

ON THE CLASSIFICATION OF IRREGULAR DIHEDRAL
BRANCHED COVERS OF FOUR-MANIFOLDS

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— *deep in whose reeds great elephants decay,*
I, an island, sail, and my shoes toss —

Ted Berrigan

for Julius Shaneson

ABSTRACT
ON THE CLASSIFICATION OF IRREGULAR DIHEDRAL BRANCHED
COVERS OF FOUR-MANIFOLDS

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I prove a necessary condition for a four-manifold Y to be homeomorphic to a p -fold irregular dihedral branched cover of a given four-manifold X , with a fixed branching set B . The branching sets considered are closed oriented surfaces embedded locally flatly in X except at one point with a specified cone singularity. The necessary condition obtained is on the rank and signature of the intersection form of Y and is given in terms of the rank and signature of the intersection form of X , the self-intersection number of B in X and classical-type invariants of the singularity.

Secondly, I show that, for an infinite class of singularities, the necessary condition is sharp. That is, if the singularity is a two-bridge slice knot, every pair of values of the rank and signature of the intersection form which the necessary condition allows is in fact realized by a manifold dihedral cover.

In a slightly more general take on this problem, for an infinite class of simply-connected four-manifolds X and any odd square-free integer $p > 1$, I give two constructions of infinite families of p -fold irregular branched covers of X . The first construction produces simply-connected manifolds as the covering spaces, while the second produces simply-connected stratified spaces with one singular stratum. The branching sets in the first of these constructions have two singularities of the

same type. In the second construction, there is one singularity, whose type is the connected sum of a knot with itself.

Contents

1	Introduction	1
2	Background	5
2.1	Basic definitions	5
2.2	Knot Theory and preliminary results	8
2.3	Brief note on notation	16
3	Irregular Dihedral Branched Covers of Four-Manifolds	17
3.1	Necessary condition for the existence of a dihedral cover	17
3.2	Sufficient condition in the case of two-bridge slice singularities	38
3.3	Construction for other singularity types	47
3.4	A family of knot invariants	56
4	Further Questions	62
	Appendix	64
	Bibliography	67

Chapter 1

Introduction

The study of branched covers of S^n dates back to the 1920s when Alexander discovered their astonishing generality: he proved that every closed orientable PL n -manifold is a PL branched cover of S^n [1]. Since this seminal work, the classification of branched covers has been an active area of research – see, for example, [2], [12], [22].

A typical question of interest is to find the minimum number of sheets, or the least complex, according to some criterion, branching set needed to realize all manifolds in a given dimension as covers of the sphere. So far, the answers are known only in dimension three: by a well-known result ([10], [15]), every closed oriented three-manifold is a *three*-fold irregular cover of S^3 branched over a *knot*.

Intuitively, a knot in S^3 is the “simplest possible” branching set over a three-dimensional base (by opposition to, say, a link or a self-intersecting curve). There-

fore, we can restate the result about covers of S^3 in the following way: allowing a complicated branching set does not enrich the family of branched covers of the three-sphere. The situation in dimension four is considerably more subtle. A four-manifold Y which can be realized as a branched cover of S^4 with branching set a locally flat oriented submanifold must have signature equal to zero (see [23]). On the other hand, by a result of Montesions [16], *every* closed oriented PL four-manifold is a four-fold cover of S^4 branched over an immersed PL surface. The middle ground between these two results remains poorly understood. For example, what manifolds can be realized as covers of S^4 branched over an embedded, but not necessarily locally flat, oriented surface of a given genus? More specifically, what singularities does the branching set need to have in order to realize a given manifold Y as a cover of S^4 ? These are among the motivating questions of this thesis.

However, the work presented here is not restricted to covers of the sphere. Given any simply-connected closed oriented topological four-manifold X , we ask: which closed oriented topological four-manifolds Y are homeomorphic to branched covers of X , and with what singularities on the branching set? In Chapter 3 we prove a necessary condition for a four-manifold Y to be homeomorphic to a p -fold irregular dihedral branched cover of the pair (X, B) , where X is a closed oriented topological four-manifold, and the branching set $B \subset X$ is an oriented surface embedded locally flatly in X except at finitely many points with specified singularity types. In a subsequent section, I show that, under some additional assumptions on the

singularity type, the necessary condition is sufficient as well.

The sufficient condition just described is obtained via a procedure for constructing an irregular branched cover with a specified slice knot singularity. Namely, fix an odd square-free integer p , a closed oriented four-manifold X and a surface B embedded locally flatly in X . The first step is to embed in X a surface B_1 , homeomorphic to B , so that the embedding $B_1 \subset X$ admits a slice singularity. The second step is to construct an irregular p -fold cover of X branched over B_1 . Third, I give a procedure for obtaining an infinite family of covers by modifying the branching set away from the singularity. If, in addition, X is simply-connected and $\pi_1(X - B, x_0) = \mathbb{Z}/2\mathbb{Z}$, the covers obtained by this method are simply-connected manifolds. A consequence of this construction is that any slice knot which itself admits an irregular p -fold dihedral cover can be realized as the unique singularity of a branched cover between four-manifolds. Whether the same is true for all knots remains an interesting open question.

In Section 3.3, I describe two more procedures for constructing infinite families of irregular p -fold branched covers over singular branching sets. These procedures admit a more general type of singularities but work over a more restricted set of four-manifolds X . The first procedure yields a manifold cover; the branching set has two components. One of the components is locally flat, and the other has two singularities of the same type, α , which is not necessarily slice. The second procedure produces a stratified space as a cover, and the branching set has two

components: one is again locally flat, and the other has one singularity of composite type, $\alpha\#\alpha$. In both cases, the assumption that α is slice is not needed.

In Section 3.4 I prove that the correction term to the signature of a branched cover which arises from the presence of a singularity α is an invariant of the knot type α . I also prove that this invariant is additive with respect to knot connected sum. One summand in the formula for this invariant is expressed in terms of linking numbers in a branched cover of α . An algorithm for computing linking numbers in a branched cover of S^3 is outlined in the Appendix. It constitutes a minor generalization of the algorithm presented in Perko's Thesis [18].

Chapter 2

Background

2.1 Basic definitions

Let X and Y be topological manifolds of the same dimension $m \geq 2$. In this text a *branched cover* $f : Y \rightarrow X$ will mean a finite-to-one surjective map which is a local homeomorphism over the complement of a codimension-two subcomplex B of X . We call B the *branching set* of f and say that f is a cover of X branched over B or simply that f covers the pair (X, B) . The restriction

$$f|_{f^{-1}(X-B)} : f^{-1}(X-B) \rightarrow X-B$$

is the *associated unbranched cover* of f . The *degree* of a branched cover is the degree of its associated unbranched cover.

When working in the smooth category, one naturally adopts a more restrictive definition of a branched cover. That is, one requires that B be a smooth submanifold

of X . In addition, if $N(B)$ is a closed tubular neighborhood of B , then for every connected component N_i of $f^{-1}(N(B))$ we require that there be an integer r such that the restriction $f|_{N_i} : N_i \rightarrow N(B)$ is a bundle map which on every (two-dimensional) fiber is the canonical n -fold cover of the punctured disk to itself. We say that n is the *branching index* of f at N . The branched covers considered in this paper will admit such a parametrization locally, except at finitely many points on the branching set, which we will call *singular points* or simply *singularities*. The type of singularities we allow are described below.

Definition 2.1.1. Let X be a topological four-manifold and let B be a closed surface embedded in X . Let $\alpha \subset S^3$ be a non-trivial knot, and let $z \in B$ be a point. Assume there exist a small open disk D_z about z in X such that there is a homeomorphism of pairs $(D_z - z, B - z) \cong (S^3 \times (0, 1), \alpha \times (0, 1))$. Then, we say that the embedding of B in X has a singularity of *type* α at z .

We will consider covers whose branching sets are closed oriented surfaces, locally flat except for finitely many singularities of the above type. In this scenario a branched cover f over (X^4, B^2) has the following description: for any $b \in B$ which is locally flat, a parametrization of f as in the smooth case exists in a neighborhood of b . For $z \in B$ a singularity of type α and D_z as in Definition 2.1.1, over D_z the map f is the cone on a cover of S^3 branched along the knot α .

If there exists a degree p branched cover $f : Y \rightarrow X$ with branching set a topologically flat, possibly disconnected, oriented submanifold B of X , a formula

of Hirzebruch's [11] generalized by Viro [23] gives:

$$\sigma(Y) = p\sigma(X) - \sum_{r=2}^p \frac{r^2 - 1}{3} e(B_r). \quad (2.1.2)$$

Here σ denotes the signature of a four-manifold, $B_r \subset Y$ is the union of components of $f^{-1}(B)$ of index r , and $e(B_r)$ is the normal Euler number of the embedding of B_r into Y . (Of course, there is also a version of this formula in which $e(B_r)$ is replaced by $\frac{1}{r}e(f(B_r))$, where $e(f(B_r))$ is the normal Euler number of the image of B_r in X .) We wish to compute the effect on this formula which of introducing a singular point to branching set. The answer to this question for a broad class of covers is the object of Theorem 3.1.1.

We will be concerned primarily with the following two types of branched covers.

Definition 2.1.3. Let $f : Y \rightarrow X$ be a branched cover of topological manifolds with branching set B . If the associated unbranched cover of f arises from a surjective homomorphism $\phi : \pi_1(X - B, x_0) \rightarrow \mathbb{Z}/p\mathbb{Z}$, we say f is a *cyclic p -fold* branched cover.

Definition 2.1.4. Let $f : Y \rightarrow X$ be a branched cover of topological manifolds with branching set B , and let p be an odd integer greater than 1. Let $\phi : \pi_1(X - B, x_0) \rightarrow D_{2p}$ be a surjective homomorphism, where D_{2p} is the dihedral group of order $2p$. If the associated unbranched cover of f corresponds to $\phi^{-1}(\mathbb{Z}/2\mathbb{Z})$ under the classification of covering spaces of $X - B$, we say that f is an *irregular p -fold dihedral* cover of X branched along B . For $z \in B$ a singularity, $f^{-1}(z)$ consists of a single point.

It is helpful to give a description of the pre-image of a point on the branching set in an irregular dihedral cover. For every locally flat point $b \in B$ the pre-image $f^{-1}(D_b)$ of a small neighborhood D_b of b in X contains $\frac{p-1}{2}$ components of branching index 2 and one component of branching index 1.

We say a pair (X, B) with $B^{n-2} \subset X^n$ admits a p -fold irregular dihedral (or cyclic) cover if there exists such a cover over X whose branching set is B . When the base manifold X is understood – primarily, when $X = S^3$ and $B \subset X$ is a knot – we may simply say that the knot type of B admits a p -fold irregular dihedral cover.

2.2 Knot Theory and preliminary results

In this section we review the knot-theoretic concepts that arise in our work, and prove a number of Lemmas which we will need later. Since the essential questions of this thesis are not addressed until the next chapter, it would not be unreasonable for the reader to skim or skip this section and refer back to it as its relevance to the subsequent results becomes clear.

Definition 2.2.1. A knot $K \subset S^3 = \partial B^4$ is called *slice* if there exists a properly embedded smooth two-disk $D \subset B^4$ with $\partial D = K$.

Definition 2.2.2. A knot $K \subset S^3 = \partial B^4$ is called *ribbon* if there exists an immersion ψ of a two-disk D into S^3 such that $\psi(\partial D) = K$, all singularities of $\psi(D)$ are simple arcs ι_1, \dots, ι_s , and for all j , $1 \leq j \leq s$, $\psi^{-1}(\iota_j) = \iota'_j \amalg \iota''_j$ such that $\iota'_j \subset D^\circ$

and $\partial v_j'' \subset \partial D$. We say that D is a *ribbon disk* for K .

All ribbon knots are slice: the interior of a ribbon disk can be pushed into the interior of B^4 without self-intersections, producing a slice disk. The converse is an old conjecture of Fox [7] which has been proved for two-bridge knots by Paolo Lisca [14].

The following property of ribbon disks will be very useful to us in constructing simply-connected four-manifolds as branched covers.

Lemma 2.2.3. *Let $K \subset S^3 = \partial B^4$ and let $D' \subset S^3$ be a ribbon disk for K . Then, there exists $D \subset B^4$, a slice disk for K , such that the map $i_* : (\pi_1(S^3 - K), x_0) \rightarrow (\pi_1(B^4 - D), x_0)$ induced by inclusion is surjective.*

Proof. The key is that we pushing the interior of D' into the interior of B^4 in such a way that the resulting slice disk D admits a Morse function g whose critical points are only saddles and minima. Computing the fundamental group of the complement of D in B^4 by cross-sections (see [6]), we start with $\pi_1(\partial B^4 - \partial D, x_0) = \pi_1(S^3 - K, x_0)$ and proceed to introduce new generators or relations at each critical point of g . Since g has no maxima, no new generators are introduced, implying that $i_* : \pi_1(S^3 - K, x_0) \rightarrow \pi_1(B^4 - D, x_0)$ is a surjection.

□

Proposition 2.2.4. *Let $K \subset S^3 \subset \partial B^4$ be a slice knot and let $D \subset B^4$ be a slice disk for K . Let $p > 1$ be an odd square-free integer. If the pair (S^3, K)*

admits an irregular p -fold dihedral cover, then the pair (B^4, D) admits one as well. Furthermore, if K is a two-bridge knot, D can be chosen in such a way that the irregular dihedral cover of B^4 branched along D is simply-connected.

Proof. Let $\Delta_K(t)$ denote the Alexander polynomial of K and $\Delta_D(t)$ that of D . Denote by \hat{S} the double branched cover of the pair (S^3, K) and by \hat{K} the pre-image of K under the covering map. Then, $|\Delta_K(-1)| = |(H_1(\hat{S}; \mathbb{Z})|$. Similarly, denote by \hat{B} the double cover of B^4 branched along D and \hat{D} is the pre-image of D . Again we have, $|\Delta_D(-1)| = |(H_1(\hat{B}; \mathbb{Z})|$.

Since K admits a dihedral cover, $H_1(\hat{S}; \mathbb{Z})$ has $\mathbb{Z}/p\mathbb{Z}$ as a subgroup. It follows that $\Delta_K(-1) \equiv 0 \pmod{p}$. Since D is a slice disk for K , by results of Fox and Milnor [8] we have $\Delta_K(-1) = \pm(\Delta_D(-1))^2$, so $(\Delta_D(-1))^2 \equiv 0 \pmod{p}$. Since p is square-free, we conclude that $\Delta_D(-1) \equiv 0 \pmod{p}$ as well. Then $H_1(\hat{B}; \mathbb{Z})$ surjects onto $\mathbb{Z}/p\mathbb{Z}$, and therefore \hat{D} admits a p -fold cyclic cover T with $\partial T = N$. This cover T is a *regular* dihedral $2p$ -fold branched cover of (B^4, D) . Let Z be the quotient of T by the action of any $\mathbb{Z}/2\mathbb{Z}$ subgroup of D_{2p} . Then Z is the desired irregular dihedral p -fold cover of (B^4, D) . Its boundary, which we denote by U , is an irregular dihedral p -fold cover of K .

So far we have shown that a dihedral presentation of the group of a slice knot extends to a dihedral presentation of the complement of a slice disk in B^4 . Now assume in addition that K is a two-bridge knot. In this case it is well-known that U is in fact S^3 . Indeed, the pre-image S^* of a bridge sphere for K is a dihedral

cover of S^2 branched over four points, so S^* has Euler characteristic

$$\chi(S^*) = p(\chi(S^2) - 4) + 4\frac{p+1}{2} = 2,$$

producing a genus-zero Heegard splitting for U . Therefore, $U \cong S^3$.

Since K is two-bridge slice, it is ribbon. Hence, by Lemma 2.2.3, the slice disk D for K can be chosen so that $0 = \pi_1(U, x_0) \xrightarrow{i_*} \pi_1(Z, x_0)$ is a surjection. Therefore, given a map $\psi : \pi_1(B^4 - D, x_0) \rightarrow D_{2p}$, the pre-image $(\psi \circ i_*)^{-1}(\mathbb{Z}/2\mathbb{Z})$ surjects onto $\psi^{-1}(\mathbb{Z}/2\mathbb{Z})$. This implies that the inclusion of the unbranched cover associated to U into the unbranched cover associated to Z induces a surjection on fundamental groups. Since the branching set of U is a subset of the branching set of Z , it follows that $0 = \pi_1(U, x_0) \twoheadrightarrow \pi_1(Z, x_0)$ is a surjection. We conclude that the irregular dihedral cover of the pair (B^4, D) is simply-connected, as desired. \square

Definition 2.2.5. Let $\alpha \subset S^3$ and $\beta \subset S^3$ be two knot types. We say that β is a *mod p characteristic knot* for α if there exists a Seifert surface V for α with Seifert pairing L_V such that $\beta \subset V^\circ \subset S^3$ represents a non-zero primitive class in $H_1(V; \mathbb{Z})$ and $L_V(\beta, \omega) + L_V(\omega, \beta) \equiv 0 \pmod{p}$ for all $\omega \in H_1(V; \mathbb{Z})$.

In [4] Cappell and Shaneson defined characteristic knots and proved that for p an odd prime and α a non-trivial knot, α admits an irregular dihedral p -fold cover if and only if there exists a mod p characteristic knot for α , which in turn exists if and only if p divides $|\det(L_V + L_V^T)|$. In this section, we exhibit a family of pairs (α, β) , where α is a two-bridge slice knot and β is a $(2, 2n + 1)$ -torus knot which can be realized as mod 3 characteristic knot for α .

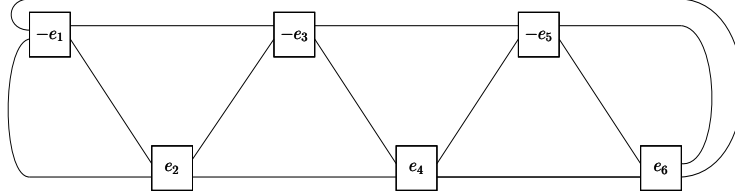


Figure 2.1: The knot $C(e_1, \dots, e_6)$. Each square represents a two-strand braid with only positive or only negative twists, according to the sign of e_i . The absolute value of e_i denotes the number of crossings.

Recall that Lisca [14] proved that, for two-bridge knots, being slice is equivalent to being ribbon. Previously, Casson and Gordon [5] gave a necessary condition for a two-bridge knot to be ribbon, and Lamm [13] listed all knots satisfying this condition. He found that for all $a \neq 0, b \neq 0$ the knots $K_1(a, b) = C(2a, 2, 2b, -2, -2a, 2b)$ and $K_2(a, b) = C(2a, 2, 2b, 2a, 2, 2b)$ are slice. Figure 2.1 recalls the notation $C(e_1, \dots, e_6)$. In Figure 2.2 we give a genus 3 Seifert surface V for the knot $\alpha = C(e_1, e_2, e_3, e_4, e_5, e_6)$. We use the surface V for all subsequent computations.

Since two-bridge slice knots are of particular interest for our construction of dihedral covers of four-manifolds, our first task is to determine the values of the parameters a and b for which the knots $K_i(a, b)$ admit three-fold dihedral covers.

Proposition 2.2.6. *A knot of the type $K_1(a, b)$ admits an irregular three-fold dihedral cover if and only if*

- (1) $a \equiv 0 \pmod{3}, b \equiv 2 \pmod{3}$ or
- (2) $a \equiv 1 \pmod{3}, b \equiv 1 \pmod{3}$.

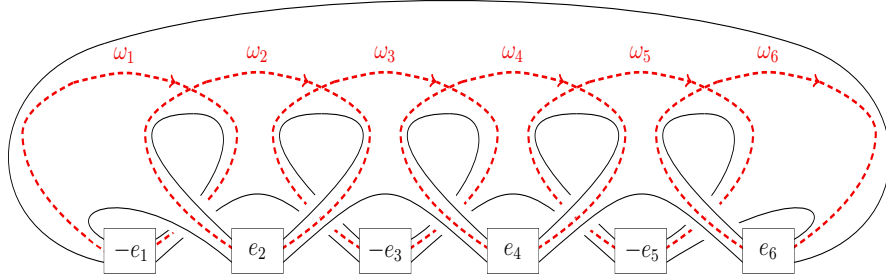


Figure 2.2: A Seifert surface for the knot $C(e_1, \dots, e_6)$, together with the set of preferred generators for its first homology.

A knot of the type $K_2(a, b)$ admits an irregular 3-fold dihedral cover if and only if

(3) $a \equiv 0 \pmod{3}, b \equiv 1 \pmod{3}$ or

(4) $a \equiv 1 \pmod{3}, b \equiv 0 \pmod{3}$.

In these cases, a curve representing the class $\beta \in H_1(V; \mathbb{Z})$ is a mod 3 characteristic knot for the corresponding $K_i(a, b)$ if and only if, with respect to the basis $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, we have, respectively,

(1) $[\beta] \equiv (1, 0, 1, 1, -1, 1) \pmod{3}$,

(2) $[\beta] \equiv (-1, 1, 1, 0, 1, 1) \pmod{3}$,

(3) $[\beta] \equiv (1, 0, 1, -1, 1, 1) \pmod{3}$,

(4) $[\beta] \equiv (-1, 1, 1, 1, 0, 1) \pmod{3}$.

Proof. Let V denote the Seifert surface for $C(e_1, e_2, e_3, e_4, e_5, e_6)$ depicted in Figure 2.2. We think of the e_i as being chosen so that the knot $C(e_1, e_2, e_3, e_4, e_5, e_6)$ is of type $K_1(a, b)$ or $K_2(a, b)$. Let L_V denote the matrix of the linking form for V

with respect to the basis $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$. The Seifert matrix for V in this basis is $L_V + L_V^T$. It has the form:

$$\begin{pmatrix} -e_1 & 1 & 0 & 0 & 0 & 0 \\ 1 & e_2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -e_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & e_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & -e_5 & 1 \\ 0 & 0 & 0 & 0 & 1 & e_6 \end{pmatrix}$$

It is sufficient to check that $\det(L_V + L_V^T) \equiv 0 \pmod{3}$ precisely in situations (1), ..., (4). For instance, in the case $C(e_1, e_2, e_3, e_4, e_5, e_6) = K_1(a, b)$, we obtain $\det(L_V + L_V^T) = -(8ab + 2b - 1)^2$. So we must solve the equation

$$8ab + 2b - 1 \equiv 0 \pmod{3}.$$

If $a \equiv 0 \pmod{3}$, the equation reduces to $2b - 1 \equiv 0 \pmod{3}$, so $b \equiv 2$. If $a \equiv 1 \pmod{3}$, then $b \equiv 1 \pmod{3}$. If $a \equiv 2 \pmod{3}$, there is no solution. The remaining computations are equally trivial, so they are omitted.

To verify that the classes $[\beta] \in H_1(V; \mathbb{Z})$ listed represent all characteristic knots, we check that for a and b as specified, we have $(L_V + L_V^T)\beta \equiv 0 \pmod{3}$ and moreover the classes β are the unique solutions mod 3. \square

More generally, we have the following:

Proposition 2.2.7. *Let $p > 1$ be an odd prime. There exists an infinite family of integer pairs (a, b) such that the two-bridge slice knot $K_1(a, b) \subset S^3$ admits an irregular dihedral p -fold cover, and similarly for $K_2(a, b)$.*

Proof. The case $p = 3$ was treated in Proposition 2.2.6, so assume $p > 3$. The determinant $D_1(a, b)$ of the Seifert matrix of the knot $K_1(a, b)$ is equal to $-(8ab + 2b - 1)^2$. Setting $a \equiv 0 \pmod{p}$, we find that $D_1(a, b) \equiv 0 \pmod{p}$ if and only if $2b \equiv 1 \pmod{p}$. Since p is odd, a solution exists. Another pair of solutions is $a \equiv 8^{-1} \pmod{p}$ and $b \equiv 3^{-1} \pmod{p}$.

Similarly, we find that the determinant $D_2(a, b)$ of the Seifert matrix of the knot $K_2(a, b)$ is $(8ab + 2a + 2b + 1)^2$. Setting $b \equiv -1 \pmod{p}$, we find that $a \equiv (-6)^{-1} \pmod{p}$. □

For any given p and any family of two-bridge slice knots $K_i(a, b)$ with a and b chosen so that $\det(L_V + L_V^T) \equiv 0 \pmod{p}$, the classes in $H_1(V; \mathbb{Z})$ represented by characteristic knots are easily computed as in Proposition 2.2.6 by solving a system of equations mod p . One can see by direct examination that if $p = 3$ each of these classes can be realized by the unknot. The same methods can be used to find knot types of characteristic knots for all p .

2.3 Brief note on notation

We denote the restriction of a map f to a part of the domain D by $f|_D$ or simply by $f|$ when the restricted domain D is understood.

The intersection of two surfaces S_1 and S_2 in a four-manifold is denoted $S_1 \cdot S_2$.

We write $\vec{u}(S)$ the push-off of a surface S along a normal \vec{u} .

The linking number of two links γ_1 and γ_2 in the three-sphere is denoted by $lk(\gamma_1, \gamma_2)$. The self-linking of a curve or link γ will be written as $lk_{\vec{v}}(\gamma, \gamma)$ if a framing is specified directly, or as $lk_F(\gamma, \gamma)$ if the framing is determined by specifying a Seifert surface F for γ .

Chapter 3

Irregular Dihedral Branched Covers of Four-Manifolds

3.1 Necessary condition for the existence of a di- hedral cover

The main result of this section is a necessary condition which the intersection form of a manifold Y must satisfy if Y is homeomorphic to an irregular dihedral branched cover of a four-manifold X with specified (oriented) singular branching set B .

Theorem 3.1.1. *Let X and Y be closed oriented four-manifolds and let p be an odd prime. Let $B \subset X$ be a closed connected and oriented surface embedded in X . Assume that $B \subset X$ is topologically locally flat except for an isolated singularity of type α . If an irregular p -fold dihedral cover $f : Y \rightarrow X$ branched along B exists,*

then α admits an irregular dihedral p -fold cover and this cover is the three-sphere.

Furthermore, the following formulas hold:

$$\chi(Y) = p\chi(X) - \frac{p-1}{2}\chi(B) - \frac{p-1}{2}, \quad (3.1.2)$$

$$\sigma(Y) = p\sigma(X) - \frac{p-1}{2}e(B) - \frac{p^2-1}{6p}L_V(\beta, \beta) - \sigma(W(\alpha, \beta)) - \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta). \quad (3.1.3)$$

Here, χ denotes the Euler characteristic, and σ is the signature of a four-manifold. For B a closed oriented surface embedded in a closed oriented four-manifold X , the self-intersection number of B in X is given by $e(B) := \langle [B]^* \cup [B]^*, [X] \rangle$, where $[B]^*$ is the Poincaré dual of the class $[B] \in H_2(X; \mathbb{Z})$, and $[X]$ is the fundamental class of X . Given a knot α , by V we denote a Seifert surface for α with Seifert pairing L_V , and we let $\beta \subset V^\circ$ be a mod p characteristic knot for α (Definition 2.2.5). Next, $\sigma_{\zeta^i}(\beta)$ denotes the Tristram-Levine ζ^i -signature of β , where ζ is a primitive p -th root of unity. Finally, the manifold $W(\alpha, \beta)$ is a cobordism between a dihedral p -fold branched cover of α and a cyclic p -fold branched cover of β . A construction of $W(\alpha, \beta)$ “by hand” was originally described in [4]; it is recalled in the proof of Proposition 3.1.14.

Remark 3.1.4. Note that when Y is a simply-connected manifold, Equation 3.1.2 is equivalent to expressing the rank of $H_2(Y; \mathbb{Z})$ in terms of data about the base manifold and branching set. This observation, however trivial, will be of much use

to us because our approach to the classification problem at hand is to pin down the intersection form of a dihedral branched cover over a given data.

Remark 3.1.5. The quantity $\frac{p^2-1}{6p}L_V(\beta, \beta) + \sigma(W(\alpha, \beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta)$, from here on denoted $\Xi_p(\alpha)$, is in fact an invariant of the knot type α . For a proof, jump to Section 3.4.

For a discussion of characteristic knots and an explanation of how to find β from a Seifert surface for α , consult Section 2.2. It is straightforward to compute $L_V(\beta, \beta)$ and $\sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta)$ from diagrams of α and β . The third term in the definition of $\Xi_p(\alpha)$, namely $\sigma(W(\alpha, \beta))$, has so far been described only abstractly, as the signature of a particular four-manifold. In Proposition 3.1.6, we compute the second homology group of this manifold in terms of the first homology of the chosen Seifert surface and characteristic knot for α . In Proposition 3.1.14, and we give an explicit formula for the term $\sigma(W(\alpha, \beta))$ in terms of linking numbers of curves in the irregular dihedral p -fold branched cover of α . A procedure for computing linking numbers in a branched cover of a knot is outlined in the Appendix.

Proposition 3.1.6. *Let $\alpha \subset S^3$ be a knot which admits a p -fold irregular dihedral cover M for some odd prime p . Let V be a Seifert surface for α and let $\beta \subset V$ be a mod p characteristic knot for α . Let Σ the p -fold cyclic cover of β . Let $W(\alpha, \beta)$, here denoted W , be the cobordism between M and Σ constructed in [4]. We denote by $V - \beta$ the surface V with a small open neighborhood of β removed, and by β_1*

and β_2 the two boundary components of $V - \beta$ that are parallel to β . Then:

$$H_2(W, M; \mathbb{Z}) \cong \mathbb{Z}^{\frac{p-1}{2}} \oplus (H_1(V - \beta; \mathbb{Z})/[\beta_1], [\beta_2])^{\frac{p-1}{2}}. \quad (3.1.7)$$

Proof. Since Cappell and Shaneson's construction of W is essential to our computation, we review it here. Let $f : \Sigma \rightarrow S^3$ be the cyclic p -fold cover of β . Since p is prime, Σ is a rational homology sphere. Let

$$f \times 1_I : \Sigma \times [0, 1] \rightarrow S^3 \times [0, 1]$$

be the induced branched cover of $S^3 \times [0, 1]$ as in [4]. Next, let

$$J := f^{-1}(V \times [-\epsilon, \epsilon])$$

be the pre-image of a closed tubular neighborhood $V \times [-\epsilon, \epsilon]$ of V in $S^3 \times \{1\}$, and let

$$T := f^{-1}(V \times \{0\}),$$

$$T \subset J \subset \Sigma \times \{1\}.$$

Then J deformation-retracts to T , and T consists of p copies of V identified along β via the identity map on S^1 and permuted cyclically by the group of covering transformations of f .

Consider the involution \bar{h} of J defined in [4] as a lift of the map

$$h : V \times [-\epsilon, \epsilon] \rightarrow V \times [-\epsilon, \epsilon],$$

$$h(u, t) \mapsto (u, -t).$$

Let q be the quotient map defined as

$$q : \Sigma \rightarrow \Sigma / (x \sim \bar{h}(x) | x \in J)$$

or, in short, $q : \Sigma \rightarrow \Sigma / \bar{h}$. Lastly, let

$$W := (\Sigma \times I) / \bar{h}.$$

As shown in [4], W is a cobordism between the cyclic p -fold cover $\Sigma = \Sigma \times \{0\}$ of β and the irregular p -fold dihedral cover of α , $M := (\Sigma / \bar{h}) \cap \partial(W)$. This completes the description of the construction of the pair (W, M) whose second homology we compute here.

Since $W = (\Sigma \times I) / \bar{h}$, where the domain of \bar{h} is $\Sigma \times \{1\}$, W is by definition the mapping cylinder of the quotient map q . Let $R := J / \bar{h}$. We have

$$H_2(W, M; \mathbb{Z}) \cong H_2(M \cup R, M; \mathbb{Z}) \cong H_2(R, M \cap R; \mathbb{Z}),$$

where the second isomorphism is excision, and the first follows from the fact that W deformation-retracts onto $\Sigma / \bar{h} = M \cup R$. Since

$$M \cap R = \partial(R) - V_0$$

(following the notation of [4], V_0 is the copy of V in T fixed by \bar{h}), we can rewrite the above isomorphism as

$$H_2(W, M; \mathbb{Z}) \cong H_2(R, \partial(R) - V_0; \mathbb{Z}).$$

The relevant portion of the long exact sequence of the pair $(R, \partial(R) - V_0)$ is:

$$H_2(R; \mathbb{Z}) \rightarrow H_2(R, \partial(R) - V_0; \mathbb{Z}) \rightarrow H_1(\partial(R) - V_0; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z}). \quad (3.1.8)$$

In Equation 3.1.11 below, we show that $H_2(R; \mathbb{Z}) = 0$. Assuming this for the moment, the above exact sequence, combined with the equation before it, gives:

$$H_2(W, M; \mathbb{Z}) \cong H_2(R, \partial(R) - V_0; \mathbb{Z}) \cong \ker(i_* : H_1(\partial(R) - V_0; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z})). \quad (3.1.9)$$

Our goal, therefore, is compute this kernel.

V is a surface with boundary and, by definition, β represents a non-zero primitive class in $H_1(V; \mathbb{Z})$. Therefore, β can be completed to a one-dimensional subcomplex $C \vee \beta$ which V deformation-retracts to. (We can assume that C is the wedge of $2g-1$ circles, where g is the genus of V .) Moreover, we can perform the deformation retraction of V onto such a one-complex simultaneously on each copy of V contained in T , fixing the curve of intersection β . Therefore, T deformation-retracts to a one-complex containing β wedged to p copies of C , where

$$H_1(C; \mathbb{Z}) \cong H_1(V; \mathbb{Z})/[\beta].$$

It follows that

$$H_2(J; \mathbb{Z}) \cong H_2(T; \mathbb{Z}) \cong 0$$

and

$$H_1(J; \mathbb{Z}) \cong H_1(T; \mathbb{Z}) \cong \mathbb{Z}^{(2g-1)p+1} \cong \oplus_p(H_1(V; \mathbb{Z})/[\beta]) \oplus \mathbb{Z},$$

where the singled-out copy of \mathbb{Z} is generated by $[\beta]$.

Furthermore, since the deformation-retraction of J onto T can be chosen to commute with \bar{h} , $J/\bar{h} = R$ deformation-retracts to T/\bar{h} , which is isomorphic to $\frac{p+1}{2}$

copies of V identified along β . (This follows from the fact that V_0 is fixed by \bar{h} , and the remaining $\frac{p-1}{2}$ copies of V in T become pairwise identified in the quotient. All copies of β are identified to a single one in both T and T/\bar{h} .) Therefore,

$$H_1(R; \mathbb{Z}) \cong \mathbb{Z}^{(2g-1)\frac{p+1}{2}+1} \cong \mathbb{Z} \oplus (H_1(V; \mathbb{Z})/[\beta])^{\frac{p+1}{2}}. \quad (3.1.10)$$

By the same reasoning as above, we can also conclude that T/\bar{h} deformation-retracts to a one-complex, so

$$H_2(R; \mathbb{Z}) = H_2(J/\bar{h}; \mathbb{Z}) \cong H_2(T/\bar{h}; \mathbb{Z}) \cong 0. \quad (3.1.11)$$

Next we examine $\partial(J)$ and $\partial(R)$. To start, $\partial(V \times [0, 1]) \cong V \cup_\alpha V$. Then $\partial(J)$ consists of p copies of $V \cup_\alpha V$, which we label $V_i^+ \cup V_i^-$, $0 \leq i < p$, with identifications we now describe. Cut each V_i^\pm along $\beta_i^\pm \subset V_i^\pm$. Now, $V_i^\pm - \eta(\beta)$ is a connected surface with three boundary components, α_i , $\beta_{i,1}^\pm$ and $\beta_{i,2}^\pm$, where the $\beta_{i,j}^\pm \subset V_i^\pm$ are labeled in such a way that the covering translation on J carries $\beta_{i,j}^\pm$ to $\beta_{i+1 \bmod p, j}^\pm$. Then we can think of $\partial(J)$ as obtained from $2p$ disjoint copies of $V - \beta$, labeled $V_i^\pm - \beta_i^\pm$, by gluing α_i^+ to α_i^- and $\beta_{i,j}^+$ to $\beta_{i+1 \bmod p, j}^-$. Thus, $\partial(J)$ is a closed surface of genus $(2g - 1)p$. In addition, we find that

$$H_1(\partial(J); \mathbb{Z}) \cong ((H_1(V - \beta; \mathbb{Z})))/([\beta_1], [\beta_2])^{2p} \oplus \mathbb{Z}^{2p}. \quad (3.1.12)$$

Recall that R is a $\mathbb{Z}/2\mathbb{Z}$ quotient of J , where the $\mathbb{Z}/2\mathbb{Z}$ action fixes $V_0 \times I$ and pairs off V_i^+ with V_{p-i}^- for $1 \leq i \leq \frac{p-1}{2}$. Thus, $\partial(R) - V_0$ is a surface of genus $p(g-1) + \frac{p+1}{2}$ and we have:

$$H_1(\partial(R) - V_0; \mathbb{Z}) \cong ((H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2]))^p \oplus \mathbb{Z}^{p+1}). \quad (3.1.13)$$

Our aim is to compute

$$\ker(i_* : H_1(\partial(R) - V_0; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z})).$$

We can now rewrite the map induced by inclusion as

$$i_* : ((H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2]))^p \oplus \mathbb{Z}^{p+1}) \rightarrow (H_1(V; \mathbb{Z})/[\beta])^{\frac{p+1}{2}} \oplus \mathbb{Z}.$$

It remains to examine i_* . It maps the copy of $(H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2]))$ coming from V_0^+ isomorphically onto its image, and it “pairs off” the remaining $p - 1$ copies of $(H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2]))$ onto $\frac{p-1}{2}$ copies of $H_1(V; \mathbb{Z})/[\beta]$ in the image. This contributes $H_1(V - \beta; \mathbb{Z})^{\frac{p-1}{2}}$ to $\ker(i_*)$. The remaining \mathbb{Z}^{p+1} in $H_1(\partial(R) - V_0; \mathbb{Z})$ is generated by $\mathbb{Z}^{\frac{p+1}{2}}$ curves which map to the single $[\beta]$ in the image, and an additional $\mathbb{Z}^{\frac{p+1}{2}}$ curves which map isomorphically to the $\frac{p+1}{2}$ classes in $H_1(V; \mathbb{Z})$ which are not in the image of $i_*H_1(V - \beta; \mathbb{Z})$. Thus, as we claimed,

$$\ker(i_*) \cong (H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2]))^{\frac{p-1}{2}} \oplus \mathbb{Z}^{\frac{p-1}{2}}.$$

□

This allows us to give a formula for the signature of W .

Proposition 3.1.14. *Adopt the assumptions and notation of Proposition 3.1.6. In addition, assume that the p -fold irregular dihedral cover of α is S^3 . Let w^1, w^2, \dots, w^r*

be a basis for $H_1(V - \beta; \mathbb{Z})/([\beta_1], [\beta_2])$. Denote by ψ the covering translation on the p -fold cyclic branched cover $f : \Sigma \rightarrow S^3$ with branching set β , and denote by $w_j^{i,\pm}, i \in \{1, \dots, r\}, j \in \{1, \dots, p\}$ the pre-images of the w^i lying in $f^{-1}(V \times [-1, 1])$ so that $w_j^{i,\pm} \subset V_j^\pm$ at $\psi V_k^\pm = V_{k+1 \bmod p}^\pm$. Lastly, denote by $\bar{w}_j^{i,\pm}$ and $\bar{\beta}_j^\pm$ the images of the corresponding curves¹ in Σ under the involution $\bar{h} : \Sigma \rightarrow S^3$. Let A be the matrix of linking numbers in S^3 of the following set of links:

$$\{\bar{w}_k^{i,+} - \bar{w}_k^{i,-}, \bar{\beta}_{k,1}^+ - \bar{\beta}_{k-1,1}^+\}_{i=1,\dots,r;k=1,\dots,\frac{p-1}{2}}.$$

Then, $\sigma(W) = \sigma(A)$.

Proof. Note that, since p is prime, Σ is a rational homology sphere. It follows that

$$H_2(W, M; \mathbb{Z}) \cong i_*(H_2(W; \mathbb{Z})) \subset (H_2(W, S^3 \cup \Sigma; \mathbb{Z})).$$

By Proposition 3.1.6 we already know that

$$H_2(W, M; \mathbb{Z}) \cong \ker(i_* : H_1(\partial(R) - V_0; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z})) =: K.$$

By the proof of the same proposition, the set of links

$$\{\bar{w}_k^{i,+} - \bar{w}_k^{i,-}, \bar{\beta}_{k,1}^+ - \bar{\beta}_{k-1,1}^+\}_{i=1,\dots,r;k=1,\dots,\frac{p-1}{2}}$$

forms a basis for K .

Recall that the isomorphism $H_2(W, M; \mathbb{Z}) \cong K$ is given by the boundary map in the long exact sequence 3.1.8. Consider any two elements u_1, u_2 in our basis for

¹Note that each $\bar{w}_j^{i,\pm}$ is a lift of $w_j^{i,\pm}$ to the irregular dihedral p -fold cover of α .

K . Each u_i is the image under the boundary map δ of a class $U_i \in H_2(W, M; \mathbb{Z})$; that is, $\partial U_i = u_i$. Now, let $u_1 = a_1 - a_2$ and $u_2 = b_1 - b_2$ with a_i, b_i curves in $M \cong S^3$. Note that $a_1 - a_2$ bounds a cylinder $S^1 \times I$ properly embedded in W . (If $a_1 - a_2 = \bar{w}_k^{i,+} - \bar{w}_k^{i,-}$, this is immediately clear since $\bar{w}_k^{i,+} - \bar{w}_k^{i,-} \subset \bar{w}_k^i \times I \subset V_k \times I$. If $a_1 - a_2 = \bar{\beta}_k^+ - \bar{\beta}_{k-1}^+$, note that $\bar{\beta}_{k-1}^+ = \bar{\beta}_k^-$, and the same argument applies.) Additionally, each of a_1, a_2 bounds a Seifert surface in S^3 ; denote the two surfaces by A_1 and A_2 , respectively. So we can compute intersections using the closed class

$$U'_1 := A_1 \cup_{\partial A_1} a_1 \times I \cup_{\partial A_2} a_2.$$

Letting B_i denote a Seifert surface for b_i , by analogy we can define

$$U'_2 := B_1 \cup_{\partial B_1} b_1 \times I \cup_{\partial B_2} B_2.$$

First, let us consider the case of self-intersection, $U'_1 \cdot U'_1$. The push-off of the link $a_1 - a_2$ along the normal in S^3 to $A_1 \cup A_2$ extends to $a_1 \times I$. Indeed, the obstruction to the existence of such an extension lies in $H^1(R, \partial(R) - V_0; \mathbb{Z}) \cong H_2(R, V_0; \mathbb{Z}) = 0$. Therefore,

$$U'_1 \cdot U'_1 = lk_{A_1 \cup A_2}(a_1 - a_2, a_1 - a_2).$$

Similarly, if $a_i \neq b_i$, we have,

$$U'_1 \cdot U'_2 = (A_1 \cup A_2) \cap (B_1 \cup B_2) = lk(a_1 - a_2, b_1 - b_2).$$

Therefore, the matrix of linking numbers between elements of our basis for K is also the intersection matrix for $(W(\alpha, \beta))$. This completes the proof. \square

The Proof of Proposition 3.1.6 also allows us to compute the fundamental group of the manifold $W(\alpha, \beta)$ for knots α which can arise as singularities of dihedral branched covers between four-manifolds.

Corollary 3.1.15. *Let p be an odd prime and let α be a knot which admits a p -fold irregular dihedral cover. Assume moreover that this cover is homeomorphic to S^3 . Let β be a characteristic knot for α and let $W(\alpha, \beta)$ be the cobordism between S^3 and the p -fold cyclic cover of β constructed in [4]. Then $W(\alpha, \beta)$ is simply-connected.*

Proof. We assume the notation of the proof of Proposition 3.1.6. (In this notation, the additional assumption of this Corollary is that $M \cong S^3$.) We have seen that $W(\alpha, \beta)$ is homotopy equivalent to $M \cup R$ and that $M \cap R = \partial R - V_0$. We also know that $i_* : \pi_1(\partial R - V_0, a_0) \rightarrow \pi_1(R, a_0)$ is surjective. On the other hand, any loop in $\pi_1(\partial R - V_0, a_0) = \pi_1(M \cap R, a_0)$ is contractible in M since $\pi_1(M; a_0) = 0$. Therefore, by van Kampen's Theorem, $\pi_1(M \cup R, a_0) = 0 = \pi_1(W(\alpha, \beta), a_0)$. \square

Finally, we prove the Main Theorem of this section.

Proof of Theorem 3.1.1. The existence of a p -fold dihedral cover $f : Y \rightarrow X$ over the pair (X, B) implies straight away that the knot α itself admits a p -fold dihedral cover M . Indeed, simply consider the restriction of f to $f^{-1}(\partial N(z))$, where $z \in B \subset X$ is the singular point on the branching set and $N(z)$ denotes a small neighborhood. Since by assumption there is a homeomorphism of pairs

$$(\partial N(z), B \cap \partial N(z)) \cong (S^3, \alpha),$$

this restriction of f to $f^{-1}(\partial N(z))$ is the desired dihedral cover. The fact that M is homeomorphic to a three-sphere follows from the assumption that the cover Y is a manifold: simply recall that over $N(z)$ lies the cone on M . This proves the first assertion.

We proceed to derive the formula for the Euler characteristic of Y . Let $N(B)$ denote a small tubular neighborhood of B in X . Then, we can write

$$X = (X - N(B)) \bigcup_{\partial N(B)} N(B).$$

Since $\partial N(B)$ is a closed oriented three-manifold, we know that $\chi(\partial N(B)) = 0$.

This gives:

$$\chi(X) = \chi(X - N(B)) + \chi(N(B)) = \chi(X - B) + \chi(B).$$

We can further break down this equation as

$$\chi(X) = \chi(X - B) + \chi(B - z) + 1.$$

Similarly, letting B' denote $f^{-1}(B)$ and $z' := f^{-1}(z)$, we have:

$$\chi(Y) = \chi(Y - B') + \chi(B' - z') + 1.$$

Since $f|_{Y-B'} : Y - B' \rightarrow X - B$ is a p -to-one covering map and $f|_{B'-z'} : B' - z' \rightarrow B - z$ is a $\frac{p+1}{2}$ -to-one covering map, we conclude that

$$\chi(Y) = p\chi(X - B) + \frac{p+1}{2}(\chi(B) - 1) + 1 = p\chi(X) - \frac{p-1}{2}\chi(B) - \frac{p-1}{2},$$

as claimed.

The computation of $\sigma(Y)$ is considerably more intricate. Our strategy for carrying it out will be to reduce to the case of a branched cover with locally flat branching set, at which point the signature of the cover can be computed by the well-known formula we recalled in Equation 2.1.2. By keeping track of the changes of signature produced in the process, we will be able to compute the defect to the signature that arises from the presence of a singularity on the branching set.

We resolve the singularity in two stages. At the start, the branching set has one singular point, in a neighborhood of which the branching set can be described in terms the knot α . Our first step will be to replace this singularity by a curve's worth of "standard" (that is, independent of the knot type α) non-manifold points on the branching set. The second step will be to excise these "standard" singularities and construct a new cover whose branching set is a locally flat submanifold of the base. We carry out these two steps in detail below, and we calculate the effect each of them has on the signatures of the four-manifolds involved.

Step 1. Let $D_z \subset X$ be a neighborhood of the singular point z such that $(D_z \cap B) \subset D_z$ is the cone on α . As we already established, α admits a p -fold dihedral cover. Equivalently, if V is any Seifert surface for α , there exists a mod p characteristic knot $\beta \subset V$ (see Definition 2.2.5). Let $W(\alpha, \beta)$ be the manifold constructed in [4] as a cobordism between a p -fold dihedral cover of α and a p -fold cyclic cover of (S^3, β) . By construction, there is a p -fold branched covering map

$$h_1 : W(\alpha, \beta) \rightarrow S^3 \times [0, 1].$$

Secondly, let

$$h_2 : Q \rightarrow D^4$$

be a p -fold cover of the closed four-ball branched over a Seifert surface V' for β , as constructed in Theorem 5 of [3]. Let Σ be the p -fold cyclic cover of β . By construction, $\partial Q \cong \Sigma$ and, similarly, $W(\alpha, \beta)$ has one boundary component homeomorphic to Σ . Note that, for $i = 1, 2$, the map

$$h_i|_{\Sigma} : \Sigma \rightarrow S^3$$

is the p -fold cyclic cover branched along β . Therefore, we can construct a branched cover

$$h_1 \cup h_2 : W(\alpha, \beta) \bigcup_{\Sigma} Q \longrightarrow S^3 \times [0, 1] \bigcup_{S^3 \times \{1\}} D^4. \quad (3.1.16)$$

We denote $W(\alpha, \beta) \bigcup_{\Sigma} Q$ by W for short, and the map $h_1 \cup h_2$ by h . Thus, we can rewrite Equation 3.1.16 as

$$h : W \rightarrow D^4.$$

This map is a p -fold branched cover whose restriction to the boundary of W a p -fold irregular dihedral cover of the pair (S^3, α) . So, denoting the branching set of h by T , there is a homeomorphism of pairs

$$(\partial D^4, \partial T) \cong (S^3, \alpha).$$

Furthermore,

$$T \cong \alpha \times [0, \frac{1}{2}] \bigcup_{\alpha \times \{\frac{1}{2}\}} V \bigcup_{\beta \times \{\frac{1}{2}\}} \beta \times [\frac{1}{2}, 1] \bigcup_{\beta \times \{1\}} V'.$$

We see from this description that T is a two-dimensional subcomplex of D^4 which is a manifold away from the curve $\beta \times \{\frac{1}{2}\}$. As evident from the above homeomorphism, the branching set is homeomorphic to the Cartesian product of S^1 and the letter “Y” in a small neighborhood of the curve $\beta \times \{\frac{1}{2}\}$.

We shall use the map h to construct a new cover of the manifold X which will differ from f only in a neighborhood of the singularity $z \in B$. Specifically, let $D'_z := f^{-1}(D_z)$ and observe that the restrictions of the maps f and h to the boundaries of $Y - D_z$ and W , respectively, are the p -fold irregular dihedral branched cover² of (S^3, α) , which is again S^3 . Therefore, we can define a new branched covering map

$$f \cup h : (Y - D'_z) \bigcup_{S^3} W \longrightarrow (X - D_z) \bigcup_{S^3} D^4.$$

Denote the manifold $(Y - D'_z) \bigcup_{S^3} W$ by Y_1 and the map $f \cup h$ by f_1 . Note that, by Novikov additivity [17], $\sigma(Y_1) = \sigma(Y) + \Sigma(W, M)$. Of course,

$$X - D_z \bigcup_{S^3} D^4 \cong X,$$

so we continue to denote the base space by X . We denote the branching set of f_1 by B_1 and note that

$$B_1 \cong B - N(z) \bigcup_{\alpha} T.$$

²We use the phrase “*the* dihedral cover of α ” somewhat liberally here. Dihedral covers of α are in bijective correspondence with equivalence classes of characteristic knots β . Naturally, if α admits multiple non-equivalent dihedral covers, we choose the one determined by f to construct W .

As prescribed, B_1 has a circle's worth of non-manifold points regardless of the choice of the knot α .

Step 2. Denote by β^* the curve of non-manifold points of T above, so that $\beta^* \subset S^3 \times \frac{1}{2}$ and $\beta^* \subset T \subset X$. Let $N(\beta^*) \cong_\psi S^1 \times B^3$ be a small tubular neighborhood of β^* in X . To construct the homeomorphism $\psi : N(\beta^*) \rightarrow S^1 \times B^3$, we choose a frame $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ for the normal bundle of β . For every $b \in \beta^*$, let $\vec{n}_1(b)$ be the normal to β in V at the point b , $\vec{n}_2(b)$ the normal to V in $S^3 \times \{\frac{1}{2}\}$, and $\vec{n}_3(b)$ the normal to S^3 in the product structure $S^3 \times I$. Clearly, $\{\vec{n}_1(b), \vec{n}_2(b), \vec{n}_3(b)\}$ are linearly independent for all $b \in \beta^*$.

We can now construct a new closed oriented four-manifold, denoted X_2 , as follows:

$$X_2 = (X - N(\beta^*)) \bigcup_{S^1 \times S^2} (X - N(\beta^*)).$$

The identification of the two copies of $\partial(X - N(\beta^*))$ is done by a homeomorphism

$$\phi : S^1 \times S^2 \rightarrow S^1 \times S^2$$

given by the formula

$$\phi(e^{i\theta}, y) = (e^{-i\theta}, y).$$

In particular, ϕ reverses orientation on $S^1 \times S^2$, so the manifold X_2 can be given an orientation which restricts to the original orientations on both copies of $X - N(\beta^*)$.

Therefore, by Novikov additivity we obtain:

$$\sigma(X_2) = 2\sigma(X - N(\beta^*), \partial) = 2\sigma(X). \quad (3.1.17)$$

Note that, since ϕ acts as the identity on the S^2 factor, it identifies the boundary of the branching set $T - N(\beta^*)$ in one copy of $X - N(\beta^*)$ with the boundary of branching set in the other copy of $X - N(\beta^*)$. Thus, the image of the branching set after this identification has the form

$$(B_1 - N(\beta^*)) \bigcup_{3S^1} (B_1 - N(\beta^*)). \quad (3.1.18)$$

Here the fact that the union of the two copies of $T - N(\beta^*)$ is taken along three circles corresponds to the fact that a neighborhood of the singular curve β^* intersects T in three closed curves (one for each “vertex” of the letter “Y”).

Denote the surface constructed in Equation 3.1.18 by B_2 . The careful reader will have noticed that B_2 is disconnected; we will describe its two connected components in more detail shortly. Since ϕ reverses the orientation on each boundary circle, the orientations of the two copies of $(B_1 - N(\beta^*))$ can be combined to obtain a compatible orientation on B_2 . Furthermore, by our choice of \vec{n}_3 , $N(\beta^*) \cap S^3 \times \{\frac{1}{2}\}$ is precisely the normal neighborhood of β^* in $S^3 \times \{\frac{1}{2}\}$ framed by $\{\vec{n}_1, \vec{n}_2\}$, the normals to β^* in V and to V in $S^3 \times \{\frac{1}{2}\}$. Consequently,

$$(S^3 \times \{\frac{1}{2}\}) \cap \partial(N(\beta^*)) \cong \partial((S^3 \times \{\frac{1}{2}\}) \cap N(\beta^*)) \cong \partial(\beta^* \times D^2) \cong S^1 \times S^1. \quad (3.1.19)$$

In particular, the restriction of ϕ to the boundary of the normal neighborhood of β^* in $S^3 \times \{\frac{1}{2}\}$ also reverses orientation. This implies that the positive normal to the oriented surface $(V - N(\beta^*)) \cup_{\phi_1} (V - N(\beta^*))$ inside the three-manifold $(S^3 \times \{\frac{1}{2}\} - N(\beta^*)) \cup_{\phi_1} (S^3 \times \{\frac{1}{2}\} - N(\beta^*))$ restricts to the normals of V in each corresponding copy of S^3 . This observation will be very useful shortly.

Recalling the definition of B_1 , namely $B_1 = (B - D_z) \cup_{\alpha} (T - N(\beta^*))$, we can break down $B_2 = (B_1 - N(\beta^*)) \cup_{3S^1} (B_1 - N(\beta^*))$ into

$$B_2 = ((B - D_z) \cup_{\alpha} (T - N(\beta^*))) \bigcup_{3S^1} ((B - D_z) \cup_{\alpha} (T - N(\beta^*))). \quad (3.1.20)$$

By construction, B_2 is embedded locally flatly in X_2 – that is, all singularities have been resolved. Also, as we indicated previously, B_2 has two connected components, since removing a neighborhood of β^* disconnects T . Attaching along the three curves in $S^1 \times S^2$ pairs off each of the four surfaces with boundary and its homeomorphic copy, producing two closed surfaces which we denote B'_2 and B''_2 . Here, B'_2 is the component of B_2 obtained by identifying two copies of $(B - D_z) \cup_{\alpha} (V - \beta)$ along $S^1 \amalg S^1$, and B''_2 is the component of B_2 obtained by identifying two copies of V' along S^1 . By construction, the cover over B'_2 is p -fold dihedral, whereas the cover over B''_2 is p -fold cyclic. That is, a point in B'_2 has $\frac{p+1}{2}$ pre-images, all but one of branching index 2, whereas a point in B''_2 has one pre-image of index p . This distinction will be relevant to our computation shortly.

Now our aim is to construct a p -fold branched cover of (X_2, B_2) from the covers f of (X, B) and h of (D^4, T) . We are helped greatly in this task by the observation that

$$h^{-1}(N(\beta^*)) \cong S^1 \times B^3$$

(a nice explanation of this rather surprising fact can be found on p.173-174 of [4]).

³It would be more consistent with our earlier notation to say that B''_2 is obtained from two copies of $\beta \times [0, 1] \cup_{\beta \times \{1\}} V'$, which, of course, is a surface homeomorphic to V' .

Therefore, denoting $h^{-1}(N(\beta^*))$ by N' , we can form the covering manifold as:

$$Y_2 := (Y_1 - N') \bigcup_{S^1 \times S^2} (Y_1 - N').$$

Here, the identification along the boundary $S^1 \times S^2$ is also done by ϕ , so, again, Y_2 can be given an orientation which restricts to each copy of $(Y_1 - N')$ to the orientation compatible with the given orientation on Y . In particular,

$$\sigma(Y_2) = \sigma((Y_1 - N') \cup_{S^1 \times S^2} (Y_1 - N')) = 2\sigma(Y_1) = 2(\sigma(Y) + \sigma(W, M)). \quad (3.1.21)$$

Because Y_2 and X_2 were constructed from copies of $(Y_1 - N')$ and $(X - N(\beta^*))$ by gluing via ϕ , the restrictions of f_1 to the two copies of $(Y_1 - N')$,

$$f_1| : (Y_1 - N') \rightarrow (X - N(\beta^*)),$$

can be glued to obtain a map

$$f_2 : ((Y_1 - N') \cup_{S^1 \times S^2} (Y_1 - N')) \rightarrow (X - N(\beta^*)) \cup_{S^1 \times S^2} (X - N(\beta^*)),$$

written for short as

$$f_2 : Y_2 \rightarrow X_2.$$

To complete the proof, what remains is to compute the effect this surgery has on the signatures of the base and covering manifolds. By Equation 2.1.2,

$$\sigma(Y_2) = p\sigma(X_2) - \frac{p-1}{2}e(B'_2) - \frac{p^2-1}{3p}e(B''_2).$$

Recall that from Equations 3.1.17 and 3.1.21 we have

$$\sigma(X_2) = 2\sigma(X)$$

and

$$\sigma(Y) = \frac{1}{2}\sigma(Y_2) - \sigma(W, M).$$

Also, by Novikov additivity,

$$\sigma(W, M) = \sigma(W(\alpha, \beta), M \cup \Sigma) + \sigma(Q, \Sigma) = \sigma(W(\alpha, \beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta).$$

In the last step, we have expressed the signature of Q in terms of Tristram-Levine signatures of β , using Theorem 5 of [3]. We have also shortened $\sigma(W(\alpha, \beta), M \cup \Sigma)$ to $\sigma(W(\alpha, \beta))$. Now we combine the last four equations and simplify. The result is:

$$\sigma(Y) = p\sigma(X) - \frac{1}{2}\left(\frac{p-1}{2}e(B'_2) - \frac{p^2-1}{3p}e(B''_2)\right) - \sigma(W(\alpha, \beta)) - \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta). \quad (3.1.22)$$

To complete the proof, we need to compute the self-intersection numbers of B'_2 and B''_2 in X_2 and relate them to that of B in X .

Recall that we denote the push-off of a surface S along a normal \vec{u} by $\vec{u}(S)$, and, as before, we denote self-intersection by “ \bullet ”. For brevity, we also denote $B - D_z$, the complement in B of a neighborhood of the singularity z , by B_z .

Note that if \vec{v} is an extension (not necessarily non-vanishing) to B_z of the normal to V in $S^3 \times \frac{1}{2}$ such that B_z and $\vec{v}(B_z)$ are transverse, then by definition

$$e(B) = (B_z \cup_{\alpha} V) \bullet \vec{v}(B_z \cup_{\alpha} V).$$

Since V is disjoint from both $\vec{v}(V)$ and $\vec{v}(B_z)$, and B_z is disjoint from $\vec{v}(V)$, the above equation simplifies to

$$e(B) = B_z \bullet \vec{v}(B_z). \quad (3.1.23)$$

Recall that the surface B'_2 is obtained from two copies of $B_z \cup_\alpha (V - \beta)$ attached by a homeomorphism ϕ_1 on their boundary $\beta_1 \amalg \beta_2$. Recall also that \vec{n}_2 , the restriction to β^* of the positive normal to V in $S^3 \times \frac{1}{2}$ (and thus of \vec{v}), is preserved by the gluing homeomorphism ϕ_1 . Therefore, the two copies of the normal \vec{v} to $B_z \cup_\alpha (V - \beta)$ can be combined obtain a normal, which we also denote \vec{v} , to B'_2 in X_2 . We have:

$$B'_2 = B_z \cup_\alpha (V - \beta) \cup_{\beta_1 \amalg \beta_2} (V - \beta) \cup_\alpha B_z. \quad (3.1.24)$$

Since $V - \beta$ and $\vec{v}(V - \beta)$ contribute nothing to the self-intersection $B'_2 \cdot \vec{v}(B'_2)$,

$$e(B'_2) = 2(B_z \cdot \vec{v}(B_z)) = 2e(B). \quad (3.1.25)$$

Similarly, if \vec{v} is an extension (not necessarily nowhere-zero) to V' of the normal \vec{n}_2 to the boundary β^* of V' such that V' and $\vec{v}(V')$ are transverse, we have:

$$e(B''_2) = 2(V' \cdot \vec{v}(V')) = 2lk_{\vec{v}}(\beta, \beta) = L_V(\beta, \beta). \quad (3.1.26)$$

Here, L_V denotes the Seifert form on V , the Seifert surface for α . The last equality follows from the fact that \vec{v} is an extension of the normal to V in $S^3 \times \frac{1}{2}$ and V' is a Seifert surface for β .

Putting everything together, we can rewrite Equation 3.1.22 as:

$$\sigma(Y) = p\sigma(X) - \frac{p-1}{2}e(B) - \frac{p^2-1}{6p}L_V(\beta, \beta) - \sigma(W(\alpha, \beta)) - \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta). \quad (3.1.27)$$

With that, the proof is complete. \square

Remark 3.1.28. The property that a p -fold dihedral cover of a knot α is homeomorphic to the three-sphere can be regarded as a condition for α to be an allowable

singularity on the branching set of an irregular p -fold dihedral cover between four-manifolds. The condition is satisfied, for example, for all two-bridge knots and any odd p (see the proof of Proposition 2.2.4) and can be disregarded if one allows the covering space to be a stratified space, rather than necessarily a manifold, or if one considers a slightly more general notion of branched cover (see Remark 3.3.3).

Remark 3.1.29. We note that the techniques used in the proof of Theorem 3.1.1 are purely local. Using the same methods, one can just as easily compute the correction to the signature and Euler characteristic of a branched cover Y resulting from the presence of multiple singularities on the branching set. There is also an interesting connection between the cover obtained by branching over two singularities and the cover obtained by branching over their connected sum. See Remark 3.4.5.

3.2 Sufficient condition in the case of two-bridge slice singularities

In this section, we describe a method for constructing an irregular p -fold dihedral cover of a general simply-connected four-manifold X . The main theorem of this section establishes that, for a certain class of singularities, all pairs of integers (σ, χ) afforded by the necessary condition (Theorem 3.1.1) as the signature and Euler characteristic of a p -fold irregular dihedral cover of a given base manifold X with specified branching set B are indeed realized as the signature and Euler

characteristic of a p -fold irregular dihedral cover over (X, B) .

Theorem 3.2.1. *Let X be a simply-connected four-manifold. Let $B \subset X$ be an oriented surface embedded topologically locally flatly in X and such that $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. Let p be an odd square-free integer, and let α be a two-bridge slice knot which admits a p -fold dihedral cover. If σ and χ are two integers such that*

$$\chi = p\chi(X) - \frac{p-1}{2}\chi(B) - \frac{p-1}{2} \quad (3.2.2)$$

and

$$\sigma(T) = p\sigma(X) + \frac{p-1}{2}e(B) - \Xi_p(\alpha), \quad (3.2.3)$$

then there exists a simply-connected four-manifold Y such that $\sigma(Y) = \sigma$, $\chi(Y) = \chi$ and Y is homeomorphic to an irregular dihedral p -fold cover of X . The branching set of this covering map is a surface $B_1 \cong B$, embedded in X with an isolated singularity z of type α and such that $e(B_1) = e(B)$.

Before we present the proof, we establish two preliminary results.

Proposition 3.2.4. *Let X be four-manifold and let $B \subset X$ be an embedded oriented surface of genus g such that $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. Then X admits a simply-connected double cover with branching set B .*

Proof. Since $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$, a double cover of X branched along B exists; we show that it is simply-connected. We denote the cover by \hat{X} and we denote by \hat{B} the (homeomorphic) pre-image of B under the covering map. We apply van

Kampen's theorem to $\hat{X} = (\hat{X} - \hat{B}) \cup_{\partial N(\hat{B})} N(\hat{B})$, where $N(\hat{B})$ denotes a small tubular neighborhood of \hat{B} . Being the universal cover of $(X - B)$, $(\hat{X} - \hat{B})$ is simply connected, so $i_* : \pi_1(\partial N(\hat{B}), b_0) \rightarrow \pi_1(\hat{X} - \hat{B}, b_0)$ is the zero map. In addition, $i_* : \pi_1(\partial N(\hat{B}), b_0) \rightarrow \pi_1(N(\hat{B}), b_0)$ is surjective. It follows from van Kampen's Theorem that \hat{X} is simply-connected. \square

Next, we prove a lemma concerning the singularity we are about to introduce to the branching set. This will allow us to construct the desired dihedral cover.

Lemma 3.2.5. *Let $p > 1$ be an odd square-free integer and let $K \subset S^3$ be a slice knot such that the pair (S^3, K) admits an irregular p -fold dihedral cover. Then there exists an embedded two-sphere $S^2 \subset S^4$ such that the pair (S^4, S^2) admits an irregular p -fold dihedral cover W and $S^2 \subset S^4$ is locally flat except at one point where it has a singularity of type K . Moreover, if K a two-bridge knot, W is a simply-connected topological manifold.*

Proof. Let $D_1^2 \subset B_1^4$ be a slice disk for K . Denote the cone on the pair (S^3, K) by (B_2^4, D_2^2) . It has the property that D_2^2 is a locally flat submanifold of B_2^4 except at the cone point x , where by construction D_2^2 has a singularity of type K . Identifying the two pairs (B_1^4, D_1^2) and (B_2^4, D_2^2) via the identity map along the two copies of (S^3, K) lying on their boundaries, we obtain an embedding of a two-sphere $S := D_1^2 \cup_K D_2^2$ in $S^4 = B_1^4 \cup_{S^3} B_2^4$ such that S has a unique singularity of type K at x .

By Proposition 2.2.4, the pair (B_1^4, D_1^2) admits an irregular dihedral p -fold cover W whose boundary M is the irregular dihedral p -fold cover of the pair (S^3, K) .

Since (B_2^4, D_2^2) is a cone, its irregular dihedral p -fold cover is simply the cone on M .

Thus, the pair (S^4, S) admits a cover

$$Z := W \bigcup_{\partial W \sim M \times \{0\}} (M \times [0, 1] / M \times \{1\})$$

as claimed. If, in addition, K is a two-bridge knot, by Proposition 2.2.4 we know that M is the three-sphere and moreover that we can pick the disk D_1^2 to be ribbon so that W is simply-connected. Thus, Z is a simply-connected manifold. \square

Proof of Theorem 3.2.1. The proof is as follows: first, we modify the branching set B by introducing a singular point of an appropriate type to the embedding of B in X ; next, we construct the desired covering space Y by pasting together several manifolds along their boundaries; we check that Y is indeed a p -fold irregular dihedral cover of X with the specified branching set; lastly, we verify that Y is a simply-connected manifold.

We begin by modifying the surface $B \subset X$ by introducing a singularity of type α . Let $S^2 \subset S^4$ be an embedded two-sphere with a unique singularity of type α constructed as in Lemma 3.2.5.

Let $y \in S^2 \subset S^4$ be any locally flat point with $N(y)$ a neighborhood of y not containing the singular point x . We use $N(y)$ to form the connected sum of pairs $(X, B) \# (S^4, S^2) =: (X, B_1)$. By construction, B_1 is homeomorphic to B but is embedded in X in such a way as to admit a unique singularity of type α . Furthermore, it is easy to compute, for example by a Mayer-Vietoris sequence, that

$H_1(X - B; \mathbb{Z}) \cong H_1(X - B_1; \mathbb{Z})$ and the latter group is $\mathbb{Z}/2\mathbb{Z}$ by assumption. Hence, X admits a double cover $f : \hat{X} \rightarrow X$ branched along B_1 .

Since $y \in S^4$ is a locally flat point, $B \cap \partial N(y)$ is the unknot. Now viewing $\partial N(y)$ as embedded in B_1 , we note that the restriction of f to $f^{-1}(\partial N(y))$ is a double branched cover of the trivial knot, whose total space is again S^3 . It follows that the double branched cover \hat{X} of the pair (X, B_1) is the connected sum (along S^3 viewed as a double cover of the unknot) of the double branched covers of a punctured (X, B) and $(S^4, S^2) - N(y)$. We denote by $f_0 : \hat{X}_0 \rightarrow (X - N(x))$ the restriction of f to the pre-image \hat{X}_0 of $X - N(x)$; in other words, f_0 is a double branched cover of a punctured (X, B) .

Next, consider the irregular dihedral p -fold cover $g : Z \rightarrow S^4$ of (S^4, S^2) constructed as in Lemma 3.2.5. For y as above, $g^{-1}(\partial N(y))$ is the irregular dihedral p -fold cover of the unknot, which consists of the disjoint union of $\frac{p+1}{2}$ copies of S^3 , $\frac{p-1}{2}$ of which are double covers and one a single cover. Therefore, $g^{-1}(S^4 - N(y))$ is an irregular dihedral p -fold cover of (B^4, D^2) . Its boundary consists of $\frac{p+1}{2}$ copies of S^3 . Of those, $\frac{p-1}{2}$ double-cover the complement of the unknot and one is mapped homeomorphically by g . Now we form the manifold Y which we will show is a dihedral cover of X . We attach to $g^{-1}(S^4 - N(y))$ a copy of \hat{X}_0 along each boundary S^3 which double-covers the complement of the unknot and a punctured copy of X along the boundary S^3 which is a cover of index 1. The map

$$h := g \cup_{\frac{p-1}{2}} f_0 \cup 1_{X-N(*)} : Y \rightarrow X$$

is a branched cover of (X, B_1) . By construction, h satisfies the property that for all points $z \in B - x$, if $N(z)$ is a small neighborhood of z in X not containing x , then $h^{-1}(N(z))$ has $\frac{p-1}{2}$ components of index 2 and one component of index 1. So Y is the desired dihedral cover.

Finally, we observe that Y consists of simply-connected manifolds joined together via homeomorphisms on their boundaries. Indeed, X is simply-connected by assumption, and \hat{X} is simply-connected by Proposition 3.2.4. The irregular dihedral cover Z of S^4 is simply-connected by Lemma 3.2.5, and, therefore, so is $g^{-1}(S^4 - N(y))$. We concluded that Y is simply-connected, which completes the proof.

□

Remark 3.2.6. Certain interesting variations on this result are not hard to obtain. For instance, if we do not require that our construction produce a simply-connected cover, we can relax the condition that $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$ and use for our branching set any surface B which represents an even class in $H_2(X; \mathbb{Z})$. For another result in the simply-connected realm, assume that a pair of integers (σ, χ) satisfy Equations 3.2.2 and 3.2.3 for some given X, B, α and p . Then, if $\chi' = \chi + (p-1)k$ for a natural number k , we can find a manifold Y' which is homeomorphic to a p -fold irregular dihedral cover of X and satisfies $\sigma(Y') = \sigma, \chi(Y') = \chi'$. This follows from the proof of Theorem 3.2.1, together with the following Lemma.

Lemma 3.2.7. *Let $B^2 \subset X^4$ be an oriented surface of genus g embedded locally*

flatly in X such that $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. Then, there exists a smoothly embedded oriented surface C of genus $g + 1$ in X such that $\pi_1(X - C, x_0) \cong \mathbb{Z}/2\mathbb{Z}$, and such that $e(B) = e(C)$, where e denotes the self-intersection number of a submanifold.

Proof. Let $T \subset S^4$ be an unknotted embedding of the two-torus in the four-sphere. That is, assume that $S^1 \times S^1 \cong T \subset S^3 \times [0, 1] \subset S^4$ is such that:

$$\begin{aligned} T \cap S^3 \times \{0\} &\cong T \cap S^3 \times \{1\} \cong \{*\}, \\ T \cap S^3 \times \{\frac{1}{3}\} &\cong T \cap S^3 \times \{\frac{2}{3}\} \cong \{S^1 \vee S^1\}, \\ T \cap S^3 \times \{t\} &\cong S^1, t \in (0, \frac{1}{3}) \cup (\frac{2}{3}, 1), \end{aligned}$$

and

$$T \cap S^3 \times \{t\} \cong S^1 \amalg S^1, t \in (\frac{1}{3}, \frac{2}{3}).$$

Moreover, assume that for all t the corresponding level set S^1 or $S^1 \vee S^1$ or $S^1 \amalg S^1$ bounds D^2 or $D^2 \vee D^2$ or $D^2 \amalg D^2$, respectively, inside the corresponding $S^3 \times \{t\}$. Using Fox's method (detailed in [6]) for computing the fundamental group of a surface complement in S^4 by cross-sections, we find that $\pi_1(S^4 - T) \cong \mathbb{Z}$, generated by a meridian of T in S^4 .

Now consider the connected sum of pairs $(X, B) \# (S^4, T)$ and let $C = B \# T \subset X \# S^4 \cong X$. Since a meridian m_1 of T in S^4 becomes identified under the connected sum with a meridian m_2 of B in X , it follows that the fundamental group of $(X - C)$ is isomorphic to $\langle m_1, m_2 | m_1 = m_2, m_2^2 = 0 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Finally, under the isomorphism of pairs $(X, B) \# (S^4, T) \cong (X, C)$, the class $[C] \in H_2(X; \mathbb{Z})$ corresponds to the class $[B \# T] \in H_2(X \# S^4; \mathbb{Z})$. Since $[T] = 0 \in H_2(X \# S^4; \mathbb{Z})$, indeed $e(B) = e(C)$.

□

The next theorem establishes the richness of the family of covers one can obtain by introducing a slice knot singularity to a surface (later to become the branching set) embedded in a four-manifold.

Theorem 3.2.8. *Let X be a simply-connected closed oriented four-manifold whose intersection form is indefinite and whose second Betti number is positive. For any odd square-free integer p , there exists an infinite family of simply-connected closed oriented four-manifolds $\{Y_i\}$, each of which is homeomorphic to an irregular p -fold cover of X branched over an oriented surface embedded in X with an isolated slice knot singularity.*

Proof. Let $B \subset X$ be a closed surface, embedded topologically locally flatly in X and such that $\pi_1(X - B; x_0) \cong \mathbb{Z}/2\mathbb{Z}$. Since X is indefinite and its second Betti number is positive, such a surface exists.

Let α be a two-bridge slice knot which admits an irregular dihedral p -fold cover. Such a knot α exists by Proposition 2.2.7. Following the steps of the proof of Theorem 3.2.1, we embed a two-sphere $S \subset S^4$ locally flatly except for one singularity of type α ; next, we construct a p -fold irregular dihedral cover of the pair $(X, B) \# (S^4, S) \cong (X, B)$, as in the proof of Theorem 3.2.1.

Using the same knot α as a singularity, by Lemma 3.2.7, we can increase the genus of the branching set B to obtain an infinite family of such covers. These covers are distinguished by their Euler characteristic. Using knots for which the values of Ξ_p differ, it is possible to obtain covers distinguished by their signatures as well. \square

We now turn to the question of determining when a particular manifold Y can be realized as a p -fold dihedral cover over a given base data (X, B, α) . Since our approach is to analyze a dihedral cover in terms of its signature and Euler characteristic, we will restrict our attention to situations where the manifold Y is determined (or nearly determined) by the rank and signature of its intersection form. The case of odd indefinite manifolds yields a particularly satisfying conclusion.

Theorem 3.2.9. *Let X and Y be simply-connected closed oriented four-manifolds whose intersection forms are odd and indefinite and whose Kirby-Siebenmann invariants are equal. Fix an odd square-free integer p and a two-bridge slice knot α . Let $B \subset X$ be an embedded surface such that $\pi_1(X - B, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. If the Euler characteristic and signature of Y satisfy the formulas in Theorem 3.1.1 with respect to X , B and α , then Y is homeomorphic to an irregular p -fold dihedral cover of X .*

Proof. We follow the steps used in the proof of Theorem 3.2.1 to construct a p -fold irregular dihedral cover of X branched over a surface $B_1 \cong B$ which is embedded in X with a singularity of type α . Call this cover Z . Since α is a two-bridge slice knot, by Theorem 3.2.1 we know that Z is a simply-connected manifold. We will

prove that the intersection form of Z is isomorphic to that of Y .

Being a dihedral cover of X , Z satisfies the equations set forth in Theorem 3.1.1. By assumption, Y satisfies these equations as well, so $\sigma(Y) = \sigma(Z)$ and $\chi(Y) = \chi(Z)$. Since Y is a simply-connected four-manifold, the rank of $H_2(Y; \mathbb{Z})$ is $\chi(Y) - 2$, and the analogous statement holds for Z . In other words, we can conclude that the intersection forms of Y and Z have the same signature and rank. The intersection form of Y is odd indefinite by assumption. The intersection form of Z is also odd and indefinite because by construction Z has a copy of X as a connected summand and X itself is odd indefinite. Therefore, the intersection forms of Y and Z are both indefinite and have the same signature, rank and parity. By Serre's classification of unimodular integral bilinear forms, they are isomorphic.

Finally, since Z is an odd-fold cover of X , the Kirby-Siebenmann invariants of X and Z are equal, hence so are the Kirby-Siebenmann invariants of Z and Y . Therefore, by Freedman's classification of simply-connected four-manifolds [9], Y and Z are homeomorphic. \square

3.3 Construction for other singularity types

We now describe a more general construction of dihedral covers, in which the condition that the singularity is slice is relaxed. Let p be an odd integer, and let α be a knot such that the pair (S^3, α) admits an irregular dihedral p -fold cover and, moreover, this cover is S^3 (for example α could be a two-bridge knot). In this sec-

tion we define an infinite set of four-manifolds \mathcal{M} such that for each $X \in \mathcal{M}$, and for every odd integer p , two families of infinite p -fold irregular branched covers of X are constructed. The first construction, given in Theorem 3.3.1, yields manifolds as the covers; the branching set in each case is a disconnected oriented surface with two singularities of type α . The second construction, Theorem 3.3.2, is derived from the first by establishing that it is possible to amalgamate the two singularities to produce, over the same set of base manifolds, infinite families of irregular p -fold covers with only one singularity of type $\alpha\#\alpha$. Reducing the number of singularities on the branching set causes the new covers obtained to be non-manifold: each cover is a stratified space with one singular point. In a final twist, we can resolve the singular point in the cover by blowing up. This allows us to produce a manifold as a cover over the same base, now with one singularity of type $\alpha\#\alpha$ on the branching set. This last idea requires us to relax the condition that a branched cover be finite-to-one and to allow the pre-image of the singularity on the branching set to be infinite (see Remark 3.3.3). The reason that the cover thus obtained is no longer a manifold is that if α is an admissible singularity type of a p -fold irregular dihedral cover, then $\alpha\#\alpha$ is not (for details, jump to Lemma 3.3.5).

Theorem 3.3.1. *Let X' be a simply-connected, closed, oriented four-manifold which admits an embedded locally flat surface B' with $\pi_1(X' - B', x_0) \cong \mathbb{Z}/2\mathbb{Z}$. Denote*

$$S^4 - S^1 \times B^3 \quad \bigcup_{S^1 \times S^2 \sim_{\phi} S^1 \times S^2} \quad S^4 - S^1 \times B^3$$

by \mathcal{S} , where $\phi(e^{i\theta}, x) = (e^{-i\theta}, x)$. For any prime p , the manifold $X := X'\#\mathcal{S}$

admits an infinite family of p -fold branched covers $Y_{p,j}$, where each $Y_{p,j}$ is a simply-connected manifold. For each j , the branching set of the covering map $f : Y_{p,j} \rightarrow X$ is an oriented surface B with two connected components, one of which is locally flat, and the other has two singularities of the same type.

Theorem 3.3.2. *Let X and p be as in Theorem 3.3.1. Then, X admits an infinite family of simply-connected p -fold branched covers $Z_{p,j}$ with the following properties. For each j , the branching set of the covering map $f : Z_{p,j} \rightarrow X$ is an oriented surface B with two connected components, one of which is locally flat, and the other has one singularity whose type is a composite knot $\alpha\#\alpha$. Furthermore, each $Z_{p,j}$ is a stratified space with one singular point, whose type is the p -fold irregular dihedral cover of $\alpha\#\alpha$.*

Remark 3.3.3. (“Theorem 3.3.2a”) Consider any one of the maps $f : Z_{p,j} \rightarrow X$ whose existence is established by the previous theorem. Denote the singular point on the branching set of f by x , and let z be the singularity of $Z_{p,j}$, so that $f^{-1}(x) = z$. Note that $Z_{p,j}$ itself can be covered by a manifold $Z'_{p,j}$, where $Z'_{p,j}$ is obtained by blowing-up $Z_{p,j}$ at z . Then, there exists a covering map $f' : Z'_{p,j} \rightarrow X$ which is a dihedral branched cover over $X - x$. The map f' is not a branched cover in the traditional sense, however, because the pre-image of the point x is not finite.

Proof of Theorem 3.3.1. We prove the theorem in four steps: (1) construct a manifold which is a p -fold branched cover of \mathcal{S} ; (2) use this construction to produce a p -fold branched cover of X ; (3) describe a method to obtain an infinite family of

covers from the first; (4) check that the manifolds constructed are simply-connected.

(1) Fix p odd. There exists a two-bridge knot α , not necessarily slice, which admits an irregular dihedral p -fold cover. To verify this, let e_1 and e_2 be two integers, not both of them odd, such that $e_1 e_2 \equiv 1 \pmod{p}$ (for example, take $e_1 = e_2 = p+1$). Then, in the notation of Figure 2.1, the two-bridge knot $C(e_1, e_2)$ admits a surjective presentation onto D_{2p} and hence a dihedral cover. Fix a two-bridge knot α which admits such a presentation.

We will now construct an irregular p -fold cover of \mathcal{S} such that the branching set will have two singularities of type α . Let $f : W(\alpha, \beta) \cup_{\Sigma} Q \rightarrow B^4$ be the p -fold branched cover used in the proof of Theorem 3.1.1. Recall that the branching set of f is a two-complex of the form

$$V^* := V \cup_{\beta \times \{0\}} \beta \times [0, 1] \cup_{\beta \times \{1\}} V', \quad (3.3.4)$$

where V is a Seifert surface for α , $\beta \subset V$ is a mod p characteristic knot, and V' is a Seifert surface for β . Recall also that the restriction of f to the boundary of $W(\alpha, \beta) \cup_{\Sigma} Q$ is a p -fold irregular cover of (S^3, α) . Cone off the boundaries of the pairs (B^4, V^*) and $(W(\alpha, \beta) \cup_{\Sigma} Q, f^{-1}(V^*))$ to and denote the respective closed manifolds by \mathcal{S}' and \mathcal{W}' . Extend f in the obvious way to produce a p -fold branched cover $f' : \mathcal{W}' \rightarrow \mathcal{S}'$. The branching set of f' is a two-complex with one singularity of type α and a circle's worth of non-manifold points (corresponding to $\beta \times \{0\}$ in Equation 3.3.4). Excise a neighborhood of this circle of singular points from \mathcal{S}' and glue two copies of the resulting manifold by the homeomorphism ϕ defined

in the proof of Theorem 3.1.1. The resulting manifold is \mathcal{S} . Denote by \mathcal{V} the two-subcomplex obtained by this process. \mathcal{V} has two connected components – one locally flat and one with two singularities of type α . Do the analogous construction with \mathcal{W}' : that is, excise the pre-image under f' of a neighborhood of the singular set, and glue two copies of the resulting manifold along their boundaries via ϕ . Note that ϕ is an orientation-reversing homeomorphism compatible with the restriction of f' . Denote the resulting closed manifold by \mathcal{W} . We combine the two copies of $f' : \mathcal{W}' \rightarrow \mathcal{S}'$ to construct a p -fold branched cover $f'' : \mathcal{W} \rightarrow \mathcal{S}$ whose branching set is \mathcal{V} . This completes the first step.

(2) Let X' and B' be as in the hypotheses of the Theorem. We wish to introduce an appropriate singularity to B' which will allow us to construct an irregular p -fold cover of $X' \# \mathcal{S}$. This step is very similar to the construction performed in the proof of Theorem 3.2.1. As in said proof, let (\hat{X}, \hat{B}) be the two-fold branched cover of (X', B') .

We wish to construct a branched cover of $X' \# \mathcal{S}$. Let $x \in \mathcal{V} \subset \mathcal{S}$ be a locally flat point on the singular component of \mathcal{V} . We delete a small neighborhood D_x of x to form the connected sum of pairs $(X', B') \# (\mathcal{S}, \mathcal{V})$. We remark that, as prescribed $B' \# \mathcal{V} =: \mathcal{B}$ has two connected components, one locally flat and one with two singularities of type α . By construction, $f''^{-1}(D_x)$ has $\frac{p-1}{2}$ components of branching index 2 and one component of branching index 1. Delete these from $(\mathcal{W}, f''^{-1}(\mathcal{V}))$ and form the connected sums with $\frac{p-1}{2}$ copies of (\hat{X}, \hat{B}) and one copy

of (X, B) . The manifold thus obtained, which we denote \mathcal{Q} , is the desired p -fold irregular branched cover of (X, \mathcal{B}) .

(3) In order to obtain an infinite family of covers, note that for any positive integer k we can increase the genus of B by k (see Lemma 3.2.7) before performing the procedure described in (2). Equation 3.1.2 shows that the covers obtained in this manner are all distinct.

(4) It remains to prove that the covers given by this method are simply-connected. Each $Y_{p,j}$ is constructed from \mathcal{W} , X and (several copies of) \hat{X} by connected sums. X is simply-connected by assumption, and \hat{X} by Proposition 3.2.4. We now show that \mathcal{W} is simply-connected as well. Recall that \mathcal{W} was constructed from two copies of $W(\alpha, \beta) \cup_{\Sigma} Q$ by removing an $S^1 \times B^3$ from each copy, gluing the two $W(\alpha, \beta) \cup_{\Sigma} Q - S^1 \times B^3$ along $S^1 \times S^2$, and coning off the remaining boundary components, each of which is homeomorphic to S^3 . Naturally, attaching copies of D^4 to the boundaries does not change the fundamental group. Note also that $W(\alpha, \beta) \cup_{\Sigma} Q - S^1 \times B^3$ is homotopy equivalent to $W(\alpha, \beta) \cup_{\Sigma} Q - S^1$. Since removing a circle has no effect on the fundamental group of a four-manifold, it's sufficient to show that $W(\alpha, \beta) \cup_{\Sigma} Q$ is simply-connected. The fact that Q is simply-connected was proved in [4], and the simply-connectedness of $W(\alpha, \beta)$ is Corollary 3.1.15. Therefore, \mathcal{W} is simply-connected, and, consequently, so is $Y_{p,j}$.

□

Our next order of business is to prove Theorem 3.3.2. For this purpose, we

describe a procedure by which, starting with any one of the covers $Y_{p,j} \rightarrow X$ constructed in Theorem 3.3.1, we can “amalgamate” the two singularities on the branching set to construct a new p -fold branched cover over the same manifold X . That is, the branching set of the new cover will contain only one singular point. The total space of the cover will be a stratified space with one singular point, the pre-image under the covering map of the singularity in X . In this construction, we make use of the following Lemma.

Lemma 3.3.5. *Let $\alpha_1 \subset S^3$ and $\alpha_2 \subset S^3$ be two knots which admit dihedral presentations $\phi_i : \pi_1((S^3 - \alpha_i), a_i) \rightarrow D_{2p}$ for some odd integer $p > 0$. Denote by M_i the corresponding irregular dihedral p -fold cover of (S^3, α_i) . Then, the knot $\alpha_1 \# \alpha_2$ admits a p -fold irregular dihedral cover homeomorphic to*

$$\left(M_1 - \amalg_{\frac{p+1}{2}} B^3 \right) \bigcup_{\amalg_{\frac{p+1}{2}} S^2} \left(M_2 - \amalg_{\frac{p+1}{2}} B^3 \right).$$

Here, the manifolds $(M_i - \amalg_{\frac{p+1}{2}} B^3)$ are attached to each other by the identity homeomorphism on their boundary $\amalg_{\frac{p+1}{2}} S^2$.

Proof. The key is to show that the knot connected sum $\alpha_1 \# \alpha_2$ can be formed in a way compatible with the two presentations ϕ_1 and ϕ_2 .

Let $x_i \in \alpha_i$, $i = 1, 2$, be any two points, and denote by g_i the homotopy class of the meridian of α_i based at a_i and going once along the boundary⁴ a small

⁴Since ϕ_i maps all meridians to elements of order two, there is no need to worry about the orientation of this loop.

normal disk intersecting α_i at x_i . Without loss of generality, we can assume that $\phi_1(g_1) = \phi_2(g_2)$. (Proof: if the two elements are not equal, they are conjugate. In this case, we can compose ϕ_2 with an automorphism ψ of D_{2p} sending $\phi_2(g_2)$ to $\phi_1(g_1)$. The cover of (S^3, α_2) corresponding to $\psi \circ \phi_2$ is homeomorphic to M_2 .) Use neighborhoods of the points x_1 and x_2 in the two copies of S^3 to form the knot connected sum $\alpha_1 \# \alpha_2$. By van Kampen's Theorem,

$$\pi_1((S^3 - \alpha_1 \# \alpha_2), a_0) \cong \pi_1((S^3 - \alpha_1), a_1) * \pi_1((S^3 - \alpha_2), a_2) / \langle g_1 = g_2 \rangle.$$

Since $\phi_1(g_1) = \phi_2(g_2)$, the group of $\alpha_1 \# \alpha_2$ admits a presentation to D_{2p} which extends both ϕ_1 and ϕ_2 . Let the corresponding irregular dihedral p -fold cover of $\alpha_1 \# \alpha_2$ be $f : M \rightarrow S^3$. Formally decompose the base pair $(S^3, \alpha_1 \# \alpha_2)$ as

$$(S^3, \alpha_1 \# \alpha_2) \cong (S_1^3, \alpha_1) \# (S_2^3, \alpha_2).$$

That is, think of each $S_i^3 - N(x_i)$, the complement of a small neighborhood of x_i , as embedded in the base. Then, we have

$$f^{-1}(S_i^3 - N(x_i)) \cong M_i - \amalg_{\frac{p+1}{2}} B^3.$$

Also, the pre-image under f of the pair (S^2, S^0) along which the connected sum of pairs $(S_1^3, \alpha_1) \# (S_2^3, \alpha_2)$ is taken consists of the boundaries of the $\frac{p+1}{2}$ three-balls which appear in the last equation above. Lastly,

$$(S_1^3 - \overline{N(x_1)}) \cap (S_2^3 - \overline{N(x_2)}) = \emptyset$$

and

$$(S_1^3 - N(x_1)) \cup (S_2^3 - N(x_2)) = S^3.$$

We conclude that, as we claimed,

$$M \cong \left(M_1 - \amalg_{\frac{p+1}{2}} B^3 \right) \bigcup_{\amalg_{\frac{p+1}{2}} S^2} \left(M_2 - \amalg_{\frac{p+1}{2}} B^3 \right).$$

□

Proof of Theorem 3.3.2. Let $f : Y_{p,j} \rightarrow X$ be one of the branched covers constructed in Theorem 3.3.1. Denote the branching set of f by V and the two singularities of V by z_1 and z_2 . Fix two small neighborhoods $N_i = N(z_i) \subset X$ and denote the knot $\partial N_i \cap V \subset \partial N_i$ by α_i . By construction, α_1 and α_2 are the same knot type, also denoted α , and moreover the restrictions of f to $f^{-1}(N_1)$ and $f^{-1}(N_2)$ arise from the same conjugacy class of presentations of the group of α to D_{2p} .

Let a_1 and a_2 be two points in α_1 and α_2 , respectively, and let $\gamma \subset V$ be a simple path from a_1 to a_2 which does not intersect the interiors of N_1 and N_2 . Denote by N_γ a small neighborhood of γ in V . Changing our choice of a_2 if necessary, we can assume without loss of generality that taking the connected sum of α_1 and α_2 along γ is compatible with the two presentations to D_{2p} (see Lemma 3.3.5). Now let $N(\gamma)$ be a small neighborhood of γ in X . Then

$$f| : f^{-1}(N(\gamma)) \rightarrow N(\gamma)$$

is a p -fold irregular branched cover of (B^3, I) , so it consists of $\frac{p+1}{2}$ disjoint copies of B^3 , one mapped by homeomorphically f and the rest of branching index two. Denote the four-ball $N_1 \cup N(\gamma) \cup N_2$ by N . Secondly, by a harmless abuse of notation, denote the connected sum $\alpha_1 \# \alpha_2$ along γ by $\alpha_1 \cup \gamma \cup \alpha_2$. Then, there is

a homeomorphism of pairs

$$(\partial N, \alpha_1 \cup \gamma \cup \alpha_2) \cong (S^3, \alpha \# \alpha)$$

and $f^{-1}(\partial N)$ is the p -fold irregular cover of $\alpha_1 \# \alpha_2$ compatible with the two original covers.

Consider the manifold $X - N$. Its boundary is a three sphere which, by construction, intersects the boundary of $V - N$ in $\alpha_1 \# \alpha_2$. Taking the cone on the pair $(X - N, V - N)$ produces a simply-connected manifold \bar{X} homeomorphic to the original manifold X . The cone on $V - N$ is a surface homeomorphic to V embedded in \bar{X} with a singularity of type $\alpha_1 \# \alpha_2$.

We mimic this procedure in the cover. That is, consider $Y_{p,j} - f^{-1}(N)$. It is a simply-connected four-manifold with boundary the p -fold irregular dihedral cover of $\alpha_1 \# \alpha_2$. We cone off its boundary to construct $Z_{p,j}$, a stratified space with one singular point. Extending $f| : Y_{p,j} - f^{-1}(N) \rightarrow X - N$ over the two cones in the obvious way produces the desired p -fold branched cover $f' : Z_{p,j} \rightarrow X$. Since $Y_{p,j}$ is simply-connected, so is $Z_{p,j}$, as desired. \square

3.4 A family of knot invariants

In this section we study $\Xi_p(\alpha)$, the “defect” to the signature of a branched cover which results from the presence of a singularity of type α on the branching set.

Proposition 3.4.1. *Let p be an odd square-free integer, and let $\alpha \subset S^3$ be knot*

which arises as the singularity of an irregular dihedral p -fold cover between four-manifolds. Assume that p^2 does not⁵ divide $\Delta(-1)$, where $\Delta(t)$ is the Alexander polynomial of α . In the notation of Theorem 3.1.1, the integer

$$\Xi_p(\alpha) := -\frac{p^2-1}{6p}L_V(\beta, \beta) - \sigma(W(\alpha, \beta)) - \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta) \quad (3.4.2)$$

is an invariant of the knot type α .

Proof. Since α arises as a singularity of an irregular dihedral p -fold cover, by Theorem 3.1.1, α itself admits an irregular dihedral p -fold cover. Since p^2 does not divide $\Delta(-1)$, this cover is unique (see footnote on p. 166 of [4]).

When both α and β are fixed, it is clear that each of the terms $\frac{p^2-1}{6p}L_V(\beta, \beta)$, $\sigma(W(\alpha, \beta))$ and $\sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta)$ is well-defined. Our goal is to show that their sum is in fact independent of the choice of β .

Let $f : Y \rightarrow X$ be an irregular dihedral p -fold cover, branched over an oriented surface $B \subset X$, embedded in X with a unique singularity of type α . Such a cover exists by assumption. Then

$$\Xi_p \alpha = p\sigma(X) - \frac{p-1}{2}e(B) - \sigma(Y),$$

a formula independent of the choice of β .

A priori, however, it might be possible for another branched cover $f' : Y' \rightarrow X'$, whose branching set also has a singularity of type α , to produce a different value of

⁵One could allow p^2 to divide $\Delta(-1)$. In this case, Ξ_p would not necessarily be an invariant of the knot type α but, rather, of α together with a specified presentation of $\pi_1(S^3 - \alpha, x_0) \rightarrow D_{2p}$.

Ξ_p . This does not occur. By the proof of Theorem 3.1.1, any choice of characteristic knot β can be used to compute the defect $\Xi_p(\alpha)$ to the signature of Y . Using the same β and Equation 3.4.2 to compute this signature defect for two different covers, for instance Y and Y' , shows that $\Xi_p(\alpha)$ does not vary with the choice of branched cover and indeed depends only on α . \square

It is evident from Equation 3.1.3 that the possible values of the Ξ_p invariant play a key part in determining the possible values of the signatures of branched covers over a given base. Therefore, it is of interest to study the properties and possible values of this invariant. The rest of this section is dedicated to proving that Ξ_p is additive with respect to knot connected sum.

Proposition 3.4.3. *If α_1 and α_2 are two knots for which Ξ_p is defined⁶, then*

$$\Xi_p(\alpha_1 \# \alpha_2) = \Xi_p(\alpha_1) + \Xi_p(\alpha_2).$$

Here, $\#$ denotes knot connected sum and

$$\Xi_p(\alpha) := \frac{p^2 - 1}{6p} L_V(\beta, \beta) + \sigma(W(\alpha, \beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta),$$

in the notation of Theorem 3.1.1.

Proof. Additivity of Ξ_p with respect to knot connected sum can be deduced from two simple observations. The first is that a characteristic knot for the connected

⁶As in Corollary 3.4.1, it is easiest, if not necessary, to assume in addition that each α_i admits a unique dihedral cover.

sum of two knots can be obtained by taking the connected sum of two individual characteristic knots. We proceed to prove this assertion. Let α_1 and α_2 be two knots, each of which admits a p -fold dihedral cover. Choose a Seifert surface V_i for α_i , and let L_i denote the matrix (with respect to some basis) of the corresponding linking form. Let $\beta_i \subset V_i$ be a characteristic knot. Then $V_1 \# V_2$ is a Seifert surface for $\alpha_1 \# \alpha_2$ and, with respect to the obvious basis, its Seifert matrix is
$$\begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix}.$$
 Because $\beta_1 \subset V_1$ is a $\text{mod } p$ characteristic knot, we know that each entry of $[\beta_1](L_1 + L_1^T)$ is congruent to $0 \text{ mod } p$, where by $[\beta_1]$ we denote the homology class of β_1 with respect to the chosen basis for $H_1(V_1; \mathbb{Z})$. The analogous statement holds for $[\beta_2]$ and $L_2 + L_2^T$. We wish to show that every entry of

$$[\beta_1 \# \beta_2] \begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix} \tag{3.4.4}$$

is also congruent to $0 \text{ mod } p$. Because $[\beta_1 \# \beta_2] = [\beta_1] + [\beta_2]$, we have

$$[\beta_1 \# \beta_2] \begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix} = [\beta_1] \begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix} + [\beta_2] \begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix}.$$

Since $\beta_1 \subset V_1$, the coordinates of $[\beta_1]$ corresponding to the basis elements that generate $H_1(V_2; \mathbb{Z})$ are all 0. (Put differently, $\beta_1 \subset V_1^\circ$ does not link any curve in V_2° .) Therefore, the components of the vector

$$[\beta_1] \begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix}$$

are those of $[\beta_1](L_1 + L_1^T)$, followed by zeros. The analogous statement holds for

$$[\beta_2] \begin{bmatrix} L_1 + L_1^T & 0 \\ 0 & L_2 + L_2^T \end{bmatrix}.$$

This shows that, indeed, every entry of the vector given by Equation 3.4.4 is $0 \pmod p$. Furthermore, since β_i represents a primitive class in $H_1(V_i; \mathbb{Z})$, $[\beta_1 \# \beta_2] \in H_1(V_1 \# V_2; \mathbb{Z})$ is primitive as well. Therefore, $\beta_1 \# \beta_2 \subset V_1 \# V_2$ is a characteristic knot for $\alpha_1 \# \alpha_2$.

Tristram-Levine signatures are additive with respect to knot connected sum and, by the same reasoning as above,

$$L_{V_1 \# V_2}(\beta_1 \# \beta_2, \beta_1 \# \beta_2) = L_{V_1 \# V_2}(\beta_1 + \beta_2, \beta_1 + \beta_2) = L_{V_1}(\beta_1, \beta_1) + L_{V_2}(\beta_2, \beta_2).$$

This proves the additivity of the first and third terms in the definition of Ξ_p .

The second observation is that the p -fold dihedral cover of $\alpha_1 \# \alpha_2$ is composed of the two individual dihedral covers by what we might call a “repeat connected sum” taken by removing $\frac{p+1}{2}$ balls from each manifold, one for each branch curve (see the proof of Lemma 3.3.5). Consequently, if we use the same basis for $H_1(V_1 \# V_2)$ as above, the intersection matrix for $W(\alpha_1 \# \alpha_2, \beta_1 \# \beta_2)$ will be block-diagonal, with the intersection matrices of $W(\alpha_1, \beta_1)$ and $W(\alpha_2, \beta_2)$ on the diagonal. This proves additivity of the third term. \square

Remark 3.4.5. At first glance, the additivity of Ξ_p with respect to knot-connected sum may appear to imply that introducing a singularity of type $\alpha_1 \# \alpha_2$ results in

a branched cover homeomorphic to the one obtained with two singularities α_1 and α_2 . Indeed, Proposition 3.4.3 implies that the “defect” to the signature of a branched cover arising from a singularity of type $\alpha_1\#\alpha_2$ is equal to that arising from the presence of two singularities, α_1 and α_2 . However, introducing two singularities α_1 and α_2 to a pair (X^4, B^2) results in a covering space which is in fact distinct from the cover obtained by introducing one singularity of type $\alpha_1\#\alpha_2$ to the same base. A simple way to see this is to note that the Euler characteristic of the cover does depend on the number of singularities. The formula for the Euler characteristic of a p -fold dihedral cover of (X^4, B^2) with m singular points is

$$\chi(Y) = p\chi(X) - \frac{p-1}{2}\chi(B) - \frac{p-1}{2}m.$$

There is also a more subtle – and more essential – distinction to be made. As seen from the Proof of Lemma 3.3.5, the knot types α_1 , α_2 and $\alpha_1\#\alpha_2$ can not simultaneously be admissible singularity types for a p -fold irregular branched cover between four-manifolds. That is, if the cover with singularity types α_1 and α_2 is a manifold, then the cover with one singularity of type $\alpha_1\#\alpha_2$ is not.

Chapter 4

Further Questions

We conclude with a short list of research questions that emerge from this study.

1. *Singularity types.* We have provided a method of constructing a branched cover with one slice singularity, as well as a method for constructing a cover with two isolated singularities of “arbitrary” type. This raises the following natural questions. First, can our method be generalized to construct a cover with *one* singularity of arbitrary type? (This is related to the following question: can a presentation of a knot group to a dihedral group D_{2p} be extended to a presentation of the fundamental group of the complement in D^4 of some Seifert surface for the knot?) Second, can every cover between simply-connected four-manifolds be realized by our construction?
2. *Knot invariants.* A question closely related to the above is this: which knots are admissible singularity types for a dihedral cover between four-manifolds?

We know that two-bridge knots provide an example. However, there is no known necessary and sufficient geometric criterion for determining whether the irregular dihedral p -fold cover of a given knot is S^3 . Since the class of admissible singularities effectively determines the variety of dihedral covers over a given base, such a criterion would be manifestly useful. On the flip side, it would be of interest to establish if a certain class of knots could account for all irregular dihedral covers between four-manifolds. Can every cover be realized with a two-bridge slice singularity? A study of the invariant Ξ_p would provide a first clue in this direction.

3. *Intersection forms and branched covers.* Our strategy in determining whether a given simply-connected topological four-manifold Y is homeomorphic to an irregular dihedral branched cover of another simply-connected four-manifold X has been to study the intersection forms of possible covers of X . The fact that an indefinite unimodular integral bilinear form is determined by its rank, signature and parity has allowed us to arrive at the bulk of our conclusions by relying, almost entirely, on studying the behavior of the signature and rank of intersection forms under dihedral branched covers. As a result, apart from an obstruction, in all probability rather coarse, definite four-manifolds have so far evaded our classification. Considerable refinements of our results could be achieved by a study of the behavior of the intersection forms themselves under dihedral covers.

Appendix

Let $\alpha \subset S^3$ be a knot, and let $f : M \rightarrow S^3$ be a cover branched along α , arising from a presentation $\psi : \pi_1(S^3 - \alpha, x_0) \rightarrow S_n$. The linking numbers (when defined) between the various components of $f^{-1}(\alpha)$ are a subtle knot invariant studied extensively by Perko [19], [21]. He used linking numbers to distinguish knots up to 11 crossings [20] and to detect non-amphichiral knots [18], among other applications.

In his undergraduate thesis [18], Perko described a procedure for computing these linking numbers. His method is, to this day, the most efficient algorithm known for computing linking numbers of branch curves. We give a very short summary of Perko's method for computing linking numbers in a branched cover. Our aim is to provide just enough detail to be able to describe a slight modification of his algorithm which allows us to calculate the linking numbers of other curves, as needed for evaluating the component of $\Xi_p(\alpha)$ which is expressed in terms of linking.

Perko's procedure for computing linking numbers between branch curves in a branched cover $f : M \rightarrow S^3$ with branching set α :

1. Use a diagram for α to endow S^3 with a cell structure. The two-skeleton here is the cone on α , and there is a single three-cell.
2. Use lifts $f^{-1}(e_k^j)$ of the various cells in S^3 , together with information about how the meridians of α permute the interiors of the three-cells in the cover, to obtain a cell structure on M .
3. Compute the boundaries of all two-cells of M . This step is non-trivial: “over-passing” two-cells accrue additional boundary components determined by the action of meridians of α on the three-cells.
4. Solve a system of linear equations to determine, for each component α_i of $f^{-1}(\alpha)$, a two-chain with boundary α_i .
5. For each pair (α_i, α_j) , examine the signed intersection numbers of α_i with a two-chain with boundary α_j to evaluate $lk(\alpha_i, \alpha_j)$.

In order to compute the linking numbers of other curves in M , we introduce an appropriate subdivision of the cell structure used by Perko. Consider a curve $\gamma \subset (S^3 - \alpha)$ whose lifts to M are of interest. We add the cone on γ to the two-skeleton of S^3 . In order to be able to lift this new cell structure to a cell structure on M , we treat γ as a “pseudo-branch curve” of the map f . That is, we think of the presentation $\pi_1(S^3 - \alpha) \rightarrow S_n$ as a presentation $\pi_1(S^3 - (\alpha \cup \gamma)) \rightarrow S_n$ in which meridians of γ map to the trivial permutation. (Naturally, this can be done for

multiple curves γ_i simultaneously.) Linking numbers can be computed by following steps 3, 4 and 5 above.

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