PROFILES OF LARGE COMBINATORIAL STRUCTURES

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Graduate school takes a long time and is a very stressful experience. I've had the pleasure of being able to attend graduate school close to my family and having a family that I actually want to be close to. I would like to apologize to my cousin, John DeCaro for not inventing the "Italian restaurant process" or naming any object in this thesis after a type of pasta. I thank my parents, Janis and Albert Lugo. Long ago they tried to teach me what square numbers were by rearranging pennies on a kitchen table; I asked, innocently, if there were "twiangle numbers". There are, of course; this was just an early example of a seemingly endless stream of questions that they endured for the most part with good humor. They have supported me through good times and bad and have always provided a home for me. This thesis

is as much their achievement as mine. Finally, my grandmother, Josephine DeCaro, suffers from Alzheimer's disease, and can no longer fully appreciate the importance of this moment in my life. But I know she would be proud.

ABSTRACT

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We derive limit laws for random combinatorial structures using singularity analysis of generating functions. We begin with a study of the Boltzmann samplers of Flajolet and collaborators, a useful method for generating large discrete structures at random which is useful both for providing intuition and conjecture and as a possible proof technique. We then apply generating functions and Boltzmann samplers to three main classes of objects: permutations with weighted cycles, involutions, and integer partitions. Random permutations in which each cycle carries a multiplicative weight σ have probability $(1-\gamma)^\sigma$ of having a random element be in a cycle of length longer than γn; this limit law also holds for cycles carrying multiplicative weights depending on their length and averaging σ. Such permutations have number of cycles asymptotically normally distributed with mean and variance $\sim \sigma \log n$. For permutations with weights $\sigma_k = 1/k$ or $\sigma_k = k$, other limit laws are found; the prior have finitely many cycles in expectation, the latter around $\sqrt{n}$. Compositions of uniformly chosen involutions of $[n]$, on the other hand, have about $\sqrt{n}$ cycles on average. These can be modeled as modified 2-regular graphs. A composition of two random involutions in S_n typically has about $n^{1/2}$ cycles, characteristically of length

$n^{1/2}$. The number of factorizations of a random permutation into two involutions appears to be asymptotically lognormally distributed, which we prove for a closely related probabilistic model. We also consider connections to pattern avoidance, in particular to the distribution of the number of inversions in involutions. Last, we consider integer partitions. Various results on the shape of random partitions are simple to prove in the Boltzmann model. We give a (conjecturally tight) asymptotic bound on the number of partitions $p_M(n)$ in which all part multiplicities lie in some fixed set n, and explore when that asymptotic form satisfies $\log p_M(n) \sim \pi\sqrt{Cn}$ for rational C. Finally we give probabilistic interpretations of various pairs of partition identities and study the Boltzmann model of a family of random objects interpolating between partitions and overpartitions.

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Chapter 1

Introduction

1.1 What is analytic combinatorics?

Analytic combinatorics is an approach to combinatorics that treats the generating function as the central object. Furthermore, the generating function is not just viewed as a convenient bookkeeping device (a formal power series), but is considered as an analytic object in its own right. Generating functions are tremendously useful, because they enable combinatorialists to harness the rich tools of complex analysis. There are generally two attitudes towards generating functions. The first is to treat them as formal power series; the second is to treat them as analytic objects. The formal power series view is nice for computation – one does not have to worry about convergence! Using this approach it is possible to derive explicit formulas, new recurrences, and the like for many combinatorially defined sequences. It is tempting to omit the analytic parts, but as Wilf puts it [Wil94, Preface], "To omit those parts of the subject, however, is like listening to a stereo broadcast of, say, Beethoven's Ninth Symphony, using only the left audio channel." Complex-analytic methods enable us to extract information that would be extraordinarily elusive by purely algebraic means.

Recent decades have seen substantial growth in combinatorics and discrete mathematics in general, much of which is motivated by applications to computer science. The "gold standard" for proof in combinatorics has long been the bijective proof – we feel that we really understand why two sets are the same size when we can pair up elements in one with elements in the other. But there are few general methods for *finding* bijections, so combinatorics is often seen as a collection of ad hoc tricks, or "theorems in search of a theory". As combinatorics has grown from a bag of tricks to a full-fledged branch of mathematics, there have been various efforts to create such unifying theories. The analytic approach taken here is one. A more algebraic approach – rather reminiscent of category theory – is the "theory of species" of the French-Canadian school [Joy81, BLL98]. This theory has the particular advantage of making clear the concept of a "natural isomorphism" in combinatorics. Polya's theory of enumeration under group action [PR87] has been another powerful theory, particularly for the enumeration of unlabelled objects.

A particular strength of generating function methods is that different combinatorial classes have similar generating functions and therefore similar asymptotic properties; these are what one might call *universality phenomena*, a term borrowed from statistical physics. For example, in the enumeration of trees, where generating functions satisfy certain polynomial relations, the number of trees of size n with a finite set of allowed node degrees always has the form $C \cdot A^n n^{-3/2}$ regardless of the finite set in question; the height is proportional to the square root of size; and the number of leaves is normally distributed in the limit. The logarithmic combinatorial structures of [ABT03] are another example; cycle structure of permutations, factorizations of polynomials in finite fields, connected components of certain forests, prime factorizations of integers, and a variety of other combinatorial objects have a number of components which is logarithmic in their size and a largest component which makes up an appreciable fraction of the entire structure. These are all in some sense tied together by their generating functions, which resemble $(1-z)^{-\theta}$ where θ is a parameter that controls the shape of the structure. Later we will see hints of other such classes of structures, such as the "square-root structures" which have generating functions like $\exp(\sigma z/(1-z))$.

Within the framework of analytic combinatorics, many asymptotic enumeration results become quasi-routine. There is a long history of collecting asymptotic results in combinatorics, going back to [PR87] and [HP73] among others. The survey papers of Bender and Odlyzko [Ben74, Odl95] summarize many of these methods; textbooks covering asymptotic enumeration include [GK90, GKP94, Wil94]. Another excellent exposition of asymptotic methods, not just restricted to combinatorics, is [dB81]. Asymptotic enumeration has proven to be a tremendously useful tool in the analysis of algorithms; see for example [FS95]. The main results of analytic combinatorics are treated in the recent treatise of Flajolet and Sedgewick [FS09]. This book is largely divided into two parts. The first part explains how to derive generating functions for combinatorial objects; the second part shows how to extract asymptotic information from such generating functions. Both of these are essential parts of any problem in asymptotic combinatorics.

The first sort of results in asymptotic combinatorics are essentially those of *asymptotic enumeration.* Stanley [Sta97, p. vii] tells us that "Enumerative combinatorics is concerned with counting the number of elements of a finite set S. This definition, as it stands, tells us very little about the subject since virtually any mathematical problem can be cast in these terms. In a genuine enumerative problem, the elements of S will usually have a rather simple combinatorial definition and very little additional structure." The central problem of asymptotic enumeration, then, is to approximately solve this problem for a sequence of finite sets $S_1, S_2, S_3, \ldots$, where S_k is the set of objects of "size" k. Then we would like an approximate formula for $|S_n|$ in terms of simple functions of n.

One simple example comes from tilings of 2-by-n boards with 1-by-2 and 2-by-1 dominoes. The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Each such tiling has at its left end either one vertical or two horizontal dominoes, so these tilings satisfy the same recurrence. Checking the initial conditions, we see that a 2-by-n board has F_{n+1} tilings. In chapter 2 we will use (very

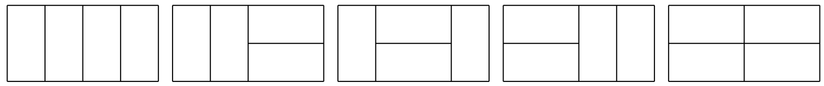

Figure 1.1: The five tilings of a 2-by-4 board by dominoes.

simple!) asymptotic methods to derive the classic formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

This is an example of an exact formula for a combinatorially defined sequence; asymptotic formulae like $F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$, however, are often accessible even when the exact forms are not.

The next natural question of asymptotic combinatorics is "what does a typical object look like?" There are several ways to answer this question. The most crude is simply to determine the mean of some statistic on the objects. (Whenever we speak about the mean of some combinatorial statistic, it will be over objects chosen from the *uniform* distribution on the set in question, unless stated otherwise.) More complicated statistical information – higher moments and variances of statistics – are also accessible. (The variance, however, is a difference of two quantities which in many cases are of the same order, so terms beyond the first order are needed.) Often the first two moments are enough; the Gaussian distribution is ubiquitous in limit laws for random structures. Intuitively this is true because the statistics are sums of indicator variables which are "almost independent". Various discrete distributions also occur frequently. Two examples from permutation enumeration which will figure prominently in this thesis are the number of cycles of a random permutation of $[n]$, and the number of cycles of length k of a random permutation of $[n]$; the former is asymptotically normally distributed with mean and variance $\log n$, and the latter is asymptotically Poisson with mean $1/k$. We will determine many such distributions, and attempt to explain them via appeals to probabilistic intuition.

Again we return to the Fibonacci example. In this case we can answer the following simple probabilistic questions, which will feature as an example in Chapter 2 and resurface in another guise in Chapter 6:

- The probability that the leftmost tile in a large tiling is vertical is $(\sqrt{5}-1)/2$;
- The distribution of the number of vertical dominoes at the "left end" of a tiling is geometric;
- The distribution of the number of vertical dominoes in the entire tiling is asymptotically normally distributed, with mean $n/\sqrt{5}$ and variance $4\sqrt{5}/25 \cdot n$.

Our main method for showing distributional results such as these will be *bivariate* generating functions, which track objects both by their size and by the statistics of interest; this principle will become particularly important in Chapter 4.

Analytic combinatorics has become quite useful in the analysis of algorithms, figuring quite prominently in books such as [Knu, FS95, GK90]. It is particularly useful in average-case analysis of algorithms and in randomized algorithms. Traditionally analysis of algorithms has focused on worst-case results, asking how much computing time, memory, or other resources will be used given the worst possible input to a program. Such analyses are therefore geared towards constructing exceptional cases which do not often occur in practice. Randomized algorithms, such as those in the book [MR95], do away with this difficulty. Perhaps the simplest example is the *quick-sort* algorithm, which runs on average in time $O(n \log n)$ on lists of size n, but actually takes $O(n^2)$ time on lists that are already sorted. If one wishes to create a general-purpose sorting routine that runs quickly on almost-sorted inputs – which would be desirable in many cases occurring in practice – then one thing to do is to shuffle the input randomly and then run quicksort. Some critics of average-case analysis have said that it reveals more information about the distribution of inputs than the actual performance of algorithms – it is common to choose simple input distributions, such as the uniform distribution, which may or may not correspond with distributions that occur in practice. But this flaw from the computer science point of view does not matter to the mathematician.

Analytic combinatorics can also be useful for statistical testing. In statistical testing one often wants to test the hypothesis that certain objects are drawn from a certain distribution on all such objects. In a typical situation there are a very large number of possible objects – perhaps we are picking random binary trees, or random permutations – or it may be difficult to generate samples from the distribution in question, and so tests such as the χ-squared test on the entire distribution are not useful. In many cases of interest we may only have a sample of size one. For example, the human genome can be considered as a very long string of the letters A, C, G, and T. Given a snippet of the genome, how can we tell if it "does something"? If a segment of genome has no function, it can be modeled as a string of A, C, G, and T chosen uniformly and independently at random; if it *does* have a function, then this is not true. Thus one wants to know the probability that a certain pattern (say CAT) appears in a long string with some large frequency. There also exist combinatorial models of secondary structures in RNA – RNA is single-stranded but base-pairing can occur among segments of the same strand. (See for example [Neb02].) Analytic combinatorics allows these structures to be enumerated (they look rather like trees) and it should be possible to determine the expected "amount" of secondary structure in random strands of RNA; in this way sections of the genome which have particularly low or high amounts of secondary structure could be identified.

On a more frivolous note, the same techniques can be used to determine if messages are hidden in (ordinary-language) texts. In the text of *Hamlet*, there are approximately 1.63×10^{39} hidden occurrences of the word "combinatorics", in which the letter c appears, followed somewhere later by the letter o, and so on. (Of course these overlap.) Does this mean Shakespeare is sending a message about combinatorics? Of course not. If a monkey typed letters at random, with the letter frequencies of English, roughly the same count would occur. This example is from [FS09, Example I.11], based on the papers [BV02, FSV06]. It is related to the famous and controversial "Bible Code" [WRR94, MBNBHK99], in which hidden messages were found in the Bible using similar methods; what would be most interesting in such a case would be a *lack* of patterns.

This understanding of patterns and lack of patterns is a goal of analytic combinatorics. We want to know what "typical" objects look like. The discovery of universality classes of combinatorial objects is especially tantalizing. Many *solvable* combinatorial models fall in these classes. It is tempting to suspect that more complicated combinatorial systems, perhaps "naturally occurring" systems that cannot be completely analyzed but are of practical interest, also fall into such classes. We might call this a "physics" of random structures, and attempt to form laws about combinatorial structures that are ignorant of fine details of the underlying mathematics. With such a classification in progress – partially rigorously, partially by building up a library of examples and recognizing patterns – we are coming to an understanding of the large-scale laws that govern all random structures. Herein we add to that library of examples and aim to give some intuition on how small-scale randomness gives rise to some sort of large-scale order.

1.2 Statements of results

This thesis is organized as follows. Chapter 2 is a compilation of results that will be useful in the remainder of the thesis. We begin with a brief overview of the theory of generating functions, covering the different types of generating functions that we will need in the sequel, and explain how probabilistic information can be extracted from combinatorial generating functions. We then show how generating functions for many sequences of combinatorial interest can easily be derived via the "symbolic method", which constructs combinatorial objects recursively from atoms using a few basic combinatorial building blocks. After this we give a brief exposition of singularity analysis, which is our main technique for extracting asymptotic information from generating functions. Here we will recall various *ad hoc* results of singularity analysis – partial fraction expansion, Hayman's method, saddle-point methods, and the theorems of Meinardus and Wright – and we will give a more systematic treatment based upon the Flajolet-Odlyzko transfer theorems. Finally we recall miscellaneous results: the Euler-Maclaurin theorem, central limit theorems, and results from the method of moments.

Chapter 3 explains the Boltzmann sampling methodology. Boltzmann samplers are a method for sampling objects at random from a combinatorial class, with a given approximate size; they are often much faster than methods for generating objects of a fixed size, but at the cost of approximation. These samplers are also much easier to analyze than fixed-size samplers because of their recursive structure, and because dependence between different parts of the structure is reduced. We begin by defining Boltzmann samplers and showing how to construct them for various recursively specified combinatorial classes. After this we give formulas for some statistics of such samplers. It appears that results about Boltzmannized objects are very similar to results about the corresponding fixed-size objects if and only if the distribution of sizes of the Boltzmannized objects is concentrated. We give examples of this; the most striking result is that the size of Boltzmannized *partitions*, tuned to have mean size n, has standard deviation of order $n^{3/4}$. We also explain how the Boltzmann sampler gives a method for creating models of "combinatorial objects of infinite size" which is useful in the sequel.

In Chapter 4, parts of which are adapted from the paper [Lug09], we consider the combinatorics of permutations with restricted cycle structure. Given the set $[n] = \{1, 2, \ldots, n\}$, a permutation is a bijective function $f : [n] \to [n]$ – in such a function, each element of $[n]$ occurs exactly once among $f(1), f(2), \ldots, f(n)$. Permutations can naturally be decomposed into their cycles, and it has long been known that a "typical" permutation of n objects has approximately $\log n$ cycles [Gon44]. Furthermore, the distribution of the number of cycles of length k in a random large permutation approaches a Poisson distribution with mean $1/k$. In this chapter this work is extended to random choices from some restricted classes of permutations – for example, those in which all cycle lengths are even, which for large n have number of cycles nearly normally distributed with mean and variance $\frac{1}{2} \log n$. The statistics arising when permutations are weighted depending on their cycle structure are also of interest; this is a generalization of the Ewens sampling formula of population genetics [Ewe72]. Restricted and weighted permutations turn out to be quite similar, as is seen in section 4.4.

In Section 4.5 we proceed to another specific case, that of permutations with periodic weighting sequences. These obey the same limit laws but the asymptotic enumeration of such permutation introduces new factors. In Section 4.6 we consider the weighting scheme $\sigma_j = 1/j$; in this weight scheme, permutations have one long cycle and, on average, $\pi^2/6$ short cycles. Section 4.7 considers permutations having square roots or more generally mth roots; this is a natural example of a permutation model with restricted multiplicities which nonetheless strongly resembles the weighted models. In Section 4.8 we consider the weighting scheme $\sigma_j = j$, which corresponds to "sets of lists"; a set of lists in $[n]$ usually has about $\sqrt{n}$ components, of typical size $\sqrt{n}$, which is a combinatorial consequence of the generating function $\exp(z/(1-z))$ of "exponential of a pole" type. In Section 4.9 we show that the number of cycles of a permutation of $[n]$ of length in $[\gamma n, \delta n]$ obeys a limit law. Finally Sections 4.10 and 4.11 consider connections between random permutations and, respectively, stochastic processes and number theory.

In Chapter 5, we consider the cycle structure of compositions of involutions. An *involution* on $[n]$ is a permutation in which all cycles have length 1 or 2, and thus involutions have order 1 or 2 as elements of the symmetric group; thus this is an attempt to look more closely at algebraic structure. An involution in S_n can be viewed as a partial matching on $[n]$. Thus a *composition* of two involutions can be viewed as a *superposition* of two partial matchings, which is a graph (with colored edges) in which each vertex has degree at most two. These graphs have components which are either cycles or paths and can be enumerated by the exponential formula. In Section 5.2 we find generating functions for these graphs counting them by their size and number of cycles and paths; these can be reinterpreted in terms of permutation cycle structure. In particular, as shown in Section 5.3, the number of k-cycles of a composition of two random involutions of $[n]$ converges in distribution, as $n \to \infty$, to $A_k + 2B_k$ where A_k is $Poisson(1)$ and B_k is $Poisson(1/2k)$; the expected *total* number of cycles is $\sim \sqrt{n}$, as seen in Section 5.5. The first of these facts can be predicted by looking at cycles and paths as "rare events". Finally we address the class multiplication problem for involutions: in how many ways can a permutation be written as a product of two involutions? An n-cycle π can be factored into two involutions in n ways, and a permutation consisting of two n-cycles has n^2+n factorizations into involutions. These building blocks lead to Theorem 5.7.1, which gives the total number of solutions to $\pi = \sigma \circ \tau$ where σ, τ are involutions. This leads to Theorem 5.7.5, in which we show that in a certain stochastic model of permutations the number of factorizations is lognormally distributed; the logarithm of the number of factorizations of a random permutation into two involutions has mean $(\log n)^2/2$ and variance $(\log n)^3/3$. In particular, the median number of factorizations of a permutation is near $\exp((\log n)^2/2)$ but the mean is of larger order, near $\exp(2\sqrt{n})$. This is a hint that the measure defined by compositions of random involutions looks quite different than the uniform measure on S_n.

For large n, the number of involutions of $[n]$ is asymptotic to $\sqrt{n!}$ multiplied by a subexponential factor [MW55], roughly the square root of the number of all permutations. So we explore ways in which involutions are a "square root" of permutations. The Stanley-Wilf conjecture (proven by Marcus and Tardos [MT04]) states that the number of π-avoiding permutations of n, $S_n(\pi)$, satisfies $\lim_{n\to\infty} S_n(\pi)^{1/n} = L(\pi)$ for some constant π. For involutions we can define $I_n(\pi)$ similarly; we have $\lim_{n\to\infty} I_n(\pi)^{1/n} = \sqrt{L(\pi)}$ in cases where both limits are known. Probabilistically, this means that π-avoiding involutions are much *more* common than π-avoiding permutations. This motivates counting the occurrences of patterns in involutions; in Section 5.9 we show that the number of inversions in a random involution has the same *mean* as the number of inversions in a random permutation, but twice the variance. Finally, in Section 5.10, using saddle-point methods we prove that the number of permutations in which all cycles have length in a given finite set S, with $m = \max S$, is asymptotic to $(n!)^{1-1/m}$ times a subexponential factor, and that the expected number of k-cycles in such a permutation, chosen uniformly at random, is asymptotic to $n^{k/m}/k$. The main term $\sqrt{n!}$ for involutions can be explained by noting that a permutation corresponds to an ordered pair of involutions, both via the RSK correspondence and since the graph of an involution σ – that is, the set $\{i, \sigma(i) : i \in [n]\}$ – is symmetric across the diagonal. The cycle structure can be explained by considering Boltzmann samplers.

In Chapter 6 we consider Boltzmann sampling as applied to partitions of integers. Section 6.1 is devoted to recovering classical results about partitions from careful consideration of the Boltzmann sampler. In particular we show that the mean number of parts of partitions and partitions into distinct parts can be predicted from the Boltzmann samplers, giving an explanation for results of [EL41]. We also derive results on the average shape of the Young diagrams of partitions which echo [DVZ00]. Finally, we determine the number of parts of different multiplicities which occur in the Boltzmann model for partitions with restrictions on part multiplicities. In Section 6.2 we enumerate families of partitions for which the generating function has form $\prod_{k=1}^{\infty} g(z^k)$, where $g(z) = \prod_{k=1}^{\infty}(1 - z^k)^{-b_k}$. These encompass many, but not all, classes of multiplicity-restricted permutations. In Section 6.3 we enumerate similar families in which the generating function does not have such a simple form. The enumeration involves dilogarithms of the root of the generating polynomial for the allowed multiplicities. This work is motivated by Subbarao's identity [Sub71]: the number of partitions into parts of multiplicities 2, 3, or 5 is equal to number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 mod 12, so we can associate the constant $5/12$ with the set $\{2, 3, 5\}$. There appears to be no such similar identity, and no such rational constant, for partitions into parts of multiplicity 2 or 3. In Section 6.4 we interpret various pairs of partition identities in terms of probabilities. From the Rogers-Ramanujan identities we can show that the probability a partition of n into nonconsecutive parts contains no part equal to 1 approaches $(\sqrt{5} - 1)/2$ as $n \to \infty$; this is, rather unexpectedly, connected to the Fibonacci numbers. Similar connections to combinatorics on words occur for the Gollnitz-Gordon identities and Gordon's identities. Finally, we consider in Section 6.5 the probabilistic aspects of overpartitions, which are partitions in which the last occurrence of each part can be barred. We show that a typical overpartition of n has barred parts summing to $n/3$. We review results on random overpartitions from [CH04, CGH06], and define w-overpartitions, which are a class of weighted objects interpolating between partitions and overpartitions. Boltzmannization of w-overpartitions gives formulas interpolating between known statistics of partitions and of overpartitions.

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Chapter 2

Background and singularity analysis

In this chapter we collect the background results necessary for this thesis. We begin by defining the various types of generating functions to be used and show how probabilistic information can be extracted from generating functions. We then explain the symbolic method for deriving generating functions of combinatorial classes. Next we give a primer on singularity analysis, which is used for extracting asymptotics of the coefficients of generating functions. We close with statements of some miscellaneous results.

2.1 Generating functions

We will deal with several different types of univariate generating functions in this thesis: the *ordinary*, *exponential*, and *Dirichlet* generating functions. In addition we will consider certain multivariate generating functions.

Given a sequence $\{a_n\}_{n=0}^{\infty}$, with $a_n \in \mathbb{C}$, its *ordinary generating function* is $A(x) = \sum_{n\geq 0} a_n x^n$. This can be viewed in two ways: as a formal power series in $\mathbb{C}[x]$, or as a function $A : \mathbb{C} \to \mathbb{C}$. We will generally use lowercase letters to denote a sequence and uppercase letters to denote the corresponding generating function. The *exponential generating function* of $\{a_n\}$ is $A(x) = \sum_{n\geq 0} a_n x^n/n!$. We will not consider both the ordinary and exponential generating functions of the same sequence, so the difference will be clear in context. We let $[z^n]A(z)$ denote the coefficient of z^n in $A(z)$, and $[z^n/n!]A(z)$ denote $n!$ times the coefficient of z^n in $A(z)$.

If $\{a_n\}$ does not grow too quickly – faster than any exponential r^n in the ordinary case, or faster than any function of the form $r^n n!$ in the exponential case – then $A(z)$ is an analytic function in some neighborhood of zero.

In general we will use ordinary generating functions to count *unlabelled* combinatorial objects, and exponential generating functions to count *labelled* combinatorial objects. There are usually many more labelled objects of a given type than unlabelled objects, due to symmetry considerations.

The *Dirichlet generating function* of a sequence $\{a_n\}$, often called its *Dirichlet series*, is given by $\sum_{n\geq 1} a_n n^{-s}$. The Dirichlet series is particularly well suited for number-theoretic problems, because it is adapted to the Dirichlet convolution: given two sequences $\{a_n\}$ and $\{b_n\}$, with Dirichlet generating functions $A(s)$ and $B(s)$ respectively, let $c_n = \sum_{d|n} a_d b_{n/d}$, and let $C(s) = \sum_{n\geq 1} c_n n^{-s}$. Then $C(s) = A(s)B(s)$. The Dirichlet generating function of the all-ones sequence is $\sum_{n\geq 0} n^{-s} = \zeta(s)$. We will make use of Dirichlet series in Chapter 6.

We will also consider multivariate generating functions. Algebraically these are objects in $\mathbb{C}[x_1, \ldots, x_n]$; analytically they are functions from $\mathbb{C}^n$ to $\mathbb{C}$, which are analytic in some polydisc centered at the origin. Most of our generating functions will be asymmetric, in the following sense: one variable will keep track of the *size* of the combinatorial object under consideration, while the others will mark certain statistics. We will generally indicate the size variable by the letter z and the statistic-tracking variables by $u_1, u_2, \ldots$ or by u and v.

Consider an array of numbers $a_{n,k_1,\ldots,k_r}$. The ordinary $(r+1)$-variate generating function of this array is given by

$$A(z, u_1, \ldots, u_r) = \sum_{n\geq 0} \sum_{k_i \geq 0 \forall i} a_{n,k_1,\ldots,k_r} z^n u_1^{k_1} \cdots u_r^{k_r}$$

and the exponential $(r+1)$-variate generating function is given by

$$A(z, u_1, \ldots, u_r) = \sum_{n\geq 0} \sum_{k_i \geq 0 \forall i} \frac{a_{n,k_1,\ldots,k_r}}{n!} z^n u_1^{k_1} \cdots u_r^{k_r}.$$

Specializations of these generating functions will also be useful. In particular, the univariate function $A(z, 1, \ldots, 1)$ satisfies

$$[z^n/\omega_n]A(z, 1, \ldots, 1) = \sum_{(k_1,\ldots,k_r)\in\mathbb{Z}_+^r} a_{n,k_1,\ldots,k_r}$$

where ω_n is 1 in the ordinary case and $n!$ in the exponential case. This sum is the number of objects of size n where the parameters under consideration are arbitrary; thus $A(z, 1, \ldots, 1)$ counts objects accordingly only to their size. If some variable is set to 0, this has the effect of *excluding* objects for which the corresponding statistic is nonzero. Further "semi-combinatorial" specializations are also possible. For example, in the case $r = 1$ (the bivariate case), $[z^n/\omega_n]A(z, -1)$ is the difference between the number of A-objects with an even u-statistic and with an odd u-statistic. Letting u equal other roots of unity gives other linear combinations.

A simple example is as follows: the generating function for permutations counted by their size and number of fixed points is

$$\begin{aligned} P(z,u) &= \exp\left(uz + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right) = \exp\left((u-1)z + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right) \\ &= \exp\left((u-1)z + \log\frac{1}{1-z}\right) = \frac{1}{1-z}\exp(u-1)z. \end{aligned}$$

Thus $P(z, 1) = 1/(1 - z)$ and $P(z, 0) = e^{-z}/(1 - z)$. The prior is the generating function of permutations regardless of their number of fixed points; the latter is the generating function of permutations without fixed points, or derangements. We will later see that $\lim_{n\to\infty}[z^n]P(z, 0) = e^{-1}$; thus the probability that a random permutation is a derangement is e^{-1}. Such a statement should be understood to be an abbreviation for "the limit of the probability that a permutation of $[n]$ chosen uniformly at random is a derangement, as $n \to \infty$, is e^{-1}". We also have $P(z, -1) = e^{-2z}/(1 - z)$. The number of permutations of $[n]$ with an even number of fixed points is thus $[z^n/n!](P(z, -1) + P(z, 1))/2$; for large n this is very near $(1 + e^{-2})/2$ times the number of permutations. And in fact $(1+e^{-2})/2$ is the probability that a Poisson random variable with mean 1 is even.

We will not have to do asymptotics for multivariate generating functions. Finding asymptotic information on the coefficients of multivariate generating functions is a quite delicate operation, and an area of active research. The underlying principle for asymptotic work is that the type and location of the singularity closest to the origin governs the asymptotic behavior of the coefficients. In the univariate case, singularities are points; in the multivariate case, one must deal with a singular variety. However, we will be able to extract the probabilities we seek from bivariate generating functions by considering only their univariate specializations.

Let $\mathcal{A}$ be a combinatorial class, so $\mathcal{A} = \cup_{n=0}^{\infty} \mathcal{A}_n$ with $|\mathcal{A}_n| = a_n < \infty$. The *uniform probability distribution* over $\mathcal{A}_n$ assigns to any $\alpha \in \mathcal{A}_n$ the same probability, namely $1/A_n$. We will let $\mathbb{P}$, $\mathbb{P}_{\mathcal{A}_n}$ or $\mathbb{P}_n$ denote the probability relative to this uniform distribution. In general, the symbol $\mathbb{P}$ will denote probabilities, and subscripts on it will denote the particular probabilistic model under consideration.

Consider a *parameter* or *statistic* χ, which associates to every object $\alpha \in \mathcal{A}$ an integer value $\chi(\alpha)$. The parameter χ determines a discrete random variable on the probability space $\mathcal{A}_n$. The probability generating function of this random variable is $p(u) = \sum_k \mathbb{P}(\chi = k)u^k$. The following proposition follows immediately from the definition.

Proposition 2.1.1. *[FS09, Prop. III.1] Let $A(z, u)$ be the bivariate generating func-*

tion of a parameter χ defined over a combinatorial class $\mathcal{A}$. The probability generating function of χ over $\mathcal{A}_n$ is given by

$$\sum_k \mathbb{P}_{\mathcal{A}_n}(\chi = k)u^k = \frac{[z^n]A(z,u)}{[z^n]A(z,1)}.$$

We note that $[z^n]A(z,u)$ is in general a function of u; if we intend the coefficient of $z^n u^0$, we will use the notation $[z^n u^0]$. That is, $[z^2](3z^2 + z^2u + z^3) = 3 + u$, and $[z^2u^0](3z^2 + z^2u + z^3) = 3$.

Proposition 2.1.2. *[FS09, Prop. III.2] The factorial moment of order r of a parameter χ is determined from the bivariate generating function $A(z,u)$ by r-fold differentiation followed by evaluation at 1:*

$$\mathbb{E}_{\mathcal{A}_n}((\chi)_r) = \frac{[z^n]\ \partial_u^r A(z,u)|_{u=1}}{[z^n]A(z,1)}.$$

Proof. From Proposition 2.1.1, we have the probability generating function of $\mathcal{A}_n$. The effect of differentiation is as follows:

$$\left(\frac{\partial}{\partial u}\right)^r \sum_k \mathbb{P}(\chi = k)u^k = \sum_k \mathbb{P}(\chi = k)(k)_r u^{k-r}.$$

Setting $u = 1$, we have $\partial_u^r p(u) = \sum_k \mathbb{P}(\chi = k)(k)_r$; the right-hand side is just the rth factorial moment of χ. $\square$

In particular, the first two moments satisfy

$$\mathbb{E}_{\mathcal{A}_n}(\chi) = \frac{[z^n]A_u(z,1)}{[z^n]A(z,1)}, \mathbb{E}_{\mathcal{A}_n}(\chi^2) = \frac{[z^n]A_{uu}(z,1)}{[z^n]A(z,1)} + \frac{[z^n]A_u(z,1)}{[z^n]A(z,1)}$$

and the variance satisfies $\mathbb{V}(\chi) = \mathbb{E}(\chi^2) - \mathbb{E}(\chi)^2$.

2.2 The symbolic method in combinatorics

In this section we give a brief exposition of the *symbolic method* in combinatorics. This is essentially a device for the specification of combinatorial objects which are recursively built up from simpler objects. We will build up a library of recursive constructions used in such constructions, which are called *admissible* constructions, and show how such constructions are translated into operations on generating functions. This makes the determination of the generating functions counting such objects, which include many of the generating functions occurring naturally in combinatorics, fairly routine, which frees us up to concentrate on the analysis of the generating functions.

An (unlabelled) *combinatorial class* is nothing more than a countable union of finite sets. We write $\mathcal{A} = \bigcup_{n \geq 0} \mathcal{A}_n$, where $\mathcal{A}_n$ is the number of objects of size n; we then write $a_n = |\mathcal{A}_n|$ for the number of objects of size n. We can endow a class A with a multidimensional parameter $\chi = (\chi_1, \ldots, \chi_d)$, which is a function from $\mathcal{A}$ to the set of d-tuples of nonnegative integers. We say such a parameter is d-valued.

We now define labelled combinatorial classes. A *weakly labelled object* is a graph whose vertices are a subset of the positive integers. An object of size n is said to be well-labelled, or labelled, if it is weakly labelled and its collection of labels is the set $\{1, 2, \ldots, n\}$. A labelled class is a combinatorial class consisting of well-labelled objects. The restriction to graphs may seem overly restrictive, but all "natural" labelled objects can be encoded as graphs. In our case, labelled objects will be permutations; a permutation σ can be encoded as a directed graph with edges $i \to \sigma(i)$

for each i. Labelled combinatorial classes can carry parameters just as unlabelled ones do.

The essence of the symbolic method is that we can recursively specify combinatorial classes, in such a way that they are naturally built up from atoms by applying a few basic constructions. Those consructions which are adapted to the generating-function approach, we call admissible. Formally, let Φ be a construction that associates to m combinatorial classes a new class: $\mathcal{B} = \Phi[\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{m)}]$. Then Φ is called admissible iff the sequence $(B_n)_{n=1}^{\infty}$ depends only on the sequences $(A_n^{(1)})_{n=1}^{\infty}, \cdots, (A_n^{(m)})_{n=1}^{\infty}$. That is, the generating function $B(z)$ depends only on the $A^{(k)}(z)$. (This is from [FS09, Def. I.5].)

The classes $\{\epsilon\}$ and $\mathcal{Z}$, which have one element of size 0 and size 1 respectively, will be our building blocks for all other classes. We call $\{\epsilon\}$ a *neutral class* and $\mathcal{Z}$ an *atom*. We will proceed by listing some admissible constructions.

Disjoint unions. Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes, and let ϕ and χ be the corresponding parameters, both d-valued. Then define $\mathcal{C} = \mathcal{A} + \mathcal{B}$ by letting $\mathcal{C}_n$ be the disjoint union of $\mathcal{A}_n$ and $\mathcal{B}_n$. Formally,

$$\mathcal{C}_n = \{(\alpha, 1) : \alpha \in \mathcal{A}_n\} \cup \{(\beta, 2) : \beta \in \mathcal{B}_n\}$$

where 1 and 2 are simply tags in case $\mathcal{A}_n$ and $\mathcal{B}_n$ have elements in common. Let ψ be a d-valued parameter for $\mathcal{C}$, with $\psi((\alpha, 1)) = \phi(\alpha)$ and $\psi((\beta, 2)) = \chi(\beta)$. Then the generating functions of these classes, either ordinary or exponential, satisfy $C(z, u_1, \ldots, u_r) = A(z, u_1, \ldots, u_r) + B(z, u_1, \ldots, u_r)$. The proof is straightforward:

an object $\gamma \in C$ with $\psi(\gamma) = \vec{k}$ is either a pair $(\alpha, 1)$ with $\phi(\alpha) = \vec{k}$ or a pair $(\beta, 2)$ with $\chi(\beta) = \vec{k}$.

Cartesian products and labelled products. The Cartesian product of two combinatorial classes, $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, is defined in the ordinary set-theoretic way. Parameters are inherited by addition: if $\gamma = (\alpha, \beta)$, then $\psi(\gamma) = \phi(\alpha) + \chi(\beta)$. Then the ordinary generating functions satisfy $C(z, \vec{u}) = A(z, \vec{u})B(z, \vec{u})$. To see this, consider objects $\gamma \in C$ with $|\gamma| = n, \psi(\gamma) = \vec{k}$. These are of the form (α, β) with $|\alpha| + |\beta| = n, \phi(\alpha) + \chi(\beta) = \vec{k}$. Thus the product form for the generating function follows immediately from the multiplication process.

The exponential case is slightly more complicated. We must define the *labelled product* of two labelled combinatorial classes $\mathcal{A}$ and $\mathcal{B}$, which we denote $\mathcal{A} \star \mathcal{B}$. We first must define relabellings. Given a weakly labelled structure α of size n, we denote by $\rho(\alpha)$ its *reduction*, which is the same object with the labels reduced to the standard set $[n]$ and kept in the same order. Then given two labelled objects $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, we let $\alpha \star \beta$ be the set of ordered pairs that reduce to (α, β):

$$\alpha \star \beta = \{(\alpha', \beta') : (\alpha', \beta') \text{ is well-labelled }, \rho(\alpha') = \alpha, \rho(\beta') = \beta\}.$$

Then the labelled product of classes is given by

$$\mathcal{A} \star \mathcal{B} = \bigcup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \alpha \star \beta.$$

If $\mathcal{C} = \mathcal{A} \star \mathcal{B}$, we have that $C(z, \vec{u}) = A(z, \vec{u})B(z, \vec{u})$. To see this, consider objects $\gamma \in \mathcal{C}$ with $|\gamma| = n, \psi(\gamma) = \vec{k}$. These are of the form (α', β') with $|\alpha'| + |\beta'| =$

$n, \phi(\alpha') + \chi(\beta') = \vec{k}$. Furthermore $(\alpha', \beta') \in \alpha \star \beta$. Thus α' and β' together contain all the labels $1, 2, \ldots, n$, with none repeated.

Now we refine based on $|\alpha'|$. We can construct $\binom{n}{j} a_j b_{n-j}$ pairs $(\alpha', \beta') \in \mathcal{A} \star \mathcal{B}$ in which $|\alpha'| = j, |\beta'| = n - j$ – we choose the j labels which will be used in α' and then pick α' with size j and β' with size $n - j$, with suitable labels. Summing over j, the total number of pairs (α', β') with $|\alpha'| + |\beta'| = n$, that satisfy the labelling conditions, is

$$\sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j} = n! \sum_{j=0}^{n} \frac{a_j}{j!} \frac{b_{n-j}}{n-j!}.$$

This is just $n![z^n]A(z)B(z)$. So we have $[z^n/n!]C(z) = n![z^n]A(z)B(z)$, or $[z^n/n!]C(z) = [z^n/n!]A(z)B(z)$.

Sequences. We can form a combinatorial class by taking sequences of elements from an already-specified class *which contains no elements of size zero.* We denote the class obtained in this way by $\mathcal{B} = \textsc{Seq}(\mathcal{A})$. This is an abbreviation for a combination of sums and products:

$$\textsc{Seq}(\mathcal{A}) = \{\epsilon\} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \cdots$$

where ϵ is a structure of size 0, corresponding to the empty sequence. Then we have the generating function identity

$$B(z, \vec{u}) = 1 + A(z, \vec{u}) + A(z, \vec{u})^2 + A(z, \vec{u})^3 + \cdots = \frac{1}{1 - A(z, \vec{u})}.$$

The geometric series converges as a formal power series, since $[z^0]A(z) = 0$ by assumption.

Sets (for labelled structures). Denote by $\textsc{Set}_k(\mathcal{A})$ the class of k-sets formed from $\mathcal{A}$. Formally we write $\textsc{Set}_k(\mathcal{B}) = \textsc{Seq}_k(\mathcal{B})/\sim$. Here $\sim$ is the equivalence relation in which two sets are equivalent if the components of one are a permutation of the components of the other. Therefore $\sim$ partitions $\textsc{Seq}_k(\mathcal{A})$ into orbits of size $k!$. So if $\mathcal{B} = \textsc{Set}_k(\mathcal{A})$, we have $B(z) = A(z)^k/k!$, and similarly for the parameter-enriched version $B(z, \vec{u}) = A(z, \vec{u})/k!$.

We then define the set construction by

$$\textsc{Set}(\mathcal{A}) = \{\epsilon\} + \mathcal{A} + \textsc{Set}_2(\mathcal{A}) + \cdots = \bigcup_{k \geq 0} \textsc{Set}_k(\mathcal{A}).$$

Translating into generating functions, where $\mathcal{B} = \textsc{Set}(\mathcal{A})$, we get

$$B(z, \vec{u}) = \sum_{k \geq 0} \frac{A(z, \vec{u})^k}{k!} = \exp A(z, \vec{u}) \tag{2.1}$$

Thus taking sets corresponds to exponentiation; this is a form of the well-known "exponential formula".

Multisets (for unlabelled structures). For a finite combinatorial class $\mathcal{A}$, with $\mathcal{A}_0$ empty, the multiset class $\mathcal{B} = \textsc{Mset}(\mathcal{A})$ can be defined by

$$\textsc{Mset}(\mathcal{A}) = \prod_{\alpha \in \mathcal{A}} \textsc{Seq}(\{\alpha\}).$$

That is, let $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Then we can write a multiset of elements of $\mathcal{A}$ as a sequence of repeated elements α_1, followed by a sequence of repeated elements α_2, and so on. The generating function of $\textsc{Seq}(\{\alpha\})$ is $1/(1 - z^{|\alpha|})$, and so we get

$$B(z) = \prod_{\alpha \in A} 1/(1 - z^{|\alpha|}) = \prod_{n=1}^{\infty} (1 - z^n)^{-A_n}.$$

This can be written in the form $B(z) = \exp \sum_{k=1}^{\infty} A(z^k)/k$, which does not require coefficient extraction, but we will not need this.

Cycles. We proceed to define cycles as we did for sets. We have $\textsc{Cyc}_k(\mathcal{A}) = \textsc{Seq}_k(\mathcal{A})/\sim$, where now $\sim$ identifies two sequences when the components of one are a cyclic permutation of the components of the other. Thus $\sim$ partitions $\textsc{Seq}_k(\mathcal{A})$ into orbits of size k. Thus if $\mathcal{B} = \textsc{Cyc}_k(\mathcal{A})$, we have $B(z) = A(z)^k/k$, again with possible parameter enrichment. We now define $\textsc{Cyc}(\mathcal{A}) = \bigcup_{k\geq 1} \textsc{Cyc}_k(\mathcal{A})$. If $\mathcal{B} = \textsc{Cyc}(\mathcal{A})$, we thus have the generating function relation

$$A(z) = \sum_{k=1}^{\infty} \frac{1}{k} B(z)^k = \log \frac{1}{1 - B(z)}. \tag{2.2}$$

A note on notation. We will often use subscripts to denote a set of allowed sizes in the constructions $\textsc{Set}, \textsc{Mset}, \textsc{Cyc}$. For example, $\textsc{Cyc}_{\leq 3}(\mathcal{Z})$ would denote the combinatorial class of cycles of length less than or equal to 3, and so $\textsc{Set}(\textsc{Cyc}_{\leq 3}(\mathcal{Z}))$ is the combinatorial class of permutations with all cycle lengths at most 3. The specification $\textsc{Set}_e(\textsc{Cyc}(\mathcal{Z})$, in which the e stands for even, corresponds to permutations with an even number of cycles; similarly $\textsc{Set}(\textsc{Cyc}_o(\mathcal{Z})$, with o for odd, corresponds to permutations with all cycles having odd length. The corresponding generating functions can be found by summing over only the set of allowed sizes in an analogue of (2.1) or (2.2).

Marking components. The multivariate generating functions under consideration in this thesis are in general "asymmetric": one variable indicates size, and other variables indicate various statistics of these objects. In particular these statistics tend to be smaller than the size. We will generally arrive at such multivariate generating functions by attaching *marks* – these are objects of size 0 which are attached to atoms in specifications for combinatorial structures. Generally we will denote marks by the symbols μ and ν and they will be translated into variables u, v. For example, compositions of integers can be specified by $\mathcal{C} = \textsc{Seq}(\textsc{Seq}_{\geq 1}(\mathcal{Z}))$ and therefore have the generating function $1/(1-z/(1-z)) = (1-z)/(1-2z)$. We may insert a "mark" in front of each part in order to get the bivariate generating function for compositions counted by size and number of parts; we get the specification $\mathcal{C} = \textsc{Seq}(\mu\, \textsc{Seq}_{\geq 1}\mathcal{Z}))$, and so the generating function is

$$C(z,u) = \frac{1}{1-\frac{uz}{1-z}} = \frac{1-z}{1-(u+1)z}.$$

Similarly, we can specify permutations as $\mathcal{P} = \textsc{Set}(\textsc{Cyc}(\mathcal{Z}))$. This gives the exponential generating function $P(z) = \exp(\log 1/(1-z))) = 1/(1-z)$. If we mark cycles of length k, then we have $\mathcal{P} = \textsc{Set}(\textsc{Cyc}_{\neq k}(\mathcal{Z}) + \mu\textsc{Cyc}_k(\mathcal{Z}))$. Therefore permutations counted by size and number of cycles of length k have the generating function

$$\begin{aligned} P(z,u) &= \exp\left(\left(\sum_{j\neq k}\frac{z^j}{j}\right) + u\frac{z^k}{k}\right) \\ &= \exp\left(\log\frac{1}{1-z} - \frac{z^k}{k} + u\frac{z^k}{k}\right) = \frac{\exp((u-1)z^k/k)}{1-z}. \end{aligned}$$

Factors of $u-1$ often appear when marking; they arise since in marking we often replace a term in a series with the same term multiplied by u. We will sometimes write expressions like $\mathcal{P} = \textsc{Set}(\textsc{Cyc}(\mathcal{Z}) + (\mu - 1)\textsc{Cyc}_k\mathcal{Z})$ despite the fact that the symbol $-$ is technically meaningless in our combinatorial specification language; in such cases we will always be “subtracting” a set from a set which it is contained in.

2.3 Singularity analysis

In previous sections of this chapter we have seen that it is possible to write enumerative and probabilistic information about a combinatorial class in terms of generating functions associated with that class. We would like to extract asymptotic enumerative results and probabilistic limit laws from generating functions. In order to do this we will consider the generating function as an analytic object. In this section we compile various results used in this thesis for extracting coefficients, which are adapted to various types of singularities.

The big picture. First we consider how the radius of convergence of a generating function is linked to the growth of its coefficients.

Theorem 2.3.1 (Hadamard)**.** *The radius of convergence of the Taylor series* $a_0 + a_1 z + a_2 z^2 + \cdots$ *is given by*

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

It is a classical fact that such a function must have a singularity on its circle of convergence, $|z| = R$. Furthermore, in the “combinatorial case” where $a_n \geq 0$ for all n, we have

Theorem 2.3.2 (Pringsheim)**.** *If the Taylor series of* $f(z)$ *at the origin has nonnegative coefficients and radius of convergence* R*, then the point* $z = R$ *is a singularity.*

We call this singularity the *dominant singularity* of a combinatorial generating function. This name is apt because of the following formula:

Theorem 2.3.3 (Exponential growth formula)**.** *[FS09, Thm. IV.7] If $f(z)$ is analytic at 0 and has all coefficients nonnegative, and R is the modulus of the singularity nearest to the origin in the sense that*

$$R = \sup\{r \geq 0 : f \text{ is analytic at all points of } 0 \leq z \leq r\}$$

then the coefficient $f_n = [z^n]f(z)$ satisfies $f_n = R^{-n}\theta(n)$, where $\theta(n)$ is a subexponential factor, i. e. $\limsup |\theta(n)|^{1/n} = 1$.

This is an example of what Flajolet and Sedgewick call the first principle of coefficient asymptotics: "The *location* of a function's singularities dictate the *exponential growth* (A^n) of its coefficients." Their second principle is "the *nature* of a function's singularities determines the associate *subexponential factor* ($\theta(n)$)". The second principle is rather opaque at this point but we will learn much more about it; for now, observe that since $z = 1/\phi$ is a pole of order 1 of $F(z)$, the subexponential factor $\theta(n)$ is in fact a constant.

Rational functions. Singularity analysis is simplest for rational functions; in this case it can be reduced to the partial fraction decompositions familiar from calculus. We will consider in some depth the example of tilings of 2-by-n boards with dominoes from Section 1.1. This will enable us to use the tools we already have to answer some probabilistic questions in a setting where coefficient extraction is simple and exact formulas for coefficients can be found, before we begin to concern ourselves with the machinery of singularity analysis proper.

We begin by observing that the Fibonacci numbers have the generating function

$$F(z) = \sum_{n \geq 0} F_n z^n = \frac{z}{1 - z - z^2}.$$

To see this, we begin with the recurrence

$$F_n = F_{n-1} + F_{n-2} + [\![n = 1]\!]$$

where $[\![\cdot]\!]$ are the "Iverson bracket": $[\![n = 1]\!]$ is 1 if $n = 1$ and 0 otherwise. (Knuth [Knu92] advocates using $[\cdot]$ in this way but this conflicts with our notation for coefficient extraction.)

We can multiply both sides by z^n and sum over n to get

$$\sum_{n \geq 0} F_n z^n = \sum_{n \geq 0} F_{n-1} z^n + \sum_{n \geq 0} F_{n-2} z^n + \sum_{n \geq 0} [\![n = 1]\!] z^n.$$

The sum on the left-hand side is $F(z)$. The first sum on the right-hand side can be rewritten, letting $m = n - 1$:

$$\sum_{n \geq 0} F_{n-1} z^n = \sum_{m \geq -1} F_m z^{m+1} = z \sum_{m \geq -1} F_m z^m = zF(z)$$

and similarly the second sum on the right-hand side is $z^2 F(z)$. Finally, $\sum_{n \geq 0} [\![n = 1]\!] z^n = z$. So we have $F(z) = (z + z^2)F(z) + z$; solving for $F(z)$ gives $F(z) = z/(1 - z - z^2)$.

To derive the exact formula for the Fibonacci numbers, we can write $F(z)$ as a sum of partial fractions. Let $\phi = (1 + \sqrt{5})/2$ and $\tau = (1 - \sqrt{5})/2$; then we have

$$\frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi z} - \frac{1}{1 - \tau z} \right).$$

Finally, $[z^n]1/(1-\phi z) = \phi^n$ and $[z^n]1/(1-\tau z) = \tau^n$, so extracting z^n coefficients we get

$$[z^n]\frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}}(\phi^n - \tau^n).$$

So we have the classical formula for the Fibonacci numbers.

Now, the generating function $F(z) = z/(1-z-z^2)$, when treated as a complex-analytic function, has singularities at $z = 1/\phi$ and $z = 1/\tau$, and these singularities are poles. The function $F(z)$ is analytic everywhere else in the complex plane. The singularity at $1/\phi$ is closest to the origin, and F_n grows like ϕ^n. This is what happens in general, although in most cases some "subexponential factor" contributes to the asymptotics.

We return to the Fibonacci example. We can answer the following questions:

- What is the probability that the leftmost tile in a large tiling is vertical?
- What is the distribution of the number of vertical dominoes at the "left end" of a tiling?
- What is the distribution of the number of vertical dominoes in tilings of the 2-by-n board?

(The first and second of these questions will be revisited in Section 6.4.)

For the first question, we note that the number of tilings of a 2-by-n board is F_{n+1}. Tilings of 2-by-n boards in which the leftmost tile is vertical can be identified with tilings of 2-by-$(n-1)$ boards, so there are F_n of them. Therefore the probability that the leftmost tile in a 2-by-n tiling is vertical is F_n/F_{n+1}; as $n \to \infty$ this approaches $1/\phi = (\sqrt{5}-1)/2$. Note that we have written a probability as the ratio of the answer to a problem in combinatorial enumeration, evaluated at two different points; we will see this principle again, particularly in the limit laws of Chapter 4.

We can continue in the same manner to get an answer to the second question. Tilings of the 2-by-n board which "begin" with k vertical dominoes followed by a pair of horizontal dominoes correspond with tilings of the 2-by-$(n-(k+2))$ board. The probability that a random tiling begins with exactly k vertical dominoes is therefore F_{n-k-1}/F_{n+1}; as $n \to \infty$ this approaches $\phi^{-(k+2)}$. The distribution of the number of initial vertical tiles, then, is geometric. (Such a refinement according to the number of initial vertical tiles also provides a proof of the identity $F_{n+1} = F_{n-1} + F_{n-2} + \cdots + F_1 + F_0$; see [BQ03] for many more proofs of combinatorial identities of this type.)

This probabilistic interpretation does not give the probability that a random tile in the "interior" of a tiling is vertical, though. Domino tilings of a 2-by-n board in which the kth column contains a vertical tile can be identified with pairs consisting of a tiling of the 2-by-$(k-1)$ board and one of the 2-by-$(n-k)$ board; thus the probability that a random 2-by-n tiling has a vertical domino in the kth column is $F_k F_{n-k+1}/F_{n+1}$. If we assume k and $n-k$ are both large, then we can replace each Fibonacci number with the leading term of the explicit formula; thus this is approximately

$$\frac{\frac{1}{\sqrt{5}}\phi^k \cdot \frac{1}{\sqrt{5}}\phi^{n-k+1}}{\frac{1}{\sqrt{5}}\phi^{n+1}} = \frac{1}{\sqrt{5}}.$$

Thus the probability that a random "interior" column contains a vertical domino is $1/\sqrt{5}$, and we expect that horizontal dominoes slightly predominate.

Indeed they do. The main tool here is a *bivariate* generating function, which counts tilings according not just to their size, but also according to their number of vertical dominoes. This is

$$P(z,u) = \frac{1}{1 - uz - z^2}$$

since we can write $\mathcal{P} = \textsc{Seq}(\mu\square + \boxminus)$, or more conventionally $\mathcal{P} = \textsc{Seq}(\mu\mathcal{Z} + \mathcal{Z}\times\mathcal{Z})$. The coefficient $[z^n u^k]P(z,u)$ is the number of tilings of the 2-by-n board with k vertical dominoes. The series begins

$$P(z,u) = 1 + uz + (1+u^2)z^2 + (2u+u^3)z^3 + (1+3u^2+u^4)z^4 + \cdots$$

and indeed the number of tilings of the 2-by-4 board containing $0, 2, 4$ vertical dominoes are $1, 3, 1$. The mean number of vertical dominoes is given by the quotient $[z^n]P_u(z,1)/[z^n]P(z,1)$; we note that the numerator just counts the total number of vertical dominoes in all the tilings, and the denominator counts their number. We have $P_u(z,1) = z/(1-z-z^2)^2$; expanding this into partial fractions gives

$$\frac{z}{(1-z-z^2)^2} = \frac{A}{1-\phi z} + \frac{B}{(1-\phi z)^2} + \frac{C}{(1-\tau z)} + \frac{D}{(1-\tau z)^2}$$

where $A = -\sqrt{5}/50 - 1/10, B = (1+\sqrt{5})/10, C = \sqrt{5}/50 - 1/10, D = (1-\sqrt{5})/10$. From this we can derive an exact formula for the coefficient $[z^n]z(1-z-z^2)^{-2}$, namely

$$[z^n]\frac{z}{(1-z-z^2)^2} = A\phi^n + B(n+1)\phi^n + C\tau^n + D(n+1)\tau^n.$$

In particular $P_u(z,1) \sim Bn\phi^n$ as $n \to \infty$. We can extract this information knowing *only* the coefficient B above; in complex-analytic terms, then, we only need to know that $z = 1/\phi$ is a pole of order 2 with residue B.

This coefficient is the total number of vertical tiles in all tilings. The mean number of vertical tiles in a tiling, then, is asymptotic to $(Bn\phi^n/F_{n+1} \sim (Bn\phi^n)/(\phi^{n+1}/\sqrt{5}) = B\sqrt{5}/\phi)n = n/\sqrt{5}$. In this case there is an exact formula for the coefficients in terms of Fibonacci numbers, as well; it is given in [Slo10] – but the power of this approach is that exact formulas are not necessary. Similarly we can extract the variance of the number of vertical tiles in a random tiling. It turns out to be asymptotic to $(4\sqrt{5}/25)n$ as $n \to \infty$.

Saddle-point bounds. For functions which are not rational, coefficient extraction is not quite so simple as before. Our principal tool for coefficient extraction in the remainder of this section will be Cauchy's integral formula, applied on judiciously chosen contours. Saddle-point methods are a broad class of methods for extracting asymptotic information from analytic generating functions. A crude class of these are generally useful for extracting upper bounds on the coefficients of generating functions; it often turns out that these upper bounds are reasonably close to the correct answer.

We begin by recalling Cauchy's integral formula. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, analytic in an open neighborhood containing the closed disc $|z| \leq \rho$. Then f has a power series expansion at 0, $f(z) = \sum_{n \geq 0} f_n z^n$. Cauchy's integral formula states that

$$f_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} \, dz$$

where γ is the circle $|z| = \rho$, traversed in the counterclockwise direction.

Now, if we take the absolute value of the right-hand side, we get

$$f_n \leq \frac{1}{2\pi} \int_\gamma \frac{|f(z)|}{\rho^{n+1}} \, dz \leq \rho^{-n} \max_{|z|=\rho} f(z).$$

If we choose ρ so that $\rho^{-n} \max_{|z|=\rho} f(z)$ is close to its minimum, this is often a reasonably tight bound for f_n. For contours passing through saddle points, most of the contribution to the integral comes from the region near the saddle point, often of width $1/\sqrt{n}$; thus bounds which are off by a factor of $n^{1/2}$ are common. We will in particular need the following lemma, which can be found in Odlyzko's survey [Odl95, Lemma 8.1].

Lemma 2.3.4. *Suppose that $f(z)$ is analytic in $|z| < R$, and that $[z^n]f(z) \geq 0$ for all $n \geq 0$. Then for any x, $0 < x < R$, and any $n \geq 0$, $[z^n]f(z) \leq x^{-n} f(x)$.*

Proof. Note that for $0 < x < R$, the term $f_n x^n$ is less than $f(x)$ itself, by nonnegativity of the coefficients. Rearrange to get $f_n \leq f(x)/x^n$, as desired. $\square$

Hayman's method. Hayman's method, which is essentially a saddle-point method, is one of the first methods for "routinizing" the extraction of coefficients from combinatorial generating functions. The previous subsection shows how to get *bounds* for coefficients from considering saddle points. Hayman's method gives a means of extracting leading-term asymptotics.

We call a function $f(z) = \sum_{n\geq 0} f_n z^n$ *Hayman-admissible* (or just *admissible*) in the disc $z < R$ if it satisfies certain complex-analytic conditions. Instead of reproducing those conditions here, we give some sufficient conditions for admissibility, from [Wil94, p. 184]:

- If f is admissible, then so is $\exp f$.
- If f and g are both admissible in the disc $|z| < R$, so is their product fg.
- Let f be admissible in $|z| < R$. Let P be a polynomial with real coefficients and positive leading coefficient; if $R \leq \infty$, further assume $P(R) > 0$. Then the product fP is admissible in $|z| < R$.
- Let P be a polynomial with real coefficients, and let f be admissible in $|z| < R$. Then $f + P$ is admissible, and $P(f(z))$ is admissible if P has positive leading coefficient.
- If P is a nonconstant polynomial with real coefficients, $f(z) = \exp P(z)$, and $[z^n]f(z) > 0$ for all sufficiently large n, then $f(z)$ is admissible in the plane.

Now we define auxiliary functions $a(r) = rf'(r)/f(r)$, and

$$b(r) = ra'(r) = r\frac{f'(r)}{f(r)} + r^2\frac{f''r}{f(r)} - r^2\left(\frac{f'(r)}{f(r)}\right)^2.$$

Under these conditions we have the following asymptotic estimate.

Theorem 2.3.5 (Hayman)**.** *[Hay56, Wil94] Let $f(z) = \sum f_n z^n$ be an admissible function. Let r_n be the positive real root of the equation $a(r_n) = n$, for each positive* *integer n. Then*

$$f_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}$$

as $n \to \infty$.

Hayman initially provided this estimate in order to derive Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$. If we take $f(z) = \exp(z)$ then we have $f_n = 1/n!$, and $\exp(z)$ is admissible. We will use this estimate in Section 5.10 to derive estimates for the number of permutations with all cycle lengths in some finite set S; these have generating function which are $\exp P(z)$ for some polynomial P, and are therefore admissible in the case where the members of S do not have a nontrivial common multiple.

Flajolet-Odlyzko transfer theorems. Given two real numbers ϕ, R with $R > 1$ and $0 < \phi < \pi/2$, the open domain $\Delta(\phi, R)$ is defined as

$$\Delta(\phi, R) = \{z : |z| < R, z \neq 1, |\arg(z-1)| > \phi\}.$$

A domain is a Δ-domain at 1 if it is $\Delta(\phi, R)$ for some choice of ϕ and R. For $\zeta \in \mathbb{C} \setminus \{0\}$, a Δ-domain at ζ is the image of a Δ-domain at 1 under multiplication by ζ. A function is Δ-analytic if it is analytic in some Δ-domain.

Theorem 2.3.6 (Flajolet-Odlyzko)**.** *[FS09, Thm. VI.3] Let α, β be arbitrary real numbers. Let $f(z)$ be a function which is Δ-analytic. If $f(z)$ satisfies in the intersection of a neighborhood of 1 with its Δ-domain the condition*

$$f(z) = O\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right)$$

then $[z^n]f(z) = O(n^{\alpha-1}(\log n)^{\beta})$. The same result holds if O is replaced by o.

This is proved by applying Cauchy's integral formula on a well-chosen contour. The following corollary is immediate:

Corollary 2.3.7. *Let f be a Δ-analytic function, and let $\alpha \notin \{0, -1, -2, \ldots\}$. Suppose $f(z) \sim (1-z)^{-\alpha}$ as $z \to 1$ with $z \in \Delta$. Then the coefficients of f satisfy $[z^n]f(z) \sim n^{\alpha-1}/\Gamma(\alpha)$.*

These results are referred to as "transfer theorems", as they allow us to transfer knowledge about the asymptotics of a function near its singularity to the asymptotics of its coefficients. A simple example is the asymptotics of the number of 2-regular graphs, following [FS09, p. 395]. We note that 2-regular graphs have the combinatorial specification $\mathcal{R} = \textsc{Set}(\mathrm{U}\textsc{Cyc}_{\geq 3}(\mathcal{Z}))$, where $\mathrm{U}\textsc{Cyc}$ is an *undirected* cycle construction. Thus there are $2k$ ordered k-sequences corresponding to a single k-cycle, so in (2.2) we see that $\mathcal{A} = \mathrm{U}\textsc{Cyc}(\mathcal{B})$ translates to $A(z) = \sum_{k\geq 1} \frac{1}{2k} B(z)^k = 1/2 \cdot \log(1-B(z))^{-1}$. This gives the generating function

$$R(z) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}}$$

for 2-regular graphs. This function is Δ-analytic – in fact it is analytic in the complex plane with the set $\{z \in \mathbb{R} : z \geq 1\}$ removed. Furthermore $R(z) \sim e^{-3/4}/\sqrt{1-z}$ as $z \to 1$. We can immediately read off from Corollary 2.3.7 that $[z^n]R(z) \sim e^{-3/4}n^{-1/2}\Gamma(1/2)^{-1} = e^{-3/4}/\sqrt{\pi n}$.

To obtain more refined asymptotics of $[z^n]f(z)$ for functions f which are analytic at $z=0$, it often suffices to obtain asymptotic expansions for $f(z)$ in terms of well-understood functions and apply the transfer theorems.

We begin by defining asymptotic expansions. A sequence of functions $\omega_0, \omega_1, \ldots$ is said to constitute an asymptotic scale if all functions ω_j exist in a common neighborhood of some point s_0, and if they satisfy there $\omega_{j+1}(s) = o(\omega_j(s))$, that is, $\lim_{s \to s_0} \omega_{j+1}(s)/\omega_j(s) = 0$. (We may have $s_0 = \infty$.) Given such a scale, a function f is said to admit an asymptotic expansion in $\omega_0, \omega_1, \ldots$ if there exist complex coefficients $\lambda_0, \lambda_1, \ldots$ such that for each integer m,

$$f(s) = \sum_{j=0}^{m} \lambda_j \omega_j(s) + O(\omega_{m+1}(s)) \tag{2.3}$$

as $s \to s_0$. We can write $f(s) \sim \sum_{j=0}^{\infty} \lambda_j \omega_j$ in this case; sometimes we will explicitly indicate the error term, analogously to (2.3), especially if we wish to emphasize that certain of the λ_j are zero.

One particularly useful asymptotic scale is the functions of the form $(1 - z)^{-\alpha}(\log(1/(1-z)))^{\beta}$, which we will call the *standard scale*. The following theorems are from [FS09, Sec. VI.2], which also includes a table of asymptotic forms of various commonly occurring functions.

Theorem 2.3.8. *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in $f(z) = (1-z)^{-\alpha}$ admits for large n a complete asymptotic expansion in descending powers of n,*

$$[z^n]f(z) \sim \frac{n^{a-1}}{\Gamma(a)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k} \right)$$

where e_k is a polynomial in α of degree $2k$. In particular $e_1 = \alpha(\alpha-1)/2, e_2 = \alpha(\alpha-1)(\alpha-2)(3\alpha-1)/24, e_3 = \alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)/48$.

This can be viewed as a refinement of the binomial theorem with negative exponent,

$$[z^n](1-z)^{-\alpha} = (-1)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$$

where $\Gamma(n+\alpha)/\Gamma(n+1) \sim n^{\alpha-1}$ from Stirling's formula.

In the cases where logarithms occur, we have a series in descending powers of $\log n$:

Theorem 2.3.9. *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in the function $f(z) = (1-z)^{-\alpha}(1/z \cdot \log 1/(1-z))^{\beta}$ admits a full asymptotic expansion in descending powers of $\log n$,*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}\left[1 + \frac{C_1}{\log n} + \frac{C_2}{\log^2 n} + \cdots\right]$$

where $C_k = \binom{\beta}{k}\Gamma(\alpha) \left.\frac{d^k}{ds^k}\frac{1}{\Gamma(s)}\right|_{s=\alpha}$.

We can use these results to obtain an asymptotic expansion for the number of 2-regular graphs on n vertices. Note that $R(z) = \exp(-z/2 - z^2/4)/\sqrt{1-z}$, the exponential generating function of such graphs, is Δ-analytic. We take the Taylor series of $\exp(-z/2 - z^2/4)$ at $z = 1$ to get

$$e^{-z/2-z^2/4} = e^{-3/4} + e^{-3/4}(1-z) + \frac{e^{-3/4}}{4}(1-z)^2 - \frac{e^{-3/4}}{12}(1-z)^3 + O((1-z)^4)$$

and so

$$R(z) \sim e^{-3/4}(1-z)^{-1/2} + e^{-3/4}(1-z)^{1/2} + \frac{e^{-3/4}}{4}(1-z)^{3/2} - \frac{e^{-3/4}}{12}(1-z)^{5/2} + O((1-z)^{7/2}).$$

We now apply Theorem 2.3.8 to each term with absolute error $O(n^{-9/2})$:

$$\begin{aligned}
[z^n](1-z)^{-1/2} &= \frac{1}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} + O(n^{-4})\right)\\
[z^n](1-z)^{1/2} &= \frac{1}{\sqrt{\pi n^3}}\left(\frac{-1}{2} + \frac{3}{16n} + \frac{25}{256n^2} + O(n^{-3})\right)\\
[z^n]\frac{1}{4}(1-z)^{3/2} &= \frac{1}{\sqrt{\pi n^5}}\left(\frac{3}{16} + \frac{45}{128n} + O(n^{-2})\right)\\
[z^n]\frac{-1}{12}(1-z)^{5/2} &= \frac{1}{\sqrt{\pi n^7}}\left(\frac{-1}{9} + O(n^{-1})\right)
\end{aligned}$$

Adding these together (after multiplication by $e^{-3/4}$) gives an asymptotic series for $[z^n]R(z)$,

$$[z^n]R(z) = \frac{e^{-3/4}}{\sqrt{\pi n}}\left(1 - \frac{5}{8n} + \frac{49}{128n^2} + \frac{3161}{9216n^3} + O(n^{-4})\right).$$

This illustrates the general principles for deriving asymptotic series. First, fix the desired level of accuracy, and expand the function in question around its singularity, obtaining all terms which after transferring will contribute at this level or higher (above, $O(n^{-9/2})$). Then obtain the asymptotic expansion of the Taylor coefficients of each term, again only to the necessary level of accuracy; finally add all the series together.

Finally, in some cases there are finitely many singularities at the same distance; these are all "dominant singularities". The result is that we take the separate contributions from each singularity on the circle of convergence and add them together. Formally this is given by the following theorem.

Theorem 2.3.10 (Singularity analysis for multiple singularities)**.** *[FS09, Thm. VI.5] Let $f(z)$ be analytic on $|z| < \rho$ and have a finite number of singularities on the circle*

$|z| = \rho$, at points $\zeta_1, \ldots, \zeta_r$. Assume that there exists a Δ-domain Δ_0 such that $f(z)$ is analytic in the indented disc $D = \bigcap_{j=1}^{r}(\zeta_j \cdot \Delta_0)$, where $\zeta \cdot \Delta_0$ is the image of Δ_0 under multiplication by ζ. Assume that there exist r functions $\sigma_1, \ldots, \sigma_r$, which are each a linear combination of functions from the standard scale, and a function τ from the standard scale such that

$$f(z) = \sigma_j(z/\zeta_j) + O(\tau(z/\zeta_j))$$

as $z \to \zeta_j$ in D. Then the coefficients of $f(z)$ satisfy the asymptotic estimate

$$f_n = \sum_{j=1}^{r} \zeta_j^{-n}\sigma_{j,n} + O(\rho^{-n}\tau_n^*)$$

where each $\sigma_{j,n}$ has its coefficients determined by Theorems 2.3.8, 2.3.9, and $\tau_n^* = n^{a-1}(\log n)^b$ if $\tau(z) = (1-z)^{-a}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^b$.

Meinardus' theorem. Meinardus' theorem can be used to extract the asymptotics of infinite product generating functions of the form $\prod_{n\geq 1}(1-z^n)^{-a_k}$, where the sequence of a_k has a reasonably nice structure. This theorem is originally due to [Mei54a, Mei54b]. An English-language exposition can be found in [And98, Ch. 6]

The coefficient $[z^n]f(z)$ is the number of partitions of n into parts in which there are a_k parts of type k for each k. (In most cases that we will consider, the a_k are positive integers and it is practical to think of these types as "colors".)

To this product we associate the Dirichlet series $\alpha(s) = \sum_{n\geq 1} a_n/n^s$. Assume that $\alpha(s)$ can be analytically continued to a meromorphic function on the half-plane $Re(s) \geq -C_0$ for some $C_0 > 0$, and that in this half-plane α is analytic except for a simple pole at $\rho > 0$ with residue A. Furthermore, we must have the following "concentration conditions":

- $\alpha(s) = O(|t|^{C_1})$ uniformly in $\sigma \geq -C_0$ as $|t| \to \infty$, where $s = \sigma + it$ with σ, t real and C_1 is a fixed positive real number.

- Let $g(\tau) = \sum_{n=1}^{\infty} a_n e^{-\tau n}$. Then if $\tau = y + 2\pi i x$, and for $|\arg \tau| > \pi/4$ and $|x| \leq 1/2$, we have $Re(g(\tau)) - g(y) \leq -C_2 y^{-\epsilon}$ for small enough y, where $\epsilon > 0$ is arbitrary and C_2 depends on ϵ.

Given these conditions, we have Meinardus' theorem:

Theorem 2.3.11 (Meinardus)**.** *As $n \to \infty$,*

$$r(n) = Cn^{\kappa} \exp(Kn^{\rho/(\rho+1)})(1 + O(n^{-\kappa_1}))$$

where the constants in the asymptotic form are

$$\begin{aligned} K &= (1 + \rho^{-1})(A\Gamma(\rho + 1)\zeta(\rho + 1))^{1/(\rho+1)} \\ \kappa &= \frac{\alpha(0) - 1 - \rho/2}{1 + \rho} \\ C &= e^{\alpha'(0)}(2\pi(1 + \rho))^{-1/2}[A\Gamma(\rho + 1)\zeta(\rho + 1)]^{(1-2\alpha(0))/(2\rho+2)} \end{aligned}$$

and the exponent in the relative error is

$$\kappa_1 = \frac{\alpha}{\alpha + 1} \min\left(\frac{C_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta\right)$$

for an arbitrary real number δ.

In particular, if S is a periodic subset of the positive integers – that is, if S is can be written as the set of integers congruent to one of $\{r_1, \ldots, r_a\}$ modulo k – then these concentration conditions holds [Bre86].

Wright's expansions. In Chapter 5 there are many functions the coefficients of which are determined from the fact that they resemble $\exp(\sigma/(1-z))$ for some constant $\sigma > 0$. The leading-term asymptotics of their coefficients were explicitly given by E. M. Wright.

Theorem 2.3.12 (Wright)**.** *[Wri32, Thm. 2 and Thm. 3]*

(a) The leading-term asymptotics for

$$c_n = [z^n](1-z)^\beta \Phi(z) \exp\left(\frac{\sigma}{1-z}\right)$$

where β is a complex number, $\Phi(z)$ is regular in the unit disk, and σ is a real number, are given by

$$c_n = \frac{1}{n^{\beta/2+3/4}} \left[\exp(2\sqrt{\sigma n}) \frac{1}{2\sqrt{\pi}} \Phi(1) e^{\sigma/2} \sigma^{\beta/2+1/4}\right] (1 + O(n^{-1/2})).$$

(b) The leading-term asymptotics for

$$[z^n] \left(\log \frac{1}{1-z}\right)^k (1-z)^\beta \Phi(z) \exp\left(\frac{1}{1-z}\right)$$

with k a positive integer can be derived from that for the $k = 0$ case by differentiating k times with respect to β and switching signs if k is odd.

In particular, in the $k = 1$ case we have

$$c_n = \frac{\log n}{2n^{\beta/2+3/4}} \left[\exp(2\sqrt{n}) \frac{1}{2\sqrt{\pi}} \Phi(1) e^{1/2}\right] (1 + O(n^{-1/2})).$$

2.4 Other miscellaneous results

The Euler-Maclaurin formula. One central theme of this thesis is the approximation of the discrete by the continuous. We seek limit laws for large combinatorial structures, and often statistics of these large combinatorial structures are given by sums. It is natural to approximate these sums by integrals. The Euler-Maclaurin formula makes such approximation rigorous, and in addition allows us to derive a full asymptotic series for such sums in which the leading term is the corresponding integral.

Define the *Bernoulli numbers* B_k by giving their exponential generating function $t/(e^t - 1) = \sum_{m=0}^{\infty} B_m t^m/m!$. In particular we have $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30$, and $B_{2j+1} = 0$ for $j \geq 1$. Then we have

Theorem 2.4.1 (Euler-Maclaurin)**.** *[GKP94] Let f be a smooth function defined on the reals. Then we have the asymptotic series*

$$\sum_{n=a}^{b} f(n) \sim \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)$$

where a and b are integers.

In a typical case we will hold a constant and let $b \to \infty$. In this case it is enough to have f defined on the interval $[a, \infty)$.

Central limit theorems. We will often be dealing with distributions which come from adding up a large number of small, independent or "almost independent" contributions; we will want to prove that limit distributions arising in these cases are normal. For this purpose we will need the following central limit theorems.

We begin with the Lyapunov central limit theorem. This theorem shows that the partial sums of certain sequences of independent random variables with finite mean and variance, once standardized, converge to the standard normal. The Lyapunov condition (2.4) amounts to showing that no single summand dominates the sum.

Theorem 2.4.2 (Lyapunov)**.** *[Dur04] Let $Y_1, Y_2, \ldots$ be independent random variables with finite mean and variance, $\mathbb{E}(Y_n) = \mu_n$ and $\mathbb{V}(Y_n) = \sigma_n^2$. Let $s_n^2 = \sum_{k=1}^n \sigma_k^2$. If for some $\delta > 0$, $\mathbb{E}(|Y_k|^{2+\delta})$ is finite for $k = 1, 2, \ldots$ and the Lyapunov condition*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} \mathbb{E}(|Y_k - \mathbb{E}Y_k|^{2+\delta}) = 0 \tag{2.4}$$

is satisfied, then the standardization $(\sum_{k=1}^n (Y_n - \mu_n))/s_n$ converges in distribution to a standard normal random variable as $n \to \infty$.

Theorem 2.4.3 (Lindeberg-Feller)**.** *For each n, let $X_{n,m}$, $1 \le m \le n$, be independent random variables with expectation 0. Suppose that:*

(i) $\lim_{n\to\infty} \sum_{m=1}^n \mathbb{E}X_{n,m}^2 = \sigma^2$ for some positive constant σ.

(ii) For all $\epsilon > 0$, the truncated expectation $\lim_{n\to\infty} \sum_{m=1}^n \mathbb{E}(X_{n,m}^2 [\![X_{n,m} > \epsilon]\!]) = 0$.

Then let $S_n = X_{n,1} + \cdots + X_{n,n}$. Then the S_n converge in distribution to a standard normal with mean 0 and standard deviation σ as $n \to \infty$.

This says that the sum of a large number of small independent effects has approximately normal distribution. The condition in (ii), like the Lyapunov condition, amounts to showing that no single summand dominates the sum. In fact the Lindeberg condition follows from the Lyapunov condition [Bil95, p. 362].

Theorem 2.4.4 (Renewal CLT)**.** *Let $Y_1, Y_2, \ldots$ be iid positive random variables, with $\mathbb{E}Y_i = \mu$ and $\mathbb{V}Y_i = \sigma^2$ positive real numbers. Let $S_n = Y_1 + \cdots + Y_n$ and let $N_t = \sup\{m : S_m \leq t\}$. Then as $n \to \infty$,*

$$\frac{N_t - t/\mu}{\sqrt{\sigma^2 t/\mu^3}} \xrightarrow{d} N(0,1).$$

That is, the time until the sum of the Y_i reaches t is asymptotically normally distributed, with mean t/μ and variance $\sigma^2 t/\mu^3$.

Method of moments. Finally, many distributions to be considered in the text will be found by the method of moments: we will argue that a random variable has certain moments and then that this suffices to specify the random variable in question.

Theorem 2.4.5 (Stieltjes moment problem)**.** *Let $\nu_0, \nu_1, \ldots$ be a sequence of positive real numbers. If $\limsup_{k\to\infty} \nu_k^{1/2k}/2k < \infty$, then there is at most one distribution on $[0,\infty)$ with kth moment equal to ν_k.*

Proposition 2.4.6. *The moments of a distribution with finite support uniquely determine the distribution.*

See [FS09, p. 778] for a proof.

Proposition 2.4.7. *If $F_n(x)$ for $n = 0, 1, 2, \ldots$ are the distribution functions of random variables and*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (x)_k dF_n(x) = \int_{-\infty}^{\infty} (x)_k dF(x)$$

and F is characterized by its moments, then the F_n converge in distribution to F.

See [Bil95, Thm. 30.2] for a proof.

Chapter 3

Boltzmann samplers

3.1 Definition of Boltzmann samplers

Boltzmann samplers are a family of algorithms, first given in [DFLS04], used to generate random combinatorial structures. The classical paradigm for the generation of random combinatorial structures, as exemplified by [NW78], has concentrated on generating objects of *fixed* size. In Boltzmann sampling, on the other hand, a measure is specified on all the members of a combinatorial class, of *any* size; by making a small sacrifice in precision one is able to create much faster and easier-to-implement algorithms. Furthermore, we will see in the remainder of this thesis that Boltzmann samplers enable one to guess quite easily various probabilistic results on random combinatorial structures.

Definition 3.1.1. *Let $\mathcal{C}$ be a combinatorial class with generating function $C(x) =$*

$\sum_{n\geq 0} \frac{C_n}{\omega_n} x^n$, *where either* $\omega_n \equiv 1$ *for all* n *(the ordinary case) or* $\omega_n = n!$ *for all* n *(the exponential case). Then the Boltzmann distribution on* $\mathcal{C}$ *with positive real parameter* x*, where the sum giving* $C(x)$ *converges, assigns to each object* $\gamma \in \mathcal{C}$ *the probability*

$$\mathbb{P}_x(\gamma) = \frac{x^{|\gamma|}}{\omega_{|\gamma|}} \frac{1}{C(x)}.$$

A Boltzmann sampler $\Gamma C(x)$ *for a class* $\mathcal{C}$ *is a procedure which generates objects from* $\mathcal{C}$ *according to the Boltzmann distribution.*

We must show, of course, that Boltzmann samplers can be constructed. In Section 2.2 we saw an introduction to the symbolic method in combinatorics. The symbolic method allows us to recursively specify combinatorial classes, building up each class from atoms via a few basic constructions. We now show how these constructions can be transformed into Boltzmann samplers.

Unlabelled objects. Many unlabelled combinatorial classes are built up from simpler classes using the operations of disjoint union, Cartesian product, and sequence. So, given combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ with Boltzmann samplers $\Gamma A, \Gamma B$, we must construct Boltzmann samplers $\Gamma(A + B), \Gamma(A \times B), \Gamma(\textsc{Seq}(A))$.

Disjoint union. Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes, with $\mathcal{C} = \mathcal{A} + \mathcal{B}$; objects in $\mathcal{C}$ inherit their sizes from $\mathcal{A}$ and $\mathcal{B}$. By disjointness, $C_n = A_n + B_n$, and so the generating functions satisfy $C(z) = A(z) + B(z)$.

Now consider a random element of $\mathcal{C}$ from the Boltzmann distribution with parameter x. The probability that such an element comes from $\mathcal{A}$ is

$$\sum_{\alpha \in \mathcal{A}} \frac{x^{|\alpha|}}{C(x)} = \sum_{n \geq 0} \frac{A_n x^n}{C(x)} = \frac{A(x)}{C(x)}$$

and, conditioned on coming from $\mathcal{A}$, the distribution of objects is exactly the Boltzmann-x distribution on $\mathcal{A}$. Therefore a Boltzmann sampler on $\mathcal{C}$ with parameter x is as follows:

- Generate a Bernoulli random variable with mean $A(x)/C(x)$.
- If this Bernoulli has value 1, return the output of $\Gamma A(x)$, otherwise return the output of $\Gamma B(x)$.

Cartesian product. Again let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes, with $\mathcal{C} = \mathcal{A} \times \mathcal{B}$; if $\gamma = (\alpha, \beta)$ for $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, then $|\gamma| = |\alpha| + |\beta|$. Then $C(z) = A(z)B(z)$. The probability of $\gamma \in \mathcal{C}$ in the Boltzmann model is then

$$\mathbb{P}_x(\gamma) = \frac{x^{|\gamma|}}{C(x)} = \frac{x^{|\alpha|}}{A(x)} \frac{x^{|\beta|}}{B(x)} = \mathbb{P}_x(\alpha)\mathbb{P}_x(\beta).$$

Therefore a Boltzmann sampler $\Gamma C(x)$ on $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ can be constructed by calling $\Gamma A(x)$ and $\Gamma B(x)$ independently.

Sequence. Let $\mathcal{A}$ be a combinatorial class, and $\mathcal{C} = \textsc{Seq}(\mathcal{A})$. Then

$$\textsc{Seq}(\mathcal{A}) = 1 + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + \cdots = \sum_{n \geq 0} \mathcal{A}^n.$$

The generating functions satisfy

$$C(z) = 1 + A(z) + A(z)^2 + \cdots = \sum_{n \geq 0} A(z)^n = \frac{1}{1 - A(z)}.$$

Although this is an infinite sum we can treat it analogously to finite sums. A Boltzmann sampler $\Gamma C(x)$ is obtained by calling $\Gamma A^n(x)$ with probability $A(x)^n/C(x) = A(x)^n(1-A(x))$, for each n. That is, call $\Gamma A^N(x)$ where N is a geometric random variable with rate $A(x)$. The probability of obtaining the sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is therefore

$$A(x)^n(1-A(x))\frac{x^{|\alpha_1|}}{A(x)}\cdots\frac{x^{|\alpha_n|}}{A(x)} = (1-A(x))x^{|\alpha_1|+\cdots+|\alpha_n|} = \frac{x^{|\alpha|}}{C(x)}$$

in accordance with the definition.

Alternatively, we can proceed from first principles and note that $\mathcal{C} = 1 + \mathcal{A} \times \mathcal{C}$, where 1 represents the empty sequence. Therefore an alternative implementation of $\Gamma C(x)$ is as follows: return the empty sequence with probability $1/C(x) = 1 - A(x)$, and with probability $A(x)$ return the pair $(\Gamma A(x), \Gamma C(x))$. The formulation in terms of geometric random variables essentially "unrolls" this construction. In particular, we note that in the Boltzmann distribution on the class $\mathcal{C} = \textsc{Seq}(\mathcal{A})$, the *number of parts* of an object has a geometric distribution.

This does *not*, however, mean that the corresponding fixed-size objects have a geometric distribution for their number of parts; we will see this for compositions. Rather, it follows from the fact that Boltzmann samplers for sequences tend to have size which is roughly geometrically distributed, and the number of parts is near some constant multiple of the size; compositions are an example.

Labelled structures. For labelled structures we can construct Boltzmann samplers in the same way as for unlabelled structures; in this case we must use exponential generating functions instead of ordinary generating functions. The samplers for disjoint union and sequence carry over unchanged. We replace the Cartesian product $A \times B$ with the labelled product $A \star B$ previously defined; then a Boltzmann sampler $\Gamma C(x)$ for $C = A \star B$ can still be constructed by calling $\Gamma A(x)$ and $\Gamma B(x)$ independently, and completing with a randomly chosen relabeling. *Sequences* in the labelled world are constructed from sums and products, and thus proceed in the same manner as in the unlabelled world.

Set. Let $\mathcal{C} = \textsc{Set}(\mathcal{A})$, and assume that a Boltzmann sampler $\Gamma A(x)$ exists. The sampler $\Gamma C(x)$ will work by calling $\Gamma A(x)$ repeatedly. Recall that $C(x) = \exp A(x)$. Now, the probability that an element of $\mathcal{C}$ chosen from the Boltzmann-x distribution consists of k components is

$$\frac{1}{C(x)} \frac{A(x)^k}{k!} = e^{-A(x)} \frac{A(x)^k}{k!}$$

since k-component sets drawn from A have the generating function $A(x)^k/k!$. This is the probability that a Poisson random variable with mean $A(x)$ takes the value k. So for $\mathcal{C} = \textsc{Set}(\mathcal{A})$, the sampler $\Gamma C(x)$ works by sampling a Poisson random variable of mean $A(x)$, which we call k, and then calling $\Gamma A(x)$ k times.

Cycle. Let $\mathcal{C} = \textsc{Cyc}(\mathcal{A})$; as with sets, a Boltzmann sampler for cycles will work by calling the Boltzmann sampler for the components repeatedly. Recall that $C(x) = \log(1/(1 - A(x))$. The probability that an element of $\mathcal{C}$ chosen from the

Boltzmann-x distribution consists of k components is

$$\frac{1}{C(x)} \frac{A(x)^k}{k} = \frac{1}{\log\left(\frac{1}{1-A(x)}\right)} \frac{A(x)^k}{k}.$$

This is the probability that a "logarithmic" random variable with rate $A(x)$ takes the value k. The logarithmic random variable with rate λ has law

$$\mathbb{P}(X = k) = \frac{1}{\log \frac{1}{1-\lambda}} \frac{\lambda^k}{k}$$

For example, the class of permutations has the specification $\textsc{Set}(\textsc{Cyc}(\mathcal{Z}))$ – that is, permutations are sets of cycles of atoms. Since the outer construction here is $\textsc{Set}$, Boltzmann-sampled permutations with parameter x have a number of cycles which is Poisson-distributed. The mean of this Poisson is given by evaluating the generating function of cycles, $\log 1/(1-x)$. This gives a first example of the use of Boltzmann samplers for approximate statistics of combinatorial classes; as we will see the mean size of a Boltzmann-x permutation is $1/(1-x)$, so we can quickly predict that a random permutation of $[n]$ has about $\log n$ cycles.

We will rarely explicitly use the cycle construction. Rather, we prefer to write $\textsc{Cyc} = \textsc{Cyc}_1 + \textsc{Cyc}_2 + \cdots$, which gives for example

$$\begin{aligned} \textsc{Set}(\textsc{Cyc}(\mathcal{Z})) &= \textsc{Set}(\textsc{Cyc}_1(\mathcal{Z}) + \textsc{Cyc}_2(\mathcal{Z}) + \cdots) \\ &= \textsc{Set}(\textsc{Cyc}_1(\mathcal{Z})) \times \textsc{Set}(\textsc{Cyc}_2(\mathcal{Z})) \times \cdots \end{aligned}$$

Thus we can create a set of cycles by creating a set of cycles of each possible length and juxtaposing all the cycles thus obtained. So we need a Boltzmann sampler for

$\textsc{Cyc}_k(\mathcal{A})$ given one for $\mathcal{A}$. It suffices to use the Boltzmann sampler for $\textsc{Seq}_k(\mathcal{A}) = \mathcal{A}^k$ – that is, to generate k objects from $\mathcal{A}$ in sequence – and consider two sequences to be the same cycle if they are equivalent up to cyclic permutation. This provides the Boltzmann sampler for permutations in terms of individual cycle lengths. The exponential generating function of $\textsc{Cyc}_k(\mathcal{Z})$ is x^k/k, so to generate a permutation, generate $\mathcal{P}(x^k/k)$ cycles of length k, independently for each k.

3.2 Some philosophy

Boltzmann models for the analysis of combinatorial objects derive much of their power from the simplicity of the Boltzmann sampler for Cartesian or labelled products. Therefore in cases where it is possible to write a combinatorial class as a product of other combinatorial classes, we can treat the "factor" classes as independent.

One example of this can be seen in the cycle structure of permutations. Consider permutations of $[n]$ chosen uniformly at random. Let X_k, a random variable, be the number of cycles of length k in such permutations. Then it is well-known that as $n \to \infty$ with k fixed, X_k converges in distribution to the Poisson with mean $1/k$. Furthermore the pair (X_k, X_l) converges in distribution to a pair of independent Poissons with means $1/k, 1/l$, and similarly for larger tuples.

The (joint) Poisson distribution follows from the following lemma [Wat74]:

Proposition 3.2.1. *For nonnegative integers $k_1, \ldots, k_r$,*

$$\mathbb{E}_n\left(\prod_{j=1}^{r}(X_j)_{k_j}\right) = \left(\prod_{j=1}^{r}\left(\frac{1}{j}\right)^{k_j}\right)[\![\sum_{j=1}^{r} jk_j \leq n]\!]$$

Proof. Consider the generating function counting permutations by their total size and number of cycles of each length $1, 2, \ldots, r$, marked by z and $u_1, u_2, \ldots, u_r$. This generating function is

$$P(z, u_1, \ldots, u_r) = \frac{\exp\left(\sum_{j=1}^{r}\frac{(u_j-1)z^j}{j}\right)}{1-z}$$

as can be seen from the class specification $\mathcal{P} = \textsc{Set}(\textsc{Cyc}_{>r}(\mathcal{Z}) + \mu_1\textsc{Cyc}_1(\mathcal{Z}) + \cdots + \mu_r\textsc{Cyc}_r(\mathcal{Z}))$. The desired moment can be obtained by differentiation:

$$\mathbb{E}_n\left(\prod_{j=1}^{r}(X_j)_{k_j}\right) = \frac{[z^n]\ \frac{\partial^{k_1}}{\partial u_1^{k_1}}\cdots\frac{\partial^{k_r}}{\partial u_r^{k_r}}P(z, u_1, \ldots, u_r)\Big|_{u_1=\cdots=u_r=1}}{[z^n]P(z, 1, \ldots, 1)}.$$

We have $P(z, 1, \ldots, 1) = 1/(1-z)$, so the denominator is 1. The numerator is

$$[z^n]\frac{\prod_{j=1}^{r}\left(\frac{z^j}{j}\right)^{k_j}}{1-z} = \prod_{j=1}^{r}\frac{1}{j^{k_j}}[z^n]\frac{z^{\sum_j jk_j}}{1-z}.$$

This coefficient is 1 if $n \geq \sum_j jk_j$ and 0 otherwise, giving the desired result. □

To keep the notation reasonably clean, we find joint moments of the tuple $(X_1, \ldots, X_r)$; but we can of course fix any of the k_j to be zero, so this proposition actually includes all joint moments of any of the X_i. For n large enough, these moments are exactly the joint factorial moments of the Poisson distribution $(\mathcal{P}(1), \mathcal{P}(1/2), \ldots, \mathcal{P}(1/r))$. In fact, these joint factorial moments are exactly those of the joint Poisson exactly when n is "large enough" to fit k_j cycles of length j for each j, therefore permitting the factorial moment to be larger than zero. From the method of moments, the cycle counts of an n-permutation converge in distribution to independent Poissons.

The Boltzmann sampler for permutations, on the other hand, assigns $\mathcal{P}(x^k/k)$ cycles to each length k, *independently*. The asymptotic independence seen in the fixed-case model is replaced by exact independence. Furthermore in the "critical" case where $k = 1$, which corresponds to permutations of large sets, the distributions of the cycle counts in the Boltzmann sampler are exactly the limiting distribution from the fixed-size model.

The Boltzmann model originates in statistical mechanics. In statistical mechanics, certain systems are said to satisfy *Maxwell-Boltzmann statistics*. This occurs in the classical (non-quantum) situation in which temperature is high enough and density low enough that quantum effects are negligible. In such systems, configurations with energy equal to E have a probability of occurrence proportional to $e^{-E/k_B T}$, where E is energy, k_B is the Boltzmann constant, and T is inverse temperature. If we set $k_B \equiv 1$ (equivalent to a change of units) and $T = 1/\beta$, then the probability of occurrence of states with energy E is proportional to $e^{-\beta E}$. Setting $x = e^{-\beta}$ and identifying the size of a combinatorial configuration with the energy of a thermodynamical system, we recover the Boltzmann model. Note that the "atoms" in our combinatorial objects, whether the structures are labelled or unlabelled, still satisfy the Maxwell-Boltzmann statistics. There is not an immediately obvious analogue to the Bose-Einstein or Fermi-Dirac distributions occurring in quantum statistical mechanics, in which particles become quantum-mechanically indistinguishable.

3.3 Formulas for the mean and variance of object size

The Boltzmann framework gives sampling algorithms that run quickly, are easy to program, and are easy to analyze – but with the tradeoff of not generating objects all of the same size. Thus it is useful to quantify exactly how much of a tradeoff this is.

Proposition 3.3.1. *[DFLS04, Thm. 2.1] The size of objects in a class $\mathcal{C}$ produced from the Boltzmann distribution with parameter x has first and second moments satisfying*

$$\mathbb{E}_x(N) = \frac{xC'(x)}{C(x)}, \mathbb{E}_x(N^2) = \frac{x^2C''(x) + xC'(x)}{C(x)}.$$

Proof. The probability generating function of the random size N is

$$\sum_{n\geq 0} \mathbb{P}_x(N=n)z^n = \frac{C(xz)}{C(x)}.$$

This gives the factorial moments

$$\mathbb{E}_x((N)_j) = \left(\frac{\partial^j}{\partial z^j}\frac{C(xz)}{C(x)}\right)_{z=1} = \frac{x^jC^{(j)}(x)}{C(x)}.$$

In particular we have for $j = 1, 2$

$$\mathbb{E}_x(N) = \frac{xC'(x)}{C(x)}, \mathbb{E}_x(N(N-1)) = \frac{x^2C''(x)}{C(x)}$$

and adding these gives $\mathbb{E}_x(N^2)$ by linearity of expectation. $\square$

The variance of the size, then, is given by

$$\mathbb{V}_x(N) = x\frac{d}{dx}\mathbb{E}_x(N) = \frac{\left(x\frac{d}{dx}\right)^2 C(x)}{C(x)} - \left(\frac{x\frac{d}{dx}C(x)}{C(x)}\right)^2.$$

We can see that $\mathbb{E}_x(N)$ is an increasing function of x, as long as $\mathcal{C}$ contains objects of at least two different sizes. Since $\mathbb{V}_x(N) = x\frac{d}{dx}\mathbb{E}_x(N)$ and $\mathbb{V}_x(N) > 0$, we have that $\frac{d}{dx}\mathbb{E}_x(N) > 0$ as well; thus $\mathbb{E}_x(N)$ is increasing.

On the other hand, $\mathbb{V}_x(N)$ is not necessarily an increasing function of x, for $0 < x < x_c$. The simplest case is $C(x) = 1 + x$. In this case N is Bernoulli with mean $x/(1+x)$, and therefore $\mathbb{V}_x(N)$ is maximized when $x/(1+x) = 1/2$, i. e. when $x = 1$. The variance $\mathbb{V}_x(N)$ is an increasing function of N in cases where there are "enough" objects of large size, however. One example is the case $C(x) = \exp A(x)$, where $A(x)$ has all Taylor coefficients nonnegative; this corresponds to $\mathcal{C} = \textsc{Set}(\mathcal{A})$. In this case we have

$$\mathbb{V}_x(N) = xA'(x) + x^2 A''(x), \frac{d}{dx}\mathbb{V}_x(N) = A'(x) + 3xA''x + x^2 A'''(x).$$

Since all the Taylor coefficients of $A(x)$ are nonnegative, the same is true for $d/dx\mathbb{V}_x(N)$.

3.4 The rule of thumb

Let X_n, for $n = 1, 2, 3, \ldots$, be a family of random variables. Let $\mu_n = \mathbb{E}X_n$ and $\sigma_n = \sqrt{\mathbb{V}X_n}$ be the mean and variance of X_n. We recall that if $\sigma_n = o(\mu_n)$ as

$n \to \infty$, then the distribution of X_n is concentrated around its mean; that is,

$$\lim_{n\to\infty} \mathbb{P}\left(1-\epsilon \leq \frac{X_n}{\mu_n} \leq 1+\epsilon\right) = 1$$

as $n \to \infty$. (See [FS09, Prop. III.3].)

Nothing in this definition requires n to be an integer; thus we can define concentration of a family of random variables indexed by the positive real numbers in this way.

Now, fix a combinatorial class $\mathcal{A}$, and let $\mu(x), \sigma(x)$ denote the mean and standard deviation of the size of Boltzmann-x $\mathcal{A}$-objects. These are both increasing functions of x, and so $\sigma(\mu^{-1}(n))$ is also an increasing function. This gives the standard deviation of the size of Boltzmannized $\mathcal{A}$-objects, where the Boltzmann parameter has been chosen to make the mean object size n.

In the previous section we derived formulas for the mean and variance of the size of Boltzmann-sampled objects. We can apply these results to Boltzmann-sampled objects and distinguish between combinatorial classes for which the size of the Boltzmannized objects is concentrated and those for which it is not. It appears that for classes for which the size of the Boltzmannized objects are concentrated, results on Boltzmannized objects translate well into results on fixed-size combinatorial objects; the translation does not work as well for classes for which the size of Boltzmannized objects is not concentrated.

Involutions. Involutions have the exponential generating function $A(x) =$

$\exp(x + x^2/2)$. From this we have

$$\mu(x) = \frac{xA'(x)}{A(x)} = x + x^2, \sigma^2(x) = \frac{(x\partial_x)^2 A(x)}{A(x)} - \mu(x)^2 = x + 2x^2.$$

Thus we have $\mu^{-1}(n) = (\sqrt{1+4n} - 1)/2$, and so

$$\sigma^2(\mu^{-1}(n)) = 2n - \frac{\sqrt{1+4n} - 1}{2} \sim 2n.$$

So involutions are a concentrated class.

More generally, for permutations with all cycle lengths in some finite set S, let $A(x) = \exp(P(x))$, where $P(x) = \sum_{s \in S} x^s$ is the generating polynomial of S. Then we have

$$\mu(x) = \frac{xA'(x)}{A(x)} = \frac{xP'(x)e^{P(x)}}{e^{P(x)}} = xP'(x) = \sum_{s \in S} s x^s$$

and

$$\begin{aligned} \sigma^2(x) &= \frac{(x\partial_x)^2 A(x)}{A(x)} - \mu(x)^2 = \frac{(xP' + x^2 P'' + x^2 (P')^2) e^P}{e^P} - (xP')^2 \\ &= xP'(x) + x^2 P''(x) \\ &= \sum_{s \in S} s x^s + \sum_{s \in S} s(s-1) x^s = \sum_{s \in S} s^2 x^s. \end{aligned}$$

In particular, $\mu^{-1}(n) \sim (n/m)^{1/m}$ as $n \to \infty$, where $m = \max S$. We have $\sigma^2(n) \sim m^2 x^m$ as $x \to \infty$, so $\sigma^2(\mu^{-1}(n)) \sim m^2(n/m) = mn$. Thus permutations with their cycle lengths restricted to any finite set are a concentrated class.

Partitions. Consider the Boltzmann sampler for partitions into distinct parts. This includes a part of size k with probability $x^k/(1+x^k)$.

Proposition 3.4.1. *The mean number of parts of a partition drawn from the Boltzmann sampler of parameter x is asymptotic to* $(1-x)^{-1} \log 2$, *as* $x \to 1^-$.

Proof. The mean number of parts is given by

$$\sum_{k\geq 1} \frac{x^k}{1+x^k}.$$

We can approximate this sum by the integral

$$\int_0^\infty \frac{x^k}{1+x^k}\,dk$$

Now, we can do a change of variable in order to find this integral: let $u = x^k$, so $k = (\log u)/(\log x)$ and $dk = du/(u \log x)$. This gives

$$\int_1^0 \frac{u}{1+u}\,\frac{du}{u\log x} = \frac{1}{\log x}\int_1^0 \frac{1}{1+u}\,du = \frac{-\log 2}{\log x}.$$

Since $\log x \sim x - 1$ as $x \to 1^-$, we get

$$\int_0^\infty \frac{x^k}{1+x^k}\,dk = \frac{-\log 2}{\log x}.$$

We next need to check how well the sum is approximated by the integral. Let $f(k) = x^k/(1+x^k)$. Then we have the Euler-Maclaurin expansion

$$\sum_{k\geq 0} \frac{x^k}{1+x^k} = \int_0^\infty \frac{x^k}{1+x^k}\,dk + \frac{f(0)+f(\infty)}{2} + \sum_{j=1}^\infty \frac{B_{2j}}{(2j)!}(f^{(2j-1)}(\infty) - f^{(2j-1)}(0))$$

where $f^{(m)}(\infty) := \lim_{z\to\infty} f^{(m)}(z)$. We then have

$$f^{(j)}(k) = \frac{-(\log^j x)E_j(-x^k)}{(1+x^k)^{j+1}} \tag{3.1}$$

where E_j is an Eulerian polynomial of degree j: $E_1(z) = z, E_2(z) = z + z^2, E_3(z) = z + 4z^2 + z^3, E_4(z) = z + 11z^2 + 11z^3 + z^4, \ldots$.

The Eulerian polynomials count permutations by their number of *descents*. We say a permutation σ written in the one-line notation has a *descent* whenever $\sigma(i) >$

$\sigma(i+1)$. Then let $A(n,k)$ be the number of permutations of n with $k-1$ descents, and $E_j(x) = \sum_{k=1}^{\infty} A(j,k)x^k$. We prove (3.1) by induction. The $k=0$ case is clear. We can compute

$$\frac{d}{dk}\left(\frac{-(\log^j x)E_j(-x^k)}{(1+x^k)^{j+1}}\right) = \frac{\log^{j+1} x}{(1+x^k)^{j+2}}\left(E_j'(-x^k)(1+x^k) + (j+1)E_j(-x^k)\right)x^k$$

and so it suffices to show

$$E_{j+1}(z) = z(E_j'(z)(1-z) + (j+1)E_j(z)).$$

But this is standard; see [Com74, p. 292].

In particular $f^{(j)}(0) = -(\log^j x)E_j(-1)2^{-(j+1)}$. This gives

$$\sum_{k\geq 0} \frac{x^k}{1+x^k} = \frac{-\log 2}{\log x} + \frac{1}{4} - \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\frac{(\log^{2j-1} x)E_{2j-1}(-1)}{4^j}.$$

The $k=0$ term of the sum is exactly $1/2$, so we can subtract $1/2$ from both sides to get

$$\sum_{k\geq 1} \frac{x^k}{1+x^k} = \frac{-\log 2}{\log x} - \frac{1}{4} - \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\frac{(\log^{2j-1} x)E_{2j-1}(-1)}{4^j}$$

and we note that $E_{2j-1}(-1)$ forms the sequence of signed *tangent numbers*: $E_1(-1) = -1, E_3(-1) = 2, E_5(-1) = -16, E_7(-1) = 272, \ldots$. This follows from the generating functions $E(u,t) = \sum_n E_j(u)t^n/n! = (1-u)/(1-ue^{t(1-u)})$ for the Eulerian numbers and $T(x) = \sum_{k=1}^{\infty} T_k x^{2k-1}/(2k-1)! = (1-e^{2t})/(1+e^{2t})$ for the signed tangent numbers, from which $E(-1,t) = 1 + T(t)$.

Thus we have

$$\begin{aligned}\sum_{k\geq 1} \frac{x^k}{1+x^k} &= \frac{-\log 2}{\log x} - \frac{1}{4} + \frac{1/6}{2!}\frac{(\log x)\cdot(-1)}{4} \\ &+ \frac{-1/30}{4!}\frac{\log^3 x\cdot(2)}{16} + \frac{1/42}{6!}\frac{\log^5 x\cdot(-16)}{64} + \cdots\end{aligned}$$

and recalling that $\log x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \cdots$, we get

$$\begin{aligned}\sum_{k\geq 1} \frac{x^k}{1+x^k} &= \frac{\log 2}{1-x} - \left(\frac{1}{2}\log 2 + \frac{1}{4}\right) + \left(\frac{1}{48} - \frac{1}{12}\log 2\right)(1-x) \\ &+ \left(\frac{1}{96} - \frac{1}{24}\log 2\right)(1-x)^2 + O((1-x)^3);\end{aligned}$$

the series can be continued to any desired accuracy.

□

Proposition 3.4.2. *The mean size of a partition into distinct parts drawn from the Boltzmann sampler of parameter x is asymptotic to $(1-x)^{-2} \cdot \pi^2/12$, as $x \to 1^-$.*

Proof. Proceeding as before, we have the integral

$$\int_0^\infty \frac{kx^k}{1+x^k}\, dk.$$

Again changing variables, this is

$$\int_1^0 \frac{\frac{\log u}{\log x} u}{1+u} \frac{du}{u \log x} = \frac{1}{\log^2 x} \int_1^0 \frac{\log u}{1+u}\, du.$$

The integral is improper – as $u \to 0^+$ the integrand blows up – and evaluates to

$$\frac{1}{\log^2 x}\left[\lim_{u\to 0}(Li_2(1+u) + \log u \log(1+u)) - Li_2(2)\right]$$

and recalling that $Li_2(2) = -\pi^2/12$ gives the leading term. The Euler-Maclaurin formula gives, with $f(k) = kx^k/(1+x^k)$,

$$\sum_{k\geq 0} \frac{kx^k}{1+x^k} \sim \int_0^\infty \frac{kx^k}{1+x^k}\, dk + \frac{f(0)+f(\infty)}{2} + \sum_{k=1}^\infty \frac{B_{2j}}{(2j)!}(f^{(2j-1)}(\infty) - f^{(2j-1)}(0)).$$

Now, $f(k) - k/2$ is an even function of k, so $f^{(3)}(0) = f^{(5)}(0) = \cdots = 0$. From this we can find

$$\sum_{k\geq 0} \frac{kx^k}{1+x^k} \sim \frac{\pi^2}{12\log^2 x} + \frac{0+0}{2} + \frac{1/6}{2!}(0-1/2) + \sum_{j=2}^{\infty} \frac{B_{2j}}{(2j)!}(0-0).$$

Thus we have that the mean size is given by $\frac{\pi^2}{12\log^2 x} - \frac{1}{24} + o((1-x)^k)$ for any positive integer k. $\square$

Proposition 3.4.3. *The variance of the size of a partition into distinct parts drawn from the Boltzmann sampler of parameter x is asymptotic to $\frac{\pi^2}{6}(1-x)^{-3}$ as $x \to 1^-$.*

Proof. Let $X_k = Be(x^k/(1+x^k))$; we are computing

$$\mathbb{V}\left(\sum_{k\geq 1} X_k\right) = \sum_{k\geq 1} \mathbb{V}(kX_k) = \sum_{k\geq 1} k^2 \frac{x^k}{(1+x^k)^2} \sim \int_0^{\infty} \frac{k^2x^k}{(1+x^k)^2}.$$

Changing variables, then, the variance of the size is

$$\frac{1}{\log^3 x} \int_1^0 \frac{\log^2 u}{(1+u)^2}\, du$$

This integral is $-\pi^2/12$. First, integrating by parts twice,

$$\int u^n \log^2 u\, du = \frac{u^{n+1}\log^2 u}{(n+1)} - \frac{2u^{n+1}\log u}{(n+1)^2} + \frac{2u^{n+1}}{(n+1)^3}$$

and so

$$\int_1^0 u^n \log^2 u\, du = \frac{-2}{(n+1)^3}.$$

Now, since $(1+u)^{-2} = 1 - 2u + 3u^2 - \cdots$, we have

$$\begin{aligned}
\int_1^0 \frac{\log^2 u}{(1+u)^2}\,du &= \sum_{n=0}^{\infty}(-1)^n(n+1)\int_1^0 u^n \log^2 u\,du \\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)\frac{-2}{(n+1)^3} \\
&= 2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2} = 2\left(\frac{-\pi^2}{12}\right) = \frac{-\pi^2}{6}.
\end{aligned}$$

Now, as before, we have

$$\sum_{k\geq 1}\frac{k^2x^k}{(1+x^k)^2} \sim \int_0^{\infty}\frac{k^2x^k}{(1+x^k)^2}\,dk + \frac{f(0)+f(\infty)}{2} + \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}(f^{(2j-1)}(\infty) - f^{(2j-1)}(0)).$$

In this case $f(0) = 0$ by inspection, and $f^{(2j-1)}(0) = 0$ for all integers j since f is an even function. Thus we find that

$$\sum_{k\geq 1}\frac{k^2x^k}{(1+x^k)^2} = \int_0^{\infty}\frac{k^2x^k}{(1+x^k)^2}\,dk + o((1-x)^k)$$

for any positive integer k. In powers of $1-x$, then, the variance of the size of a Boltzmann-x partition into distinct parts is

$$\frac{\pi^2}{6}(1-x)^3 - \frac{\pi^2}{4}(1-x)^2 + \frac{\pi^2}{12}(1-x) - O((1-x)^{-1}).$$

$\square$

Permutations. Permutations have the (exponential) generating function $1/(1-x)$. Therefore the mean and variance of the size of Boltzmann-sampled permutations, with parameter x, are $x/(1-x)$ and $x/(1-x)^2$. For x near 1, the variance is roughly the square of the mean.

Consider the family of random variables $B_x = \sum_{k=1}^{\infty} kY_k$, where Y_k is Poisson with mean x^k/k; note that B_x is the size of a random Boltzmann-x permutation.

Proposition 3.4.4. *As $x \to 1^-$, $(1-x)B_x$ converges in distribution to an exponential random variable with mean* 1*.*

Proof. The sum B_x is, as previously seen, the size of permutations chosen from the Boltzmann distribution with parameter x. The nth moment of the size of such permutations is therefore given by $(x\partial_x)^n(1/(1-x))/(1/(1-x))$.

Since the operator $(x\partial_x)$ multiplies the coefficient of x^k by k, we have

$$(x\partial_x)^n(1/(1-x)) = \sum_{k=1}^{\infty} k^n x^k.$$

The generating function of the sequence $\{k^n\}_{n=1}^{\infty}$ is

$$\sum_{k=1}^{\infty} k^n x^k = \frac{\sum_{m=0}^{n-1} A(n,m)x^{m+1}}{(1-x)^{n+1}}$$

where $A(n,m)$ is an Eulerian number, the number of permutations of n with exactly m ascents. Therefore we have

$$\mathbb{E}\left[((1-x)B_x)^n\right] = (1-x)^n \frac{\frac{\sum_{m=0}^{n-1} A(n,m)x^{m+1}}{(1-x)^{n+1}}}{1/(1-x)} = \sum_{m=0}^{n-1} A(n,m)x^{m+1}.$$

As $x \to 1^-$, then, the right-hand side approaches $\sum_{m=0}^{n-1} A(n,m)$. This sum is just $n!$ since it enumerates permutations by their number of ascents. By the Stieltjes moment problem (Thm. 2.4.5), this suffices to specify the limiting distribution. Finally, we note that if B is exponential with mean 1, then $\mathbb{E}(B^n) = n!$. $\square$

This is similar to the "sharp-cutoff" model for random permutations on n elements, in which we model the cycle type of a random permutation on n elements by the sequence $(X_1, \ldots, X_n)$ where X_k is Poisson with mean $1/k$ and the different X_k

are independent. In [ABT03, Lemma 4.7] it is shown that the sum $\left(\sum_{k=1}^{n} kX_k\right)/n$ has a limiting distribution as $n \to \infty$; however this distribution can only be given in terms of iterated integrals.

3.5 The critical sampler and objects of infinite size

So far we have dealt with Boltzmann samplers where the Boltzmann parameter x is a point at which $C(x)$, the generating function of the class being sampled, converges. By Pringsheim's theorem, if all the coefficients of $C(x)$ are positive, then the Taylor series for $C(x)$ diverges when x is the radius of convergence of C. But if we forge ahead and run the Boltzmann sampler at this critical value nonetheless, we obtain useful models for thinking about large combinatorial objects.

For example, in the Boltzmann sampler for permutations, we have $\mathcal{P}(x^k/k)$ objects of size k, for each k. The exponential generating function for permutations is $1/(1-x)$, so the critical value is $x_c = 1$. This suggests that we think of very large permutations as having $\mathcal{P}(1/k)$ cycles of length k, for each k. And indeed this is the limiting distribution for the number of cycles of length k in a random permutation. Furthermore, just as the number of cycles of each length in this infinite Boltzmann sampler are actually independent, the number of cycles of each length in actual permutations are asymptotically independent. This philosophy even extends to the limiting distribution of the number of cycles of length between γn and δn in a random permutation of $[n]$, as we will see in Section 4.9 – although this distribution is not Poisson, its kth moments are those of the Poisson distribution for sufficiently small k.

For weighted permutations we can proceed in much the same way. We will consider the weighted permutation model in which cycles of size k get a weight σ_k, the weight of a permutation is the product of the weights of its cycles, and the probability of picking a permutation is proportional to its weight. We will see more of this in Chapter 4. The Boltzmann sampler for such objects takes $\mathcal{P}(\sigma_k x^k/k)$ cycles of size k, for each k; thus in the limit there are $\mathcal{P}(\sigma_k/k)$ cycles for each k. Sets of lists, or permutations in which each cycle has a distinguished "first" element, can be viewed as the case $\sigma_k = k$; thus we can think of a vector of infinitely many $\mathcal{P}(1)$ random variables as the cycle type of a very large set of lists.

In some cases, however, the sum $\sum_{k \geq 1} k\sigma_k$ converges, and therefore the expected size of the Boltzmannized objects is *finite*; in these cases the structure of the short cycles in the permutation are given by the Boltzmann sampler, and there is one very long cycle.

For partitions of integers, the critical Boltzmann sampler does not make sense; setting $x = 1$, we find that the number of parts of length 1 is a geometric random variable with failure rate zero. But a critical sampler for partitions into distinct parts is possible. The sampler with parameter x includes a part k with probability $x^k/(1+x^k)$. Letting $x = 1$, then, the critical sampler for partitions into distinct parts includes each part with probability $1/2$. That is, a very large partition into distinct parts can be modeled as a sequence of independent fair coin flips! In Section 6.4 this model will be useful for probabilistic interpretation of some partition identities.

Compositions. Consider the Boltzmann sampler for compositions, which are sequences of sequences of atoms. This Boltzmann sampler works as follows: fix a parameter x with $0 < x < 1/2$. Generate a Bernoulli random variable $Be(x/(1-x))$. If this is 0, stop. If it is 1, then generate a part which is equal to k with probability $x^{k-1}(1-x)$.

The critical sampler for compositions, with $x = 1/2$, is therefore as follows. In the "outer loop" we never stop, since it only terminates when a $Be(1)$ random variable takes the value 0. Second, each part is equal to k with probability $1/2^k$. Alternatively, we can interpret this in the "balls and bars" model: we generate a random composition of infinity by generating infinitely many balls, where between each two balls we have probability 1/2 of having a bar.

The average part is equal to $\sum_{k\geq 1} k/2^k = 2$; this suggests that compositions of n have about $n/2$ parts. In fact, we can apply the renewal central limit theorem to see that the time at which a sum of geometric random variables with mean $\mu = 2$ (and variance $\sigma^2 = 2$) reaches n is asymptotically normally distributed with mean $n/\mu = n/2$ and variance $\sigma^2 n/\mu^3 = n/4$.

This is in fact the case. Consider compositions of $[n]$, counted according to their size and number of parts. These are counted by

$$P(z,u) = \frac{1}{1-(uz+uz^2+uz^3+\cdots)} = \frac{1}{1-\frac{uz}{1-z}} = \frac{1-z}{1-(1+u)z}.$$

Differentiating, we have

$$P_u(z,1) = \frac{z(1-z)}{(1-2z)^2} = \frac{-1}{4} + \frac{1}{4(1-2z)^2}$$

and so, for $n \geq 1$ (so we can ignore the $-1/4$),

$$[z^n]P_u(z,1) = \frac{1}{4}\binom{-2}{n}(-2)^n = \frac{1}{4}(-1)^n(n+1)(-2)^n = (n+1)2^{n-2}.$$

The average number of parts of a composition of n is therefore $\mu_n := (n+1)2^{n-2}/2^{n-1} = (n+1)/2$, as is predicted by the balls-and-bars model.

Similarly, we can find the variance of the number of parts. We have

$$P_{uu}(z,1) = \frac{2(1-z)z^2}{(1-2z)^3} = \frac{1}{4}\left(1 - \frac{1}{1-2z} - \left(\frac{1}{1-2z}\right)^2 + \left(\frac{1}{1-2z}\right)^3\right)$$

and so, for $n \geq 1$,

$$\begin{aligned}
[z^n]P_{uu}(z,1) &= \frac{1}{4}\left(-(2^n) - \binom{-2}{n}(-2)^n + \binom{-3}{n}(-2)^n\right) \\
&= \frac{1}{4}\left(-2^n - (-1)^n(n+1)(-2)^n + (-1)^n\frac{(n+1)(n+2)}{2}(-2)^n\right) \\
&= 2^{n-2}\left(-1-(n+1)+(n+1)(n+2)/2\right)
\end{aligned}$$

from which we find that the variance of the number of parts of a random composition is

$$\frac{P_{uu}(z,1)}{P(z,1)} + \mu_n - \mu_n^2 = \frac{n^2+n-2}{4} + \frac{n+1}{2} - \left(\frac{n+1}{2}\right)^2 = \frac{n-1}{4}.$$

Now consider only parts of length k. The generating function for compositions by their size and number of parts equal to k is

$$P^{(k)}(z,u) = \frac{1}{1 - \left(\frac{z}{1-z} + (u-1)z^k\right)}$$

Therefore the total number of k-parts in all compositions of n is

$$[z^n]P_u^{(k)}(z,1) = [z^n]\frac{z^k(1-z)^2}{(1-2z)^2} = [z^{n-k}]\left(\frac{1-z}{1-2z}\right)^2.$$

We can easily find that $[z^r]\left(\frac{1-z}{1-2z}\right)^2 = (r+3)2^{r-2}$; therefore the total number of k-parts in all compositions of n is $(n-k+3)2^{n-k-2}$. This fact can also be proven combinatorially. We will count compositions with distinguished k-parts. The distinguished k-part either comes at the beginning of a composition, at the end, or at one of $n-k-1$ intermediate positions. If the distinguished part is at the beginning or end the composition is completed by generating a composition of $n-k$, which can be done in 2^{n-k-1} ways. If the distinguished part starts l units from the beginning of the composition, then the composition is completed by generating a composition of l and a composition of $n-k-l$, which can be done in $2^{l-1}2^{n-k-l-1} = 2^{n-k-2}$ ways. Thus there are a total of $2 \cdot 2^{n-k-1} + (n-k-1)2^{n-k-2} = (n-k+3)2^{n-k-2}$ compositions with a single distinguished k-part.

The mean number of k-parts in all compositions of n is therefore

$$\frac{[z^n]P_u^{(k)}(z,1)}{[z^n]P^{(k)}(z,1)} = \frac{(n-k+3)2^{n-k-2}}{2^{n-1}} = \frac{n-k+3}{2^{k+1}}.$$

As $n \to \infty$ with k fixed, this is asymptotic to $n/2^{k+1}$, which is the prediction from the critical Boltzmann sampler.

Proceeding similarly, we can work out the variance of the number of k-parts of a random composition. We have

$$P_{uu}^{(k)}(z,1) = \frac{2z^{2k}(1-z)^3}{(1-2z)^3}$$

and we observe that $[z^n](1-z)^3/(1-2z)^3 = 2^{n-3}(n+2)(n+7)/2$. Therefore we have

$$[z^n]P_{uu}^{(k)}(z,1) = 2^{n-2k-3}(n-2k+2)(n-2k+7)$$

from which we compute that the variance of the number of k-parts is

$$\frac{2^{n-2k-3}(n-2k+2)(n-2k+7)}{2^{n-1}} + \frac{n-k+3}{2^{k+1}} - \left(\frac{n-k+3}{2^{k+1}}\right)^2$$

which is linear in n. These results can be obtained directly from a balls-and-bars model, but one must carefully sum covariances in a tedious case analysis. The generating function method is more systematic and does not require treating so many different terms separately.

3.6 Historical antecedents

In combinatorial work, there are two principal papers that have used special cases of the Boltzmann sampler. The first is that of Shepp and Lloyd [LS66]; this paper gives the distribution of the lengths of the rth shortest or rth longest cycle of a random permutation of $[n]$. They study a sequence of independent random variables $(\alpha_1, \alpha_2, \ldots)$ where α_j is Poisson of mean z^j/j, where z is a parameter strictly between 0 and 1; let $\mathbb{P}_z$ denote probabilities with respect to this model. They show that

$$\mathbb{P}_z\left(\alpha_1 = a_1, \alpha_2 = a_2, \ldots\right) = (1-z)z^{\sum_{j=1}^{\infty} ja_j} \prod_{j=1}^{\infty} \frac{(1/j)^{a_j}}{a_j!}$$

and therefore that

$$\mathbb{P}_z\left(\alpha_1 = a_1, \alpha_2 = a_2, \ldots\right) = \prod_{j=1}^{\infty} \frac{(1/j)^{a_j}}{a_j!}$$

when $\sum_{j=1}^{\infty} ja_j = n$ and zero otherwise. They proceed to study this model by observing that if Φ is a function of the cycle type $(\alpha_1, \alpha_2, \ldots)$ of a random permutation, then $\mathbb{E}_z(\Phi)/(1-z) = \sum_{n\geq 0} \mathbb{E}_n(\Phi)z^n$. The left-hand side of this relation is a function of independent random variables and is therefore easily understood; they perform the coefficient extraction by use of Tauberian theorems.

The limiting distribution of the length of the rth shortest cycle is discrete and supported on the integers; the limiting distribution of the length of the rth longest cycle, after rescaling by a factor of $[n]$, is a nontrivial continuous distribution. The limiting distribution of the length of the rth shortest cycle is what one would predict directly from the use of Boltzmann samplers. Let $\mathbb{P}_n$ denote uniform measure on permutations of $[n]$, and let S_r denote the length of the rth shortest cycle in a permutation. Then Shepp and Lloyd give the formula [LS66, p. 349]

$$\lim_{n\to\infty} \mathbb{P}_n(S_r = j) = \int_{H_{j-1}}^{H_j} \frac{t^{r-1}}{(r-1)!} e^{-t}\, dt$$

where $H_j = \sum_{k=1}^{j} 1/k$ is a harmonic number. This follows from the fact that this integral is equal to $\lim_{x\to 1} \mathbb{P}_x(S_r = j)$, where P_x is the Boltzmann-x measure on permutations, and a Tauberian side condition. In the case where $r = 1$, for example, we get

$$\lim_{n\to\infty} \mathbb{P}_n(S_1 = j) = \exp(-H_{j-1}) - \exp(-H_j) = \exp(-H_{j-1})(1 - e^{-1/j}).$$

We have $S_1 = j$ exactly when there are 0 cycles of length $1, 2, \ldots, j-1$ and at least 1 cycle of length j. In the critical Boltzmann sampler the cycle counts are independent Poissons, from which this result follows.

The other principal area of application is to integer partitions. In this case the Boltzmann sampler originally appears in a paper of Fristedt [Fri93]. Fristedt considers a random partition model denoted by Q_q, where q is a parameter; this model assigns to the partition λ the probability $Q_q(\lambda) := q^{|\lambda|} \prod_{k=1}^{\infty}(1-q^k)$. Then letting X_k denote the number of parts of size k in a random partition, we see that [Fri93, Prop. 4.1]

$$Q_q(X_1 = x_1, X_2 = x_2, \ldots) = \prod_{k=1}^{\infty}(1-q^k)q^{kx_k}.$$

where $(x_1, x_2, \ldots)$ is a sequence of nonnegative integers. From this Fristedt proceeds to derive many properties of the structure of partitions of large integers. This has been extended by Vershik and collaborators [DVZ00, Ver96] to find the limiting shape of the Young diagram of integer partitions, and the same conditioning trick was used in [CPSW99] to study the multiplicity of parts in random partitions.

3.7 Algorithmic uses

Although we use the Boltzmann sampler as a device for *analysis* of random combinatorial structures, it was introduced (at least under this name) as a means for the *generation* of random combinatorial structures. Two paradigms are possible. One is the approximate-size paradigm, in which it suffices to generate objects with size in some interval $[(1-\epsilon)n, (1+\epsilon)n]$. The other is a fixed-size paradigm, in which we require objects of size *exactly* n to be generated. In many cases approximate-size generation is possible in one trial for large objects, since the distribution of sizes is concentrated (i. e. has variance smaller than the mean).

If approximate-size generation is possible, then so is fixed-size generation; to generate objects of size exactly n from a combinatorial class $\mathcal{A}$, we simply generate objects from the Boltzmann-x distribution on $\mathcal{A}$ until we find one which is of the right size. This method succeeds regardless of the choice of x, since the Boltzmann-x distribution restricted to $\mathcal{A}_n$ is uniform for any choice of x and n; however we will choose x so that $\mu_{\mathcal{A}}(x) \approx n$. In this case the number of trials needed to get an object of exact size n is proportional to $\sigma_{\mathcal{A}}(\mu_{\mathcal{A}}^{-1}(n))$, where $\sigma_{\mathcal{A}}(x)$ is the variance of the size of $\mathcal{A}$-objects chosen from the Boltzmann-x distribution. For example, if $\mathcal{A}$ is Hayman-admissible, then rejection sampling [DFLS04, Thm. 6.2] takes a mean number of trials asymptotic to $\sqrt{2\pi}\sigma_{\mathcal{A}}(\mu_{\mathcal{A}}^{-1}(n))$. This follows from the fact that if $\mathcal{A}$ has a Hayman-admissible generating function, then the distribution of sizes of objects generated by the Boltzmann-x sampler is asymptotically normal as x approaches its critical value or $n \to \infty$.

One particularly interesting application is to random sampling of plane partitions [BFP07]. A plane partition is a two-dimensional array of integers $(a_{i,j})_{i,j\geq 1}$, adding up to n, which is weakly decreasing both in rows and columns. Plane partitions have the generating function $P(x) = \prod_{r\geq 1}(1-x^r)^{-r}$, due to MacMahon. The simple form of this generating function calls for a combinatorial interpretation. But plane partitions do not seem to be specifiable in terms of admissible constructions starting from atoms. However Pak [Pak02] gives a bijection between plane partitions and the class $\mathcal{M} = \textsc{Mset}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2)$. The right-hand side can be thought of as multisets of ordered pairs of nonnegative integers, where the pair (k, l) has weight $k + l + 1$ and the weight of a multiset is the sum of the weights of its elements (with multiplicity); then this bijection takes a multiset with weight n to a plane partition of n. Then in [BFP07] this bijection is used to give an algorithm for the generation of plane partitions, starting with the Boltzmann sampler for $\textsc{Mset}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2)$; this algorithm is faster than previously known algorithms for random sampling from plane partitions. In general, it may be possible to do some sort of "post-processing" on the output of Boltzmann samplers to use them for generation of random objects which are not easily specified. For example, there does not seem to be a "nice" combinatorial specification of integer partitions such that the difference between any two parts is at least 2. But there is a nice specification of a class equinumerous with these, namely partitions with all parts congruent to 1 or 4 modulo 5. Equinumerosity here is a consequence of the Rogers-Ramanujan identities, for which there are bijective proofs; it may be possible to use one of these proofs to determine the "average shape" of the partitions counted by the Rogers-Ramanujan identities.

Chapter 4

Profiles of permutations

4.1 Introduction

In this chapter we study the cycle structure of random permutations in which the lengths of all cycles are constrained to lie in some infinite set S, and permutations may be made more or less likely to be chosen through multiplicative weights placed on their cycles. Cycle structures viewed in this manner are a special case of certain measures on S_n which are conjugation-invariant and assign a weight to each element of S_n based on its cycle structure.

Definition 4.1.1. *Let $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots)$ be an infinite sequence of nonnegative real numbers. Then the weight of the permutation $\pi \in S_n$, with respect to $\vec{\sigma}$, is*

$$w_{\vec{\sigma}}(\pi) = \prod_{i=1}^{n} \sigma_i^{c_i(\pi)}$$

where $c_i(\pi)$ is the number of cycles of length i in π.

Informally, each cycle in a permutation receives a weight depending on its length, and the weight of a permutation is the product of the weights of its cycles. The sequence $\vec{\sigma}$ is called a *weighting sequence*.

For each positive integer n, let $(\Omega^{(n)}, \mathcal{F}^{(n)})$ be a probability space defined as follows. Take $\Omega^{(n)} = S_n$, the set of permutations of $[n]$, and let $\mathcal{F}^{(n)}$ be the set of all subsets of S_n. Endow $(\Omega^{(n)}, \mathcal{F}^{(n)})$ with a probability measure $\mathbb{P}_{\vec{\sigma}}^{(n)}$ for each weighting sequence $\vec{\sigma}$ as follows. Let $\mathbb{P}_{\vec{\sigma}}^{(n)}(\pi) = w_{\vec{\sigma}}(\pi)/\sum_{\pi' \in S_n} w_{\vec{\sigma}}(\pi')$; that is, each permutation has probability proportional to its weight. Extend $\mathbb{P}_{\vec{\sigma}}^{(n)}$ to all subsets of S_n by additivity. To streamline the notation, we will sometimes write $\mathbb{P}_{\vec{\sigma}}(\pi)$ for $\mathbb{P}_{\vec{\sigma}}^{(n)}(\pi)$. The sum of the weights of $\vec{\sigma}$-weighted permutations of $[n]$ is

$$\sum_{\pi \in S_n} w_{\vec{\sigma}}(\pi) = n![z^n] \exp\left(\sum_{k \geq 1} \sigma_k z^k / k\right)$$

by the exponential formula for labelled combinatorial structures.

We fix some notation. Define the random variable $X_k^{(n)} : \Omega^{(n)} \to \mathbb{Z}^+$ by setting $X_k^{(n)}(\pi)$ equal to the number of k-cycles in the permutation π. Let $X^{(n)}(\pi) = \sum_{k=1}^{n} X_k^{(n)}(\pi)$ be the total number of cycles. We will often suppress π and (n) in the notation, and we will write (for example) $\mathbb{P}_{\vec{\sigma}}(X_1 = 1)$ as an abbreviation for $\mathbb{P}_{\vec{\sigma}}(\{\pi : X_1^{(n)}(\pi) = 1\})$. Let $Y_k = kX_k$. We define Y_k in order to simplify the statement of some results.

This model incorporates various well-known classes of permutations, including generalized derangements (permutations in which a finite set of cycle lengths is prohibited), and the Ewens sampling formula from population genetics [Ewe72], which corresponds to the weighting sequence $(\sigma, \sigma, \sigma, \ldots)$. If $\vec{\sigma}$ is a 0-1 sequence with finitely many 1s, then this model specializes to random permutations of which all cycle lengths lie in a finite set. These have a fascinating structure studied by Benaych-Georges [BG07] and Timashev [Tim08]; a typical permutation of $[n]$ with cycle lengths in a finite set S has about $\frac{1}{k}n^{k/\max S}$ k-cycles, for each k in S. In particular, most cycles are of length $\max S$, which may be unexpected at first glance. Analytically, this situation is studied via the asymptotics of $[z^n]e^{P(z)}$ where P is a polynomial, as done by Wilf [Wil86]. Yakymiv [Yak00] has studied the case, alluded to by Bender [Ben74], in which $\vec{\sigma}$ is a sequence of 0s and 1s with a fixed density σ of 1s; the behavior of such permutations is in broad outline similar to that of the Ewens sampling formula with parameter σ. An "enriched" version of the model has been studied by Ueltschi and coauthors [GRU07, UB08]. In their model, permutations are endowed with a spatial structure. Each element of the ground set of the permutation is a point in the plane, and weights involve distances between points. Their "simple model of random permutations with cycle weight" [UB08, Sec. 2] is the model used here, where $\sigma_i = e^{-\alpha_i}$.

There are other combinatorially interesting conjugation-invariant measures on S_n, including permutations with all cycle lengths distinct [GK90], and permutations with kth roots for some fixed k [FFG$^+$06, Pou02]. However the generating functions counting these classes are not exponentials of "nice" functions and thus different techniques are required.

Throughout this chapter, we often implicitly assume that permutations under the uniform measure on S_n are the "primitive" structure, and weighted permutations are a perturbation of these. Here we follow Arratia *et al.* in [ABT97, ABT03], in embracing a similar philosophy and viewing the permutation as the archetype of a class of "logarithmic combinatorial structures", and Flajolet and Soria's definition of functions of logarithmic type [FS90].

It will be convenient to use bivariate generating functions which count permutations by their size and number of cycles. In general, we take $F(z,u) = \sum_{n,k} f_{n,k} \frac{z^n}{n!} u^k$ to be the bivariate generating function, exponential in z and ordinary in u, of a combinatorial class $\mathcal{F}$, where $f_{n,k}$ is the number of objects in $\mathcal{F}$ of size n and with a certain parameter equal to k. In our case n will be the number of elements of a permutation, and k the total number of cycles or the number of cycles of a specified size. Then $[z^n] \left. \frac{\partial}{\partial u} F(z,u) \right|_{u=1} / [z^n] F(z,1)$ gives the expected value of the parameter k for an object of size n selected uniformly at random. The following lemma will frequently be useful, as it reduces the bivariate analysis to a univariate analysis.

Lemma 4.1.2. *Let $f(z)$ be the exponential generating function of permutations with weight sequence $\vec{\sigma}$. Then the expected number of k-cycles in a permutation chosen according to the measure $\mathbb{P}_{\vec{\sigma}}^{(n)}$ is*

$$\mathbb{E}_{\vec{\sigma}}^{(n)} X_k = \frac{\sigma_k}{k} \frac{[z^{n-k}] f(z)}{[z^n] f(z)}.$$

Proof. The bivariate generating function counting the cycles of such permutations is

$$\sigma_1 z + \sigma_2 \frac{z^2}{2} + \cdots + \sigma_{k-1} \frac{z^{k-1}}{k-1} + u \sigma_k \frac{z^k}{k} + \sigma_{k+1} \frac{z^{k+1}}{k+1} + \cdots$$

and this can be rewritten as $(u-1)\frac{\sigma_k z^k}{k} + \sum_{j\geq 1} \frac{\sigma_j z^j}{j}$. Thus, from the exponential formula, the bivariate generating function counting such permutations is

$$P(z,u) = \exp\left((u-1)\frac{\sigma_k z^k}{k} + \sum_{j\geq 1} \frac{\sigma_j z^j}{j}\right).$$

The expected number of cycles in a random permutation is $[z^n]P_u(z,1)/[z^n]P(z,1)$, giving the result. $\square$

The structure of this chapter is as follows. In Section 4.2 we give exact formulas and asymptotic series (Propositions 4.2.2 and 4.2.3) for the mean and variance of the number of cycles of permutations chosen from the Ewens distribution. We also consider the average number of k-cycles in such permutations of $[n]$ for fixed k (Propositions 4.2.4 and 4.2.5) and for $k = \alpha n$ (Proposition 4.2.6). An "integrated" version of these results, Theorem 4.2.7, is one of the main results; this is a limit law for the probability that a random element of a weighted permutation is in a cycle within a certain prescribed range of lengths. In Section 4.3 we derive similar results for permutations in which all cycle lengths have the same parity. In addition, we determine the mean and variance of the number of cycles of such permutations (Theorem 4.3.6 treats the odd case, and Theorem 4.3.8 treats the even case). In Section 4.4 we explore connections to the generation of random objects by Boltzmann sampling. The main theorem of this section, Theorem 4.4.3, states that the Boltzmann-sampled permutations of a certain class of approximate size n, including the Ewens and parity-constrained cases, have their number of cycles distributed with mean and variance approximately a constant multiple of $\log n$.

In Section 4.5 we proceed to another specific case, that of permutations with periodic weighting sequences. These obey the same limit laws but the asymptotic enumeration of such permutation introduces new factors. In Section 4.6 we consider the weighting scheme $\sigma_j = 1/j$; in this weight scheme, permutations have one long cycle and, on average, $\pi^2/6$ short cycles. Section 4.7 considers permutations having square roots or more generally mth roots; this is a natural example of a permutation model with restricted multiplicities which nonetheless strongly resembles the weighted models. In Section 4.8 we consider the weighting scheme $\sigma_j = j$, which corresponds to "sets of lists"; a set of lists in $[n]$ usually has about $\sqrt{n}$ components, of typical size $\sqrt{n}$, which is a combinatorial consequence of the generating function $\exp(z/(1-z))$ of "exponential of a pole" type. In Section 4.9 we show that the number of cycles of a permutation of $[n]$ of length in $[\gamma n, \delta n]$ obeys a limit law. Finally Sections 4.10 and 4.11 consider connections between random permutations and, respectively, stochastic processes and number theory.

4.2 The Ewens sampling formula and Bernoulli decomposition

The *Ewens distribution* [Ewe72] on permutations of $[n]$ with parameter σ gives to each permutation π probability proportional to $\sigma^{X(\pi)}$. This corresponds to the weighting sequence $\vec{\sigma} = (\sigma, \sigma, \sigma, \ldots)$; we will write $\mathbb{P}_\sigma^{(n)}, \mathbb{E}_\sigma^{(n)}$ for $\mathbb{P}_{\vec{\sigma}}^{(n)}, \mathbb{E}_{\vec{\sigma}}^{(n)}$, and call a random permutation selected in this manner a *σ-weighted permutation*. In this section we derive formulas for the mean and variance of the number of cycles of permutations chosen from the Ewens distribution. Note that the number of cycles can be decomposed into a sum of independent Bernoulli random variables. Similar decompositions are due to Arratia et al. in [ABT03, Sec. 5.2] for general σ, and Feller [Fel45, (46)] for $\sigma = 1$; the fact that the number of cycles is normally distributed is seen in [FS90, Example 1]. Thus this section is largely expository; the proofs are provided for the purpose of comparison with other proofs to be given below. The asymptotic series for $\mathbb{E}_\sigma^{(n)}$ and $\mathbb{V}_\sigma^{(n)}$ appear to be new.

Theorem 4.2.1. *[Pit06, Exercise 3.2.3] The distribution of the random variable X under the measure $\mathbb{P}_\sigma^{(n)}$ is that of the sum $\sum_{k=1}^n Z_k$, where the Z_k are independent random variables and Z_k has the Bernoulli distribution with mean $\sigma/(\sigma + k - 1)$.*

Proof. The generating function of permutations of $[n]$ counted by their number of cycles is $\sum_{k=1}^n S(n,k)u^k = u(u+1)(u+2)\cdots(u+n-1)$, where $S(n,k)$ are the Stirling cycle numbers. Replacing u with σu and normalizing gives the probability generating function for the number of cycles,

$$\sum_{k=1}^n S(n,k)\sigma^k u^k = \frac{\sigma u}{\sigma}\frac{\sigma u + 1}{\sigma + 1}\cdots\frac{\sigma u + n - 1}{\sigma + n - 1},$$

and each factor is the probability generating function for a Bernoulli random variable. $\square$

Combinatorially, we can envision this Bernoulli decomposition as follows. We imagine forming a permutation of $[n]$ by placing the elements $1, \ldots, n$ in cycles in turn. When the element k is inserted, with probability $\sigma/(\sigma+k-1)$ it is placed in a new cycle, and with probability $1/(\sigma+k-1)$ it is placed after any of $1, 2, \ldots, k-1$ in the cycle containing that element. Then the probability of obtaining any permutation with c cycles is $\sigma^c/(\sigma(\sigma+1)\cdots(\sigma+n-1))$, which is exactly the measure given to this permutation by $\mathbb{P}_\sigma^{(n)}$. This is an instance of the Chinese Restaurant Process [Pit06, Sec. 3.1].

From this decomposition into Bernoulli random variables, we can derive formulas for the mean and variance of the number of cycles under the measure $\mathbb{P}_\sigma^{(n)}$. In particular we note that since X is a sum of Bernoulli random variables with small mean, the variance of X is very close to its mean. Let ψ denote the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$; this has an asymptotic series $\psi(z) = \log z - \frac{1}{2}z^{-1} - \frac{1}{12}z^{-2} + O(z^{-4})$ as $z \to \infty$. Let $H_n = \sum_{k=1}^n \frac{1}{k}$ be the nth harmonic number and let $\gamma = 0.57721\ldots$ be the Euler-Mascheroni constant.

Proposition 4.2.2. *The expected number of cycles of a random σ-weighted permutation of $[n]$ is $\mathbb{E}_\sigma^{(n)} X = \sigma(\psi(n+\sigma) - \psi(\sigma))$; in particular if σ is a positive integer we have*

$$\mathbb{E}_\sigma^{(n)} X = \sigma \log n + (\sigma\gamma - \sigma H_{\sigma-1}) + (\sigma^2 - \sigma/2)n^{-1} + O(n^{-2}). \tag{4.1}$$

Proof. From Theorem 4.2.1 we have

$$\mathbb{E}_\sigma^{(n)} X = \sum_{k=1}^n \frac{\sigma}{\sigma+k-1} = \sigma \sum_{k=1}^n \frac{1}{\sigma+k-1}.$$

Now, $\psi(z+1) - \psi(z) = 1/z$; thus

$$\begin{aligned}
\psi(n+\sigma) - \psi(\sigma) &= (\psi(n+\sigma) - \psi(n+\sigma-1)) + \cdots + (\psi(\sigma+1) - \psi(\sigma)) \\
&= \frac{1}{n+\sigma-1} + \frac{1}{n+\sigma-2} + \cdots + \frac{1}{\sigma} \\
&= \sum_{k=1}^{n} \frac{1}{\sigma+k-1}.
\end{aligned}$$

This proves that $\mathbb{E}_\sigma^{(n)} X = \sigma(\psi(n+\sigma) - \psi(\sigma))$. The asymptotic series follows from that for $\psi(z)$ where we have used the fact that $\psi(n) = H_{n-1} - \gamma$ when n is a positive integer. □

Proposition 4.2.3. *The variance of the number of cycles of a random σ-weighted permutation of $[n]$ is*

$$\sigma^2 (\psi'(n+\sigma) - \psi'(\sigma)) + \sigma(\psi(n+\sigma) - \psi(\sigma)); \tag{4.2}$$

this has an asymptotic series,

$$\mathbb{V}_\sigma^{(n)} X = \sigma \log n + (-\sigma^2 \psi'(\sigma) - \sigma\psi(\sigma)) + \frac{4\sigma^2 - 1}{2} n^{-1} + O(n^{-2}) \tag{4.3}$$

The proof is similar to that of the previous proposition, noting that the variance of a Bernoulli random variable with mean p is $p - p^2$.

From (4.3) we can also derive for integer σ the explicit formula (not involving ψ)

$$\mathbb{V}_\sigma^{(n)} X = -\sigma^2 \sum_{j=\sigma}^{\sigma+n-1} \frac{1}{j^2} + \sigma(\log n + \gamma - H_{\sigma-1}) + O(1/n)$$

which holds as $n \to \infty$. It suffices to show that

$$\psi'(n+\sigma) - \psi'(\sigma) = -\sum_{j=\sigma}^{\sigma+n-1} \frac{1}{j^2}. \tag{4.4}$$

To see this, recall the identity $\psi(x+1) - \psi(x) = 1/x$; differentiating gives $\psi'(x+1) - \psi'(x) = -1/x^2$. Summation over $x = \sigma, \sigma+1, \ldots, \sigma+n-1$ gives (4.4).

Finally, we recall a normal distribution result for the total number of cycles [ABT03, (5.22)]. Let $\hat{X} = \frac{X - \sigma \log n}{\sqrt{\sigma \log n}}$ be the standardization of X. Then $\lim_{n\to\infty} \mathbb{P}_\sigma^{(n)}(\hat{X} \leq x) = \Phi(x)$, where $\Phi(x)$ is the cumulative distribution function of a standard normal random variable. This follows from Theorem 4.2.1 and the Lindeberg-Feller central limit theorem.

We have thus far looked at the total number of cycles of σ-weighted permutations. These distributions, suitably scaled, are continuous in the large-n limit. In contrast, looking at each cycle length separately, we approach a discrete distribution. More specifically, the number of k-cycles of σ-weighted permutations of $[n]$, for large n, converges in distribution to $\mathcal{P}(\sigma/k)$, where $\mathcal{P}(\lambda)$ denotes a Poisson random variable with mean λ; here we consider how quickly $\mathbb{E}_\sigma^{(n)} X_k$ approaches σ/k. Recall that X_k is a random variable, with $X_k(\pi)$ the number of k-cycles of a permutation π.

Proposition 4.2.4. *[AT92, (37)][Wat74] The average number of k-cycles in a σ-weighted permutation of $[n]$ is*

$$\mathbb{E}_\sigma^{(n)} X_k = \frac{\sigma}{k} \frac{(n)_k}{(n+\sigma-1)_k} \tag{4.5}$$

where $(n)_k = n(n-1)\ldots(n-k+1)$ is the "falling power".

We provide a new proof in terms of generating functions.

Proof. The bivariate generating function counting σ-weighted permutations by their size and number of k-cycles is $P(z,u) = (1-z)^{-u\sigma} \exp(\sigma(u-1)z^k/k)$. The mean number of k-cycles is given by

$$\frac{[z^n]\, \partial_u P(z,u)|_{u=1}}{[z^n]P(z,1)} = \frac{[z^n]\frac{\sigma z^k}{k}(1-z)^{-u\sigma}}{[z^n](1-z)^{-\sigma}} = \frac{\sigma}{k}\frac{[z^{n-k}](1-z)^{-\sigma}}{[z^n](1-z)^{-\sigma}}$$

and the binomial formula gives (4.5). $\square$

When σ is an integer, a combinatorial proof can be obtained by considering σ-weighted permutations as permutations where each cycle is colored in one of σ colors.

Proposition 4.2.5. *There is an asymptotic series for* $\mathbb{E}_\sigma^{(n)} X_k$,

$$\mathbb{E}_\sigma^{(n)} X_k = \frac{\sigma}{k}\left(1 - \frac{(\sigma-1)k}{n} + O(n^{-2})\right).$$

Proof. The numerator and denominator of (4.5) are polynomials in n of degree k; write the two highest-degree terms of each explicitly and divide. $\square$

Proposition 4.2.6. *Fix* $0 < \alpha \leq 1$. *The expected number of elements in* αn*-cycles of a random* σ*-weighted permutation satisfies, as* $n \to \infty$,

$$\mathbb{E}_\sigma^{(n)} Y_{\alpha n} = \sigma(1-\alpha)^{\sigma-1} + O(n^{-1})$$

(Here we have assumed for simplicity that αn is an integer.)

Proof. Let $\beta = 1-\alpha$. We have from Proposition 4.2.4 that

$$\mathbb{E}_\sigma^{(n)} Y_{\alpha n} = \sigma \frac{(n)_{\alpha n}}{(n+\sigma-1)_{\alpha n}} = \sigma \frac{n!(\beta n + \sigma - 1)!}{(\beta n)!(n+\sigma-1)!} = \sigma \frac{n!}{(n+\sigma-1)!}\frac{(\beta n+\sigma-1)!}{(\beta n)!}$$

We now note that $(n+r)!/n! = n^r(1+O(n^{-1}))$, for constant r as $n \to \infty$, from Stirling's formula. Applying this twice with $r = \sigma - 1$ gives the result. $\square$

It would be of interest to determine the limiting distribution of the number of cycles with length between γn and δn for constants γ and δ. There can be at most $\lfloor \gamma^{-1} \rfloor$ such cycles, so this random variable is supported on $0, 1, \ldots, \lfloor \gamma^{-1} \rfloor$. Thus to determine the limiting distribution it suffices to determine the 0th through $\lfloor \gamma^{-1} \rfloor$th moments of this random variable. The $\sigma = 1$ case will be treated in Section 4.9.

We can essentially integrate the result of Proposition 4.2.5 to determine the number of elements in cycles with normalized length in a specified interval. However, this can be done in a more general framework. Recall the definition of a Δ-domain from Section 2.3: for constants $R > 1$ and $\phi > 0$ we define a Δ-domain as a set of the form

$$\Delta(\phi, R) = \{z : |z| < R, z \neq 1, |\arg(z-1)| > \phi\}.$$

Theorem 4.2.7. *Let $\sum_k \sigma_k z^k/k = \sigma \log \frac{1}{1-z} + K + o(1)$ be analytic in its intersection with some Δ-domain, for some constants σ and K. Then the probability that a uniformly chosen random element of a random $\vec{\sigma}$-weighted permutation of $[n]$ lies in a cycle of length between γn and δn approaches $(1-\gamma)^\sigma - (1-\delta)^\sigma$ as $n \to \infty$.*

Note that analyticity in the slit plane suffices; this is the case $\phi = 0$. We begin by stating two lemmas needed in the proof.

Lemma 4.2.8. *Let $\{\sigma_k\}_{k=1}^\infty$ be a sequence of nonnegative real numbers with mean σ, i. e. $\sum_{k=1}^\infty \sigma_k = \sigma n + o(n)$ as $n \to \infty$. Fix constants $0 \leq \gamma < \delta < 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=\lceil \gamma n \rceil}^{\lfloor \delta n \rfloor} \sigma_k \left(1 - \frac{k}{n}\right)^{\sigma - 1} = (1-\gamma)^\sigma - (1-\delta)^\sigma.$$

Proof. We rewrite the sum as an integral,

$$\sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \left(1-\frac{k}{n}\right)^{\sigma-1} = \int_{\gamma n}^{\delta n} \left(1-\frac{k}{n}\right)^{\sigma-1} d\mu(k)$$

where $\mu(x) = \sum_{j=1}^{\lfloor x\rfloor} \sigma_j$. Integrating by parts gives

$$(1-\delta)^{\sigma-1}\mu(\delta n) - (1-\gamma)^{\sigma-1}\mu(\gamma n) - \int_{\gamma n}^{\delta n} \mu(k) d\left(1-\frac{k}{n}\right)^{\sigma-1}. \tag{4.6}$$

Differentiation allows us to rewrite the integral in (4.6) as a Riemann integral,

$$\int_{\gamma n}^{\delta n} \mu(k) d\left(1-\frac{k}{n}\right)^{\sigma-1} = \frac{1-\sigma}{n}\int_{\gamma n}^{\delta n} \mu(k)\left(1-\frac{k}{n}\right)^{\sigma-2} dk. \tag{4.7}$$

Let $\tau(k) = \mu(k) - \sigma k$. Then the integral on the right-hand side of (4.7) becomes

$$\frac{1-\sigma}{n}\left(\int_{\gamma n}^{\delta n} \sigma\left(1-\frac{k}{n}\right)^{\sigma-2} dk + \int_{\gamma n}^{\delta n} \tau(k)\left(1-\frac{k}{n}\right)^{\sigma-2}\right) dk. \tag{4.8}$$

We perform the first integral in (4.8) and note that $\mu(\delta n) \sim \sigma\cdot\delta n, \mu(\gamma n) \sim \sigma\cdot\gamma n$ in (4.6). This gives

$$\frac{1}{n}\sum_k \left(1-\frac{k}{n}\right)^{\sigma-1} \sim (1-\gamma)^{\sigma} - (1-\delta)^{\sigma} + \frac{1-\sigma}{n^2}\int_{\gamma n}^{\delta n} \tau(k)\left(1-\frac{k}{n}\right)^{\sigma-2} dk. \tag{4.9}$$

So it suffices to show that the final term in (4.9) is negligible, i. e.

$$\int_{\gamma n}^{\delta n} \tau(k)\left(1-\frac{k}{n}\right)^{\sigma-2} dk = o(n^2).$$

Since $\{\sigma_k\}_{k=1}^{\infty}$ has mean σ, we have $\sum_{k=1}^{n}\sigma_k = \sigma n + o(n)$. Thus $\tau(k) = o(n)$. On $[\gamma n, \delta n]$, $(1-k/n)^{\sigma-2}$ is bounded. So the integrand above is $o(n)$, and the integral is $o(n^2)$ as desired. □

Lemma 4.2.9. *Say* $[z^n]P(z) = Cn^{\sigma-1}(1+o(1))$ *uniformly in* n*, for some positive constants* C, σ*. Then*

$$\sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \frac{[z^{n-k}]P(z)}{[z^n]P(z)} \sim \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \left(1-\frac{k}{n}\right)^{\sigma-1}$$

as $n\to\infty$*, for any* $0 \le \gamma < \delta < 1$*.*

Proof. From the hypothesis that $[z^n]P(z) \sim Cn^{\sigma-1}$, we get

$$\frac{[z^{n-k}]P(z)}{[z^n]P(z)} \sim \frac{C(n-k)^{\sigma-1}}{Cn^{\sigma-1}} = \left(1-\frac{k}{n}\right)^{\sigma-1}$$

uniformly as $n, k \to \infty$ with $0 \le k < \delta n$. Therefore

$$\begin{aligned}
\sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \frac{[z^{n-k}]P(z)}{[z^n]P(z)} &= \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \left(1-\frac{k}{n}\right)^{\sigma-1} (1+o(1)) \\
&= \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \left(1-\frac{k}{n}\right)^{\sigma-1} + \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \cdot o(1) \cdot \left(1-\frac{k}{n}\right)^{\sigma-1}.
\end{aligned}$$

The first sum in the previous equation is $\Theta(n)$. The second sum has $\Theta(n)$ terms; since $(1-k/n)^{\sigma-1}$ and σ_k can both be bounded above on the interval $[\gamma n, \delta n]$ each term is $o(1)$. Thus the second sum is $o(n)$. So

$$\begin{aligned}
\sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \frac{[z^{n-k}]P(z)}{[z^n]P(z)} &= \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \left(\sigma_k \left(1-\frac{k}{n}\right)^{\sigma-1}\right) + o(n) \\
&= \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \left(\sigma_k \left(1-\frac{k}{n}\right)^{\sigma-1}\right) (1+o(1))
\end{aligned}$$

as desired. ∎

Proof of Theorem 4.2.7. This probability can be written as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k \frac{[z^{n-k}]P(z)}{[z^n]P(z)}.$$

Now, recall

$$\sum_k \sigma_k z^k/k = \sigma \log \frac{1}{1-z} + K + o(1)$$

by hypothesis. Thus the generating function $P(z)$ of $\vec{\sigma}$-weighted permutations is

$$P(z) = \exp\left(\sum_k \sigma_k z^k/k\right) = \exp\left(\sigma \log \frac{1}{1-z} + K + o(1)\right) = (1-z)^{-\sigma} e^K (1 + o(1)).$$

Applying the Flajolet-Odlyzko transfer theorem (Corollary 2.3.7), $[z^n]P(z) = Cn^{\sigma-1}(1+o(1))$ for some positive real constant C. Thus $P(z)$ satisfies the hypotheses of Lemma 4.2.9. Applying that lemma, we see that this sum is asymptotic to $n^{-1}\sum_{k=\lceil \gamma n\rceil}^{\lfloor \delta n\rfloor} \sigma_k (1-k/n)^{\sigma-1}$; the desired result then follows from Lemma 4.2.8. □

The hypotheses, and hence the conclusions, of Theorem 4.2.7 hold for many weight sequences $\sigma_1, \sigma_2, \ldots$ with $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \sigma_k = \sigma$; that is, for weight sequences averaging σ. In particular, we have the following special case.

Corollary 4.2.10. *Fix constants $0 \le \gamma \le \delta \le 1$. Let $p_\sigma(n;\gamma,\delta)$ be the probability that the element 1, in a σ-weighted permutation of $[n]$, lies in a cycle of length in the interval $[\gamma n, \delta n]$. Then*

$$\lim_{n\to\infty} p_\sigma(n;\gamma,\delta) = (1-\gamma)^\sigma - (1-\delta)^\sigma.$$

Proof. We have the cycle generating function $\sum_{k=1}^\infty \sigma z^k/k = \sigma \log \frac{1}{1-z}$; apply Theorem 4.2.7. □

For example, setting $\sigma = 1/2, \gamma = 0.99, \delta = 1$, we see that for large n, 10% of elements of 1/2-weighted permutations are in cycles of length at least $0.99n$. If we define the "co-length" of a cycle of a permutation to be the number of elements *not* in that cycle, a cleaner statement of the theorem becomes possible. The proportion of elements of σ-weighted permutations in cycles of co-length at most ζn is ζ^σ.

It would be desirable to replace the condition in the hypothesis of Theorem 4.2.7 with the less restrictive

$$\sum_k \frac{\sigma_k z^k}{k} = \sigma \log \frac{1}{1-z} \cdot (1 + o(1));$$

it seems likely that this suffices to prove a limit law but the proof does not easily adapt to that case.

4.3 Permutations with all cycle length of the same parity

This section is devoted to results on random permutations in which all cycle lengths have the same parity; that is, they are either all even or all odd. We adopt the notation $\mathbb{P}_e^{(n)}$ for the family of measures $\mathbb{P}_{\vec{\sigma}}^{(n)}$ where $\vec{\sigma} = (0, 1, 0, 1, \ldots)$, and similarly $\mathbb{P}_o^{(n)}$ for the family with $\vec{\sigma} = (1, 0, 1, 0, \ldots)$; these are the measures corresponding to permutations with all cycle lengths even and with all cycle lengths odd, respectively.

The results obtained here resemble those for the Ewens sampling formula with parameter 1/2. A heuristic explanation for this phenomenon is as follows. Let us produce a permutation of $[n]$ from the Ewens distribution with parameter 1/2 by first picking a permutation π uniformly at random from S_n, and then flipping a fair coin for each cycle of π. If all the coins come up heads we keep π; otherwise we "throw back" the permutation π and repeat this process until we have a trial in which all coins come up heads. The number and normalized size of cycles of permutations obtained in this manner should be similar to those of permutations with all cycle lengths even, since for large permutations the parity constraint is essentially equivalent to a coin flip.

Proposition 4.3.1. *The expected number of elements in k-cycles of a permutation of $[n]$ with all cycle lengths even is*

$$\mathbb{E}_e^{(n)} Y_k = \frac{n(n-2)\cdots(n-k+2)}{(n-1)(n-3)\cdots(n-k+1)}$$

if k is even, and 0 if k is odd.

Proof. By Lemma 4.1.2, we have $\mathbb{E}_e^{(n)} Y_k = [z^{n-k}](1-z^2)^{-1/2}/[z^n](1-z^2)^{-1/2}$; we apply the binomial theorem and simplify. $\square$

For example, when $n = 10$ we have

$$\begin{aligned} \left(\mathbb{E}_e^{(10)} Y_2, \mathbb{E}_e^{(10)} Y_4, \ldots, \mathbb{E}_e^{(10)} Y_{10}\right) &= (10/9, 80/63, 32/21, 128/63, 256/63) \\ &\approx (1.11, 1.27, 1.52, 2.03, 4.06) \end{aligned}$$

and we observe that most entries are in the longer cycles. For $n = 100$ this is illustrated in the figure above.

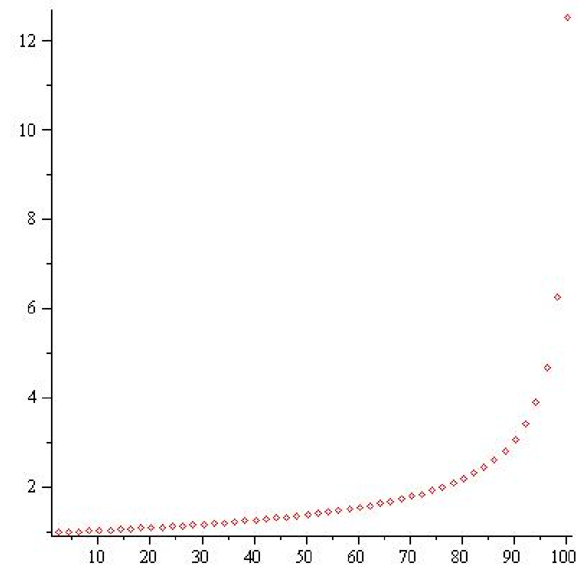


Figure 4.1: $\mathbb{E}_e^{(100)} Y_k$ for $k = 2, 4, \ldots, 100$

.

Proposition 4.3.2. *The expected number of elements in k-cycles of a random permutation of $[n]$ with all cycle lengths odd is*

$$\mathbb{E}_o^{(n)} Y_k = \frac{n(n-2)\cdots(n-k+1)}{(n-1)(n-3)\cdots(n-k)}$$

if n is even, and

$$\mathbb{E}_o^{(n)} Y_k = \frac{(n-1)(n-3)\cdots(n-k+2)}{(n-2)(n-4)\cdots(n-k+1)}$$

if n is odd.

Proof. The generating function of permutations with all cycle lengths odd, counted by their number of cycles, is $P(z,u) = ((1+z)/(1-z))^{u/2}$. We use Lemma 4.1.2 to see that the mean number of elements in k-cycles is given by $\mathbb{E}_o^{(n)} Y_k =$

$[z^{n-k}]\sqrt{\frac{1+z}{1-z}}/[z^n]\sqrt{\frac{1+z}{1-z}}$. We recall that $[z^n]\sqrt{\frac{1+z}{1-z}}$ is $\frac{(n-1)!!^2}{n!}$ if n is even and $\frac{n!!(n-2)!!}{n!}$ if n is odd; substituting and simplifying gives the result. □

By similar methods, we can obtain formulas for the exact number of permutations of $[n]$ with all cycle lengths divisible by a, and the exact expected number of k-cycles in such permutations for integers k which are divisible by a. These permutations have exponential generating function $(1 - z^a)^{-1/a}$. Permutations with all cycle lengths congruent to k mod a for some nonzero k are more difficult to deal with, as it appears that the generating function cannot be written in an elementary form except when a is even and $k = a/2$. (See [Sac97, Sec. 5.0.3] for the relevant generating functions.)

Proposition 4.3.3. *(a) The number of elements of k-cycles in a permutation of $[n]$ with all cycle lengths odd, for fixed odd k, satisfies $\mathbb{E}_o^{(n)} Y_k = 1 + \frac{k+1}{2n} + O(n^{-2})$ as n approaches ∞ through even values, and $\mathbb{E}_o^{(n)} Y_k = 1 + \frac{k-1}{2n} + O(n^{-2})$ as n approaches ∞ through odd values.*

(b) The number of elements of k-cycles in a permutation of $[n]$ with all cycle lengths even, for fixed even k, satisfies $\mathbb{E}_e^{(n)} Y_k = 1 + \frac{k}{2n} + O(n^{-2})$ as n approaches infinity through even values.

Proof. To prove (a), from Proposition 4.3.2 we have previous formulas for $\mathbb{E}_o^{(n)} Y_k$ depending on the parity of n. These are fractions which have numerators and denominators which are polynomials in n; we can write out the two highest-degree terms of each polynomial and simplify. To prove (b) we proceed similarly from Proposition

4.3.1. $\square$

Note that the expected number of elements in k-cycles of permutations with all cycle lengths even (or odd) approaches 1 as n gets large, if k has the appropriate parity. Assume we are dealing with permutations with all cycle lengths even. Naively, we might add the limits of the expected number of elements in 2-cycles, 4-cycles, ..., n-cycles, and expect to get n. But these are each 1; their sum is $n/2$. Since each element is in a cycle, we must have $\sum_{k=1}^{n} \mathbb{E}_e^{(n)} Y_k = n$. The difficulty is that the convergence of $\mathbb{E}_e^{(n)} Y_k$ as $n \to \infty$ is not uniform over k. Under the correct scaling, then, subtler phenomena can be seen; the "missing" elements end up disproportionately in the longer cycle lengths for permutations with all cycle lengths even. We note that similar phenomena of nonuniform convergence have previously been observed in random mappings, for example in [FO90].

Proposition 4.3.4. *Fix $\epsilon \in (0,1)$. The expected number of elements in k-cycles in a random permutation of $[n]$ with all cycle lengths even satisfies uniformly*

$$\mathbb{E}_e^{(n)} Y_k \to \left(1 - \frac{k}{n}\right)^{1/2}$$

as $k, n \to \infty$ with $0 < k/n < 1 - \epsilon$.

Proof. The result of Proposition 4.3.1 can be rewritten in terms of factorials as

$$\mathbb{E}_e^{(n)} Y_k = 2^k \left(\frac{(n/2)!}{((n-k)/2)!}\right)^2 \frac{(n-k)!}{n!}$$

and by Stirling's approximation and routine simplifications, we have

$$\mathbb{E}_e^{(n)} Y_k = \left(1 - \frac{k}{n}\right)^{-1/2} \frac{1 + \frac{1}{4n} + O(n^{-2})}{1 + \frac{1}{4(n-k)} + O((n-k)^{-2})}. \tag{4.10}$$

Let $n, k \to \infty$ with $0 < k/n < 1 - \epsilon$. Then we have $1/(4(n-k)) \in [(4n)^{-1}, (4\epsilon n)^{-1})]$, and so $O((n-k)^{-2}) = O(n^{-2})$. Furthermore $1/(4(n-k)) = O(n^{-1})$, with the constant implicit in the O-notation being $(4\epsilon)^{-1}$. Therefore

$$\mathbb{E}_e^{(n)} Y_k = \left(1 - \frac{k}{n}\right)^{-1/2} (1 + O(n^{-1}))$$

uniformly, as $k, n \to \infty$ with $0 < k/n < 1 - \epsilon$. $\square$

Furthermore, we can essentially integrate the result of Proposition 4.3.4 to determine the cumulative distribution function of the length of the cycle containing a random element of a random permutation with all cycle lengths even (or odd). This is the content of the next theorem.

Theorem 4.3.5. *Fix constants* $0 \leq \gamma \leq \delta \leq 1$. *Let* $p_e(n; \gamma, \delta)$ *be the probability that* 1 *is contained in a cycle of length between* γn *and* δn *of a permutation chosen uniformly at random from all permutations of* $[n]$ *with all cycle lengths even. Then*

$$\lim_{n \to \infty} p_e(n; \gamma, \delta) = \sqrt{1 - \gamma} - \sqrt{1 - \delta}.$$

Since the measure $\mathbb{P}_e$ is invariant under conjugation, this is the probability that an element of $[n]$ chosen uniformly at random is in a cycle of length between γn and δn in a random permutation of $[n]$ with all cycle lengths even.

Proof. Note that

$$\sum_{2|k} \frac{z^k}{k} = \frac{1}{2} \log \frac{1+z}{1-z} = \frac{1}{2} \log \frac{1}{1-z} + \log 2 + o(1)$$

and apply Theorem 4.2.7. $\square$

The same is true for permutations with all cycle lengths odd; like those with all cycle lengths even they fall in the "$\sigma = 1/2$ class".

We now move to consider the mean and variance of the total number of cycles of all lengths.

Theorem 4.3.6. *The mean number of cycles of a randomly chosen permutation of* $[n]$ *with all cycle lengths odd is, as* $n \to \infty$*,*

$$\frac{1}{2}\log n + \frac{\gamma + 3\log 2}{2} \pm \frac{\gamma + \log n}{8n} + O\left(\frac{\log n}{n^2}\right)$$

where we take the $+$ *sign if* n *is odd and the* $-$ *sign if* n *is even. The variance of the number of cycles is, as* $n \to \infty$*,*

$$\frac{1}{2}\log n + \frac{\gamma + 3\log 2 - 4\pi^2}{8} + O\left(\frac{\log^2 n}{n}\right).$$

Proof. We have the exponential generating function counting such permutations by size and number of cycles, $\left(\frac{1+z}{1-z}\right)^{u/2}$. We can differentiate to obtain the mean and variance of the number of cycles. These are given by

$$\mu_n := \frac{[z^n]\frac{1}{2}\sqrt{\frac{1+z}{1-z}}\log\frac{1+z}{1-z}}{[z^n]\sqrt{\frac{1+z}{1-z}}}, \sigma_n^2 := \frac{[z^n]\frac{1}{4}\sqrt{\frac{1+z}{1-z}}\log^2\frac{1+z}{1-z}}{[z^n]\sqrt{\frac{1+z}{1-z}}} + \mu_n - \mu_n^2.$$

Let $f_r(z) = \sqrt{\frac{1+z}{1-z}}\log^r\frac{1+z}{1-z}$ for $r = 0, 1, 2$ and let $a_r(n) = [z^n]f_r(z)$ for $r = 0, 1, 2$. Then we have

$$\mu_n = \frac{a_1(n)}{2a_0(n)}, \sigma_n^2 = \frac{a_2(n)}{4a_0(n)} + \mu_n - \mu_n^2 \tag{4.11}$$

and we need to find asymptotic series for the $a_r(n)$ as $n \to \infty$. We observe that $a_0(n)$ is the number of permutations of $[n]$ with all cycle lengths odd, which is $(n-1)!!^2/n!$

if n is even and $n!!(n-2)!!/n!$ if n is odd; Stirling's formula gives an asymptotic expansion, depending on the parity of n. To find a series for $a_1(n)$ as $n \to \infty$, we expand $f_1(z) = \sqrt{\frac{1+z}{1-z}}$ in a series with terms which are half-integral powers of $1-z$. From this we derive a series for $\sqrt{\frac{1+z}{1-z}} \log \frac{1+z}{1-z}$ with terms of the form $(1-z)^{i-1/2}L^j$ where $L = \log 1/(1-z)$. The function being expanded is analytic in the complex plane slit along the real half-line $\{z \in \mathbb{R} : z \geq 1\}$; by Theorem 2.3.6, an error of $O((1-z)^{i-1/2}L)$ in the series for $f_1(z)$ leads to an error $O(n^{-i-1/2}\log n)$ in the series for $a_1(n)$. We can thus transfer an asymptotic expansion for $f_1(z)$ near $z = 1$ to give an expansion for $a_1(n)$ as $n \to \infty$, and similarly for $f_2(z)$ and $a_2(n)$. Combining these series as specified by (4.11) gives the result. □

We observe that this is $(\log 2) + o(1)$ more than the number of cycles of a permutation of $[n]$ with all cycle lengths even.

The following two results give a decomposition of the number of cycles of permutations with all cycle lengths even into a sum of Bernoulli random variables.

Theorem 4.3.7. *The generating function of permutations of* $[2n]$ *with all cycle lengths even, counted by their number of cycles, is*

$$p_{2n}(u) = [u(u+2)(u+4)\cdots(u+(2n-2))] \cdot (2n-1)!! \tag{4.12}$$

Proof. The bivariate generating function for permutations with all cycle lengths even, counted by their size and number of cycles, is $(1-z^2)^{-u/2}$. Let $p_k(u)$ be the desired generating function. Then we have

$$(1-z^2)^{-u/2} = p_0(u) + p_1(u)z + p_2(u)\frac{z^2}{2!} + \cdots$$

and it is clear that p_k is the zero polynomial for odd k. For even k, the binomial theorem gives

$$(1-z^2)^{-u/2} = 1 + \binom{-u/2}{1}(-z^2) + \binom{-u/2}{2}(-z^2)^2 + \cdots$$

and so we have $p_{2n}(u) = (2n)!\binom{-u/2}{n}$ by comparing coefficients; this can be expanded to give the expression above. □

A combinatorial proof is also possible. Recall that we can write a permutation π of $[n]$ in terms of its *inversion table*, a sequence of integers $a_1, a_2, \ldots, a_n$, with $a_i = |\{j : j < i, \pi(j) > \pi(i)\}|$. The number of zeros in the sequence $(a_1, \ldots, a_n)$ is the number of left-to-right maxima of π. The "fundamental correspondence" between permutations written in cycle notation and in one-line notation takes permutations with k cycles to those with k left-to-right maxima; furthermore, permutations with all cycle lengths even are taken to those with all left-to-right maxima in odd positions, and conversely. Thus it suffices to show that $p_n(u)$ is the generating function of permutations with all left-to-right maxima in odd positions, counted by their number of maxima; this is done by considering the inversion table.

Theorem 4.3.8. *The number of cycles C_n of a random permutation of $[2n]$ with all cycle lengths even, as $n \to \infty$, is asymptotically normally distributed with*

$$\mathbb{E}(C_n) = \frac{1}{2}\log n + \left(\frac{1}{2}\gamma + \log 2\right) + O(n^{-1}) \tag{4.13}$$

and

$$\mathbb{V}(C_n) = \frac{1}{2}\log n + \left(\frac{1}{2}\gamma + \log 2 - \frac{\pi^2}{8}\right) + O(n^{-1}) \tag{4.14}$$

Proof. Let $n = 2m$. From Theorem 4.3.7, we have $C_m = \sum_{k=1}^m X_{m,k}$ where the $X_{m,k}$ are independent Bernoulli random variables with $\mathbb{P}(X_{m,k} = 1) = 1/(2k-1)$. The formula (4.13) for the expectation follows from the asymptotic series for the harmonic numbers. The variance is given by

$$\mathbb{V}C_m = \sum_{k=1}^m \left(\frac{1}{2k-1} - \left(\frac{1}{2k-1}\right)^2\right) = \mathbb{E}C_m - \sum_{k=1}^m \frac{1}{(2k-1)^2}.$$

and we need to consider the second sum. We have $\sum_{j=1}^m \frac{1}{j^2} = -\psi'(m+1) + \pi^2/6$, so

$$\mathbb{V}C_m = \mathbb{E}C_m - \left(-\psi'(2n+1) + \frac{1}{4}\psi'(n+1) + \frac{\pi^2}{8}\right).$$

But $\psi'(m) = O(m^{-1})$, so in fact we get $\mathbb{V}C_m = \mathbb{E}C_m - \frac{\pi^2}{8} + O(1/m)$; from this and (4.13) we get (4.14). Asymptotic normality follows from the Lindeberg-Feller central limit theorem (Theorem 2.4.3). □

There is not such a simple decomposition for permutations with all cycle lengths odd. However, it appears that the polynomials counting permutations of $[n]$ with all cycle lengths odd by their number of cycles have only pure imaginary roots. If this is true, then the number of cycles of a random permutation of $[n]$ with all cycle lengths odd can be decomposed into a sum of $\lfloor n/2 \rfloor$ independent $\{0,2\}$-valued random variables, plus 1 if n is odd. It may be of interest to study the zeros of these polynomials.

4.4 Boltzmann sampling

We have at this point seen substantial similarities between permutations with all cycle lengths having the same parity and permutations with cycle weights 1/2. This suggests that an average of weights is in some sense a more fundamental parameter than the individual weights. This has been anticipated by the notion of a *function of logarithmic type* [FS90], which has been used in the study of permutations [Han94]. Let $\Delta_0(\rho, \eta) = \{z : |z| < \rho + \eta, z \notin [\rho, \rho + \eta]\}$. A function $G(z)$ is called *logarithmic* if it is of the form

$$G(z) = a \log \frac{1}{1 - z/\rho} + R(z)$$

for some constant multiplier a and function $R(z)$, where $R(z)$ is analytic in Δ_0 and satisfies $R(z) = K + o(1)$ for some constant K as $z \to \rho$ in Δ_0, and ρ is the unique dominant singularity of G on its circle of convergence. In [FS90, Prop. 1] structures having components enumerated by a function of logarithmic type $G(z)$ are considered; for such structures of size n, the expected number of cycles is $a \log n + O(1)$, as is the variance. However, the structures considered in this chapter have not all had components counted by functions of logarithmic type. For example, the components of permutations with all cycle lengths even are counted by the exponential generating function $\frac{1}{2} \log \frac{1}{1-z^2}$, which has singularities at $z = \pm 1$ and thus does not have a unique dominant singularity.

The following conjecture, in the light of these averaging phenomena, seems natural. It is supported by Theorem 4.4.3, an analogous result on "Boltzmannized" permutations.

Conjecture 4.4.1. *Let $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots)$ be a sequence of nonnegative real numbers with mean α, that is, with $\lim_{n\to\infty} \frac{1}{n} \left(\sum_{k=1}^{n} \sigma_k\right) = \alpha$. Then permutations of $[n]$ selected according to the weights $\vec{\sigma}$ have an asymptotically Gaussian number of cycles as $n \to \infty$, with mean and variance asymptotic to $\alpha \log n$.*

Definition 4.4.2. *Let $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots)$ be a weighting sequence, and let x be a positive real parameter. Let $|\pi|$ denote the size of the ground set of a permutation π. Then we define the $\vec{\sigma}$-weighted Boltzmann measure with parameter x on permutations, a probability measure on $\bigcup_{k=0}^{\infty} S_k$, by*

$$\mathbb{P}_{\vec{\sigma},x}(\pi) = \frac{w_{\vec{\sigma}}(\pi) \cdot \frac{x^{|\pi|}}{|\pi|!}}{\exp\left(\sum_{k\geq 1} \sigma_k x^k / k\right)}$$

(See Definition 4.1.1 for the weight $w_{\vec{\sigma}}(\pi)$.)

For any choice of $\vec{\sigma}$ and x, $\mathbb{P}_{\vec{\sigma},x}$ is a probability measure. It suffices to show that $\mathbb{P}_{\vec{\sigma},x}$ has total mass 1. But $\sum_{k\geq 1} \sigma_k x^k / k$ is the weighted generating function of cycles, and we can apply the exponential formula. Thus $\mathbb{P}_{\vec{\sigma},x}$ is a straightforward weighted generalization of the Boltzmann measure on labelled objects studied in Chapter 3. We also retain the formulas from Proposition 3.3.1 for the expected size and the variance of the size of the objects chosen according to this measure. Let $C(x)$ be the exponential generating function of a labelled combinatorial class, and N the size of a random object chosen from that class according to the Boltzmann measure. Then

$$\mathbb{E}_{\vec{\sigma},x}(N) = \frac{x \frac{d}{dx} C(x)}{C(x)}, \mathbb{E}_{\vec{\sigma},x}(N^2) = \frac{\left(x \frac{d}{dx}\right)^2 C(x)}{C(x)}.$$

We now assemble a sequence of lemmas. These lemmas will be used to prove the following theorem, which is the main result of this section.

Theorem 4.4.3. *Let $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots)$ be a weighting sequence with mean α. Let $x = x(\mu)$ be chosen so that $\mathbb{E}_{\vec{\sigma},x}(N) = \mu$. Let X be a random variable denoting the number of cycles of a permutation. Then $\mathbb{E}_{\vec{\sigma},x}(X) = \mathbb{V}_{\vec{\sigma},x}(X) \sim \alpha \log \mu$ as $x \to 1^-$ or $\mu \to \infty$.*

The main analytic result needed follows.

Lemma 4.4.4. *[PS98, Exercise I.88] Let $b_0, b_1, \ldots$ be positive real numbers, such that $\sum_{n=0}^{\infty} b_n$ is divergent, and $\sum_{k\geq 0} b_k t^k$ is convergent for $0 \leq t < 1$. Then*

$$\lim_{n\to\infty} \frac{a_0 + a_1 + \cdots + a_n}{b_0 + b_1 + \cdots + b_n} = s \text{ implies } \lim_{t\to 1^-} \frac{\sum_{k\geq 0} a_k t^k}{\sum_{k\geq 0} b_k t^k} = s.$$

Lemma 4.4.5. *Let $\sigma_1, \sigma_2, \sigma_3, \ldots$ be a sequence of real numbers such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma_k = \alpha.$$

Then $\sum_{k=1}^{\infty} \sigma_k x^k = \frac{\alpha}{1-x} + o((1-x)^{-1})$.

Proof. Apply Lemma 4.4.4 with $a_k = \sigma_k, b_k = 1$. □

Lemma 4.4.6. *Let $\{\sigma_k\}_{k=1}^{\infty}$ be a sequence of nonnegative real numbers, bounded above, such that $\sum_{k=1}^{n} \sigma_k \sim \alpha n$ as $n \to \infty$, for some constant $\alpha > 0$. Then $\sum_{k=1}^{n} \frac{\sigma_k}{k} \sim \alpha \log n$ as $n \to \infty$.*

Proof. We begin by showing that if $\int_1^n f(x)\,dx \sim n$ as $n \to \infty$ for some function f such that $f(x)/x$ is integrable on $[1, \infty)$, then $\int_1^n \frac{f(x)}{x}\,dx \sim \log n$ as $n \to \infty$. Let $F(x) = \int_1^n f(x)\,dx$. We integrate $\int_1^n \frac{f(x)}{x}\,dx$ by parts, getting

$$\int_1^n \frac{f(x)}{x}\,dx = \frac{F(n)}{n} - \frac{F(1)}{1} + \int_1^n \frac{F(x)}{x^2}\,dx.$$

Clearly $F(1) = 0$, and $F(n) \sim n$ by assumption, so

$$\int_1^n \frac{f(x)}{x}\,dx = 1 + \int_1^n \frac{F(x)}{x^2}\,dx + o(1)$$

Since $F(x) \sim x$ as $x \to \infty$, the integrand satisfies $F(x)/x^2 \sim 1/x$, and so

$$\int_1^n \frac{F(x)}{x^2}\,dx \sim \int_1^n \frac{1}{x}\,dx = \log n,$$

proving the claim.

Now, we need to check that this statement about integrals translates into an analogous one about sums. Let $\{\sigma_k\}_{k=1}^{\infty}$ be as in the hypothesis, and let $f(x) = \sigma_{\lfloor x \rfloor}$. Then we want to show that

$$\int_1^{n+1} \frac{f(x)}{x}\,dx - \sum_{k=1}^{n} \frac{\sigma_k}{k} = o(\log n)$$

as $n \to \infty$. We have

$$\begin{aligned}
\int_1^{n+1} \frac{f(x)}{x}\,dx - \sum_{k=1}^{n} \frac{f(k)}{k} &= \sum_{k=1}^{n} \left(\int_k^{k+1} \frac{f(x)}{x}\,dx - \frac{f(k)}{k} \right) \\
&= \sum_{k=1}^{n} f(k) \left(\log\left(1 + \frac{1}{k}\right) - \frac{1}{k} \right)
\end{aligned}$$

and so, since $|\log(1 + 1/k) - 1/k| \leq 1/2k^2$ for positive integer k,

$$\left| \int_1^{n+1} \frac{f(x)}{x}\,dx - \sum_{k=1}^{n} \frac{f(k)}{k} \right| \leq \left| \sum_{k=1}^{n} \frac{f(k)}{2k^2} \right|.$$

Since $\{f(k)\}_{k=1}^{\infty}$ is bounded, the sum on the right-hand side is convergent. We have $\int_1^{n+1} f(x)/x \, dx \sim \log n$, so $\sum_{k=1}^{n} f(k)/k \sim \log n$ as well. Thus we have proven the lemma for $\alpha = 1$. Multiplying through by α gives the desired result. $\square$

Lemma 4.4.7. *Let $\sigma_1, \sigma_2, \sigma_3, \ldots$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma_k = \alpha$. Then*

$$\sum_{k=1}^{\infty} \frac{\sigma_k x^k}{k} = \alpha \log \frac{1}{1-x} + o\left(\log \frac{1}{1-x}\right) \tag{4.15}$$

Proof. Applying Lemma 4.4.6 to the hypothesis, $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\sigma_k}{k} = \alpha$. We apply Lemma 4.4.4 with $a_k = \sigma_k/k, b_k = 1/k$. This gives us

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \sigma_k/k}{1 + H_n} = \lim_{x\to 1^-} \frac{\sum_{k\geq 1} \sigma_k x^k/k}{\sum_{k\geq 1} x^k/k}.$$

Now, $\sum_{k\geq 1} x^k/k = \log(1/(1-x))$, and $1 + H_n \sim \log n$, so we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{\sigma_k}{k} = \lim_{x\to 1^-} \frac{\sum_{k\geq 1} \sigma_k x^k/k}{\log(1/(1-x))}.$$

Thus the right-hand side here has value α, proving (4.15). $\square$

Proof of Theorem 4.4.3. Note that for the Boltzmann measure with parameter x and weight sequence $\vec{\sigma}$, we have $C(x) = \exp\left(\sum_{k\geq 1} \sigma_k x^k/k\right)$. Thus the distribution of sizes N under this measure has expectation

$$\mathbb{E}_{\vec{\sigma},x}(N) = x \frac{d}{dx} \left(\sum_{k\geq 1} \frac{\sigma_k x^k}{k} \right) = \sum_{k\geq 1} \sigma_k x^k$$

Furthermore, the Boltzmann distribution $\mathbb{P}_{\vec{\sigma},x}$ can be obtained by taking $\mathcal{P}(\sigma_k x^k/k)$ cycles of length k for each $k \geq 1$ and forming uniformly at random a permutation with the resulting cycle type. The mean and variance of the number of cycles chosen from the distribution $\mathbb{P}_{\vec{\sigma},x}$ is thus exactly $\sum_{k\geq 1} \sigma_k x^k/k$. Since we have $\mathbb{E}_x(N) \sim \alpha/(1-x)$ as $x \to 1^-$ by Lemma 4.4.5, we can solve for x to see that $1 - \frac{\alpha}{\mathbb{E}_x(N)} \sim x$ as $x \to 1^-$. Therefore

$$\sum_{k\geq 1} \sigma_k x^k/k \sim \alpha \log \frac{1}{1-x} \sim \alpha \log \frac{1}{1 - \left(1 - \frac{\alpha}{\mathbb{E}_x(N)}\right)} = \alpha \log \frac{\mathbb{E}_x(N)}{\alpha} \sim \alpha \log \mathbb{E}_x(N)$$

which is the desired result. □

It would be desirable to translate Theorem 4.4.3 into a result about permutations of a fixed size selected uniformly at random; this is one possible way of proving Conjecture 4.4.1. Note that $\mathbb{P}_{\vec{\sigma},x}$ is a mixture of the various $\mathbb{P}_{\vec{\sigma}}^{(n)}$. It is often possible to prove results about a family of measures P_λ, parametrized by λ, which are mixtures of well-understood measures $\mathbb{P}^{(n)}$, where we draw from $\mathbb{P}^{(n)}$ with probability $e^{-\lambda}\lambda^n/n!$; this goes by the name of analytic de-Poissonization [JS98, Szp01]. Informally, we pick from $\mathbb{P}^{(N)}$ where N is Poisson with parameter λ. In the case described here we can get results on permutations chosen from $\mathbb{P}^{(N)}$ where N is the size of objects from a Boltzmann distribution; thus techniques of "analytic de-Boltzmannization" will be necessary to achieve this goal.

4.5 Periodic sequences of weights

Many of the results of Section 4.3 can be easily generalized to random permutation models with periodic weighting sequences. In particular, consider the case where the weight sequence is $\vec{\sigma} = (\sigma, \tau, \sigma, \tau, \ldots)$ – that is, where cycles receive a weight determined only by their parity. These have the class decomposition $\textsc{Set}(\sigma\textsc{Cyc}_e(\mathcal{Z}) + \tau\textsc{Cyc}_o(\mathcal{Z}))$. The exponential generating function counting $\vec{\sigma}$-permutations is

$$\exp\left(\tau\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) + \sigma\left(\frac{z^2}{2} + \frac{z^4}{4} + \frac{z^6}{6} + \cdots\right)\right) = \frac{(1+z)^{(\tau-\sigma)/2}}{(1-z)^{(\tau+\sigma)/2}}.$$

This function has singularities at $z = 1$, and at $z = -1$ if $\tau - \sigma$ is not a positive even integer. By Theorem 2.3.10 we can analyze each singularity separately. Around $z = 1$ the function resembles $\frac{2^{(\tau-\sigma)/2}}{(1-z)^{(\tau+\sigma)/2}}$. The contribution of this singularity to the sum of the weights of all permutations of $[n]$ is therefore asymptotic to $2^{(\tau-\sigma)/2}[z^n](1-z)^{-(\tau+\sigma)/2}n!$, or

$$\frac{2^{(\tau-\sigma)/2}}{\Gamma((\tau+\sigma)/2)} n^{(\tau+\sigma)/2-1} n!.$$

Around $z = -1$ the function resembles $(1+z)^{(\tau-\sigma)/2}/2^{(\tau+\sigma)/2}$; the contribution from this singularity to the nth coefficient is therefore asymptotic to $2^{-(\tau+\sigma)/2}[z^n](1+z)^{(\tau-\sigma)/2}$. This is an alternating sequence with nth term having absolute value of order $n^{(\tau-\sigma)/2-1}$, and is therefore negligible compared to the previous term unless $\sigma = 0$. In this case, however, we are actually dealing with permutations with all cycle lengths even.

Thus the sum of all weights of $\vec{\sigma}$-permutations is

$$\frac{2^{(\tau-\sigma)/2}}{\Gamma((\tau+\sigma)/2)} n^{(\tau+\sigma)/2-1} n!(1 + O(n^{-\sigma}))$$

More generally, consider $\vec{\sigma}$-weighted permutations for an r-periodic sequence $\vec{\sigma}$.

Proposition 4.5.1. *Let $\vec{\sigma}$ be an r-periodic sequence with mean σ, and let $\tau_s = \sigma_s - \sigma$. The total weight of all $\vec{\sigma}$-weighted permutations of $[n]$ is asymptotic to*

$$n!\frac{n^{\sigma-1}}{\Gamma(\sigma)} \exp\left(-\frac{1}{r}\sum_{s=1}^{r} \tau_s \psi(s/r)\right)$$

as $n \to \infty$.

Proof. Such permutations have the generating function $\exp \sum_{k\geq 1} \sigma_k z^k/k$. We consider the coefficients $[z^n]\left(\exp \sum_{k\geq 1} \sigma_k z^k/k\right)$. The dominant singularities of this function are at the rth roots of unity. In particular at $z = 1$, this function behaves like $(1-z)^{-\sigma} e^K$, where $K = \sum_{k\geq 1} \frac{(\sigma_k - \sigma)}{k}$. For ease of notation let $\tau_k = \sigma_k - \sigma$. Now

$$\sum_{k=0}^{n-1} \frac{1}{kr+s} = \frac{1}{r}\left(\psi\left(n+\frac{s}{r}\right) - \psi\left(\frac{s}{r}\right)\right)$$

and therefore

$$\sum_{k=1}^{rn} \frac{\tau_k}{k} = \frac{1}{r}\sum_{s=1}^{r} \tau_s \psi(n+s/r) - \frac{1}{r}\sum_{s=1}^{r} \tau_s \psi(s/r).$$

As $n \to \infty$ the first sum on the right-hand side is $O(n^{-1})$. This can be seen from the fact that $\psi(n+\alpha) = \log n + O(n^{-1})$ as $n \to \infty$ and that $\sum_{s=1}^{r} \tau_s = 0$. Therefore

$$\sum_{k=1}^{rn} \frac{\tau_k}{k} = -\frac{1}{r}\sum_{s=1}^{r} \tau_s \psi(s/r) + O(n^{-1})$$

and at last

$$\sum_{k=1}^{\infty} \frac{\tau_k}{k} = -\frac{1}{r}\sum_{s=1}^{r} \tau_s \psi(s/r).$$

The asymptotics are thus those of $[z^n](1-z)^{-\sigma} e^K$; recalling that $[z^n](1-z)^{-\sigma} \sim n^{\sigma-1}/\Gamma(\sigma)$ gives the result. $\square$

Finally, to write explicit formulas for the number of $\vec{\sigma}$-weighted permutations when $\vec{\sigma}$ is periodic, we can use Gauss's digamma theorem [Knu, vol. 1, p. 94] for evaluating ψ at rational arguments:

$$\psi(p/q) = -\gamma - \log(2q) - \frac{\pi}{2}\cot\frac{p\pi}{q} + 2\sum_{k=1}^{\lceil q/2\rceil - 1} \cos\frac{2\pi pk}{q}\log\left(\sin\frac{\pi k}{q}\right). \tag{4.16}$$

These explicit formulas are particularly appealing when the trigonometric functions in (4.16) have simple values. In the case with 3-periodic weights, let $\vec{\sigma} = (\sigma + a, \sigma + b, \sigma + c, \sigma + a, \sigma + b, \sigma + c, \ldots)$ with $a + b + c = 0$. Then the total weight of all $\vec{\sigma}$-permutations of n is asymptotic to

$$\frac{n^{\sigma-1}}{\Gamma(\sigma)} n! 3^{(a+b)/2} \exp\left(\frac{1}{18}\sqrt{3}\pi(a-b)\right).$$

In the 4-periodic case, with $\vec{\sigma} = (\sigma + a, \sigma + b, \sigma + c, \sigma + d)$ and $a + b + c + d = 0$, the total weight is asymptotic to

$$\frac{n^{\sigma-1}}{\Gamma(\sigma)} n! 2^{(3a+2b+3c)/4} \exp\left(\frac{\pi}{8}(a-c)\right).$$

And in the 6-periodic case, the total weight is asymptotic to

$$\frac{n^{\sigma-1}}{\Gamma(\sigma)} n! 2^{(a+c+e)/3} 3^{(a+b+d+e)/4} \exp\left((3a + b - d - 3e)\frac{\pi\sqrt{3}}{36}\right).$$

Example 4.5.2. *The probability that a permutation has all its cycle lengths congruent to* 1 *or* 5 *mod* 6 *can be obtained from the last of these formulas. The weight vector is* $\vec{\sigma} = (1, 0, 0, 0, 1, 0, \ldots)$ *with period* 6*. So we have* $\sigma = 1/3$*, the mean of these numbers. We have* $a = e = 1 - \sigma = 2/3$ *and* $b = c = d = f = 0 - \sigma = -1/3$*. This gives*

$$[z^n]\exp\left(\sum_{\substack{k=1 \\ k\equiv\pm1 \pmod 6}}^{\infty} \frac{z^k}{k}\right) \sim \frac{n^{-2/3}}{\Gamma(2/3)} 2^{1/3} 3^{1/6}.$$

In practice this is difficult to observe; a full asymptotic expansion will include terms which oscillate modulo 6 and decay very slowly.

From these enumerations it is possible to prove analogues of some of the results of previous sections. For example an analogue of Theorem 4.2.6, which states that $\mathbb{E}_\sigma^{(n)} Y_{\alpha n} \sim \sigma(1-\alpha)^{\sigma-1}$, holds, where σ is now the mean weight. An analogue of Theorem 4.2.7 – the integrated version of Theorem 4.2.6 – likely also holds, as do results on the normal distribution of the number of cycles.

4.6 Permutations with reciprocal weights

Consider weighted permutation models with cycle weights σ_k. Then a recent theorem of Betz et al. [BUV09, Thm. 3.1] states:

Theorem 4.6.1 (Betz, Ueltschi, Velenik)**.** *Assume that $\sigma_{n-j}\sigma_j/\sigma_n \le c_j$ for all n and for $1 \le j \le n/2$, for constants c_j satisfying $\sum_{j\ge 1} c_j/j < \infty$. Let ℓ_1 be the length of the cycle containing* 1*. Then*

$$\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{P}(l_1 > n-m) = 1.$$

We consider the particular model with $\sigma_k = 1/k$. Then we have $\sigma_{n-j}\sigma_j/\sigma_n = n/(j(n-j))$; recalling that $n \ge 2j$, we may take $c_j = 2/j$. Thus $\sum_{j\ge 1} c_j/j = \pi^2/3 < \infty$. Therefore the conclusion of the theorem holds with these weightings; that is, almost all indices belong to a single giant cycle.

Betz et al. ask if short cycles occur at all; their Theorem 3.2 states that under the conditions of the previous theorem, with probability bounded away from zero there are no short cycles. They also show that there exists some $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, random permutations of $[n]$ with cycle weights $\sigma_j = \lambda/j$ have cycles of length 1 with probability bounded away from zero. However, due to the generality of their paper they do not make these bounds explicit.

These permutations have generating function $\exp(\sum_k z^k/k^2) = \exp L(z)$, and the total number of cycles in all such permutations (with the permutations counted according to their weight) has generating function $L(z) \exp L(z)$. Unfortunately the dilogarithm $L(z)$, customarily expressed as an analytic continuation of this sum, causes some difficulty as it has a branch cut starting at $z = 1$ and going infinitely to the right. Thus the generating functions $\exp L(z)$ and $L(z) \exp L(z)$ are problematic for singularity analysis. Graham, Knuth, and Patashnik [GKP94, pp. 464-466 and Exercise 9.23] give the formula

$$b_n := [z^n] \exp L(z) = e^{\pi^2/6} \left(\frac{n + 2\log n + O(1)}{n^3} \right)$$

The expected number of m-cycles of a permutation of $[n]$ chosen with these weights is

$$\lim_{n\to\infty} \frac{[z^n] \frac{z^m}{m^2} \exp\left(\sum_k z^k/k^2\right)}{[z^n] \exp\left(\sum_k z^k/k^2\right)} = \frac{1}{m^2} \frac{b_{n-m}}{b_n}.$$

Now, $b_{n-1}/b_n = 1 + O(1/n)$ as $n \to \infty$, and so $b_{n-m}/b_n = 1 + O(1/n)$ as $n \to \infty$ for any positive integer m. Thus we see that the expected number of m-cycles in a large permutation is $1/m^2$.

Proposition 4.6.2. *The limiting distribution of the number of k-cycles in weighted permutations with $\sigma_k = 1/k$ is Poisson with mean $1/k^2$.*

Proof. The generating function $\exp(L(z)+(u-1)z^k/k^2)$ counts weighted permutations with their k-cycles marked. The probability that such a permutation has exactly m cycles of length k is given by

$$\frac{[z^n u^m] \exp(L(z) + (u-1)z^k/k^2)}{[z^n] \exp L(z)}.$$

The asymptotics of the denominator are known. The numerator is in fact

$$\frac{1}{m!k^{2m}}[z^{n-km}] \exp(L(z) - z^k/k^2).$$

and so the desired limiting probability is

$$\frac{1}{m!k^{2m}} \lim_{n\to\infty} \frac{[z^{n-km}] \exp(L(z) - z^k/k^2)}{[z^n] \exp(L(z)}.$$

Now, $[z^n] \exp(L(z) - z^k/k^2) = \Theta(n^{-2})$ and so the numerator is slowly varying. Thus this limit is the same as

$$\frac{1}{m!k^{2m}} \lim_{n\to\infty} \frac{[z^n] \exp(L(z) - z^k/k^2)}{[z^n] \exp(L(z)}.$$

By the following lemma, this limit is e^{-1/k^2}, so

$$\mathbb{P}_n(X_k = m) = \frac{1}{m!k^{2m}} e^{-1/k^2} = \mathbb{P}(\mathcal{P}(1/k^2) = m).$$

□

Lemma 4.6.3.

$$\lim_{n\to\infty} \frac{[z^n] \exp(L(z) - z^k/k^2)}{[z^n] \exp L(z)} = e^{-1/k^2}.$$

Proof. We follow the proof that $[z^n]\exp L(z) \sim e^{\pi^2/6}/n^2$ given by Graham, Knuth, and Patashnik. Let $H(z) = \exp(L(z) - z^k/k^2)$. We begin with the equation $H(z) = \exp \sum_{k\neq j} 1/k^2$ and differentiate both sides. Equating coefficients in the result gives the recurrence

$$nh_n = \sum_{k<n, k\neq n-j} \frac{h_k}{n-k}$$

Now, let $G(z) = \exp L(z) = \sum_{n\geq 0} g_n z^n$. It is known that $g_n = O(n^{-2}\log^2 n)$. Since $0 \leq h_n \leq g_n$, we have $h_n = O(n^{-2}\log^2 n)$ as well. At this point we can write

$$\begin{aligned} nh_n &= \sum_{k<n, k\neq n-j} \frac{h_k}{n-k} \\ &= \sum_{k<n} \frac{h_k}{n-k} - \frac{h_{n-j}}{j} \\ &= \frac{1}{n}\sum_{k\geq 0} h_k - \frac{1}{n}\sum_{k\geq n} h_k + \frac{1}{n}\sum_{k<n} \frac{kh_k}{n-k} - \frac{h_{n-j}}{j}. \end{aligned}$$

The first sum is $H(1) = \exp(\pi^2/6 - 1/j^2)$. The second and third sums are $O((\log n)^2/n)$ and $O((\log n)^3/n)$ by corresponding bounds on g_k given in [GKP94]. The term h_{n-j}/j is $O(n^{-2}\log^2 n)$. Thus we have $nh_n = n^{-1}\exp(\pi^2/6 - 1/j^2) + O(n^{-2}\log^3 n)$, which gives $h_n \sim \exp(\pi^2/6 - 1/j^2)n^{-2}$. (More accurate bounds are possible but not needed for our purposes; the earlier bounds on h_n were just scaffolding.) Division gives the desired limit. $\square$

This naturally leads to the conjecture that the expected number of cycles in such permutations is $\sum_{m\geq 1} 1/m^2 = \pi^2/6$. However, these cycles are in general not enough to fill the n-element space. The typical structure of such permutations appears to be *Poisson*$(\pi^2/6)$ "short" (essentially $O(1)$ in length) cycles and one "long" cycle of length $n - O(1)$. More generally, we conjecture that whenever the series $\sum_{k\geq 1} \sigma_k/k$ converges, the number of cycles of a random $\vec{\sigma}$-weighted permutation converges in distribution to $1 + \mathcal{P}(\sum_{k\geq 1} \sigma_k/k)$.

4.7 Permutations with roots

One model of random permutations which is not a weighted model, but which has properties in common with many weighted models, are the *permutations with square roots.*

Proposition 4.7.1. *A permutation σ has a square root – that is, $\sigma = \tau^2$ has a solution – if and only if the numbers of cycles of σ that have each even length are even numbers.*

Proof. Consider a permutation τ. Squaring τ takes each cycle of odd length in τ to a different cycle of odd length, and each cycle of even length to two cycles of half that length. A cycle of even length in τ^2 is therefore the result of splitting a cycle of twice its length in τ into two cycles. So cycles of each even length come in pairs, and the total number of cycles of even length is even.

Given a permutation σ having an even number of cycles of each even length, we can construct a square root τ of σ. If $(a_1, a_2, \ldots, a_{2m+1})$ is a cycle in σ, then let $(a_1, a_{m+1}, a_2, a_{m+3}, \cdots, a_{2m+1}, a_{m+1})$ be a cycle in τ. If $(a_1, \ldots, a_{2k})$ and $(b_1, \ldots, b_{2k})$ are cycles in σ of the same length, then let $(a_1 b_1 a_2 b_2 \cdots a_{2k} b_{2k})$ be a cycle in τ. $\square$

Note that permutations can have multiple square roots; this construction just gives one of them.

At this point it is simple to give the generating function for the number of permutations having a square root. These permutations form the class $\textsc{Set}(\textsc{Cyc}_o(\mathcal{Z})) \times \prod_{k=1}^{\infty} \textsc{Set}_e(\textsc{Cyc}_{2k}(\mathcal{Z}))$ and therefore have the exponential generating function

$$\left(\exp\left(\sum_{j=0}^{\infty} \frac{z^{2j+1}}{2j+1}\right)\right) \prod_{k=1}^{\infty} \cosh\left(\frac{z^{2k}}{2k}\right) = \sqrt{\frac{1+z}{1-z}} \prod_{k=1}^{\infty} \cosh \frac{z^{2k}}{2k}.$$

Pouyanne [Pou02] has shown more generally that a permutation $\sigma \in \mathcal{S}_n$ has an mth root – that is, that $\sigma = \tau^m$ has a solution – if and only if its number of l-cycles is a multiple of $l^\infty \wedge m := \lim_{n\to\infty} \gcd(l^n, m)$. We see that $l^\infty \wedge m = 1$ only when l and m are relatively prime. So in the general case, permutations having mth roots are unrestricted in their cycles of length relatively prime to m.

Pouyanne then shows that the probability that a random permutation has an mth root approaches, as $n \to \infty$, $\pi_m n^{\phi(m)/m-1}$ where

$$\pi_m = \frac{1}{\Gamma(\phi(m)/m)} \prod_{k|m} k^{-\mu(k)/k} \prod_{\substack{l\geq 1 \\ \gcd(l,m)\neq 1}} e_{l^\infty \wedge m}\left(\frac{1}{l}\right).$$

Here μ is the number-theoretic Mobius function and ϕ is the Euler totient function. The probability that a random permutation has all its cycle lengths relatively prime to m is also proportional to $n^{\phi(m)/m-1}$, and so there is some nondegenerate (i. e. neither 0 nor 1) limiting probability that a permutation with an mth root has all cycle lengths relatively prime to m.

Proposition 4.7.2. *The expected number of j-cycles in a permutation of $[n]$ having* *an mth root, for j relatively prime to m, approaches $1/j$ as $n \to \infty$ with j fixed.*

Proof. The generating function counting permutations with mth roots by their size and number of j-cycles is

$$P(z,u) = \exp\left(\frac{(u-1)z^j}{j}\right) P(z)$$

where $P(z)$ is the generating function counting permutations by their size and number of j-cycles. The mean number of j-cycles is found by differentiating, and is

$$\frac{[z^n]\, \partial_u P(z,u)|_{u=1}}{[z^n]P(z,1)} = \frac{1}{j}\frac{[z^{n-j}]P(z)}{[z^n]P(z)}.$$

The coefficients $[z^n]P(z)$ are slowly varying as a function of n, giving the desired result. □

However, the scaling has been chosen poorly here. Making the correct choice of scaling we have

Proposition 4.7.3. *The expected number of elements of αn-cycles in a permutation of $[n]$ admitting an mth root, where αn is relatively prime to m, is asymptotic to $(1-\alpha)^{\phi(m)/m-1}$ as $n \to 0$.*

Proof. We proceed as in the previous proof; the mean number of αn-cycles is

$$\frac{1}{\alpha n}\frac{[z^{(1-\alpha)n}]P(z)}{[z^n]P(z)}.$$

Recalling that $[z^n]P(z) \sim \pi_m n^{\phi(m)/m-1}$ and simplifying gives the desired result. □

If instead we consider those cycle lengths the occurrence of which is restricted, we get a much different result.

Proposition 4.7.4. *The expected number of j-cycles in a permutation of $[n]$ having an mth root, where* $\gcd(j,m) > 1$*, approaches the limit*

$$\frac{1}{j}\frac{e'_{j^\infty \wedge m}(1/j)}{e_{j^\infty \wedge m}(1/j)}$$

as $n \to \infty$. Here $e_d(z) = \sum_{d \geq 0} z^{nd}/(nd)!$ and $'$ denotes differentiation.

Proof. The generating function for permutations with mth roots, with j-cycles marked, is

$$P_m(z,u) = \left(\prod_{k|m}(1-z^k)^{-\mu(k)/k}\right)\left(\prod_{gcd(l,m)\geq 1} e_{l^\infty \wedge m}(z^l/l)\right)\left(\frac{e_{j^\infty \wedge m}(uz^j/j)}{e_{j^\infty \wedge m}(z^j/j)}\right).$$

Differentiating, we find

$$[z^n]\partial_u \, P_m(z,u)\big|_{u=1} = \frac{1}{j}[z^{n-j}]\left(\frac{e'_{j^\infty \wedge m}(z^j/j)}{e^{j^\infty \wedge m}(z^j/j)}\right) P_m(z,1).$$

So it suffices to show that

$$[z^n]\left(\frac{e'_{j^\infty \wedge m}(z^j/j)}{e_{j^\infty \wedge m}(z_j/j)}\right) P_m(z,1) \sim \frac{e'_{j^\infty \wedge m}(1/j)}{e_{j^\infty \wedge m}(1/j)}[z^n]P_m(z,1)$$

We denote the left-hand side by $[z^n]\tilde{P}_m(z)$. Then this follows from the proof on [Pou02, p. 9]. In particular, if we write $C_m(z) = \prod_{k|m}(1-z^k)^{-\mu(k)/k}$ and $R_m(z) = \prod_{\gcd(l,m)\geq 1} e_{l^\infty \wedge m}(z^l/l)$, so $P_m(z) = C_m(z)R_m(z)$. Similarly, write $\tilde{R}_m(z) = e'_{j^\infty \wedge m}(z^j/j)/e_{j^\infty \wedge m}(z^j/j) \cdot R_m(z)$, then $\tilde{P}_m(z) = C_m(z)\tilde{R}_m(z)$. Let $p_n, c_n, r_n, \tilde{p}_n, \tilde{r}_n$ denote the coefficients of the corresponding (uppercase) generating functions. Then $\tilde{p}_n = \sum_k c_{n-k}\tilde{r}_k$. We want to show that $\tilde{\pi}_m = \kappa_m \tilde{R}_m(1)$. It suffices to show that $\lim_{n\to\infty} \sum_{k=0}^n c_{n-k}/c_n \cdot r_k = \sum_{n\geq 0} r_n$, which in fact follows only from properties of $C_m(z)$ (not $R_m(z)$). ◻

In the particular case $m = 2$, $e_2(z) = \cosh z$, and so we get that the expected number of j-cycles in a permutation having a square root is $1/j \tanh 1/j$ when j is even. The expected value of a random variable X with $\mathcal{P}(\lambda)$ distribution, conditioned on X taking even value, is

$$\frac{\sum_{2|n} \frac{ne^{-\lambda}\lambda^n}{n!}}{\sum_{2|n} \frac{e^{-\lambda}\lambda^n}{n!}} = \frac{e^{-\lambda}\lambda \sinh \lambda}{e^{-\lambda}\cosh \lambda} = \lambda \tanh \lambda.$$

For contrast, the *square* of a random permutation has a much different structure. Let Y_k be the number of k-cycles in the square of a permutation chosen uniformly at random. Then $\mathbb{E}_n Y_k = (1/k)([\![2 \nmid k]\!] + [\![k \leq n/2]\!])$. This holds since $Y_k = X_k + 2X_{2k}$ if k is odd, and $2X_{2k}$ if k is even. Recalling that $\mathbb{E}_n X_k = \frac{1}{k}[\![k \leq n]\!]$ gives the desired result. It follows that if $f(\alpha)$ is the piecewise linear function going from $(0,0)$ to $(1/2, 3/4)$ to $(1,1)$, then as $n \to \infty$,

$$\frac{1}{n}\mathbb{E}_n \left(\sum_{k=\alpha n}^{\beta n} kY_k \right) \to f(\beta) - f(\alpha).$$

That is, the probability that in the square of a permutation chosen uniformly at random, a uniform random element of $\{1, \ldots, n\}$ lies in a cycle of length in $[\alpha n, \beta n]$ approaches $f(\beta) - f(\alpha)$. Furthermore, we can easily see that the number of cycles of odd length in a permutation is asymptotically normally distributed with mean and variance $(1/2)\log n$, and the number of cycles of even length has the same distribution. The number of cycles in σ^2 is the number of odd-length cycles of σ, plus twice the number of even-length cycles; thus the number of cycles in the square of a random permutation is asymptotically normal with mean $(3/2)\log n$ and variance $(5/2)\log n$.

This is *more* cycles than a uniform random permutation, while a random permutation which is a square has *less* cycles than a uniform random permutation, about $(\log n)/2$. This is a consequence of the fact that permutations with many cycles have more square roots than permutations with few cycles.

4.8 Sets of lists

One interesting case is *sets of lists*, *permutations of rooted cycles*, or *fragmented permutations*. These are all names for the combinatorial class specified by $\textsc{Set}(\textsc{Seq}(\mathcal{Z}))$. Such objects have the generating function $\exp(z/(1-z))$. If we instead consider the objects in which each sequence has weight σ, then they have the generating function $\exp(\sigma z/(1-z))$. In the weighted permutation model, these correspond to permutations with weighted *cycles* in which k-cycles have weight σk.

We can compute from Wright's theorem (Theorem 2.3.12) the number of these objects. We have

$$[z^n] \exp\left(\frac{\sigma z}{1-z}\right) = e^{-\sigma}[z^n] \exp\left(\frac{\sigma}{1-z}\right)(1+O(n^{-1/2}))$$

and this satisfies the hypotheses of Wright's theorem with $\beta = 0, \Phi(z) = z$. Thus we find

$$[z^n] \exp\left(\frac{\sigma}{1-z}\right) = \frac{\exp(2\sqrt{\sigma n} + \sigma/2)\sigma^{1/4}}{2\sqrt{\pi} n^{3/4}}.$$

and so

$$[z^n] \exp\left(\frac{\sigma z}{1-z}\right) = \frac{\exp(2\sqrt{\sigma n} - \sigma/2)\sigma^{1/4}}{2\sqrt{\pi} n^{3/4}}.$$

Now to find mean cycle counts, we note that the generating function for σ-weighted sets of lists with lists of size k marked – the combinatorial class $\textsc{Set}(\sigma(\textsc{Seq}(\mathcal{Z}) + (\mu - 1)\textsc{Seq}_k(\mathcal{Z})))$ – is $F(z, u) = \exp(\sigma z/(1 - z)) \exp(\sigma(u - 1)z^k)$. By differentiating, we find that

$$\mathbb{E}_n X_k = \frac{[z^n]\partial_u\, F(z,u)|_{u=1}}{[z^n]F(z,1)} = \frac{\sigma[z^n]z^k \exp\left(\frac{\sigma z}{1-z}\right)}{[z^n]\exp\left(\frac{\sigma z}{1-z}\right)} = \sigma\frac{[z^{n-k}]\exp\left(\frac{\sigma z}{1-z}\right)}{[z^n]\exp\left(\frac{\sigma z}{1-z}\right)}$$

where $\mathbb{E}_n$ denotes probabilities with respect to the σ-weighted measure on sets of lists, and X_k is the number of k-cycles. From this we can compute asymptotic formulas for $\mathbb{E}_n X_k$ in the cases where k is constant, a constant multiple of n, or a constant multiple of $\sqrt{n}$.

Proposition 4.8.1. *We have*

$$\mathbb{E}_n X_k = \sigma - \sigma^{3/2} k n^{-1/2} + O(n^{-1})$$

as $n \to \infty$ with k held constant.

Proof. Let $f_n = [z^n]\exp(z/(1 - z))$. Then we have $\mathbb{E}_n X_k = \sigma f_{n-k}/f_n$. After some simplification we get

$$\mathbb{E}_n X_k = \sigma\frac{\exp(2\sqrt{\sigma})(\sqrt{n-k} - \sqrt{n})}{\left(\frac{n-k}{n}\right)^{3/4}}(1 + O(n^{-1}).$$

(A word on the relative error is in order: Wright's theorem actually gives a full asymptotic series for f_n in descending powers of $n^{-1/2}$.) The denominator is in fact $1 + O(n^{-1})$, and this gives $\mathbb{E}_n X_k = \sigma\exp(2\sqrt{\sigma}(\sqrt{n-k} - \sqrt{n})$. The exponent is $-\sqrt{\sigma/n}k + O(n^{-3/2})$, and simplification gives the result. $\square$

In particular, as $k \to \infty$ the number of components of size k approaches the constant σ.

Similarly, we have that $\mathbb{E}_n X_{\alpha n} = \sigma f_{(1-\alpha)n}/f_n$. A simple calculation shows that

$$\mathbb{E}_n X_{\alpha n} = \sigma(1-\alpha)^{-3/4} \exp(2\sqrt{\sigma n}(\sqrt{1-\alpha}-1))(1+O(n^{-1/2})).$$

Ignoring the $(1-\alpha)^{-3/4}$ factor, we see that the number of cycles of length n dies off exponentially fast with n.

The most interesting case comes with the correct choice of scaling, $k = t\sqrt{n}$.

Proposition 4.8.2.

$$\mathbb{E}_n X_{t\sqrt{n}} = \sigma \exp(-t\sqrt{\sigma}) \left(1 + \frac{3t}{4\sqrt{n}} + O(n^{-1})\right).$$

Proof. We have the quotient

$$\begin{aligned} E_n X_{t\sqrt{n}} &= \sigma \frac{f_{n-t\sqrt{n}}}{f_n} \\ &= \sigma \frac{\exp(2\sqrt{\sigma n}(1-t/\sqrt{n})^{1/2}}{(1-t/\sqrt{n})^{3/4}} \frac{1+a_1(n-t\sqrt{n})^{-1/2}+O(1/n)}{1+a_1 n^{-1/2}+O(1/n)} \end{aligned}$$

and now we can simplify the exponential. Note that $\sqrt{n}(1-t/\sqrt{n})^{1/2} = \sqrt{n}(1 - t/2\sqrt{n} + O(1/n))$. Therefore the exponential is equal to $\exp(-t\sqrt{\sigma})(1+O(1/n))$. This is the dominant term. The denominator $(1-t/\sqrt{n})^{3/4}$ is easily seen to be $1 - 3t/4n^{-1/2} + O(n^{-1})$. The fractional error factor is $1+O(n^{-1})$. Division gives the result. □

This naturally leads us to believe that the expected number of cycles in a σ-weighted set of lists is about $\sqrt{\sigma n}$. If we ignore the error terms, $\mathbb{E}_n X_{t\sqrt{n}} \approx \sigma e^{-t\sqrt{\sigma}}$.

So we have $\mathbb{E}_n X_k \approx \sigma e^{-k\sqrt{\sigma/n}}$ where k is of order $\sqrt{n}$. Integrating with respect to k gives the result. This is not a proof, but we can prove analogous results.

Proposition 4.8.3. *The mean number of cycles in a σ-weighted set of lists is asymptotic to $\sqrt{\sigma n}$, as $n \to \infty$.*

Proof. The generating function for σ-weighted sets of lists, marked by their size and number of components, is $\exp(\sigma uz/(1-z))$. Differentiating, the mean number of components in a σ-weighted set of lists of $[n]$ is

$$f_n^{-1}[z^n]\frac{\sigma z}{1-z}\exp\left(\frac{\sigma z}{1-z}\right)$$

The coefficient here is

$$\sigma e^{-\sigma}[z^n]\frac{z}{1-z}\exp\frac{\sigma}{1-z}$$

and by Wright's theorem this is asymptotic to $\sigma^{3/4}e^{-\sigma/2}\frac{1}{n^{1/4}2\sqrt{\pi}}\exp(2\sqrt{\sigma n})$. Division by the known form of f_n gives the desired result. $\square$

To put this in context, note that the number of sets of lists on $[n]$ having k parts is $\binom{n-1}{k-1}n!/k!$; these are the *Lah numbers* $L_{n,k}$ [Com74, p. 135]. A combinatorial proof of this count is as follows. To construct a set of k lists, first list all n elements in a single list, and then cut into k pieces by choosing $k-1$ of the possible $n-1$ cut points. This can be done in $n!\binom{n-1}{k-1}$ ways. But each element appears $k!$ times, so there are $\frac{n!}{k!}\binom{n-1}{k-1}$ queues in total. The total weight of σ-weighted sets of lists on $[n]$ with k parts is $\sigma^k L_{n,k}$. Let $M_k = \sigma^k L_{n,k}$. Then we note that $M_{n,k+1}/M_{n,k} = \sigma(n-k)/(k^2+k)$. Thus the sequence $M_{n,1}, \ldots, M_{n,n}$ is unimodal, with maximum where $\sigma(n-k)/(k^2+k) = 1$; this occurs near $k = \sqrt{\sigma n}$.

Sets of lists are an example of what we might call *square-root combinatorial structures*, by analogy with logarithmic combinatorial structures. For objects of size n, these have on the order of $\sqrt{n}$ components, and the typical size of a component is $\sqrt{n}$. Other examples of these include compositions of random involutions, with the components being cycles, and integer partitions. All of these have generating functions which are, loosely speaking, the exponential of some function with a pole.

4.9 Bulk results

In this section we will consider the number of cycles of length between γn and δn in a permutation of $[n]$ selected uniformly at random. Recall that the number of k-cycles in a permutation of $[n]$, for a fixed k, converges to a Poisson distribution with mean $1/k$ as $k \to \infty$. If instead of holding k constant we let it vary with n, the number of αn-cycles in permutations of $[n]$ approaches zero as $n \to \infty$ with α fixed. So to investigate the number of cycles of long lengths, we must rescale and look at many cycle lengths at once. The expectation of the number of cycles with length in this interval is $\sum_{k=\gamma n}^{\delta n} 1/k$, which approaches the constant $\log \delta/\gamma$ as n grows large. We will see that the number of cycles converges to a well-defined limiting distribution.

We recall from Proposition 3.2.1 that if $k_1, k_2, \ldots, k_s$ are distinct integers in $[1, n]$, and $r_1, \ldots, r_s$ are positive integers, then

$$\mathbb{E}\left(\prod_{i=1}^{s}\left(X_{k_i}^{(n)}\right)_{r_i}\right) = \prod_{i=1}^{s}\frac{1}{k_i^{r_i}}$$

if $n \geq \sum_{i=1}^{s} k_i r_i$, and zero otherwise.

In particular this provides a proof that the limits $\mu_k = \lim_{n\to\infty} X_k^{(n)}$ are the moments of a Poisson random variable,with mean $1/k$. Our major tool is the following theorem, which expresses the rth factorial moment of the number of cycles of a random permutation of $[n]$ with length in $[\gamma n, \delta n]$ as a certain r-fold integral.

Theorem 4.9.1. *Fix $0 \leq \gamma < \delta \leq 1$. Let $X^{(n)}$ be the number of cycles in a random permutation of $[n]$ having length in the interval $[\gamma n, \delta n]$. Then*

$$\lim_{n\to\infty} \mathbb{E}(X^{(n)})_r = \int_{\substack{z_1+\ldots+z_r \leq 1 \\ z_i \in [\gamma,\delta]}} \frac{1}{z_1 \cdots z_r}\, dz_1 \cdots dz_r.$$

Proof. Let $X_k^{(n)}$ be the number of k-cycles of a random permutation of $[n]$. Then $X^{(n)} = \sum_{k=\gamma n}^{\delta n} X_k^{(n)}$ and we can take the expectations of rth factorial moments to get

$$\mathbb{E}\left(X^{(n)}\right)_r = \mathbb{E}\left(\left(\sum_{k=\gamma n}^{\delta n} X_k^{(n)}\right)_r\right).$$

This sum can be expanded using the multinomial theorem for falling powers. We get

$$\mathbb{E}\left(X^{(n)}\right)_r = \mathbb{E}\left(\sum_{l_{\gamma n}+\cdots+l_{\delta n}=r} (X_{\gamma n})_{l_{\gamma n}} \cdots (X_{\delta n})_{l_{\delta n}} \binom{r}{l_{\gamma n}, \cdots, l_{\delta n}}\right)$$

and we can bring the expectation inside the sum. The termwise expectations are known from Proposition 3.2.1, and so we have

$$\mathbb{E}(X^{(n)})_r = \sum_{\substack{l_{\gamma n}+\cdots+l_{\delta n}=r \\ \sum_{k=\gamma n}^{\delta n} k l_k \leq n}} \left[\binom{r}{l_{\gamma n}, \cdots, l_{\delta n}} \prod_{k=\gamma n}^{\delta n}\left(\frac{1}{k}\right)^{l_k}\right].$$

Now, we consider the multinomial expansion

$$\left(\sum_{k=\gamma n}^{\delta n} \frac{1}{k}\right)^r = \sum_{l_{\gamma n}+\cdots+l_{\delta n}=r} \left[\binom{r}{l_{\gamma n}, \cdots, l_{\delta n}} \prod_{k=\gamma n}^{\delta n} \left(\frac{1}{k}\right)^{l_k}\right].$$

The expansion has a term $1/(k_1 \dots k_r)$ for each r-tuple $(k_1, \dots, k_r)$ in $[\gamma n, \delta n]^r$. This can be interpreted as a Riemann sum for the r-fold integral

$$\int_{\gamma n}^{\delta n} \cdots \int_{\gamma n}^{\delta n} \frac{1}{w_1 \cdots w_r} \, dw_1 \cdots dw_r.$$

The restriction $\sum_k kl_k \leq n$ cuts off that part of the region of summation where $w_1 + \cdots + w_r > n$. Thus the actual sum (4.9) is a Riemann sum for

$$\int \cdots \int \frac{1}{w_1 \cdots w_r} \, dw_1 \cdots dw_r$$

where the r-fold integral is over $w_1 + \ldots + w_n \in [\gamma n, \delta n], w_1 + \ldots + w_r \leq n$. The change of variables $z_i = w_i/n$ gives the desired result. □

Proposition 4.9.2. *Fix $\alpha > 1/2$. As $n \to \infty$, the probability that a randomly chosen permutation of $[n]$ has a cycle of length at least αn approaches $-\log \alpha$.*

Proof. We apply Theorem 4.9.1 to get

$$\lim_{n \to \infty} \mathbb{E}(X^{(n)}) = \int_{\alpha}^{1} \frac{1}{z} \, dz = -\log \alpha.$$

A permutation of $[n]$ can have at most one cycle of length longer than $n/2$, so the probability of having such a cycle is equal to the expected number of them. □

This is the simplest example of our general method. We know that the distribution of X is concentrated on two values; thus knowing $\mathbb{E}(X^{(n)})_0$ and $\mathbb{E}(X^{(n)})_1$ suffices to give the distribution of X. In general, if we know that X is concentrated on k values, finding $\mathbb{E}(X^{(n)})_0, \mathbb{E}(X^{(n)})_1, \ldots, \mathbb{E}(X^{(n)})_{k-1}$ gives a system of k linear equations in k unknowns which can be solved to determine the distribution of X. In order to make stating results easier, we make the following definition.

Definition 4.9.3. *We say a random variable X has quasi-Poisson(r, λ) distribution if $\mathbb{E}((X)_k) = \lambda^k$ for $k = 0, 1, \ldots, r$ and X is supported on $\{0, 1, \ldots, r\}$.*

The kth factorial moment of a Poisson(λ) random variable is λ^k. So in a sense, the quasi-Poisson random variables are trying to be Poisson, subject to an upper limit on their value. Let $\pi_i(r, \lambda)$ be the probability that a quasi-Poisson(r, λ) random variable has value i. Our knowledge of the moments allows us to set up a system of equations to find $\pi_i(r, \lambda)$. The solution is given in the following theorem.

Theorem 4.9.4. *The probability that a quasi-Poisson(r, λ) random variable has value i is*

$$\pi_i(r, \lambda) = \sum_{j=i}^{r} \binom{j}{i} \frac{1}{j!} (-1)^{j-i} \lambda^j.$$

We begin by recalling the following lemma.

Lemma 4.9.5. *Let $M = M_n, N = N_n$ be $(n+1)$ by $(n+1)$ matrices such that $M_{ij} = \binom{j}{i}, N_{ij} = \binom{j}{i}(-1)^{j+i}$, where the rows and columns of M and N are indexed by $0, 1, \ldots, n$. Then $MN = I$, the identity matrix.*

For a proof, see [Sta99, p. 66-67].

Proof of Theorem 4.9.4. The factorial moments specified in the definition of quasi-Poisson random variables give

$$\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^r \end{pmatrix} = \begin{pmatrix} (0)_0 & (1)_0 & \cdots & (r)_0 \\ (0)_1 & (1)_1 & \cdots & (r)_1 \\ & \vdots & & \\ (0)_r & (1)_r & \cdots & (r)_r \end{pmatrix} \begin{pmatrix} \pi_0(r,\lambda) \\ \pi_1(r,\lambda) \\ \vdots \\ \pi_r(r,\lambda) \end{pmatrix}. \tag{4.17}$$

The kth entry when the right-hand side of (4.17) is $\sum_{k=0}^{r}(k)_i \pi_k(r,\lambda)$, which is the expectation of $(X)_i$ when X is quasi-Poisson. This matrix is obtained from the M_r of Lemma 4.9.5 by multiplying all the entries in column i by $i!$. By Lemma 4.9.5 its inverse is obtained from N_r by dividing all the entries in row j by $j!$. Thus, we have

$$B_r(1,\lambda,\ldots,\lambda^r)^T = (\pi_0(r,\lambda), \pi_1(r,\lambda), \ldots, \pi_r(r,\lambda))^T$$

where $B_r = N_r^{-1}$. Thus $(B_r)_{ij} = \binom{j}{i}\frac{1}{j!}(-1)^{j+i}$ and this is the desired result in matrix form. $\square$

The sum (4.9.4) giving $\pi_i(r,\lambda)$ consists of the first $r-i$ nonzero terms of the Maclaurin series for $(z^i/i!)e^{-z}$, evaluated at $z=\lambda$. Thus if r is large, then $\pi_i(r,\lambda)$ approximates the corresponding probability for Poisson random variables. The quasi-Poisson$(r,1)$ distribution is well-known under another name in the study of permutations. It is the distribution of the number of fixed points of a permutation of $[r]$.

To show that a sequence of random variables converges to a given distribution, we will use the method of moments. In particular, given a sequence $X_1, X_2, \ldots$ of random variables, if the limits $\mu_k = \lim_{n\to\infty}(X_n)_k$ exists for each nonnegative integer k and the sequence $\mu_1, \mu_2, \ldots$ characterizes a distribution, then the X_n converge in distribution to that limiting distribution. (See Section 2.4.)

Theorem 4.9.6. *Fix γ, δ such that $\frac{1}{k+1} \leq \gamma < \delta \leq \frac{1}{k}$ for some integer k. (Alternatively, $\lfloor \delta^{-1} \rfloor + 1 = \lceil \gamma^{-1} \rceil$.) Let $X^{(n)}$ be a random variable on S_n with uniform measure, with $X^{(n)}(\pi)$ equal to the number of cycles of the permutation π with length in $[\gamma n, \delta n]$. Then as $n \to \infty$, $X^{(n)}$ converges in distribution to the quasi-Poisson$(k, \log \delta/\gamma)$ distribution.*

Proof. It suffices to show that $\lim_{n\to\infty} \mathbb{E}((X^{(n)})_r) = (\log \delta/\gamma)^r$.. We apply Theorem 4.9.1; the desired limit is

$$\int_{\substack{z_1+\ldots+z_r\leq 1 \\ z_i\in[\gamma,\delta]}} \frac{1}{z_1 \cdots z_r} \, dz_1 \cdots dz_r$$

and this integral is actually over an r-dimensional box $[\gamma, \delta]^r$, since the condition $z_1 + \cdots + z_r \leq 1$ is always satisfied. The integral factors, giving the desired result. $\square$

For example, from this theorem one can compute that in a random permutation of $[n]$, for n large, the probabilities of having $0, 1, 2$ or 3 cycles of length between $n/4$ and $n/3$ are approximately $0.7497, 0.2168, 0.0295, 0.0040$.

One shortcoming of Theorem 4.9.6 (and, implicitly, Theorem 4.9.2), which the reader may have noted, is that we require γ and δ to be in the same interval of the form $[\frac{1}{k+1}, \frac{1}{k}]$ for some integer k. This is not accidental; the expressions for the limiting probabilities become much more complicated if this is not the case. Such expressions can still be found, for example, in the case where $1/3 \leq \gamma \leq 1/2 \leq \delta \leq 1$. This distribution is supported on $\{0, 1, 2\}$; its kth moments $q_k(\gamma, \delta)$ satisfy $q_0(\gamma, \delta) = 1$, $q_1 = \log \frac{\delta}{\gamma}$, and

$$q_2(\gamma, \delta) = \log \frac{1-\delta}{\gamma} \log \frac{\delta^2}{(1-\delta)\gamma} - \log \delta \log(1-\delta) - Li_2(1-\delta) + Li_2(\delta) + (\log(1-\delta))^2$$

if $\gamma + \delta < 1$, and

$$q_2(\gamma, \delta) = -\log \gamma \log(1-\gamma) - Li_2(\gamma) + Li_2(1-\gamma) + (\log \gamma)^2$$

if $\gamma + \delta \geq 1$. From these one can find explicit formulas for the probabilities $p_k(\gamma, \delta)$ that a permutation has k cycles of length between γn and δn by solving a system of linear equations.

In the case of the Ewens distribution, the following conjecture seems reasonable:

Conjecture 4.9.7. *The expected number of cycles of length in $[\gamma n, \delta n]$ of a permutation of $[n]$ chosen from the Ewens distribution approaches*

$$\lambda = \int_\gamma^\delta \frac{1}{x}(1-x)^{\sigma-1}\, dx$$

as $n \to \infty$. Furthermore, in the case where $1/(k+1) \leq \gamma < \delta < 1/k$ for some positive integer k, the distribution of the number of cycles converges in distribution to quasi-Poisson(k, λ).

We previously showed that the proportion of elements of a random permutation of n selected according to the Ewens distribution which are in cycles of length in $[\gamma n, \delta n]$ approaches $(1-\gamma)^\sigma - (1-\delta)^\sigma$ as n gets large, and with $\sigma = 1/2$ the same is true for permutations of n selected uniformly from all those with all cycle lengths even, or from all those with all cycle lengths odd. It seems reasonable to conjecture that this correspondence should hold at least so far as to give that these classes of permutations satisfy the previous conjecture with $\sigma = 1/2$, and perhaps for other logarithmic combinatorial structures.

4.10 Connections to stochastic processes

Many of our results on the cycle structure of random permutations can be explained by renewal theory. Recall the renewal central limit theorem from Section 2.4. By suitable reparametrizations, we can rephrase many of our results in such terms. Recall the limit law for the length of the cycle containing 1 in a Ewens-θ permutation. As n approaches ∞, the probability that this cycle is of length between γn and δn approaches $(1-\gamma)^\theta - (1-\delta)^\theta$. Alternatively, the probability that the *complement* of the cycle containing 1 has length between λn and μn approaches $\mu^\theta - \lambda^\theta$. The cumulative density function of the normalized colength of the cycle containing 1 is therefore $f(x) = x^\theta$.

Thus, the probability that the normalized colength is less than x is x^θ. The probability that the *negative logarithm* of the normalized colength is at least z is therefore $e^{-z\theta}$. We recognize, then, that the negative logarithm of the normalized colength approaches an exponential random variable with mean $1/\theta$ as $n \to \infty$. Such a random variable has variance $1/\theta^2$.

Once we remove the cycle containing 1, what remains is a Ewens-θ permutation on a smaller set, and thus satisfies the same limit law. The negative logarithm of the normalized length remaining after the first k cycles (sorted by their minimal element) are stripped off therefore has distribution which is the sum of k exponential-$1/\theta$ random variables.

Now, we have found *all* the cycles of the permutation when the normalized length gets below $1/n$, or alternatively when its negative logarithm gets above $\log n$. We can predict when this occurs using the renewal central limit theorem. We apply the renewal central limit theorem with Y_i exponential with mean $1/\theta$. This gives

$$\frac{N_{\log n} - \theta \log n}{\sqrt{\theta \log n}} \xrightarrow{d} N(0,1).$$

By reparametrizing in other ways, we can derive similar results for other random permutation models. Consider sets of lists; for full generality we consider the model in which cycles of length k have multiplicative weight σk. Recall that the expected number of $t\sqrt{n}$-lists in a random set of σ-weighted lists is

$$\sigma e^{-t\sqrt{\sigma}} \left(1 + \frac{3t}{4} n^{-1/2} + O(n^{-1})\right).$$

The expected number of elements in $t\sqrt{n}$-lists in such a random set is therefore $\sigma t\sqrt{n} e^{-t\sqrt{\sigma}}(1 + o(1))$ and in particular the probability that an element chosen uniformly at random is in a $t\sqrt{n}$-list is $\sigma t n^{-1/2} e^{-t\sqrt{\sigma}}(1 + o(1))$. As with profiles of permutations, this can be integrated. First, letting $x = t\sqrt{n}$, this is

$$\frac{\sigma x}{n} e^{-x\sqrt{\sigma/n}}.$$

Integrating, the probability that a random element is in a cycle of length at least $t\sqrt{n}$ is

$$\int_{t\sqrt{n}}^{\infty} \frac{\sigma x}{n} e^{-x\sqrt{\sigma/n}}\, dx.$$

Letting $u = x\sqrt{\sigma/n}$ and simplifying, this is $\int_{t\sqrt{\sigma}}^{\infty} ue^{-u}\, du$. Finally, let $u = v\sqrt{\sigma}$; then we can rewrite this as $\int_{v}^{\infty} v\sigma e^{-v\sqrt{\sigma}}\, dv$ The integrand is the PDF of a $\Gamma(2, 1/\sqrt{\sigma})$ random variable, which has mean $2/\sqrt{\sigma}$ and variance $2/\sigma$; denote this distribution by X. Therefore the length of the cycle containing 1, divided by $\sqrt{n}$, approaches in distribution X as $n \to \infty$..

So consider $\sqrt{n} - \sqrt{n - X\sqrt{n}}$. If we assume that $X\sqrt{n}$ is much smaller than n, and approximate the square root by a linear function near n, the distribution of this looks like that of $X/2$, which has mean $1/\sqrt{\sigma}$ and variance $1/(2\sigma)$. Consider the process which starts at $\sqrt{n}$ and subtracts a random variable with distribution $X/2$ at each step; how long does this process take to reach 0? By the renewal central limit theorem, this time is asymptotically normally distributed with mean $\sqrt{n\sigma}$ and variance $\sqrt{n\sigma}/2$.

4.11 Connections to number theory

Finally, there are many connections between the cycle structure of permutations and the prime factorizations of integers. This analogy has been pointed out by [Gra09], from a number-theoretic point of view; here we look at the analogy from the combinatorialist's perspective. We begin by recalling the Erdős-Kac theorem: informally, this theorem states that integers near n have a number of prime factorizations which is normally distributed, with mean and variance $\log \log n$. Somewhat more formally, we have:

Theorem 4.11.1. *[EK40] If $\omega(n)$ is the number of distinct prime factors of n, then for any fixed $a < b$,*

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \leq x : a \leq \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq b \right\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2}\, dt.$$

The same is true if we consider the number of prime factors counted with multiplicity.

Now, recall that the number of cycles of a permutation of n is asymptotically normally distributed with mean and variance $\log n$. If we consider the natural logarithm of a number to be its "size", then we see that an integer of size n (that is, an integer near e^n) has number of prime factors normally distributed with mean and variance $\log n$.

Indeed, a wide variety of facts about permutations are echoed by facts about prime factorizations, and conversely. Since the usual methods of proof in combinatorics and in analytic number theory are different, some results will be closer to the surface in one subject than the other.

For example, consider the usual probabilistic interpretation of the prime number theorem: integers near n have "probability" $1/\log n$ of being prime. That is, integers of "size" x have probability $1/x$ of being prime. The permutation analogue is that permutations of n have probability $1/n$ of being cycles, which is exactly true.

Also note that the expected number of cycles in a permutation of $[n]$ which are longer than αn is asymptotic to $-\log\alpha$ as $n \to \infty$. Since the expected number of k-cycles in a permutation of n, with $1 \leq k \leq n$, is $1/k$, the expected number of cycles longer than αn is $H_n - H_{\lfloor \alpha n \rfloor}$, where $H_n = \sum_{k=1}^{n}$ is the nth harmonic number. As $n \to \infty$ this approaches $-\log\alpha$ from the usual asymptotic series for the harmonic numbers. We also see that the expected number of prime factors of an integer n which are greater than n^α is $-\log\alpha$. The asymptotic density of positive integers n with kth largest factor smaller than $n^{1/\alpha}$ is $\rho_k(\alpha)$, where we have $L_0(\alpha) = [\![\alpha > 0]\!]$ and

$$L_k(\alpha) = [\![\alpha \geq k]\!] \int_k^\alpha L_{k-1}(t-1)\,\frac{dt}{t},$$

and $1 - \rho_k(\alpha) = \sum_{n=0}^{\infty} \binom{-k}{n} L_{n+k}(\alpha)$ [Rie94, p. 162]. The density of positive integers with kth largest factor *larger* than $n^{1/\alpha}$ is therefore $1 - \rho_k(\alpha)$, and so the expected number of factors larger than $n^{1/\alpha}$ is $\sum_{k\geq 1}(1-\rho_k(\alpha))$. Therefore the expected number of such factors is

$$\sum_{k\geq 1}\sum_{n\geq 0}\binom{-k}{n} L_{n+k}(\alpha).$$

Letting $n + k = j$ we can rewrite this sum as

$$\sum_{j\geq 1}\sum_{n=0}^{j-1}\binom{n-j}{n} L_j = \sum_{j\geq 1} L_j \left(\sum_{n=-0}^{j-1}(-1)^n\binom{j-1}{n}\right)$$

and the inner sum is 0 except when $j = 1$, when it is 1. So the expected number of factors larger than $n^{1/\alpha}$ is $L_1(\alpha)$; this is $\log\alpha$.

Similarly, the Dickman function, as defined by [Dic30], tells us the distribution of the largest prime factor of a random integer. Let $\psi(x, y)$ denote the number of integers less than or equal to x with all prime factors less than or equal to y. Then $\psi(x, x^u) \sim x\rho(1/u)$ as $x \to \infty$, where ρ is the function defined by $\rho(u) = 1$ for $0 \leq u \leq 1$ and $\rho(u) = \frac{1}{u}\int_{u-1}^{u} \rho(t)\, dt$ for all $u > 1$. In particular $\rho(u) = 1 - \log u$ for $1 \leq u \leq 2$. This result has been extended in [KP77], where it is shown that there are $\sim \rho_k(u)x$ integers with kth largest prime factor less than $x^{1/u}$, where $\rho_k(u) = 1$ for $u \leq 1$, and

$$\rho_k(u) = 1 - \int_1^u (\rho_k(t-1) - \rho_{k-1}(t-1))\frac{dt}{t}.$$

They also showed that $\rho_k(u)$ is the probability that the kth longest cycle in a random permutation of N letters has length less than N/u. This extends work of [LS66] on the length of the longest cycle.

The results of Section 4.9 also are connected to number theory. The number of cycles in a permutation of $[n]$ of length between γn and δn is analogous to the number of prime factors of an integer near n between n^γ and n^δ. The case $\gamma = 0$ (that is, integers with all factors less than n^δ) was considered by Dickman [Dic30], and that of $\delta = 1$ (all factors greater than n^γ) by Buchstab [Buc49]; the general case was treated by Friedlander [Fri76]. There do not seem to be results considering the probability that an integer near n has a specified *number* of prime factors in $[n^\gamma, n^\delta]$.

Instead of looking at the sizes of cycles of prime factors, we can look at multiplicities. The "probability" that an integer is squarefree is $6/\pi^2 = 1/\zeta(2)$; that is, the number of squarefree integers less than n which are squarefree is asymptotic to $6n/\pi^2$. The analogue of a squarefree integer is a permutation with no cycle length repeated.

The probability that a random permutation of $[n]$ has no repeated cycle lengths is $e^{-\gamma}$ where γ is the Euler-Mascheroni constant [FFG+06]. These probabilities are given by the infinite products

$$\prod_{p \geq 1} \left(1 - \frac{1}{p^2}\right), \prod_{k \geq 1} \left(1 + \frac{1}{k}\right) e^{-1/k}$$

respectively, where the former is over primes and the latter is over all integers. The result in the squarefree case can be predicted from the "Cramer model" of prime factorizations, in which integers are assumed to be divisible by p with probability $1/p$ and divisibility by each prime is independent; in the distinct-cycle-length case, we can predict the result from the fact that a random permutation has $\mathcal{P}(k)$ cycles of length k, since $(1 + 1/k)e^{-1/k}$ is the probability that a $\mathcal{P}(1/k)$ random variable is either 0 or 1.

Finally, we can seek analogues of our results on the parity of cycle lengths occurring in the factorizations of integers. The analogue of parity of cycle lengths is not parity of primes. Perhaps the most natural way to divide the primes in half is to split them into classes congruent to $4n \pm 1$ or to $6n \pm 1$. In this case we can consider the following theorem of Spearman and Williams.

Theorem 4.11.2. *[SW07, Thm. 1.1, case $\lambda = 1$]. Let $S := S(l_1, l_2, \ldots, l_r, k)$, be the set of integers with all prime factors congruent to one of $l_1, l_2, \ldots, l_r$ modulo k, and let $0 < \epsilon < 1$. Then there exists a positive constant C such that*

$$\sum_{\substack{n \leq x \\ n \in S}} 1 = \frac{r}{\phi(k)} Cx(\log x)^{r/\phi(k)-1} + O((\log x)^{r/\phi(k)-2+\epsilon})$$

where the constant implicit in the O notation depends at most on $\epsilon, k, l_1, \ldots, l_r, \lambda$.

The constant C is a certain product over Dirichlet characters modulo k which we do not give explicitly. The version of the theorem we state here is for the "counting" sums $\sum_{n \leq x, n \in S} 1$; in fact Spearman and Williams considered sums of the form $\sum_{\substack{n \leq x \\ n \in S}} n^\alpha$ for all $\alpha \geq -1$. This theorem has a probabilistic interpretation: the probability that an integer near x has all its prime factors congruent to one of $l_1, l_2, \ldots, l_r$ modulo k is proportional to $(\log x)^{r/\phi(k)-1}$. That is, integers of "size" n have all their prime factors in one of the allowed residue classes with probability proportional to $n^{r/\phi(k)-1}$.

Now, $r/\phi(k)$ is the relative density of these primes in the set of all primes, by the prime number theorem for arithmetic progressions. So this is exactly an analogue of results in Section 4.5, in which we show that a permutation of $[n]$ has probability proportional to $n^{r/s-1}$ of having all its cycle lengths in one of r specified residue classes modulo s. In particular, the number of integers less than x with all prime factors congruent to 1 mod 4, or to 3 mod 4, are both $\Theta(x/\sqrt{\log x})$; these are analogous to results on permutations with all cycle lengths of the same parity.

Finally, Landau [Lan08] showed that the number of integers less than x that can be written as a sum of two squares, denoted by $S(x)$, satisfies $\lim_{x\to\infty} \sqrt{\log x}/x \cdot S(x) = K$ where K is a constant. Probabilistically, numbers near x have probability $1/\sqrt{\log x}$ of being expressible as a sum of two squares. We recall that a positive integer can be written as a sum of two squares if and only if all primes of form $4k+3$ occur to an even power in its prime factorization. So positive integers which can be written as sums of two squares are analogous to permutations in which cycles of even length occur with even multiplicity – that is, to permutations with square roots. Pouyanne [Pou02] gives an asymptotic expression for the probability that a permutation of $[n]$ has an mth root, for any fixed m as $n \to \infty$; for $m = 2$ this is asymptotic to $C/\sqrt{n}$ for a constant C.

Chapter 5

Cycle structure of compositions of involutions

5.1 Introduction

In this chapter we study the cycle structure of compositions of involutions. Recall that an *involution* is a permutation with all cycles having length 1 or 2. Let a_n be the number of involutions in the symmetric group S_n. Then as $n \to \infty$,

$$a_n \sim \sqrt{n!}e^{\sqrt{n}}(8\pi en)^{-1/4}. \tag{5.1}$$

This form involving $\sqrt{n!}$ is due to [FS09, p. 583]; see [MW55] for the result in another form, and [Pem09, Example 3.2] for details of the asymptotic analysis by the saddle-point method. The factor $\sqrt{n!}$ is much faster-growing than $e^{\sqrt{n}}(8\pi en)^{-1/4}$. So in a logarithmic sense the number of involutions of $[n]$ is approximately the square root of the number of permutations of $[n]$. Thus the number of pairs of involutions of $[n]$ is logarithmically near $n!$. This suggests identifying permutations with pairs of involutions. A natural way to combine two involutions to form a permutation is composition, so we study compositions.

We then proceed to represent involutions graph-theoretically as partial matchings; thus compositions of two involutions can be identified with graphs having 2-colored edges, where each vertex has at most one incident edge of each color. The components of such graphs are paths and cycles, so we easily find generating functions involving them. This is our principal tool for extracting information on the cycle structure of these graphs and the corresponding permutations. In particular, if σ and τ are random involutions of $[n]$, then as $n \to \infty$:

- The distribution of the number of k-cycles of $\tau \circ \sigma$ converges in distribution to $\mathcal{P}(1) + 2\mathcal{P}(1/2k)$ (Theorem 5.3.1);
- The mean number of cycles of $\tau \circ \sigma$ (of all lengths) is $\sqrt{n} + \frac{1}{2}\log n + O(1)$ (Theorem 5.5.4);
- If σ and τ are constrained to be fixed-point-free, then the distribution of the number of cycles of $\tau \circ \sigma$ is asymptotically normal with mean $\log n$ and variance $2\log n$ (Proposition 5.6.2).

Next, we consider the number of factorizations of a permutation into involutions. The mean number of factorizations into two involutions of a permutation $\pi \in S_n$

chosen uniformly at random – that is, solutions to $\pi = \tau \circ \sigma$, with τ and σ involutions of $[n]$ – is $e^{2\sqrt{n}}/\sqrt{8\pi e n}(1+o(1))$. We derive a formula (Theorem 5.7.1) for the number of factorizations of $\pi \in S_n$ into two involutions, in terms of the cycle type of π. This is a product over cycle lengths. In a model of random permutations in which there are $\mathcal{P}(1/k)$ cycles of length k for $k = 1, \ldots, n$, the number of factorizations of a random permutation π is lognormally distributed (Theorem 5.7.5). If $\mathbb{P}_n^*$ denotes this probability measure, $F(\pi)$ the number of factorizations of π, and Φ the standard normal cdf, then

$$\lim_{n\to\infty} \mathbb{P}_n^* \left(\frac{\log F(\pi) - \frac{1}{2}(\log n)^2}{\frac{1}{3}(\log n)^3} \leq x \right) \to \Phi(x).$$

In particular the median number of factorizations of π is near $\exp((\log n)^2/2)$, much smaller than the mean. This is one of many indications that the measure on S_n coming from compositions of involutions chosen uniformly at random is much different from the uniform measure on S_n.

After this, we consider pattern avoidance in involutions. Stanley-Wilf limits for various classes of pattern-avoiding permutations are known; in those cases where a Stanley-Wilf limit for the corresponding pattern-avoiding involutions exists, the former is the square of the latter. In the simplest case, that of 21-patterns or inversions, it is also possible to enumerate involutions on $[n]$ by their number of inversions; we compare the distribution of the number of inversions in involutions with the number of inversions in ordinary permutations.

Finally, we asymptotically enumerate permutations with all cycle lengths in a finite set S; involutions are the case $S = \{1, 2\}$. Call a permutation with all cycle lengths in S an S-permutation. Let $p_n^{(S)}$ be the probability that a permutation of $[n]$ selected uniformly at random is an S-permutation. Then $\lim_{n\to\infty} \frac{\log p_n^{(S)}}{\log n!} = -1/(\max S)$; a refinement of this is Theorem 5.10.1 below. In particular the number of k-cycles of a typical S-permutation is near $\frac{1}{k} n^{k/(\max S)}$, generalizing the result that a typical involution of $[n]$ has $\sqrt{n}$ fixed points.

5.2 Graph-theoretic decomposition

An involution σ can be represented as a partial matching on the set $[n]$, where k and l are matched if $\sigma(k) = l$ (and therefore $\sigma(l) = k$). We can view this matching as a graph, by drawing an edge between k and l when $\sigma(k) = l$. A pair of partial matchings or involutions, (σ, τ), can be identified with a graph on the vertex set $[n]$ with 2-colored edges, where we color the edges solid or dotted according to whether they are from σ or from τ. We write $\sigma \cup \tau$ for this graph, and refer to it as a *superposition*; we write $\tau \circ \sigma$ for the corresponding permutation.

Figure 5.1 illustrates such a superposition, where the solid edges represent the matching or involution σ and the dotted edges represent the matching or involution τ, with

$$\sigma = (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)(11\,12)(13)(14\,15)(16\,17)(18\,19),$$

$$\tau = (1\,4)(2\,3)(5)(6\,7)(8)(9)(10\,11)(12\,13)(14\,19)(15\,16)(17\,18).$$

To read the product permutation from the figure, start at any vertex, follow the solid

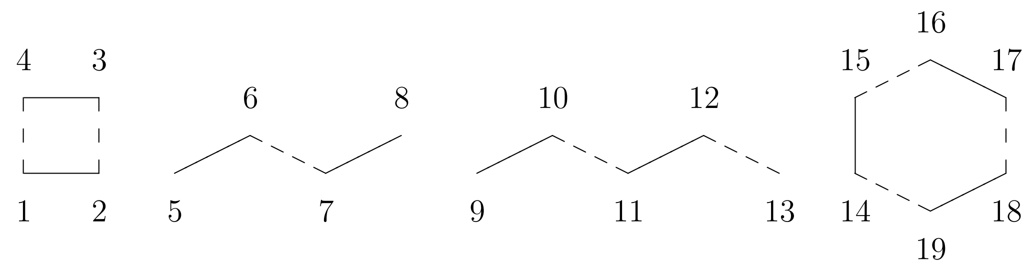


Figure 5.1: Two matchings on 19 vertices.

edge at that vertex and then the dotted edge at that vertex. (If only one of those edges exists, follow it; if neither exists, the original vertex is a fixed point.) In this example, we have

$$\tau \circ \sigma = (1\,3)(2\,4)(5\,7\,8\,6)(9\,11\,13\,12\,10)(14\,16\,18)(15\,17\,19).$$

Theorem 5.2.1. *The trivariate generating function for pairs of partial matchings (σ, τ), counted according to the size of the ground set (indicated by the variable z) and number of paths and cycles in $\sigma \cup \tau$ (indicated by u and v respectively), exponential in z and ordinary in u and v, is*

$$Q(z,u,v) = \frac{\exp(uz/(1-z))}{(1-z^2)^{v/2}} \tag{5.2}$$

That is, $n![z^n u^k v^l]Q(z,u,v)$ is the number of pairs of partial matchings on n vertices with k paths and l cycles.

Proof. We enumerate the possible connected components of a pair of partial matchings and apply the exponential formula.

The connected components of such a graph are cycles of even length and paths, with the edges alternating in color. These are the only possible components since if colors are ignored, all vertices must have degree at most two. We note that the degenerate path (a single vertex) and the degenerate cycle (two vertices connected by a solid edge and a dotted edge) are both possible components. We consider the length of a path to be its number of vertices, so a single vertex is a path of length 1.

First we count the possible labelled cycles. Since the edges must alternate in color, only cycles of even length are possible. Now, an unlabelled cycle with even length n, like the cycle on the right in Figure 5.2, can be labelled in $n!$ ways. But there are n symmetries of this cycle that take solid edges to solid edges, corresponding to the dihedral group with n elements. So there are $(n-1)!$ possible labelled cycles of length n, for each even n, and none for odd n. Thus the exponential generating function (egf) for properly colored cycles is

$$1!\frac{z^2}{2!} + 3!\frac{z^4}{4!} + 5!\frac{z^5}{6!} + \cdots = \frac{z^2}{2} + \frac{z^4}{4} + \frac{z^6}{6} + \cdots = \frac{1}{2}\log\frac{1}{1-z^2}.$$

Next we count labelled paths with edges alternating color. If such a path has an even number of edges, like the leftmost path in Figure 5.2, then there are $n!$ ways to label it – each permutation of $[n]$ corresponds to a labelling, by writing that permutation (in one-line notation) starting at the end of the path with a solid edge. If there are an odd number of edges, as in the middle path in Figure 5.2, then the ends of the path are indistinguishable. We first choose whether the two terminal edges of the path are solid or dotted (they must be the same color); then there are only $n!/2$ ways to label a solid-dotted-solid path or a dotted-solid-dotted path. So there are $n!$ paths with n vertices, with vertices having distinct labels in $[n]$ and edges having

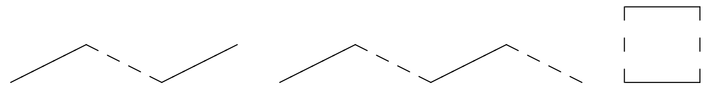


Figure 5.2: Unlabelled paths of length four and five, and an unlabelled cycle of length four.

alternating colors. The egf for properly colored paths is $z/(1-z)$.

The generating function of components marked according to their type (path or cycle) is therefore $u \cdot \frac{z}{1-z} + v \cdot \frac{1}{2} \log \frac{1}{1-z^2}$, and applying the exponential formula gives (5.2). $\square$

We quickly derive two corollaries more relevant to permutation enumeration.

Corollary 5.2.2. *The exponential generating function of pairs of involutions is* $P(z) = \exp(z/(1-z))/\sqrt{1-z^2}$.

Proof. Take the specialization $u = 1, v = 1$ in Theorem 5.2.1. This gives the exponential generating function for pairs of partial matchings, which we identify with pairs of involutions. $\square$

Corollary 5.2.3. *The semi-exponential generating function of pairs of involutions* (σ, τ)*, counted by the size of the ground set and the number of permutation cycles in the composition* $\tau \circ \sigma$*, is*

$$R(z,u) = \frac{\exp\left(\frac{uz}{1-z}\right)}{(1-z^2)^{u^2/2}},$$

Proof. Consider a pair of perfect matchings (σ, τ). Each connected component of the corresponding graph $\sigma \cup \tau$ gives rise to either one or two cycles in $\tau \circ \sigma$. Each $2k$-cycle in the graph $\sigma \cup \tau$ gives rise to two k-cycles in the permutation $\tau \circ \sigma$, corresponding to half of the vertices. Each k-path in $\sigma \cup \tau$ gives rise to a k-cycle in $\tau \circ \sigma$. To count by permutation cycles, then, we need $z^n u^k v^l$ in $Q(z, u, v)$ to be mapped to $z^n u^{k+2l}$; thus we take the specialization $R(z, u) = Q(z, u, u^2)$ in Theorem 5.2.1. □

5.3 Asymptotic distribution of the number of k-cycles

In this section we show

Theorem 5.3.1. *The distribution of the number of k-cycles of the composition of a pair of random involutions of $[n]$ converges in distribution to the distribution of $A_k + 2B_k$ as $n \to \infty$, where A_k and B_k are independent, A_k is Poisson with mean 1, and B_k is Poisson with mean $1/(2k)$.*

We need the following more general result. Recall that a sequence $\{b_n\}$ is *slowly varying* if $\lim_{n\to\infty} b_{n-1}/b_n = 1$.

Lemma 5.3.2. *Let $P(z, u)$ be the generating function*

$$P(z, u) = \sum_{n,k \geq 0} P_{n,k} \frac{z^n}{n!} u^k$$

where $P_{n,k}$ is the number of objects in a combinatorial class $\mathcal{P}$ of size n with some parameter χ equal to k. Assume $P(z, u) = Q(z)e^{R(z,u)}$ with R a polynomial, $[z^n]Q(z)$ is slowly varying as $n \to \infty$, and $R(1, t)$ is the factorial moment generating function *of some distribution which is determined by its moments. For each n, let $\mathbb{P}_n(\chi = k) = P_{n,k}/\sum_k P_{n,k}$ define a probability distribution on the positive integers. Then as $n \to \infty$, the sequence of distributions of χ on $\mathbb{P}_n$ converges in distribution to the distribution with factorial moment generating function* $\exp R(1,t)$.

Proof. Let j be the degree of R in the variable u. We can show by induction that $\partial_u^r P(z,1) = P(z,1)T_r(z)$, where $T_r(z)$ is a polynomial of degree jr. Then we have

$$\mathbb{E}_n((\chi)_r) = \frac{[z^n]\ \partial_u^r P(z,u)|_{u=1}}{[z^n]P(z,1)} = \frac{[z^n]P(z,1)T_r(z)}{[z^n]P(z,1)}.$$

Now, $\lim_{n\to\infty} [z^{n-s}]P(z,1)/[z^n]P(z,1) = 1$ from the condition on slow variation. Then

$$\frac{[z^n]P(z,1)T_r(z)}{[z^n]P(z,1)} = \sum_{s=0}^{jr} [z^s]T_r(z) \frac{[z^{n-s}]P(z,1)}{[z^n]P(z,1)}$$

and taking limits as $n \to \infty$ gives

$$\lim_{n\to\infty} \frac{[z^n]P(z,1)T_r(z)}{[z^n]P(z,1)} = T_r(1).$$

So $\lim_{n\to\infty} \mathbb{E}_n((\chi)_r) = T_r(1)$. Now let $F(t) = R(1,t)$. The rth factorial moment of the distribution with factorial mgf $F(t)$ is $F^{(r)}(1)$. This is $\frac{\partial^r}{\partial u^r} \exp R(z,u)\big|_{z=u=1}$ and we recall that $\partial_u^r Q(z) \exp R(z,u)|_{u=1} = P(z,1)T_r(z)$ by definition. Therefore we have

$$\frac{\partial^r}{\partial u^r} \exp R(z,u)\bigg|_{z=u=1} = \frac{P(1,1)T_r(1)}{Q(1)} = T_r(1)$$

which is what we wanted. $\square$

Proof of Theorem 5.3.1. First, find the semi-exponential generating function for compositions of involutions, with k-cycles marked. That is the generating function for superpositions of partial matchings, as found in the proof of Theorem 5.2.1, with paths of length k singly marked and cycles of length $2k$ doubly marked. This gives

$$P(z,u) = \frac{\exp(z/(1-z))}{\sqrt{1-z^2}} \exp\left((u-1)z^k + (u^2-1)\frac{z^{2k}}{2k}\right)$$

and we apply Lemma 5.3.2. The slow variation hypothesis holds since $[z^n]\exp(z/(1-z))/\sqrt{1-z^2} = a_n^2/n! = e^{2\sqrt{n}}(8\pi en)^{-1/2}(1+o(1))$. We have $R(z,u) = \exp((u-1)z^k + (u^2-1)z^{2k}/(2k))$; this is the factorial moment generating function of $A_k + 2B_k$, which follows from the fact that $Poisson(\lambda)$ has factorial mgf $\exp(\lambda(t-1))$. Finally, we recall that if the moment generating function of a random variable has positive radius of convergence, then the random variable is determined by its moments [Bil95, Thm 30.1]. $A_k + 2B_k$ has mgf $\exp(e^t - 1 + (e^{2t}-1)/2k)$, which is entire. □

The sum of Poissons given in Theorem 5.3.1 is quite natural. There are two types of components in superpositions of partial matchings on $[n]$ that can lead to k-cycles of the corresponding permutations: paths of length k (which induce one permutation k-cycle) and cycles of length $2k$ (which induce two permutation k-cycles). For large n and fixed k, the expected number of k-paths approaches 1 and the expected number of $2k$-cycles approaches $1/k$. Furthermore, the sites in which individual cycles can appear are each rare, so it is not surprising to see an independent Poisson distribution for each type of component.

5.4 Partial matchings with a specified number of fixed points

In this section we consider superpositions of partial matchings, $\sigma \cup \tau$, where σ is chosen uniformly from all partial matchings on $[n]$ with k fixed points, and τ is chosen uniformly from all partial matchings with l fixed points.

Proposition 5.4.1. *The expected number of r-paths in $\sigma \cup \tau$ is*

$$kl\frac{(\frac{n-k}{2})_{(r-1)/2}(\frac{n-l}{2})_{(r-1)/2}2^{r-1}}{(n)_r} \tag{5.3}$$

if r is odd, and

$$\frac{(k(k-1)(\frac{n-k}{2})_{r/2-1}(\frac{n-l}{2})_{r/2} + l(l-1)(\frac{n-k}{2})_{r/2}(\frac{n-l}{2})_{r/2-1})2^{r-1}}{2(n)_r} \tag{5.4}$$

if r is even.

Proposition 5.4.2. *The expected number of r-cycles in $\sigma \cup \tau$, is*

$$\frac{\left(\frac{n-k}{2}\right)_{r/2}\left(\frac{n-l}{2}\right)_{r/2}2^r}{r(n)_r} \tag{5.5}$$

for even r.

These statements can be easily verified. For odd paths, we compute the probability that a path occurs which traverses the edges $1, 2, \ldots, r$ in that order, and multiply by the number of possible paths. The argument is similar for even paths, except we must handle the cases where the two ends of the path are fixed points in σ and fixed points in τ separately. Finally, we do this for cycles; the most interesting feature is the factor of r in the denominator which arises from the symmetry of cycles. This model of random involutions, with $n \to \infty$ and k, l varying with n in such a way that $k + l = \Omega(1)$ and $k + l = o(n)$ simultaneously, has been considered in [RV09] in the context of dynamical systems.

Corollary 5.4.3. *The expected number of paths of length r in $\sigma \cup \tau$, the superposition of two randomly selected perfect matchings on $[n]$, where σ and τ each have pn fixed points, is asymptotic to $p^2(1-p)^{r-1}$ as $n \to \infty$. The expected number of cycles of length r (if r is even) approaches $(1-p)^r/r$ as $n \to \infty$.*

Corollary 5.4.4. *Let $r = O(\sqrt{n})$ as $n \to \infty$. Then the mean number of r-paths in $\sigma \cup \tau$, where σ and τ are randomly selected involutions with $\sqrt{n}$ fixed points each, is asymptotic to $\exp(-r/\sqrt{n})$ as $n \to \infty$, and the mean number of r-cycles is asymptotic to $\exp(-r/\sqrt{n})/r$.*

These follow from Propositions 5.4.1 and 5.4.2 by making appropriate substitutions, and applying Stirling's formula in the case of Corollary 5.4.4. The number of r-paths decays exponentially in r.

Finally, we can translate Corollary 5.4.4 back into the terminology of involutions. To get a better sense of the scaling behavior of cycle sizes, we look at the expected number of $\alpha\sqrt{n}$-cycles of a composition of two random involutions. The expected number of k-cycles is $\left(b_{n-k} + \frac{1}{2k} b_{n-2k}\right)/b_n$, where $b_n = [z^n]\exp(z/(1-z))/\sqrt{1-z^2}$. Recall that $b_n \sim e^{2\sqrt{n}}(8\pi e n)^{-1/2}$. Let $k = \alpha\sqrt{n}$ grow with n. We can compute that $b_{n-\alpha\sqrt{n}}/b_n \sim e^{-\alpha}$ as $n \to \infty$; this is the limit of the number of $\alpha\sqrt{n}$-cycles as $n \to \infty$. Recall that such square-root scaling is typical of structures counted by generating functions which are the exponential of a function with a simple pole, the simplest example of which are the "fragmented permutations" (permutations with rooted cycles) or "sets of lists" counted by $\exp(z/(1-z))$, and discussed in Section 4.8.

5.5 The total number of cycles

The function $R(z,u)$ given in Corollary 5.2.3 will be our jumping-off point for asymptotic results on cycle structure. We will need Theorem 2.3.12 to derive asymptotic results. We begin by observing that applying Theorem 2.3.12(a) with $\Phi(z) = e^{-z}/\sqrt{1+z}, \beta = -1/2$ gives $[z^n]P(z) = \frac{1}{\sqrt{8\pi e n}} \exp(2\sqrt{n})(1 + O(n^{-1/2}))$. This is consistent with the known number of involutions in (5.1).

Proposition 5.5.1. *The mean number of components which are paths in a superposition of two partial matchings on $[n]$ selected uniformly at random is $\sqrt{n} + O(1)$.*

Proof. The bivariate generating function counting superpositions of partial matchings by size and number of paths is $Q(z,u,1) = \exp(uz/(1-z))/\sqrt{1-z^2}$, obtained by setting the variable which marks cycles in Theorem 5.2.1 equal to 1. From Proposition 2.1.2, the mean number of components which are paths in a superposition of two partial matchings is

$$\frac{[z^n]Q_u(z,1,1)}{[z^n]Q(z,1,1)} = \frac{[z^n]\frac{z}{(1-z)\sqrt{1-z^2}}\exp(z/(1-z))}{[z^n]\exp(z/(1-z))}. \tag{5.6}$$

Applying Theorem 2.3.12(a) with $\beta = -3/2, \Phi(z) = e^{-1}z/\sqrt{1+z}$ gives

$$[z^n]\frac{z}{(1-z)\sqrt{1-z^2}} = \frac{\exp(2\sqrt{n})}{\sqrt{8\pi e}}(1+O(n^{-1/2})).$$

Since $Q(z,1,1) = P(z)$, the denominator in (5.6) is $\exp(2\sqrt{n})/\sqrt{8\pi e n}\cdot(1+O(n^{-1/2}))$, giving the desired result. □

Proposition 5.5.2. *The mean number of components which are cycles in a superposition of two partial matchings on $[n]$ selected uniformly at random is $\frac{1}{4}\log n - \frac{1}{2}\log 2 + O(n^{-1/2}\log n)$.*

Proof. The bivariate generating function counting superpositions of partial matchings by size and number of cycles is $Q(z,1,v) = \exp(z/(1-z))(1-z^2)^{-v/2}$, obtained by setting the variable marking paths in Theorem 5.2.1 equal to 1.

By Proposition 2.1.2, then, the mean number of cycles is given by $[z^n]\partial_v Q(z,1,1)/[z^n]Q(z,1,1)$. The numerator is

$$[z^n]\frac{1}{2}\exp\left(\frac{z}{1-z}\right)\log\left(\frac{1}{1-z^2}\right)(1-z^2)^{-1/2}$$

and we can write the logarithm as a sum to get

$$\begin{aligned}[z^n]\partial_v Q(z,1,1) &= [z^n]\frac{1}{2e}\exp\left(\frac{1}{1-z}\right)\log\left(\frac{1}{1-z}\right)(1-z^2)^{-1/2} \\ &+ [z^n]\frac{1}{2e}\exp\left(\frac{1}{1-z}\right)\log\left(\frac{1}{1+z}\right)(1-z^2)^{-1/2}.\end{aligned}$$

The asymptotics of each term can be derived from Theorem 2.3.12. For the first term, we have $k = 1, \beta = -1/2, \Phi(z) = 1/(2e\sqrt{1+z})$; thus the first term is

$$\frac{\log n \exp 2\sqrt{n}}{\sqrt{n}} \frac{1}{2^{7/2}\sqrt{e\pi}}(1 + O(n^{-1/2})).$$

The second term, with $k = 0, \beta = -1/2, \Phi(z) = 1/(2e\sqrt{1+z}) \log(1/(1+z))$, is

$$\frac{\exp 2\sqrt{n}}{\sqrt{n}} \frac{-\log 2}{2^{5/2}\sqrt{e\pi}}(1 + O(n^{-1/2})).$$

Putting these together, the mean number of cycles is given by

$$\frac{\frac{\exp 2\sqrt{n}}{\sqrt{n}}\left(\frac{\log n}{2^{7/2}\sqrt{e\pi}} - \frac{\log 2}{2^{5/2}\sqrt{e\pi}} + O\left(\frac{\log n}{\sqrt{n}}\right)\right)}{\frac{1}{2^{3/2}\sqrt{e\pi n}}\exp(2\sqrt{n})(1 + O(n^{-1/2}))}$$

which simplifies to the desired result.

□

Proposition 5.5.3. *The mean number of elements in cycles in a superposition of two random partial matchings of* $[n]$ *is* $\frac{1}{2}\sqrt{n} + O(1)$.

Proof. The generating function counting pairs of matchings by their size and number of elements in cycles is

$$S(z, u) = \exp\left(\frac{z}{1-z} + \frac{u^2z^2}{2} + \frac{u^4z^4}{4} + \frac{u^6z^6}{6} + \cdots\right) = \frac{\exp(z/(1-z))}{\sqrt{1-u^2z^2}}.$$

By now the pattern of proof is clear; we want

$$\frac{[z^n]S_u(z,1)}{[z^n]S(z,1)} = \frac{[z^n]\frac{z^2}{(1-z^2)^{3/2}}\exp(z/(1-z))}{[z^n]\exp(z/(1-z))/\sqrt{1-z^2}}.$$

The denominator is known. The numerator can be found from Theorem 2.3.12 with $k = 0, \beta = -3/2, \Phi(z) = z^2/(e(1+z)^{3/2})$. □

Theorem 5.5.4. *The mean number of cycles in a composition of two uniform random involutions on* $[n]$ *is* $\sqrt{n} + \frac{1}{2}\log n + O(1)$.

Proof. A superposition of partial matchings with k paths and l (graph) cycles is identified with a composition of involutions having $k+2l$ (permutation) cycles. Therefore, the mean number of cycles in a composition of two uniform random involutions is the sum of:

- the mean number of paths in a superposition of partial matchings, from Proposition 5.5.1, and
- twice the mean number of cycles in a superposition of partial matchings, from Proposition 5.5.2.

$\square$

Proposition 5.5.5. *The probability that a superposition of two partial matchings of* $[n]$ *selected uniformly at random has no cyclic components is* $\sqrt{2}n^{-1/4} + O(n^{-3/4})$ *as* $n \to \infty$.

Proof. Partial matchings with no cyclic components have generating function $Q(z, 1, 0) = \exp(z/(1-z))$; thus the probability in question is

$$\frac{[z^n]\exp\left(\frac{z}{1-z}\right)}{[z^n]\exp\left(\frac{z}{1-z}\right)/\sqrt{1-z^2}}$$

By Theorem 2.3.12 the numerator is $e^{2\sqrt{n}}/(2n^{3/4}\sqrt{e\pi})(1+O(n^{-1/2}))$; the denominator is $e^{2\sqrt{n}}/\sqrt{8\pi en}(1+O(n^{-1/2}))$, giving the desired result. $\square$

5.6 Fixed-point-free involutions

Proposition 5.6.1. *The number of pairs of fixed-point-free involutions* (σ, τ) *on* $[2n]$ *such that* $\pi = \tau \circ \sigma$ *has* $2c_k$ k*-cycles for each* k *is the same as the number of permutations of* $[2n]$ *which have* c_k $2k$*-cycles for each* k*, and no cycles of odd length.*

Proof. We construct a bijection between the two sets. Given such a pair of fixed-point-free involutions, the graph of $\sigma \cup \tau$ consists of c_k graph cycles of length $2k$, with the edges alternately solid and dotted. From each graph cycle we construct a permutation cycle. We need only make a choice of direction, say by starting at the smallest element and following the solid edge out of that element. This operation is clearly reversible; given a permutation with only even cycles we can reconstruct the graph $\sigma \cup \tau$ of a pair of fixed-point-free involutions. □

For example, the pair of involutions $\sigma = (12)(34)(56), \tau = (16)(23)(45)$, with $\tau \circ \sigma = (135)(264)$, corresponds to a graphical 6-cycle; this cycle can be read as the permutation (123456).

Proposition 5.6.2. *The number of cycles in a composition of two fixed-point-free involutions on* $[2n]$ *chosen uniformly at random has the distribution of* $2\sum_{k=1}^{n} X_k$*, where* X_k *is Bernoulli with mean* $1/(2k-1)$ *and the* X_k *are independent.*

Proof. From Theorem 4.3.7, the distribution of the number of cycles of a permutation of $[2n]$ with all cycle lengths even is that of $\sum_{k=1}^{n} X_k$, where X_k is Bernoulli with mean $1/(2k-1)$ and the X_k are independent. From Proposition 5.6.1, there are exactly the same number of permutations of $[2n]$ with $2j$ cycles, all of even length, as there are pairs of fixed-point-free involutions $(\sigma, \tau) \in S_{2n} \times S_{2n}$ with $\tau \circ \sigma$ having j cycles. $\square$

Note that the expected number of cycles in a composition of two fixed-point-free involutions of $[2n]$ is $2H_{2n} - H_n = \log n + (2 \log 2 + \gamma) + O(n^{-2})$, which differs from the expected number of cycles in a random permutation of $[2n]$ by $\log 2 + O(n^{-1})$. However, compositions of fixed-point-free involutions do not "look like" random permutations. Most obviously, since cycles come in pairs of the same length, a composition of fixed-point-free involutions of $[n]$ has no cycles longer than $n/2$. Cycle lengths satisfy the following limit law.

Proposition 5.6.3. *Fix constants* $0 \leq \gamma \leq \delta \leq 1/2$*. Let* $p_i(n; \gamma, \delta)$ *be the probability that* 1 *is contained in a cycle of* $\tau \circ \sigma$ *of length between* γn *and* δn*, where* σ *and* τ *are fixed-point-free involutions on* $[n]$ *chosen uniformly at random. Then* $\lim_{n \to \infty} p_i(n; \gamma, \delta) = \sqrt{1 - 2\gamma} - \sqrt{1 - 2\delta}$*.*

Proof. Call a permutation with all cycle lengths even an E-permutation, and a composition of fixed-point-free involutions an I-permutation. The number of $2k$-cycles in E-permutations of $[n]$ is half the number of k-cycles in I-permutations of $[n]$, by Proposition 5.6.1. In particular the number of elements of $2k$-cycles in E-permutations of $[n]$ and the number of elements of k-cycles in I-permutations of $[n]$ are equal. So the probability that a random element of a random I-permutation of $[n]$ is in a cycle of length in $[\gamma n, \delta n]$ is equal to the probability that a random element of a random E-permutation of $[n]$ is in a cycle of length in $[2\gamma n, 2\delta n]$. By Theorem 4.3.5, the latter probability approaches $\sqrt{1-2\gamma} - \sqrt{1-2\delta}$ as $n \to \infty$. □

Proposition 5.6.4. *Fix $\epsilon \in (0, 1/2)$. The expected number of elements in k-cycles in a composition of two random fixed-point-free involutions of $[n]$ converges uniformly to $(1-2k/n)^{-1/2}$ as $k/n \to \infty$ with $0 < k/n < 1/2 - \epsilon$.*

Proof. By Proposition 5.4.2 the expected number of elements in r-cycles in a superposition $\sigma \cup \tau$ of fixed-point-free perfect matchings is

$$\frac{(n/2)!^2}{((n-r)/2)!^2} 2^r \frac{(n-r)!}{n!}.$$

In the case $r = \alpha n$, this is asymptotic to $1/\sqrt{1-\alpha}$ as $n \to \infty$, with uniform convergence over $0 < \alpha < 1$; this is shown in Proposition 4.3.4, where the same expression occurs in relation to permutations with all cycle lengths even. Noting that elements in r-cycles in a pair of perfect matchings give rise to elements in $r/2$-cycles of the corresponding permutation gives the desired result. □

5.7 The number of factorizations of a permutation into involutions

Let $\pi \in S_n$ be a permutation, and let λ, μ be partitions of the integer n. The problem of determining the number of solutions to $\pi = \tau \circ \sigma$ in which σ has cycle type λ and τ has cycle type μ has been called the "class multiplication problem" by Stanley [Sta09]. One case of substantial prior interest has been the case in which π is a full cycle and λ, μ are generic; see for example [Bia04, Irv06, PS02]. Bóna and Flynn [BF08] asked for the probability that two fixed elements of $[n]$ lie in the same cycle of the product of two random n-cycles; this is $1/2$ if n is odd, as shown in [Sta09]. We note that this also requires at least one of the permutations involved to be an n-cycle. Involutions, we will see, are another extreme case of the class multiplication problem; furthermore the techniques used here to enumerate factorizations of permutations into involutions are purely enumerative, as opposed to the more algebraic approaches of previous work.

The square of the number of involutions of $[n]$ is a bit larger than $n!$; we have

$$a_n^2 \sim n! \cdot \frac{e^{2\sqrt{n}}}{\sqrt{8\pi e n}}.$$

The mean number of factorizations of a random permutation into a product of involutions is just the second factor. The number of factorizations can be as large as a_n for the identity permutation, since $id = \sigma^2$ for any involution σ, and as small as $n-1$ for those permutations which consist of an $(n-1)$-cycle and a 1-cycle.

Theorem 5.7.1. *Define the function*

$$f(r,k) = \sum_{j=0}^{\lfloor r/2 \rfloor} \frac{r!}{(r-2j)!j!2^j} k^{r-j}.$$

Let π be a permutation of $[n]$ with c_k cycles of length k, for each k. Then

$$F(\pi) = \prod_{k=1}^{n} f(c_k, k)$$

is the number of factorizations of π into two involutions, i. e. the number of solutions of $\pi = \tau \circ \sigma$ with σ and τ involutions.

We remark that $f(r,k)$ is the number of partial matchings of $[r]$ with k-colored components. This interpretation is key to the proof, which works by pairing up some of the k-cycles and then assigning one of k partial factorizations to each unpaired k-cycle or pair of k-cycles.

We begin with the following special case.

Lemma 5.7.2. *The number of ways to factor an n-cycle π into two involutions is n.*

Proof. Without loss of generality let $\pi = (123\cdots n)$. We construct a corresponding pair of partial matchings (σ, τ). This must be a path of length n, since cycles in $\sigma \cup \tau$ give rise to pairs of permutation cycles. So we consider an unlabeled path of length n with alternating solid and dotted edges, and attempt to label it. If n is odd, we begin by labelling some vertex by 1. Then follow the solid edge at that vertex, followed by the dotted edge at the next vertex, to determine the site of 2; repeat to determine the sites of 3, 4, and so on. The remaining vertices can therefore be labelled in exactly one way. If n is even, then there are only $n/2$ inequivalent sites at which to begin this process, but there are two ways to color the unlabelled path. $\square$

For example, the cycle (1234) has the factorizations

$$\begin{aligned}(\sigma,\tau) &= ((1)(24)(3),(12)(34)),((13)(2)(4),(14)(23)),\\ &\quad ((12)(34),(2)(13)(4)),((14)(23),(24)(1)(3)).\end{aligned}$$

These correspond to the labelled paths illustrated in Figure 5.7.

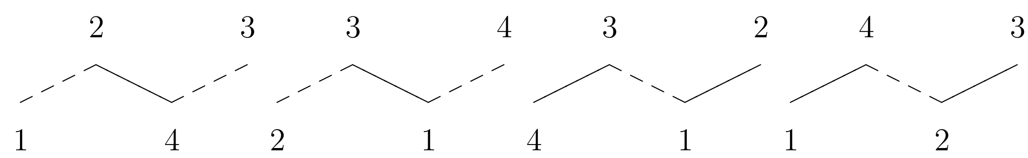


Figure 5.3: Factorizations of (1234) into involutions.

This is also a special case of the following theorem of Goupil and Schaeffer for the number of factorizations of an n-cycle into permutations of types λ and μ.

Theorem 5.7.3. *[GS98, Thm. 2.1] Let $\ell(\lambda)$ denote the number of parts of the partition λ; let $(n_1, \ldots, n_k) \vDash n$ mean $n_1 + \ldots + n_k$ is a composition of n. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be any two partitions of n with $g(\lambda, \mu) = g$, where $\ell(\lambda) + \ell(\mu) = n + 1 - 2g(\lambda, \mu)$. Then the number of factorizations of an n-cycle into a permutation of type λ and a permutation of type μ is*

$$c^n_{\lambda\mu} = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{g_1+g_2=g} (l+2g_1-1)!(m+2g_2-1)! \sum_{\substack{(i_1,\ldots,i_l)\vDash g_1 \\ (j_1,\ldots,j_m)\vDash g_2}} \prod_r \binom{\lambda_r}{2i_r+1} \prod_r \binom{\mu_r}{2j_r+1}$$

where $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$ for a partition $\lambda = 1^{\alpha_1} \cdots n^{\alpha_n}$.

Proof of Lemma 5.7.2 from Theorem 5.7.3. We observe that if the product of two involutions of type λ and μ is to be a single n-cycle, if $n = 2k+1$, then λ and μ each have type $2^k 1$; if $n = 2k$, either λ has type $2^{k-1}1^2$ and μ has type 2^k, or vice versa.

First we consider the case where $n = 2k+1$ is odd. We thus have $\lambda_1 = \cdots = \lambda_k = \mu_1 = \cdots = \mu_k = 2, \lambda_{k+1} = \mu_{k+1} = 1$. In this case, we compute $g(\lambda, \mu) = 0$, $z_\lambda = z_\mu = k!2^k$. Thus the outer sum has the single term $g_1 = g_2 = 0$, and the inner double sum also has the single term $i_1 = \cdots = i_l = j_1 = \cdots = j_m = 0$. The formula of Theorem 5.7.3 thus becomes

$$c_{\lambda\mu}^{2k+1} = \frac{2k+1}{k!^2 4^k}(k!)(k!)\left[\binom{2}{1}^k \binom{1}{1}\right]^2 = 2k+1.$$

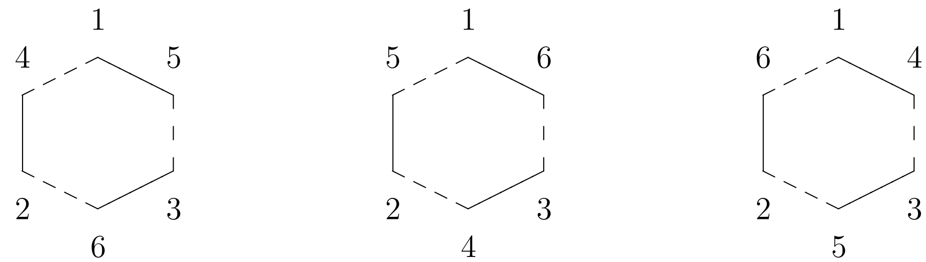
In the case where $n = 2k$ is even, note that Theorem 5.7.3 is symmetric under interchanging λ and μ. So it suffices to compute $c_{\lambda\mu}^{2k}$ in the case $\lambda = 2^{k-1}1^2, \mu = 2^k$ and double the result. Here we have $z_\lambda = 2^k(k-1)!, z_\mu = 2^k k!$, and again $g = 0$; again the outer sum has the single term $g_1 = g_2 = 0$ and the inner double sum has the single term corresponding to the pair of empty compositions. We get

$$c_{\lambda\mu}^{2k} = \frac{2k}{4^k(k-1)!k!}k!(k-1)!\binom{2}{1}^{k-1}\binom{1}{1}^k\binom{2}{1}^k = k.$$

The actual number of factorizations is $c_{\lambda\mu}^{2k} + c_{\mu\lambda}^{2k} = 2c_{\lambda\mu}^{2k} = 2k$, as desired. □

Lemma 5.7.4. *The number of ways to factor a permutation π of $[2n]$ consisting of two n-cycles into two involutions σ, τ, such that the corresponding graph $\sigma \cup \tau$ is a $2n$-cycle, is n.*

Proof. Without loss of generality, let $\pi = (1, 2, \ldots, n)(n+1, n+2, \ldots, 2n)$ in cycle notation. We draw a graphical cycle with $2n$ vertices, with edges alternately solid and dotted. Label some arbitrary vertex with 1; follow solid and dotted edges alternately around the cycle to place $2, 3, \ldots, n$. Then label some arbitrary unlabeled vertex with $n+1$ and follow solid and dotted edges alternately around the cycle to place $n+2, \ldots, 2n$. There are $2n^2$ ways to carry out this procedure. However, the unlabeled $2n$-cycle with alternately colored edges has $2n$ symmetries. So there are $(2n^2)/(2n) = n$ distinct labellings; each one corresponds to a factorization. □



(15)(24)(36) ∘ (14)(26)(35) (16)(25)(34) ∘ (15)(24)(36) (14)(26)(35) ∘ (16)(25)(34)

Figure 5.4: Factorizations of (123)(456) into involutions, which correspond to graphical 6-cycles.

Proof of Theorem 5.7.1. Given an arbitrary permutation π to be factored into involutions with $\pi = \tau \circ \sigma$, we can consider the cycles of each length separately. Consider the cycles of length k; assume there are r of these. We pair up some of the k-cycles with each other, representing that they come from the same cycle in the graph $\sigma \cup \tau$. Those cycles which remain unpaired arise from paths, not cycles, in $\sigma \cup \tau$. We then factor each unpaired cycle according to Lemma 5.7.2, and each pair of cycles according to Lemma 5.7.4. If there are j pairs of cycles, then there are $r-2j$ unpaired cycles, and thus $r-j$ total components to factor; thus the number of such factorizations, once the cycles are paired up, is k^{r-j}. The number of ways to find j disjoint pairs of cycles, with order irrelevant, is

$$\frac{\binom{r}{2}\binom{r-2}{2}\cdots\binom{r-2j+2}{2}}{j!} = \frac{r!}{(r-2j)!j!2^j}.$$

Summing over j gives the function $f(r,k)$ defined in the theorem. $\square$

For example, consider the factorizations of the permutation $\pi = (12)(345)$ into involutions. We have $f(1,k) = k$, and so the number of factorizations of π into involutions is $F(\pi) = f(1,2)f(1,3) = 2 \cdot 3 = 6$. We note that $(12) = \tau \circ \sigma$, where τ and σ are involutions, has the solutions

$$(\sigma, \tau) \in \{((12), id), (id, (12))\}. \tag{5.7}$$

Similarly, (345) has factorizations

$$(\sigma, \tau) \in \{((35), (45)), ((34), (35)), ((45), (34))\}. \tag{5.8}$$

We can combine the factorizations in (5.7) and (5.8) to get the factorizations of π:

$$\begin{aligned}(\sigma, \tau) \quad &\in \quad \{((12)(35), (45)), ((12)(34), (35)), ((12)(45), (34)),\\ &= \quad ((35), (12)(45)), ((34), (12)(35)), ((45), (12)(34))\}.\end{aligned}$$

Finally, we can consider the distribution of the number of factorizations of a random permutation of $[n]$ into involutions. We consider the following probability model, which we call the *sharp-cutoff model*. Let $X_k = \mathcal{P}(1/k)$ for $k = 1, 2, \ldots, n$, where n is a positive integer parameter. Let $m = X_1 + 2X_2 + \cdots + nX_n$, and take a permutation of $[m]$ with X_k cycles of length k for each k, chosen uniformly at random from all permutations of that cycle type. We denote the corresponding measure on the set of all permutations by $\mathbb{P}_n^*$. Then $\mathbb{E}\left(\sum_{k=1}^n kX_k\right) = n$. This model generates each permutation of $[m]$ having all cycle lengths less than or equal to n with probability $e^{-H_n}/m!$, where $H_n = \sum_{k=1}^n 1/k$ is a harmonic number; in particular for each $m \leq n$, each permutation of $[m]$ occurs with the same probability.

Theorem 5.7.5. *As $n \to \infty$,*

$$\lim_{n\to\infty} \mathbb{P}_n^* \left(\frac{\log(F(\pi)) - \frac{1}{2}(\log n)^2}{\frac{1}{3}(\log n)^3} \le x \right) \to \Phi(x)$$

Proof. First, we show that $\mathbb{E}(\log f(X_k, k)) = \log(k)/k + O(k^{-3})$. Let $\mu_k = \mathbb{E}(\log f(X_k, k))$. We can write the expectation as a sum over possible values of X_k, giving

$$\mu_k = e^{-1/k} \sum_{r\ge 1} \frac{1}{r!k^r} \log f(r,k). \tag{5.9}$$

We can derive an asymptotic series for $\log f(r, k)$ from the Taylor series for $\log(1+x)$ around $x = 0$; this gives an asymptotic series for the rth term in (5.9), which is of order $k^{-r}\log k$. Adding these gives $\mu_k = (\log k)/k + (1/2k^3) + O(k^{-4})$. Similarly let $h_k = \mathbb{E}((\log f(X_k, k))^2)$; then in like manner we can derive the series $h(k) = (\log k)^2/k + (\log k)^2/k^2 + 2\log k/k^3 + O(k^{-4})$. The variance is given by $\sigma_k^2 = \mathbb{V}(\log f(X_k, k)) = h(k) - \mu_k^2$, and we find $\sigma_k^2 = (\log k)^2/k + 2\log k/k^3 + O(\log k/k^4)$.

Next we show that $\sum_{k=1}^n \mu_k \sim (\log n)^2/2$ and $\sum_{k=1}^n \sigma_k^2 \sim (\log n)^3/3$ as $n \to \infty$. We have $\mu_k = (\log k)/k + O(k^{-3})$. Now, $\sum_{k=1}^n (\log k)/k \sim \int_1^n (\log k)/k\, dk = \frac{1}{2}(\log n)^2$, where the asymptotic equality can be justified by the Euler-Maclaurin summation formula. Expanding the big-O notation, $|\mu_k - (\log k)/k| \le Ck^{-3}$ for some constant C, so $\sum_{k=1}^\infty \mu_k - (\log k)/k$ converges. Therefore $\sum_{k=1}^n \mu_k \sim \sum_{k=1}^n (\log k)/k \sim \frac{1}{2}(\log n)^2$. The proof for $\sum_{k=1}^n \sigma_k^2$ is similar. Note that $\sum_{k=1}^n \mu_k = \mathbb{E}(\log F(\pi))$ and $\sum_{k=1}^n \sigma_k^2 = \mathbb{V}(\log F(\pi))$. Finally, we apply Lyapunov's central limit theorem (Theorem 2.4.2) to show that $\log F(\pi)$ is asymptotically normal. We will take $\delta = 1$, and $Y_k = \log f(X_k, k)$. As previously shown, $s_n^2 \sim (\log n)^3/3$, so $s_n^3 \sim (\log n)^{9/2}/(3\sqrt{3})$. We also observe $\mathbb{E}(|Y_k|^3)$ is finite for each k. To check (2.4), first note that

$$\mathbb{E}(|Y_k - \mathbb{E}Y_k|^3) = \sum_{r\geq 1} \left[(\log f(r,k) - \mathbb{E}Y_k)^3 \mathbb{P}(X_k = r)\right] + (\mathbb{E}Y_k)\,\mathbb{P}(X_k = 0).$$

Since $\mathbb{E}Y_k$ is positive, this is less than

$$\left[\sum_{r\geq 1} (\log f(r,k))^3 \mathbb{P}(X_k = r)\right] + (\mathbb{E}Y_k)\,\mathbb{P}(X_k = 0).$$

The first term in this equation is in fact $\mathbb{E}(Y_k^3)$. (The sum giving $\mathbb{E}(Y_k^3)$ should naturally be over $r \geq 0$, but $f(0,k) = 1$ and so the $r = 0$ term does not contribute to the sum.) Therefore we have

$$\mathbb{E}(|Y_k - \mathbb{E}Y_k|^3) \leq \mathbb{E}(Y_k^3) + (\mathbb{E}Y_k)\mathbb{P}(X_k = 0) \leq \mathbb{E}(Y_k^3) + \mathbb{E}(Y_k).$$

But $\mathbb{E}(Y_k^3) \sim (\log k)^3/k$ and $\mathbb{E}Y_k \sim (\log k)/k$ as $k \to \infty$. so $\mathbb{E}(|Y_k - \mathbb{E}Y_k|^3) \sim (\log k)^3/k$. Therefore we have

$$\frac{1}{s_n^3} \sum_{k=1}^{n} \mathbb{E}(|Y_k - \mathbb{E}Y_k|^3) \sim \frac{3^{3/2}}{(\log n)^{9/2}} \frac{(\log n)^4}{4} = \frac{3^{3/2}/4}{\sqrt{\log n}}$$

and in particular this goes to 0 as $n \to \infty$, so (2.4) is satisfied. Therefore the standardization of $\log F(\pi)$ converges in distribution to the standard normal, as desired. $\square$

It is natural to suspect that the distribution of the number of factorizations of a permutation of fixed size, chosen uniformly at random, also has a lognormal distribution. Generating large numbers of random permutations lends some support to such a conjecture. The mean of the logarithm of the number of factorizations of a random permutation appears to be near $(\log n)^2/2$; the variance appears to be of order

$(\log n)^3$, but with a smaller constant than in Theorem 5.7.5, between about 0.1 and 0.2. This seems plausible, as permutations from the sharp-cutoff model vary in size, and this variation in size contributes to the variation in number of factorizations.

Finally, we can refine Theorem 5.7.1 to count the number of factorizations $\pi = \tau \circ \sigma$ where σ and τ are involutions with s and t fixed points, respectively. This requires determining all the possible unlabeled graphs on $[n]$ with properly 2-colored edges which can be labeled to give two involutions which compose to a permutation with the cycle type of π, and then counting the labellings which actually give π. This is impractical for large s and t. The fixed-point-free case, though, is straightforward.

Proposition 5.7.6. *Let $c_1, c_2, \ldots, c_n$ be nonnegative even integers with $\sum_{k=1}^{n} kc_k = n$. Then the number of factorizations of a permutation π of type $1^{c_1} \ldots n^{c_n}$ into two fixed-point-free involutions is*

$$\prod_{k=1}^{n} (c_k - 1)!!k^{c_k/2}$$

where we adopt the convention $(-1)!! = 1$.

Proof. The graph $\sigma \cup \tau$ corresponding to such a factorization consists of c_k cycles of length $2k$, for each k. The permutation k-cycles can be paired up into graphical cycles in $(c_k - 1)!!$ ways. Each pair of permutation k-cycles thus obtained can be used to label a graphical cycle in any of k ways, following Lemma 5.7.4. Thus the number of ways to arrange the elements of k-cycles of π in the graphical representation is $(c_k - 1)!!k^{c_k/2}$. The total number of factorizations is just the product over cycle lengths. □

We note that if any of the c_k are odd, then π has *no* factorizations into fixed-point-free involutions. Furthermore, the proportion of permutations of $[n]$ having all c_k even (that is, an even number of cycles of each length) is $\Theta(n^{-2})$.

5.8 Pattern avoidance

To provide further evidence that involutions are in some sense a “square root” of permutations, we consider pattern avoidance in permutations. (A useful introduction to pattern avoidance is [Bón04, Ch. 4-5].)

Let $\pi \in S_n$ and $\sigma \in S_m$ be permutations. The pattern σ is said to occur in the permutation π, or π is said to contain σ, if there exist $1 \leq \rho(1) < \dots \rho(m) \leq n$ such that $\pi(\rho(i)) < \pi(\rho(j))$ if and only if $\sigma(i) < \sigma(j)$. That is, there is some subsequence of π (written in the one-line notation) that has the same order type as σ. For example, the permutation 5**7**21**3**6**4** contains the pattern 312, as indicated by the bolded elements. If the pattern σ does not occur in π, then π is said to be σ-avoiding. For example, the permutation 2431765 is 312-avoiding.

Let $S_n(\pi)$ denote the set of π-avoiding permutations of $[n]$. We say two patterns π and ρ are *Wilf-equivalent* if $|S_n(\pi)| = |S_n(\rho)|$ for all $n \geq 0$. The Stanley-Wilf conjecture (now the Marcus-Tardos theorem [MT04]) on pattern avoidance gives the possible growth rates of sequences $\{|S_n(\pi)|\}_{n\geq 0}$. Stanley and Wilf conjectured, and Marcus and Tardos proved, that for each pattern π, $|S_n(\pi)|$ is bounded above by C^n, for some constant C depending on π. We call the smallest such C the *growth rate* of the pattern π and denote it by $L(\pi)$. Arratia [Arr99] has shown that the Marcus-Tardos theorem is equivalent to the existence of the limit $\lim_{n\to\infty} |S_n(\pi)|^{1/n}$, which equals $L(\pi)$, for all patterns π.

Now let $I_n(\pi)$ denote the set of π-avoiding *involutions* of $[n]$. Then we can define the *involutory growth rate* of a pattern, $L_i(\pi) = \lim_{n\to\infty} |I_n(\pi)|^{1/n}$. This limit may not exist in general, but it does in some special cases, leading to the following conjecture.

Conjecture 5.8.1. *Let π be a permutation pattern. Then $L_i(\pi)$ exists and $L_i(\pi)^2 = L(\pi)$.*

Table 5.1 shows $I_n(\pi)$, $S_n(\pi)$, and the ratio of their squares in cases when both are known. We note in particular that the conjecture is true for all patterns of length at most 3. We also note that Wilf-equivalence of two patterns is not the same as "involutory Wilf-equivalence". In particular $|S_n(\pi^r)| = |S_n(\pi)|$, where π^r is the reversal of π, but it is not necessarily true that $|I_n(\pi^r)| = |I_n(\pi)|$. Counterexamples include $\pi = 132$ and $\pi = 12345$; we have $|I_n(12345)| \sim (\pi^3/8)4^n n^{-3}$ [Reg81] but $I_n(54321) \sim \frac{32}{\pi} 4^n n^{-3}$ [BM03].

The pattern 1342 has growth rate 8 and the pattern 12453 has growth rate $(1 + \sqrt{8})^2$ [Bón05]; the latter is the first known example of a pattern with non-integer growth rate. Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a pattern, and let $\pi' = 1(\pi_1 + 1) \cdots (\pi_n + 1)$ be another pattern. Bóna has shown [Bón05, Lemma 5.4] that if $L(\pi) = g^2$, then $L(\pi') = (g + 1)^2$. In other words, this operation raises the square root of the growth rate by 1; thus there is some precedent for studying $\sqrt{L(\pi)}$. Perhaps in general

π	$I_n(\pi)$	$S_n(\pi)$	$I_n(\pi)^2/S_n(\pi)$
$12\dots k$	$\sim a_k(k-1)^n$ $(1/n)^{(k-1)(k-2)/4}$ [Reg81, 4.5 Case 1], [Ges90]	$\sim b_k(k-1)^{2n}$ $(1/n)^{k^2/2-k}$ [Reg81, 4.5 Case 2]	$\sim c_k n^{-1+k/2}$
1234, 2143, 3412, 4321, 1243	$M_n \sim \sqrt{\frac{27}{4\pi}}3^n n^{-3/2}$ [EM04]	$\sim \frac{81\sqrt{3}}{16\pi}9^n n^{-4}$ [Wes90, Cor. 3.1.7] shows patterns are Wilf-equivalent	$\frac{4\sqrt{3}}{9}n$
123, 132, 213, 321	$\binom{n}{\lfloor n/2\rfloor} \sim 2^n/\sqrt{\pi n}$ [SS85]	$C_n \sim 4^n/\sqrt{\pi n^3}$	$\sqrt{n/\pi}$
231, 312	2^{n-1} [SS85]	$C_n \sim 4^n/\sqrt{\pi n^3}$	$\sqrt{\frac{\pi}{16}}n^{3/2}$
54321	$C_{\lceil n/2\rceil}C_{1+\lfloor n/2\rfloor} \sim \frac{32}{\pi}\frac{4^n}{n^3}$ [BM03]	$2^{25/2}3\pi^{-3/2}16^n n^{-15/2}$ [Reg81] and symmetry	$\frac{1}{24}\sqrt{\frac{2}{\pi}}n^{3/2}$

Table 5.1: Table of patterns for which the ordinary and involutory growth rates are both known. C_n and M_n are the Catalan and Motzkin numbers, respectively; a_k, b_k, c_k are constants depending on k.

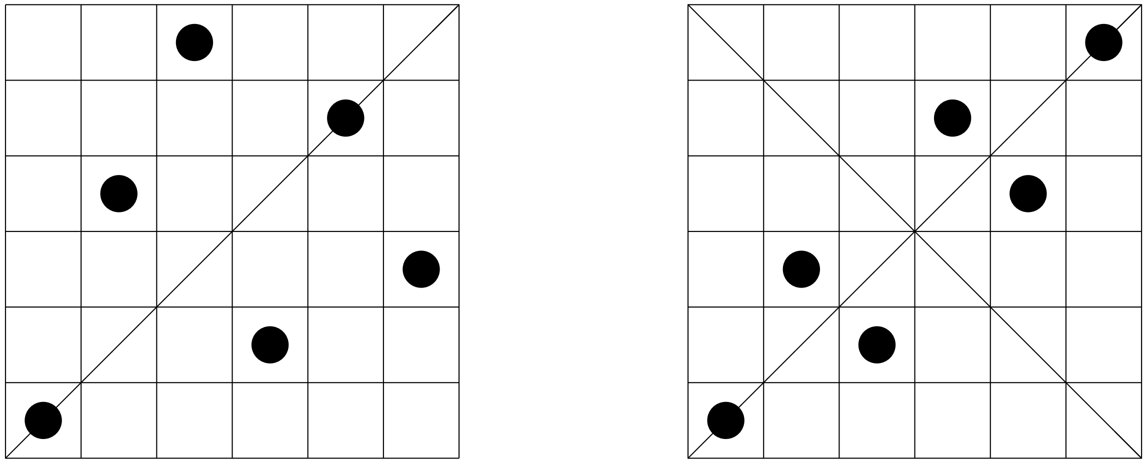


Figure 5.5: The graph of the involution $146253 \in S_6$, and the reverse-complement-invariant involution $132546 \in S_6$.

$L_i(\pi') = L_i(\pi) + 1$.

Conjecture 5.8.1 can be restated probabilistically. The probability that a random permutation of $[n]$ is π-avoiding seems to be the square of the probability that a random involution of $[n]$ is π-avoiding, multiplied by some asymptotically subexponential factor. (In the few known cases this factor is Cn^{-k} for some real constant C and nonnegative rational number k.) Thus involutions are, in general, more likely to avoid patterns than ordinary permutations. This is because an involution is, in a sense, half a permutation. The RSK algorithm [Sta99] takes a permutation π to a pair of Young tableaux (P, Q); if π is an involution then $P = Q$, so involutions can be identified with individual Young tableaux. The "graph" of a permutation π is the set of points $\{(i, \pi(i)) : 1 \leq i \leq n\}$ and an involution can be specified by fixing only the points on or below the diagonal, identifying involutions with half-graphs. See Figure 5.5 for an illustration of the graph of an involution.

Finally, Egge has studied permutations with graphs which are symmetric under other reflections or rotations [Egg07]. One might hope these lead to further generalizations of Conjecture 5.8.1. For example, consider the *reverse complement* map on permutations, $\pi \to \pi^{rc}$, which takes $\pi_1 \dots \pi_n$ to $(n+1-\pi_n) \dots (n+1-\pi_1)$; this map corresponds to rotation of the graph of π by a half-rotation. Involutions invariant under the reverse complement are therefore invariant under both rotation by a half-rotation and reflection over the diagonal, and so are determined by one-fourth of their graph. See Figure 5.5 for an example of such a permutation; note that the permutations can be reconstructed from any of the four quadrants into which the dotted lines split the diagram. The number of 132-avoiding, rc-invariant involutions of $[2n]$ or $[2n+1]$ is 2^n [Egg07, Cor. 3.5]. Up to polynomial factors this is the fourth root of the Catalan number C_{2n} or C_{2n+1}, which is the number of 132-avoiding permutations.

Wulcan [Wul02] has enumerated involutions avoiding generalized patterns, including all the generalized patterns of length 3; at this point no systematic review of the growth rates of the corresponding patterns in permutations has been undertaken.

5.9 Inversions in involutions

A well-known result from the folklore is the asymptotic distribution of the number of inversions in a random permutation of $[n]$. For a permutation σ, these are pairs i, j such that $i < j$ and $\sigma(i) > \sigma(j)$. The distribution can be found by noting that the distribution of the number of inversions of a permutation of $[n]$ is that of

$U_1 + U_2 + \cdots + U_n$, where U_k is a uniform random variable on $\{0, 1, \ldots, k-1\}$. We have $\mathbb{E}U_k = (k-1)/2$ and $\mathbb{V}U_k = (k^2-1)/12$. Thus

$$\mathbb{E}(U_1 + \ldots + U_n) = \frac{n^2-1}{4}, \mathbb{V}(U_1 + \ldots + U_n) = \frac{2n^3+3n^2-5n}{72}.$$

We can check that the distribution is normal using the Lyapunov theorem; alternatively, Bóna [Bón08, Thm. 10] has proven that the standardization of the number of occurrences of a fixed pattern in a random permutation of n converges to the standard normal, and inversions are just 21-patterns.

Theorem 5.9.1. *The mean number of inversions in an involution of $[n]$ chosen uniformly at random is $\frac{1}{2}\binom{n}{2} - a_{n-3}/a_n\binom{n}{3}$, where a_n is the number of involutions of $[n]$.*

Proof. It will suffice to count the number of involutions σ of $[n]$ for which $\sigma(i) > \sigma(j)$, for each i, j satisfying $1 \leq i < j \leq n$. These involutions arise in four ways.

1. (ij) is a cycle of σ. There are a_{n-2} ways to complete this to get an involution.

2. i is a fixed point of σ and j is not. To build an involution, first we fix $\sigma(j)$. Since $\sigma(i) = i$ and $\sigma(j) = k$, in order to have an inversion we must have $k < i$. There are thus $i-1$ ways to choose k. There are then a_{n-3} ways to construct an involution on the remaining $n-3$ elements.

3. j is a fixed point, and i is not. We begin by fixing $k = \sigma(i)$; we must have $j < k$ and so there are $n-j$ ways to choose k. We then must construct an involution on the remaining $n-3$ elements, in one of a_{n-3} ways.

4. Neither i nor j are fixed points. Let $\sigma(i) = k, \sigma(j) = l$. There are $\binom{n-2}{2}$ ways to pick k, l so that $\sigma(i) > \sigma(j)$. There are then a_{n-4} ways to complete $(ik)(jl)$ to an involution on $[n]$.

The number of involutions with an inversion at (i, j) – that is, with $i < j$ and $\sigma(i) > \sigma(j)$ – is therefore

$$a_{n-2} + a_{n-3}(n + i - j - 1) + a_{n-4}\binom{n-2}{2}.$$

Now, recall the recurrence $a_n = a_{n-1} + (n-1)a_{n-2}$, which we rewrite in the form $a_{n-2} = (a_n - a_{n-1})/(n-1)$. In the case where $j = i + 1$, then, the number of involutions with an inversion at (i, j) is

$$\begin{aligned}
& a_{n-2} + a_{n-3}(n-2) + a_{n-4}\frac{(n-2)(n-3)}{2} \\
= \quad & a_{n-2} + a_{n-3}(n-2) + \frac{a_{n-2} - a_{n-3}}{n-3}\frac{(n-2)(n-3)}{2} \\
= \quad & a_{n-2}\frac{n}{2} + a_{n-3}\frac{n-2}{2} \\
= \quad & a_{n-2}\frac{n}{2} + \frac{a_{n-1} - a_{n-2}}{n-2}\binom{n-2}{2} \\
= \quad & \frac{1}{2}\left(a_{n-1} + (n-1)a_{n-2}\right) = \frac{1}{2}a_n.
\end{aligned}$$

Therefore the number of involutions with an inversion at (i, j) is

$$\frac{1}{2}a_n - a_{n-3}(j - i - 1).$$

The total number of inversions among all involutions of $[n]$ is therefore

$$\frac{1}{2}a_n\binom{n}{2} - a_{n-3}\sum_{1 \le i < j \le n} j - i - 1 = \frac{1}{2}a_n\binom{n}{2} - a_{n-3}\binom{n}{3}.$$

Dividing by a_n gives the mean number of inversions. □

We note that $\frac{a_{n-3}}{a_n} \sim n^{-3/2}$. Thus the mean number of inversions of an involution is nearly $n^{3/2}/6$ less than the mean number of inversions of a permutation; this is one standard deviation below the mean.

Let $b_n = \frac{1}{2}a_n\binom{n}{2} - a_{n-3}\binom{n}{3}$ be the total number of inversions in involutions of n. Let $c_n = \sum_{\sigma \in S_n, \sigma^2=1}(inv(\sigma))(inv(\sigma)-1)$. For future use, we can find the generating function of the b_n.

Proposition 5.9.2. *The b_n have exponential generating function*

$$\sum_{n\geq 0} b_n \frac{z^n}{n!} = \frac{z^2(6+4z+3z^2)}{12} e^{z+z^2/2}.$$

Proof. This is a simple calculation. We have

$$\sum_{n\geq 0} \frac{1}{2}\binom{n}{2} a_n \frac{z^n}{n!} = \frac{1}{4}\sum_{n\geq 2} \frac{a_n z^n}{(n-2)!} = \frac{1}{4}\sum_{n\geq 0} a_{n+2}\frac{z^{n+2}}{n!} = \frac{z^2 A''(z)}{4}$$

and

$$\sum_{n\geq 0} a_{n-3}\binom{n}{3}\frac{z^n}{n!} = \frac{1}{6}\sum_{n\geq 3}\frac{a_{n-3}z^n}{n!} = \frac{1}{6}z^3 A(z).$$

Putting these together gives

$$\sum_{n\geq 0} b_n \frac{z^n}{n!} = \frac{1}{4}z^2 A''(z) - \frac{1}{6}z^3 A(z) = \frac{z^2(6+4z+3z^2)}{12}e^{z+z^2/2}$$

as desired. □

This is an example indicating that involutions are not especially good at avoiding patterns; the probability of an inversion in any site is asymptotically 1/2, as for ordinary permutations. However, the variance is larger, so the probability of having very few inversions is large compared to that for ordinary permutations, and the probability of having very *many* inversions is large as well.

In particular, we have the following:

Theorem 5.9.3. *The variance of the number of inversions of an involution of* $[n]$ *chosen uniformly at random is asymptotic to* $n^3/18$ *as* $n \to \infty$.

Our starting point for this proof is the following proposition of Dukes:

Proposition 5.9.4 (Dukes)**.** *[Duk07, Prop. 2.8] Let* $I_n(q) = \sum_{\sigma \in S_n, \sigma^2=1} q^{inv(\sigma)}$. *Then*

$$I_{n+2}(q) = I_{n+1}(q) + (q + q^3 + \cdots + q^{2n+1}) I_n(q) \tag{5.10}$$

with $I_0(q) = I_1(q) = 1$.

These polynomials are q-analogues of the Hermite polynomials, as shown by Désarménien [Dés82].

Proof of Theorem 5.9.3. We observe that $a_n = I_n(1), b_n = I_n'(1), c_n = I_n''(1)$. Differentiating (5.10) twice, we get

$$\begin{aligned} I_{n+2}''(q) &= I_{n+1}''(q) + (q + q^3 + \cdots + q^{2n+1})'' I_n(q) \\ &+ 2(q + q^3 + \cdots + q^{2n+1})' I_n'(q) + (q + q^3 + \cdots + q^{2n+1}) I_n''(q). \end{aligned}$$

Substituting $q = 1$ gives the recurrence

$$I_{n+2}''(1) = I_{n+1}''(1) + \frac{5n + 9n^2 + 4n^3}{3} I_n(1) + 2(n^2 + 2n + 1) I_n'(1) + (n+1) I_n''(1).$$

We can rewrite this as

$$c_{n+2} = c_{n+1} + \frac{5n + 9n^2 + 4n^3}{3} a_n + 2(n^2 + 2n + 1)b_n + (n+1)c_n. \tag{5.11}$$

Finally, we multiply through by $z^n/n!$ and sum over $n \geq 0$. Let $C(z) = \sum_{n\geq 0} z^n/n!$ denote the exponential generating function of $\{c_n\}$. Thus c_{n+2} and c_{n+1} in (5.11) become $C''(z)$ and $C'(z)$ respectively. The term involving a_n becomes

$$\frac{5(z\partial_z) + 9(z\partial_z)^2 + 4(z\partial_z)^3}{3} A(z) = \frac{z(18 + 60z + 58z^2 + 45z^3 + 12z^4 + 4z^5)}{3} e^{z+z^2/2}.$$

The term involving b_n becomes

$$(2(z\partial_z)^2 + 4(z\partial_z) + 2)B_z = \frac{z^2(54 + 106z + 165z^2 + 89z^3 + 53z^4 + 10z^5 + 3z^6)}{6} e^{z+z^2/2}.$$

The term involving c_n becomes $(z\partial_z + 1)C(z) = zC'(z) + C(z)$.

So we get the differential equation

$$C''(z) = C'(z) + P(z)e^{z+z^2/2} + zC'(z) + C(z)$$

where

$$P(z) = \frac{36z + 174z^2 + 222z^3 + 255z^4 + 113z^5 + 61z^6 + 10z^7 + 3z^8}{6}.$$

We have the initial conditions $C(0) = C'(0) = 0$, since $C(0) = c_0, C'(0) = c_1$. Write C in the form $C(z) = Q(z)\exp(z + z^2/2)$. Then our differential equation can be rewritten as

$$Q''(z) + (1+z)Q'(z) = P(z)$$

by differentiating and dividing through by $\exp(z+z^2/2)$. Now, write $Q = q_0 + q_1 z + q_2 z^2 + \cdots$; then we have

$$(2q_2+q_1)+(6q_3+q_1+2q_2)z+(12q_4+2q_2+3q_3)z^2+(20q_5+3q_3+4q_4)z^3+\cdots = P(z).$$

In addition, the initial conditions $C(0) = C'(0)$ become $Q(0) = Q'(0) = 0$, and so $q_0 = q_1 = 1$. This gives $q_2 = 0, q_3 = 1$, and so on; eventually we find $q_9 = q_{10} = q_{11} = 0$. Since the z^n coefficient of $P(z)$, which is an eighth-degree polynomial, is a linear combination of q_n, q_{n+1}, q_{n+2}, it follows that all higher q_k are zero; so Q is itself an eighth-degree polynomial. This gives the solution

$$C(z) = \frac{z^3(240+520z+304z^2+220z^3+40z^4+15z^5)}{240}e^{z+z^2/2}.$$

Therefore we can write c_n in terms of the a_n as

$$c_n = (n)_3 a_{n-3} + \frac{13}{6}(n)_4 a_{n-4} + \frac{19}{15}(n)_5 a_{n-5} + \frac{11}{12}(n)_6 a_{n-6} + \frac{1}{6}(n)_7 a_{n-7} + \frac{1}{16}(n)_8 a_{n-8}.$$

Now, rewrite the recurrence $a_n = a_{n-1} + (n-1)a_{n-2}$ as $a_{n-2} = (a_n - a_{n-1})/(n-1)$. Thus we can rewrite a_{n-8} in terms of a_{n-6} and a_{n-7}; iterating this process eventually gives the formula

$$c_n = \frac{1}{240}(2388n-1269n^2+10n^3+15n^4)a_n+\frac{1}{240}(-1164n-572n^2+612n^3-20n^4)a_{n-1}.$$

We can write b_n in such a way as well:

$$\begin{aligned}
b_n &= \frac{1}{2}\binom{n}{2}a_n - a_{n-3}\binom{n}{3} \\
&= \frac{1}{2}\binom{n}{2}a_n - \frac{a_{n-1}-a_{n-2}}{n-2}\binom{n}{3} \\
&= \frac{1}{2}\binom{n}{2}a_n - a_{n-1} - a_{n-2}\frac{n(n-1)}{6} \\
&= \frac{1}{2}\binom{n}{2}a_n - \frac{n(n-1)}{6}a_{n-1} + \frac{n(n-1)}{6}a_{n-2} \\
&= \frac{1}{2}\binom{n}{2}a_n - \frac{n(n-1)}{6}a_{n-1} + \frac{n(n-1)}{6}\frac{a_n - a_{n-1}}{n-1} \\
&= \frac{1}{2}\binom{n}{2}a_n - \frac{n(n-1)}{6}a_{n-1} + \frac{n}{6}a_n - a_{n-1} \\
&= \left(\frac{1}{2}\binom{n}{2} + \frac{n}{6}\right)a_n - \left(\frac{n(n-1)}{6} + \frac{n}{6}\right)a_{n-1} \\
&= \frac{3n^2-n}{12}a_n - \frac{n^2}{6}a_{n-1}.
\end{aligned}$$

Combining the formulas for b_n and c_n in terms of a_n, a_{n-1}, we get

$$\begin{aligned}
\sigma_2^n &= \frac{c_n}{a_n} + \frac{b_n}{a_n} - \left(\frac{b_n}{a_n}\right)^2 \\
&= -\frac{1}{36}n^4q_n^2 + \left(\frac{-97}{20}n - \frac{51}{20}n^2 + \frac{227}{90}n^3\right)q_n + \left(\frac{148}{15}n - \frac{227}{45}n^2 + \frac{1}{12}n^3\right)
\end{aligned}$$

where $q_n = a_{n-1}/a_n$.

Now, $q_n \sim n^{-1/2}$ as $n \to \infty$, so the asymptotically dominant terms are the terms in $n_4q_n^2$ and n^3; we have

$$\sigma_n^2 = -\frac{1}{36}n^4q_n^2 + \frac{1}{12}n^3 + O(n^{5/2}) = -\frac{1}{36}n^3 + \frac{1}{12}n^3 + O(n^{5/2}) = \frac{1}{18}n^3 + O(n^{5/2})$$

as desired. $\square$

Dukes also counts fixed-point-free involutions by number of inversions. Let $J_n(q)$ be the polynomial counting fixed-point-free involutions of $[n]$ by number of inversions;

then

$$J_{n+2}(q) = (q + q^3 + \cdots + q^{2n+1})J_n(q) \tag{5.12}$$

with $J_0(q) = 1$. Probabilistically, this means that the number of inversions of a random fixed-point-free involution of $[2m]$ is given by the sum $Y_m = X_1 + \cdots + X_m$, where the X_k are independent, and X_k is a uniform random variable on $\{1, 3, 5, \cdots, 4k+1\}$. We have $\mathbb{E}X_k = 2k+1$ and $\mathbb{V}X_k = \frac{(2k+1)^2-1}{3}$. The number of inversions of a random fixed-point-free involution of $n = 2m$ therefore has mean

$$\sum_{k=1}^{n/2}(2k+1) = \frac{n^2+4n}{4}$$

and variance

$$\sum_{k=1}^{n/2} \frac{(2k+1)^2-1}{3} = \frac{n(n+2)(n+4)}{18}.$$

So we have, asymptotically, the same mean and variance as in the general case of involutions.

Finally, we can give combinatorial derivations of the generating polynomials (5.10) and (5.12). The fixed-point-free case (5.12) is the simpler of the two. We will construct a bijection between the set $\mathcal{J}_{2n+2}$ of fixed-point-free involutions of $[2n+2]$ and $2n+1$ copies of the set $\mathcal{J}_{2n}$ of fixed-point-free involutions of $[2n]$. We denote elements of one of the copies by (σ, r) where $\sigma \in \mathcal{J}_{2n}$ and $r \in \{2, 3, \ldots, 2n+2\}$. Then $f : \mathcal{J}_{2n} \times \{2, \ldots, 2n+2\} \to \mathcal{J}_{2n+2}$ operates as follows, given $\sigma \in \mathcal{J}_{2n}, r \in \{2, \ldots, 2n+2\}$:

- Write σ in cycle notation as $(a_1 a_2)(a_3 a_4) \cdots (a_{2n-1} a_{2n})$.

- Let $a'_k = a_k + 1$ for $1 \le k \le r-2$, and $a'_k = a_k + 2$ for $r-1 \le k \le 2n$.

- Take $f(\sigma, r) = (1r)(a'_1 a'_2)(a'_3 a'_4)\cdots(a'_{2n-1}a'_{2n})$.

For example, $f((13)(26)(45), 4) = (14)(25)(38)(67)$. Essentially, to find $f(\sigma, r)$ we insert the cycle $(1r)$ into σ and rename the elements of σ accordingly. Now, we compute $inv(f(\sigma, r))$. The inversions of $f(\sigma, r)$ come in three types:

- Inversions inherited from σ; there are $inv(\sigma)$ of these.

- Inversions which have r as the element appearing first. There are $r-1$ of these, since r appears before all of $1, 2, \ldots, r-1$.

- Inversions which have 1 as the element appearing second. There are $r-1$ of these, since 1 appears in the rth position.

But the second and third types overlap; the inversion formed by r and 1 is counted twice. Therefore $inv(f(\sigma, r)) = inv(\sigma) + 2r - 3$. Therefore, for $\sigma \in \mathcal{J}_{2n}$, the permutations $f(\sigma, 2), \cdots, f(\sigma, 2n+2)$ have $1, 3, \cdots, 4n+1$ more inversions than σ. So the generating polynomial of the $f(\sigma, r)$ by number of inversions is exactly $(q+q^3+\cdots+q^{4n+1})$ times the generating polynomial of the σ by number of inversions, which is what we wanted to show.

For the case of involutions in general, we will construct a bijection between the set $\mathcal{I}_{n+2}$ of involutions of $[n+2]$ and the union of $\mathcal{I}_{n+1}$ and $n+1$ copies of $[n]$. We can break up the set $\mathcal{I}_{n+2}$ into those involutions of $[n+2]$ that fix 1 and those that do not. Those that fix 1 clearly are in bijection with $\mathcal{I}_{n+1}$, and furthermore this bijection fixes the number of inversions. Those that do not fix 1 are in bijection with $n+1$

copies of $\mathcal{I}_n$, following the previous paragraph.

5.10 The number of permutations with all cycle lengths in some finite set

The fact that the number of involutions of $[n]$ is approximately $\sqrt{n!}$ can be generalized to permutations with cycle lengths lying in any finite set. We call a permutation with all cycle lengths lying in the set S an S-permutation. The logarithmic asymptotics of S-permutations are governed by the largest element of S.

Theorem 5.10.1. *Let S be a finite set of positive integers, with $m = \max S$, and such that the elements of S do not all have a common factor. Let $n!p_n^{(S)}$ be the number of S-permutations of $[n]$. Then*

$$p_n^{(S)} n!^{1/m} \sim C_S n^{-1/2+1/2m} \exp(f_S(n^{1/m}))$$

for some polynomial f_S of degree $m-1$ and constant C_S which can be explicitly computed. In particular,

$$\lim_{n\to\infty} \frac{\log p_n^{(S)}}{\log n!} = -1/m.$$

The condition $\gcd S = 1$ is a technical one required so that $\exp(\sum_{s\in S} z^s/s)$ is Hayman-admissible.

Proof. We apply Hayman's method (Theorem 2.3.5) to the generating function $f(z) =$

$\exp\left(\sum_{s\in S} z^s/s\right)$. We have

$$p_n^{(S)} \sim \frac{f(r_n)}{r_n^n\sqrt{2\pi b(r_n)}}$$

where $a(z) = \sum_{s\in S} z^s$, r_n is the positive real root of $a(z) = n$, and $b(z) = \sum_{s\in S} sz^s$. Using the Lagrange inversion formula, we can find an asymptotic series for r_n in descending powers of $n^{1/m}$. (See [Wil86] for details.) From this we can determine the leading-term asymptotic behavior of $f(r_n)$ and r_n^n; we get $f(r_n) = \exp(n/m + c_1 n^{(m-1)/m} + \cdots + c_m n^0 + O(n^{-1/m}))$ and $r_n^n = n^{n/m}\exp(d_1 n^{(m-1)/m} + d_2 n^{(m-2)/m} + \cdots + d_m n^0 + O(n^{-1/m}))$ for constants c_k, d_k depending on S. Finally, $b(r_n) \sim mn$. So

$$p_n^{(S)} \sim \frac{\exp(n/m + c_1 n^{(m-1)/m} + \cdots + c_m n^0 + O(n^{-1/m})}{n^{n/m}\exp(d_1 n^{(m-1)/m} + \cdots + d_m n^0 + O(n^{-1/m})}$$

and applying Stirling's approximation gives the result. $\square$

To illustrate the theorem, consider $S = \{1, 2, 3\}$, so r_n is the positive real root of $z + z^2 + z^3 = n$. This has asymptotic series $r_n = n^{1/3} - \frac{1}{3} - \frac{2}{9}n^{-1/3} + \frac{7}{81}n^{-2/3} + O(1/n)$ for large n, which can be computed by a method of undetermined coefficients.

From this we can find the leading terms $r_n^n \sim n^{n/3}\exp(-n^{2/3}/3 - 5n^{1/3}/18)$ and $f(r_n) \sim \exp(n/3 + n^{2/3}/6 + 5n^{1/3}/9 - 5/18)$. Thus

$$p_n^{(S)} \sim \frac{\exp\left(\frac{n}{3} + \frac{1}{2}n^{2/3} + \frac{5}{6}n^{1/3} - \frac{5}{18}\right)}{n^{n/3}\sqrt{6\pi n}}$$

and finally

$$p_n^{(S)} \cdot n!^{1/3} \sim (e^5 2^6 3^9 \pi^6)^{-1/18} n^{-1/3}\exp\left(\frac{1}{2}n^{2/3} + \frac{5}{6}n^{1/3}\right)$$

Corollary 5.10.2. *The expected number of cycles of length k in an S-permutation chosen uniformly at random, where $k \in S$ and $m = \max S$, is $n^{k/m}/k \cdot (1 + o(1))$ as $n \to \infty$.*

This has also been shown by Benaych-Georges [BG07] and Timashev [Tim08].

Proof. Let $a_n = n! p_n^{(S)}$ be the number of S-permutations of $[n]$. The generating function of S-permutations by their size and number of k-cycles is

$$G^{(S)}(z, u) = \exp\left(\left(\sum_{s \in S} z^s/s\right) + (u-1)z^k/k\right).$$

The mean number of k-cycles in S-permutations of $[n]$ is therefore

$$\frac{[z^n]\left(\frac{\partial}{\partial z} G^{(S)}(z,u)\big|_{u=1}\right)}{[z^n]G^{(S)}(z,1)} = \frac{[z^n]\frac{1}{k}z^k G^{(S)}(z,1)}{[z^n]G^{(S)}(z,1)} = \frac{1}{k}\frac{p_{n-k}}{p_n}.$$

Now, $p_{n-1}^{(S)}/p_n^{(S)} \sim (n-1)!^{-1/m}/n!^{-1/m} = n^{1/m}$, the subexponential factor in Theorem 5.10.1 being slowly varying. So the mean number of k-cycles is asymptotic to $\frac{1}{k}(n^{1/m})^k$, as desired. □

The Boltzmann sampler for S-permutations provides an explanation for Corollary 5.10.2. To generate random S-permutations, we fix a positive real parameter x and then pick a cycle type by taking $\mathcal{P}(x^k/k)$ cycles of length k for each $k \in S$. The cycles themselves are then populated with elements uniformly at random. Fixing x to be the positive root of $\sum_{k \in S} z^k = n$ – that is, $x = r_n$ – gives permutations of expected size n, and all S-permutations of the same size are equally likely to be generated. The expected number of k-cycles of a permutation generated by this process is r_n^k/k.

To make this connection more precise, we can derive asymptotic series for the number of S-permutations of $[n]$, as $n \to \infty$; these lead to asymptotic series for the expected number of k-cycles in such permutations. We consider the case of involutions. We begin with the leading-term asymptotics for the number of involutions, $a_n = f(n)(1 + o(1))$, where

$$f(n) = 2^{-1/2} e^{-1/4} (n/e)^{n/2} e^{\sqrt{n}} (1 + o(1)).$$

We also have the recurrence relation $a_n = a_{n-1} + (n-1)a_{n-2}$, which we can write as $a_{n-1}/a_n + (n-1)a_{n-2}/a_n = 1$. Now, $a_n = f(n)(1 + An^{-1/2} + O(1/n))$ for some constant A. This is true since the exponential of a Hayman-admissible function is what is called HS-admissible (after Harris and Schoenfeld [HS68]), as shown by Odlyzko and Richmond [OR85, Thm. 4]. An HS-admissible function admits an asymptotic series in descending powers of β_n [HS68, Thm. 1], where in this case $\beta_n = u_n/2 + u_n^2$ and u_n is the positive root of $z + z^2 = n + 1$; this can be rewritten as a series in descending powers of $n^{1/2}$. We can bootstrap this to find an asymptotic series, by plugging $a_n = f(n)(1 + An^{-1/2} + O(1/n))$ into the recurrence relation. This gives

$$\frac{f(n-1)\left(1+\frac{A}{\sqrt{n-1}}+O(1/n)\right)+(n-1)f(n-2)\left(1+\frac{A}{\sqrt{n-2}}+O(1/n)\right)}{f(n)} = 1 \qquad (5.13)$$

We can derive an asymptotic series for $f(n-1)/f(n)$, which begins

$$\frac{f(n-1)}{f(n)} = n^{-1/2} - \frac{1}{2}n^{-1} + \frac{3}{8}n^{-3/2} - \frac{13}{48}n^{-2} + O(n^{-5/2})$$

and similarly

$$\frac{f(n-2)}{f(n)} = n^{-1} - n^{-3/2} + \frac{3}{2}n^{-2} - \frac{5}{3}n^{-5/2} + \frac{53}{24}n^{-3} + O(n^{-7/2}).$$

These can be used to derive an asymptotic series for the left-hand side of (5.13). The series begins

$$1+\frac{1}{24}(24A-7)n^{-3/2}+O(n^{-2})$$

but we know that this must be 1; thus $A = 7/24$. Repeating the process with $a_n = f(n)(1+(7/24)n^{-1/2}+Bn^{-1}+O(n^{-3/2})$ gives

$$1+\frac{1152B+119}{576}n^{-2}+O(n^{-5/2})=1$$

and so we get $B = -119/1152$. Continuing in this way, we can derive the series

$$\begin{aligned} a_n &= f(n)\cdot\left(1+\frac{7}{24}n^{-1/2}-\frac{119}{1152}n^{-1}-\frac{7933}{414720}n^{-3/2}\right.\\ &+ \left.\frac{1967381}{39813120}n^{-2}-\frac{57200419}{1337720832}n^{-5/2}+\frac{6340449533}{687970713600}n^{-3}+O(n^{-7/2})\right) \end{aligned}$$

with relative error $O(n^{-7/2})$. From this, we can derive series for the mean and variance of the number of fixed points of an involution; these are

$$n^{1/2}-1/2+\frac{3}{8}n^{-1/2}-\frac{1}{8}n^{-1}-\frac{1}{128}n^{-3/2}+\frac{3}{32}n^{2}-\frac{85}{1024}n^{-5/2}+O(n^{-3})$$

and

$$n^{1/2}-1+\frac{5}{8}n^{-1/2}-\frac{1}{4}n^{-1}-\frac{13}{128}n^{-3/2}+\frac{1}{4}n^{-2}+O(n^{-5/2}).$$

respectively.

Now, consider the Boltzmann sampler for involutions, tuned to have average size n. The expected size of the Boltzmannized involutions with parameter x is $x+x^2$; thus x is the positive solution of $x+x^2=n$, namely $x=(\sqrt{1+4n}-1)/2$. This has the asymptotic series

$$n^{1/2}-1/2-\frac{1}{8}n^{-1/2}+\frac{1}{128}n^{3/2}-\frac{1}{1024}n^{-5/2}+O(n^{-7/2})$$

which can be found either by the binomial theorem, or as a Puiseux series. This is both the mean and variance of the number of fixed points of Boltzmann-x involutions, since the number of fixed points is Poisson with mean x.

In the case of Boltzmann samplers for involutions, tuned to give average size x, the mean and variance of the number of fixed points are the same. But when we consider involutions proper, the variance of the number of fixed points is smaller than the mean, by $1/2 + O(n^{-1/2})$. Conditioning the size of the permutation compresses the distribution by some small but non-negligible amount.

Chapter 6

Partitions

We now move on to some more number-theoretic results, from the theory of integer partitions. The first application of the Boltzmann sampling methodology to integer partitions appears to come from Fristedt [Fri93]. We recall that the Boltzmann sampler for partitions of integers has X_j parts equal to j, where $\mathbb{P}(X_j = k) = x^{jk}(1-x^k)$. In Fristedt's paper this appears as an ad hoc trick. Some later authors found it useful: for example Corteel, Pittel, Savage, and Wilf use it in the paper [CPSW99] which considers the expected number of parts of different multiplicities in partitions, and Vershik and collaborators in [Ver96, DVZ00] use this to determine the limiting shape of the Young diagram of a partition. We continue in this tradition, proving such results for the Boltzmannized models themselves. In Section 6.1 we will prove Boltzmannized analogues of various classical results on random partitions. In Section 6.2 we will define "rational classes" of partitions. These are certain classes of partitions with restrictions on the multiplicities of the parts, which are enumerated by functions q satisfying $\log q(n) \sim \log p(An)$, where A is a rational number; this appears to be a fairly strong restriction on the set of allowed multiplicities. In Section 6.3 we extend the results of the previous section, on partitions with restricted part multiplicities, to more general classes, for which we obtain similar relations but with irrational constants A. In Section 6.4 we use partition identities which enumerate sets of partitions, one of which is a subset of the other, to determine some limiting probabilities in partitions. These limiting probabilities can be connected to Boltzmann samplers for such partitions and to the combinatorics of words. Finally, in Section 6.5 we consider some probabilistic aspects of overpartitions, which are partitions in which the last occurrence of each part can be barred, and examine the statistics of a family of weighted objects which interpolate between partitions and overpartitions.

6.1 Recovering classical results from Boltzmann samplers

We have seen previously, in Chapter 3, that the mean number of parts of a partition into distinct parts drawn from the Boltzmann sampler with parameter x is asymptotic to $(1-x)^{-1}\log 2$, as $x \to 1^-$. The mean size of partitions drawn from the same distribution is asymptotic to $(1-x)^{-2}(\pi^2/12)$, as $x \to 1^-$.

We set $N = (1-x)^{-2}(\pi^2/12)$, and solve for $1-x$; this gives $1-x = \sqrt{\pi^2/12N}$.

Substituting this into the asymptotic form for the number of parts gives

$$\sqrt{\frac{12N}{\pi^2}}\log 2, \text{ or } \frac{2\sqrt{3}\log 2}{\pi}N.$$

This is, in fact, the mean number of parts of a partition into distinct parts; see [EL41].

We can proceed similarly for unrestricted partitions. The Boltzmann sampler for ordinary partitions has $\mathbb{P}_x(P_k = j) = x^{jk}(1 - x^k)$. The distribution of sizes of Boltzmann-x partitions is therefore the distribution of $\sum_{k\geq 1} kP_k$, and the mean size is $\sum_{k\geq 1} \frac{kx^k}{1-x^k}$. We can approximate this by the corresponding integral

$$\int_0^\infty \frac{kx^k}{1-x^k}\,dk$$

and making the change of variables $u = x^k$, this is

$$\frac{1}{(\log x)^2}\int_1^0 \frac{\log u}{1-u}\,du. \tag{6.1}$$

The *dilogarithm* is given by the sum and integral

$$Li_2(z) = \sum_{k=1}^\infty \frac{z^k}{k^2} = \int_z^0 \frac{\log(1-t)}{t}\,dt$$

and will be very useful in this chapter. By making the change of variables $t = 1 - u$, we see that (6.1) becomes $(\log x)^{-2}Li_2(1)$. Since $Li_2(1) = \sum_{k=1}^\infty k^{-2} = \pi^2/6$, we finally have that the mean size of Boltzmann-x partitions is $\pi^2/6 \cdot (1-x)^{-2}$. Setting N equal to this and solving for $1 - x$ gives $1 - x = \pi/\sqrt{6N}$.

The mean *number of parts* of a Boltzmann-x partition can be found similarly; it is of course $\sum_{k\geq 1} \mathbb{E}P_k = \sum_{k\geq 1} x^k/(1-x^k)$. Again approximating by an integral, and making the change of variables $u = x^k$, this is

$$\int_x^0 \frac{u}{1-u} \frac{du}{u \log x} = \frac{1}{\log x} \int_x^0 \frac{du}{1-u} = \frac{\log 1 - x}{\log x}.$$

As $x \to 1^-$, this is asymptotic to $\frac{1}{1-x} \log \frac{1}{1-x}$. Letting $1 - x = \pi/\sqrt{6N}$, this mean number of parts is $\sqrt{3N/2\pi^2} \log N$, which is indeed the asymptotic mean number of parts of a partition of N [EL41]. The mean number of *distinct* parts, in contrast, is asymptotic to $\sqrt{6N/\pi}$ [EL41, Wil83]. The sum $\sum_{k \geq 1} \mathbb{P}(P_k \geq 1) = \sum_{k \geq 1} x^k$ gives the number of distinct parts; this is of course $x/(1-x)$, which with $1 - x = \pi/\sqrt{6N}$ has the expected asymptotics. We can then ask how many of these parts occur with multiplicity m, as in [CPSW99]; the expected number of such parts in the Boltzmann-x model is

$$\sum_{k \geq 1} \mathbb{P}(P_k = m) = \sum_{k \geq 1} x^{mk}(1 - x^k) = \sum_{k \geq 1} x^{mk} - \sum_{k \geq 1} x^{(m+1)k} = \frac{x^{mk}}{1 - x^{mk}} - \frac{x^{(m+1)k}}{1 - x^{(m+1)k}}.$$

Now, with $N \to \infty$ and $1 - x = \pi/\sqrt{6N}$, we see that $x^{mk}/(1 - x^{mk}) \sim \sqrt{6N/\pi} m$ as $m \to \infty$; the second term is treated similarly, so the expected number of parts with multiplicity m is $\sqrt{6N/\pi}(1/m - 1/(m+1))$, or $1/(m(m+1))$ times the number of parts. This is the content of [CPSW99, Thm. 3].

Proceeding a bit further, we can recover the limiting shape of the Young diagram of a partition. Vershik and collaborators [DVZ00, Ver96] have shown that most integer partitions have a well-defined "profile". The outer boundary of the Young diagram of a random partition of n, scaled by a factor of $\sqrt{n}$, tends to the continuous plane curve given by $\exp(-\alpha x) + \exp(-\alpha y) = 1$, where $\alpha = \pi/\sqrt{6}$. In particular we note that this limiting shape is unchanged when x and y are interchanged.

Fix $N = \pi^2/6 \cdot (1-x)^{-2}$. We will show that the expected number of parts in a Boltzmann-sampled partition which are greater than $r\sqrt{N}$ is asymptotic to $s\sqrt{N}$, where r and s satisfy $e^{-\alpha r} + e^{-\alpha s} = 1$. Fix a positive real constant r. Then the expected number of parts in a Boltzmann-x partition which are greater than $r\sqrt{n}$ is given by the sum $\sum_{k \geq r\sqrt{n}} \mathbb{E}P_k$, which is approximated by the integral

$$\int_{r\sqrt{n}}^{\infty} \frac{x^k}{1-x^k}\,dk.$$

We note that $r\sqrt{n} = r\pi/\sqrt{6} \cdot (1-x)^{-1}$. Now, we make the change of variables $u = x^k$ to get

$$\int_0^{x^{r\pi/(\sqrt{6}(1-x))}} \frac{du}{(1-u)\log x}.$$

As $x \to 1^-$, the upper limit of this integral approaches $\exp(-r\pi/\sqrt{6})$. So we replace the upper limit with this, and integrate. We get, as $x \to 1^-$,

$$\sum_{k \geq r\sqrt{n}} \mathbb{E}P_k \sim \frac{-\log\left(1 - \exp(-r\pi/\sqrt{6})\right)}{1-x}.$$

Now, recalling that $1 - x = \pi/\sqrt{6n}$, we get

$$\sum_{k \geq r\sqrt{n}} \mathbb{E}P_k \sim -\log(1 - e^{-r\pi/\sqrt{6}})\sqrt{\frac{6n}{\pi}}.$$

In particular, if $f(r) = \mathbb{E}_x(\text{number of parts greater than } r\sqrt{n})/\sqrt{n}$, then we have $\exp(-\pi/\sqrt{6} \cdot r) + \exp(-\pi/\sqrt{6} \cdot f(r)) = 1$, as desired.

Some other results are more straightforward. For example, from the Boltzmann sampler for partitions into distinct parts, the probability that a partition contains a part k, in the large-n limit, is $1/2$, and that the probabilities of containing various parts are independent. This is in fact true for fixed-size partitions as well:

Proposition 6.1.1. *Fix positive integers $j_1, \ldots, j_r, k_1, \ldots, k_t$, with none equal. The probability that a partition of n into distinct parts contains none of $j_1, \ldots, j_r$ and all of $k_1, \ldots, k_t$ approaches $2^{-(r+t)}$ as $n \to \infty$.*

Proof. We note that the generating function for partitions into distinct parts having none of the parts $j_1, \ldots, j_r$ is

$$\sum_{k=1}^{\infty} \frac{1-z^k}{1-z^{2k}} \times \prod_{i=1}^{r} \frac{1-z^{j_i}}{1-z^{2j_i}}.$$

The associated Dirichlet series is therefore

$$\alpha(s) = (1-2^{-s})(\zeta(s) - \sum_{i=1}^{r} j_i^{-s}).$$

Now, $\alpha(s)$ has dominant pole at $1/2$ with residue 1; furthermore $\alpha(0) = 0$ and $\alpha'(0) = (\log(2))(-1/2-r)$. Thus it follows from Meinardus' theorem that the number of such partitions is asymptotic to

$$2^{-r}\frac{3^{3/4}}{12}n^{-3/4}\exp(\pi\sqrt{n/3}). \tag{6.2}$$

In particular, the $r = 0$ case is just that of partitions into distinct parts, so this asymptotic form is 2^{-r} times the number of partitions of n into distinct parts. This proves the result in the case where $t = 0$. In the case where $t > 0$, note that the number of partitions of n into distinct parts, which do not contain any of $j_1, \ldots, j_r$ and do contain $k_1, \ldots, k_t$, is the same as the number of partitions of $n-(k_1+\cdots+k_t)$ which do not contain any of $j_1, \ldots, j_r, k_1, \ldots, k_t$. The latter has the asymptotic form of (6.2) with r replaced by $r+t$, since $p_d(n) \sim p_d(n-C)$ for any constant C. □

In contrast, for ordinary partitions, we have:

Proposition 6.1.2. *The probability that an ordinary partition of n chosen uniformly at random has none of the parts $j_1, \ldots, j_r$ is asymptotic to $j_1 \ldots j_r(\pi/\sqrt{6n})^r$ as $n \to \infty$.*

Proof. Note that such partitions have the associated Dirichlet series

$$\alpha(s) = \zeta(s) - (j_1^{-s} + \cdots + j_r^{-s}).$$

This has a simple pole of residue $A = 1$ at $\rho = 1$. Thus, applying Meinardus' theorem, the partitions are asymptotically counted by $Cn^\kappa \exp(K\sqrt{n})$ where $K = \pi\sqrt{2/3}$,

$$\kappa = \frac{\alpha(0) - 3/2}{2} = \frac{\left(\frac{-1}{2} - r\right) - \frac{3}{2}}{2} = -1 - \frac{r}{2}$$

and, noting that $\alpha'(0) = \frac{-1}{2}\log(2\pi) + \log j_1 + \cdots + \log j_r$,

$$C = e^{\alpha'(0)}(4\pi)^{-1/2}\left(\frac{\pi^2}{6}\right)^{\frac{3+r}{4}} = \frac{j_1 \cdots j_r}{\pi\sqrt{8}}\left(\frac{\pi^2}{6}\right)^{\frac{1+r}{2}}$$

Dividing this by the asymptotic form $p(n) \sim \exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$ for the number of ordinary partitions gives the desired result. $\square$

Now, partitions selected according to the Boltzmann distribution with parameter x have probability $(1-x^{j_1})(1-x^{j_2})\cdots(1-x^{j_r})$ of having none of the parts $j_1, \ldots, j_r$. As $x \to 1^-$ this is asymptotic to $j_1 j_2 \ldots j_r(1-x)^r$. If we choose x to generate partitions of expected size about n – that is, if we take $1 - x = \pi/\sqrt{6n}$ – then we recover an analogue of the above result.

Proposition 6.1.3. *Let $f(z)$ be the generating function of a finite multiset S which includes zero. The expected number of different parts having multiplicity m in a Boltzmann-x partition with part multiplicities chosen from S is asymptotic to*

$$\frac{1}{1-x}\int_0^1 \frac{v^{m-1}}{f(v)}\,dv$$

and their sum is asymptotic to

$$\frac{1}{(1-x)^2}\int_0^1 \frac{-mv^m \log v}{f(v)}\,dv$$

as $x \to 1^-$.

Proof. The expected number of different parts having multiplicity m in such a partition is given by $\sum_{k\geq 1} \frac{x^{mk}}{f(x^k)}$. This can be approximated by the integral $\int_0^\infty \frac{x^{mk}}{f(x^k)}\,dk$ and making the change of variables $v = x^k$, this becomes $\frac{-1}{\log x}\int_0^x \frac{v^{m-1}}{f(v)}\,dv$. As x approaches 1, we can replace $-1/(\log x)$ with $1/(1-x)$ to get the integral above. Similarly, the expected sum of all the parts with multiplicity m is given by

$$\int_1^\infty \frac{mkx^{mk}}{f(x^k)}\,dk \sim \frac{1}{(1-x)^2}\int_0^1 \frac{-mv^m \log v}{f(v)}\,dv.$$

□

In the particular case where $f(z) = 1+z+z^2$ we can obtain explicit results. Here the expected number of parts of multiplicity 1 is asymptotic to

$$\frac{1}{1-x}\int_0^1 \frac{1}{1+v+v^2}\,dv = \frac{1}{1-x}\frac{\pi\sqrt{3}}{9}$$

and the expected number of parts of multiplicity 2 is asymptotic to

$$\frac{1}{1-x}\int_0^1 \frac{v}{1+v+v^2}\,dv = \frac{1}{1-x}\left(-\frac{\pi\sqrt{3}}{18} + \frac{1}{2}\log 3\right).$$

The expected sum of all parts having multiplicity 1 can be written in terms of dilogarithms and is about $0.7813(1-x)^{-2}$; the expected sum of the multiplicity-2 parts is about $0.3153(1-x)^{-2}$. These add up to $\frac{\pi^2}{9}(1-x)^{-2}$. In general we see that the parts of multiplicity 1 predominate. This is true even though the "critical" Boltzmann sampler for such partitions consists of taking each part to have multiplicity 0, 1, or 2 with equal probability.

6.2 Rational classes of partitions

Recall the asymptotic formula for the number of integer partitions of n: $p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$. Many classes of partitions have size $p(n)$ satisfying $\log p(n) \sim \pi\sqrt{A \cdot 2n/3}$. We will say that such a class of partitions is "in the A class".

Thus we suspect that partitions are a "square-root structure" in the sense of Chapter 4. Indeed, the generating function $\prod_{k \geq 1} 1/(1-x^k)$ grows like

$$P(x) = \exp\left(\frac{\pi^2}{6(1-x)}(1+o(1))\right)$$

as $x \to 1^-$ along the real axis, thus resembling the "exponential-of-a-pole" generating functions previously seen. However in this case the circle is a natural boundary for the generating function.

Now recall the result that the number of partitions of n into *odd* parts and into *distinct* parts are the same. There are many bijective proofs of this fact, which is originally due to Euler; see [Pak06, Sec. 3] for these bijections, perhaps the most appealing of which is Glaisher's bijection. Let $\lambda = 1^{m_1}3^{m_3}\cdots$ be a partition with odd parts. Then let $\varphi(\lambda)$ contain the part $i \cdot 2^r$ if and only if m_i written in binary has 1 in the rth position.

In fact, the Boltzmann samplers for partitions into odd parts and into distinct parts are particularly well-behaved with respect to this bijection. Boltzmann-x partitions into odd parts satisfy $\mathbb{P}_x^o(P_k = j) = x^{jk}(1-x^k)$ for each odd k, where P_k is the number of k-parts. Boltzmann-x partitions into distinct parts satisfy $\mathbb{P}_x^d(P_k = 1) = x^k/(1+x^k)$. So in the distinct case, we have for example

$$\mathbb{P}_x^d(P_3 = 1, P_{12} = 1, P_6 = P_{24} = P_{48} = \cdots = 0) = \frac{x^{15}}{(1+x^3)(1+x^6)(1+x^{12})\cdots}$$
$$= \frac{x^{18}}{(1-x^3)^{-1})} = x^{15} - x^{18}$$

and similarly $\mathbb{P}_x^o(P_3 = 5) = x^{15} - x^{18}$. More generally, the probability that a Boltzmann-x partition into distinct parts has its parts of form $k \cdot 2^r$ summing to jk is the same as the probability that a Boltzmann-x partition into odd parts has j parts equal to k.

The generating function proof is perhaps one of the best-known generating function arguments. Partitions into distinct parts are counted by $P_d(z) = \prod_{k\geq 1}(1+z^k)$. We can rewrite $1+z^k$ as $(1-z^{2k})/(1-z^k)$, so $\mathbb{P}_d(z) = \prod_{k\geq 1}\frac{1-z^{2k}}{1-z^k}$. Among the factors in the denominator, $1-z^j$ is cancelled out by a factor in the numerator if j is even but survives if j is odd, so

$$\mathbb{P}_d(z) = \prod_{\substack{k\geq 1 \\ k\equiv 1 \pmod 2}} \frac{1}{1-z^k}$$

and the right-hand side here is clearly the generating function for partitions with all parts odd.

This is the first of many results in which partitions with certain restrictions on the multiplicities of their parts are equinumerous with partitions with certain restrictions on the allowed parts. In the case of partitions with unrestricted multiplicity, Andrews [And69] defined the notion of an *Euler pair*. This is a pair of sets (S_1, S_2) such that for all natural numbers n, the number of partitions of n into distinct parts taken from S_1 is equal to the number of partitions of n into S_2. Andrews shows that (S_1, S_2) is an Euler pair if and only if $2S_1 \subset S_1$ and $S_2 = S_1 - 2S_1$. In particular, if S_1 is a set with asymptotic density α, then S_2 has asymptotic density $\alpha/2$.

We also take inspiration from Subbarao's identity:

Theorem 6.2.1 (Subbarao)**.** *[Sub71] The number of partitions of n into parts with multiplicities 2, 3, or 5 is equal to the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 mod 12.*

Proof. The generating function of partitions with parts of multiplicities 2, 3, or 5 is $\prod_{n\geq 1}(1+z^{2n}+z^{3n}+z^{5n})$. We can factor this polynomial to get $\prod_{n\geq 1}(1+z^{2n})(1+z^{3n})$ and we would like to write this polynomial as a product of terms of the form $1-z^k$. This can be rewritten as the quotient

$$\prod_{n\geq 1} \frac{(1-z^{4n})(1-z^{6n})}{(1-z^{2n})(1-z^{3n})}.$$

We now ask how many times the factor $1-z^k$ appears in this product, counting appearances in the denominator as positive and in the numerator as negative. The number of such appearances is $[\![2|k]\!] + [\![3|k]\!] - [\![4|k]\!] - [\![6|k]\!]$. This function is periodic with period 12, and from direct computation is 1 if k is congruent to $2, 3, 6, 9$ or 10 mod 12 and 0 otherwise. $\square$

Thus partitions with parts having multiplicity 2, 3, or 5 are in the "5/12 class". It is natural to ask how the coefficient 5/12 can be extracted from the set $\{2, 3, 5\}$.

Theorem 6.2.2. *Let $b_1, b_2, \ldots$ be an m-periodic sequence of integers. Then $P(z) = \prod_{k\geq 1}(1-z^k)^{-b_k}$ satisfies*

$$\log([z^n]P(z)) \sim \pi\sqrt{\frac{2}{3}}\sum_{k=1}^{m}\frac{b_1 + \cdots + b_m}{m}.$$

In the case where all the b_k are equal to 1 or 0, then $P(z)$ trivially counts the number of partitions with all parts lying in some set of congruence classes modulo m.

Proof. We will apply Meinardus' theorem. We have, for $1 \leq k \leq m$,

$$\sum_{n\equiv k(m)} n^{-s} = \sum_{a\geq 0}\frac{1}{(am+k)^s} = m^{-s}\sum_{a\geq 0}\frac{1}{(a+k/m)^s} = m^{-s}\zeta(s, k/m)$$

where ζ is the Hurwitz zeta function. This gives the associated Dirichlet series

$$\alpha(s) = m^{-s}\sum_{k=1}^{m} b_k\zeta(s, k/m).$$

The residue of $\zeta(s, v)$ as a function of s, at $s = 1$, is 1. Therefore $\alpha(s)$ has residue $m^{-1}\sum_{k=1}^{m} b_k$ at 1; call this number A. Note that A is the *average* of the b_k. Thus we can apply Meinardus' theorem to see that the coefficients of P satisfy $\log([z^n]P(z)) \sim Kn^{1/2}$, where

$$K = (1+\rho^{-1})(A\Gamma(\rho+1)\zeta(\rho+1))^{1/(\rho+1)} = 2\left(\frac{1}{m}\sum_{k=1}^{m} b_k \cdot \frac{\pi^2}{6}\right)^{1/2}$$

as desired. $\square$

In particular, we see that if the sequence of $\{b_k\}$ is periodic, then the partitions enumerated by $\prod_{k\geq 1}(1-z^k)^{-b_k}$ are in the A class, where A is the mean of the b_k. In the case where b_k is a periodic sequence of 0s and 1s, and thus $\prod_{k\geq 1}(1-z^k)^{-b_k}$ counts partitions with all parts lying in a union of arithmetic progressions, the class of that set of partitions is just the density of the corresponding arithmetic progression. Nathanson [Nat00] has given an alternate proof of the latter fact.

Now we will consider partitions with *restricted* multiplicities. Let $\mathcal{M} = (M_1, M_2, M_3, \ldots)$ be a sequence of multisets of nonnegative integers, with each set including 0. Then the product

$$\prod_{j\geq 1}\left(\sum_{k\in M_j} x^{jk}\right)$$

is the generating function of partitions with the number of parts equal to j in the set M_j, which we will call $\mathcal{M}$-partitions. We will show that if the sequence $M_1, M_2, M_3, \ldots$ is periodic, and each of its members fall in a certain class of "rational sets", then $\mathcal{M}$ is a rational class of partitions.

Definition 6.2.3. *We call a set M a rational set if its generating function $\sum_{m\in M} z^m$ can be written in the form $\prod_{k=1}^{r}(1-z^k)^{-b_k}$. The weight of the set M, denoted $w(M)$, is the sum $\sum_{k\geq 1} b_k/k$.*

Note that we have defined the product to be *finite*.

Some examples of rational sets occurring in partition problems are the following:

Empty partitions. These have $M = \{0\}$, with generating function 1; thus $M = \{0\}$ is rational with weight 0.

Unrestricted partitions. These have $M = \mathbb{Z}^+$, with generating function $1/(1-z)$. Thus the set $\mathbb{Z}^+$ is rational with weight 1.

Partitions with all parts having multiplicity less than r. In this case $M = \{0, 1, \ldots, r-1\}$, and so M has generating function

$$1 + z + \cdots + z^{r-1} = \frac{1-z^r}{1-z}.$$

Thus $M = \{0, 1, \ldots, r-1\}$ is rational with weight $1 - 1/r$. In particular, if $r = 2$, we are considering partitions into distinct parts, and the weight is $1/2$.

Overpartitions, or signed partitions. These are partitions in which the last occurrence of each part can be overlined. Thus we have $M = \{0, 1, 1, 2, 2, 3, 3, \ldots\}$, with generating function

$$1 + 2z + 2z^2 + 2z^3 + \cdots = \frac{1+z}{1-z} = \frac{(1-z^2)}{(1-z)^2}$$

and thus $b_1 = 2, b_2 = -1$, and the rational weight of the set is $3/2$. The *signed partitions* introduced by Andrews [And07], which are partitions in which some parts may be negative integers but parts $+k$ and $-k$ cannot both occur, fall in the same class.

Partitions with designated summands. These partitions defined in [ALL02] are partitions in which exactly one occurrence of each part must be overlined. These have the multiplicity multiset $M = 0^1 1^1 2^2 3^3 4^4 \cdots$ (with exponents indicating multiplicity in M), with generating function

$$g(z) = 1 + \sum_{k \geq 1} kz^k = 1 + \frac{z}{(1-z)^2} = \frac{1-z+z^2}{(1-z)^2} = \frac{(1-z^6)}{(1-z)(1-z^2)(1-z^3)}$$

and therefore have rational weight 5/3.

If instead we require that at most one occurrence of each part is overlined, then we have $M = 0^1 1^2 2^3 \cdots$, with generating function $1/(1-z)^2$, and in fact such partitions are in bijection with pairs of ordinary partitions.

Singleton-free partitions. Partitions with no parts appearing exactly once have

$$g(z) = 1 + z^2 + z^3 + z^4 + \cdots = 1 + \frac{z^2}{1-z} = \frac{1+z^3}{1-z^2} = \frac{(1-z^6)}{(1-z^2)(1-z^3)}$$

and thus have rational weight $1/2 + 1/3 - 1/6 = 2/3$. More generally, partitions with no part appearing with any multiplicity $1, 3, \ldots, 2r-1$ have multiplicity generating function

$$\begin{aligned} g(z) &= 1 + z^2 + \cdots + z^{2r-2} + z^{2r} + z^{2r+1} + z^{2r+2} + \cdots \\ &= \frac{1-z^{2r}}{1-z^2} + \frac{z^{2r}}{1-z} = \frac{1-z^{4r+2}}{(1-z)(1-z^{2r+1})} \end{aligned}$$

and therefore have weight $1/2 + 1/(2r+1) - 1/(4r+2) = (r+1)/(2r+1)$.

Dilations of rational sets. Let M be a rational set. Then aM, the set obtained from M by multiplying every element by k, is also a rational set. It has weight $w(aM) = w(M)/a$. To see this, note that if the generating function of M is $g_M(z) = \prod_{k \geq 1}(1-z^k)^{-b_k}$, then the generating function of aM is $g_{aM}(z) = \prod_{k \geq 1}(1-z^{ak})^{-b_k}$.

For example, the set $M = \{0, r\}$ is a rational set with weight $1/(2r)$.

Convolutions of rational sets. Let M and N be rational sets. Let their *convolution* be the set $M \otimes N = \{m + n : m \in M, n \in N\}$, counted with multiplicity. Then $M \otimes N$ is a rational set, with weight $w(M) + w(N)$. This is true because the generating function of $M \otimes N$ is the product of the generating functions of M and N.

For example, the set $\{0, 2\}$ and $\{0, 3\}$ are rational sets, with weights $1/2$ and $1/3$ respectively. Therefore the set $\{0, 2, 3, 5\}$, their convolution, is rational with weight $5/12$.

We claim the following theorem:

Theorem 6.2.4. *Let $M_1, M_2, \ldots$ be a sequence of rational sets of nonnegative integers, each containing zero. Assume this sequence is r-periodic. Let $p_M(n)$ be the number of partitions of n in which the multiplicity of j is an element of M_j for each j. Then $\log p_M(n) \sim \sqrt{n} \times \pi\sqrt{2A/3}$ as $n \to \infty$, where $A = (w(M_1) + w(M_2) + \cdots + w(M_r))/r$ is the average of the weights.*

Proof. Let $f(z) = \sum_{n \geq 0} p_M(n) z^n$ be the generating function of M-partitions. It suffices to show that $f(z) = \prod_{k \geq 1}(1 - z^k)^{-c_k}$ where the sequence $\{c_k\}$ is periodic and has mean A.

Let the generating function of M_i be $\prod_{k=1}^{\infty}(1 - z^k)^{-b_{ik}}$. Since M_i is a rational set, only finitely many of the b_{ik} are different than zero for any given i.

We then have the generating function

$$f(z) = \prod_{i=1}^{r} \prod_{j \equiv i(r)} \prod_{k \geq 1} (1 - z^{jk})^{-b_{ik}}.$$

We would like to know the opposite of the sum of the exponents with which $1 - z^s$ appears, for any given s. Call this $e(s)$, so we will have $f(z) = \prod_{s=1}^{\infty}(1 - z^s)^{-e(s)}$. Consider the sequence $\{e(s)\}_{s=1}^{\infty}$. The factors in the triple product having b_{ik} in the exponent are those with $s = jk$ where $j \equiv i \pmod r$; that is, those with $s \equiv ik \pmod{kr}$. Thus we have

$$e(s) = \sum_{i=1}^{r} \sum_{k \geq 1} b_{ik} [\![s \equiv ik \pmod{kr}]\!]$$

and in particular the sequence $\{e(s)\}_{s=1}^{\infty}$ is the sum of finitely many periodic sequences (recall that the inner sum is actually finite). Thus $e(s)$ is a periodic function of s. The mean of the sequence $\{e(s)\}$ is the sum of the means of these periodic sequences, and the sequence corresponding to the (i, k) term has mean $b_{ik}/(kr)$. Therefore the mean of the sequence $\{e(s)\}$ is

$$\sum_{i=1}^{r} \sum_{k \geq 1} \frac{b_{ik}}{kr} = \frac{1}{r} \sum_{i=1}^{r} \sum_{k \geq 1} \frac{b_{ik}}{k} = \frac{1}{r} \sum_{i=1}^{r} w(M_i)$$

as desired. $\square$

For example, consider partitions in which parts congruent to 1 mod 4 cannot be singletons, parts congruent to 2 mod 4 must occur with multiplicity $0, 2, 3$ or 5, parts congruent to 3 mod 4 cannot be repeated , and parts congruent to 4 mod 4 must have even multiplicity. These are $\mathcal{M}$-partitions with $M_1 = \{0, 2, 3, 4, \ldots\}$, $M_2 = \{0, 2, 3, 5\}$,

$M_3 = \{0,1\}$, $M_4 = \{0,2,4,6,\ldots\}$, and $M_{4+r} = M_r$ for $r \geq 1$. Each of these sets is rational, with $w(M_1) = 2/3, w(M_2) = 5/12, w(M_3) = 1/2, w(M_4) = 1/2$. The average of these weights is 25/48, and so we have $\log p_M(n) \sim \sqrt{n} \times \pi\sqrt{(2/3)(25/48)}$.

These classification results can be seen as analogous to those in [Nat00]. In this paper it is shown that given a set of integers A with $\gcd(A) = 1$, and $p_A(n)$ the partition function of A, then if A has asymptotic density α, then $\log p_A(n) \sim \pi\sqrt{2/3} \cdot \sqrt{\alpha n}$ and conversely. These results can be viewed as a generalization of the forward direction of Nathanson's result to partitions with multiplicity restrictions.

6.3 Tauberian theorems and irrational weights

In contrast to the results of the previous section, consider partitions with all parts having multiplicities 0, 2 or 3. These are not equinumerous with partitions with their parts lying in some set. To see this, we first compute the number of such partitions of each size, from the generating function:

$$\prod_{k\geq 1}(1 + z^{2k} + z^{3k}) = z^2 + z^3 + z^4 + 3z^6 + z^7 + 3z^8 + 3z^9 + 3z^{10} + 2z^{11} + 7z^{12} + \cdots$$

Now we attempt to build S such that partitions with parts in S are counted by this same series. So there should be zero S-partitions of 1; thus $1 \notin S$. There should be one S-partition of 2, so $2 \in S$. There should also be one S-partition of 3, so $3 \in S$. Without considering 4, there is already an S-partition of 4, namely $2 + 2$, so $4 \notin S$. Finally, without considering 5, there is an S-partition of 5, namely $2 + 3$. We want there to be zero S-partitions of 5, so we conclude there is no set with the desired property.

Therefore the techniques of the previous section will not work for enumerating partitions with restricted multiplicities when we do not have fortuitous factorizations of the generating functions of the multiplicities. Using saddle point methods, we will find an upper bound on the logarithm of the number of partitions with part multiplicities restricted to some set M; these bounds will be expressed in terms of the roots of the generating polynomial of M. We begin with the following lemma on the rate of growth of generating functions of such partitions.

Lemma 6.3.1. *Let M be a finite set of nonnegative integers, including zero. Let $f_M(x) = \sum_{m \in M} x^m$ be its generating function. Let $P_M(x) = \prod_{k \geq 1} f_M(x^k)$ be the generating function of partitions with all parts having multiplicity in M. Then $\log P_M(x) \sim C/(1-x)$ as $x \to 1^-$, where $C = \sum_{j=1}^{k} -Li_2(1+\alpha_j)$ and the α_j satisfy $f_M(-1/\alpha_j) = 0$.*

Proof. Fix M. Let $g(s) = \sum_{k=1}^{\infty} \log f_M(e^{-ks})$. (Since M will be fixed throughout the proof, we suppress it in the notation.) We replace this sum by the integral

$$I(s) = \int_1^{\infty} \log f_M(e^{-us})\, du$$

which we will later show does not seriously affect our asymptotic results. Now, since f_M is a polynomial with $f(0) = 1$, we can factor it, with $f_M(z) = \prod_{j=1}^{m}(1 + \alpha_j z)$

where m is the degree of f_M, or alternatively the largest element of M. This gives

$$I(s) = \int_1^\infty \sum_{j=1}^m \log(1 + \alpha_j e^{-us})\, du.$$

The inner sum is finite, so we can interchange sum and integral to get

$$I(s) = \sum_{j=1}^m \int_1^\infty \log(1 + \alpha_j e^{-us})\, du.$$

Finally, we have

$$\int \log(1 + \alpha e^{-us})\, du = \frac{1}{s} Li_2(1 + \alpha e^{-us})$$

and using this to evaluate the definite integral we get

$$I(s) = \sum_{j=1}^m \frac{-Li_2(1 + \alpha_j e^{-s})}{s}.$$

Thus as $s \to 0^+$, we have $I(s) \sim -s^{-1} \sum_{j=1}^m Li_2(1 + \alpha_j)$.

Next, we need to show that $I(s) \sim g(s)$. From the Euler-Maclaurin formula (see [Odl95, (5.32)] for the precise form used here) we have

$$g(s) = I(s) + O\left(\int_1^\infty \frac{d}{du} \log p(e^{-us})\, du\right).$$

The integrand here is

$$\frac{-\sum_{m \in M} m s e^{-mus}}{\sum_{m \in M} e^{-mus}}.$$

In particular, there are $|M| - 1$ terms in the numerator. Each term in the numerator is bounded in absolute value by $m_- s e^{-m_+ us}$, where $m_+ = \max M$ and m_- is the smallest nonzero element of M. The denominator includes the term 1 and other positive terms, so is bounded below by 1. Therefore the integrand satisfies

$$\left| \frac{d}{du} \log p(e^{-us}) \right| \leq |M| \cdot |m_- s e^{-m_+ us}|$$

and so the integral satisfies

$$\int_1^\infty \frac{d}{du} \log p(e^{-us})\, du \le |M| m_- s \int_1^\infty e^{-m_+ us}\, du = \frac{|M| m_-}{m_+} e^{-m_+ s}.$$

Thus $g(s) = I(s) + O(e^{-m_+ s})$. So $g(s) \sim C/s$, with C as defined above. Setting $x = e^{-s}$ gives the desired result. $\square$

We now can obtain an asymptotic upper bound for the number of partitions with all multiplicities in M.

Proposition 6.3.2. *Let M be a set of nonnegative integers, including zero. Then we have the bound*

$$\log p_M(n) \le (2 + o(1))\sqrt{Cn}$$

as $n \to \infty$, where C is as defined in the previous lemma.

Proof. We now apply the bound from Lemma 2.3.4. We set $x = e^{-s}$ where $s = \sqrt{C/n}$. This gives

$$p_M(n) \le e^{sn} P_M(e^{-s})$$

where $p_M(n)$ is the number of partitions of n with multiplicities restricted to the set M. Taking logarithms,

$$\log p_M(n) \le sn + \log P_M(e^{-s}).$$

We know that $\log P_M(x) \sim C/(1-x)$ as $x \to 1^-$ from the previous lemma. Thus $\log P_M(e^{-s}) \sim C/s$ as $s \to 0^+$. This gives $\log p_M(n) \le sn + (1 + o(1))C/s$, and recalling $s = \sqrt{C/n}$, we get the bound $\log p_M(n) \le (2 + o(1))\sqrt{Cn}$. $\square$

Unfortunately this does not give us the actual rate of growth. It appears that in fact $\log p_M(n) = (2 + o(1))\sqrt{Cn}$. We recall the following "Hardy-Ramanujan" Tauberian theorem, as stated in [HLR04].

Proposition 6.3.3. *Let $H(x) = \sum_{n=0}^{\infty} b_n x^n$ where the b_n form a positive, non-decreasing sequence. Suppose $\log H(x) \sim C/(1-x)$ as $x \to 1^-$. Then $\log b_n \sim 2\sqrt{Cn}$ as $n \to \infty$.*

But if we have $b_n = p_M(n)$, then the sequence $\{b_n\}$ is increasing. It appears, however, that if $\gcd(M) = 1$ then this sequence is "eventually increasing", i. e. $p_M(n+1) \geq p_M(n+1)$ for all large enough n. We cannot prove this but we can prove that certain subsequences of $\{b_n\}$ are increasing.

Proposition 6.3.4. *Let M be a finite set of nonnegative integers including zero, and let h be the least common multiple of the nonzero elements of M. Then $p_M(n+h) \geq p_M(n)$ for all $n \geq 0$.*

Proof. It suffices to give an injection ϕ from M-partitions of n to M-partitions of $n+h$. Let $\lambda = \lambda_j^{m_j} \lambda_{j-1}^{m_{j-1}} \cdots \lambda_1^{m_1}$, where $\lambda_j > \lambda_{j-1} > \cdots > 1$ and $m_1, \ldots, m_j \in M \setminus \{0\}$. Then $\phi(\lambda) = (\lambda_j + h/m_j)^{m_j} \lambda_{j-1}^{m_{j-1}} \cdots \lambda_1^{m-1}$. That is, we increase all occurrences of the largest part of λ by h/m_j, and keep all other parts constant. Clearly $\phi(\lambda)$ is an M-partition of $n + h$, and this transformation is one-to-one. $\square$

Proposition 6.3.5. *Let $q_M(n) = \sum_{j=0}^{h-1} p_M(n-j)$, where we take $p_M(0) = 1$ and $p_M(n) = 0$ for $n < 0$. Then $q_M(n)$ is a weakly increasing function of n.*

Proof. Observe that $q_M(n+1) - q_M(n) = p_M(n+1) - p_M(n-h+1)$. This difference is nonnegative by the previous proposition. $\square$

We will now apply the Hardy-Ramanujan theorem to the sequence $q_M(n)$.

Proposition 6.3.6. *Let M be a finite set of nonnegative integers including 0, and define q_M as in Corollary 6.3.5. Then we have $\log q_M(n) = (2 + o(1))\sqrt{Cn}$, where C is defined as in Lemma 6.3.1.*

Proof. Let f_M be the generating polynomial of M. Then the generating function of $\{q_M(n)\}$ is

$$Q_M(z) = \sum_{n \geq 0} q_M(n) z^n = (1 + z + \cdots + z^{h-1}) \prod_{k=1}^{\infty} f_M(z^k).$$

By Lemma 6.3.1, we have $\log \prod_{k=1}^{\infty} f_M(z^k) \sim C/(1-z)$ as $z \to 1^-$. Clearly $\log(1 + z + \cdots + z^{h-1}) \sim \log h$ as $z \to 1^-$. Thus $\log Q_M(z) \sim C/(1-z)$ as $z \to 1^-$. The numbers $\{q_M(n)\}$ form a non-decreasing sequence. By the Hardy-Ramanujan Tauberian theorem, $\log q_M(n) \sim 2\sqrt{Cn}$. $\square$

In particular, if it could be shown that $p_M(n)$ does not oscillate too wildly, then we would have $\log p_M(n) \sim 2\sqrt{Cn}$. It should also be noted that when the generating polynomial of M has a "nice" factorization, the sums of dilogarithms appearing here are what would be expected from the results of Section 6.2.

Finally, we can derive identities satisfied by the dilogarithm from the work of the preceding two sections. An example is the following identity:

Proposition 6.3.7. *Let ζ_r be a primitive rth root of unity. Then*

$$\sum_{k=1}^{r-1} -Li_2(1-\zeta_r^k) = \frac{r-1}{r}\frac{6}{\pi^2}.$$

Proof. Consider partitions of n in which no part has multiplicity r or greater; let the number of such partitions be $p_r(n)$. From Theorem 6.2.4, the number of such partitions satisfies $\log p_r(n) \sim \pi\sqrt{2(r-1)/3r \cdot n}$. But $p_r(n)$ is increasing, since it is equal to the number of partitions of n into parts not divisible by r. So by Lemma 6.3.1 and the Hardy-Ramanujan Tauberian theorem,

$$\log p_r(n) \sim \left(\sum_{k=1}^{r-1} -Li_2(1-\zeta_r^k)\right)^{1/2} \cdot 2\sqrt{n}.$$

Combining these, we have

$$\pi\sqrt{\frac{2(r-1)n}{3r}} \sim \left(\sum_{k=1}^{r-1} -Li_2(1-\zeta_r^k)\right)^{1/2} \cdot 2\sqrt{n}$$

and so the coefficients on each side must be equal. Rearranging gives the desired result. □

6.4 Probabilistic interpretations of partition identities

Many well-known partition identities occur in families. In the first member of such a family partitions satisfying some condition on consecutively occurring parts are enumerated. (Usually they are shown to be equinumerous with some family which is easier to enumerate.) In other members of the family, some subset of this set is enumerated. By comparing these two results, we can compute the probability that large partitions have certain properties.

An example is given by the Rogers-Ramanujan identities.

Proposition 6.4.1 (First Rogers-Ramanujan). *The number of partitions of an integer n in which the difference between any two parts is at least 2 is the same as the number of partitions into parts congruent to 1 or 4 modulo 5.*

Proposition 6.4.2 (Second Rogers-Ramanujan). *The number of partitions of an integer n in which the difference between any two parts is at least 2 and no part is equal to 1 is the same as the number of partitions into parts congruent to 2 or 3 modulo 5.*

We call the number of partitions of the types given in these identities $r_1(n)$ and $r_2(n)$, respectively. From Theorem 2.3.11 it follows that

$$r_a(n) \sim \frac{\csc(a\pi/5)}{4\pi \cdot 15^{1/4}} n^{-3/4} \exp\left(2\pi\sqrt{n/15}\right)$$

for $a = 1, 2$. Therefore

Corollary 6.4.3. *As $n \to \infty$, we have*

$$\frac{r_2(n)}{r_1(n)} \to \frac{\csc(2\pi/5)}{\csc(\pi/5)} = \frac{\sqrt{5}-1}{2}.$$

Now, $r_2(n)/r_1(n)$ is the probability that a partition of n in which the difference between any two parts is at least 2 contains no 1. Why is this probability algebraic? And why does the "golden ratio" appear here?

Recall the Boltzmann sampler for partitions into distinct parts. When the Boltzmann parameter is x, this sampler includes a part k with probability $x^k/(1+x^k)$. For large partitions we take $x \to 1^-$ and so each part is included independently with probability $1/2$. Therefore we can model large partitions as random bit strings, and large partitions with no two consecutive parts as random bit strings with no two consecutive 1s. So, heuristically, the probability that a partition with no two consecutive parts contains no 1 is the probability that a random bit string with no two consecutive 1s starts with 0.

Formally speaking, this probability does not exist, since a random bit string has no two consecutive 1s with probability zero. But consider random bit strings of length n, with no two consecutive 1s. The number of these is the Fibonacci number F_{n+2}, and the number starting with a 0 is F_{n+1}; their ratio is F_{n+1}/F_{n+2}, which approaches $(\sqrt{5}-1)/2$ as $n \to \infty$. In fact the probability that a uniformly chosen partition with no two consecutive parts *and no part greater than* n contains no 1 is exactly F_{n+1}/F_{n+2}.

We can construct a similar interpretation for the Gollnitz-Gordon identities.

Proposition 6.4.4 (Gollnitz-Gordon)**.** *The number of partitions in which there are no consecutive summands, and furthermore the difference between any even summands is at least 4, is equal to the number of partitions of* n *into parts of the form* $8m+1, 8m+4, 8m+7$. *The number of these which furthermore contain no parts 1 or 2 is equal to the number of partitions of* n *into parts of the form* $8m+3, 8m+4, 8m+5$.

From Meinardus' theorem the number of partitions of the first type is asymptotic to

$$\frac{1}{4}\cos\frac{\pi}{8}n^{-3/4}\exp(\pi/2\sqrt{n})$$

and the number of partitions of the second type is asymptotic to

$$\frac{1}{4}\cos\frac{3\pi}{8}n^{-3/4}\exp(\pi/2\sqrt{n}).$$

It then follows that the probability that a "Gollnitz-Gordon partition" of n contains no 1s or 2s approaches $(\cos 3\pi/8)/(\cos\pi/8) = \sqrt{2}-1$ for large n. To construct an interpretation in terms of words is a bit more involved than the Rogers-Ramanujan identities. We consider words on the alphabet $\{a,b,c\}$ where c must be followed by a whenever it occurs. Let A_n, B_n, C_n be the number of such words of length n beginning with a, b, c respectively; let W_n be the total number of words. Then $A_n = B_n = W_{n-1}, C_n = A_{n-1} = W_{n-2}$. The number of such words must grow exponentially with n – that is, $W_n \sim r \cdot s^n$ for some constants r and s. So $A_n = B_n \sim rs^{n-1}$ and $C_n \sim rs^{n-2}$; we conclude $s^2 = 2s+1$, from which $s = \sqrt{2}+1$. The probability that a long word begins with a is thus $s^{-1} = \sqrt{2}-1$. Now we interpret a, b, c as $00, 10, 01$ respectively. For example, the word *babcabbab* is identified with the bit string 10 00 10 01 00 10 10 00 10, and thus the partition $1+5+8+11+13+17$. Now, the restrictions on words, when translated into restrictions on bit strings, are exactly the restrictions on Gollnitz-Gordon partitions. We do not allow a fourth letter corresponding to 11 since consecutive parts are never allowed. Also, whenever 01 occurs in the bit string – which corresponds to the letter c – an even part $2k$

occurs. This means that parts $2k+1$ and $2k+2$ cannot occur, the first by the restriction on consecutive summands and the second by the restriction on consecutive even summands. So c must be followed by a, and Gollnitz-Gordon partitions with largest part at most $2n$ can be identified with words of length n.

The ratios $(\sqrt{5}-1)/2$ and $\sqrt{2}-1$ also appear in a probabilistic model of partitions due to [MO09]. Their model is the following: let $0 < p < 1$, and let $C_1, C_2, \ldots$ be independent events with $\mathbb{P}_p(C_n) = 1 - p^n$. Let $A(r,t)$ be the set of sequences where C_n occurs if n is not congruent to $\pm r$ mod t. Then [MO09, Thm. 1.3]

$$\lim_{p \to 1} \frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))} = \frac{-1+\sqrt{5}}{2}, \lim_{p \to 1} \frac{\mathbb{P}_p(A(3,8))}{\mathbb{P}_p(A(1,8))} = \sqrt{2}-1$$

which Masri and Ono prove using modular forms.

The Rogers-Ramanujan identities are special cases of a theorem of Gordon [Gor61].

Proposition 6.4.5 (Gordon). *Let $B_{k,i}(n)$ be the number of partitions of n written as $(b_1, \ldots, b_s)$ with $b_j - b_{j+k-1} \geq 2$ and at most $i-1$ parts equal to 1. Let $A_{k,i}(n)$ be the number of partitions of n into parts not congruent to 0 or $\pm i$ mod $2k+1$. Then $A_{k,i}(n) = B_{k,i}(n)$ for all n.*

We can extract from this the distribution of the number of 1s:

Proposition 6.4.6. *The probability that a partition $b_1 + b_2 + \ldots$ of n (in decreasing order) with $b_j - b_{j+k-1} \geq 2$ for each j, has exactly r ones approaches*

$$\frac{\sin\left(\frac{(r+1)\pi}{2k+1}\right) - \sin\left(\frac{r\pi}{2k+1}\right)}{\sin\left(\frac{k\pi}{2k+1}\right)}.$$

for $r = 0, 1, \ldots, k-1$, as $n \to \infty$.

The case $k = 2$ is Corollary 6.4.3.

Proof. It suffices to show that the limiting probability of having less than r ones, for $r = 1, \ldots, k$, is $\left(\sin \frac{r\pi}{2k+1}\right) / \left(\sin \frac{k\pi}{2k+1}\right)$. By the previous theorem, the desired partitions with less than r ones are equinumerous with partitions having no parts congruent to 0 or $\pm r$ mod $2k+1$. We can now apply Meinardus' theorem with

$$\alpha(s) = (1 - (2k+1)^{-s})\zeta(s) - \zeta(s, r/(2k+1)) - \zeta(s, 1 - r/(2k+1)).$$

The number of such partitions is, from Meinardus' theorem, asymptotic to

$$e^{\alpha'(0)}(4\pi)^{-1/2}(\pi^2 A/6)^{1/4} n^{-3/4} \exp(\pi\sqrt{2An/3}) \tag{6.3}$$

with $A = 1 - \frac{3}{2k+1}$. Differentiating and using the identity $\zeta'(0, v) = \log(\Gamma(v)/\sqrt{2\pi})$, we get

$$e^{\alpha'(0)} = \frac{1}{\sqrt{2k+1}} \frac{2\pi}{\Gamma\left(\frac{r}{2k+1}\right)\Gamma\left(1 - \frac{r}{2k+1}\right)}.$$

The reflection formula $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ finally gives

$$e^{\alpha'(0)} = \frac{2}{\sqrt{2k+1}} \sin\frac{\pi r}{2k+1}.$$

Thus the number of the partitions counted by $B_{k,r}(n)$ is asymptotic to $\sin \pi r/(2k+1)$ multiplied by a function of k and n. The desired limiting probability is

$$\lim_{n\to\infty} B_{k,r}(n)/B_{k,k}(n)$$

; the numerator and denominator involve the same function of k, n and cancel, leaving the quotient of sines as the limit. $\square$

We can observe the similarity to the following result from the combinatorics of words.

Proposition 6.4.7. *The probability that a word of length n over the alphabet $\{0, 1, \ldots, k-1\}$, in which the sum of any two consecutive letters is at most $k-1$, begins with the letter r approaches*

$$\frac{\sin \frac{(r+1)\pi}{2k+1} - \sin \frac{r\pi}{2k+1}}{\sin \frac{k\pi}{2k+1}}$$

as $n \to \infty$.

Proof. Let M_k be the $k \times k$ matrix which has 1s on and above the main antidiagonal and 0s for all other entries. Then the numbers of words of length n beginning with $0, 1, \ldots, k-1$ and satisfying this condition are given by the vector $M_k^{n-1}\vec{1}$, where $\vec{1}$ is the column vector of k 1s. The limiting distribution is therefore given by the eigenvector corresponding to the largest eigenvalue of M_k.

Let $P_k(x)$ be the characteristic polynomial of M_k. Then assume for the moment the identity

$$P_k(x) \cdot P_k(-x) = x^{2k} U_{2k}(1/(2x)) \tag{6.4}$$

where U is the Chebyshev polynomial of the second kind. We will prove this identity later. The zeroes of U_n are at $\cos(j\pi/(n+1))$ for $j = 1, 2, \ldots, n$; thus the zeroes of $x^{2k}U_{2k}(1/(2x))$ are at $\frac{1}{2} \sec \frac{j\pi}{2k+1}$, with $j = 1, 2, \ldots, 2k$. The two largest of these zeroes in absolute value are those with $j = k, k+1$, which are negatives of each other; one is a zero of $P_k(x)$ and the other is a zero of $P_k(-x)$.

Assuming (6.4), the largest eigenvalue of M_k is $\frac{1}{2}\sec\frac{k\pi}{2k+1}$. We want to show that

$$v_k = \left(\sin\frac{\pi}{2k+1}, \sin\frac{2\pi}{2k+1} - \sin\frac{\pi}{2k+1}, \ldots, \sin\frac{k\pi}{2k+1} - \sin\frac{(k-1)\pi}{2k+1}\right)^T$$

is the corresponding eigenvector, that is, that $M_k v = \frac{1}{2}\sec\frac{k\pi}{2k+1}\vec{v}$. Performing the matrix multiplication, it suffices to show that

$$2\cos\frac{k\pi}{2k+1}\sin\frac{j\pi}{2k+1} = \sin\frac{(k+1-j)\pi}{2k+1} - \sin\frac{(k-j)\pi}{2k+1}. \tag{6.5}$$

Now, we recall the identity $2\cos\phi\sin\theta = \sin(\theta+\phi) + \sin(\theta-\phi)$. Applying this to the left-hand side of (6.5), we get

$$2\cos\frac{k\pi}{2k+1}\sin\frac{j\pi}{2k+1} = \sin\frac{(j+k)\pi}{2k+1} + \sin\frac{(j-k)\pi}{2k+1}. \tag{6.6}$$

From the right-hand side of (6.6) we can get the right-hand side of (6.5) by noting that sin is 2π-periodic (for the first term) and an odd function (for the second term). □

Proof of (6.4). . We want to show that $P_k(x)P_k(-x) = x^{2k}U_{2k}(1/(2x))$, where $t \to P_k(t)$ is the characteristic polynomial of the matrix which has 1 in all positions on or above the main antidiagonal. We will do this by showing both sides of this equation satisfy the same recurrence. First, $P_k(x)$ and $P_k(-x)$ can both be written as determinants. In the case $k = 3$ these determinants are

$$P_3(x) = \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 0 \\ 1 & 0 & -x \end{vmatrix}, P_3(-x) = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 0 \\ 1 & 0 & x \end{vmatrix}$$

(We will in general illustrate the proofs with small matrices instead of writing the matrices in a fully general form.) Write $Q_k(x) = P_k(x)P_k(-x)$. Then $P_k(x)P_k(-x)$

is a product of determinants, and thus a determinant of products. By matrix multiplication, in the $k = 3$ case we have

$$Q_3(x) = \begin{vmatrix} 3-x^2 & 2 & 1 \\ 2 & 2-x^2 & 1 \\ 1 & 1 & 1-x^2 \end{vmatrix}$$

We claim that $Q_k(x) = (1-2x^2)Q_{k-1}(x) - x^4 Q_{k-2}(x)$ for $k \geq 2$. This follows from simple properties of determinants. In the case $k = 4$ we have

$$Q_4(x) = \begin{vmatrix} 4-x^2 & 3 & 2 & 1 \\ 3 & 3-x^2 & 2 & 1 \\ 2 & 2 & 2-x^2 & 1 \\ 1 & 1 & 1 & 1-x^2 \end{vmatrix}$$

and we first subtract the last row from all other rows, and then subtract the second-to-last row from the bottom row, we get

$$Q_4(x) = \begin{vmatrix} 3-x^2 & 2 & 1 & x^2 \\ 2 & 2-x^2 & 1 & x^2 \\ 1 & 1 & 1-x^2 & x^2 \\ 0 & 0 & x^2 & 1-x^2 \end{vmatrix}.$$

We expand by minors along the bottom row. This gives

$$Q_4(x) = (1-2x^2)\begin{vmatrix} 3-x^2 & 2 & 1 \\ 2 & 2-x^2 & 1 \\ 1 & 1 & 1-x^2 \end{vmatrix} - x^2 \begin{vmatrix} 3-x^2 & 2 & x^2 \\ 2 & 2-x^2 & x^2 \\ 1 & 1 & x^2 \end{vmatrix}.$$

The first determinant on the left-hand side is $(1-2x^2)Q_3(x)$. To find the second determinant, we subtract the bottom row from all other rows to get

$$\begin{vmatrix} 2-x^2 & 1 & 0 \\ 1 & 1-x^2 & 0 \\ 1 & 1 & x^2 \end{vmatrix}$$

and finally expanding by minors along the rightmost column gives $xQ_2(x)$. Thus we have $Q_4(x) = (1-2x^2)Q_3(x) - x^4 Q_2(x)$; the same is true for larger matrices.

We now show that the polynomials $V_k(x) = x^{2k}U_{2k}(1/(2x))$ satisfy the same recurrence. Let $W_k(x) = U_k(1/x)x^k$. Then consider the recurrence for the Chebyshev U polynomials, $U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$. Let $x = 1/t$ to get $U_n(1/x) = 2/xU_{n-1}(1/x) - U_{n-2}(1/x)$. Note that $U_k(1/x) = x^{-k}W_k(x)$ and clear factors of x to get

$$W_n(x) = 2W_{n-1}(x) - t^2 W_{n-2}(x).$$

From this we can read off the generating function for the polynomials W,

$$\sum_{n\geq 0} W_n(x)t^n = \frac{1}{1-2t+t^2x^2}.$$

Substituting $2x$ for x gives

$$\sum_{n\geq 0} W_n(2x)t^n = \frac{1}{1-2t+4t^2x^2} \tag{6.7}$$

and substituting $-t$ for t gives

$$\sum_{n\geq 0} (-1)^n W_n(2x)t^n = \frac{1}{1+2t+4t^2x^2}. \tag{6.8}$$

Averaging (6.7) and (6.8) gives

$$\sum_{n\geq 0,\, 2|n} W_n(2x)x^n = \frac{1+4t^2x^2}{1+8t^2x^2-4t^2+16t^4x^4}.$$

Rewrite the sum with $k = n/2$, and substitute $\sqrt{t}/2$ for t, to get

$$\sum_{k\geq 0} W_{2k}(2x)4^{-k}x^k = \frac{1+tx^2}{1+2tx^2-t+t^2x^4}$$

We note that $V_k(x) = W_{2k}(2x)/2^{2k}$, so this gives the generating function for the V_k. The desired recurrence follows immediately.

Finally, we can check that $Q_1(x) = V_1(x) = 1-x^2$ and $Q_2(x) = V_2(x) = x^4-3x^2+1$, so $\{Q_k\}$ and $\{V_k\}$ have the same initial values. Thus $Q_k(x) = V_k(x)$, establishing the result. □

Finally, we note that this method of modeling the multiplicities of parts of partitions by Markov chains appears to work even when corresponding pairs of partition identities do not exist. For example, consider partitions of n into nonconsecutive parts which contain no k. These can be identified with bit strings which contain no consecutive 1s and have a 0 in the kth position. Now, the number of bit strings of length n with no two consecutive 1s and a zero in the kth position is $F_{k+1}F_{n-k+2}$. So the probability that a random bit string of length n with no two consecutive 1s has a 0 in the kth position is

$$\frac{F_{k+1}F_{n-k+2}}{F_{n+2}} \sim \frac{F_{k+1}\phi^{n-k+2}}{\phi^{n+2}} = \frac{F_{k+1}}{\phi^k}.$$

When $k = 1, 2, 3$ these are $(\sqrt{5}-1)/2 \approx .618, 3-\sqrt{5} \approx .764, 3\sqrt{5}-6 \approx .708$ respectively.

These appear to be the limiting probabilities that are obtained in actual partitions. Let $P_k(n)$ denote the number of partitions of n into nonconsecutive parts which contain no k. (In particular $P_1(n)$ is the same as $r_2(n)$; both count the partitions which occur in the second Rogers-Ramanujan identity.) Let $p(n,j)$ denote the number of partitions of n into j nonconsecutive distinct parts, and let $q(n,j)$ denote the number of partitions of n into j parts. Then $p(n,j) = q(n-j(j-1),j)$. From a partition $n = n_1+n_2+\cdots+n_j$ into nonconsecutive distinct parts, we can obtain an unrestricted partition $n-j(j-1) = (n_1-2(j-1))+(n_2-2(j-2))+\cdots+(n_{j-1}-2)+n_j$, and this correspondence is reversible.

Furthermore, we can write $P_2(n)$ in terms of the ps and therefore the qs. First, we note that

$$P_2(n) = \sum_{k\geq 1} p(n-(3k+2),k).$$

To see this, consider partitions of n into nonconsecutive distinct parts, which contain a 2. Such partitions contain no 1s and no 3s. From such a partition with $k+1$ parts, we can obtain a partition of $n-(3k+2)$ with k nonconsecutive distinct parts by removing the 2 and subtracting 3 from each other part. By the equivalence between ps and qs we have

$$P_2(n) = \sum_{k\geq 1} q\left(n-(k^2+2k+2),k\right).$$

Note that this sum is actually finite. Finally, the $q(n,j)$ are easily computed. For example, we have the well-known recurrence

$$Q(n,k) = Q(n,k-1) + Q(n-k,k)$$

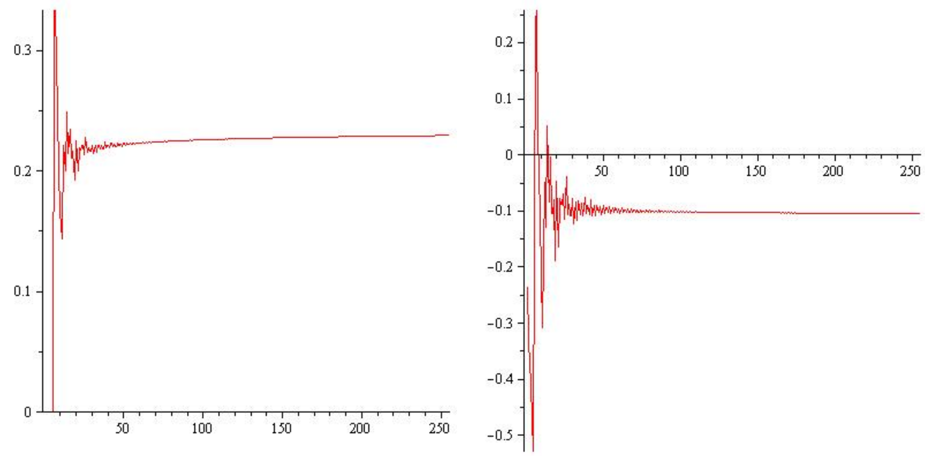


Figure 6.1: Left panel: $P_2(n)/r_1(n)$ for $n = 1, 2, \ldots, 255$. Right panel: $\sqrt{n}(P_2(n)/r_1(n)) - (\sqrt{5} - 2))$ for the same values of n.

with $Q(n, 0) = 0, Q(1, k) = 1$, where $Q(n, k)$ is the number of partitions of n into at most k parts; then $q(n, k) = Q(n, k) - Q(n, k-1)$. Therefore we can compute $P_2(n)$, and the ratio $P_2(n)/r_1(n)$, for each n. We conjecture that $\lim_{n\to\infty} P_2(n)/r_1(n) = \sqrt{5} - 2$, and the convergence is of square-root speed; see Figure 6.4.

6.5 Probabilistic aspects of overpartitions

Overpartitions are a particular type of partition-like object in which the last occurrence of each part can be barred. For example, the overpartitions of 4 are $4, \bar{4}, 3+1, \bar{3}+1, 3+\bar{1}, \bar{3}+\bar{1}, 2+2, 2+\bar{2}, 2+1+1, \bar{2}+1+1, 2+1+\bar{1}, \bar{2}+1+\bar{1}, 1+1+1+1, 1+1+1+\bar{1}$. We let $\bar{p}(n)$ denote the number of overpartitions of n. It is clear that an overpartition is the disjoint union of a standard partition and a partition into distinct parts.

Therefore overpartitions have the generating function

$$\prod_{n\geq 0}\frac{1+z^n}{1-z^n} = 1+2z+4z^2+8z^3+14z^4+24z^5+40z^6+64z^7+100z^8+154z^9+232z^{10}+\cdots.$$

In the nomenclature of Section 6.2, overpartitions are of the class 3/2. We note that we can rewrite the generating function of overpartitions as

$$\prod_{n\geq 0}\frac{(1-z^{2n})}{(1-z^n)^2} = \prod_{n\geq 0}(1-z^n)^{-b_n}$$

where b_n is 1 when n is even, and 2 when n is odd. "By inspection" we can see that since the b_n average 3/2, overpartitions fall in the 3/2 class. More rigorously, the associated Dirichlet series is $\alpha(s) = (2-2^{-s})\zeta(s)$, which has simple pole at $\rho = 1$ with residue $A = 3/2$. From Meinardus' theorem, we find

$$\bar{p}(n) \sim \frac{1}{8n}\exp(\pi\sqrt{n}).$$

This result is due to Hardy and Ramanujan [HR18].

Much work on overpartitions has been arithmetic in nature, for example [Mah04]. The original motivation for studying overpartitions comes from the *jagged partitions* of [FJM05], which have their origins in statistical physics; these are certain sequences of numbers which are "almost" weakly decreasing, and are equinumerous with overpartitions.

Random overpartitions have previously been studied [CH04, CGH06]; these papers give results about the number of parts, number of parts of various multiplicities, and so on of random overpartitions. In this section we will define a family of weighted objects which interpolate between partitions and overpartitions.

First, an overpartition consists of a partition and a partition into distinct parts. In terms of combinatorial classes, we have $\mathcal{O} = \mathcal{P} \times \mathcal{D}$, where $\mathcal{O}, \mathcal{P}, \mathcal{D}$ are the classes of overpartitions, partitions, and partitions into distinct parts, respectively. So the Boltzmann sampler for overpartitions with parameter x simply generates a partition and a partition into distinct parts, both with parameter x. We recall that the Boltzmann sampler for partitions into distinct parts generates partitions with mean size asymptotic to $\frac{\pi^2}{12}(1-x)^{-2}$, and the Boltzmann sampler for partitions generates partitions with mean size asymptotic to $\frac{\pi^2}{6}(1-x)^{-2}$. Thus the mean size of Boltzmann-x *overpartitions* is $\frac{\pi^2}{4}(1-x)^{-2}$. The variance of this size is the sum of the variances originating from the component Boltzmann samplers, which are $\frac{\pi^2}{6}(1-x)^{-3}$ and $(\pi^2/3 - 2\zeta(3))/(1-x)^3$, so the distribution of sizes is concentrated. Therefore a "typical" overpartition of n has barred parts summing to $n/3$.

Alternatively, recall the asymptotic results $p_d(n) \sim \frac{1}{4(3n)^{3/4}} \exp \pi\sqrt{n/3}, p(n) \sim \frac{1}{4n\sqrt{3}} \exp \pi\sqrt{2n/3}$. The "typical" number of distinct parts of an overpartition of n, then, is that x for which $p_d(x)p(n-x)$ is maximized; this is maximized when x is near $n/3$. This is essentially the method of proof used in [CGH06];

Corteel, Hitczenko, and Goh [CH04, CGH06] have undertaken the study of certain statistics of random overpartitions. Here we define a common generalization of partitions and overpartitions, which interpolates between these two random objects. We then construct Boltzmann samplers for these objects, and analyze the Boltzmann samplers to determine how these parameters vary.

In particular, as observed in [CH04], overpartitions can be understood as partitions counted with the weight 2^k where k is the number of part sizes. Alternatively, they can be understood as overpartitions counted with the weight $(2-1)^l$ where l is the number of barred parts. So we define a random object, a w-overpartition of n (for $w > 0$), as a partition counted with the weight w^k where k is the number of part sizes, or alternatively in the case $w > 1$ as an overpartition counted with the weight $(w-1)^l$ where l is the number of bars. Thus a random 2-overpartition is an overpartition chosen uniformly at random, and a random 1-overpartition is a partition chosen uniformly at random.

We note that w-overpartitions are counted by the generating function

$$(1 + wz + wz^2 + wz^3 + \cdots)(1 + wz^2 + wz^4 + wz^6 + \cdots)\cdots = \prod_{j \geq 1} \frac{1 + (w-1)z^j}{1 - z^j}.$$

So we can construct a Boltzmann sampler with parameter x by taking P_j parts of size j, where P_j is an integer-valued random variable with $\mathbb{P}(P_j = 0) = \frac{1-x^j}{1+(w-1)x^j}$ and $\mathbb{P}(P_j = k) = \frac{wx^{jk}(1-x^j)}{1+(w-1)x^j}$ for $k > 0$. and the P_j are independent.

As for partitions and overpartitions, the critical value for the Boltzmann sampler for w-overpartitions is at $x = 1$. The expected size of a random w-overpartition with Boltzmann parameter x is given by

$$\mathbb{E}(\sum_{j \geq 1} jP_j) = \sum_{j \geq 1} \frac{jwx^j}{(1-x^j)(1-(w-1)x^j)}.$$

We can approximate this sum by the corresponding integral over j, from 1 to ∞, giving

$$\mathbb{E}(\sum_{j\geq 1} jP_j) \sim \frac{\log(1-x) - \log(1+(w-1)x)}{\log x} + \frac{Li_2(1-x) - Li_2(1+(w-1)x)}{(\log x)^2}$$

where Li_2 is the dilogarithm. As $x \to 1^-$ with fixed w, this has the asymptotic form

$$\frac{\pi^2/6 - Li_2(w)}{(1-x)^2} + O((1-x)^{-1}).$$

This is the asymptotic expected size of a random w-overpartition with Boltzmann parameter x. Thus we take $N = f(w)(1-x)^{-2}$ where $f(w) = \pi^2/6 - Li_2(w)$.

The average number of *parts* of a partition is similarly obtained by integration. This is $E\sum_j P_j$; we have

$$\mathbb{E}(P_j) = \frac{wx^j}{(1-x^j)(1+(w-1)x^j)}$$

and, integrating again,

$$\mathbb{E}(\sum_{j\geq 1} P_j) \sim \frac{\log(x-1) - \log(1+(w-1)x)}{\log x}.$$

As $x \to 1^-$, this has the asymptotic form $(1-x)^{-1}\log(1-x)^{-1} + O((1-x)^{-1})$. Recalling the relationship between x and N, we see that $1/(1-x) \sim \sqrt{N/f(w)}$. Thus we have:

Proposition 6.5.1. *The average number of parts in a random w-overpartition, with Boltzmann parameter chosen to give w-overpartitions of expected size N, is asymptotic to*

$$\frac{\sqrt{N}\log N}{2\sqrt{\pi^2/6 - Li_2(w)}}.$$

In particular, in the case $w = 1$ we recover a Boltzmannized version of the Erdos-Lehner result on the number of parts of a random partition; in the case $w = 2$ case we recover a Boltzmannized version of the corresponding result of Corteel and Hitczenko [CH04, Thm. 1.3], namely that the average number of parts in a random Boltzmannized overpartition, tuned to have expected size N, is $\pi^{-1}\sqrt{N}\log N$.

We can get a similar result if we consider the number of expected part *sizes* instead of the expected number of *parts*. Let P_j be defined as before, and let $Q_j = \min(P_j, 1)$. Then $\mathbb{E}(Q_j) = 1 - \mathbb{P}(P_j = 0) = wx^j/(1 + (w-1)x^j))$. Integrating over j gives $\sum_j \mathbb{E}Q_j \sim \frac{-w\log(1+(w-1)x)}{(w-1)\log x}$, and as $x \to 1^-$, this is asymptotic to $\frac{w\log w}{w-1}(1-x)^{-1}$.. The average number of part sizes in a Boltzmann-x w-overpartition, where x has been tuned to give average size n, is thus asymptotic to

$$\frac{w\log w}{(w-1)\sqrt{f(w)}}\sqrt{N}.$$

In the case $w = 1$ we take a limit to find that the coefficient is $\sqrt{6}/\pi$; this reproduces a result of [EL41]. In the case $w = 2$ this coefficient is $(4\log 2)/\pi$, reproducing [CH04, Thm. 1.1]. Finally, we observe that

$$\lim_{w\to\infty} \frac{w\log w}{(w-1)\sqrt{f(w)}} = \sqrt{2}$$

which we might interpret as "an ∞-overpartition has $\sqrt{2N}$ distinct part sizes on average". In fact, this makes sense. Whatever an ∞-overpartition is, it is an object biased very heavily towards having as many distinct part sizes as possible. The partitions with the most part sizes for their sum are the "triangular" partitions $1 + 2 + \ldots + k$, and a triangular partition of N has size near $\sqrt{2N}$.

Now, [CGH06, Thm. 3] give a result on the sum of the barred parts of an overpartition:

Theorem 6.5.2. *Let $\bar{W}_n$ be the sum of the barred parts in a random overpartition of n. For $k = o(n)$,*

$$\mathbb{P}(\bar{W}_n = \lfloor n/3 \rfloor \pm k) = \frac{3}{4n^{3/4}} \exp\left(\frac{-9\pi k^2}{16n^{3/2}}\right)(1 + o(1)).$$

However, the Boltzmann method does not reproduce this result. We give here the results that lead to this conclusion, as they are interesting results about the Boltzmann method applied to partitions into distinct parts.

Proposition 6.5.3. *The expected number, expected sum, and variance of the sum of the overlined parts of a Boltzmannized w-overpartition with parameter x are, as $x \to 1^-$,*

$$\frac{\log w}{1-x}, \frac{-Li_2(w)}{(1-x)^2}, \frac{-2Li_2(w)}{(1-x)^3}$$

Proof. Let R_j be Bernoulli with mean $(w-1)x^j/(1+(w-1)x^j)$; this is the probability that a random Boltzmann-x w-overpartition has an overlined part equal to j. The expected number of overlined parts is then

$$\mathbb{E}(\sum_j R_j) \sim \int_0^\infty \frac{(w-1)x^j}{1+(w-1)x^j}\, dj = \frac{-\log w}{\log x} \sim \frac{\log w}{1-x}.$$

Their expected sum is

$$\mathbb{E}(\sum_j jR_j) \sim \int_0^\infty \frac{j(w-1)x^j}{1+(w-1)x^j}\, dj = \frac{-Li_2(w)}{(\log x)^2} \sim \frac{-Li_2(w)}{(1-x)^2}.$$

Finally, since the R_j are independent, so are the jR_j, and thus the variance of their sum is the sum of their variances. So

$$\begin{aligned}
\mathbb{V}(\sum_j jR_j) &\sim \int_0^\infty j^2 \frac{(w-1)x^j}{(1+(w-1)x^j} \left(1 - \frac{(w-1)x^j}{(1+(w-1)x^j}\right) dj \\
&= \int_0^\infty \frac{j^2(w-1)x^j}{(1+(w-1)x^j)^2}\, dj \\
&= \frac{2Li_2(w)}{(\log x)^3} \sim \frac{-2Li_2(w)}{(1-x)^3}.
\end{aligned}$$

$\square$

Now, we choose x so that the expected size of a w-overpartition with Boltzmann parameter x is near N; that is, take $1 - x = \sqrt{f(w)/N}$. Then we get

$$\mathbb{E}(\sum_j jR_j) \sim \frac{-Li_2(w)}{(1-x)^2} = \frac{Li_2(w)}{Li_2(w) - \pi^2/6} N.$$

In the case $w = 2$, the expected sum of overlined parts is $N/3$, corroborating [CH04, Thm. 1.4].

But with the same parameter x, we get

$$\mathbb{V}(\sum_j jR_j) \sim \frac{-2Li_2(w)}{(1-x)^3} = \frac{-2Li_2(w)}{(\pi^2/6 - Li_2(w))^{3/2}} N^{3/2}.$$

In the case $w = 2$, this is $\frac{4}{3\pi}N^{3/2}$. But [CGH06, Corollary 1] gives the variance $\frac{8}{9\pi}N^{3/2}$ for the fixed-size N, two-thirds the variance in the Boltzmannized case. This gives an indication of the limitations of naive use of Boltzmann samplers. Two-thirds of the variance of the number of overlined parts in the Boltzmannized case comes from variance occurring for fixed N; one-third comes from the variance of the size of the Boltzmannized object itself.

To make this more precise, we conjecture that the variance of the sum of the overlined parts in a Boltzmannized w-overpartition tuned to size N is equal to the sum of:

1. the variance of the sum of the overlined parts in a fixed-size w-overpartition of size N, and

2. the *square* of the expected proportion of parts of a w-overpartition which are overlined, multiplied by the variance of the size of a Boltzmannized w-overpartition tuned to size N.

Implicitly we have assumed here that these two sources of variance are independent. The first item in this list is the only one of the terms here not yet known, so we can solve for it. Under these assumptions, we find that the variance of the sum of the overlined parts in a fixed-size w-overpartition is

$$-2Li_2(w)f(w)^{3/2}N^{3/2} - \left(\frac{-Li_2(w)}{f(w)}\right)^2 \left(\frac{2}{\sqrt{f(w)}}\right) N^{3/2}$$

and after some simplification, this is

$$\frac{-\pi^2}{3}\frac{Li_2(w)}{f(w)^{5/2}}$$

which matches the result of [CGH06].

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