Twisted spectral data and singular monopole

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ABSTRACT

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We study higher dimensional versions of monopoles with Dirac singularities on manifolds which are principal circle bundles over a smooth complex projective variety. We interpret such generalized monopoles in terms of twisted spectral data on a companion algebraic vareity. We conjecture that this correspondence is bijective under certain stability condition, and thus gives an algebraic construction of singular monopoles.

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Chapter 1

Generalized monopole

1.1 Monopole equation

Bogomolny's equation describes static monopoles in Yang-Mills theory. Given a three dimensional Riemannian manifold Y, let E denote a rank n Hermitian vector bundle $E \to Y$, and let A be a unitary connection on E, let ϕ be a skew Hermitian section of the bundle $End(E)$, the section ϕ is called a Higgs field on E (with trivial coefficients). We say that the triple (E, A, ϕ) satisfies Bogomolny equation if

$$F_A = *D_A\phi \tag{1.1.1}$$

We first study the equation when $Y = S^1 \times \Sigma$, where Σ is a Riemann surface. In this case, Bogomolny equation comes from dimension reduction of the anti-self-duality equation: consider $M = \mathbb{R} \times Y = \mathbb{R} \times S^1 \times \Sigma$, fix a complex structure on Σ, let z be its local coordinate, we use s to parameterize $\mathbb{R}$ and t to parameterize

S^1, then $w = s + it$ is a holomorphic coordinate on $\mathbb{R} \times S^1$, thus give M a complex structure so that $M \cong \mathbb{C}^\times \times \Sigma$. In terms of these coordinates, a connection form A on Y is expressed as

$$A = A_t \,\mathrm{d}\, t + A_z \,\mathrm{d}\, z + A_{\bar{z}} \,\mathrm{d}\, \bar{z}$$

Given a Higgs field ϕ on Y, let $\hat{A} = \phi \,\mathrm{d}\, s + A$ in this local trivialization, then $\hat{A}$ glues into a connection on M, and we have

Proposition 1.1.1. (A, ϕ) *satisfies Bogomolny equation if and only if* $\hat{A}$ *satisfies anti-self-duality equation:*

$$* F_{\hat{A}} = -F_{\hat{A}} \tag{1.1.2}$$

Fix a Kähler metric on Σ, and denote its Kähler form by ω. We can extend the metric to a Kähler metric on M, and denote its Kähler form as Ω. Split $F_{\hat{A}}$ with respect to the complex structure on M, and denote by $\Lambda F_{\hat{A}}$ the curvature component proportional to Ω, we see that the anti-self-duality equation is equivalent to

$$F^{0,2}_{\hat{A}} = F^{2,0}_{\hat{A}} = 0,\ \Lambda F_{\hat{A}} = 0 \tag{1.1.3}$$

A useful generalization of the anti-self-duality equation is Hermit-Einstein equation

$$F^{0,2}_{\hat{A}} = F^{2,0}_{\hat{A}} = 0,\ \Lambda F_{\hat{A}} = \sqrt{-1} C\ \mathrm{id}_E \tag{1.1.4}$$

with constant C. To generalize Bogomolny's equation, we apply dimension reduciton to the equation 1.1.4: let X be a k dimensional smooth projective variety, with Kähler form ω, let Y be a S^1 bundle over X, together with a Hermitian vector bundle E over Y, a connection A and a Higgs field ϕ. In a local chart U of X where we can trivialize Y as $S^1 \times U$, use $z_1, \cdots, z_k$ for the holomorphic coordinates on U, and use t for the coordinate of S^1, suppose

$$\omega = \sum_{1 \le i,j \le k} g_{i\bar{j}} \,\mathrm{d}\, z_i \wedge \mathrm{d}\, \bar{z}_j$$

$$A = A_t \,\mathrm{d}\, t + \sum_{j=1}^{k} (A_j \,\mathrm{d}\, z_j + A_{\bar{j}} \,\mathrm{d}\, \bar{z}_j)$$

and denote

$$\nabla_j = \Big(\frac{\partial}{\partial z_j} + A_j\Big) \,\mathrm{d}\, z_j, \quad 1 \le j \le k$$

$$\nabla_{\bar{j}} = \Big(\frac{\partial}{\partial \bar{z}_j} + A_{\bar{j}}\Big) \,\mathrm{d}\, \bar{z}_j, \quad 1 \le j \le k$$

$$\nabla_t = \Big(\frac{\partial}{\partial t} + A_t\Big) \,\mathrm{d}\, t$$

$$F_{ab} = [\nabla_a, \nabla_b], \quad a, b \in \{1, \cdots, k, \bar{1}, \cdots \bar{k}, t\}$$

equation 1.1.4 is equivalent to:

$$\begin{cases} F_{jl} = 0, \quad F_{\bar{j}\bar{l}} = 0 \\ F_{tj} = \sqrt{-1} \nabla_j \phi \wedge \mathrm{d}\, t, \quad F_{t\bar{j}} = -\sqrt{-1} \nabla_{\bar{j}} \phi \wedge \mathrm{d}\, t \\ g^{j\bar{l}} F_{j\bar{l}} - \nabla_t \phi = \sqrt{-1} C I \end{cases} \tag{1.1.5}$$

Definition 1.1.2. The triple (E, A, ϕ) is a generalized monopole on Y with slope C if for each $U \subset X$ where Y trivializes, equation 1.1.5 holds.

Remark 1.1.3. We will allow singular solutions, and in that case, we define generalized monopole on an open submanifold $Y_0 \subset Y$ in the same way.

Definition 1.1.4. For a generalized monopole (E, A, ϕ) on Y with slope C, we say that a generailzed monopole (E', A', ϕ') on Y with slope C' is a sub-monopole of (E, A, ϕ) if E' is a sub-bundle of E preserved by A and ϕ, and $A_{|E'} = A'$, $\phi_{|E'} = \phi'$.

Remark 1.1.5. A sub-monopole must have the same slope, i.e., $C' = C$.

Definition 1.1.6. A generalized monopole (E, A, ϕ) on Y is irreducible if its only sub-monopoles are the trivial ones, i.e. $(0, 0, 0)$ and (E, A, ϕ).

In local coordinates, denote

$$\nabla_X^{0,1} = \sum_{j=1}^{k} \nabla_{\bar{j}}$$

$$F_X^{0,2} = [\nabla_X^{0,1}, \nabla_X^{0,1}]$$

then the equation 1.1.5 is equivalent to

$$\begin{cases} F_X^{0,2} = 0 \\ \left[\nabla_X^{0,1}, \nabla_t - \sqrt{-1}\phi\right] = 0 \\ g^{j\bar{l}} F_{j\bar{l}} - \nabla_t \phi = \sqrt{-1} C I \end{cases} \tag{1.1.6}$$

Let $E_t = E_{|\{t\} \times U}$, the first equation in 1.1.6 implies that $\nabla_X^{0,1}$ defines an integrable complex structure on E_t.

Definition 1.1.7. The scattering map from t_0 to t_1

$$R_{t_0, t_1} : E_{t_0} \to E_{t_1}$$

is defined by parallel transport with respect to $A_t - i\phi$, namely, for each $\sigma_0 \in E_{|(t_0, z)}$ there is a unique solution $\sigma(t)$ of

$$\frac{\mathrm{d}\,\sigma}{\mathrm{d}\,t} + (A_t - i\phi)\sigma = 0$$

with initial value $\sigma(t_0) = \sigma_0$. We set $R_{t_0,t_1}(\sigma_0) = \sigma(t_1)$.

According to the second equation in 1.1.6, the scattering map defines a holomorphic isomorphism from E_{t_0} to E_{t_1}.

1.2 Dirac singularity

We consider solutions with Dirac type singularities, the prototype is a family of $U(1)$ monopoles on the three dimensional manifold

$$B^3_0(\epsilon) := \{(r, \theta, \psi) \mid 0 < r < \epsilon,\ \theta \in [0, \pi],\ \psi \in [0, 2\pi)\}$$

with the standard volume form on $B^3_0(\epsilon)$:

$$\mathrm{d}\,\mu = r^2 \sin\theta \,\mathrm{d}\, r \wedge \mathrm{d}\,\theta \wedge \mathrm{d}\,\psi$$

Projecting along radial direction yields a map from $B^3_0(\epsilon)$ to the unit sphere S^2, denote by L_k the Hermitian line bundle on $B^3_0(\epsilon)$ obtained by pulling the line bundle of degree k on S^2. Explicitly, we can trivialize line bundles on the open sets

$$U_0 = \{\theta \neq 0\}, \quad U_\pi = \{\theta \neq \pi\}$$

and glue them into L_k via the transition function given by $g_{\pi 0} = e^{\sqrt{-1}k\psi}$. Consider the connection form A on L_k defined by

$$A_0 = \frac{\sqrt{-1}k}{2}(\cos\theta + 1)\,\mathrm{d}\,\psi \quad \text{on} \quad U_0$$

$$A_\pi = \frac{\sqrt{-1}k}{2}(\cos\theta - 1)\,\mathrm{d}\,\psi \quad \text{on} \quad U_\pi$$

on the overlap these forms satisfy

$$A_\pi + g_{\pi 0}^{-1} \,\mathrm{d}\, g_{\pi 0} = A_0$$

so A is a $U(1)$ connection on L_k. In this case, the Higgs field will be a function taking value in purely imaginary numbers, and if we choose

$$\phi = \frac{\sqrt{-1}k}{2r}$$

it is easy to verify that (L_k, A, ϕ) satisfies Bogomolny equation on $B_0^3(\epsilon)$. Such a solution has boundary behavior that has a nice geometric interpretation, it is first explored in the paper of Witten and Kapustin, they showed that scattering map cross Dirac type singularities plays a role of Hecke modification in geometric Langlands program.

We extend this picture to higher dimensions. Let Y be an m-dimensional Riemannian manifold, $Z \subset Y$ is a codimension 3 compact sub-manifold. For every $p \in Z$, we can choose local coordinates $(x_1, \cdots, x_m)$ so that:

- $p = (0, \cdots, 0)$
- Z is given by $x_1 = x_2 = x_3 = 0$
- the metric is of the form $I_m + O(R)$ as $R := \sqrt{x_1^2 + \cdots + x_m^2} \to 0$

we call $(x_1, \cdots, x_m)$ regular system of coordinates of Y with respect to Z centered at p.

Definition 1.2.1. With the above settings, let (E, A, ϕ) be a generalized $U(n)$ monopole on Y, we say the monopole has a Dirac singularity along Z with weights $(k_1, \cdots, k_n)$ if for every $p \in Z$, under every regular coordinates $(x_1, \cdots, x_m)$ of Y with respect to Z centered at p, such that

1. There is a unitary isomorphism α of the restriction of the bundle E to

$$B_\epsilon := \{(x_1, x_2, x_3, 0, \cdots, 0) \mid 0 < R < \epsilon\} \cong B_0^3(\epsilon)$$

with a direct sum of line bundles $L_{k_1} \oplus L_{k_2} \oplus \cdots \oplus L_{k_n}$ on $B_0^3(\epsilon)$.

2. Identify B_ϵ with $B_0^3(\epsilon)$, and identify E with direct sum of vector bundle $L_{k_1} \oplus L_{k_2} \oplus \cdots \oplus L_{k_n}$ on $B_0^3(\epsilon)$ under the bundle isomorphism α, in the trivializations of E over the two open subsets $\{\theta \neq 0\}$ and $\{\theta \neq \pi\}$ induced by the standard trivializations of the line bundles L_{k_i} the trivializations have transition function $\mathrm{diag}(e^{\sqrt{-1}k_1\psi}, \cdots, e^{\sqrt{-1}k_n\psi})$, in both of the trivializations, ϕ and A have asymptotic behaviors as follow:

$$\phi = \frac{\sqrt{-1}}{2r}\mathrm{diag}(k_1, \cdots, k_n) + O(1), \quad D_A(r\phi) = O(1) \quad \text{as} \quad r \to 0.$$

Chapter 2

Charbonneau-Hurtubise theorem

In the paper [**?**] Charbonneau and Hurtubise study generalized monopoles of Dirac singularity on a product of a circle and a Riemann surface, I will summarize their results in this chapter. Let Σ denote a compact Riemann surface and fix a Kähler metric on it, with Kähler form ω. Consider monopoles on $Y = S^1 \times \Sigma$. In this case, the monopole equations 1.1.5 can be writen simply as:

$$F_A - \sqrt{-1}C\mathrm{id}_E\omega = *D_A\phi \tag{2.0.1}$$

and the singularities occur at a discrete set of points.

2.1 Scattering map

Now parameterize $S^1 = \mathbb{R}/\mathbb{Z}$ by $t \in [0, 1]$, and consider monopole (E, A, ϕ) with singularities at $p_j = (t_j, z_j) \in S^1 \times \Sigma$ with weights $\vec{k}_j = (k_{j1}, k_{j2}, \cdots, k_{jn})$, $j =$

$1, 2, \cdots, N$, furthermore, we require that $0 < t_1 < \cdots < t_N < 1$. Write $\mathcal{E}_t = E_{|\{t\}\times\Sigma}$. Regarding the scattering map $R_{t,t'}$, we have

Proposition 2.1.1. *Away from t_i, $\mathcal{E}_t$ is a holomorphic bundle on Σ, and*

1. *If there is no singular time t_i between t and t', the scattering map $R_{t,t'} : \mathcal{E}_t \to \mathcal{E}_{t'}$ is an holomorphic isomorphism.*

2. *If only one t_i lies in between t and t', then $c_1(\mathcal{E}_{t'}) - c_1(\mathcal{E}_t) = \mathrm{Tr}(\vec{k}_i)$, where*

$$\mathrm{Tr}(\vec{k}_i) = \sum_{j=1}^{n} k_{ij}$$

The map $R_{t,t'}$ is a meromorphic map which is an isomorphism away from z_i, and near z_i there exist trivializations of $\mathcal{E}_t, \mathcal{E}_{t'}$ such that $R_{t,t'}$ is given by

$$\mathrm{diag}((z - z_i)^{k_{i1}}, \cdots, (z - z_i)^{k_{in}})$$

Consider the map $R_{0,1} : \mathcal{E}_0 \to \mathcal{E}_1$, since $\mathcal{E}_1 = \mathcal{E}_0$, we get a meromorphic endomorphism of $\mathcal{E}_0$ which is holomorphic outside of z_i's.

Definition 2.1.2. A bundle pair $(\mathcal{E}, \rho)$ consists of a holomorphic bundle $\mathcal{E}$ on Σ and a meromorphic endomorphism $\rho : \mathcal{E} \to \mathcal{E}$ such that ρ is an isomorphism outside of a finite set of points.

Thus we get a map from generailzed monopoles to bundle pairs:

$$\Theta : (E, A, \phi) \mapsto (\mathcal{E}_0, R_{0,1})$$

In general, consider an arbitrary pair $(\mathcal{E}, \rho)$ consisting of a rank n holomorphic vector bundle $\mathcal{E}$ over Σ, and a meromorphic bundle automorphism $\rho : \mathcal{E} \to \mathcal{E}$. Near a singular point p of ρ, choose a coordinate z centered at p and a trivialization of $\mathcal{E}$

Proposition 2.1.3 (Iwahori et al. [?]). *There are invertible holomorphic $n \times n$ matrices $F(z), G(z)$ and integers $\vec{k} = \{k_1, \cdots, k_n\}$, such that in this trivialization,*

$$\rho = F(z)\,\mathrm{diag}(z^{k_1}, \cdots, z^{k_n})G(z)$$

and the set of integers $\{k_1, \cdots, k_n\}$ is independent of the choice of F and G.

Definition 2.1.4. In the above setting, we say that the bundle pair $(\mathcal{E}, \rho)$ has singularity type $\vec{k}$ at p.

It is clear that the bundle pair $(\mathcal{E}_0, R_{0,1})$ obtained from above has singularity type $\vec{k}_i$ at z_i.

Notice that by proposition 2.1.1, we have

$$c_1(\mathcal{E}_1) - c_1(\mathcal{E}_0) = \sum_{j=1}^{N} \mathrm{Tr}(\vec{k}_j),$$

since $\mathcal{E}_1 = \mathcal{E}_0$, a bundle pair corresponding to a generalized monopole must satisfy

$$\sum_{i=1}^{N} \mathrm{Tr}(\vec{k}_i) = 0.$$

Proposition 2.1.5. *For a singular $U(n)$ generalized monopole with slope C, we have*

$$\sum_{i=1}^{N} \mathrm{Tr}(\vec{k}_i) t_i = c_1(\mathcal{E}_0) + \frac{nC}{2\pi}\,\mathrm{vol}(\Sigma).$$

2.2 Stability

Consider a bundle pair $(\mathcal{E}, \rho)$ on Σ, with singular type $\vec{k}_i$ at z_i, $i = 1, 2, \cdots, N$. Given a list of numbers $\vec{t} = (t_1, t_2, \cdots, t_N) \in T^N$, where T^N denotes the torus formed by the product of N circles of circumference 1, for the bundle pair $(\mathcal{E}, \rho)$, we define

Definition 2.2.1. $\vec{t}-$degree

$$\delta_{\vec{t}}(\mathcal{E}, \rho) = c_1(\mathcal{E}) - \sum_{j=1}^{N} t_j \operatorname{Tr}(\vec{k}_j)$$

and $\vec{t}$-slope

$$\mu_{\vec{t}}(\mathcal{E}, \rho) = \delta_{\vec{t}}(\mathcal{E}, \rho)/\text{rank } \mathcal{E}$$

Definition 2.2.2. A bundle pair $(\mathcal{E}, \rho)$ is $\vec{t}$-stable if any proper non-trivial ρ-invariant subbundle has a strictly smaller $\vec{t}$-slope.

Proposition 2.2.3. *If $(\mathcal{E}, \rho)$ is the image of a irreducible generalized monopole under the map Θ, then it is a $\vec{t}$-stable bundle pairs for some $\vec{t} \in T^N$.*

Theorem 2.2.4 (Charbonneau-Hurtubise)**.** *Given $\vec{t}$-stable bundle pair $(\mathcal{E}, \rho)$ on Σ, with singularity type $\vec{k}_i$ at z_i, $i = 1, 2, \cdots, N$, such that all $\operatorname{Tr}(\vec{k}_i)$ sum to 0. Given $0 < t_1 \leq t_2 \leq \cdots \leq t_N < 1$ and denote $p_i = (t_i, z_i) \in S^1 \times \Sigma$, there is a generalized monopole (E, A, ϕ) on $S^1 \times \Sigma$ with Dirac-type singularities of weight $\vec{k}_i$ at p_i, for which $\Theta(E, A, \phi) = (\mathcal{E}, \phi)$.*

Given $\vec{k}_i$, $1 \le i \le N$, such that $\sum \mathrm{Tr}(\vec{k}_i) = 0$, denote

$$\mathcal{M}^{ir}(S^1 \times \Sigma, p_1, \cdots, p_N, \vec{k}_1, \cdots, \vec{k}_N)$$

as the moduli of irreducible generalized monopole on $S^1 \times \Sigma$ with Dirac singularity of type $\vec{k}_i$ at $p_i = (t_i, z_i)$, and denote

$$\mathcal{M}_s(\Sigma, z_1, \cdots, z_N, \vec{k}_1, \cdots, \vec{k}_N, \vec{t})$$

as the moduli of $\vec{t}$-stable bundle pair with singularity of type $\vec{k}_i$ at z_i, Charbonneau-Hurtubise theorem says that the map Θ

$$\Theta : \mathcal{M}^{ir}(S^1 \times \Sigma, p_1, \cdots, p_N, \vec{k}_1, \cdots, \vec{k}_N) \to \mathcal{M}_s(\Sigma, z_1, \cdots, z_N, \vec{k}_1, \cdots, \vec{k}_N, \vec{t})$$

$$(E, A, \phi) \mapsto (\mathcal{E}_0, R_{0,1})$$

is a bijection.

Chapter 3

Spectral data

3.1 Spectral cover of bundle pair

From any bundle pair $(\mathcal{E}, \rho)$ over Riemann surface Σ, we can associate to spectral data $(\widetilde{\Sigma}, \mathcal{L})$, where $\widetilde{\Sigma}$ is a branched cover of $\Sigma \setminus \{z_1, \cdots, z_N\}$ and $\mathcal{L}$ is a sheaf on $\widetilde{\Sigma}$. Let $\Sigma_0 = \Sigma \setminus \{z_1, \cdots, z_N\}$.

Definition 3.1.1. The spectral curve $\widetilde{\Sigma} \subset \mathbb{C}^\times \times \Sigma_0$ is defined to be

$$\tilde{\Sigma} = \{(\lambda, z) \in \mathbb{C}^* \times \Sigma_0 \mid \det(\lambda I - \rho(z)) = 0\}$$

where $I \in End(\mathcal{E})$ is identity over each $z \in \Sigma$.

Denote $p : \mathbb{C}^* \times \Sigma \to \Sigma$ to be the projection,

Definition 3.1.2. The spectral sheaf $\mathcal{L}$ is a subsheaf of $p^*\mathcal{E}$ restricted on $\tilde{\Sigma}$ which assigns to (λ, z) the eigenspace of $\ker(p^*\rho(z) - \lambda \cdot \mathrm{id})$.

Thus we get a map Ξ from bundle pair $(\mathcal{E}, \rho)$ to its spectral data $(\tilde{\Sigma}, \mathcal{L})$.

If $\rho(z)$ is regular everywhere (i.e. it has distinct eigenvalues for different Jordan blocks), then $\mathcal{L}$ is a line bundle on $\widetilde{\Sigma}$. Note that being nonregular is a complex codimension three condition, and as in the case of Riemann surface, a generic ρ is regular.

The spectral data $(\widetilde{\Sigma}, \mathcal{L})$ reconstructs the bundle pair $(\mathcal{E}, \rho)$. Indeed, if we denote $\pi : \widetilde{\Sigma} \to \Sigma_0$ as the restriction of p on $\widetilde{\Sigma}$, and let η be the tautological section of $p^*(\mathbb{C}^* \times \Sigma) \cong \mathbb{C}^* \times \mathbb{C}^* \times \Sigma$ consisting of points (λ, λ, z) in local coordinates, then we have

$$\mathcal{E} = \pi_* \mathcal{L}, \quad \rho = \pi_* \eta$$

In this way, we see that Ξ is invertible, so we get a bijection between meromorphic bundle pair and its spectral data.

We interpret stability as numerics controlling singularities of the spectral data. Near a singular point z_i, $\rho(z)$ is asymptotic to

$$\mathrm{diag}(z^{k_{i1}}, z^{k_{i2}}, \cdots, z^{k_{in}}),$$

hence the spectral cover has several branches approaching to 0 or to ∞, and the growth rate is controlled by the power k_{ij}.

Definition 3.1.3. Let X be a smooth projective variety, we say that $\widetilde{X}$ is a spectral cover of X if there exist a matrix of meromorphic function ρ on X, such that

$$\widetilde{X} = \{(z, \lambda) \in X \times \mathbb{C}^\times |\ \det(\lambda I - \rho(z)) = 0\}$$

Given an n-sheet spectral cover $\widetilde{\Sigma}$ of $\Sigma - \{z_1, \cdots, z_N\}$, in a small punctured neighborhood of z_i, we have n disjoint sheets covering it, in this neighborhood, we can parameterize Σ by holomorphic coordinate z, assume the j-th sheet is given by $(z, g_j(z))$, where $g_j(z)$ is a holomorphic function.

Definition 3.1.4. We say that the cover has singularity type $\vec{k}_i = (k_{i1}, \cdots, k_{in})$ if the leading term of Laurent expansion for the $g_j(z)$ is $z^{k_{ij}}$.

Definition 3.1.5. Consider an n-sheet branched cover $\widetilde{\Sigma} \to \Sigma$ with singularity type $\vec{k}_i$ at z_i, given a vector $\vec{t} = (t_1, \cdots, t_N) \in T^N$, we define its $\vec{t}$-degree as

$$\delta_{\vec{t}} = c_1(\mathcal{L}) - \sum_{i=1}^{N} t_i \operatorname{Tr}(\vec{k}_i)$$

and define its $\vec{t}$-slope as

$$\mu_{\vec{t}} = \frac{\delta_{\vec{t}}}{n}$$

Definition 3.1.6. The spectral data $(\widetilde{\Sigma}, \mathcal{L})$ is $\vec{t}$-stable if for any subsheaf $\mathcal{F} \subset \imath_{\widetilde{\Sigma}*}\mathcal{L}$ of pure dimension one, the $\vec{t}$-slope of $\mathcal{F}$ is strictly smaller than that of $(\widetilde{\Sigma}, \mathcal{L})$.

Now we can consider the moduli of stable spectral data $(\widetilde{\Sigma}, \mathcal{L})$, such that $\widetilde{\Sigma}$ is an n-sheeted branched cover of Σ with singularity type $\vec{k}_i$ over z_i, and $\mathcal{L}$ is in the Prym variety of $p : \widetilde{\Sigma} \to \Sigma$. Denote this moduli as following:

$$\mathcal{M}_s^{sp}(\Sigma, z_1, \cdots, z_N, \vec{k}_1, \cdots, \vec{k}_N, \vec{t})$$

In the language of spectral data, we can restate the Charbonneau-Hurtubise theorem as

Theorem 3.1.7. *The composition map*

$$\Xi \circ \Theta : \mathcal{M}^{ir}(S^1 \times \Sigma, p_1, \cdots, p_N, \vec{k}_1, \cdots, \vec{k}_N) \to \mathcal{M}_s^{sp}(\Sigma, z_1, \cdots, z_N, \vec{k}_1, \cdots, \vec{k}_N, \vec{t})$$

$$(E, A, \phi) \mapsto (\widetilde{\Sigma}, \mathcal{L})$$

is a bijection.

Notice that if we replace Riemann surface Σ by a smooth projective variety X, and consider generalized monopoles on $S^1 \times X$ with Dirac singularities, we can still get bundle pair $(\mathcal{E}, \rho)$ via the scattering map $R_{0,1}$, and thus get its spectral data $(\widetilde{X}, \mathcal{L})$.

3.2 Fourier-Mukai transform

We observe that for monopole without singularities, the map $\Xi \circ \Theta$ can be constructed directly, via Fourier-Mukai transform.

Let $\mathscr{P}$ be the trivial line bundle on $S^1 \times \mathbb{C}^\times$, and $\nabla_{\mathscr{P}}$ be the connection on $\mathscr{P}$ given by the connection form $A_{\mathscr{P}} = z \, \mathrm{d}\, t$, where t and z parameterize S^1 and $\mathbb{C}^\times$ respectively. Recall that Fourier-Mukai transform identifies local system (E, A_E) on S^1 with skyscraper sheaf $\mathcal{L}$ in $\mathbb{C}^\times$, in the following way:

Denote by

$$p_1 : S^1 \times \mathbb{C}^\times \to S^1, \;\; p_2 : S^1 \times \mathbb{C}^\times \to \mathbb{C}^\times$$

the natural projections. Let $A^r = A_E + A_{\mathscr{P}}$, consider it as a connection form on

p_1^*E, it gives a partial connection ∇^r, locally

$$\nabla^r = \partial_t + A^r \,\mathrm{d}\, t : \ p_1^*E \to p_1^*E \otimes \Omega^1_{S^1}$$

then $R^0 p_{2*}(\ker \nabla^r) = 0$, $\mathcal{L} = R^1 p_{2*}(\ker \nabla^r)$ is a skyscraper sheaf on $\mathbb{C}^\times$, and $\dim H^0(\mathbb{C}^\times, \mathcal{L}) = \mathrm{rank}(E)$.

Let U be quasi-projective variety, consider the trivial line bundle $\mathscr{P}$ on $S^1 \times U \times \mathbb{C}^\times$ with connection $A_{\mathscr{P}}$ given by the form $z \,\mathrm{d}\, t$. We apply Fourier-Mukai transform fiberwise, this discussion together with the Charbonneau-Hertubise theorem now give the following

Proposition 3.2.1. *Fourier-Mukai transform with kernel* $(\mathscr{P}, A_{\mathscr{P}})$ *converts smooth monopole data* $(E, A - \sqrt{-1}\phi)$ *on* $S^1 \times U$ *to its spectral data* $(\widetilde{U}, \mathcal{L})$*.*

On the other hand, given spectral data $(\widetilde{U}, \mathcal{L})$, on each fiber, we consider

$$E = p_{1*}(p_2^*\mathcal{L})$$

which is a locally free sheaf of constant rank, thus a vector bundle on $S^1 \times U$. Furthermore, the construction equips E with a canonical flat partial connection ∇^r_E. Choose a unitary connection A on E, subtracting it from ∇^r_E gives a Higgs field $\phi = -\sqrt{-1}(\nabla^r_E - A_t)$, so that (E, A, ϕ) is triple that can be a candidate of monopole. Though this may not always be a monopole, we conjecture that when the spectral data satisfies certain stability condition, we can get a monopole in this way.

On non-trivial S^1 bundles, we can still apply Fourier-Mukai transform locally and get twisted spectral data by gluing, this enables us to extend the theorem of Charbonneau and Hurtubise. We will interpret twisted spectral data as a geometric object – a line bundle on an analytic gerbe.

3.3 Analytic gerbe

Let X be a complex manifold and let $\mathcal{O}$ denote the sheaf of holomorphic functions on X. We can consider a cohomology class in $H^2(X, \mathcal{O}^*)$ as a geometric object – an $\mathcal{O}^*$-gerbe, much in the same way as we interpret elements in $H^1(X, \mathcal{O}^*)$ as holomorphic line bundles. We first explain this in Čech cohomology:

Definition 3.3.1. Given an open cover $\{U_\alpha\}$ of X, an $\mathcal{O}^*$-gerbe g is an assignment to each threefold intersection $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ an invertible holomorphic function $g_{\alpha\beta\gamma}$, such that

$$g_{\alpha\beta\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\gamma\beta\alpha}$$

and on each fourfold intersection

$$\delta(g)_{\alpha\beta\gamma\delta} = g_{\beta\gamma\delta} \cdot g^{-1}_{\alpha\gamma\delta} \cdot g_{\alpha\beta\delta} \cdot g^{-1}_{\alpha\beta\gamma} = 1$$

Definition 3.3.2. A trivialization of gerbe g on U is defined by holomorphic functions

$$f_{\alpha\beta} = f^{-1}_{\beta\alpha} : U_{\alpha\beta} \cap U \to \mathbb{C}^*$$

on twofold intersections in U such that in $U_{\alpha\beta\gamma} \cap U$,

$$g_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha}$$

Definition 3.3.3. A gerbe is trivial if it has a global trivilization. Two gerbes are equivalent if their difference is trivial.

Given two trivializations $f_{\alpha\beta}$ and $f'_{\alpha\beta}$ on U, their difference is measured by

$$h_{\alpha\beta} = f_{\alpha\beta}/f'_{\alpha\beta}$$

we have $\delta(h) = 1$, so h is a Čech 1-cocycle, and the difference between two trivializations is a line bundle on U. Notice that over each open set U_α, we can set $f_{\beta\gamma} = g_{\alpha\beta\gamma}$ for $\beta, \gamma \neq \alpha$, and set $f_{\alpha\beta} = 1$, by cocycle condition on g, we have

$$g_{\beta\gamma\delta} = f_{\beta\gamma} f_{\gamma\delta} f_{\delta\beta}$$

therefore we get a trivialization over U_α. On the intersection $U_\alpha \cap U_\beta$, the two trivializations differ by a line bundle $L_{\alpha\beta}$. This leads us to an equivalent definition of gerbe:

Definition 3.3.4. An $\mathcal{O}^*$-gerbe is defined by the following data:

- A line bundle $L_{\alpha\beta}$ on each $U_{\alpha\beta}$
- An isomorphism $L_{\alpha\beta} \cong L_{\beta\alpha}^{-1}$
- A trivialization $\theta_{\alpha\beta\gamma}$ of $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ on $U_{\alpha\beta\gamma}$
- The trivialization $\theta_{\alpha\beta\gamma}$ satisfies $\delta(\theta) = 1$ on $U_{\alpha\beta\gamma\delta}$

Geometrically, we can consider such gerbes as analytic stacks: denote by $B\mathcal{O}^*$ the classifying stack of $\mathcal{O}^*$, i.e., it assigns to each open set V a category, whose objects are $\mathcal{O}^*$-torsors on V and morphisms are isomorphisms of torsors. Then an $\mathcal{O}^*$-gerbe on X is a $B\mathcal{O}^*$ torsor on X, i.e. a stack of groupoids over X, which admits a principal hemogeneous action of $B\mathcal{O}^*$. In this way, a gerbe $\mathcal{H}$ on X comes with a projection $\mathcal{H} \to X$, and is classified by an element in $H^1(X, B\mathcal{O}^*)$, This is consistent with the previous discussion since $B\mathcal{O}^* = \mathcal{O}^*[1]$ and so $H^1(X, B\mathcal{O}^*) = H^1(X, \mathcal{O}^*[1])$ is isomorphic to $H^2(X, \mathcal{O}^*)$.

3.4 Twisted spectral data

Suppose Y is a principal S^1 bundle over X, let (E, A, ϕ) be a singular monopole on Y, with singularities located in the fiber over divisors $Z_1, \cdots, Z_N \subset X$. Take an open cover $\{U_i\}$ of $X_0 := X \setminus \{Z_1, \cdots Z_N\}$, so that for each U_i the fiber bundle $Y_i := Y_{|U_i} \to U_i$ trivializes. Using this trivialization we construct Poincaré bundle with connection $(\mathscr{P}_i, A_i)$ on $Y_i \times \mathbb{C}^\times$ as in Proposition 3.2.1. The Fourier-Mukai transform with kernel $(\mathscr{P}_i, A_i)$ converts the monopole $(E, A, \phi)_{|Y_i}$ into spectral data on $U_i \times \mathbb{C}^\times$. The spaces $U_i \times \mathbb{C}^\times$ glue into $X \times \mathbb{C}^\times$, the spectral cover glue into $\widetilde{X} \subset X_0 \times \mathbb{C}^\times$, but the Poincaré bundle with connection $(\mathscr{P}_i, A_i)$ does not glue, they only glue into an $\mathcal{O}^*$ gerbe $\mathcal{H}$ with conneciton whose 3-curvature is represented the De Rham cohomology class

$$\frac{1}{4\pi i} p_X^*\big(c_1(Y)\big) \wedge p_{\mathbb{C}^\times}^*\Big(\frac{\mathrm{d}\, z}{z} - \frac{\mathrm{d}\, \bar{z}}{\bar{z}}\Big) \in H^3(X \times \mathbb{C}^\times, \mathbb{Z})$$

Tautologically the locally defined Poincare line bundles with connections glue into a global pair $(\mathscr{P}, A_{\mathscr{P}})$ on the gerbe $Y \times_X \mathcal{H}$, and we get the following:

Proposition 3.4.1. *Fourier-Mukai transform associated to $(\mathscr{P}, A_{\mathscr{P}})$ converts the monopole data $(E, A - \sqrt{-1}\phi)$ on Y to spectral data $(\widetilde{X}, \mathcal{L})$, where $\mathcal{L}$ is a weight one line bundle on the restriction of $\mathcal{H}$ to $\widetilde{X}$, or equivalently a trivialization of the gerbe $\mathcal{H}_{|\widetilde{X}}$.*

Definition 3.4.2. A twisted spectral data on X corresponding to gerbe $\mathcal{H}$, with singularities along $Z_1, \cdots Z_N$, is a pair $(\widetilde{X}, \mathcal{L})$ consisting of a spectral cover $\widetilde{X} \subset X \times \mathbb{C}^\times$ of $X \setminus \{Z_1, \cdots, Z_N\}$, and a weight one line bundle $\mathcal{L}$ on the restriction of the gerbe $\mathcal{H}$ to $\widetilde{X}$.

Theorem 3.4.3. *On a Riemann surface, twisted spectral data always exists.*

Proof. Let Σ be a Riemann surface, $z_1, \cdots, z_N \in \Sigma$, and $\mathcal{H}$ be a gerbe on $\Sigma \times \mathbb{C}^\times$. Choose any spectral cover $\widetilde{\Sigma}$ of $\Sigma \setminus \{z_1, \cdots, z_N\}$, by exponential exact sequence:

$$\cdots \to H^2(\widetilde{\Sigma}, \mathcal{O}) \to H^2(\widetilde{\Sigma}, \mathcal{O}^*) \to H^3(\widetilde{\Sigma}, \mathbb{Z}) \to \cdots$$

we have $H^2(\widetilde{\Sigma}, \mathcal{O}^*) = 0$ since the terms before and after it are 0. Therefore the restriction of gerbe $\mathcal{H}$ on $\widetilde{\Sigma}$ is always trivial. A line bundle on a trivial gerbe is the same as a trivialization of the gerbe. Therefore twisted spectral data exist. □

Let X be a smooth complex projective variety of complex dimension k, we consider the case when the circle bundle $Y \to X$ sits inside a line bundle $\mathcal{O}_X(D)$ as its unit circle bundle, where D is an effective divisor on X. Then the Fourier-Mukai transform gives us a gerbe $\mathcal{H}$ on $X \times \mathbb{C}^\times$. Let $Z_1, \cdots Z_N$ be irreducible divisors on X, we have

Theorem 3.4.4. *Suppose $\mathcal{O}(D)$ is ample, and set $\{Z_1, \cdots Z_N\}$ contains all of the irreducible components of D, then twisted spectral data corresponding to $\mathcal{H}$ with singularities along $Z_1, \cdots, Z_N$ exists.*

Proof. Let $X_0 = X \setminus \{Z_1, \cdots Z_N\}$, $M = X \times \mathbb{C}^\times$, and $M_0 = X_0 \times \mathbb{C}^\times$. Consider the map

$$\alpha: \ H^2(M, \mathcal{O}^*) \to H^2(M_0, \mathcal{O}^*)$$

we claim that $\alpha(\mathcal{H}) = 0$.

Notice that we can fit α into exponential exact sequence:

$$\begin{array}{ccc} H^2(M,\mathcal{O}) & \longrightarrow & H^2(M_0,\mathcal{O}) \\ \downarrow & & \downarrow \\ H^2(M,\mathcal{O}^*) & \xrightarrow{\alpha} & H^2(M_0,\mathcal{O}^*) \\ \downarrow \delta_1 & & \downarrow \delta_2 \\ H^3(M,\mathbb{Z}) & \xrightarrow{\beta} & H^3(M_0,\mathbb{Z}) \end{array}$$

Then the gerbe $\mathcal{H} \in H^2(M, \mathcal{O}^*)$ maps via δ_1 to $p_X^*(\omega) \wedge p_{\mathbb{C}^\times}^*(\tau)$, where $p_X, p_{\mathbb{C}^\times}$ are the natural projections from M to X and $\mathbb{C}^\times$ respectively, ω is the De Rham cohomology class in $H^2(X, \mathbb{Z})$ representing the Chern class of $\mathcal{O}(D)$ and τ is the De Rham cohomology class that generates $H^1(\mathbb{C}^\times, \mathbb{Z})$.

We know that $H^i(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z}$ when $i = 0, 1$, and $H^i(\mathbb{C}^\times, \mathbb{Z}) = 0$ when $i > 1$.

By Kunneth formula for $M = X \times \mathbb{C}^\times$,

$$H^3(M,\mathbb{Z}) \cong H^3(X,\mathbb{Z}) \otimes H^0(\mathbb{C}^\times,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \otimes H^1(\mathbb{C}^\times,\mathbb{Z})$$

therefore $\delta_1(\mathcal{H}) \in H^2(X,\mathbb{Z}) \otimes H^1(\mathbb{C}^\times,\mathbb{Z})$ and it maps to ω under the natural isomorphism $H^2(X,\mathbb{Z}) \otimes H^1(\mathbb{C}^\times,\mathbb{Z}) \cong H^2(X,\mathbb{Z})$.

Similarly, we have

$$H^3(M_0,\mathbb{Z}) \cong H^3(X_0,\mathbb{Z}) \otimes H^0(\mathbb{C}^\times,\mathbb{Z}) \oplus H^2(X_0,\mathbb{Z}) \otimes H^1(\mathbb{C}^\times,\mathbb{Z})$$

and $\beta \circ \delta_1(\mathcal{H}) \in H^2(X_0,\mathbb{Z}) \otimes H^1(\mathbb{C}^\times,\mathbb{Z}) \cong H^2(X_0,\mathbb{Z})$. The composition $\beta \circ \delta_1(\mathcal{H})$ is induced by the restriction map

$$\sigma : H^2(X,\mathbb{Z}) \to H^2(X_0,\mathbb{Z})$$

Let $W = Z_1 \cup \cdots \cup Z_N$, then $X_0 = X \setminus W$, the map σ fits into the Gysin sequence:

$$H^0(W,\mathbb{Z}) \to H^2(X,\mathbb{Z}) \to H^2(X_0,\mathbb{Z}) \to \cdots \tag{3.4.1}$$

its Poincaré dual gives

$$H_{2k-2}(W,\mathbb{Z}) \to H_{2k-2}(X,\mathbb{Z}) \to H_{2k-2}(X_0,\mathbb{Z}) \to \cdots$$

Since the Pioncaré dual of ω is represented by the divisor D, and D is spanned by irreducible components of W, so D is in the image of first map $H_{2k-2}(W,\mathbb{Z}) \to H_{2k-2}(X,\mathbb{Z})$, therefore the second map $H_{2k-2}(X,\mathbb{Z}) \to H_{2k-2}(X_0,\mathbb{Z})$ sends D to 0. Go back to the sequence 3.4.1, we see that $\sigma(\omega) = 0$, therefore $\beta \circ \delta_1(\mathcal{H}) = 0$.

Now consider $\alpha(\mathcal{H})$, since $\delta_2 \circ \alpha(\mathcal{H}) = \beta \circ \delta_1(\mathcal{H}) = 0$, we have $\alpha(\mathcal{H})$ comes from $H^2(M_0,\mathcal{O})$. Since $\mathcal{O}(D)$ is ample, and D is an non-negative integer combination of

Z_i's, so $\mathcal{O}(Z_1 + \cdots Z_N)$ is ample, therefore X_0 is a Stein manifold, hence $H^i(X_0, \mathcal{O}) = 0$ for $i > 0$, then by Kunneth formula, $H^2(M_0, \mathcal{O}) = 0$. So the pre-image of $\alpha(\mathcal{H})$ in $H^2(M_0, \mathcal{O})$ can only be 0, thus $\alpha(\mathcal{H}) = 0$.

So we see that the gerbe $\mathcal{H}$ restricts to a trivial gerbe on $\widetilde{X}$, hence twisted spectral data exist. □

Chapter 4

Conjecture

In the light of Kobayashi-Hitchin correspondence, we conjecture that monopole data corresponds to twisted spectral data with some stability conditions.

To be completed.