4.3 that

$$\varphi_K^*((B_g)^*(\bar{y}_i)) = \varphi_G^*(\bar{y}_i) = \varphi_H^*((B_{f_q})^*(\bar{y}_i)).$$

We have already computed $(B_g)^*$ and $(B_{f_q})^*$, in particular for $q = (0, 0, 0, 1)$.

We may now compute the Pontrjagin classes of N^{11}, and in particular p_1. Given the vector bundles α_G, α_K and α_H defined as before,

$$\begin{aligned} p_1(N^{11}) &= p_1(\alpha_G) - p_1(\alpha_K) - p_1(\alpha_H) \\ &= \varphi_G^*\left(\sum_{\alpha_i \in \Delta_G^+} \alpha_i^2\right) - \varphi_K^*\left(\sum_{\beta_j \in \Delta_K^+} \beta_j^2\right) - \varphi_H^*\left(\sum_{\gamma_j \in \Delta_H^+} \gamma_j^2\right). \end{aligned}$$

The positive roots of $G = SO(8)$ are $t_i \pm t_j$, $1 \le i < j \le 4$. We have already shown that $\sum_{\alpha_i} \bar{\alpha}_i^2 = 6\bar{y}_1$. Now $\varphi_G^*(\bar{y}_1) = \varphi_H^*((B_{f_q})^*(\bar{y}_1)) = \sigma_1(q^2)\varphi_H^*(\bar{u}^2)$. Hence $p_1(\alpha_G) = 6\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)$.

The positive roots of $K = G_2$ are

$$s_1,\ s_2,\ -s_3,\ s_1 - s_3,\ s_2 - s_1,\ s_2 - s_3,$$

where $\sum s_i = 0$. Then $\sum_{\beta_j} \bar{\beta}_j^2 = 4\bar{x}_1$ and $p_1(\alpha_K) = 4\sigma_1(q^2)\varphi_H^*(\bar{u}^2)$ as before.

There is only one positive root for $SO(3)$, namely u. Therefore $\sum_{\gamma_j} \bar{\gamma}_j^2 = \bar{u}^2$ and, since $q = (0, 0, 0, 1)$,

$$\begin{aligned} p_1(N^{11}) &= p_1(\alpha_G) - p_1(\alpha_K) - p_1(\alpha_H) \\ &= \varphi_H^*(\bar{u}^2) \in H^4(N^{11}). \end{aligned}$$

It remains to show that $\varphi_H^*(\bar{u}^2) \neq 0$. Consider the Serre spectral sequence for the fibration $G \longrightarrow N^{11} \longrightarrow B_{HK}$. By our earlier computations of $(B_{f_q})^*$ and $(B_g)^*$

the generator $y_1 \in H^3(G) = E_2^{0,3} = E_3^{0,3}$ gets mapped under d_3 to

$$d_3(y_1) = (B_{f_q})^*(\bar{y}_1) - (B_g)^*(\bar{y}_1) = \bar{u}^2 - \sigma_1(\bar{s}^2) \in E_3^{4,0} = E_2^{4,0} = H^4(B_{HK}).$$

But $H^4(B_{HK})$ has generators $\bar{u}^2$ and $\sigma_1(\bar{s}^2)$, both of which are mapped to zero by d_3. Thus, in E_4 the $E_4^{4,0}$ term is a $\mathbb{Z}_p$ generated by $\bar{u}^2$ and survives to E_∞. Now the edge homomorphism $(\varphi_H^*, \varphi_K^*)$ is given by

$$(\varphi_H^*, \varphi_K^*) : H^i(B_{HK}) = E_2^{i,0} \twoheadrightarrow E_\infty^{i,0} \hookrightarrow H^i(H\backslash G/K).$$

Therefore $(\varphi_H^*, \varphi_K^*)(\bar{u}^2) = \varphi_H^*(\bar{u}^2) \neq 0$. □

Recall that we have computed p_1 using $\mathbb{Z}_p$ coefficients, $p \geq 3$. Therefore, as in Theorem 5.4.3, we have proved only that, for integral coefficients, $p_1(N^{11}) = \pm 2^k \in \mathbb{Z}$, for some $k \in \mathbb{Z}$, $k \geq 0$. However, recall that all Pontrjagin classes for spheres are trivial and that integral Pontrjagin classes are homeomorphism invariants. Hence

Corollary 5.4.5. *N^{11} is not homeomorphic to $S^4 \times S^7$.*

Remark 5.4.6. Since H^8 and H^{12} are trivial for each of the manifolds M^{13} and N^{11}, we have in fact computed their total Pontrjagin classes $p = 1 + p_1$ in $\mathbb{Z}_p$ coefficients.

We return now to the problem of distinguishing M^{13} and $\mathbb{C}P^3 \times S^7$. We will do this by "hot-wiring" the technique for computing Pontrjagin classes in the absence of torsion in the cohomology groups so that we can compute the integral Pontrjagin class of M^{13}.

Before we begin we establish some topological statements which will be used in the proof of Theorem 5.4.9. From now on we will always assume that our cohomology groups have integral coefficients, and by spectral sequence we will always mean Serre spectral sequence.

Proposition 5.4.7. *B_{G_2}, the classifying space of G_2, has low-dimensional integral cohomology groups $H^1 = H^2 = H^3 = H^5 = 0$ and $H^4 = \mathbb{Z}$ with generator $\bar{x} = \frac{1}{2}\sigma_1(\bar{s}^2)$, where $\sigma_1(\bar{s}^2) := \sigma_1(\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2)$, $\sum \bar{s}_i = 0$, and $\bar{s}_i \in H^2(B_{T_{G_2}})$, $i = 1, 2, 3$, are the transgressions of the elements $s_i \in H^1(T_{G_2})$, $i = 1, 2, 3$, which span the integral lattice of G_2.*

Proof. Consider the universal bundle $G_2 \longrightarrow E_{G_2} \longrightarrow B_{G_2}$ where E_{G_2} is contractible. From [Wh], page 360, we know that $H^j(G_2) = 0$, $j = 1, 2, 4, 5$, and $H^3(G_2) = \mathbb{Z}$. Let x be a generator of $H^3(G_2)$. Since E_{G_2} is contractible all entries in the spectral sequence for the fibration $G_2 \longrightarrow E_{G_2} \longrightarrow B_{G_2}$ must get killed off. Since $d_4 : E_4^{0,3} \longrightarrow E_4^{4,0}$ is the only possible non-trivial differential with domain $E_4^{0,3}$ it must map $x \in H^3(G_2)$ to a generator $\bar{x}$ of $H^4(B_{G_2})$, and so $H^4(B_{G_2}) = \mathbb{Z}$. Similarly it is clear from the spectral sequence that $H^j(B_{G_2}) = 0$ for $j = 1, 2, 3, 5$.

Now consider the fibration $S^6 = G_2/SU(3) \longrightarrow B_{SU(3)} \longrightarrow B_{G_2}$. The spectral sequence associated to this fibration shows that $\bar{x} \in E_2^{4,0} = H^4(B_{G_2})$ survives to E_∞. Thus, since there are no other non-zero entries on the corresponding diagonal in E_∞, we see that $H^4(B_{G_2}) = H^4(B_{SU(3)})$. Recall that $H^*(B_{SU(3)})$ is a polynomial algebra generated by the elementary symmetric polynomials $\sigma_i(\bar{s}) = \sigma_i(\bar{s}_1, \bar{s}_2, \bar{s}_3)$,

$i = 2, 3$, in the transgressions $\bar{s}_j$ of $s_j \in H^1(T_{SU(3)})$, $j = 1, 2, 3$, where the s_j span the integral lattice of $SU(3)$. Note that $\sum s_j = 0$, $T_{G_2} = T_{SU(3)}$ and $\deg(\sigma_i(\bar{s})) = 2i$. Therefore $H^4(B_{G_2})$ is generated by $\sigma_2(\bar{s})$. However, by Lemma 5.4.2 we see that $\sigma_2(\bar{s}) = -\frac{1}{2}\sigma_1(\bar{s}^2)$. We set $\bar{x} = \frac{1}{2}\sigma_1(\bar{s}^2)$. □

Proposition 5.4.8. *The manifold $SO(8)/G_2 = (S^7 \times S^7)/\mathbb{Z}_2$ has low-dimensional integral cohomology groups $H^j(SO(8)/G_2) = H^j(\mathbb{R}P^7)$, $0 \leq j \leq 6$.*

Proof. Consider the spectral sequence for the fibration

$$\mathbb{R}P^7 = SO(7)/G_2 \longrightarrow SO(8)/G_2 \longrightarrow SO(8)/SO(7) = S^7.$$

Recall that

$$H^j(\mathbb{R}P^7) = \begin{cases} \mathbb{Z} & \text{if } j = 0, 7 \\ \mathbb{Z}_2 & \text{if } j = 2, 4, 6 \\ 0 & \text{if } j = 1, 3, 5. \end{cases}$$

It is clear that each $E_2^{0,j} = H^j(\mathbb{R}P^7)$, $j \leq 5$, survives to E_∞. For $E_2^{0,6} = H^6(\mathbb{R}P^7) = \mathbb{Z}_2$ notice that there are no non-trivial homomorphisms $\mathbb{Z}_2 \longrightarrow \mathbb{Z}$ and so the differential $d_7 : E_7^{0,6} = \mathbb{Z}_2 \longrightarrow E_7^{7,0} = \mathbb{Z}$ must be trivial. Therefore $E_2^{0,6} = H^6(\mathbb{R}P^7)$ also survives to E_∞. Since there are no other non-zero entries on the corresponding diagonals we get the desired result. □

We are now in a position to complete the proof of Theorem A(v).

Theorem 5.4.9. *The first integral Pontrjagin class of $M^{13} = SO(8)/\!\!/(S^1 \times G_2)$ is given by*

$$|p_1(M^{13})| = 8y^2,$$

where y is a generator of $H^2(M^{13}) = \mathbb{Z}$.

In particular, M^{13} is not homeomorphic to $\mathbb{C}P^3 \times S^7$.

Proof. Recall that diagram (5.4.1) is

$$\begin{array}{ccc}
G \times E_{GG} & \longrightarrow & G \times E_{GG} \\
\downarrow & & \downarrow \\
G \times_U E_{GG} & \xrightarrow{\varphi_G} & G \times_{GG} E_{GG} = B_{\Delta G} \\
\varphi_U \downarrow & & \downarrow B_\Delta \\
B_U & \xrightarrow[B_\iota]{} & B_{GG}
\end{array}$$

which, up to homotopy, is the same as

$$\begin{array}{ccc}
G & \longrightarrow & G \\
\downarrow & & \downarrow \\
G/\!\!/U & \xrightarrow{\varphi_G} & B_{\Delta G} \\
\varphi_U \downarrow & & \downarrow B_\Delta \\
B_U & \xrightarrow[B_\iota]{} & B_{GG}
\end{array}$$

where $G = SO(8)$, $U = HK = S^1 \times G_2$, and $G/\!\!/U = M^{13}$. We have altered the previous notation slightly so that $\varphi_U = (\varphi_H, \varphi_K)$ and ι is the embedding $(f_q, g) : U \hookrightarrow GG$ for $q = (0, 0, 0, 1)$. In the proofs of Theorem 5.4.3 and Theorem 5.4.4 we followed the usual techniques of [BH], [E2] and [Si] when there is no torsion in cohomology, namely we computed B_ι and B_Δ and then used the fact that the diagram commutes in order to compute the $\mathbb{Z}_p$, $p \geq 3$, Pontrjagin class. However, since $SO(8)$ and G_2 have torsion in integral cohomology, we need to adopt a different approach in order to compute the integral Pontrjagin class. Since $H^8(M^{13}) = H^{12}(M^{13}) = 0$ we can restrict our attention to the first integral Pontrjagin class $p_1(M^{13}) \in H^4(M^{13})$. The key idea to be taken from the proofs of Theorem 5.4.3 and Theorem 5.4.4 is that we computed the first Pontrjagin classes of some vector bundles over $B_{\Delta G}$ and B_U, then pulled them back to M^{13} under the classifying maps φ_G and φ_U respectively. As it turns out, the first Pontrjagin classes of the vector bundles over $B_{\Delta G}$ and B_U are the same in integral coefficients as in $\mathbb{Z}_p$ coefficients, $p \geq 3$. Our strategy, therefore, is to compute the maps $\varphi_U^* : H^4(B_U) \longrightarrow H^4(M^{13})$ and $\varphi_G^* : H^4(B_{\Delta G}) \longrightarrow H^4(M^{13})$ and pull back the respective first Pontrjagin classes.

As a first step in computing $\varphi_U^* : H^4(B_U) \longrightarrow H^4(M^{13})$ we notice that $H^*(U) = H^*(S^1) \otimes H^*(G_2)$ and $H^*(B_U) = H^*(B_{S^1}) \otimes H^*(B_{G_2})$ since $H^*(S^1)$ and $H^*(B_{S^1})$ are torsion-free. Therefore

$$H^j(U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z} = \langle w \rangle & \text{if } j = 1 \\ \mathbb{Z} = \langle x \rangle & \text{if } j = 3 \\ 0 & \text{if } j = 2, 4, 5 \end{cases}$$

where w is a generator of $H^1(S^1)$ and x is a generator of $H^3(G_2)$, and applying

Proposition 5.4.7

$$H^j(B_U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z} = \langle \bar{w} \rangle & \text{if } j = 2 \\ \mathbb{Z} \oplus \mathbb{Z} = \langle \bar{w}^2 \rangle \oplus \langle \bar{x} \rangle & \text{if } j = 4 \\ 0 & \text{if } j = 1, 3, 5 \end{cases}$$

where $\bar{w}$ is the transgression of w resulting from the spectral sequence for the universal bundle of S^1 and generates $H^2(B_{S^1})$ (hence generates $H^*(B_{S^1}) = \mathbb{Z}[\bar{w}]$), and $\bar{x}$ is the transgression of x resulting from the spectral sequence for the universal bundle of G_2 and generates $H^4(B_{G_2})$.

Recall that $\varphi_U : G/\!\!/U \longrightarrow B_U$ is the classifying map since we have the following diagram of principal U-bundles

$$\begin{array}{ccc} U & \longrightarrow & U \\ \downarrow & & \downarrow \\ G \times E_U & \xrightarrow{\pi_2} & E_U \\ \downarrow & & \downarrow \\ G \times_U E_U & \xrightarrow[\pi_2]{} & B_U \end{array}$$

where π_2 denotes projection onto the second factor and $U \longrightarrow E_U \longrightarrow B_U$ is the universal bundle. Since E_U is contractible, projection onto the first factor gives homotopy equivalences $G \times E_U \simeq G$ and $G \times_U E_U \simeq G/\!\!/U$. Then φ_U is the resulting map $G/\!\!/U \longrightarrow B_U$ and so is the classifying map. Therefore, up to homotopy, we may consider the following commutative diagram of fibrations

$$\begin{array}{ccc} U & \longrightarrow & U \\ \downarrow & & \downarrow \\ G & \longrightarrow & E_U \\ \downarrow & & \downarrow \\ G/\!\!/U & \xrightarrow[\varphi_U]{} & B_U \end{array}$$

Consider first the spectral sequence for the fibration on the left. Recall that $H^*(M^{13}) = H^*(\mathbb{C}P^3 \times S^7)$. Hence

$$H^j(G/\!\!/U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z} = \langle y \rangle & \text{if } j = 2 \\ \mathbb{Z} = \langle y^2 \rangle & \text{if } j = 4 \\ 0 & \text{if } j = 1, 3, 5. \end{cases}$$

Since $G = SO(8)$ we have from [CMV] that

$$H^j(G) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ \mathbb{Z}_2 = \langle r \rangle & \text{if } j = 2 \\ \mathbb{Z} = \langle z \rangle & \text{if } j = 3 \\ \mathbb{Z}_2 = \langle r^2 \rangle & \text{if } j = 4. \end{cases}$$

Since $H^1(G) = 0$ we see that $d_2 : E_2^{0,1} = \langle w \rangle \longrightarrow E_2^{2,0} = \langle y \rangle$ must have trivial kernel, i.e. $d_2(w) = ky$ for some $k \in \mathbb{Z}$, $k \neq 0$. Then $E_3^{0,2} = \langle y \rangle / \langle ky \rangle$ survives to E_∞ and since $H^2(G) = \mathbb{Z}_2$ we must therefore have $k = 2$, i.e. $d_2(w) = 2y$.

On the other hand, the spectral sequence shows that on the E_4-page we have the differential $d_4 : E_4^{0,3} = \langle x \rangle \longrightarrow E_4^{0,4} = \langle y^2 \rangle / \langle 2y^2 \rangle$. However, since $H^3(G) = \mathbb{Z}$ and $H^4(G) = \mathbb{Z}_2$, we must have $d_4(x) = 0 \in \langle y^2 \rangle / \langle 2y^2 \rangle$.

Since E_U is contractible it is clear from the spectral sequence for the fibration on the right that $d_2 : E_2^{0,1} = \langle w \rangle \longrightarrow E_2^{2,0} = \langle \bar{w} \rangle$ is an isomorphism with $d_2(w) = \bar{w}$, and $d_4 : E_4^{0,3} = \langle x \rangle \longrightarrow E_4^{4,0} = \langle \bar{w}^2 \rangle \oplus \langle \bar{x} \rangle$ is given by $d_4(x) = \bar{x}$.

By naturality of the spectral sequence we thus have for the left-hand fibration that $d_2(w) = \varphi_U^*(\bar{w}) \in \langle y \rangle$ and $d_4(x) = \varphi_U^*(\bar{x}) \in \langle y^2 \rangle / \langle 2y^2 \rangle$. Therefore, since we have already shown that $d_2(w) = 2y \in \langle y \rangle$ and $d_4(x) = 0 \in \langle y^2 \rangle / \langle 2y^2 \rangle$, we find

$$\varphi_U^*(\bar{w}) \;=\; 2y \in H^2(G/\!\!/U) = \langle y \rangle \quad \text{and} \tag{5.4.11}$$

$$\varphi_U^*(\bar{x}) \;=\; 2ky^2 \in H^4(G/\!\!/U) = \langle y^2 \rangle, \quad \text{for some } k \in \mathbb{Z}. \tag{5.4.12}$$

We now turn our attention to computing $\varphi_G^* : H^4(B_{\Delta G}) \longrightarrow H^4(M^{13})$. In order to show that $\varphi_G : G/\!\!/U \longrightarrow B_{\Delta G}$ is the classifying map consider the commutative diagram of principal G-bundles

$$\begin{array}{ccc}
G & \longrightarrow & G \\
\downarrow & & \downarrow \\
GG \times_U E_{GG} & \longrightarrow & GG \times_{GG} E_{GG} \\
\downarrow & & \downarrow \\
(\Delta G \backslash GG) \times_U E_{GG} & \xrightarrow[\varphi_G]{} & (\Delta G \backslash GG) \times_{GG} E_{GG}
\end{array}$$

Since $GG \times_{GG} E_{GG} = E_{GG}$ and $(\Delta G \backslash GG) \times_{GG} E_{GG} = G \times_{GG} E_{GG} = B_{\Delta G}$ we see that the fibration on the right-hand side is the universal bundle for G. On the left-hand side we have $(\Delta G \backslash GG) \times_U E_{GG} = G \times_U E_{GG}$, and projection onto the first factor gives homotopy equivalences $GG \times_U E_{GG} \simeq GG/U$ and $G \times_U E_{GG} \simeq G/\!/U$. Thus up to homotopy the diagram becomes

$$\begin{array}{ccc} G & \longrightarrow & G \\ \downarrow & & \downarrow \\ GG/U & \longrightarrow & E_{GG} \\ \downarrow & & \downarrow \\ G/\!/U & \xrightarrow[\varphi_G]{} & B_G \end{array}$$

as desired. Recall that $H^3(G) = \langle z \rangle$. The cohomology of B_G is described in [Br] and [F], but for our purposes we need only that

$$H^j(B_G) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1,2 \\ \mathbb{Z}_2 & \text{if } j = 3 \\ \mathbb{Z} & \text{if } j = 4 \\ \mathbb{Z}_2 & \text{if } j = 5. \end{cases}$$

Whilst proving Proposition 3.6 in [GZ2] the authors show that, since $E = E_{GG}$ is contractible, in the spectral sequence for the bundle $G \longrightarrow E \longrightarrow B_G$ the differential $d_4 : E_4^{0,3} = \langle 2z \rangle \longrightarrow E_4^{4,0} = H^4(B_G)$ is an isomorphism, i.e. $2z$ gets mapped to a generator $\bar{z}$ of $H^4(B_G) = \mathbb{Z}$. This follows from the facts that $E_2^{2,2} = \mathbb{Z}_2$ by the Universal Coefficient Theorem, and that $d_2 : E_2^{0,3} = \langle z \rangle \longrightarrow E_2^{2,2} = \mathbb{Z}_2$ must be onto.

Therefore naturality of the spectral sequence implies that $d_4(2z) = \varphi_G^*(\bar{z})$ in the spectral sequence for the left-hand fibration $G \longrightarrow GG/U \longrightarrow G/\!\!/U$, where $H^3(G) = \langle z \rangle$ and $H^4(B_G) = \langle \bar{z} \rangle$.

In order to determine the exact value of $\varphi_G^*(\bar{z}) \in H^4(G/\!\!/U)$ we need to examine the spectral sequence for the left-hand fibration. First we must compute the cohomology of GG/U in low-dimensions. Recall that $GG/U = V_{8,6} \times SO(8)/G_2$, where $V_{8,6}$ is the Stiefel manifold $SO(8)/SO(2)$. From [CMV] we find that

$$H^j(V_{8,6}) = \begin{cases} \mathbb{Z} & \text{if } j = 0, 2 \\ 0 & \text{if } j = 1, 3, 5 \\ \mathbb{Z}_2 & \text{if } j = 4. \end{cases}$$

We computed the low-dimensional cohomology groups of $SO(8)/G_2$ in Proposition 5.4.8. From the general Künneth formula for cohomology ([Sp], page 247) we find that

$$H^j(GG/U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1, 3 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } j = 2 \\ \mathbb{Z}_2^3 & \text{if } j = 4 \\ \mathbb{Z}_2 & \text{if } j = 5. \end{cases}$$

Since $H^4(GG/U) = \mathbb{Z}_2^3$, in the spectral sequence for $G \longrightarrow GG/U \longrightarrow G/\!\!/U$ the differential $d_2 : E_2^{0,3} = H^3(G) = \langle z \rangle \longrightarrow E_2^{2,2} = \mathbb{Z}_2$ must be trivial, i.e. $E_2^{2,2} = \mathbb{Z}_2$ must survive to E_∞. It thus follows that $E_2^{i,j} = E_3^{i,j} = E_4^{i,j}$ for $i \leq 5, j \leq 4$.

Since $H^3(GG/U) = 0$ the differential $d_4 : E_4^{0,3} = \langle z \rangle \longrightarrow E_4^{4,0} = H^4(G/\!\!/U) = \langle y^2 \rangle$ must be given by $d_4(z) = ny^2$ for some non-zero $n \in \mathbb{Z}$. On the other hand, since $H^4(GG/U) = \mathbb{Z}_2^3$, $E_4^{0,4} = E_4^{2,2} = \mathbb{Z}_2$ and $E_4^{1,3} = E_4^{3,1} = 0$, the filtration for the spectral sequence shows that $n = 2$, i.e. $d_4(z) = 2y^2$. But we have already shown that $d_4(2z) = \varphi_G^*(\bar{z})$. Therefore

$$\varphi_G^*(\bar{z}) = 4y^2 \in H^4(G/\!\!/U) = \langle y^2 \rangle.$$

Furthermore, while proving Lemma 5.4 in [GZ2] the authors show that, by considering the spectral sequences of the fibrations $SO(8)/SO(3) \longrightarrow B_{SO(3)} \longrightarrow B_{SO(8)}$ and $SO(3)/SO(2) \longrightarrow B_{SO(2)} \longrightarrow B_{SO(3)}$, we can let $\bar{z} = \sigma_1(\bar{t}^2) = \sigma_1(\bar{t}_1^2, \bar{t}_2^2, \bar{t}_3^2, \bar{t}_4^2)$, where $(t_1, \ldots, t_4)$ are the coordinates of a maximal torus T_G of G and by abuse of notation we identify $t_i \in H^1(T_G)$ with $\bar{t}_i \in H^2(B_{T_G})$ via transgression.

We are now in a position to compute the first Pontrjagin class of $M^{13} = G/\!\!/U$. Let τ be the tangent bundle of $G/\!\!/U$. Consider the following vector bundles over $G/\!\!/U$. Let $\alpha_U := G \times_U \mathfrak{u}$, where $U = S^1 \times G_2$ acts on G via the biquotient action, and on the Lie algebra $\mathfrak{u}$ of U via Ad_U. Let $\alpha_G := (GG/U) \times_G \mathfrak{g}$, where G acts on GG/U diagonally on the left and on $\mathfrak{g}$ via Ad_G. Since U acts freely on G we have, via a similar argument to that in [Si],

$$\tau \oplus \alpha_U = \alpha_G.$$

Recall that if V, W are vector bundles over some manifold M, $p(V \oplus W) =$

$p(V) \smile p(W)$. Hence in our case

$$p(\tau)p(\alpha_U) = p(\alpha_G).$$

Recall from [BH] that, in the absence of torsion, the Pontrjagin class of a vector bundle $\alpha_L = P \times_L V$ associated to the L-principal bundle $P \longrightarrow B := P/L$ is given by

$$p(\alpha_L) = 1 + p_1(\alpha_L) + p_2(\alpha_L) + \cdots = \varphi_L^*(a), \quad a := \prod_{\alpha_i \in \Delta_L^+} (1 + \bar{\alpha}_i^2) \in H^*(B_{T_L})^{W_L},$$

where Δ_L^+ is the set of positive weights of the representation of L on V, $\varphi_L : B \longrightarrow B_L$ is the classifying map of the L-principal bundle, and W_L is the Weyl group of L. Note that in this situation $H^1(T_L) \cong H^2(B_{T_L})$, and hence $a \in H^*(B_{T_L})^{W_L} \cong H^*(B_L)$

In our case, even though we have torsion in cohomology, we are fortunate in that $H^4(B_G) \cong H^4(B_{T_G})^{W_G}$ and $H^4(B_U) \cong H^4(B_{T_U})^{W_U}$ since the generators are $\bar{z} = \sigma_1(\bar{t}^2)$ and $\bar{x} = \frac{1}{2}\sigma_1(\bar{s}^2)$ respectively. Moreover the vector bundles α_U and α_G are associated to the principal bundles $U \longrightarrow G \longrightarrow G/\!/U$ and $G \longrightarrow GG/U \longrightarrow G/\!/U$ respectively, and the weights are the roots of the corresponding Lie group.

Hence we may write

$$\begin{aligned} p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_U) \\ &= \varphi_G^* \left(\sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2 \right) - \varphi_U^* \left(\sum_{\beta_j \in \Delta_U^+} \bar{\beta}_j^2 \right). \end{aligned}$$

The positive roots of $G = SO(8)$ are $t_i \pm t_j$, $1 \leq i < j \leq 4$. Hence, as in the proof of Theorem 5.4.3,

$$\sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2 = 6 \sum_{i=1}^{4} \bar{t}_i^2 = 6\sigma_1(\bar{t}^2) = 6\bar{z}.$$

But $\varphi_G^*(\bar{z}) = 4y^2$. Hence $p_1(\alpha_G) = 6\varphi_G^*(\bar{z}) = 24y^2 \in H^4(G/\!\!/U)$.

From our earlier description of the roots of G_2, and since S^1 has no roots, the positive roots of U are

$$s_1,\ s_2,\ -s_3,\ s_1 - s_3,\ s_2 - s_1,\ s_2 - s_3,$$

where $\sum s_i = 0$ and $s_i = \frac{1}{3}(2t_i - t_j - t_k)$, $\{i, j, k\} = \{1, 2, 3\}$. Then, as in the proof of Theorem 5.4.3,

$$\sum_{\beta_j \in \Delta_U^+} \bar{\beta}_j^2 = 4\sigma_1(\bar{s}^2) = 8\bar{x}.$$

Thus, since $\varphi_U^*(\bar{x}) = 2ky^2$, $p_1(\alpha_U) = 8\varphi_U^*(\bar{x}) = 16ky^2 \in H^4(G/\!\!/U)$.

Therefore

$$\begin{aligned} p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_U) \\ &= 8(3 - 2k)y^2 \in H^4(G/\!\!/U). \end{aligned}$$

From Theorem 5.4.3 we know that $p_1(\tau) = p_1(G/\!\!/U)$ is divisible only by 2. Therefore $k = 1$ or $k = 2$ since $3 - 2k$ is odd, which in turn implies $p_1(G/\!\!/U) = \pm 8y^2$ as desired. □

Remark 5.4.10. It is tempting to suggest that $p_1(M^{13}) = -8y^2$ since we know $\varphi_G^*(\bar{z}) = \varphi_U^*(\bar{w}^2) = 4y^2$ and hence one might expect that $\varphi_U^*(\bar{x}) = 4y^2$ (as opposed to $2y^2$) purely based on the commutativity of the diagram of fibrations and the validity of the analogous statement in our $\mathbb{Z}_p$ argument.

However, in order to make this assertion one would need to compute the maps $(B_\iota)^* : H^4(B_{GG}) \longrightarrow H^4(B_U)$ and $(B_\Delta)^* : H^4(B_{GG}) \longrightarrow H^4(B_{\Delta G})$. One can easily use the Künneth formula ([Sp], page 247) to compute the low-dimensional cohomology groups of B_{GG}. Unfortunately, for example, the spectral sequence of the right-hand fibration $G \longrightarrow B_{\Delta G} \longrightarrow B_{GG}$ is rather unwieldy and so the computation of $(B_\Delta)^* : H^4(B_{GG}) \longrightarrow H^4(B_{\Delta G})$ is quite difficult.

Chapter 6

Torus quotients of $S^3 \times S^3$

6.1 Free and almost free T^2 actions on $S^3 \times S^3$

Wilking, [Wi], has shown that a particular circle action on $S^3 \times S^3$ induces almost positive curvature on $S^3 \times S^2$. This, together with the description in [To] of $\mathbb{C}P^2 \# \mathbb{C}P^2$ as a biquotient $S^3 \times S^3 /\!\!/ T^2$, suggests that it may be beneficial to study T^2 actions on $S^3 \times S^3$. We are, of course, interested in finding new examples of biquotients with almost and quasi-positive curvature. Recall that a bi-invariant metric on $S^3 \times S^3$ is simply a product of bi-invariant metrics on each factor. Suppose we use a Cheeger deformation from the bi-invariant metric to equip $S^3 \times S^3$ with a left-invariant metric which is right-invariant under our T^2 action. If we allow such isometric torus actions to be arbitrary on the right-hand side of $S^3 \times S^3$ then, since Im $\mathbb{H}$ is 3-dimensional, at every point of $S^3 \times S^3$ we will be able to obtain a horizontal zero-curvature plane of the form Span $\{(v,0),(0,w) \mid v,w \in \mathrm{Im}\,\mathbb{H}\}$, which hence will project to a zero-curvature plane in $S^3 \times S^3 /\!\!/ T^2$. Therefore we shall restrict our attention to a special subfamily of torus actions which act arbitrarily on the left, but diagonally on the right of $S^3 \times S^3$.

Let $G = S^3 \times S^3$. As we are interested in biquotient actions, we need to consider homomorphisms

$$f:\ T^2 \longrightarrow T^2 \subset T^2 \times T^2 \subset G \times G$$

such that $f(T^2)$ is diagonal in the second factor, i.e. the projection onto the second factor is either trivial or one-dimensional. Hence all tori $f(T^2)$ must have either one or two-dimensional projections onto the first factor. If we perform the appropriate reparametrizations we may thus assume without loss of generality and up to a reordering of factors that the torus $f(T^2) \subset G \times G$ has one of the forms

$$U_L := \left\{ \left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \;\Big|\; z,w \in S^1 \right\}; \quad \text{or} \tag{6.1.1}$$

$$U_c := \left\{ \left(\begin{pmatrix} z \\ z^c \end{pmatrix}, \begin{pmatrix} w \\ w \end{pmatrix} \right) \;\Big|\; z,w \in S^1 \right\}, \quad c \in \mathbb{Z}; \quad \text{or} \tag{6.1.2}$$

$$U_{a,b} := \left\{ \left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z^a w^b \\ z^a w^b \end{pmatrix} \right) \;\Big|\; z,w \in S^1 \right\}, \quad a,b \in \mathbb{Z}. \tag{6.1.3}$$

It is clear that U_L acts effectively and freely on G. We are interested in determining when the other actions are free. First we will examine the effectiveness of these actions. It is possible that there is an ineffective kernel for the actions of U_c and $U_{a,b}$. However, the existence of an ineffective kernel will have no impact on our curvature computations in §6.2 and so we will perform these computations with respect to the actions of U_c and $U_{a,b}$ defined above.

Consider $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ and recall that $jz = \bar{z}j$ for all $z \in \mathbb{C}$. Therefore, given some $q \in S^3 \subset \mathbb{H}$,

$$\begin{aligned} & z^k w^\ell q \bar{z}^m \bar{w}^n = q \\ \iff \quad & (z^k w^\ell) 1 (\bar{z}^m \bar{w}^n) = 1 \quad \text{and} \quad (z^k w^\ell) j (\bar{z}^m \bar{w}^n) = j \\ \iff \quad & z^{k-m} w^{\ell-n} = 1 \quad \text{and} \quad z^{k+m} w^{\ell+n} = 1. \end{aligned} \tag{6.1.4}$$

Lemma 6.1.1. *U_c and $U_{a,b}$ act effectively on G when $c+1$ and $a+b+1$ respectively are odd. In the event that $c+1$ and $a+b+1$ are even, then U_c and $U_{a,b}$ respectively act on G with ineffective kernel $\Delta\mathbb{Z}_2 := \{\pm(1,1)\}$.*

Proof. Let us first consider the action of U_c. By equation (6.1.4) every point (q_1, q_2) in G is fixed by U_c if and only if

$$z\bar{w} = 1, \quad zw = 1, \quad z^c\bar{w} = 1, \quad \text{and} \quad z^c w = 1.$$

This is then equivalent to $z = w$, $z^2 = 1$, and $z^{c+1} = 1$. It is clear that if $c+1$ is odd then $z = w = 1$ and the action is effective. If $c+1$ is even then $z = w = \pm 1$ and it is easy to check that $z = w = -1$ indeed fixes each point of G. Thus $\{(z, w) \mid z = w = \pm 1\}$ is the ineffective kernel as desired.

Again by equation (6.1.4), $U_{a,b}$ fixes all of G if and only if

$$z^{1-a}\bar{w}^b = 1, \quad z^{1+a}w^b = 1, \quad \bar{z}^a w^{1-b} = 1. \text{ and } \; z^a w^{1+b} = 1.$$

If we multiply the first two equations together we find that $z^2 = 1$. Combining the first and the third equations we see that $z = w$. Thus $z^{1-a-b} = 1$. Since $z^2 = 1$ this is equivalent to $z^{1+a+b} = 1$. Hence if $1 + a + b$ is odd the action of $U_{a,b}$ is effective. If $1+a+b$ is even we have $z = w = \pm 1$. Again it is easy to check that $z = w = -1$ fixes all of G and thus the ineffective kernel of the $U_{a,b}$ action is $\Delta\mathbb{Z}_2$. $\square$

If the action of $U_\bullet = f(T^2)$ is ineffective we get an induced effective torus action by $\widetilde{U}_\bullet := \tilde{f}(\mathcal{T}^2) \cong U_\bullet$ from the commutative diagram

$$\begin{array}{ccc} T^2 & \xrightarrow{\quad f \quad} & U_\bullet\,, \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\tilde{f}} & \\ & \mathcal{T}^2 = T^2/\Delta\mathbb{Z}_2 & \end{array}$$

where

$$\begin{aligned} \pi : T^2 &\longrightarrow \mathcal{T}^2 \\ z &\longmapsto z^2 =: \xi \\ w &\longmapsto zw =: \zeta. \end{aligned}$$

In the case of $c + 1 = 2\ell$ the torus $\widetilde{U}_c$ has the form

$$\widetilde{U}_c = \left\{ \left(\begin{pmatrix} \xi^{\frac{1}{2}} \\ \xi^{\frac{c}{2}} \end{pmatrix}, \begin{pmatrix} \bar{\xi}^{\frac{1}{2}}\zeta \\ \bar{\xi}^{\frac{1}{2}}\zeta \end{pmatrix} \right) \;\middle|\; \xi, \zeta \in S^1 \right\}, \quad c = 2\ell - 1,\ \ell \in \mathbb{Z}.$$

Similarly when $1 + a + b$ is even we may write the torus $\widetilde{U}_{a,b}$ in the form

$$\widetilde{U}_{a,b} = \left\{ \left(\begin{pmatrix} \xi^{\frac{1}{2}} \\ \bar{\xi}^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} \bar{\xi}^{\frac{a-b}{2}}\zeta^b \\ \bar{\xi}^{\frac{a-b}{2}}\zeta^b \end{pmatrix} \right) \;\middle|\; \xi, \zeta \in S^1 \right\}, \quad a - b = 2k + 1,\ k \in \mathbb{Z}.$$

It is now a simple exercise to check that the only points which can possibly be fixed by the actions of U_c, $\widetilde{U}_c$, $U_{a,b}$ or $\widetilde{U}_{a,b}$ lie on the orbits of the points $(1, 1)$, $(1, j)$, $(j, 1)$ and (j, j). Therefore we need only examine these points in order to determine when the actions are free.

Lemma 6.1.2. *The action of U_c, c even, is free if and only if $c = 0$. The action of $\widetilde{U}_c$, c odd, is never free.*

Proof. Consider first the action of U_c for c even. By the discussion above and from equation (6.1.4) we find that

$$\begin{aligned}
(1,1) \text{ fixed} &\iff z\bar{w} = 1 \;\;\&\;\; z^c\bar{w} = 1 \iff z = w \;\;\&\;\; z^{c-1} = 1;\\
(1,j) \text{ fixed} &\iff z\bar{w} = 1 \;\;\&\;\; z^c w = 1 \iff z = w \;\;\&\;\; z^{c+1} = 1;\\
(j,1) \text{ fixed} &\iff zw = 1 \;\;\&\;\; z^c\bar{w} = 1 \iff z = \bar{w} \;\;\&\;\; z^{c+1} = 1;\\
(j,j) \text{ fixed} &\iff zw = 1 \;\;\&\;\; z^c w = 1 \iff z = \bar{w} \;\;\&\;\; z^{c-1} = 1.
\end{aligned}$$

Thus we see that the action is free if and only if $c \pm 1 = \pm 1$. But c is even and hence $c = 0$ is the only value for which we obtain a free action.

Consider now $\widetilde{U}_c$ for c odd. We again apply equation (6.1.4) and find

$$\begin{aligned}
(1,1) \text{ fixed} &\iff \xi\bar{\zeta} = 1 \;\;\&\;\; \xi^{\frac{c+1}{2}}\bar{\zeta} = 1 \iff \xi = \zeta \;\;\&\;\; \xi^{\frac{c-1}{2}} = 1;\\
(1,j) \text{ fixed} &\iff \xi\bar{\zeta} = 1 \;\;\&\;\; \xi^{\frac{c-1}{2}}\zeta = 1 \iff \xi = \zeta \;\;\&\;\; \xi^{\frac{c+1}{2}} = 1;\\
(j,1) \text{ fixed} &\iff \zeta = 1 \;\;\&\;\; \xi^{\frac{c+1}{2}}\bar{\zeta} = 1 \iff \zeta = 1 \;\;\&\;\; \xi^{\frac{c+1}{2}} = 1;\\
(j,j) \text{ fixed} &\iff \xi\bar{\zeta} = 1 \;\;\&\;\; \xi^{\frac{c-1}{2}}\zeta = 1 \iff \zeta = 1 \;\;\&\;\; \xi^{\frac{c-1}{2}} = 1.
\end{aligned}$$

Therefore the action is free (namely $\xi = \zeta = 1$ in each case) if and only if $\frac{c\pm 1}{2} = \pm 1$, i.e. if and only if $c \pm 1 = \pm 2$. But c is odd and there is no value for which we obtain both $c + 1 = \pm 2$ and $c - 1 = \pm 2$. Hence we will always have a fixed point and so the action of $\widetilde{U}_c$, c odd, is never free.

□

Lemma 6.1.3. *The action of $U_{a,b}$, $a+b$ even, is free if and only if $a=b=0$. The action of $\widetilde{U}_{a,b}$, $a+b$ odd, is never free.*

Proof. Consider first the action of $U_{a,b}$ for $a+b$ even. As in the proof of the previous lemma, equation (6.1.4) yields

$$\begin{array}{llllll}
(1,1) \text{ fixed} & \Longleftrightarrow & z^{1-a}\bar{w}^b = 1 \ \& \ \bar{z}^a w^{1-b} = 1 & \Longleftrightarrow & z = w \ \& \ z^{1-a-b} = 1; \\
(1,j) \text{ fixed} & \Longleftrightarrow & z^{1-a}\bar{w}^b = 1 \ \& \ z^a w^{1+b} = 1 & \Longleftrightarrow & z = \bar{w} \ \& \ z^{1-a+b} = 1; \\
(j,1) \text{ fixed} & \Longleftrightarrow & z^{1+a}w^b = 1 \ \& \ \bar{z}^a w^{1-b} = 1 & \Longleftrightarrow & z = \bar{w} \ \& \ z^{1+a-b} = 1; \\
(j,j) \text{ fixed} & \Longleftrightarrow & z^{1+a}w^b = 1 \ \& \ z^a w^{1+b} = 1 & \Longleftrightarrow & z = w \ \& \ z^{1+a+b} = 1.
\end{array}$$

Thus we see that the action is free (namely $z = w = 1$ in each case) if and only if $1 \pm a \pm b = \pm 1$. But $a+b$ is even, hence $\pm a \pm b$ is even, and so $a = b = 0$ is the only situation in which we can obtain a free action.

Consider now $\widetilde{U}_{a,b}$ for $a+b$ odd. We yet again apply equation (6.1.4) and find

$$\begin{array}{llllll}
(1,1) \text{ fixed} & \Longleftrightarrow & \xi^{\frac{1+a-b}{2}}\bar{\zeta}^b = 1 \ \& \ \bar{\xi}^{\frac{-1-a+b}{2}}\zeta^{1-b} = 1 & \Longleftrightarrow & \xi = \zeta \ \& \ \xi^{\frac{1+a-3b}{2}} = 1; \\
(1,j) \text{ fixed} & \Longleftrightarrow & \xi^{\frac{1+a-b}{2}}\bar{\zeta}^b = 1 \ \& \ \bar{\xi}^{\frac{-1+a-b}{2}}\zeta^{1+b} = 1 & \Longleftrightarrow & \zeta = 1 \ \& \ \xi^{\frac{1+a-b}{2}} = 1; \\
(j,1) \text{ fixed} & \Longleftrightarrow & \xi^{\frac{1-a+b}{2}}\zeta^b = 1 \ \& \ \bar{\xi}^{\frac{-1-a+b}{2}}\zeta^{1-b} = 1 & \Longleftrightarrow & \zeta = 1 \ \& \ \xi^{\frac{1-a+b}{2}} = 1; \\
(j,j) \text{ fixed} & \Longleftrightarrow & \xi^{\frac{1-a+b}{2}}\zeta^b = 1 \ \& \ \bar{\xi}^{\frac{-1+a-b}{2}}\zeta^{1+b} = 1 & \Longleftrightarrow & \xi = \zeta \ \& \ \xi^{\frac{1-a+3b}{2}} = 1.
\end{array}$$

Therefore the action is free (namely $\xi = \zeta = 1$ in each case) if and only if $1+a-3b = \pm 2$, $1+a-b = \pm 2$, $1-a+b = \pm 2$, and $1-a+3b = \pm 2$. It is a simple exercise to check that there are no values of a and b which satisfy all four equations simultaneously.

Hence we will always have a fixed point and so the action of $\widetilde{U}_{a,b}$, $a+b$ odd, is never free. □

If we combine the last two lemmas we notice that we have in fact proved:

Proposition 6.1.4. *Up to a change of coordinates or reordering of factors, the only free T^2 actions on $S^3 \times S^3$ which are diagonal on the right are given by $U_L = U_{a,b}$, $a = b = 0$, and U_c, $c = 0$. The actions are, respectively,*

$$(z,w) \star \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} zq_1 \\ wq_2 \end{pmatrix}, \ z,w \in S^1, q_1, q_2 \in S^3; \quad \text{and} \tag{6.1.5}$$

$$(z,w) \star \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} zq_1\bar{w} \\ q_2\bar{w} \end{pmatrix}, \ z,w \in S^1, q_1, q_2 \in S^3. \tag{6.1.6}$$

The resulting manifolds are $S^2 \times S^2$, $\mathbb{C}P^2 \# -\mathbb{C}P^2$ respectively.

For those actions which are not free we may consider the equations obtained in the proofs of Lemmas 6.1.2 and 6.1.3 in order to write down explicitly the isotropy groups $\Gamma_{(q_1,q_2)}$ of singular points, which we recall can only be the T^2-orbits of the points $(q_1, q_2) = (1,1), (1,j), (j,1), (j,j) \in S^3 \times S^3$. The isotropy groups for each action are collected in Table 6.1. By considering the groups in this table we can easily find examples which have only one or two singular points and small isotropy groups at these points. In the event that they arise, $\mathbb{Z}_0$ and $\mathbb{Z}_1$ denote S^1 and $\{1\}$ respectively. We include some examples in Tables 6.2, 6.3, 6.4 and 6.5.

Table 6.1: Isotropy groups of the T^2 actions U_c, $\widetilde{U}_c$, $U_{a,b}$ and $\widetilde{U}_{a,b}$

T^2	$\Gamma_{(q_1,q_2)}$ at:			
	$(1,1)$	$(1,j)$	$(j,1)$	(j,j)
U_c, c even	$\mathbb{Z}_{\|c-1\|}$	$\mathbb{Z}_{\|c+1\|}$	$\mathbb{Z}_{\|c+1\|}$	$\mathbb{Z}_{\|c-1\|}$
$\widetilde{U}_c$, c odd	$\mathbb{Z}_{\frac{1}{2}\|c-1\|}$	$\mathbb{Z}_{\frac{1}{2}\|c+1\|}$	$\mathbb{Z}_{\frac{1}{2}\|c+1\|}$	$\mathbb{Z}_{\frac{1}{2}\|c-1\|}$
$U_{a,b}$, $a+b$ even	$\mathbb{Z}_{\|1-a-b\|}$	$\mathbb{Z}_{\|1-a+b\|}$	$\mathbb{Z}_{\|1+a-b\|}$	$\mathbb{Z}_{\|1+a+b\|}$
$\widetilde{U}_{a,b}$, $a+b$ odd	$\mathbb{Z}_{\frac{1}{2}\|1+a-3b\|}$	$\mathbb{Z}_{\frac{1}{2}\|1-a+b\|}$	$\mathbb{Z}_{\frac{1}{2}\|1+a-b\|}$	$\mathbb{Z}_{\frac{1}{2}\|1-a+3b\|}$

Table 6.2: Some special cases of the action of U_c, c even

c	$\Gamma_{(q_1,q_2)}$ at:			
	$(1,1)$	$(1,j)$	$(j,1)$	(j,j)
2	$\{1\}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\{1\}$
-2	$\mathbb{Z}_3$	$\{1\}$	$\{1\}$	$\mathbb{Z}_3$

Table 6.3: Some special cases of the action of $\widetilde{U}_c$, c odd

c	$\Gamma_{(q_1,q_2)}$ at:			
	$(1,1)$	$(1,j)$	$(j,1)$	(j,j)
3	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{1\}$
-2	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$

Table 6.4: Some special cases of the action of $U_{a,b}$, $a+b$ even

(a,b)	$\Gamma_{(q_1,q_2)}$ at:			
	$(1,1)$	$(1,j)$	$(j,1)$	(j,j)
$(1,1)$	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}_3$
$(1,-1)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_3$	$\{1\}$
$(-1,1)$	$\{1\}$	$\mathbb{Z}_3$	$\{1\}$	$\{1\}$
$(-1,-1)$	$\mathbb{Z}_3$	$\{1\}$	$\{1\}$	$\{1\}$

Table 6.5: Some special cases of the action of $\widetilde{U}_{a,b}$, $a+b$ odd

(a,b)	$\Gamma_{(q_1,q_2)}$ at:			
	$(1,1)$	$(1,j)$	$(j,1)$	(j,j)
$(-6,-3)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$
$(3,0)$	$\mathbb{Z}_2$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
$(-3,0)$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$	$\mathbb{Z}_2$
$(6,3)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

6.2 Curvature on $(S^3 \times S^3)/\!\!/T^2$

Let $K = \Delta S^3 \subset G = S^3 \times S^3$, and let $\langle\ ,\rangle_0$ be the bi-invariant product metric on G. The Lie algebras of G and K are denoted $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to $\langle\ ,\rangle_0$. Notice that $(\mathfrak{g}, \mathfrak{k})$ is a rank one symmetric pair. Define a new left-invariant, right K-invariant metric on G via:

$$\langle X, Y\rangle_1 = \langle X, \Phi(Y)\rangle_0,$$

where $\Phi(Y) = Y_{\mathfrak{p}} + \lambda Y_{\mathfrak{k}}$, $\lambda \in (0,1)$. By Lemma 2.2.2 we know that a plane $\sigma = \text{Span}\,\{\Phi^{-1}(X), \Phi^{-1}(Y)\} \subset \mathfrak{g}$ has zero-curvature with respect to $\langle\ ,\rangle_1$ if and only if

$$0 = [X,Y] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}].$$

Hence, for G and K as above, a zero-curvature plane must be of the form

$$\sigma = \text{Span}\,\{\Phi^{-1}(v,0), \Phi^{-1}(0,v) \mid v \in \text{Im}\ \mathbb{H}\}. \tag{6.2.1}$$

Since we are considering T^2 actions which are diagonal on the right of G, it is clear that the actions are by isometries and hence induce a metric on $G/\!\!/T^2$.

Theorem 6.2.1. *$(G, \langle\ ,\rangle_1)/\!\!/T^2$ has almost positive curvature if and only if the action is not free.*

Proof. By O'Neill's formula it is sufficient to show that points in G with horizontal zero-curvature planes lie on a hypersurface. Recall that the existence of an ineffective kernel will have no impact on our curvature computations. We therefore need only consider torus actions of the form

$$\begin{aligned} U_{a,b} &= \left\{ \left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z^a w^b \\ z^a w^b \end{pmatrix} \right) \;\middle|\; z, w \in S^1 \right\}, \quad a, b \in \mathbb{Z}; \\ U_c &= \left\{ \left(\begin{pmatrix} z \\ z^c \end{pmatrix}, \begin{pmatrix} w \\ w \end{pmatrix} \right) \;\middle|\; z, w \in S^1 \right\}, \quad c \in \mathbb{Z}, \end{aligned}$$

and notice that the U_L action of (6.1.1) is the special case $(a, b) = (0, 0)$ of $U_{a,b}$.

Consider first the action by $U_{a,b}$. The vertical subspace at (q_1, q_2), left translated to $(1, 1)$, is given by

$$\mathcal{V}_{(q_1,q_2)} = \left\{ \frac{1}{2} \begin{pmatrix} \theta \operatorname{Ad}_{\bar{q}_1} i - (a\theta + b\varphi)i \\ \varphi \operatorname{Ad}_{\bar{q}_2} i - (a\theta + b\varphi)i \end{pmatrix} \;\middle|\; \theta, \varphi \in \mathbb{R} \right\}.$$

Thus the horizontal subspace with respect to $\langle\ ,\ \rangle_1$ is

$$\mathcal{H}_{(q_1,q_2)} = \left\{ \Phi^{-1}(v, w) \;\middle|\; \begin{array}{l} \operatorname{Ad}_{q_1} v - a(v + w) \perp i \\ \\ \operatorname{Ad}_{q_2} w - b(v + w) \perp i \end{array} \right\}.$$

Hence, by equation (6.2.1), a zero-curvature plane $\sigma = \operatorname{Span}\{\Phi^{-1}(v, 0), \Phi^{-1}(0, v)\}$ is horizontal if and only if

$$\operatorname{Ad}_{q_1} v - av \perp i, \tag{6.2.2}$$

$$av \perp i, \tag{6.2.3}$$

$$\operatorname{Ad}_{q_2} v - bv \perp i, \quad \text{and} \tag{6.2.4}$$

$$bv \perp i. \tag{6.2.5}$$

We want to show that $v, \operatorname{Ad}_{q_1} v, \operatorname{Ad}_{q_2} v \perp i$ since this is equivalent to $v \perp i, \operatorname{Ad}_{\bar{q}_1} i$, $\operatorname{Ad}_{\bar{q}_2} i$. This will imply that $v = 0$ unless $i, \operatorname{Ad}_{\bar{q}_1} i$, and $\operatorname{Ad}_{\bar{q}_2} i$ are linearly dependent, which in turn would imply positive curvature at the point $[(q_1, q_2)] \in G/\!\!/U_{a,b}$. It is clear that this situation arises if and only if $(a,b) \neq (0,0)$, i.e. if and only if the action of $U_{a,b}$ is free. Suppose $(a,b) \neq 0$. Then $i, \mathrm{Ad}_{\bar{q}_1} i$, and $\mathrm{Ad}_{\bar{q}_2} i$ are linearly dependent if and only if

$$\det \begin{pmatrix} \langle \mathrm{Ad}_{\bar{q}_1} i, j\rangle & \langle \mathrm{Ad}_{\bar{q}_1} i, k\rangle \\ \langle \mathrm{Ad}_{\bar{q}_2} i, j\rangle & \langle \mathrm{Ad}_{\bar{q}_2} i, k\rangle \end{pmatrix} = 0, \tag{6.2.6}$$

which defines a hypersurface in G. Note that equation (6.2.6) is invariant under the action of $U_{a,b}$ since $\mathrm{Ad}_{z^k w^\ell q \bar{z}^m \bar{w}^n} i = \mathrm{Ad}_{z^k w^\ell q} i$ and $\langle \mathrm{Ad}_q i, j\rangle = 2\,\mathrm{Re}\,(\bar{u}vi)$, $\langle \mathrm{Ad}_q i, k\rangle = 2\,\mathrm{Re}\,(\bar{u}v)$, for $z, w \in S^1$, $q = u + vj \in S^3, u, v \in \mathbb{C}$. Thus we have a hypersurface in $G /\!\!/ U_{a,b}$ defined by (6.2.6) on which points with zero-curvature planes must lie.

We now turn our attention to the action by U_c. The vertical subspace at (q_1, q_2), left translated to $(1,1)$, is given by

$$\mathcal{V}_{(q_1,q_2)} = \left\{ \frac{1}{2} \begin{pmatrix} \theta\,\mathrm{Ad}_{\bar{q}_1} i - \varphi i \\ c\,\theta\,\mathrm{Ad}_{\bar{q}_2} i - \varphi i \end{pmatrix} \,\middle|\, \theta, \varphi \in \mathbb{R} \right\}.$$

Thus the horizontal subspace with respect to $\langle\ ,\ \rangle_1$ is

$$\mathcal{H}_{(q_1,q_2)} = \left\{ \Phi^{-1}(v,w) \,\middle|\, \begin{array}{c} \mathrm{Ad}_{q_1} v + c\,\mathrm{Ad}_{q_2} w \perp i \\ v + w \perp i \end{array} \right\}.$$

Hence, by (6.2.1), a zero-curvature plane $\sigma = \mathrm{Span}\,\{\Phi^{-1}(v,0), \Phi^{-1}(0,v)\}$ is horizontal if and only if

$$\mathrm{Ad}_{q_1} v \perp i, \tag{6.2.7}$$

$$c\,\mathrm{Ad}_{q_2} v \perp i, \quad \text{and} \tag{6.2.8}$$

$$v \perp i. \tag{6.2.9}$$

It is clear that the only situation in which we do not get $v, \mathrm{Ad}_{q_1} v, \mathrm{Ad}_{q_2} v \perp i$ is when $c = 0$, i.e. when the action is free. In all other situations we have almost positive curvature by the same argument as for $U_{a,b}$. $\square$

Remark 6.2.2. In all cases we get a horizontal zero-curvature plane at (q_1, q_2) if either

- $\mathrm{Ad}_{\bar{q}_1} i = \pm i \iff q_1 \in \mathbb{C}$ or $\mathbb{C}j$,
- $\mathrm{Ad}_{\bar{q}_2} i = \pm i \iff q_2 \in \mathbb{C}$ or $\mathbb{C}j$, or
- $\mathrm{Ad}_{\bar{q}_1} i = \pm \mathrm{Ad}_{\bar{q}_2} i \iff q_1 \perp q_2, iq_2$ or $q_1 \perp jq_2, kq_2$.

Thus we will always have a zero-curvature plane at the singular points when the action is not free. Moreover, in the free cases we have a zero-curvature plane at every point. More precisely:

- The action $U_{a,b}$ with $a = b = 0$ yields $\mathrm{Ad}_{\bar{q}_1} i, \mathrm{Ad}_{\bar{q}_2} i \perp v$, which implies that there is a unique horizontal zero-curvature plane when $\mathrm{Ad}_{\bar{q}_1} i$ and $\mathrm{Ad}_{\bar{q}_2} i$ are linearly independent, and there is an S^1 worth of zero-curvature planes when $\mathrm{Ad}_{\bar{q}_1} i = \pm \mathrm{Ad}_{\bar{q}_2} i$, i.e. when $q_1 \perp q_2, iq_2$ or $q_1 \perp jq_2, kq_2$;
- The action U_c with $c = 0$ yields $i, \mathrm{Ad}_{\bar{q}_1} i \perp v$, which implies that there is a unique horizontal zero-curvature plane when $q_1 \notin \mathbb{C}$ or $\mathbb{C}j$, and there is an S^1 worth of zero-curvature planes when $q_1 \in \mathbb{C}$ or $\mathbb{C}j$.

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