ELLIPTIC INVOLUTIVE STRUCTURES AND GENERALIZED HIGGS ALGEBROIDS

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ABSTRACT

ELLIPTIC INVOLUTIVE STRUCTURES AND GENERALIZED HIGGS ALGEBROIDS

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We study the module theory of two types of Lie algebroids: elliptic involutive structures (EIS) (which are equivalent to transversely holomorphic foliations) and what we call twisted generalized Higgs algebroids (TGHA). Generalizing the well-known results in the extremal cases of flat vector bundles and holomorphic vector bundles, we prove that there is an equivalence between modules over an EIS and locally free sheaves of modules over the sheaf of functions that are constant along the EIS. We define Atiyah like characteristic classes for such modules. Modules over a TGHA give a simultaneous generalization of Higgs bundles and generalized holomorphic vector bundles. For general Lie algebroids, we define a higher direct image construction of modules along a submersion. We also specialize to Higgs bundles, where we define Dolbeault cohomology valued secondary characteristic classes. We prove that these classes are compatible with the non-abelian Hodge theorem and the characteristic classes of flat vector bundles. We use these secondary classes to state and prove a refined Grothendieck-Riemann-Roch theorem for the pushforward of a Higgs bundle along a projection whose fiber is Kähler.

Contents

Chapter 1

Introduction

The language of Lie algebroids, their corresponding differential graded algebras (dga), and their modules (i.e. representations) serve as a natural and unifying framework for many geometric structures. The tangent bundle of a smooth manifold is a prototypical geometric example of a Lie algebroid. Its dga is the de Rham algebra and its modules are flat vector bundles. Similarly, the anti-holomorphic tangent bundle of a complex manifold is a Lie algebroid whose dga is the Dolbeault algebra and whose modules are holomorphic vector bundles. Foliations, Poisson manifolds, Higgs bundles, and generalized complex structures are also naturally described by Lie algebroids.

This work consists of three main parts. Part I is about Lie algebroids in general, with chapters 2 and 3 serving as a quick review of the basic notions that we will use throughout this work. In chapter 4 we introduce a higher direct image construction for Lie algebroid modules along a submersion. The last two parts deal with two specific types of Lie algebroids: elliptic involutive structures and what we call (twisted) generalized Higgs algebroids.

1.1 Higher direct image of Lie algebroid modules

In [BL95], Bismut and Lott give a geometric construction of the flat connection on the higher direct image of a flat vector bundle $E \to M$ along a submersion $M \to B$. This is done by considering the infinite rank bundle $E^\bullet_{M/B}$ over B formed by sections of the vertical de Rham complex twisted by E. The vertical exterior derivative turns this vector bundle into a complex and the underlying vector bundle of the higher direct image is the cohomology of this complex. From the point of view of B, the connection on E determines a flat superconnection on $E^\bullet_{M/B}$, whose degree 0 piece is the vertical exterior derivative. The degree 1 part induces the desired flat connection on the higher direct image.

In chapter 4 we generalize this construction to modules over general Lie algebroids. Specifically, given a smooth submersion $M \to B$ that lifts to a map of

Lie algebroids $A_M \to A_B$ we define a higher direct image of an A_M-module, which is a $\mathbb{Z}$-graded A_B-module, by taking cohomology along the vertical Lie algebroid $\ker A_M \to A_B$. We prove that this construction is natural with respect to the action of A_B-modules via tensor product. We also prove a twisted Leray-Hirsch theorem, which we will make repeated use of in later sections.

1.2 Elliptic involutive structures

An elliptic involutive structure (EIS) [Trè92, BCH08] on a smooth manifold M is an involutive complex distribution $V \subset T_{\mathbb{C}}M$ satisfying $V + \overline{V} = T_{\mathbb{C}}M$. A module over V (viewed as a Lie algebroid) is then a vector bundle with a flat partial connection along V. At one extreme, we have $V = \overline{V} = T_{\mathbb{C}}M$, which has flat vector bundles as its modules. At the other extreme we have $V \cap \overline{V} = 0$, which, along with the involutivity requirement, shows that V can be taken to be the bundle of anti-holomorphic vectors for some integrable complex structure on M. Modules in this case are holomorphic vector bundles. Between these two extremes we have a foliation $V \cap \overline{V} \cap TM$ with a transversely holomorphic structure [GM80]. Modules are then vector bundles that are flat along the leaves of the foliation and holomorphic in the transverse directions.

Elliptic involutive structures and their modules appear in many places throughout mathematics and theoretical physics, such as the algebraic K-theory of complex varieties [BMS87], generalized complex geometry [Gua11], the theory of coisotropic A-branes [KO03], and supersymmetric field theories [CDFK14]. We will see that a choice of a Borel subgroup of a semi-simple Lie group G gives an elliptic involutive structure on G as well as some of its homogeneous spaces.

We focus primarily on the module theory of EISs. Recall the well-know result that, over a manifold, there is an equivalence between flat vector bundles and locally constant sheaves of vector spaces. Similarly, over a complex manifold X there is an equivalence between vector bundles with a $\bar{\partial}$ operator and locally free sheaves of $\mathcal{O}_X$-modules. We prove that these results generalize to modules over an EIS: if $\mathcal{O}_V$ denotes the sheaf of smooth functions that are constant along V, then we have

Theorem 7. *Let $E \to M$ be a rank r vector bundle. The following data are equivalent:*

1. *A flat V-connection on E.*
2. *A locally free sheaf $\mathcal{O}(E)$ of $\mathcal{O}_V$-modules such that $E = C^\infty_M \otimes_{\mathcal{O}_V} \mathcal{O}(E)$.*
3. *A trivializing cover $\{U_j\}$ for E such that the transition functions $U_i \cap U_j \to GL(r, \mathbb{C})$ take values in $\mathcal{O}_V(U_i \cap U_j)$.*

In section 7.3 we define two types of characteristic classes for V-modules. One generalizes the Dolbeault cohomology valued Atiyah classes [Ati57] of holomorphic vector bundles. We define these using the general notion of Atiyah class for Lie algebroid pairs defined by Chen, Stiénon, and Xu [CSX12]. The second type of class generalizes the $H^{odd}(M;\mathbb{R})$ valued characteristic classes of flat vector bundles [BL95, KT75].

Chapter 8 is devoted to explicit examples. The $2n+1$ sphere S^{2n+1} can be viewed as a unitary frame bundle for the tautological line bundle over $\mathbb{C}P^n$. In this way it inherits an EIS, V. We will see that the space of all rank 1 V-modules is $\mathbb{C}$ and that the first Atiyah class of section 7.3 is a complete invariant. We then move on to the example of a compact semisimple Lie group G, which has an EIS, V, coming from the fibration over its flag variety. Here we show that the space of rank 1 V-modules is isomorphic to $\mathfrak{t}^*$, the dual of the Lie algebra of a maximal torus. We then give a description of certain subgroups H such that G/H inherits an EIS. For such subgroups we use this structure to define an induction map from representations of H to representations of G. We discuss the examples of $SU(n) \subset SU(n+1)$ and $SU(n) \subset Spin(2n)$. The last example we discuss is the EIS on the projectivized bundle of a V-module.

1.3 Generalized Higgs algebroids

The second type of Lie algebroid, which we call a (twisted) generalized Higgs algebroid ((T)GHA) has motivations coming from generalized complex geometry [Gua11] and the theory of Higgs bundles. A generalized complex structure is a simultaneous generalization of a complex structure and symplectic structure on a manifold and is determined by a certain Lie algebroid. Higgs bundles, defined as a holomorphic vector bundle $E \to X$ together with a holomorphic form $\theta \in \Omega^{0,1}(X; \operatorname{End} E)$ with $\theta \wedge \theta = 0$ can, alternatively, be defined as modules over a Lie algebroid, called the Higgs algebroid [Blo05], which is determined by the complex structure on X. Both of these Lie algebroids are special cases of a TGHA, which we define to be an elliptic complex Lie algebroid such that the kernel of the anchor map is abelian. We show that the constructions of Hitchin on co-Higgs bundles [Hit10] carryover to twisted generalized Higgs algebroids. In particular, we utilize the spectral variety construction and the notion of a transversely holomorphic gerbe (which is represented by a class in $H^2(\mathcal{O}_V^\times)$, where V is an EIS). Using these we can, following Hitchin, state a vanishing result on the cohomology of a module over a TGHA.

1.3.1 Higgs bundles

Finally, in chapter 11 we specialize to the case of Higgs bundles. This section is fairly self-contained and we think it will be of interest to those interested in Higgs bundles proper. Here we define secondary characteristic classes $a_{2j+1}(E,\theta) \in H^{j+1,j}(X)$ of a Higgs bundle (E,θ) over a complex manifold X. In the case of compact Kähler manifolds, we show that these are compatible with the non-abelian Hodge theorem of Simpson [Sim92]. Along the way we give a quick proof, using the non-abelian Hodge theorem, of Reznikov's theorem/Bloch's conjecture that the $H^{odd}(X;\mathbb{R})$-valued characteristic classes of flat vector bundles vanish when X is compact Kähler. We then examine what the pushforward construction of chapter 4 looks like in this case (the direct image of Higgs bundles has previously been described by Simpson [Sim93]) when the submersion is a Kähler fibration (a notion due to [BGS88]). We then prove a secondary Grothendieck-Riemann-Roch theorem for the secondary classes in the case of a projection $B \times Y \to B$ with Y Kähler:

Theorem 14. *Suppose B is a complex manifold, Y is Kähler and (E, θ) is a Higgs bundle over $B \times Y$. Then*

$$a_k(\mathrm{ind}(\bar{\partial}_{Y;E} + \theta_Y)) = \int_Y e(TY) a_k(E, \theta), \quad k \geq 0.$$

Part I

Lie algebroids

Chapter 2

Lie algebroid basics

We first review some of the basic theory of Lie algebroids. A good reference is [Mac05], although there only real Lie algebroids are discussed. Throughout, M is a smooth manifold, $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ the complexified tangent bundle, and $C^\infty(M)$ is the space of smooth complex valued functions on M.

Definition 1. *A (complex) Lie algebroid over M is a complex vector bundle $A \to M$ with a Lie bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(M; A)$ and a bundle map $\rho : A \to T_{\mathbb{C}}M$, called the anchor, such that*

1. *The map that ρ induces on sections is a Lie algebra homomorphism (with respect to the Lie bracket of vector fields).*
2. *For all $f \in C^\infty(M)$ and $v, w \in \Gamma(M; A)$, we have*

$$[v, fw] = (\rho(v) \cdot f)w + f[v, w],$$

where $\rho(v) \cdot f$ denotes differentiation of f along the vector field $\rho(v)$.

Associated to any Lie algebroid is the dga $(\Omega^\bullet_A(M), d_A)$, where

$$\Omega^k_A(M) = \Gamma(M; \Lambda^k A^*),$$

$$d\alpha(v_0, v_1, \ldots, v_k) = \sum_{j=0}^{k} (-1)^j \rho(v_j) \cdot \alpha(v_0, \ldots, \hat{v}_j, \ldots, v_k) + \sum_{j<l} (-1)^{j+l} \alpha([v_j, v_l], v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_l, \ldots, v_k).$$

Definition 2. *We call a Lie algebroid A elliptic if the corresponding dga is an elliptic complex. This is equivalent to having $\rho(A) + \overline{\rho(A)} = T_{\mathbb{C}}M$.*

Remark. *For real Lie algebroids, being elliptic is equivalent to being transitive (i.e. having surjective anchor map).*

Using the dga, we can give a clean definition of a morphism between Lie algebroids over different base spaces:

Definition 3. *Let $\rho : A \to TM, \rho' : A' \to TM'$ be two Lie algebroids. Then a morphism between them is a vector bundle morphism*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & A' \arrow[d] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

that induces a chain map

$$(\Omega^\bullet_{A'}(M'), d_{A'}) \to (\Omega^\bullet_A(M), d_A).$$

Examples of Lie algebroids:

1. The trivial Lie algebroid $A = T_{\mathbb{C}}M$. This is elliptic with the de Rham algebra as the corresponding dga.

2. If X is a complex manifold then $A = T^{0,1}X$ is an elliptic Lie algebroid with dga the Dolbeault algebra $\Omega^{0,\bullet}(X)$.

3. Let $P \to X$ be a principal G bundle. Quotienting the sequence $0 \to P \times \mathfrak{g} \to P \to TM$ by the G action gives the Atiyah sequence

$$0 \to P \times_{Ad} \mathfrak{g} \to TP/G \to TM \to 0.$$

   Then TP/G is an elliptic (indeed, transitive) Lie algebroid, called the Atiyah algebroid. In contrast to the previous examples, $\ker \rho$ is non-trivial.

Chapter 3

Lie algebroid modules

Definition 4. *A module over a Lie algebroid A is a vector bundle E together with flat A-connection $\nabla^{A;E}$. That is, $\nabla^{A;E}$ is an operator*

$$\Gamma(M;E) \to \Gamma(M;A^* \otimes E)$$

such that

$$\nabla^{A;E}(f\psi) = d_A f \otimes \psi + f\nabla^{A;E}\psi$$

and

$$[\nabla^{A;E}_v, \nabla^{A;E}_w] = \nabla^{A;E}_{[v,w]}.$$

This determines a complex $(\Omega^\bullet_A(M;E), d_{A;E})$ by

$$d_{A;E}(\mu \otimes \psi) = d_A\mu \otimes \psi + (-1)^{\deg \mu}\mu \wedge \nabla^A\psi, \quad \mu \in \Omega^\bullet_A(M), \psi \in \Gamma(M;E),$$

where $\Omega^\bullet_A(M;E) = \Gamma(M;\Lambda^\bullet A^* \otimes E)$.

We will often omit some of the superscripts on ∇ when it is clear from context which Lie algebroid or module we are considering.

It is a straightforward computation to verify that $d_{A;E}$ is given by an analogous equation as the definition of d_A:

Proposition 1. *For $\alpha \in \Omega^k_A(M;E)$, we have*

$$\begin{aligned} d_{A;E}\alpha(v_0, v_1, \dots, v_k) = \sum_{j=0}^{k}(-1)^j \nabla_{v_j}\alpha(v_0, \dots, \hat{v}_j, \dots, v_k) \\ + \sum_{j<l}(-1)^{j+l}\alpha([v_j, v_l], v_0, \dots, \hat{v}_j, \dots, \hat{v}_l, \dots, v_k), \end{aligned}$$

where $v_0, \dots, v_k \in \Gamma(M;A)$.

3.1 Cartan calculus

We also can define a Lie derivative operator on twisted forms:

Definition 5. *For $v \in \Gamma(M; A)$, we define the Lie derivative $\mathcal{L}_v$, which is a degree 0 operator on $\Omega^\bullet_V(M; E)$, by*

$$(\mathcal{L}_v\alpha)(w_1, \dots, w_k) = \nabla_v(\alpha(w_1, \dots, w_k)) + \sum_{j=1}^{k} (-1)^j \alpha([v, w_j], w_1, \cdots, \hat{w}_j, \cdots w_k),$$

where

$$\alpha \in \Omega^k_A(M; E), \quad w_1, \dots, w_k \in \Gamma(M; A).$$

A straightforward computation then shows that the familiar Cartan homotopy formula holds in this setting.

Proposition 2 (Cartan homotopy formula)**.** *We have*

$$\mathcal{L}_v = d_{A;E} \circ i_v + i_v \circ d_{A;E}.$$

3.2 Operations with modules

The dual, direct sum, and tensor product of modules are defined in the natural way:

Definition 6. *If E is an A-module then the dual vector bundle is naturally an A-module with connection*

$$(\nabla^{A;E^*}_v F)(\psi) = \rho(v) \cdot F(\psi) - F(\nabla^{A;E}_v \psi),$$

where $F \in \Gamma(M; E^*), \psi \in \Gamma(M; E), v \in A$.

Definition 7. *If E_1, E_2 are two modules over a Lie algebroid A, then $E_1 \oplus E_2$ and $E_1 \otimes E_2$ are modules over A with connections*

$$\nabla^{A;E_1 \oplus E_2} = \begin{pmatrix} \nabla^{A;E_1} & 0 \\ 0 & \nabla^{A;E_2} \end{pmatrix}$$

and

$$\nabla^{A;E_1 \otimes E_2}_v (\psi_1 \otimes \psi_2) = \nabla^{A;E_1}_v \psi_1 \otimes \psi_2 + \psi_1 \otimes \nabla^{A;E_2}_v \psi_2, \quad \psi_j \in \Gamma(M; E_j), v \in A.$$

Definition 8. *Suppose we have a map of Lie algebroids*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & A' \arrow[d] \\ M \arrow[r, "f"] & M' \end{tikzcd} .$$

If E is an A' module, then we denote by $\varphi^* E$ the A-module whose underlying vector bundle is $f^* E$ and whose A-connection is determined by

$$\nabla^{A;\varphi^* E}_v f^* \psi = f^* \nabla^{A';E}_v \psi, \quad v \in A, \psi \in \Gamma(M'; E).$$

3.3 Principal Lie algebroid connections

It will also be useful to have a notion of an A-connection on a principal G-bundle $P \xrightarrow{\pi} M$. This will then induce A-connections on any associated vector bundles that are compatible with the G-structure.

Definition 9 ([Fer02]). *An A-connection on P is given by a bundle map $h : \pi^*A \to TP$ such that the diagram*

$$\begin{tikzcd} \pi^*A \arrow[r, "h"] \arrow[d] & T_{\mathbb{C}}P \arrow[d] \\ A \arrow[r] & T_{\mathbb{C}}M \end{tikzcd}$$

commutes and h is G-invariant: $h(pg, v) = h(p, v)g$, where the right action of G on TP is given by the differential of the right action of G on P.

The connection is called flat if $h(\pi^*A) \subset T_{\mathbb{C}}P$ is closed under Lie brackets.

It is straightforward to verify the following

Proposition 3. *Let E_0 be a representation of G and $E = P \times_G E_0$ the associated vector bundle over M. A section $\sigma \in \Gamma(M; E)$ is equivalent to a G-equivariant map $f : P \to E_0$. Then an A-connection h on P determines an A-connection ∇ on E by*

$$\nabla_v \sigma = df(h(\pi^* v)),$$

where $v \in \Gamma(M; A)$.

3.4 Duality

We now introduce the dualizing module associated to a Lie algebroid, which will give us a pairing between certain Lie algebroid cohomology groups. This notion is originally due to [ELW99] and was further discussed in [Blo05].

For a rank r complex Lie algebroid $A \to M^n$, define

$$Q_A = \Lambda^r A \otimes \Lambda^n T^*_{\mathbb{C}} M.$$

For $v \in V$, we extend the Lie derivative $\mathcal{L}_v$ to $\Lambda^r A$ by the Leibniz rule:

$$\mathcal{L}_v(v_1 \wedge \cdots \wedge v_r) := [v, v_1] \wedge v_2 \wedge \cdots \wedge v_r + v_1 \wedge [v, v_2] \wedge v_3 \cdots \wedge v_r + \cdots .$$

Then we have

Theorem 1 ([ELW99]). *Q_A is naturally an A-module with connection*

$$\nabla_v^{A;Q_A}(W \otimes \mu) = \mathcal{L}_v W \otimes \mu + W \otimes \mathcal{L}_{\rho(v)}\mu, \quad W \in \Lambda^r A, \alpha \in \Omega^n(M),$$

where $\mathcal{L}_{\rho(v)}\mu$ denotes the usual Lie derivative of differential forms.

Theorem 2 ([ELW99, Blo05]). *Let E be an A-module. The natural map*

$$\Omega^k_A(M;E) \otimes \Omega^{r-k}_A(M;E^* \otimes Q_A) \to \Omega^n(M) \xrightarrow{\int} \mathbb{C}$$

given by the pairing of $E \otimes \Lambda^r A^* \otimes$ with $E^* \otimes \Lambda^r A$, descends to a map

$$H^k_A(M;E) \otimes H^{r-k}_A(M;E^* \otimes Q_A) \to \mathbb{C}.$$

If A is further assumed to be elliptic then this is a perfect pairing.

Chapter 4

Higher direct images and the Leray spectral sequence

We now describe a higher direct image construction that will allow us to push-forward Lie algebroid modules along compatible submersions compatible. This construction is generalized from the special case of the direct image of flat vector bundles used in [BL95]. The special case of our construction for Higgs bundles appears in [Sim93].

Suppose $M \xrightarrow{\pi} B$ is a fiber bundle and over M and B are Lie algebroids A_M, A_B compatible with the fibration, i.e. we have the following diagram of vector bundles over M,

$$\begin{tikzcd} 0 \arrow[r] & A_{M/B} \arrow[r] \arrow[d] & A_M \arrow[r, "\pi_A"] \arrow[d] & \pi^* A_B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T(M/B) \arrow[r] & TM \arrow[r, "\pi_*"] & \pi^* TB \arrow[r] & 0, \end{tikzcd} \tag{4.0.1}$$

where $T(M/B)$ is by definition the vertical vectors of the fibration and $A_{M/B} = \ker \pi_A$, which forms a Lie algebroid over M. We also assume that $A_{M/B}$ is an elliptic Lie algebroid when restricted to any fiber. Let (E, ∇^{A_M}) be an A_M-module, which becomes an $A_{M/B}$-module by restriction. We will construct a $\mathbb{Z}$-graded A_B-module, denoted $H^\bullet_{A_{M/B}}(M/B; E)$, by taking the vertical cohomology. A nice way to view this construction is via the superconnection formalism. Thus we form the infinite rank complex of vector bundles over B, denoted by $E^\bullet_{M/B}$, whose fiber over $x \in B$ is

$$(E^\bullet_{M/B})_x = \Gamma(M_x; \Lambda^\bullet A^*_{M/B} \otimes E|_{M_x}),$$

where $M_x = \pi^{-1}(x)$. The differential is given by $d_{A_{M/B};E} = d_{A_M;E}|_{A_{M/B}}$. Note that this is an endomorphism of the bundle $E^\bullet_{M/B}$ since the differentiation is happening in the vertical directions.

Choose a (vector bundle) splitting $H : \pi^* A_B \to A_M$ and for $v \in \Gamma(B; A_B)$ we write $v^H \in \Gamma(M; A_M)$ for its lift. Similarly, write $\alpha^H \in \Omega^\bullet_{A_M}(M; E)$ for the element corresponding to $\alpha \in \Omega^\bullet_{A_{M/B}}(M;E)$. Then H gives an isomorphism

$$\pi^*\Lambda^\bullet A_B^* \otimes \Lambda^\bullet A_{M/B}^* \simeq \Lambda^\bullet A_M^*,$$

and so we get the identification

$$\Omega^\bullet_{A_B}(B;E^\bullet_{M/B}) \simeq \Omega^\bullet_{A_M}(M;E).$$

Under this isomorphism, let $\mathbb{A}$ denote the operator on $\Omega^\bullet_{A_B}(B;E^\bullet_{M/B})$ that corresponds to the connection $d_{A_M;E}$. Then $\mathbb{A}$ is a flat superconnection of total degree 1 (also called a $\mathbb{Z}$-connection) and, in the language of [Blo05], gives the bundle $E^\bullet_{M/B} \to B$ the structure of a quasi-cohesive A_B-module. Decompose $\mathbb{A}$ as

$$\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \cdots$$

where

$$\mathbb{A}_{[j]} : \Gamma(B;E^\bullet_{M/B}) \to \Omega^j_{A_B}(B;E^{\bullet-j+1}_{M/B}).$$

Note that $\mathbb{A}_{[0]} = d_{A_{M/B};E}$ and $\mathbb{A}_{[1]}$ is an A_B connection on each $E^j_{M/B}$ in the usual sense but may not be flat. Indeed, from $\mathbb{A}^2 = 0$ we see that

$$[d_{A_{M/B};E}, \mathbb{A}_{[1]}] = 0,$$
$$\mathbb{A}_{[1]}^2 + [d_{A_{M/B};E}, \mathbb{A}_{[2]}] = 0,$$

where all of the brackets are supercommutators. The first equation says that $\mathbb{A}_{[1]}$ descends to give a connection on the cohomology

$$H^\bullet_{A_{M/B}}(M/B;E) := H^\bullet(E^0_{M/B} \overset{d_{A_{M/B};E}}{\longrightarrow} E^1_{M/B} \overset{d_{A_{M/B};E}}{\longrightarrow} \cdots)$$

and the second equation says that this connection is flat. By the ellipticity condition on $A_{M/B}$, each $H^j_{A_{M/B}}(M/B;E)$ is finite dimensional and so is an A_B-module. Actually, $H^j_{A_{M/B}}(M/B;E)$ will only form a vector bundle if the spaces $H^j_{A_{M/B}}(M_b;E|_{M_b})$ have constant rank over $b \in B$. This happens automatically if $A_B = T_{\mathbb{C}}M$ since we can use parallel transport to show that the vertical complexes over any two points in B are isomorphic.

While the cohomology groups $H^\bullet_{A_{M/B}}(M/B;E)$ are independent of the choice of splitting $\pi^*A_B \to A_M$, the superconnection $\mathbb{A}$ is not. However, we have

Theorem 3. *The induced A_B-connection $\mathbb{A}_{[1]}$ on $H^\bullet_{A_{M/B}}(M/B;E)$ is independent of the choice of splitting H.*

To prove this, we first need the following two lemmas.

Lemma 1. *If $v^H \in \Gamma(M;A_M)$ is a lift of $v \in \Gamma(B;A_B)$ then $[v^H, w] \in A_{M/B}$ for $w \in \Gamma(M;A_{M/B})$.*

Proof. For all $\alpha \in \Gamma(B; A_B^*)$ we have

$$\alpha_{\pi(x)}(\pi_A([v^H, w]_x)) = (\pi_A^* \alpha)_x([v^H, w]).$$

But

$$\begin{aligned} 0 = (\pi_A^* d\alpha)(v^H, w) = d\pi_A^* \alpha(v^H, w) \\ = \rho_M(v^H) \cdot \pi_A^* \alpha(w) - \rho_M(w) \pi^* \alpha(v) - \pi_A^* \alpha[v^H, w] = -\pi_A^* \alpha[v^H, w] \end{aligned}$$

so that $[v^H, w] \in Ann(\pi^* \Omega_A(B)) = A_{M/B}$. □

The next lemma shows a compatibility between the Lie derivatives on the Lie algebroids $A_{M/B}$ and A_M.

Lemma 2. *Let $\mu \in \Gamma(M; \Lambda^k A_{M/B}^*)$ and let $\tilde{H} : \pi^* A_B \to A_M$ be another lift. Then*

$$\mathcal{L}_{v^H - v^{\tilde{H}}}^{A_{M/B}} \mu = (\mathcal{L}_{v^H}^{A_M} \mu^H - \mathcal{L}_{v^{\tilde{H}}}^{A_M} \mu^{\tilde{H}})|_{A_{M/B}}.$$

Proof. For $w_1, \ldots, w_k \in A_{M/B}$, we have

$$\begin{aligned} \mathcal{L}_{v^H - v^{\tilde{H}}} \mu(w_1, \ldots, w_k) =& \nabla_{v^H - v^{\tilde{H}}}^{A_M} (\mu(w_1, \ldots, w_k)) \\ &+ \sum_{j=1}^{k} (-1)^j \mu([v^H - v^{\tilde{H}}, w_j], w_1, \ldots, \hat{w}_j, \ldots, w_k) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{v^H}^{A_M} \mu^H(w_1, \ldots, w_k) &= \nabla_{v^H}^{A_M} (\mu^H(w_1, \ldots, w_k)) \\ &\quad + \sum_{j=1}^{k} (-1)^j \mu^H([v^H, w_j], w_1, \ldots, \hat{w}_j, \ldots w_k) \\ &= \nabla_{v^H}^{A_M} (\mu(w_1, \ldots, w_k)) \\ &\quad + \sum_{j=1}^{k} (-1)^j \mu([v^H, w_j], w_1, \ldots, \hat{w}_j, \ldots w_k), \end{aligned}$$

since by the previous lemma $[v^H, w_j] \in A_{M/B}$. The same equation holds for $\tilde{H}$ and the lemma now easily follows. □

Proof of Theorem 3. A choice of splitting H gives the following commutative diagram, where we are conflating a vector bundle with its space of sections and where we write $\mathbb{A}_{[1]}^H$ for what we called $\mathbb{A}_{[1]}$ before to stress the dependence on H,

$$\begin{tikzcd} E_{M/B}^0 \arrow[r, "\nabla^{A_{M/B}}"] \arrow[d, "\mathbb{A}_{[1]}^H"] & E_{M/B}^1 \arrow[r, "d_{A_{M/B};E}"] \arrow[d, "\mathbb{A}_{[1]}^H"] & E_{M/B}^2 \arrow[r] \arrow[d, "\mathbb{A}_{[1]}^H"] & \cdots \\ E_{M/B}^0 \otimes A_B^* \arrow[r, "\nabla^{A_{M/B}}"] & E_{M/B}^1 \otimes A_B^* \arrow[r, "d_{A_{M/B};E}"] & E_{M/B}^2 \otimes A_B^* \arrow[r] & \cdots \end{tikzcd}$$

To show that the maps $\mathbb{A}^H_{[1]}$ and $\mathbb{A}^{\tilde{H}}_{[1]}$ induce the same map on cohomology, we will show that we have a homotopy operator

$$\mathscr{H} : E^\bullet_{M/B} \to E^{\bullet-1}_{M/B} \otimes A^*_B,$$

which is defined by

$$i_v \mathscr{H}(\mu) = i_{v^H - \tilde{v}^H}\mu, \quad v \in A_B, \mu \in E^\bullet_{M/B}.$$

Explicitly, $\mathscr{H}(\mu) = i_{a_j^H - a_j^{\tilde{H}}}\mu \otimes a^j$ where $\{a_j\}$ is a local frame for A_B with dual frame $\{a^j\}$. We have

$$\begin{aligned} d_{A_{M/B};E}\mathscr{H}\mu + \mathscr{H} d_{A_{M/B};E}\mu &= (d_{A_{M/B};E} i_{a_j^H - a_j^{\tilde{H}}}\mu + i_{a_j^H - a_j^{\tilde{H}}} d_{A_{M/B};E}\mu) \otimes a^j \\ &= \mathcal{L}^{A_{M/B}}_{a_j^H - a_j^{\tilde{H}}}\mu \otimes a^j, \end{aligned}$$

by Proposition 2. Now using the above lemma, we have

$$\begin{aligned} \mathcal{L}^{A_{M/B}}_{a_j^H - a_j^{\tilde{H}}}\mu &= \left(\mathcal{L}^{A_M}_{a_j^H}\mu^H - \mathcal{L}^{A_M}_{a_j^{\tilde{H}}}\mu^{\tilde{H}}\right)|_{A_{M/B}} \\ &= \Big(d_{A_M;E} \circ i_{a_j^H}\mu^H + i_{a_j^H} \circ d_{A_M;E}\mu^H \\ &\qquad - d_{A_M;E} \circ i_{a_j^{\tilde{H}}}\mu^{\tilde{H}} - i_{a_j^{\tilde{H}}} \circ d_{A_M;E}\mu^{\tilde{H}}\Big)|_{A_{M/B}} \\ &= \left(i_{a_j^H} \circ d_{A_M;E}\mu^H - i_{a_j^{\tilde{H}}} \circ d_{A_M;E}\mu^{\tilde{H}}\right)|_{A_{M/B}} \\ &= i_{a_j^H} \circ \mathbb{A}^H_{[1]}\mu^H - i_{a_j^{\tilde{H}}} \circ \mathbb{A}^{\tilde{H}}_{[1]}\mu^{\tilde{H}}. \end{aligned}$$

Thus

$$\begin{aligned} d_{A_{M/B};E}\mathscr{H}\mu + \mathscr{H} d_{A_{M/B};E}\mu &= (i_{a_j^H} \circ \mathbb{A}^H_{[1]}\mu^H - i_{a_j^{\tilde{H}}} \circ \mathbb{A}^{\tilde{H}}_{[1]}\mu^{\tilde{H}}) \otimes a^j \\ &= \mathbb{A}^H_{[1]}\mu - \mathbb{A}^{\tilde{H}}_{[1]}\mu, \end{aligned}$$

as desired. □

We note that this construction has some content even in the case of the trivial fibration $\{pt\} \to M \to M$. We have an exact sequence of Lie algebroids given by

$$\begin{tikzcd} 0 \arrow[r] & \ker\rho \arrow[r] \arrow[d] & A \arrow[r, "\pi_A"] \arrow[d] & A/\ker\rho \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & TM \arrow[r, "\pi"] & TM \arrow[r] & 0. \end{tikzcd}$$

Then $\ker\rho$ is a bundle of Lie algebras and any A-module E, upon restriction, becomes a bundle of representations of $\ker\rho$. The vertical complex is the fiberwise Chevalley-Eilenberg complex with coefficients in the module $E|_{\ker\rho}$. Taking cohomology gives a $\mathbb{Z}$-graded $A/\ker\rho$ module.

4.1 Projection formula

If we are in the situation of diagram (4.0.1), then any A_B-module E gives an A_M-module π^*E. Via tensor product, the space of A_M-modules is a module over the space of A_B-modules. The next proposition shows that our direct image construction respects this structure.

Proposition 4 (Projection formula)**.** *For any A_B-module E and A_M-module F, we have an isomorphism of A_B-modules*

$$H^\bullet_{A_{M/B}}(M/B;\pi^*E\otimes F)\simeq E\otimes H^\bullet_{A_{M/B}}(M/B;F).$$

Proof. Since $A_{M/B}=\ker\pi_A$, we have $\nabla^{A_{M/B}}\pi^*\psi=0$ for all $\psi\in\Gamma(B;E)$. From this it follows that

$$\begin{gathered}\Gamma(B;E\otimes H^\bullet_{A_{M/B}}(M/B;F))\to\Gamma(H^\bullet_{A_{M/B}}(M/B;\pi^*E\otimes F))\\ \psi\otimes[\alpha]\mapsto[\pi^*\psi\otimes\alpha]\end{gathered}$$

is the desired isomorphism as vector bundles. That this respects the A_B-module structure follows from the fact that $\nabla^{A_M}\pi^*\psi=\pi^*\nabla^{A_B}\psi$ so that $\mathbb{A}_{[1]}\pi^*\psi=\pi^*\nabla^{A_B}\psi$. $\square$

4.2 Leray spectral sequence

We still assume we are in the situation of diagram (4.0.1). Since π_A is a surjective map of Lie algebroids, we have a dga embedding $\pi_A^*\Omega^\bullet_{A_B}(B)\subset\Omega^\bullet_{A_M}(M)$. Using this we can define the Leray-Cartan filtration on $\Omega^\bullet_{A_M}(M)$:

$$F^p\Omega^\bullet_{A_M}(M)=\Omega^p_{A_B}(B)\cdot\Omega^\bullet_{A_M}(M).$$

In other words,

$$\begin{gathered}\alpha\in F^p\Omega^k_{A_M}(M)\\ \Leftrightarrow\\ i_{v_1}i_{v_2}\cdots i_{v_{k-p+1}}\alpha=0,\text{for all } v_1,\dots,v_{k-p+1}\in A_{M/B}.\end{gathered}$$

Then associated to this filtration is a first quadrant spectral sequence $E_2^{p,q}$ converging to $H^\bullet_A(M)$. The first two pages are given by

$$\begin{gathered}E_1^{p,q}\simeq\Omega^p_{A_B}(B;H^q_{A_{M/B}}(M/B))\\ E_2^{p,q}\simeq H^p_{A_B}(B;H^q_{A_{M/B}}(M/B)).\end{gathered}$$

Now suppose we have classes $\alpha_1,\dots,\alpha_d\in H^\bullet_{A_M}(M)$ that restrict to a basis of $H^\bullet_{A_{M/B}}(M_x)$ for every $x\in B$, where $M_x=\pi^{-1}(x)$. In particular, this means that the A_B-module $H^\bullet_{A_{M/B}}(M/B)$ is trivial and $E_2^{\bullet,\bullet} \simeq H^\bullet_{A_B}(B) \otimes \mathrm{span}\{\alpha_1, \ldots, \alpha_d\}$. Then the spectral sequence degenerates at the E_2 page since everything in E_2 is already represented by a global cohomology class so that $H^\bullet_{A_M}(M) \simeq H^\bullet_{A_B}(B) \otimes H^\bullet_{A_{M/B}}(M/B)$ (here we are conflating the trivial vector bundle $H^\bullet_{A_{M/B}}(M/B)$ with the vector space underlying it). More generally, the same techniques give

Theorem 4 (Twisted Leray-Hirsch)**.** *Let $E \to B$ be an A_B-module and F an A_M-module. Suppose there exist $\alpha_1, \ldots, \alpha_d \in H^\bullet_{A_M}(M; F)$ that restrict to a basis of $H^\bullet_{A_{M/B}}(M_x; F|_{M_x})$ for every $x \in B$, then*

$$H^\bullet_{A_M}(M; \pi^* E \otimes F) \simeq H^\bullet_{A_B}(B; E) \otimes H^\bullet_{A_{M/B}}(M/B; F).$$

Part II

Elliptic involutive structures

Chapter 5

General theory

In this chapter we introduce one of our main objects: elliptic involutive structures. These turn out to be equivalent to transversely holomorphic foliations. Good references for the analytical properties are [Trè92, BCH08]. Geometric expositions of transversely holomorphic foliations can be found in [GM80, Jac00] and a classification of such structures on 3-manifolds is described in [Bru96, Ghy96].

Definition 10. *[Trè92] An elliptic involutive structure over M is a subbundle $V \subset T_{\mathbb{C}}M$ such that*

1. *V is involutive: $[X, Y] \subset \Gamma(M; V)$ whenever $X, Y \in \Gamma(M; V)$.*
2. *V is elliptic: $V + \overline{V} = T_{\mathbb{C}}M$.*

In other words, an elliptic involutive structure is an elliptic complex Lie algebroid with injective anchor.

Definition 11. *The real distribution $V_{\mathbb{R}} := V \cap \overline{V} \cap TM$ is involutive and so defines a foliation of M, called the characteristic foliation.*

Dually, we may describe V in terms of its annihilator $V^{\perp} \subset T^*_{\mathbb{C}}M$. The usual arguments show that

Proposition 5. *A distribution $V \subset T_{\mathbb{C}}M$ is involutive if and only if $V^{\perp}$ generates a differential ideal and V is elliptic if and only if $V^{\perp} \cap T^*M = 0$ (note that here T^*M is the real tangent bundle).*

The ellipticity condition on V is equivalent to the dga $(\Omega^{\bullet}_V(M), d_V)$ being an elliptic complex. Thus we have the following

Proposition 6. *If M is compact, then the vector spaces $H^{\bullet}_V(M; E)$ are finite-dimensional for any V-module E.*

In chapter 7 we will give a detailed study of V-modules. For now, we just remark that V comes canonically with two modules, $V^\perp$ and $T_{\mathbb{C}}M/V$. The connection on $V^\perp$ is given by the Lie derivative and the connection on $T_{\mathbb{C}}M/V$ is given by the Lie bracket. Explicitly,

$$(\nabla_v^{V;V^\perp}\alpha)(w) = v\cdot\alpha(w) - \alpha[v,w], \quad \alpha\in\Gamma(M;V^\perp), v\in V, w\in T_{\mathbb{C}}M.$$

and

$$\nabla_v^{V;T_{\mathbb{C}}M/V}(w+V) = [v,w]+V, \quad v\in V, w+V\in\Gamma(M;T_{\mathbb{C}}M/V).$$

Recall that for a complex manifold, the tangent bundle has a canonical holomorphic structure. The V-module $T_{\mathbb{C}}M/V$ is the natural generalization of this for arbitrary elliptic involutive structures. The space

$$H_V^0(M;T_{\mathbb{C}}M/V) = \{w+V\in\Gamma(M;T_{\mathbb{C}}M/V) \mid [v,w+V]=0\ \forall v\in\Gamma(M;V)\}$$

is analogous to the space of holomorphic vector fields. Although the Lie bracket does not descend to $T_{\mathbb{C}}M/V$ it does descend to $H_V^0(M;T_{\mathbb{C}}M/V)$, giving it the structure of a finite dimensional Lie algebra.

Definition 12. *The bigraded vector space $H^\bullet(M;\Lambda^\bullet V^\perp)$ is called the Dolbeault cohomology of V.*

We have the following immediate examples of elliptic involutive structures:

1. The trivial elliptic involutive structure $V = T_{\mathbb{C}}M$.

2. By the Newlander-Nirenberg theorem, an elliptic structure with $V\cap\overline{V}=0$ is equivalent to specifying an integrable complex structure on M by taking V to be the $-i$ eigenspace. In this case we have that $H^\bullet(M;\Lambda^\bullet V^\perp) = H^{\bullet,\bullet}(M)$ is the usual Dolbeault cohomology.

3. If A is an elliptic Lie algebroid then the image $\rho(A)\subset T_{\mathbb{C}}M$ is an elliptic involutive structure.

4. Suppose $f: E\to M$ is a fiber bundle and $V\subset T_{\mathbb{C}}M$ is an elliptic involutive structure. Then $f^*V^\perp$ generates a differential ideal, which corresponds to an elliptic involutive structure on E. In particular, any fiber bundle over a complex manifold has a canonical elliptic involutive structure that is neither trivial nor complex.

5. Below, we will show that the total space of any flat complex vector bundle over an arbitrary manifold and its projectivization have natural elliptic involutive structures, which are holomorphic in the vertical directions (in contrast to the last example).

Chapter 6

The sheaf $\mathcal{O}_V$ and cohomology

A fundamental object associated to an elliptic involutive structure $V \subset T_{\mathbb{C}}M$ is the sheaf of rings $\mathcal{O}_V$ defined by

$$\mathcal{O}_V(U) = \{f \in C^\infty(M) \mid v \cdot f = 0, \text{ for all } v \in V\},$$

where $U \subset M$ is open. In the case $V = T^{0,1}M$ we have the Newlander-Nirenberg theorem and there are Poincaré lemmas for the two extreme cases $V = T^{0,1}M$ and $V = T_{\mathbb{C}}M$. We also have nice descriptions of flat and holomorphic vector bundles (i.e. $T_{\mathbb{C}}M$ and $T^{0,1}M$ modules) in terms of locally constant sheaves of modules over $\mathcal{O}_{T_{\mathbb{C}}M} = \underline{\mathbb{C}_M}$ and $\mathcal{O}_{T^{0,1}M}$, respectively. We will now show that these statements generalize to elliptic involutive structures.

Our first main analytic fact is

Theorem 5 (Newlander-Nirenberg [Trè92, BCH08])**.** *Let V be an elliptic involutive structure on M. Then, locally, there exist on M real coordinates $(t^1, \ldots, t^d)$ and complex coordinates $(z^1, \ldots, z^n)$ such that*

$$V = \operatorname{span}\left\{\frac{\partial}{\partial t^i}, \frac{\partial}{\partial \bar{z}^j}\right\} = \operatorname{span}\{dz^j\}^\perp.$$

On overlapping coordinates $(s^1, \ldots, s^d)$ and $(w^1, \ldots, w^n)$, the transition functions are given by

$$s^j = f^j(t^1, \ldots, t^d, z^1, \ldots, z^d)$$
$$w^j = g^j(z^1, \ldots, z^d)$$

where the functions g^j are holomorphic.

Corollary 1. *Elliptic involutive structures are equivalent to transversely holomorphic foliations [GM80].*

Corollary 2. *If M is compact and connected, then $H^0(\mathcal{O}_V) = \mathbb{C}$.*

Proof. The proof is essentially the same as the proof of the statement in the holomorphic case. By compactness, for any $f \in \mathcal{O}_V(M)$ the function $|f|$ obtains its maximum value, which we denote by c. Now suppose $x \in |f|^{-1}(c)$. Use the theorem to choose coordinates (t^i, z^j) on a neighborhood U around x. Since $d_V f = 0$, f is constant in the t^i directions. On the other hand, for fixed t^i we have a holomorphic function $f(t^i, \cdot)$ with $|f(t^i, \cdot)|$ attaining its maximum. By the maximum principle, $f(t^i, \cdot)$ is constant and so f is constant on U. Thus $U \subset |f|^{-1}(c)$. Therefore $|f|^{-1}(c)$ is open (and non-empty) but it is also clearly closed. By connectedness $U = M$ so that f is constant on M. □

The second analytic statement we will use is

Theorem 6 (Poincaré lemma, [BCH08] thm. VIII.3.1, [Trè92])**.** *Let $U \subset \mathbb{R}^d$ be open and convex and $W \subset \mathbb{C}^n$ be open and pseudo-convex. Then for the elliptic involutive structure $V = TU \oplus T^{0,1}W\Sigma$ on $U \times W$, we have*

$$H^k_V(U \times W) = \begin{cases} \mathbb{C}; & k = 0 \\ 0; & \textit{otherwise} \end{cases}.$$

Corollary 3. *We have*

$$H^\bullet(\mathcal{O}_V(\Lambda^\bullet V^\perp)) \simeq H^\bullet_V(V; \Lambda^\bullet V^\perp),$$

where the left hand side is sheaf cohomology and $\mathcal{O}_V(\Lambda^\bullet V^\perp)$ is the sheaf of sections of $\Lambda^\bullet V^\perp$ killed by $\nabla^{V;V^\perp}$. In particular,

$$H^\bullet(\mathcal{O}_V) \simeq H^\bullet_V(M),$$

Proof. The Poincaré lemma says that

$$0 \to \mathcal{O}_V \to \Omega^0_V \overset{d_V}{\to} \Omega^1_V \overset{d_V}{\to} \cdots$$

is a resolution of $\mathcal{O}_V$, where $\Omega^\bullet_V$ is the sheaf of sections of the bundle $\Lambda^\bullet V^*$. Further, Theorem 5 shows that $\Lambda^\bullet V^\perp$ is locally trivial as a V-module so

$$0 \to \mathcal{O}_V(\Lambda^\bullet V^\perp) \to \Omega^0_V \otimes \Lambda^\bullet V^\perp \to \Omega^1_V \otimes \Lambda^\bullet V^\perp \to \cdots$$

is a resolution (since locally it is just a direct sum of the resolution of $\mathcal{O}_V$). Since $\Omega^\bullet_V \otimes \Lambda^\bullet V^\perp$ is a sheaf of C^∞_M modules (which is fine), the sheaves $\Omega^\bullet_V$ are acyclic for the global sections functor and so we may use them to compute the cohomology of $\mathcal{O}_V$. □

Chapter 7

V-modules

Thinking of an elliptic involutive structure V as a Lie algebroid, we recall from chapter 3 that a V-module consists of a complex vector bundle $E \to M$ together with a flat partial connection

$$d_{V;E} : \Omega_V^\bullet(M;E) \to \Omega_V^{\bullet+1}(M;E), \quad d_{V;E}^2 = 0.$$

For the extreme cases $V = T_{\mathbb{C}}M$ and $V = T^{0,1}X$, there is a 1-to-1 correspondence between V-modules and locally free sheaves of $\mathcal{O}_V$-modules. This is true for general elliptic involutive structures:

Theorem 7. *Let $E \to M$ be a rank r vector bundle. The following data are equivalent:*

1. *A flat V-connection on E.*
2. *A locally free sheaf $\mathcal{O}(E)$ of $\mathcal{O}_V$-modules such that $E = C_M^\infty \otimes_{\mathcal{O}_V} \mathcal{O}(E)$.*
3. *A trivializing cover $\{U_j\}$ for E such that the transition functions $U_i \cap U_j \to GL(r,\mathbb{C})$ take values in $\mathcal{O}_V(U_i \cap U_j)$.*

Remark. *Vector bundles satisfying the third condition are discussed by Gómez-Mont [GM80], which he calls h-foliated vector bundles.*

Proof. It it easily seen that 2. and 3. are equivalent. Given transition functions as in 3., we get a flat V-connection on E by declaring the frame corresponding to $U_j \times \mathbb{C}^r$ to be parallel. Thus 3. implies 1. To show 1. implies 3., it is sufficient to prove that for any V-module (E, ∇^V) and $x \in M$, there exists a parallel local frame defined on some open neighborhood of x.

Our proof is adapted from the proof in the holomorphic setting given by [Mor07] and uses Theorem 5. Choose an arbitrary local frame σ_j of E over some $U \subset M$ containing x and write

$$\nabla\sigma_j = \tau_j^k \otimes \sigma_k, \quad \tau_j^k \in \Gamma(M;V^*),$$

where we use the summation convention to implicitly sum over a repeated upper and lower index. Also choose U small enough so that, by Theorem 5, we have

$$V|_U = \operatorname{span}_{\mathbb{C}}\{dz^1, \dots, dz^m\}^{\perp}$$

for some complex coordinates $z^1, \dots, z^m$.

On this trivialization, we have $E|_U \simeq U \times \mathbb{C}^r$. Let $u^1, \dots, u^r$ be complex coordinates on $\mathbb{C}^r$ and consider the complex distribution

$$V' = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{u}^k},\ v - \tau_j^k(v)u^j\frac{\partial}{\partial u^k} \mid v \in V \subset T_{\mathbb{C}}(U \times \mathbb{C}^r)\right\} \subset T_{\mathbb{C}}(U \times \mathbb{C}^r).$$

We claim that this distribution is integrable. We have

$$\begin{aligned}
&\left[v - \tau_i^j(v)u^i\frac{\partial}{\partial u^j}, w - \tau_k^l(w)u^k\frac{\partial}{\partial u^l}\right] \\
&= [v,w] - (v\cdot\tau_k^l(w))u^k\frac{\partial}{\partial u^l} + (w\cdot\tau_i^j(v))u^i\frac{\partial}{\partial u^j} \\
&\qquad + \tau_i^j(v)\tau_k^l(w)\left[u^i\frac{\partial}{\partial u^j}, u^k\frac{\partial}{\partial u^l}\right] \\
&= [v,w] - (v\cdot\tau_k^l(w))u^k\frac{\partial}{\partial u^l} + (w\cdot\tau_i^j(v))u^i\frac{\partial}{\partial u^j} + \tau_i^j(v)\tau_j^l(w)u^i\frac{\partial}{\partial u^l} \\
&\qquad - \tau_i^j(v)\tau_k^i(w)u^k\frac{\partial}{\partial u^j} \\
&= [v,w] - (v\cdot\tau_j^k(w))u^j\frac{\partial}{\partial u^k} + (w\cdot\tau_j^k(v))u^j\frac{\partial}{\partial u^k} + \tau_j^l(v)\tau_l^k(w)u^j\frac{\partial}{\partial u^k} \\
&\qquad - \tau_l^k(v)\tau_j^l(w)u^j\frac{\partial}{\partial u^k} \\
&= [v,w] + \left(-v\cdot\tau_j^k(w) + w\cdot\tau_j^k(v) + \tau_j^l(v)\tau_l^k(w) - \tau_l^k(v)\tau_j^l(w)\right)u^j\frac{\partial}{\partial u^k}
\end{aligned}$$

Now, the flatness condition says that

$$d_A\tau_j^k - \tau_j^l \wedge \tau_l^k = 0, \quad \forall j,k.$$

Evaluating this at v, w gives

$$0 = v\cdot\tau_j^k(w) - w\cdot\tau_j^k(v) - \tau_j^k([v,w]) - \tau_j^l(v)\tau_l^k(w) + \tau_k^l(w)\tau_l^k(v).$$

Putting this into the top gives

$$\left[v - \tau_i^j(v)u^i\frac{\partial}{\partial u^j}, w - \tau_k^l(w)u^k\frac{\partial}{\partial u^l}\right] = [v,w] - \tau_j^k([v,w])u^j\frac{\partial}{\partial u^k} \in V',$$

as desired.

Since $V' + \overline{V'} = T_{\mathbb{C}}(U \times \mathbb{C}^r)$, this distribution gives an elliptic structure on $U \times \mathbb{C}^r$. Let $\tilde{\tau}_i^j \in \Omega^1(U)$ be a lift of $\tau_i^j \in \Gamma(M; V^*)$. Then the space of differential 1-forms that annihilate this distribution is

$$\begin{aligned}(V')^{\perp} &= V^{\perp} + \operatorname{span}\{du^j + \tilde{\tau}_i^j u^i : j = 1, \ldots, r\} \\ &= \operatorname{span}\{dz^1, \ldots, dz^m\} + \operatorname{span}\{du^j + \tilde{\tau}_i^j u^i : j = 1, \ldots, r\}. \end{aligned} \tag{7.0.1}$$

Theorem 5 says that there exists coordinates $t^1, \ldots, t^d, \tilde{z}^1, \ldots, \tilde{z}^n$ for $U \times \mathbb{C}^r$ (here the t^j are real and the $\tilde{z}^j$ are complex) such that $(V')^{\perp} = \operatorname{span}\{d\tilde{z}^1, \ldots, d\tilde{z}^n\}$. Thus by eq. (7.0.1) we can write

$$d\tilde{z}^j = F_i^j dz^i + G_i^j (du^i + \tilde{\tau}_k^i u^k),$$

for some $F_i^j, G_i^j \in C^\infty(U \times \mathbb{C}^r)$.

Since the map determined by the F_i^j and G_i^j is an isomorphism, there must be some indices $j_1, \ldots, j_r$ and some neighborhood such that the matrix $[G_i^j]_{i=1,\ldots,r, j=j_1,\ldots,j_r}$ is in $GL(r, \mathbb{C})$. By rearranging indices, assume $j_1 = 1, \ldots, j_r = r$.

Differentiating the above equation for $d\tilde{z}^j$ gives

$$0 = dF_i^j \wedge dz^i + dG_i^j \wedge (du^i + \tilde{\tau}_k^i u^k) + G_i^j (d\tilde{\tau}_k^i u^k - \tilde{\tau}_k^i \wedge du^k).$$

At $u^k = 0$ we get

$$dF_i^j \wedge dz^i + dG_i^j|_{U \times \{0\}} \wedge du^i - G_i^j|_{U \times \{0\}} \tilde{\tau}_k^i \wedge du^k = 0.$$

Pulling back to $V^* \otimes (T^{1,0}\mathbb{C}^r)^*$, we see that

$$d_V G_i^j|_{U \times \{0\}} - G_k^j|_{U \times \{0\}} \tau_i^k = 0.$$

Let $\tilde{\sigma}_k$ be defined by $\sigma_j = G_j^k|_{U \times \{0\}} \tilde{\sigma}_k$ (which is possible since $[G_i^j] \in GL(r, \mathbb{C})$). Then one computes from the above equation that $\tilde{\sigma}_k$ is parallel:

$$\begin{aligned}\tau_j^k \otimes \sigma_k = \nabla \sigma_j &= d_V G_j^k \otimes \tilde{\sigma}_k + G_j^k \nabla \tilde{\sigma}_k \\ &= G_l^k \tau_j^l \otimes \tilde{\sigma}_k + G_j^k \nabla \tilde{\sigma}_k = \tau_j^l \otimes \sigma_l + G_j^k \nabla \tilde{\sigma}_k, \\ &\Rightarrow G_j^k \nabla \tilde{\sigma}_k = 0 \\ &\Rightarrow \nabla \tilde{\sigma}_k = 0.\end{aligned}$$

□

It is interesting to note that even if we carry out the above proof in the case of a flat vector bundle (where the theorem becomes the well-known result that flat vector bundles are equivalent to local systems), a vital ingredient is the Newlander-Nirenberg theorem for a non-trivial elliptic involutive structure.

Corollary 4. *If (E, ∇^V) is a V-module with $\mathcal{O}(E)$ the corresponding sheaf of $\mathcal{O}_V$ modules then*

$$H^\bullet(\mathcal{O}(E)) = H^\bullet_V(M; E),$$

where the left hand side is sheaf cohomology.

Proof. As in Corollary 3 we have a resolution

$$0 \to \mathcal{O}(E) \to \Omega^0_{V;E} \stackrel{\nabla^V}{\to} \Omega^1_{V;E} \to \cdots$$

by acyclic sheaves (here $\Omega^\bullet_{V;E}$ is the sheaf of E-valued V-forms). □

Corollary 5. *Let (E, ∇^V) be a rank k V-module. Then the total space of E itself has an elliptic involutive structure. Furthermore, the elliptic involutive structure descends to the projectivized bundle $\mathbb{P}(E)$.*

Proof. By the theorem, for any $x \in M$ there exists a neighborhood $U \ni x$ and a ∇^V-parallel frame $\{\sigma_1, \ldots, \sigma_k\}$ of E on U. We may view this frame as a map

$$\sigma : U \times \mathbb{C}^k \to E|_U = \pi^{-1}(U),$$
$$(x, u^1, \ldots, u^k) \mapsto u^j \sigma_j(x).$$

We define the elliptic involutive structure V' on the total space of E by specifying $V'|_{\pi^{-1}(U)} \subset T_{\mathbb{C}}E|_U$ to be $\sigma_*(V|_U \oplus T^{0,1}\mathbb{C}^k)$. To show that this gives a well-defined global distribution, suppose that $\tilde{U}$ is another open set containing x with a local parallel frame $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k$ for $E|_{\tilde{U}}$. Write $\sigma_i = g_i^j \tilde{\sigma}_j$ for $g_i^j \in C^\infty(U \cap \tilde{U})$. Then we have

$$\tilde{\sigma}^{-1}\sigma : U \cap \tilde{U} \times \mathbb{C}^k \to U \cap \tilde{U} \times \mathbb{C}^k,$$
$$\tilde{\sigma}^{-1}\sigma(x, u^1, \ldots, u^k) = (x, g_i^j(x) u^i).$$

But since the frames σ and $\tilde{\sigma}$ are both parallel, we have $d_V g_i^j = 0$, from which it follows that $\tilde{\sigma}^{-1}\sigma_*$ takes $V|_{U \cap V} \times T^{0,1}\mathbb{C}^k$ to itself. Thus the distributions defined by σ and $\tilde{\sigma}$ agree.

The same reasoning shows that we similarly get an involutive elliptic structure on $\mathbb{P}(E)$, which when pulled back to a fiber gives the usual complex structure on $\mathbb{C}P^{k-1}$. □

The above corollary provides many natural (and compact) examples of elliptic involutive structures that are neither trivial nor complex. For example, if $E \to M$ is any flat complex vector bundle then $\mathbb{P}(E)$ has a natural elliptic involutive structure. We will examine this structure in more detail in section 8.3.

7.1 Deformations

Generalizing the Kodaira-Spencer theory of deformations of complex structures, the infinitesimal deformations of an elliptic involutive structure V is given by the first cohomology group of the sheaf of sections of $T_{\mathbb{C}}M/V$ that commute with V [GHS83, DK84, DK79]. By Corollary 4, this is isomorphic to $H^1_V(M; T_{\mathbb{C}}M/V)$.

7.2 The Picard group

Definition 13. *The Picard group associated to the elliptic involutive structure V on M, denoted $\mathrm{Pic}_V(M)$, is the abelian group of isomorphism classes of rank 1 V-modules under tensor product.*

The usual Čech argument gives

Proposition 7. *We have a natural isomorphism $\mathrm{Pic}_V(M) \simeq H^1(\mathcal{O}_V^\times)$.*

In later sections, we will make use of the following result.

Proposition 8. *Suppose M is a manifold with elliptic involutive structure V and*

$$H^1(M;\mathbb{Z}) = 0 = H^2(M;\mathbb{Z}).$$

Then every complex line bundle is topologically trivial and we have an isomorphism $\mathrm{Pic}_V(M) \simeq H^1(\mathcal{O}_V)(\simeq H^1_V(M))$ via

$$H^1_V(M) \ni \omega \mapsto d_V - \omega,$$

the right hand side being a V-connection on the bundle $M \times \mathbb{C}$.

Proof. The assumptions say that the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_V \to \mathcal{O}_V^\times \to 0$$

gives

$$H^1_V(M) \simeq H^1(\mathcal{O}_V) \simeq H^1(\mathcal{O}_V^\times) = Pic_V(M).$$

To see what this isomorphism looks like, let $\omega \in H^1_V(M)$. Choose a good cover $\{U_\alpha\}$ of M. Then on each U_α we may find $f_\alpha \in C^\infty(U_\alpha)$ such that $d_V f_\alpha = \omega|_{U_\alpha}$. Put $h_{\alpha\beta} = f_\alpha - f_\beta \in C^\infty(U_{\alpha\beta})$. Then $d_V h_{\alpha\beta} = 0$ so that $h_{\alpha\beta}$ gives a class $h \in \check{H}^1(\mathcal{O}_V)$, which corresponds to ω under the isomorphism $H^1_V(M) \to H^1(\mathcal{O}_V)$. Then $g_{\alpha\beta} := \exp h_{\alpha\beta}$ represents a class in $H^1(\mathcal{O}_V^\times)$, which determines a V-line bundle L_ω that has parallel frames σ_α on U_α with $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on $U_{\alpha\beta}$.

Now since L_ω has to be topologically trivial (since $H^2(M;\mathbb{Z})=0$), we have a global section σ. Let $t_\alpha \in C^\infty(U_\alpha)$ be defined by $\sigma_\alpha = t_\alpha \sigma|_{U_\alpha}$. Then $t_\alpha/t_\beta = g_{\alpha\beta}$. Since σ_α is parallel we have

$$\begin{aligned}0 = \nabla^{V;L}\sigma_\alpha &= d_V t_\alpha \otimes \sigma|_{U_\alpha} + t_\alpha \nabla\sigma|_{U_\alpha}\\ \Rightarrow \nabla\sigma|_{U_\alpha} &= -d_V \log t_\alpha \otimes \sigma|_{U_\alpha},\end{aligned}$$

where $d_V \log t_\alpha = \frac{1}{t_\alpha} d_V t_\alpha$ by definition (so we do not actually need a logarithm). Now the forms $-d_V \log t_\alpha$ patch together to give a global 1-form on M since

$$d_V \log t_\alpha - d_V \log t_\beta = d_V \log(t_\alpha/t_\beta) = d_V h_{\alpha\beta} = 0.$$

It is straightforward to verify that if $\omega = \omega_1 + \omega_2$ then the local connection forms for ω are the sum of the connections forms for ω_1 and ω_2, which corresponds to taking the tensor product of V-modules. Thus the map is a homomorphism.

Since we are on a good cover, every function valued in $\mathbb{C}^\times$ has a logarithm. Then we see that

$$\exp(\log t_\alpha - \log t_\beta) = g_{\alpha\beta} = \exp(f_\alpha - f_\beta)$$

so that $\log t_\alpha - \log t_\beta = f_\alpha - f_\beta + n_{\alpha\beta}$, where $n_{\alpha\beta} \in \mathbb{Z}$. Then $n_{\alpha\beta}$ determines a Check cocycle in $\check{H}^1(X;\mathbb{Z})$. Since $H^1(X;\mathbb{Z}) = 0$ we can find m_α integers such that $m_\alpha - m_\beta = n_{\alpha\beta}$. Then $\log t_\alpha - f_\alpha - n_\alpha$ piece together to a globally defined function F. Differentiating gives

$$\begin{aligned}d_V \log t_\alpha - d_V f_\alpha &= d_V F\\ \Rightarrow -d_V \log t_\alpha &= -\omega - d_V F.\end{aligned}$$

Therefore we have the global description $\nabla^V = d_V - \omega - d_V F$. But this is gauge equivalent to $d_V - \omega$ via the gauge transformation e^{-F}. □

7.2.1 A Hirzebruch-Riemann-Roch formula

We will now use the Atiyah-Singer index theorem to give a Hirzebruch-Riemann-Roch formula for the Euler characteristic of the cohomology of a V-module. For an EIS V, the real vector bundle $TM/V_\mathbb{R}$ has a natural complex structure as follows. Since $V + \overline{V} = T_\mathbb{C}M$, we have

$$(TM/V_\mathbb{R}) \otimes \mathbb{C} = T_\mathbb{C}M/(V \cap \overline{V}) = (V/V \cap \overline{V}) \oplus (\overline{V}/V \cap \overline{V}).$$

Thus we get a complex structure by declaring $T^{1,0} := V/V \cap \overline{V}$ to be the i-eigenspace and $T^{0,1} := \overline{V}/V \cap \overline{V}$ to be the $-i$-eigenspace. Choose a metric on M that is compatible with the complex structure on $TM/V_\mathbb{R}$. Then we have the decomposition

$$T_\mathbb{C}M \simeq (V \cap \overline{V}) \oplus T^{0,1} \oplus T^{1,0}, \quad \Lambda^\bullet V^* \simeq \Lambda^\bullet(V \cap \overline{V})^* \otimes \Lambda^\bullet(T^{0,1})^*.$$

A Clifford action of $T_{\mathbb{C}}M$ on $\Lambda^\bullet V^*$ given by

$$c(v) = \begin{cases} \varepsilon(v^\flat) - i(v); & v \in V \cap \overline{V} \\ -\sqrt{2}i(v); & v \in T^{0,1} \\ \sqrt{2}\varepsilon(v^\flat); & v \in T^{1,0}, \end{cases}$$

where $\flat : TM \to T^*M$ is induced from the metric (and we recall that $(T^{1,0})^\flat = (T^{0,1})^*$).

Modulo 0th order terms, the corresponding Dirac operator is $d_V + d_V^*$, whose index is $\sum_j (-1)^j \dim H^j_V(M)$. Since the Clifford action is a direct sum of the de Rham Clifford action of $V \cap \overline{V}$ on $\Lambda^\bullet(V \cap \overline{V})^*$ and the Dolbeault Clifford action of $TM/V_{\mathbb{R}}$ on $\Lambda^\bullet(T^{0,1})^*$, the corresponding index density is the product $e(F)Td(TM/V_{\mathbb{R}})$ [BGV92]. We can also twist this construction with a V-module E, which multiplies the index density by $\mathrm{ch}(E)$. Thus the Atiyah-Singer index theorem gives us

Theorem 8. *If E is a V-module then*

$$\sum_j (-1)^j \dim H^j_V(M; E) = \int_M e(F)Td(TM/V_{\mathbb{R}})\,\mathrm{ch}(E).$$

7.3 Characteristic classes

We now describe two types of characteristic classes that can be associated to a V-module (E, ∇^V).

7.3.1 Atiyah classes

The Atiyah classes lie in $H^\bullet_V(M; \Lambda^\bullet V^\perp)$ and are constructed from the fact that we have a Lie algebroid pair $(T_{\mathbb{C}}M, V)$ [CSX12].

To define these, we first extend ∇^V to a regular connection ∇ (which can always be done by using a partition of unity). The curvature $F_\nabla \in \Omega^2(M; \mathrm{End}\, E)$ will be non-zero in general but since $(\nabla^V)^2 = 0$, we see that if $v \in V$ then $F_\nabla(v, \cdot)$ vanishes on V. Thus F_∇ defines an element in $\Omega^1_V(M; V^\perp \otimes \mathrm{End}\, E)$.

Definition 14 (Definition/Proposition [CSX12])**.** *The element F_∇ is $d_{V;\mathrm{End}\, E}$-closed and its cohomology class,*

$$\mathrm{At}(E) \in H^1_V(M; V^\perp \otimes \mathrm{End}\, E),$$

is independent of the choice of connection extending ∇^V. The kth scalar Atiyah class is defined to be

$$\mathrm{at}_k(E) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{tr}\,\mathrm{at}(E)^k \in H^k_V(M; \Lambda^k V^\perp).$$

The Chern character is defined by

$$\operatorname{ch}(E) = \operatorname{tr}\exp\left(\frac{i}{2\pi}\operatorname{At}(E)\right).$$

Proposition 9. *The Atiyah classes satisfy*

1. *Suppose $f : M \to N$ is a smooth map of EISs V_M and V_N over M and N and $E \to N$ is a V_N-module. Then*

$$\operatorname{At}(f^*E) = f^* \operatorname{At} E \in H^1_{V_M}(M; V_M^\perp \otimes f^* \operatorname{End} E).$$

2. *If*

$$0 \to E_1 \to E \xrightarrow{p} E_2 \to 0$$

is a short exact sequence of V-modules, then

$$\operatorname{ch}(E) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2).$$

3. *If E_1 and E_2 are V-modules then*

$$\operatorname{At}(E_1 \otimes E_2) = \operatorname{At} E_1 \otimes \mathbb{1}_{E_2} + \mathbb{1}_{E_1} \otimes \operatorname{At} E_2$$

and

$$\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1)\operatorname{ch}(E_2).$$

Proof. For the first statement, note that if ∇ is a connection on E compatible with $\nabla^{V_N;E}$ then the pullback connection $f^*\nabla$ on f^*E is compatible with $\nabla^{V_M;f^*E}$ and the curvatures are related by $F_{f^*\nabla} = f^*F_\nabla$. Then since $f_*V_M \subset V_N$, it follows that $\operatorname{At}(f^*E) = f^* \operatorname{At} E$.

For 2., choose connections ∇_1, ∇_2 on E_1 and E_2 that are compatible with the respective EISs. Let $s : E_2 \to E$ be a C^∞ splitting of the above sequence. Then $\nabla^V s := \nabla^{V;E} \circ s - s \circ \nabla^{V;E_2} \in \Omega^{0,1}(M; \operatorname{Hom}(E_2, E_1))$ since $p \circ \nabla^V s = 0$ (which is because $p \circ \nabla^{V;E} = \nabla^{V;E_2} \circ p$). Then the connection

$$\nabla = \begin{pmatrix} \nabla_1 & \nabla^V s \\ 0 & \nabla_2 \end{pmatrix}$$

on $E \simeq E_1 \oplus E_2$ extends $\nabla^{V;E}$ so that

$$\operatorname{At}(E) = \begin{pmatrix} \operatorname{At}(E_1) & * \\ 0 & \operatorname{At}(E_2) \end{pmatrix}.$$

Thus

$$\begin{aligned} \operatorname{ch}(E) = \operatorname{tr}\exp\left(\frac{i}{2\pi}\operatorname{At}(E)\right) &= \operatorname{tr}\exp\left(\frac{i}{2\pi}\operatorname{At}(E_1)\right) + \operatorname{tr}\exp\left(\frac{i}{2\pi}\operatorname{At}(E_2)\right) \\ &= \operatorname{ch}(E_1) + \operatorname{ch}(E_2). \end{aligned}$$

For 3., choose compatible connections ∇_1 and ∇_2. Then the tensor product connection $\nabla := \nabla_1 \otimes \mathbb{1}_{E_1} + \mathbb{1}_{E_2} \otimes \nabla_2$ is compatible with the EIS on $E_1 \otimes E_2$ and has curvature $\nabla^2 = \nabla_1^2 \otimes \mathbb{1}_{E_1} + \mathbb{1}_{E_2} \otimes \nabla_2^2$. From this the two claims easily follow. ☐

7.3.2 Real classes

There is also a theory of characteristic classes for modules over real Lie algebroids [Cra03], which generalize the $H^{odd}(M;\mathbb{R})$ valued characteristic classes of flat vector bundles [KT75, BL95]. Since any V-module is, via restriction, a module for the real Lie algebroid $V_{\mathbb{R}} = (V \cap \overline{V}) \cap TM$, we also have characteristic classes of V-modules that lie in $H^{odd}_{V_{\mathbb{R}}}(M)$. These are constructed as follows. Given a $V_{\mathbb{R}}$-module $(E, \nabla^{V_{\mathbb{R}}})$, we let h be an arbitrary hermitian metric on E and form the adjoint connection $(\nabla^{V_{\mathbb{R}}})^*$ defined by

$$d_{V_{\mathbb{R}}} h(u,v) = h(\nabla^{V_{\mathbb{R}}} u, v) + h(u, (\nabla^{V_{\mathbb{R}}})^* v), \quad u, v \in \Gamma(M; E).$$

Then $(\nabla^{V_{\mathbb{R}}})^*$ is another flat $V_{\mathbb{R}}$-connection on E. Define

$$\omega(E,h) = \frac{1}{2}(\nabla^{V_{\mathbb{R}}} - (\nabla^{V_{\mathbb{R}}})^*) \in \Omega^1_{V_{\mathbb{R}}}(M; \operatorname{End} E).$$

Then

Definition/Proposition 1 ([Cra03]). *The form* $\operatorname{tr}\omega(E,h)^{2k-1} \in \Omega^{2k-1}_{V_{\mathbb{R}}}(M)$ *is* $d_{V_{\mathbb{R}}}$*-closed and its cohomology class* $u_{2k-1} \in H^{2k-1}_{V_{\mathbb{R}}}(M)$ *is independent of the choice of* h*.*

Indeed, we have the unitary (generally non-flat) V-connection $\nabla^{V_{\mathbb{R}},u} := \frac{1}{2}(\nabla^{V_{\mathbb{R}}} + (\nabla^{V_{\mathbb{R}}})^*)$ and the fact that $\nabla^{V_{\mathbb{R}}}$ and its adjoint are flat implies that $\omega(E,h)$ is parallel with respect to $\nabla^{V_{\mathbb{R}},u}$:

$$[\nabla^{V_{\mathbb{R}},u}, \omega(E,h)] = 0.$$

This implies that the trace of any power of $\omega(E,h)$ is $d_{V_{\mathbb{R}}}$-closed.

Proposition 10. *If*

$$0 \to E_1 \to E \xrightarrow{p} E_2 \to 0$$

is a short exact sequence of V*-modules, then*

$$u_{2k-1}(E) = u_{2k-1}(E_1) + u_{2k-1}(E_2).$$

Proof. Let h be a hermitian metric on E, which determines a splitting $s : E_2 \to E$ and hermitian metrics h_1, h_2 on E_1 and E_2. Then under the isomorphism $E \simeq E_1 \oplus E_2$ we have

$$\nabla^{V_{\mathbb{R}};E} = \begin{pmatrix} \nabla^{V_{\mathbb{R}};E_1} & \nabla s \\ 0 & \nabla^{V_{\mathbb{R}};E_2} \end{pmatrix},$$

where $\nabla s = \nabla^{V_{\mathbb{R}};E} \circ s - s \circ \nabla^{V_{\mathbb{R}};E_2}$. One computes that

$$\omega(E,h) = \begin{pmatrix} \omega(E_1,h_1) & \nabla s \\ -(\nabla s)^* & \omega(E_2,h_2) \end{pmatrix}.$$

But the element $\begin{pmatrix} 0 & \nabla s \\ -(\nabla s)^* & 0 \end{pmatrix} \in \Omega^1_{V_\mathbb{R}}(M; \operatorname{End} E)$ is exact with respect to the connection $\nabla^{V_\mathbb{R};E,u}$:

$$\nabla^{V_\mathbb{R};E,u} = \begin{pmatrix} \nabla^{V_\mathbb{R};E_1,u} & \nabla s \\ (\nabla s)^* & \nabla^{V_\mathbb{R};E_2,u} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \nabla s \\ -(\nabla s)^* & 0 \end{pmatrix} = \left[\nabla^{V_\mathbb{R};E,u}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right].$$

Since $\omega(E,h)$ is parallel with respect to $\nabla^{V_\mathbb{R};E,u}$ (see the preceeding discussion), it follows that the cohomology class of $\operatorname{tr}\omega(E,h)^{2k-1}$ is unchanged by adding $\nabla^{V_\mathbb{R};E,u}$-exact terms to $\omega(E,h)$. Thus in cohomology we have

$$\begin{aligned}
[\operatorname{tr}\omega(E,h)^{2k-1}] &= \left[\operatorname{tr} \begin{pmatrix} \omega(E_1,h_1) & 0 \\ 0 & \omega(E_2,h_2) \end{pmatrix}^{2k-1} \right] \\
&= [\operatorname{tr}\omega(E_1,h_1)^{2k-1}] + [\operatorname{tr}\omega(E_2,h_2)^{2k-1}] \in H^{2k-1}_{V_\mathbb{R}}(M).
\end{aligned}$$

□

In the case of lines bundles we have the following alternative description of u_1. A rank 1 V-module is determined by a class in $H^1(\mathcal{O}_V^\times)$, which induces a class in $H^1(\mathcal{O}_{V_\mathbb{R}}^\times)$. We have the short exact sequence of sheaves

$$0 \to \mathcal{O}_{V_\mathbb{R}}^{S^1} \to \mathcal{O}_{V_\mathbb{R}}^\times \xrightarrow{\log|\cdot|} \mathcal{O}_{V_\mathbb{R}} \to 0.$$

Then u_1 corresponds to the induced map in cohomology $H^1(\mathcal{O}_{V_\mathbb{R}}^\times) \to H^1(\mathcal{O}_{V_\mathbb{R}}) \simeq H^1_{V_\mathbb{R}}(M)$.

7.4 Duality

Let V be an elliptic involutive structure of rank r. The dualizing module (section 3.4) is

$$Q_V = \Lambda^{top} V \otimes \Lambda^{top} T^*_\mathbb{C} M = \Lambda^{top} V \otimes \Lambda^{top} V^* \otimes \Lambda^{top} V^\perp = \Lambda^{top} V^\perp.$$

Then Theorem 2 gives

Theorem 9. *For any V-module E, there is a perfect pairing*

$$H^k_V(M;E) \otimes H^{r-k}_V(M; \Lambda^{top} V^\perp \otimes E^*) \to \mathbb{C}.$$

Chapter 8

Examples

We now describe several examples in detail.

8.1 Fiber bundles

Suppose B has an elliptic involutive structure and $\pi : M \to B$ is a fiber bundle. The pullback $\pi^*V^\perp \subset T^*_{\mathbb{C}}M$ is a differential ideal since $V^\perp$ is (it is also clearly elliptic) and thus V induces a elliptic involutive structure on M, which we denote by $\pi^! V$. In particular, by taking M to be a complex manifold we can get a large class of examples of elliptic involutive structures that are neither trivial nor complex structures. In this case we have a map of Lie algebroids $\pi_* : \pi^! V \to V$ whose kernel is the Lie algebroid (i.e. tangent bundle algebroid) formed by the vertical vectors. The Leray-Hirsch theorem (Theorem 4) gives us

Proposition 11. *Let $\pi : M \to B$ be a fiber bundle. If there exists $\alpha_1, \ldots, \alpha_d \in H^\bullet_{A_M}(M)$ that restrict to a basis of $H^\bullet(M_x; \mathbb{C})$ for each $x \in B$, then*

$$H^\bullet_{\pi^! V}(M; \Lambda^\bullet(\pi^! V)^\perp) \simeq H^\bullet_V(B; \Lambda^\bullet V^\perp) \otimes H^\bullet(M/B; \mathbb{C}).$$

8.1.1 Principal bundles over complex manifolds

One case where the above proposition can always be applied is for a G-frame bundle $\pi : P \to X$ corresponding to a holomorphic vector bundle $E \to X$ with a G-invariant hermitian metric. Then P has the pullback elliptic involutive structure $V_P := \pi^! T^{0,1}X$. The metric on E gives us the Chern connection (i.e. the unique metric connection whose (0,1) part is the holomorphic structure). This connection induces a principal connection on P and in particular gives us a map $\Lambda^\bullet \mathfrak{g}^* \to \Omega^\bullet(P)$, where $\mathfrak{g} = Lie(G)$. Choose $\alpha_1, \ldots, \alpha_d \in \Lambda^\bullet \mathfrak{g}^*$ that form a basis (when viewed as left-invariant forms) of $H^\bullet(G)$ and write $\tilde{\alpha}_j$ for the lifts to $\Omega^1(P)$. Since the curvature of the Chern connection is of type (1,1), the horizontal lift of $T^{0,1}X$ is an integrable distribution of $T_{\mathbb{C}}P$. It follows that $d_{V_P}\tilde{\alpha}_j = 0$ for all j so that the conditions of Proposition 11 are satisfied:

Proposition 12. *For a principal unitary G-frame bundle $\pi : P \to X$ of a holomorphic vector bundle over a complex manifold, we have*

$$H^{\bullet}_{V_P}(P; \Lambda^{\bullet} V_P^{\perp}) \simeq H^{\bullet,\bullet}(X) \otimes H^{\bullet}(\mathfrak{g}; \mathbb{C}),$$

where $V_P = \pi^! T^{0,1} X$.

We also have

Proposition 13. *The Lie algebra $H^0_{V_P}(P; T_{\mathbb{C}}P/V_P)$ is isomorphic to the Lie algebra $H^0(X; T^{1,0}X)$ of holomorphic vector fields on X.*

Proof. The horizontal distribution on TP determined by the Chern connection restricts to an isomorphism $\pi^* T^{1,0} X \to T_{\mathbb{C}}P/V$ of V_P-modules. This gives an injection $H^0(X; T^{1,0}X) \xrightarrow{\pi^*} H^0_{V_P}(P; T_{\mathbb{C}}P/V)$ of Lie algebras. The image contains those elements of $H^0_{V_P}(P; V^{\perp})$ that can be represented by horizontal lifts of vector fields on X, so to show that this map is an isomorphism we must show that everything in $H^0_{V_P}(P; T_{\mathbb{C}}P/V)$ is invariant under the right G-action. Thus let $w + V \in H^0_{V_P}(P; T_{\mathbb{C}}P/V)$. We can choose w to be in the horizontal distribution corresponding to $\pi^* T^{1,0} X$. Then since $\nabla^V(w + V) = 0$ we have, in particular, $[Y^*, w] + V = 0$ for all $Y \in \mathfrak{g}$ (where Y^* corresponds to the action vector field). Since the G-action preserves the decomposition $\pi^* T^{1,0} X \oplus \pi^* T^{0,1} X$ of the horizontal distribution, it follows that $[Y^*, w] = 0$ for all $Y \in \mathfrak{g}$. But this is just the infinitesimal condition that $R_g^* w = 0$ for all $g \in G$ so that w is G-invariant and thus a horizontal lift of a vector field on X.

□

Odd dimensional spheres

In particular, all of the odd dimensional spheres S^{2n+1} have an elliptic involutive structure since they are a $U(1)$ frame bundle for the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^n$. Writing V for $V_{S^{2n+1}}$, the proposition tells us that

$$\begin{aligned} H^q_V(S^{2n+1}; \Lambda^p V^{\perp}) &\simeq \big(H^{p,q-1}(\mathbb{C}P^n) \otimes H^1(S^1; \mathbb{C})\big) \oplus \big(H^{p,q}(\mathbb{C}P^n) \otimes H^0(S^1; \mathbb{C})\big) \\ &= H^{p,q-1}(\mathbb{C}P^n) \oplus H^{p,q}(\mathbb{C}P^n) \\ &= \begin{cases} \mathbb{C}; & 0 \le p, q \le 2n \text{ and } p = q \text{ or } p = q - 1 \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

In particular, $H^1_V(S^{2n+1}) = \mathbb{C}$ and by Proposition 8 this is isomorphic to the group $\mathrm{Pic}_V(S^{2n+1})$. Let X be a basis for $Lie(S^1)$ and X^* the corresponding vector field on S^{2n+1} that generates the S^1 foliation. Let $\mu \in \Omega^1(P; Lie(S^1))$ be a connection 1-form on $S^{2n+1} \to \mathbb{C}P^n$. Identifying $Lie(S^1)$ with $i\mathbb{R}$, we view μ as an element of $\Omega^1_V(S^{2n+1})$ and we have that $\pi^*\mathcal{O}(-1)$ is canonically trivial with V-connection

$$\nabla^{V;\pi^*\mathcal{O}(-1)} = d_V + \mu.$$

Now if $n \in \mathbb{Z}$, the rank 1 V-module with connection $d_V + n\mu$ is $\pi^*\mathcal{O}(-n)$. These must pullback to the trivial flat vector bundle over any S^1 fiber. Indeed, on any fiber μ pulls back to the Maurer-Cartan form μ_{MC} and the connection $d + c\mu_{MC}$ on S^1 has holonomy $e^{-2\pi ic}$, which is trivial if $c = n \in \mathbb{Z}$. For $c \notin \mathbb{Z}$, the V-module $d_V + c\mu$ has no global parallel sections since it does not have any global parallel sections when restricted to any S^1 fiber. Indeed, letting L_c be the V-module that is the trivial line bundle with the connection $d_V + c\mu$, we have $H^\bullet_V(S^{2n+1}/\mathbb{C}P^n; L_c) = 0$ for $c \notin \mathbb{Z}$ and thus by Theorem 4 we have

$$H^\bullet_V(S^{2n+1}; L_c) = \begin{cases} 0; & c \in \mathbb{C}\backslash\mathbb{Z}. \\ H^{0,\bullet}(\mathbb{C}P^n; \mathcal{O}(-c)); & c \in \mathbb{Z}. \end{cases}$$

Writing $\mathrm{Pic}_{T_\mathbb{C}S^1}(S^1)$ for the space of flat line bundles on S^1 (i.e. rank 1 $T_\mathbb{C}S^1$ modules), this shows that we have a short exact sequence

$$0 \to \mathrm{Pic}(\mathbb{C}P^n) \xrightarrow{\pi^*} \mathrm{Pic}_V(S^{2n+1}) \xrightarrow{i^*} \mathrm{Pic}_{T_\mathbb{C}S^1}(S^1) \to 0.$$

The bundles L_c are completely classified by the first Atiyah class (Definition 14): we can complete $d_V + c\mu$ to the connection $d + c\mu$ which has curvature $cd\mu = c\pi^*\omega$ where $\omega \in \Omega^2(\mathbb{C}P^n)$ is the Fubini-Study Kähler form. Then ω defines a class $[\omega] \in H^{1,1}(\mathbb{C}P^n)$, which pullbacks to a class $\pi^*[\omega] \in H^1_V(S^{2n+1}; V^\perp)$ giving $\mathrm{at}_1(L_c) = c\pi^*[\omega]$. We can recover c by integrating $c\mu \wedge \mathrm{at}_1(L_c)^n$ over S^{2n+1}. That is, $c\mu$ gives a class in $H^1_V(S^{2n+1})$ and $\mathrm{at}_1(L_c)^n \in H^n_V(S^{2n+1}; Q_V)$ so we can use the duality pairing, Theorem 2, to get a complex number. This will be proportional to c^{n+1}.

By Proposition 13, we have

$$H^0_V(S^{2n+1}; T_\mathbb{C}S^{2n+1}/V) \simeq H^0(\mathbb{C}P^n; T^{1,0}\mathbb{C}P^n) \simeq \mathfrak{sl}(n+1, \mathbb{C}).$$

Interestingly, while the holomorphic structure on $\mathbb{C}P^n$ is stable (i.e. $H^1(T^{1,0}_{\mathbb{C}P^n}) = 0$), this is not true for the induced EIS on S^{2n+1}. Indeed, we have a V-module isomorphism $T_\mathbb{C}S^{2n+1}/V \simeq \pi^*T^{1,0}\mathbb{C}P^n$ and so by Proposition 12 and Theorem 4 we have

$$H^1_V(S^{2n+1}; T_\mathbb{C}S^{2n+1}/V) \simeq H^0(\mathbb{C}P^n; T^{1,0}\mathbb{C}P^n) \simeq \mathfrak{sl}(n+1, \mathbb{C}).$$

8.2 Compact Lie groups and homogeneous spaces

Let G be compact semi-simple. Let T be a maximal torus. Then the flag manifold G/T has a G-invariant complex structure so that, by the previous section, G has an elliptic involutive structure V. Explicitly such structures are determined by a choice of positive root system Δ_+. We then have the decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_+} \mathbb{C}Z_\alpha \oplus \mathbb{C}\bar{Z}_\alpha,$$

where Z_α denotes the root vector for the root α, $\bar{Z}_\alpha$ the root vector corresponding to $-\alpha$, and a subscript $\mathbb{C}$ denotes complexification. The elliptic involutive structure is the left-invariant distribution

$$V = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_+} \mathbb{C}\overline{Z}_\alpha.$$

In other words, V is the left-invariant complex distribution of G determined by a Borel subalgebra.

Sine the elliptic involutive structure on G is induced from $G \xrightarrow{\pi} G/T$ being a principal frame bundle over the complex manifold G/T (section 8.1.1) Proposition 12 gives us

$$H^\bullet_V(G; \Lambda^\bullet V^\perp) \simeq H^{\bullet,\bullet}(G/T) \otimes \Lambda^\bullet \mathfrak{t}^*.$$

Assume now that G is simply connected. Then the analysis is very similar to the last section; since $H^2(G;\mathbb{Z}) = 0$ for compact semisimple Lie groups, from Proposition 8 and the above equation, we have that $\mathrm{Pic}_V(G) \simeq H^1_V(G) = \mathfrak{t}^*$ since flag manifolds only have cohomology in Dolbeault bi-degree (p,p). Let L_λ be the rank 1 V-module with V-connection $d_V + \lambda$, $\lambda \in \mathfrak{t}^*$. If λ is integral (i.e. exponentiates to a homomorphism $T \to \mathbb{C}^\times$), then L_λ is the pullback of the holomorphic vector bundle $G \times_{\exp \lambda} \mathbb{C}$. By Theorem 4 we have

$$H^\bullet_V(G; L_\lambda) = H^{0,\bullet}(G/T; G \times_\lambda \mathbb{C}),$$

which can be computed using the Borel-Weil-Bott theorem.

While the holomorphic structure on G/T is stable (i.e. $H^1(T^{1,0}_{G/T}) = 0$) [Bot57] section 14, this is not true for the EIS on G. Indeed, we have a V-module isomorphism $T_{\mathbb{C}}G/V \simeq \pi^* T^{1,0}G/T$ and so by Proposition 12 and Theorem 4 we have

$$H^1_V(G; T_{\mathbb{C}}G/V) \simeq \mathfrak{t}^*_{\mathbb{C}} \otimes H^0(G/T; T^{1,0}_{G/T})$$

and $H^0(G/T; T^{1,0}_{G/T})$ is non-zero (it contains at least $\mathfrak{g}_{\mathbb{C}}$).

8.2.1 Homogeneous G-spaces

We will now describe certain subgroups H such that G/H inherits a G-invariant elliptic involutive structure compatible with that of G's and the map $G \to G/H$.

We recall that if $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form on $\mathfrak{g}$ then $\mathfrak{m}$ is preserved by $\operatorname{Ad} H$ and

$$T(G/H) \simeq G \times_{\operatorname{Ad} H} \mathfrak{m}.$$

A G-invariant complex distribution on G/H is then equivalent to specifying an $\operatorname{Ad} H$-invariant subspace of $\mathfrak{m}_{\mathbb{C}}$.

Proposition 14. *Suppose $H \subset G$ is such that its Lie algebra is given by*

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{s}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_+(\mathfrak{h})} \mathbb{C}\overline{Z}_\alpha \oplus \mathbb{C}Z_\alpha \tag{8.2.1}$$

where $\Delta_+(\mathfrak{h})$ is some subset of Δ_+ and $\mathfrak{s}_{\mathbb{C}} \subset \mathfrak{t}_{\mathbb{C}}$. Let $\Delta_+(\mathfrak{m}) = \Delta_+ \backslash \Delta_+(\mathfrak{h})$ and suppose that $\Delta_+(\mathfrak{m})$ satisfies the following property:

$$\text{if } \alpha, \beta \in \Delta_+(\mathfrak{m}) \text{ and } \alpha + \beta \in \Delta_+, \text{ then } \alpha + \beta \in \Delta_+(\mathfrak{m}). \tag{8.2.2}$$

Then

$$V_0 = (\mathfrak{t}_{\mathbb{C}} \cap \mathfrak{m}_{\mathbb{C}}) \oplus \bigoplus_{\alpha \in \Delta_+(\mathfrak{m})} \mathbb{C}\overline{Z}_\alpha \subset \mathfrak{m}_{\mathbb{C}}.$$

descends to a left-invariant elliptic involutive structure $V_{G/H}$ on G/H.

Proof. As mentioned above, to show that V_0 gives a well-defined distribution we need to show that $\operatorname{Ad} H V_0 \subset V_0$, which is equivalent to $[\mathfrak{h}_{\mathbb{C}}, V_0] \subset V_0$. To show that this distribution is involutive, it is sufficient to show that $[V_0, V_0] \subset \mathfrak{h}_{\mathbb{C}} \oplus V_0$. For the first part, we clearly have $[\mathfrak{s}_{\mathbb{C}}, V_0] \subset V_0$. Now let $\alpha \in \Delta_+(\mathfrak{h})$ and $v \in V_0$, $[\overline{Z}_\alpha, v]$ is in $\mathfrak{m}_{\mathbb{C}}$ and a negative root vector and so much be in V_0. Thus we are left to show that $[Z_\alpha, v] \subset V_0$ for all $\alpha \in \Delta_+(\mathfrak{h}), v \in V_0$. If $v \in \mathfrak{t}_{\mathbb{C}} \cap \mathfrak{m}_{\mathbb{C}}$ then $[Z_\alpha, v] = \alpha(v) Z_\alpha \in \mathfrak{h}$ but $[Z_\alpha, v]$ must also be in $\mathfrak{m}_{\mathbb{C}}$ so that $[Z_\alpha, v] = 0$. Now suppose for contradiction that $\beta \in \Delta_+(\mathfrak{m})$ and $[Z_\alpha, \overline{Z}_\beta] \notin V_0$. Then $[Z_\alpha, \overline{Z}_\beta]$ is a root vector for the root $\alpha - \beta$ and the only way for this to not be in V_0 is if $\alpha - \beta$ is a positive root. Then $\alpha = (\alpha - \beta) + \beta$ is a root that is a sum of elements of $\Delta_+(\mathfrak{m})$. By assumption (8.2.2), we must have $\alpha \in \Delta_+(\mathfrak{m})$, contradicting that $\alpha \in \Delta_+(\mathfrak{h})$. It is also clear that condition (8.2.2) immediately implies that $[V_0, V_0] \subset V_0 \subset V_0 \oplus \mathfrak{h}_{\mathbb{C}}$. $\square$

If $H < G$ is of the above form, then $t_{\mathbb{C}} \cap \mathfrak{m}$ must commute with $\mathfrak{h}$. This is because $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ but every element of $\mathfrak{h}$ is either in $\mathfrak{t}_{\mathbb{C}}$ or a root vector for $\mathfrak{g}$. Then the maximal rank subalgebra $\mathfrak{u} := \mathfrak{h} + \mathfrak{m} \cap \mathfrak{t}$ is isomorphic to $\mathfrak{h} \oplus \mathfrak{m} \cap \mathfrak{t}$ and the proposition shows that the elliptic involutive structure on $U = \exp \mathfrak{u}$ is a complex structure. By [BH58] section 13.5, it follows that U is a centralizer of a torus in G. Conversely, suppose U is a centralizer of a torus with $\mathfrak{u} \simeq \mathfrak{h} \oplus \mathfrak{t}'$ where $\mathfrak{t}'$ is a torus. Then by [BH58], we can find a system of positive roots for $\mathfrak{g}$ such that the positive complementary roots for $\mathfrak{u}$ (which are those for $\mathfrak{h}$) are closed. Thus we have

Proposition 15. *A subgroup $H \subset G$ has the form in Proposition 14 if and only if $U = \exp(\mathfrak{h} \oplus \mathfrak{t} \cap \mathfrak{m})$ is a centralizer of a torus. In this case, the elliptic involutive structure on G/H is induced from the torus bundle $G/H \to G/U$ and the invariant complex structure on G/U.*

8.2.2 Representation theoretic aspects

Such pairs of groups $H \subset G$ give an induction procedure from finite dimensional representations of H to finite dimensional representations of G by taking global sections (or more generally cohomology) of G-equivariant $V_{G/H}$-modules. To do this, we first describe the possible G-invariant V-connections on the principal H-bundle $G \to G/H$. We recall from Definition 9 that a connection is specified by an H-invariant choice of lift of $V_{G/H}$ to TG. If we want this lift to be invariant by the left G action, then this is equivalent to an $\mathrm{Ad}\, H$-invariant subspace of $\mathfrak{g}_{\mathbb{C}}$ whose orthogonal projection onto $\mathfrak{m}_{\mathbb{C}}$ is V_0. Such subspaces are in one to one correspondence with maps $V_0 \to \mathfrak{h}_{\mathbb{C}}$ that intertwine the adjoint action of $\mathfrak{h}$. To summarize, we have

Proposition 16. *The space of G-invariant principal $V_{G/H}$ connections on the H-principal bundle $G \to G/H$ is given by*

$$Hom_H(V_0, \mathfrak{h}_{\mathbb{C}}).$$

Explicitly, the horizontal subspace corresponding to $\varphi \in Hom_H(V_0, \mathfrak{h}_{\mathbb{C}})$ is $\{v+\varphi(v) \mid v \in V_0\} \subset \mathfrak{g}_{\mathbb{C}}$. The connection is flat if

$$\varphi([v, w]) = [v, \varphi(w)] + [\varphi(v), w] + [\varphi(v), \varphi(w)], \quad v, w \in V_0.$$

Remark. *If $\mathfrak{h}$ is simple and $\dim V_0 < \dim \mathfrak{h}$ then the only invariant connection is the trivial one. This is because if $\mathfrak{h}$ is simple then the adjoint representation of $\mathfrak{h}$ is irreducible and so, besides the 0 map, $Hom_H(V_0, \mathfrak{h})$ contains only surjections.*

Definition 15. *Denote by $Hom_H^{flat}(V_0, \mathfrak{h}_{\mathbb{C}})$ the space of flat connections.*

Now we have a map

$$\mathrm{Hom}_H^{flat}(V_0, \mathfrak{h}_{\mathbb{C}}) \times R(H) \to R(G)$$

defined as follows. Let $(E_0, \rho) \in R(H)$ and let $E = G \times_H E_0$ the associated bundle. By Proposition 3, a flat $V_{G/H}$-connection on $G \to G/H$ makes E into a G-invariant $V_{G/H}$-module. Then the cohomology spaces $H^\bullet_{V_{G/H}}(G/H; E)$ form representations of G. Explicitly, we have an identification

$$\Gamma(G/H; E) \simeq \{f : G \to E_0 \mid f(gh) = \rho(h)^{-1} f(g)\}.$$

Given a connection $\omega \in \mathrm{Hom}_H^{flat}(V_0, \mathfrak{h}_{\mathbb{C}})$, the connection on $\Gamma(G/H; E)$ is defined by

$$\nabla^{V_{G/H};E}_{[g,v]} f = v \cdot f + \rho(\omega(v))f, \quad [g,v] \in V_{G/H} \simeq G \times_{Ad} V_0.$$

G acts on $\Omega^\bullet(G/H; E)$ via pullback of differential forms by the left-action of G on G/H and this action commutes with $d_{V_{G/H};E}$. Thus this action descends to a representation of G on $H^\bullet_{V_{G/H}}(G/H; E)$.

We now discuss two classes of examples.

8.2.3 $SU(n) \subset SU(n+1)$

The inclusion $\mathfrak{su}(n) \to \mathfrak{su}(n+1), A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$ satisfies the properties of Proposition 14; taking the maximal torus $\mathfrak{t} \subset \mathfrak{su}(n+1)$ formed by the diagonal elements, we see that $\mathrm{diag}(1, 1, \ldots, 1, -n)$ lies in $\mathfrak{m}_{\mathbb{C}} \cap \mathfrak{t}_{\mathbb{C}}$ and centralizes $\mathfrak{su}(n)$. Then we have

$$\begin{aligned} \Delta_+(\mathfrak{su}(n)) &= \{e^i - e^j \mid 1 \leq i < j \leq n\} \\ \subset \Delta_+(\mathfrak{su}(n+1)) &= \{e^i - e^j \mid 1 \leq i < j \leq n+1\}. \end{aligned}$$

where $e^i \in \mathfrak{t}_{\mathbb{C}}^*$ is the linear form $\mathrm{diag}(h_1, \ldots, h_{n+1}) \mapsto h_i$. Thus $\Delta_+(\mathfrak{m}) = \{e^1 - e^{n+1} : 1 \leq i \leq n\}$, which clearly satisfies the condition (8.2.2).

Of course the homogeneous space is the sphere S^{2n+1} and this involutive elliptic structure agrees with previous one we have defined (that coming from the bundle $S^{2n+1} \to \mathbb{C}P^n$). Now we have $\dim V_0 = n < n^2 - 1 = \dim \mathfrak{h}$ so by Proposition 16 and its proceeding remark, we have only one invariant connection (which is, of course, flat). Thus we have an induction map

$$R(SU(n)) \to R(SU(n+1)).$$

8.2.4 $SU(n) \subset Spin(2n)$

Recall we have the embedding

$$\mathfrak{su}(n) \to \mathfrak{so}(2n), A + iB \mapsto \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right).$$

The maximal torus for $\mathfrak{su}(n)$ given by the diagonal elements maps into the maximal torus of $\mathfrak{so}(2n)$ given by

$$\mathfrak{t}_{\mathbb{C}} = \left(\begin{array}{c|c} 0 & -\mathrm{diag}(h_1, \ldots, h_n) \\ \hline \mathrm{diag}(h_1, \ldots, h_n) & 0 \end{array}\right),$$

with orthogonal complement spanned by

$$\left(\begin{array}{c|c} 0 & -Id \\ \hline Id & 0 \end{array}\right) \in \mathfrak{t}_{\mathbb{C}} \cap \mathfrak{m}_{\mathbb{C}}.$$

This centralizes $\mathfrak{su}(n) \subset \mathfrak{so}(2n)$ and so again Proposition 14 applies. Letting $e^i \in \mathfrak{t}_{\mathbb{C}}^*$ be the map $(h_1, \ldots, h_n) \mapsto h_i$, we have

$$\Delta_+(\mathfrak{su}(n)) = \{e^i - e^j \mid 1 \leq i < j \leq n\}$$
$$\Delta_+(\mathfrak{so}(2n)) = \{e^i + e^j, e^i - e^j \mid 1 \leq i < j \leq n\}$$

so that $\Delta_+(\mathfrak{m}) = \{e^i + e^j \mid 1 \leq i < j \leq n\}$ satisfies condition 8.2.2. As in the previous example, a dimension count shows that there is only one invariant connection on $Spin(2n) \to Spin(2n)/SU(n)$ so that our induction procedure gives a map

$$R(SU(n)) \to R(Spin(2n)).$$

8.3 Projectivizations of V-modules

Recall from Corollary 5 that if $E \to M$ is a rank $k+1$ V-module, then the total spaces of E and $\mathbb{P}(E)$ each have natural elliptic involutive structures. Note that this structure is *not* the pullback structure from the projections from $E, \mathbb{P}(E)$ to M; the EIS we are considering is more refined since we are only looking at the anti-holomorphic directions of the fibers.

Consider the tautological line bundle $L \to \mathbb{P}(E)$, i.e.

$$L = \{(v, l) \in E \times_M \mathbb{P}(E) \mid v \in l\} \subset \pi^* E,$$

where $\pi : \mathbb{P}(E) \to M$ is the projection. We claim that L is a $V_{\mathbb{P}(E)}$ module. Indeed, let $\{U_\alpha \subset M\}$ be a trivializing cover of E with $\mathcal{O}_V^\times$ valued transition functions. Then over U_α, $\mathbb{P}(E)|_{U_\alpha} \simeq U \times \mathbb{C}P^k$ with the product involutive structure. Then if $\{W_j\}$ is a cover of $\mathbb{C}P^k$ on which $\mathcal{O}(-1)$ is trivial and holomorphic, the subspaces of $\mathbb{P}(E)$ corresponding to $U_i \times W_j$ give a cover of $\mathbb{P}(E)$ on which L is trivial and the transition functions have values in $\mathcal{O}^\times_{V_{\mathbb{P}(E)}}$.

Consider now the dual bundle $L^* \to \mathbb{P}(E)$. The higher direct image of this along the fibration $\mathbb{P}(E) \to M$ gives a flat (virtual) vector bundle on M. The vertical cohomology of this at any fiber is just $H^\bullet(\mathbb{C}P^k; \mathcal{O}(1))$, which is non-zero only in degree 0 where it has dimension $k+1$. Thus the direct image is a single flat vector bundle of rank $k+1$ over M. This vector bundle is canonically a V-module and we claim that it is E^*.

Indeed, any section $\alpha \in \Gamma(M; E^*)$ gives rise to an element $\tilde{\alpha} \in \Gamma(\mathbb{P}(E); L^*)$ by

$$\tilde{\alpha}(v, l) = \alpha_{\pi(v)}(v), \quad (v, l) \in L.$$

By restricting to any fiber, these give the usual holomorphic sections of L^* and so we see that $E^* \simeq H^0(\mathbb{P}(E)/M; L^*)$.

We now study the characteristic classes of L^*. Let h be a hermitian metric on E. This determines hermitian metrics on π^*E, L and L^*, which we will also call h. We can use the metric on L to extend $d_{V_{\mathbb{P}(E)}}$ to a normal connection ∇^L. Indeed, for $Z \in T^{1,0}$, ∇_Z is defined by

$$h(\nabla^L_Z v, w) = Z \cdot h(v, w) - h(v, \rfloor d_V w), \quad v, w \in \Gamma(\mathbb{P}(E), L).$$

Thus, when restricted to each fiber ∇ is the Chern connection associated to h.

Let $U \subset M$ be such that $E|_U$ has a parallel local frame $\sigma_0, \ldots, \sigma_k$ and put $h_{ij} = h(\sigma_i, \sigma_j)$. Thus we have an identification

$$U \times \mathbb{C}P^k \xrightarrow{\sim} \mathbb{P}(E)|_U, \quad (x, [z^0 : \cdots : z^k]) \mapsto [z^j \sigma_j(u)].$$

Under this correspondence, consider the open set of $\mathbb{P}(E)$ where $z^0 \neq 0$. Then L is trivialized on this set with section

$$\psi : [\sigma_0 + z^1\sigma_1 + \cdots + z^k\sigma_k] \mapsto \sigma_0 + z^1\sigma_1 + \cdots + z^k\sigma_k.$$

Define $H = h(\psi, \psi) = h_{ij} z^i \bar{z}^j$, where we put $z^0 = 1$. Then with respect to the frame ψ, the connection is just $\nabla^L = d + H^{-1}\partial H$ so that the connection for ∇^{L^*} is $d - H^{-1}\partial H$. The curvature of ∇^{L^*} is then

$$F^{L^*} = -d\partial \log H = -(d_M + \bar{\partial})\partial \log H,$$

whose class in $H^1_{V_{\mathbb{P}(E)}}(\mathbb{P}(E); (T^{1,0})^*)$ represents $\mathrm{at}_1(L^*)$.

For the real cohomology class (section 7.3.2), we have $u_1(L^*) = H^{-1}dH \in H^1_{\pi^* V_{\mathbb{R}}}(\mathbb{P}(E))$.

Along any fiber $\mathbb{C}P^k$, the form $\mathrm{at}_1(L)$ pulls back to the Atiyah class of $\mathcal{O}(-1) \to \mathbb{C}P^k$, which generates $H^\bullet(\mathbb{C}P^k; \Lambda^\bullet T^{1,0}\mathbb{C}P^k)$. Since $V^\perp_{\mathbb{P}(E)}$ restricts to $T^{1,0}\mathbb{C}P^k$ along any fiber, Theorem 4 gives

$$H^\bullet_{V_{\mathbb{P}(E)}}(\mathbb{P}(E); \Lambda^\bullet V^\perp_{\mathbb{P}(E)}) \simeq H^\bullet(M) \otimes \mathrm{span}\{\mathrm{at}_1(L), \mathrm{at}_1(L)^2, \cdots, \mathrm{at}_1(L)^k\}.$$

Part III

Generalized Higgs algebroids

Chapter 9

Basic structure of generalized Higgs algebroids

We will now begin our study of generalized Higgs algebroids. Special cases of modules over such algebroids include Higgs bundles and generalized holomorphic vector bundles. As motivation for the definition, we recall some basic facts about Higgs bundles and generalized geometry.

Following Hitchin, Simpson [Sim92] defined a Higgs bundle over a complex manifold X as a pair (E, θ) where $(E, \bar{\partial}_E) \to X$ is a holomorphic vector bundle and $\theta \in \Omega^{1,0}(X; \operatorname{End} E)$ satisfies

$$[\bar{\partial}_E, \theta] = 0, \theta \wedge \theta = 0$$
$$\Leftrightarrow$$
$$(\bar{\partial}_E + \theta)^2 = 0.$$

In [Blo05] it is shown that Higgs bundles are equivalent to modules over the so-called Higgs Lie algebroid, A_{Higgs}. This is the Lie algebroid which, as a vector bundle, is $T_{\mathbb{C}}X$ but has bracket defined by

$$[v' + v'', w' + w''] = [v'', w''] + p'([v'', w'] + [v', w'']),$$

where $v', w' \in T^{1,0}X, v'', w'' \in T^{0,1}X$ and $p' : T_{\mathbb{C}}X \to T^{1,0}X$ is the projection. The anchor is defined by the projection $T_{\mathbb{C}}X \to T^{0,1}X$. We note that this is an elliptic Lie algebroid with the property that the kernel of the anchor map, $T^{1,0}X$ is abelian (i.e. the bracket is trivial).

Generalized holomorphic vector bundles have a similar description. We first recall that for any smooth manifold M, $TM \oplus T^*M$ carries a natural bilinear form and the Courant bracket [Gua11], which is defined by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \eta - \frac{1}{2} d(i_X \eta - i_Y \xi),$$

for $X, Y \in \Gamma(M; TM), \xi, \eta \in \Omega^1(M)$. A generalized complex structure is then defined to be an isotropic subbundle $E^{0,1}$ of $(TM \oplus T^*M) \otimes \mathbb{C}$ that is closed under the Courant bracket and satisfies $E^{0,1} \cap \overline{E^{0,1}} = 0$. Despite the Courant bracket not defining a Lie bracket on $TM \oplus T^*M$, it does define one on $E^{0,1}$, which makes it into a Lie algebroid with anchor map $E^{0,1} \hookrightarrow T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M \to T_{\mathbb{C}}M$. The algebroid $E^{0,1}$ is elliptic and the kernel of the anchor map is $T^*_{\mathbb{C}}M \cap E^{0,1}$, which is abelian. $E^{0,1}$-modules are called generalized holomorphic vector bundles. An important difference between Higgs algebroids and generalized complex structures is that for A_{Higgs}, the exact sequence of Lie algebroids

$$0 \to \ker A_{Higgs} \to A_{Higgs} \xrightarrow{\rho} A_{Higgs}/\ker \rho.$$

splits.

Based on these examples, we make the following definition.

Definition 16. *A generalized Higgs algebroid is an elliptic complex Lie algebroid A such that $\ker \rho$ is abelian and the sequence*

$$0 \to \ker \rho \to A \to A/\ker \rho \to 0$$

admits a (Lie algebroid) splitting.

A twisted generalized elliptic Higgs algebroids is an elliptic complex Lie algebroid with $\ker \rho$ abelian (but the above sequence may not split).

Remark. *From the perspective of generalized geometry, we do not want the splitting to be part of the data of a (untwisted) generalized Higgs algebroid. This will give us additional algebroid automorphisms, analogous to the B-field actions in generalized geometry.*

We note that elliptic Higgs algebroids together with a splitting are in 1-to-1 correspondence with the data of an elliptic involutive structure V and a V-module E (then the corresponding generalized Higgs algebroid is the abelian extension $E \rtimes V$ of V by E).

Remark. *In the language of [Blo05], this coincides with what would be called a generalized Higgs algebroid associated to a module over an elliptic involutive structure.*

Besides Higgs algebroids and the Lie algebroid determined by a generalized complex structure, another natural example of a generalized Higgs algebroid is the Atiyah algebroid, $at(P) = TP/T$, of a principal complex torus bundle $T \to P \to M$. This fits in the sequence

$$0 \to \mathfrak{t} \to at(P) \to TM \to 0, \tag{9.0.1}$$

where $\mathfrak{t} = Lie(T)$ is the trivial vector bundle over M. This is a transitive Lie algebroid (and so in particular is elliptic) with abelian kernel so that $at(P)$ is a twisted generalized Higgs algebroid.

9.1 Extensions and gerbes

Suppose A is a twisted generalized Higgs algebroid. Then we have, in particular, an involutive elliptic structure $V := \rho(A)$ and a V-module $K := \ker \rho$. To understand the extent to which A is determined by the pair (V, K), we need to understand the theory of abelian Lie algebroid extensions. These are classified by $H^2_V(M; K)$. Given a closed form $c \in \Omega^2_V(M; K)$ we can define an abelian extension A_c of V by K as follows. As a vector bundle, $A_c = K \oplus V$ with anchor inherited from V's. The Lie bracket is given by

$$[(w_1, v_1), (w_2, v_2)] = (\nabla_{v_1} w_2 - \nabla_{v_2} w_1 + c(v_1, v_2), [v_1, v_2]).$$

That this bracket satisfies the Jacobi identity is a consequence of $d_{V;K} c = 0$. Furthermore, the isomorphism class of the extension A_c depends only on the cohomology class of c in $H^2_V(M; K)$. Conversely, given an extension A of V by K we get a class in $H^2_V(M; K)$ by choosing a vector bundle splitting $s : V \to A$ and defining $c_s \in \Omega^2_V(M; K)$ by $c_s(v_1, v_2) = [s(v_1), s(v_2)] - s[v_1, v_2] \in K$. Then one verifies that c_s is closed and its cohomology class is independent of the choice of s. The extension is trivial, i.e. $A \simeq K \rtimes V$, if and only if c is cohomologically trivial.

In the case of an untwisted Higgs algebroid, we are interested in knowing when the splitting is unique. We call two Lie algebroid splittings $s, s' : A/\ker\rho \to A$ isomorphic if they are related by an inner automorphism of A that acts trivially on $\ker\rho$. The space of inner derivations that annihilate $\ker\rho$ is given by $\{\mathrm{ad}_w \mid w \in \ker\rho\}$ [1] so that inner automorphism acting trivially on $\ker\rho$ are given by $\exp^{\mathrm{ad}_w} = 1 + \mathrm{ad}_w$. Now suppose $s_1, s_2 : V \to A$ are two Lie algebroid splittings of $0 \to K \to A \to V \to 0$. Then their difference defines a class in $H^1_V(M; K)$ and the two splittings are isomorphic if and only if this class is trivial. We summarize

Theorem 10. *Given a Lie algebroid V and a V-module K, abelian Lie algebroid extensions of V by K are in one-to-one correspondence with $H^2_V(M; K)$. The split, i.e. semi-direct product extension corresponds to the class $0 \in H^2_V(M; K)$. In this case, the isomorphism classes of splittings are an affine space over $H^1_{A/\ker\rho}(M; \ker\rho)$.*

Since the cohomology of any elliptic involutive structure is locally trivial (chapter 7), we have

Corollary 6. *Any twisted Higgs algebroid has a local splitting.*

Corollary 7. *A twisted generalized Higgs algebroid over M is determined (up to isomorphism) by a triple $(V, K, [c])$, where*

1. *V is an EIS.*

[1] this is because if $v \in \Gamma(M; A)$ with $[v, w] = 0$ for all $w \in \ker\rho$ then $[v, fw] = (\rho(v) \cdot f)w = 0$ for all $w \in \ker\rho, f \in C^\infty(M)$ so that $v \in \ker\rho$.

2. K is a V-module.

3. $[c]$ is a class in $H^2_V(M;K)$, which we call the curvature of the TGHA.

In the case of an Atiyah algebroid (9.0.1), the curvature is in $H^2(M;\mathfrak{t})$ and is represented by the curvature F of a principal connection.

9.1.1 Gerbes

Recall from Corollary 5 that K^* being a module over an elliptic involutive structure implies that the total space $\operatorname{tot} K^*$ has an elliptic involutive structure itself. Following Hitchin [Hit10], the curvature $c \in H^2_V(M;K)$ of a twisted Higgs algebroid determines a class in $H^2(\mathcal{O}^\times_{\operatorname{tot} K^*})$ as follows. Pulling back c via the projection $\pi : K^* \to M$ gives a class $\pi^* c \in H^2_{V_{\operatorname{tot} K^*}}(\operatorname{tot} K^*; \pi^* K)$. But $\pi^* K^* \to \operatorname{tot} K^*$ has a canonical section and pairing this with $\pi^* c$ gives us a class in $H^2_{V_{\operatorname{tot} K^*}}(\operatorname{tot} K^*) \simeq H^2(\mathcal{O}_{V_{\operatorname{tot} K^*}})$. Exponentiating then gives a class in $H^2(\mathcal{O}^\times_{V_{\operatorname{tot} K^*}})$. Just as a class in degree 2 sheaf cohomology with coefficients in the sheaf of locally constant non-zero functions or the sheaf of holomorphic non-zero functions classifies a flat gerbe or holomorphic gerbe, we can think of this class as defining a gerbe relative to the elliptic involutive structure.

We note that the gerbe we have defined is topologically trivial. This is because the underlying topological gerbe is classified by the element in $H^3(\operatorname{tot} K^*; \mathbb{Z})$ coming from the map

$$H^2(\mathcal{O}^\times_{V_{\operatorname{tot} K^*}}) \to H^3(\operatorname{tot} K^*; \mathbb{Z})$$

induced from the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_{V_{\operatorname{tot} K^*}} \to \mathcal{O}_{V_{\operatorname{tot} K^*}} \to 0.$$

But this element is zero since our gerbe was defined by exponentiating an element of $H^2(\mathcal{O}_{V_{\operatorname{tot} K^*}})$.

9.2 B-field actions

Once we choose a splitting $K \rtimes V \simeq A$ of an untwisted Higgs algebroid A, any closed element B of $\Omega^1_V(M;K)$ gives rise to an automorphism φ_B of A via

$$w + v \overset{\varphi_B}{\mapsto} w + i_v B + v, \quad w \in K, v \in V.$$

We call such automorphisms B-field actions since for co-Higgs bundles these correspond to B-field actions in generalized complex geometry. We emphasize that while we need a splitting to define such an action, the automorphism will not preserve the splitting in general. Indeed, the action of the B-field gives a different splitting and by the previous section the new splitting is isomorphic to the original one if and only if the class of B in $H^1_V(M;K)$ is trivial.

Because of Corollary 6 we may consider any twisted generalized Higgs algebroid as being pieced together by untwisted Higgs algebroids and B-fields relating the two algebroids on overlaps. Then we will see that many of the constructions/analyses that are done for untwisted Higgs algebroids can be done for twisted ones by working over the gerbe defined in the previous section. This is done in the case of twisted co-Higgs bundles in [Hit10].

Chapter 10

Modules over generalized Higgs algebroids

We keep the notation of the previous chapter: $A \to M$ is a (possibly twisted) generalized Higgs algebroid, $K = \ker \rho \subset A$ and $V = \rho(A) \subset T_{\mathbb{C}}M$.

10.1 The case of untwisted generalized Higgs algebroids

We first consider the case where we have a splitting $A = K \rtimes V$. Then the following is straightforward

Proposition 17. *Let $A \to T_{\mathbb{C}}M$ be an elliptic generalized Higgs algebroid. Upon choosing a splitting, an A-module is equivalent to the data of a V module (E, ∇^V) together with a "Higgs field": an element $\theta \in \Omega^1_K(M; \operatorname{End} E)$ such that*

$$[\nabla^V, \theta] = 0 \tag{10.1.1}$$

$$\theta \wedge \theta = 0. \tag{10.1.2}$$

Now let $\mathscr{E}^\bullet_V$ denote the sheaf of V-parallel elements of $\Gamma(M; \Lambda^\bullet K^* \otimes E)$. Then eq. (10.1.1) and eq. (10.1.2) imply that we have a complex of sheaves

$$\mathscr{E}^0_V \xrightarrow{\theta\wedge} \mathscr{E}^1_V \xrightarrow{\theta\wedge} \cdots .$$

The following is familiar in the case of Higgs bundles [Sim92].

Proposition 18. *The Lie algebroid cohomology computes the hypercohomology of the above sequence of sheaves:*

$$\mathbb{H}^\bullet(\mathscr{E}^\bullet_V, \theta) = H^\bullet_A(M; E).$$

Proof. By Theorem 6 and eq. (10.1.1), we have a resolution

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^1_{V;E} \arrow[u] \arrow[r, "\theta\wedge"] & \Omega^1_{V;K^*\otimes E} \arrow[u] \arrow[r, "\theta\wedge"] & \Omega^1_{V;\Lambda^2 K^*\otimes E} \arrow[u] \arrow[r, "\theta\wedge"] & \cdots \\
\Omega^1_{V;E} \arrow[u, "d_{V;E}"] \arrow[r, "-\theta\wedge"] & \Omega^1_{V;K^*\otimes E} \arrow[u, "d_{V;K^*\otimes E}"'] \arrow[r, "-\theta\wedge"] & \Omega^1_{V;\Lambda^2 K^*\otimes E} \arrow[u, "d_{V;\Lambda^2 K^*\otimes E}"'] \arrow[r, "-\theta\wedge"] & \cdots \\
\Omega^0_{V;E} \arrow[u, "d_{V;E}"] \arrow[r, "\theta\wedge"] & \Omega^0_{V;K^*\otimes E} \arrow[u, "d_{V;K^*\otimes E}"'] \arrow[r, "\theta\wedge"] & \Omega^0_{V;\Lambda^2 K^*\otimes E} \arrow[u, "d_{V;\Lambda^2 K^*\otimes E}"'] \arrow[r, "\theta\wedge"] & \cdots \\
\mathscr{E}^0_V \arrow[u] \arrow[r, "\theta\wedge"] & \mathscr{E}^1_V \arrow[u] \arrow[r, "\theta\wedge"] & \mathscr{E}^2_V \arrow[u] \arrow[r, "\theta\wedge"] & \cdots \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

is a resolution. Since the sheaves $\Omega_{V;\Lambda^\bullet K^*\otimes E}$ are soft, the hypercohomology of $(\mathscr{E}^\bullet_V, \theta)$ is equal to the cohomology of the total complex of the double complex formed by taking global sections of the resolution. But this is exactly $H^\bullet_A(M;E)$. $\square$

10.1.1 K-valued Higgs bundles

If K is a holomorphic vector bundle over X, we can form the generalized Higgs algebroid $K \rtimes T^{0,1}X$. Then a module over this algebroid is a holomorphic vector bundle $(E, \bar\partial_E) \to X$ together with $\theta \in \Gamma(X; K^* \otimes \operatorname{End} E)$ such that $\theta \wedge \theta = 0$ and $[\bar\partial_E, \theta] = 0$. Such modules are called K-valued Higgs bundles [KOP, Don95]. Of course, when $K = T^{1,0}X$ these are Higgs bundles and when $K = (T^{1,0}X)^*$ these are called co-Higgs bundles [Ray11, Hit10] and are precisely the generalized holomorphic bundles corresponding to the generalized complex structure induced by a regular complex structure.

10.1.2 Gauge transformations and the B-field action on modules

Recall from section 9.2 that any closed element $B \in \Omega^1_V(M;K)$ gives rise to an automorphism φ_B of A. We now consider the action of these elements on the space of A-modules:

$$\Omega^1_{V,cl}(M;K) \ni B : (E, \nabla^{A;E}) \mapsto (E, \varphi_B^* \nabla^{A;E}).$$

Explicitly, writing $\nabla^{A;E} = \nabla^{V;E} + \theta$, we have

$$(\varphi_B^* \nabla^{A;E})_{w+v} = \nabla^{A;E}_{w+i_vB+v} = \nabla^{V;E}_v + \theta(w + i_vB).$$

Thus B leaves the Higgs field unchanged but changes the V-module structure by

$$\nabla^{V;E} \mapsto \nabla^{V;E} + \theta \circ B.$$

It is straightforward to verify

Proposition 19. *B-field transformations commute with gauge transformations.*

We also have the following generalization of Proposition 2 of [Hit10].

Proposition 20. *Suppose the class $[B] \in H^1_V(M;K)$ vanishes, i.e. $B = \nabla^{V;K}w$ for some $w \in \Gamma(M;K)$. Then the action of B on an A-module (E,θ) corresponds to the gauge transformation $\exp(\theta(w)) \in \mathrm{Aut}(E)$.*

Proof. The condition $\theta \wedge \theta = 0$ implies that $\exp(\theta(w))$ preserves the Higgs field. Thus we just need to show that

$$\exp(-\theta(w)) \circ \nabla^V_v \circ \exp(\theta(w)) = \nabla^V_v + \theta(i_vB) = \nabla^V_v + \theta(\nabla^{V;K}_v w) \qquad (10.1.3)$$

for all $v \in V$. Now,

$$\exp(-\theta(w)) \circ \nabla^{V;E}_v \circ \exp(\theta(w)) = \nabla^{V;E}_v + \exp(-\theta(w))\nabla^{V;\mathrm{End}\,E}_v \exp(\theta(w)). \quad (10.1.4)$$

The condition (10.1.1) implies that $\theta(\nabla^{V;K}_v w) = \nabla^{V;\mathrm{End}\,E}_v(\theta(w))$ so that

$$[\nabla^{V;\mathrm{End}\,E}_v \theta(w), \theta(w)] = [\theta(\nabla^{V;K}_v w), \theta(w)] = 0,$$

since $\theta \wedge \theta = 0$. By Leibniz we then have

$$\nabla^{V;\mathrm{End}\,E}_v \exp\theta(w) = \exp(\theta(w))\nabla^{V;\mathrm{End}\,E}_v \theta(w) = \exp(\theta(w))\theta(\nabla^{V;K}_v w),$$

which, together with eq. (10.1.4), establishes eq. (10.1.3). □

10.2 Spectral varieties

The notions of spectral variety and spectral sheaf, used in the theory of Higgs bundles [Sim94, Don95, KOP], carry over to generalized Higgs algebroids.

Let (E, ∇^A) be a rank r A-module with Higgs field $\theta \in \Gamma(M; K^* \otimes \mathrm{End}\,E)$. Let $p : K^* \to M$ denote the projection and consider the pullback bundle $p^*K^* \to \mathrm{tot}\,K^*$ which has a canonical section $\lambda \in \Gamma(\mathrm{tot}\,K^*; p^*K^*)$. We have the section $\lambda \otimes Id - p^*\theta \in \Gamma(\mathrm{tot}\,K^*; p^*(K^* \otimes \mathrm{End}\,E))$.

Definition 17. *The spectral variety, $\mathcal{S}$, is the subspace of* $\operatorname{tot} K^*$ *given by the vanishing of the section*

$$\det(\lambda \otimes Id - p^*\theta) \in \Gamma(\operatorname{tot} K^*; p^* S^r K^*).$$

Generically, $\mathcal{S}$ is an r-fold cover of M.

The spectral variety is the support of a sheaf $\mathcal{L}$, which encodes the eigenspaces of θ, with the property that $p_*\mathcal{L} = E$. Generically, $\mathcal{L}$ is a line bundle.

Since $\mathcal{S}$ is a finite cover of M, it inherits an elliptic involutive structure $V_{\mathcal{S}}$ from the elliptic involutive structure V on M. If the Lie algebroid is split then E is a V-module and $\mathcal{L}$ is a $V_{\mathcal{S}}$-module. In the general twisted case, $\mathcal{L}$ will be a $V_{\mathcal{S}}$-sheaf over the gerbe defined in section 9.1.1 [Hit10].

10.3 Cohomology

Most of the results of [Hit10] on the cohomology of co-Higgs bundles carry over to generalized Higgs bundles. In particular, letting $m = \operatorname{rk} V, k = \operatorname{rk} K$, we have

Proposition 21 (Proposition 6 of [Hit10])**.** *Let $A = K \rtimes V$ be a (split) generalized Higgs algebroid. Suppose (L, θ) is a rank 1 A-module such that the section $\theta \in \Gamma(M; K^*)$ has non-degenerate zero set. Then*

$$H^j_A(M; (L, \theta)) = \begin{cases} 0; & j \neq k \\ H^0_V(\theta^{-1}(0); L \otimes \Lambda^k K^*); & j = k. \end{cases}$$

Theorem 11 (Theorem 7 of [Hit10])**.** *Let $A = K \rtimes V$ be a (split) generalized Higgs algebroid with $m \geq \dim M$. Suppose (E, θ) is an A-module such that the spectral sheaf is a line bundle and the spectral cover $\mathcal{S} \subset \operatorname{tot} K^*$ is smooth with and the zero section $Z \subset \operatorname{tot} K^*$ are in general position. Then*

$$H^j_A(M; (E, \theta)) = \begin{cases} 0; & j \neq k \\ H^0_V(\mathcal{S} \cap Z; \mathcal{L} \otimes \Lambda^k K^*); & j = k. \end{cases}$$

More generally, suppose (E, θ) is a module over a twisted Higgs algebroid such that the spectral sheaf is a line bundle over the gerbe constructed in section 9.1.1. This gerbe is canonically trivial when restricted to $Z \subset \operatorname{tot} K^*$ and so the spectral sheaf determines a regular line bundle $\mathcal{L}$ on $\mathcal{S} \cap Z$ and the above result holds in the twisted case as well (in the co-Higgs case this is Theorem 8 of [Hit10]).

Chapter 11

Higgs bundles

In this section, which will largely be self-contained, we specialize to the case of Higgs bundles over a complex manifold X. Recall that these are pairs (E, θ) where $E \to X$ is a holomorphic vector bundle and $\theta \in \Omega^1(X; \operatorname{End} E)$ satisfies $\theta \wedge \theta = 0$ and $[\bar{\partial}_E, \theta] = 0$. Equivalently, these are modules over the Higgs algebroid

$$A_{Higgs} := T^{1,0}X \rtimes T^{0,1}X.$$

Something that separates the Higgs algebroid from generalized Higgs algebroids is its close relationship with the tangent bundle algebroid. Indeed, we have the family of Lie algebroids A_t defined as follows. As a vector bundle, $A_t = T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ and the bracket and anchor are

$$\begin{aligned} [v' + v'', w' + w'']_t &= p'([v', w''] + [v'', w']) + [v'', w''] \\ &\qquad\qquad + tp''([v', w''] + [v'', w']) + t[v', w'] \\ \rho_t(v' + v'') &= tv' + v'' \end{aligned}$$

where

$$v', w' \in T^{1,0}X, v'', w'' \in T^{0,1}X, p' : T_{\mathbb{C}}X \to T^{1,0}X, p'' : T_{\mathbb{C}}X \to T^{0,1}X.$$

Then

$$A_{Higgs} = A_0, \quad (T_{\mathbb{C}}X, [\cdot,\cdot]) = A_1.$$

Furthermore, all A_t are isomorphic to $A_1 = (T_{\mathbb{C}}X, [\cdot,\cdot])$ for $t \neq 0$ via the map

$$A_1 \to A_t, \quad v' + v'' \mapsto \frac{1}{t}v' + v''.$$

Further, when X is compact Kähler the non-abelian Hodge theorem of Simpson [Sim92] gives an equivalence between the category of certain A_{Higgs}-modules and certain $T_{\mathbb{C}}X$-modules (i.e. flat vector bundles):

Theorem 12 (Non-abelian Hodge theorem [Sim92]). *For X a compact Kähler manifold, there is a 1-to-1 correspondence*

$$\left(\begin{array}{c} \textit{polystable Higgs} \\ \textit{bundles on } X \textit{ with} \\ \textit{vanishing Chern} \\ \textit{classes} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \textit{semi-simple flat} \\ \textit{vector bundles on } X \end{array} \right).$$

To go from a Higgs bundle to a flat bundle requires the existence of a Hermitian-Yang-Mills metric h. The flat connection is then given by

$$\nabla = \nabla_h + \theta + \theta^*,$$

where ∇_h denotes the Chern connection (i.e. the unique connection preserving h and having $(0,1)$ part equal to $\bar{\partial}_E$) and θ^* is defined by

$$h(\theta\psi_1, \psi_2) = h(\psi_1, \theta^*\psi_2), \quad \psi_1, \psi_2 \in \Gamma(X; E).$$

11.1 Characteristic classes of Higgs bundles

Since a Higgs bundle (E, θ) is in particular a holomorphic vector bundle, it has an Atiyah class $\mathrm{At}(E) \in H^{1,1}(X; \operatorname{End} E)$ [Ati57] (see also section 7.3.1). We can incorporate the Higgs field by noting that it defines a class in $H^{1,0}(X; \operatorname{End} E)$. We define the following characteristic classes:

Definition 18. *For $j \geq 0$, let*

$$a_j(E, \theta) = \frac{1}{j!} \left(\frac{i}{2\pi} \right)^j \operatorname{tr}(\mathrm{At}(E)^j \theta) \in H^{j+1,j}(X).$$

Proposition 22. *Suppose*

$$0 \to (E_1, \theta_1) \to (E, \theta) \to (E_2, \theta_2) \to 0$$

is an exact sequence of Higgs bundles. Then we have

$$a_j(E, \theta) = a_j(E_1, \theta_1) + a_j(E_2, \theta_2).$$

for all $j \geq 0$.

Proof. This follows from the proof of Proposition 9(2) and the fact that θ, in terms of a vector bundle splitting $E \simeq E_1 \oplus E_2$, has the form

$$\theta = \begin{pmatrix} \theta_1 & \theta_{2,1} \\ 0 & \theta_2 \end{pmatrix}$$

for some $\theta_{2,1} \in \Omega^{1,0}(X; \operatorname{Hom}(E_2, E_1))$. $\square$

11.1.1 Reznikov's theorem and the nonabelian Hodge theorem

Suppose now X is a compact Kähler manifold. By the non-abelian Hodge theorem (Theorem 12), certain Higgs bundles (E, θ) over X correspond to certain flat vector bundles (E, ∇). Flat vector bundles have characteristic classes $u_\bullet$ lying in $H^{odd}(X; \mathbb{R})$ (see section 7.3.2) and the Hodge decomposition says that we can view the characteristic classes of Higgs bundles, which lie in $H^{\bullet+1,\bullet}(X)$, as elements of $H^{odd}(X; \mathbb{C})$. We will show that the Higgs and flat characteristic classes are equivalent (up to scale) under these correspondences. We first recall a theorem of Reznikov, which says that the flat characteristic classes all vanish in degrees 3 and higher. We will give an independent proof of this using the non-abelian Hodge theorem.

Theorem 13 (Reznikov's theorem/Bloch's conjecture [Rez95, Blo78])**.** *If $E \to X$ is a flat vector bundle with X Kähler, then the classes $u_{2j+1}(E) \in H^{2j+1}(X; \mathbb{R})$ vanish for $j \geq 1$.*

Proof. Because the class u_{2j+1} vanish on short exact sequences, It suffices to show this for simple flat vector bundles, i.e. those whose monodromy representation of $\pi_1 X$ is irreducible. Indeed, if we know the result is true for simple flat vector bundles and E is not simple, then we can find $E' \subset E$ simple so that by Proposition 10 $u_{2j+1}(E) = u_{2j+1}(E/E')$. Then we can repeat this procedure, replacing E/E' with a quotient by a simple flat subbundle until we get something simple.

Thus we assume (E, ∇) is simple and appeal to the non-abelian Hodge theorem, which gives us a metric h and Higgs field $\theta \in \Omega^{0,1}(X; \operatorname{End} E)$ such that $\nabla = \nabla_h + \theta + \theta^*$. The adjoint connection with respect to h is then

$$\nabla^* = \nabla_h - \theta - \theta^*$$

so that

$$\omega(E, h) = \theta + \theta^*.$$

But since $\theta^2 = 0 = (\theta^*)^2$ we have, for $j \geq 1$,

$$\begin{aligned}
-\operatorname{tr}(\omega(E,h)^{2j+1}) &= \operatorname{tr}\left((\theta + \theta^*)^{2j+1}\right) \\
&= \operatorname{tr}([\theta, \theta^*]^j (\theta + \theta^*)) \\
&= \operatorname{tr}((\theta\theta^* \cdots \theta\theta^*)\theta) + \operatorname{tr}((\theta^*\theta \cdots \theta^*\theta)\theta^*) \\
&= \operatorname{tr}(\theta(\theta\theta^* \cdots \theta\theta^*)) + \operatorname{tr}(\theta^*(\theta^*\theta \cdots \theta^*\theta)) \\
&= 0,
\end{aligned}$$

where in the second to last line we used the fact that the trace vanishes on commutators. Thus $u_{2j+1}(E) = 0$. $\square$

We now show that the flat and Higgs characteristic classes correspond under the non-abelian Hodge theorem.

Proposition 23. *Suppose $(E, \theta) \to X$ is a polystable Higgs bundle with vanishing Chern classes over a compact Kähler manifold X and let ∇ denote the corresponding flat connection.*

1. *Using the Hodge decomposition to view $a_1(E, \theta)$ as a class in $H^1(X; \mathbb{C})$, we have*
$$\operatorname{Re} a_1(E, \theta) = \frac{1}{2}(a_1(E, \theta) + \overline{a_1(E, \theta)}) = \frac{1}{2} u_1(E, \nabla).$$
2. $a_{2j+1}(E, \theta) = 0$*, for* $j \geq 1$.

Proof. The proof of 2. is similar to the proof of Theorem 13. The condition on (E, θ) implies that if h denotes the hermitian Yang-Mills metric, then

$$\nabla = \nabla_h + \theta + \theta^*$$

Then
$$0 = (\nabla)^2 = F_h + [\nabla_h, \theta + \theta^*] + [\theta, \theta^*].$$
Since $[\bar{\partial}_E, \theta] = 0$, the $(1, 1)$ part of the above equation gives us

$$F_h = -[\theta, \theta^*].$$

Then for $j \geq 1$,

$$a_j(E, \theta, h) = (-1)^j \operatorname{tr}([\theta, \theta^*]^j \theta) = (-1)^j \operatorname{tr}((\theta\theta^* \cdots \theta\theta^*)\theta) = 0,$$

since the trace vanishes on supercommutators.

For the first part, since ∇_h preserves h we have

$$\nabla^* = \nabla_h - \theta - \theta^*$$

so that

$$\begin{aligned} u_1(E, \nabla, h) &= \frac{1}{2} \operatorname{tr}(\nabla - \nabla^*) = \frac{1}{2} \operatorname{tr}(2(\theta + \theta^*)) \\ &= \operatorname{tr}(\theta + \theta^*) = \operatorname{tr} \theta + \overline{\operatorname{tr}} \theta = a_1(E, \theta, h) + \overline{a_1(E, \theta, h)}. \end{aligned}$$

Since $a_1(E, \theta, h)$ is a $\bar{\partial}$-closed $(1, 0)$ form it is actually harmonic (since we trivially have $\bar{\partial}^* a_1(E, \theta, h)$). Thus taking cohomology and using the Hodge decomposition gives (1). □

11.2 Higher direct image and a secondary Grothendieck-Riemann-Roch theorem

We now want to apply our construction from chapter 4 to study the direct image of Higgs bundles. Thus suppose we have a fibration of complex manifolds $Z \to M \xrightarrow{\pi} B$ and a Higgs bundle $(E, \theta) \to M$. Define $T(M/B) = \ker \pi_*$ and recall the set-up of chapter 4. There we defined a complex of infinite rank bundles, $E^\bullet_{M/B}$, whose fiber over $x \in B$ is the space of E-valued differential forms on the fiber with differential given by the vertical Higgs differential $\bar{\partial}_{M/B;E} + \theta|_{M/B}$. A choice of connection (i.e. horizontal distribution) on the fiber bundle $M \to B$ gives us an identification

$$\Omega^\bullet(B; E^\bullet_{M/B}) \simeq \Omega^\bullet(M; E)$$

and so, from the point of view of $E^\bullet_{M/B} \to B$, $\bar{\partial}_E + \theta$ is a super-Higgs bundle structure of total degree 1, which we denote by $\mathbb{A}$, and whose degree 0 piece is the vertical differential. We will now examine the decomposition $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \cdots$ in the case of an especially nice type of fibration $M \to B$ with connection, which is called a Kähler fibration.

11.2.1 Kähler fibrations

In [BGS88], Bismut, Gillet, and Soule give the definition

Definition 19 (Definition 1.4 in [BGS88]). *The data $(\pi, g^{M/B}, T_H M)$, where $g^{M/B}$ is a metric on $T(M/B)$, is called a Kähler fibration if there exists $\omega \in \Omega^{1,1}(M)$ such that*

1. *ω is closed*
2. *$T_H M$ and $T(M/B)$ are orthogonal with respect to ω.*
3. *For $X, Y \in T(M/B)$, $\omega(X, Y) = g^{M/B}(X, JY)$.*

Note that for a Kähler fibration, each fiber is Kähler with Kähler form given by the restriction of ω.

When viewed as a superconnection on $E^\bullet_{M/B}$, the total exterior derivative d_E on M has the following decomposition [BGV92, BL95]

$$d_E = d_{M/B;E} + \tilde{\nabla}^{M/B;E} + \hat{S} + i_\Omega, \tag{11.2.1}$$

where Ω is the curvature of the connection on $M \to B$, $\tilde{\nabla}^{M/B;E}$ is the connection defined by

$$\tilde{\nabla}^{M/B;E}_v = \nabla^{M;E\otimes\Lambda^\bullet T^*(M/B)}_{v^H}$$

(where v^H is a horizontal lift of v and the connection on $T^*(M/B)$ is the Bismut connection corresponding to $g^{M/B}$), and $\hat{S}: TB \to \mathrm{End}(\Lambda^\bullet T^*(M/B))$ is defined on $T^*(M/B)$ by

$$\langle \hat{S}(v)\mu, X\rangle = -\alpha(\nabla^{M/B}_{v^H} X - [v^H, X]), \quad X \in \Gamma(M; T(M/B)), \mu \in T^*(M/B)$$

and is extended to be a derivation (above $\langle \cdot, \cdot \rangle$ represents the paring of $T^*(M/B)$ and $T(M/B)$). The relationship between $\hat{S}$ and the tensor S in [BGS88] is

$$\langle \hat{S}(v)\mu, X\rangle = g(S(X)\mu^\sharp, v_H), \tag{11.2.2}$$

where $\mu^\# \in T(M/B)$ corresponds to μ via $g^{M/B}$.

Proposition 24. *If $v \in T^{1,0}B$ then $\hat{S}(v)$ vanishes on $\Lambda^{0,1}T^*(M/B)$ and if $v \in T^{0,1}B$ then $\hat{S}(v)$ vanishes on $\Lambda^{1,0}T^*(M/B)$. Indeed we have,*

$$\hat{S}(T^{1,0}B) \subset \mathrm{End}(\Lambda^{1,0}T^*(M/B), \Lambda^{0,1}T^*(M/B))$$
$$\hat{S}(T^{0,1}B) \subset \mathrm{End}(\Lambda^{0,1}T^*(M/B), \Lambda^{1,0}T^*(M/B)).$$

Proof. In [BGS88] it is shown that the tensor $g(S(X)\cdot, \cdot)$ is (1,1) and that if $X \in T^{1,0}M/B, Y \in T^{0,1}M/B$ then $S(X)Y = 0 = S(Y)X$. The proposition then follows from these properties along with (11.2.2) and the fact that

$$g^{M/B} : \begin{cases} T^{1,0}M/B \xrightarrow{\sim} \Lambda^{0,1}T^*(M/B), \\ T^{0,1}M/B \xrightarrow{\sim} \Lambda^{1,0}T^*(M/B). \end{cases}$$

□

Let

$$KS := \hat{S}|_{T^{1,0}B}.$$

Then KS is a Dolbeault representative of the Kodaira-Spencer map $T^{1,0}B \to H^1(M_b, T^{1,0}_{M_b})$ (c.f. lemma 4.3 in [FS90]). Then from (11.2.1) we have the following decomposition of the Dolbeault operator on M,

$$\bar{\partial} = \underbrace{\bar{\partial}_{M/B}}_{(0,1,0,0)} + \underbrace{(\tilde{\nabla}^{M/B})^{0,1}}_{(0,0,0,1)} + \underbrace{KS}_{(-1,1,1,0)} + \underbrace{i_{\Omega^{1,0}}}_{(-1,0,1,1)}. \tag{11.2.3}$$

where the quadruple (a, b, c, d) under the operator means that it maps

$$\Lambda^{p,q}T^*(M/B) \to \Lambda^{p+a,q+b}T^*(M/B) \otimes \Lambda^{c,d}T^*B.$$

More generally, if $E \to M$ is a holomorphic vector bundle with connection ∇^E compatible with $\bar{\partial}_E$, then we may write

$$\bar{\partial}_E = \bar{\partial}_{M/B;E} + (\tilde{\nabla}^{T^*(M/B)\otimes E})^{0,1} + KS + i_{\Omega^{1,0}}. \tag{11.2.4}$$

where $\tilde{\nabla}^{T^*(M/B)\otimes E}$ is the connection on $E^\bullet_{M/B}$ induced by the connection $\nabla^{M/B} \otimes \nabla^E$ on $\Lambda^\bullet T^*(M/B) \otimes E$.

11.2.2 Secondary Grothendieck-Riemann-Roch

In this section we will prove

Theorem 14. *Suppose B is a complex manifold, Y is Kähler and (E,θ) is a Higgs bundle over $B \times Y$. Then*

$$a_k(\operatorname{ind}(\bar\partial_{Y;E} + \theta_Y)) = \int_Y e(TY) a_k(E,\theta), \quad k \geq 0.$$

In keeping with the notation of the previous section, we let $M = B \times Y$ and will write M/B for Y.

Hodge theory

The infinite rank bundle $E^\bullet_{M/B}$ has an L^2 metric induced from the metrics $g^{M/B}$ and h^E. Set $D''_{M/B} = \bar\partial_{M/B;E} + \theta_{M/B}$, the vertical Higgs differential. Then

$$(D''_{M/B})^* = \bar\partial^*_{M/B;E} + i(\theta^*_{M/B})$$

where $\theta = \theta_{M/B} + \theta_B$ with $\theta_{M/B} \in \Gamma(\Lambda^{1,0}T^*(M/B) \otimes \operatorname{End} E), \theta_B \in \Gamma(\Lambda^{1,0}T^*B \otimes \operatorname{End} E)$. Using Hodge theory we have the identification

$$H^{even}_{Dol}(M/B;E) - H^{odd}_{Dol}(M/B;E) \simeq \ker(D''_{M/B} + (D''_{M/B})^*).$$

Letting $P \in \operatorname{End}(E^\bullet_{M/B})$ be the orthogonal projection onto $\ker(\bar\partial_{M/B} + \bar\partial^*_{M/B} + \theta_{M/B} + i(\theta^*_{M/B}))$, the Higgs field on $\ker(D''_{M/B} + (D''_{M/B})^*)$ is $P \circ (\theta_B + KS)$.

Using this correspondence, the metric on $E^\bullet_{M/B}$ restricts to give a metric on $H^{even}_{Dol}(M/B;E) - H^{odd}_{Dol}(M/B;E)$, whose corresponding Chern connection is $P \circ \tilde\nabla^{T^*(M/B)\otimes E} \circ P$.

11.2.3 Index theorem

We will now prove theorem Theorem 14, following the techniques of [BL95] and [BGS88]. Let c denote the Clifford action of $T(M/B)$ on $\Lambda^{0,\bullet}T^*(M/B)$, i.e.

$$c(X) = \sqrt{2}(X^\flat_{1,0} \wedge - i_{X_{0,1}}), \quad X = X_{1,0} + X_{0,1} \in T^{1,0}(M/B) \oplus T^{0,1}(M/B),$$

where $\flat$ denotes the isomorphism $T(M/B) \to T^*(M/B)$ given by $g^{M/B}$. Then

$$\Lambda^\bullet T^*(M/B) \otimes E = \Lambda^{\bullet,0}T^*(M/B) \otimes \Lambda^{0,\bullet}T^*(M/B) \otimes E$$

is a Clifford module with twisting bundle $\Lambda^{\bullet,0}T^*(M/B) \otimes E$.

It is in the following crucial lemma that we use our simplifying hypothesis that M is a product manifold.

Lemma 3. *Let*

$$\mathbb{A}' = \bar{\partial}^*_{M/B;E} + i(\theta^*_{M/B}) + (\tilde{\nabla}^{T^*(M/B)\otimes E})^{1,0}$$
$$\mathbb{A}'' = \bar{\partial}_{M/B;E} + \theta_{M/B} + (\tilde{\nabla}^{T^*(M/B)\otimes E})^{0,1}$$
$$\Theta = \theta_B + KS.$$

Then

$$(\mathbb{A}')^2 = 0 = (\mathbb{A}'')^2$$
$$[\mathbb{A}'', \Theta] = 0.$$

Proof. From eq. (11.2.4) and the fact that $\Omega = 0$ since M is a product, we have

$$0 = (\bar{\partial}_E + \theta)^2 = (\mathbb{A}'' + \Theta)^2.$$

Then decomposing the above using the 4-part type decomposition and eq. (11.2.3) gives $(\mathbb{A}'')^2 = 0$ and $[\mathbb{A}'', \Theta] = 0$. Then $(\mathbb{A}')^2 = 0$ follows, by duality, from $(\mathbb{A}'')^2 = 0$. $\square$

Definition 20. *Let $N_{M/B}$ and N_B be the number operators on $\Lambda^\bullet T^*(M/B)$ and $\Lambda^\bullet T^* B$, respectively, i.e.*

$$N_{M/B}|_{\Lambda^i T^*(M/B)} = i, \quad N_B|_{\Lambda^i T^* B} = i.$$

For $t > 0$, define

$$\mathbb{A}_t = t^{-N_{M/B}} \circ \mathbb{A}' \circ t^{N_{M/B}} + \mathbb{A}''.$$

Proposition 25. *The element* $\operatorname{Str}(\exp(-\mathbb{A}_t^2)\Theta) \in \Omega^\bullet(B)$ *is $\bar{\partial}$-closed, its Dolbeault cohomology class is independent of t, and*

$$\lim_{t\to\infty} \operatorname{Str}\left(\exp(-\mathbb{A}_t^2)\Theta\right) = \operatorname{Str}(\exp(-(P \circ \tilde{\nabla}^{T^*(M/B)\otimes E} \circ P)^2)\Theta)$$
$$= \sum_k \frac{1}{k!} \left(a_k\left(H^{even}_{Dol}(M/B; E|_{M/B})\right) - a_k\left(H^{odd}_{Dol}(M/B; E|_{M/B})\right)\right).$$

Proof. We have

$$\bar{\partial}_B \operatorname{Str}\left(\exp(-\mathbb{A}_t^2)\Theta\right) = \operatorname{Str}\left[\mathbb{A}'', \exp(-\mathbb{A}_t^2)\Theta\right].$$

Now, $\mathbb{A}_t^2 = [\mathbb{A}_t', \mathbb{A}'']$ so that $[\mathbb{A}'', \mathbb{A}_t^2] = 0$ since $(\mathbb{A}'')^2 = 0$. Further, $[\mathbb{A}'', \Theta] = 0$ by Lemma 3 so the above vanishes. To show that the Dolbeaut cohomology class is independent of t, let $\mathbb{F}_t = \mathbb{A}_t^2$. Then by Lemma 3,

$$\mathbb{F}_t = [\mathbb{A}'', t^{-N_{M/B}} \circ \mathbb{A}' \circ t^{N_{M/B}}]$$

so that

$$\frac{\partial}{\partial t}\mathbb{F}_t = \left[\mathbb{A}'', \frac{\partial}{\partial t} t^{-N_{M/B}} \circ \mathbb{A}' \circ t^{N_{M/B}}\right].$$

Now it is easy to see that $\mathbb{B}_t := \frac{\partial}{\partial t} t^{-N_{M/B}} \circ \mathbb{A}' \circ t^{N_{M/B}}$ is $C^\infty(B)$ linear, i.e. lies in $\Omega^\bullet(B; \mathrm{End}\, E^\bullet_{M/B})$. Then, using $\dot{}$ to denote differentiation with respect to t, we have for any integer k

$$\begin{aligned}\frac{\partial}{\partial t}\operatorname{Str}(\mathbb{F}_t^k\Theta) &= \sum_{j=0}^{k}\operatorname{Str}(\mathbb{F}_t^j\dot{\mathbb{F}}_t\mathbb{F}_t^{k-j-1}\Theta) = \sum_{j=0}^{k}\operatorname{Str}(\mathbb{F}_t^j[\mathbb{A}'', \mathbb{B}_t]\mathbb{F}_t^{k-j-1}\Theta) \\ &= \sum_{j=0}^{k}\operatorname{Str}([\mathbb{A}'', \mathbb{F}_t^j\mathbb{B}_t\mathbb{F}_t^{k-j-1}\Theta] = \bar{\partial}\sum_{j=0}^{k}\operatorname{Str}(\mathbb{F}_t^j\mathbb{B}_t\mathbb{F}_t^{k-j-1}\Theta),\end{aligned}$$

where we have used the fact that $\mathbb{F}_t$ and Θ (super)commute with $\mathbb{A}''$. This shows that the cohomology class of $\operatorname{Str}(\exp(-\mathbb{A}_t^2)\Theta)$ is independent of t.

For the last part, we have

$$\begin{aligned}\operatorname{Str}\left(\exp(-\mathbb{A}_t^2)\Theta\right) &= \operatorname{Str}\left(t^{N_{M/B}/2}\exp(-\mathbb{A}_t^2)\Theta t^{-N_{M/B}/2}\right) \\ &= \operatorname{Str}\left(\exp(-(t^{N_{M/B}/2}\mathbb{A}_t t^{-N_{M/B}/2})^2)\Theta\right).\end{aligned}$$

The superconnection

$$t^{N_{M/B}/2}\mathbb{A}_t t^{-N_{M/B}/2} = \sqrt{t}(D''_{M/B} + (D''_{M/B})^*) + \tilde{\nabla}^{T^*(M/B)\otimes E}$$

is adapted to the Dirac operator $D''_{M/B} + (D''_{M/B})^*$ and so by Theorem 9.19 of [BGV92]

$$\lim_{t\to\infty}\exp(-(t^{N_{M/B}/2}\mathbb{A}_t t^{-N_{M/B}/2})^2) = \exp(-(P\circ\tilde{\nabla}^{T^*(M/B)\otimes E}\circ P)^2).$$

□

Thus, as in the heat kernel proof of the families index theorem, we will investigate the limit of $\exp(-\mathbb{A}_t\Theta)$ as $t \to 0$. Define

$$\mathbb{A} = \mathbb{A}' + \mathbb{A}'' = \bar{\partial}_{M/B;E} + \bar{\partial}^*_{M/B;E} + \theta_{M/B} + i(\theta^*_{M/B}) + \tilde{\nabla}^{T^*(M/B)\otimes E}.$$

Modulo the degree 0 part $\theta_{M/B} + i(\theta^*_{M/B})$, this is the Bismut superconnection for the Dirac operator $\bar{\partial}_{M/B;E} + \bar{\partial}^*_{M/B;E}$.

We have

$$\begin{aligned}\operatorname{Str}\left(\exp(-\mathbb{A}_t)\Theta\right) &= \exp(-(t^{N_{M/B}/2}\mathbb{A}_t t^{-N_{M/B}/2})^2) \\ &= \operatorname{Str}\left(\exp(-tt^{\frac{-N_B}{2}}\mathbb{A}^2 t^{\frac{N_B}{2}})\Theta\right) \\ &= \operatorname{Str}\left(\exp(-tt^{\frac{-N_B}{2}}\mathbb{A}^2 t^{\frac{N_B}{2}})\sqrt{t}t^{\frac{-N_B}{2}}\Theta t^{\frac{N_B}{2}}\right) \\ &= t^{\frac{-N_B}{2}}\operatorname{Str}(\exp(-t\mathbb{A}^2)\sqrt{t}\Theta).\end{aligned}$$

We will now follow the techniques and notation of [BGV92]. Let $\tilde{k}_t(\cdot,\cdot)$ be the kernel of the operator $e^{-t\mathbb{A}^2}\Theta$ and $k_t(\cdot,\cdot)$ the kernel of $e^{-t\mathbb{A}^2}$. Then since Θ has order zero, $\tilde{k}_t(x,y) = \sqrt{t}k_t(x,y)\Theta(y)$. Writing $k_t(x), \tilde{k}_t(x)$ for $k_t(x,x,), \tilde{k}_t(x,x)$, we have an asymptotic expansion [BGV92] on the diagonal

$$k_t(x) = (4\pi t)^{-n/2}\sum_{i=0}^{\infty} t^i k_i(x)$$
$$\Rightarrow \tilde{k}_t(x) = (4\pi)^{-n/2}t^{-(n-1)/2}\sum_{i=0}^{\infty} t^i k_i(x)\Theta.$$

with

$$k_i \in \Gamma(M; T^*B \otimes \mathrm{End}(\Lambda^\bullet T^*(M/B) \otimes E)).$$

Now,

$$\begin{aligned}\mathrm{End}(\Lambda^\bullet T^*(M/B)) &\simeq \mathrm{End}(\Lambda^{0,\bullet}T^*(M/B)) \otimes \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B)) \\ \simeq C(T(M/B)) \otimes \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B)) &\simeq \Lambda^\bullet T^*(M/B) \otimes \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B)).\end{aligned}$$

Thus we may also view k_i as being in $\mathbb{A}^\bullet(M; \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B) \otimes E))$.

Let $\mathrm{Str}_{\Lambda^\bullet T^*(M/B)\otimes E}$ denote the supertrace of an element in $\mathrm{End}(\Lambda^\bullet T^*(M/B))$ and let $\mathrm{Str}_{\Lambda^{\bullet,0}T^*(M/B)\otimes E}$ denote the supertrace of an element in $\mathrm{End}(\Lambda^{\bullet,0}T^*(M/B)\otimes E)$. For $a \in \mathrm{End}(\Lambda^\bullet T^*(M/B))$ write $a_{[p,n]} \in \Lambda^p T^*B \otimes \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B) \otimes E)$ for the projection of a, under the correspondence above, onto the space $\Lambda^p T^*B \otimes \Lambda^n T^*(M/B) \otimes \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B) \otimes E)$ followed by the Berezin integral. In other words, $a_{[p,n]}$ is the coefficient of a on $d\,\mathrm{vol}_{M/B}$. Then we have [BGV92]

$$\mathrm{Str}_{\Lambda^\bullet T^*(M/B)\otimes E}\, a = (-2i)^{n/2}\sum_p \mathrm{Str}_{\Lambda^{\bullet,0}T^*(M/B)\otimes E}\, a_{[p,n]}$$

and

$$\begin{aligned}\mathrm{Str}(\exp(-\mathbb{A}_t^2)\Theta) &= \int_{M/B} t^{\frac{-N_B}{2}}\,\mathrm{Str}_{\Lambda^\bullet T^*(M/B)\otimes E}\,\tilde{k}_t(x)d\,\mathrm{vol}_{M/B} \\ &= (4\pi)^{-n/2}\int_{M/B}\sum_{i=0}^{\infty} t^{-\frac{N_B+n-1}{2}+i}\,\mathrm{Str}_{\Lambda^\bullet T^*(M/B)\otimes E}(k_i(x)\Theta) \\ &= (2\pi i)^{-n/2}\int_{M/B}\sum_{i,j=0}^{\infty} t^{-\frac{j+n-1}{2}+i}\,\mathrm{Str}_{\Lambda^{\bullet,0}T^*(M/B)\otimes E}(k_i(x)\Theta)_{[j,n]} \\ &= (2\pi i)^{-n/2}\int_{M/B}\sum_{i,j=0}^{\infty} t^{-\frac{j+n-1}{2}+i}\,\mathrm{Str}_{\Lambda^{\bullet,0}T^*(M/B)\otimes E}(k_i(x)_{[j-1,n]}\Theta).\end{aligned}$$

For this to have a limit as $t \to 0$, we need

$$\mathrm{Str}(k_i(x)_{[j-1,n]}\Theta) = 0, \quad \text{if } n + (j-i) > 2i.$$

Assuming this holds, we then have

$$\lim_{t\to 0} \mathrm{Str}(\exp(-\mathbb{A}_t^2)\Theta) = (4\pi)^{-n/2} \int_{M/B} \sum_i \mathrm{Str}(k_i(x)_{[2i-n,n]}\Theta).$$

Theorem 14 will then follow from Proposition 25 and the following lemma.

Lemma 4. *We have*

$$\mathrm{Str}(k_i(x)_{[j-1,n]}\Theta) = 0, \quad \textit{if } n + (j-i) > 2i$$

and

$$\sum_i \mathrm{Str}(k_i(x)_{[2i-n,n]}\Theta) = e(TM)\,\mathrm{tr}(e^F\theta_B).$$

Proof. We follow the rescaling argument of [BGV92]. Fix $x_0 \in M$ and use normal coordinates so that $x_0 = 0$. Introduce the rescaling operator δ_u with

$$(\delta_u a)(x,t) = u^{-i/2} a(u^{1/2}x, ut)$$

for

$$a(x,t) \in \Lambda^i T^*M \otimes \mathrm{End}(\Lambda^{\bullet,0}T^*(M/B)) \subset \mathrm{End}(\Lambda^{\bullet}T^*(M/B)).$$

Let

$$r(u,t,x) = u^{n/2}(\delta_u k)(t,x)$$

be the rescaled heat kernel. Then we have

$$r(u,1,0) = (4\pi)^{-n/2} \sum_{i,j} u^{i-j/2} k_i(x_0)_{[j]},$$

where the subscript $[j]$ denotes the part that has differential form degree j. Thus the lemma will be proved by showing that $\lim_{u\to 0} r(u,1,0)$ exists and is equal to $e(TM)\,\mathrm{tr}(e^F\theta_B)$.

Now $r(u,t,x)$ satisfies

$$\left(\frac{\partial}{\partial t} + u\delta_u \mathbb{A}^2 \delta_u^{-1}\right) r(u,t,x) = 0$$

and $\lim_{u\to 0} r(u,t,x)$ is determined by $\lim_{u\to 0} u\delta_u\mathbb{A}^2\delta_u^{-1}$. Letting $\mathbb{A}_0 = \bar\partial_{M/B;E} + \bar\partial^*_{M/B;E} + \tilde\nabla^{T^*(M/B)\otimes E}$, we have

$$\mathbb{A}^2 = \mathbb{A}_0^2 + \theta_{M/B} i(\theta^*_{M/B}) + i(\theta^*_{M/B})\varepsilon(\theta_{M/B}) + \tilde\nabla\theta_{M/B} + \tilde\nabla i(\theta^*_{M/B}).$$

Now $\theta_{M/B}$ and $i(\theta^*_{M/B})$ are of degree 0 since they lie in $\mathrm{End}(\Lambda^{\bullet,0}T^*M/B)$ and $\tilde\nabla\theta_{M/B}, \tilde\nabla i(\theta^*_{M/B})$ are order 1. Thus

$$\begin{aligned} u\delta_u\mathbb{A}^2\delta_u^{-1} = {} & u\delta_u\mathbb{A}_0^2\delta_u^{-1} + u\theta_{M/B} i(\theta^*_{M/B}) + u i(\theta^*_{M/B})\varepsilon(\theta_{M/B}) \\ & + \sqrt{u}\tilde\nabla\theta_{M/B} + \sqrt{u}\tilde\nabla i(\theta^*_{M/B}), \end{aligned}$$

so that $\lim_{u\to 0} u\delta_u \mathbb{A}^2 \delta_u^{-1} = \lim_{u\to 0} u\delta_u \mathbb{A}_0^2 \delta_u^{-1}$. Thus we may use the superconnection $\mathbb{A}_0$ instead of $\mathbb{A}$ for computing $\lim_{u\to 0} r(u,1,0)$.

The lemma then follows from the standard result on the index density for the Dirac operator $\bar{\partial}_{M/B;E} + \bar{\partial}^*_{M/B;E}$ [BGV92] and the fact that the term KS does not contribute to the trace since it is the only endomorphism appearing that does not act as a degree zero operator on $\Lambda^{\bullet,0}T^*M/B$. $\square$

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