The Local Isometric Embedding in $\mathbb{R}^3$ of Two-Dimensional Riemannian Manifolds With Gaussian Curvature Changing Sign to Finite Order on a Curve

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Abstract

We consider two natural problems arising in geometry which are equivalent to the local solvability of specific equations of Monge-Ampère type. These two problems are: the local isometric embedding problem for two-dimensional Riemannian manifolds, and the problem of locally prescribed Gaussian curvature for surfaces in $\mathbb{R}^3$. We prove a general local existence result for a large class of Monge-Ampère equations in the plane, and obtain as corollaries the existence of regular solutions to both problems, in the case that the Gaussian curvature vanishes to arbitrary finite order on a single smooth curve.

0. Introduction

Let (M^2, ds^2) be a two-dimensional Riemannian manifold. A well-known problem is to ask when can one realize this, locally, as a small piece of a surface in $\mathbb{R}^3$. This question has only been partially answered.

Suppose that the first fundamental form, $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, is given in the neighborhood of a point, say $(u,v) = 0$. Let K be the Gaussian curvature, then the known results are as follows. The question is answered affirmatively in the case that ds^2 is analytic or $K(0) \neq 0$; these classical results can be found in [5], [12], and [13]. In the case that $K \geq 0$ and ds^2 is sufficiently smooth, or $K(0) = 0$ and $\nabla K(0) \neq 0$, C.-S. Lin provides an affirmative answer in [7] and [8]. If $K \leq 0$ and ∇K possesses a certain nondegeneracy, Han, Hong, and Lin [4] show that an embedding always exists. Furthermore, if $(u,v) = 0$ is a nondegenerate critical point for K and ds^2 is sufficiently smooth, then the author provides an affirmative answer in [6]. However, A. V. Pogorelov has given a counterexample in [11], where he constructs a $C^{2,1}$ metric with no C^2 isometric embedding in $\mathbb{R}^3$. Recently, very clever counterexamples have been found in the case that $ds^2 \in C^\infty$, K changes sign and vanishes to infinite order, or $K \leq 0$ and vanishes to infinite order. These interesting examples, constructed by Nadirashvili and Yuan, may be found in [9] and [10]. In this paper we prove the following,

Theorem 0.1. *Let $ds^2 \in C^r$, $r \geq 60$, and suppose that σ is a geodesic passing through the origin. If K vanishes to finite order on σ, then there exists a C^{r-36} local isometric embedding into $\mathbb{R}^3$.*

We begin by deriving the appropriate equations for study. Our goal is to find three functions $x(u,v)$, $y(u,v)$, $z(u,v)$, such that $ds^2 = dx^2 + dy^2 + dz^2$. The following strategy was first used by J. Weingarten [17]. We search for a function $z(u,v)$, with $|\nabla z|$ sufficiently small, such that $ds^2 - dz^2$ is flat in a neighborhood of the origin. Suppose that such a function exists, then since any Riemannian manifold of zero curvature is locally isometric to Euclidean space (via the exponential map), there exists a smooth change of coordinates $x(u,v)$, $y(u,v)$ such that $dx^2+dy^2 = ds^2-dz^2$, that is, $ds^2 = dx^2 + dy^2 + dz^2$. Therefore, our problem is reduced to finding $z(u,v)$ such that ds^2-dz^2 is flat in a neighborhood of the origin. A computation shows that this is equivalent to the local solvability of the following equation,

$$(z_{11}-\Gamma_{11}^i z_i)(z_{22}-\Gamma_{22}^i z_i)-(z_{12}-\Gamma_{12}^i z_i)^2 = K(EG-F^2-Ez_2^2-Gz_1^2+2Fz_1z_2), \quad (0.1)$$

where $z_1 = \partial z/\partial u$, $z_2 = \partial z/\partial v$, z_{ij} are second derivatives of z, and Γ_{jk}^i are Christoffel symbols.

Equation (0.1) is a second order Monge-Ampère equation. Another well-known and related problem, which is equivalent to the local solvability of a second order Monge-Ampère equation, is that of locally prescribing the Gaussian curvature for surfaces in $\mathbb{R}^3$. That is, given a function $K(u,v)$ defined in a neighborhood of the origin, when does there exist a piece of a surface $z = z(u,v)$ in $\mathbb{R}^3$ having Gaussian curvature K? This problem is equivalent to the local solvability of the equation

$$z_{11}z_{22} - z_{12}^2 = K(1+|\nabla z|^2)^2. \quad (0.2)$$

For this problem we obtain a similar result to that of theorem 0.1.

Theorem 0.2. *Let σ be a smooth curve passing through the origin. If $K \in C^r$, $r \geq 58$, and K vanishes to finite order on σ, then there exists a piece of a C^{r-34} surface in $\mathbb{R}^3$ with Gaussian curvature K.*

With the goal of treating both problems simultaneously, we will study the local solvability of the following general Monge-Ampère equation

$$\det(z_{ij} + a_{ij}(u,v,z,\nabla z)) = Kf(u,v,z,\nabla z), \quad (0.3)$$

where $a_{ij}(u,v,p,q)$ and $f(u,v,p,q)$ are smooth functions of p and q, $f > 0$, K vanishes to finite order along a smooth curve σ passing through the origin, and a_{ij} vanishes along σ to an order greater than or equal to one degree less than that of K. Clearly equation (0.2) is of the form (0.3), and equation (0.1) is of the form (0.3) if Γ_{jk}^i vanishes to the order of one degree less than that of K along σ, which we assume without loss of generality. More precisely, since σ is a geodesic we can introduce geodesic parallel coordinates, such that σ becomes the v-axis and $ds^2 = du^2 + h^2dv^2$, for some $h \in C^{r-1}$ satisfying

$$h_{uu} = -Kh, \quad h(0,v) = 1, \quad h_u(0,v) = 0.$$

It then follows that the Christoffel symbols vanish to the appropriate order along the v-axis. We will prove

Theorem 0.3. *Let σ be a smooth curve passing through the origin. If K, a_{ij}, $f \in C^r$, $r \geq 58$, K vanishes to finite order along σ, and a_{ij} vanishes to an order greater than or equal to one degree less than that of K along σ, then there exists a C^{r-34} local solution to* (0.3).

Equation (0.3) is elliptic if $K > 0$, hyperbolic if $K < 0$, and of mixed type if K changes sign in a neighborhood of the origin. If $K(0) = 0$ and $\nabla K(0) \neq 0$ [8], then (0.3) is a nonlinear type of the Tricomi equation. While if the origin is a nondegenerate critical point for K [6], then (0.3) is a nonlinear type of Gallerstedt's equation [3]. In our case, assuming that K vanishes to some finite order $n+1 \in \mathbb{Z}_{>0}$ along σ (ie. all derivatives up to and including order n vanish along σ), and a_{ij} vanishes at least to order n along σ, the linearized equation for (0.3) may be put into the following canonical form after adding suitable first and second order perturbation terms and making an appropriate change of coordinates,

$$Lu = y^{n+1}A_1u_{xx} + u_{yy} + y^{n-1}A_2u_x + A_3u_y + A_4u, \tag{0.4}$$

where the A_i are smooth functions and $A_1 > 0$ or $A_1 < 0$. It will be shown that this special canonical form is amenable to the making of estimates, even in the case that (0.4) changes type along the line $y = 0$.

From now on we assume that $n > 0$ is even, since the case when n is odd may be treated by the results in [7] and [4] where K is assumed to be nonnegative or nonpositive, and the case $n = 0$ may be treated by the methods of [8]. Furthermore, we assume without loss of generality that the curve σ is given by an equation $\widetilde{H}(u,v) = 0$, where $\widetilde{H} \in C^\infty$ and $\widetilde{H}_v|_\sigma \geq M_1$ for some constant $M_1 > 0$. Let ε be a small parameter and set $u = \varepsilon^2 x$, $v = \varepsilon^2 y$, $z = u^2/2 + \varepsilon^5 w$ (the x, y used here are not the same as those appearing in (0.4)). Substituting into (0.3), we obtain

$$\Phi(w) := (1 + \varepsilon w_{xx} + a_{11})(\varepsilon w_{yy} + a_{22}) - (\varepsilon w_{xy} + a_{12})^2 - Kf = 0. \tag{0.5}$$

By the assumptions of theorem 0.3 we may write $a_{ij} = \varepsilon^{2n}H^n(x,y)P_{ij}(\varepsilon, x, y, w, \nabla w)$ and $Kf = \varepsilon^{2(n+1)}H^{n+1}(x,y)P(\varepsilon, x, y, w, \nabla w)$, where $H = \varepsilon^{-2}\widetilde{H}$, $H_y|_\sigma \geq M_1$, $P \geq M_2$ for some constant $M_2 > 0$ independent of ε, and P_{ij}, P are C^r with respect to x, y and C^∞ with respect to the remaining variables. Then (0.5) becomes

$$\begin{aligned} \Phi(w) &= (1 + \varepsilon w_{xx} + \varepsilon^{2n}H^nP_{11})(\varepsilon w_{yy} + \varepsilon^{2n}H^nP_{22}) \\ &\quad -(\varepsilon w_{xy} + \varepsilon^{2n}H^nP_{12})^2 - \varepsilon^{2(n+1)}H^{n+1}P \\ &= 0. \end{aligned} \tag{0.6}$$

Choose $x_0, y_0 > 0$ and define the rectangle $X = \{(x,y) \mid |x| < x_0, |y| < y_0\}$. Then solving $\Phi(w) = 0$ in X, is equivalent to solving (0.3) locally at the origin.

In the following sections, we shall study the linearization of (0.6) about some function w. In section §1 the linearization will be reduced to the canonical form (0.4). Existence and regularity for the modified linearized equation will be obtained in section §2. In section §3 we make the appropriate estimates in preparation for the Nash-Moser iteration procedure. Finally, in §4 we apply a modified version of the Nash-Moser procedure and obtain a solution of (0.6).

1. Reduction to Canonical Form

In this section we will bring the linearization of (0.6) into the canonical form (0.4). This shall be accomplished by adding certain perturbation terms and making appropriate changes of variables. The process will entail defining a sequence of linear operators L_i, $1 \leq i \leq 7$, where L_1 is the linearization of (0.6) and L_7 is of the form (0.4); furthermore, L_{i+1} will differ from L_i by a perturbation term or by a change of variables.

Fix a constant $C > 0$, and let $w \in C^{\infty}(\mathbb{R}^2)$ be such that $|w|_{C^{16}} \leq C$. Then the linearization of (0.6) evaluated at w is given by

$$L_1(w) = \sum_{i,j} b^1_{ij} \partial_{x_i x_j} + \sum_i b^1_i \partial_{x_i} + b^1, \tag{1.1}$$

where $x_1 = x$, $x_2 = y$ and

$$\begin{aligned} b^1_{11} &= \varepsilon(\varepsilon w_{yy} + \varepsilon^{2n} H^n(x,y) P_{22}(\varepsilon, x, y, w, \nabla w)), \\ b^1_{12} = b^1_{21} &= -\varepsilon(\varepsilon w_{xy} + \varepsilon^{2n} H^n(x,y) P_{12}(\varepsilon, x, y, w, \nabla w)), \\ b^1_{22} &= \varepsilon(1 + \varepsilon w_{xx} + \varepsilon^{2n} H^n(x,y) P_{11}(\varepsilon, x, y, w, \nabla w)), \\ b^1_1 &= \varepsilon^{2n} H^n(x,y) P_1(\varepsilon, x, y, w, \nabla w), \\ b^1_2 &= \varepsilon^{2n} H^n(x,y) P_2(\varepsilon, x, y, w, \nabla w), \\ b^1 &= \varepsilon^{2n} H^n(x,y) P_3(\varepsilon, x, y, w, \nabla w), \end{aligned}$$

for some P_1, P_2, P_3. If ε is sufficiently small, we may solve for $\varepsilon w_{yy} + \varepsilon^{2n} H^n P_{22}$ in equation (0.6) to obtain

$$\varepsilon w_{yy} + \varepsilon^{2n} H^n P_{22} = \frac{1}{1 + \varepsilon Q}[(\varepsilon w_{xy} + \varepsilon^{2n} H^n P_{12})^2 + \varepsilon^{2(n+1)} H^{n+1} P + \Phi(w)], \tag{1.2}$$

where $Q(\varepsilon, x, y, w, \nabla w, \nabla^2 w) = w_{xx} + \varepsilon^{2n-1} H^n P_{11}$. Plugging (1.2) into (1.1) we have,

$$\begin{aligned} L_2(w) &:= L_1(w) - \frac{\varepsilon \Phi(w)}{1 + \varepsilon Q} \partial_{xx} \\ &= \sum_{i,j} b^2_{ij} \partial_{x_i x_j} + \sum_i b^2_i \partial_{x_i} + b^2, \end{aligned}$$

where

$$b_{11}^2 = \frac{\varepsilon(\varepsilon w_{xy} + \varepsilon^{2n} H^n P_{12})^2 + \varepsilon^{2n+3} H^{n+1} P}{1+\varepsilon Q}.$$

Next define $L_3(w)$ by,

$$\begin{aligned} L_3(w) &:= \frac{1}{\varepsilon(1+\varepsilon Q)} L_2(w) \\ &= \sum_{i,j} b_{ij}^3 \partial_{x_i x_j} + \sum_i b_i^3 \partial_{x_i} + b^3. \end{aligned} \tag{1.3}$$

To simplify (1.3), we will make a change of variables that will eliminate the mixed second derivative term. In constructing this change of variables we will make use of the following lemma from ordinary differential equations.

Lemma 1.1 [1]. *Let $G(x,t)$ be a C^l real valued function in the closed rectangle $|x-s| \le T_1$, $|t| \le T_2$. Let $T = \sup|G(x,t)|$ in this domain. Then the initial value problem $dx/dt = G(x,t)$, $x(0) = s$, has a unique C^{l+1} solution defined on the interval $|t| \le \min(T_2, T_1/T)$. Moreover, $x(s,t)$ is C^l with respect to s.*

We now construct the desired change of variables. For any domain $\Omega \subset \mathbb{R}^2$, and constant μ, let $\mu\Omega = \{\mu(x,y) \mid (x,y) \in \Omega\}$.

Lemma 1.2. *For ε sufficiently small, there exists a C^r diffeomorphism*

$$\xi = \xi(x,y), \ \eta = y,$$

of a domain X_1 onto $\mu_1 X$, where $\mu_1 > 1$, such that in the new variables (ξ, η), $L_3(w)$ is denoted by $L_4(w)$ and is given by

$$L_4(w) = \sum_{i,j} b_{ij}^4 \partial_{x_i x_j} + \sum_i b_i^4 \partial_{x_i} + b^4,$$

where $x_1 = \xi$, $x_2 = \eta$, and

$$\begin{aligned} b_{11}^4 &= \varepsilon^{2(n+1)} H^{n+1} P_{11}^4, \\ b_{12}^4 = b_{21}^4 &\equiv 0, \\ b_{22}^4 &\equiv 1, \\ b_1^4 &= \varepsilon^{2n} H^n P_1^{41} + n\varepsilon^{2n} H^{n-1} P_1^{42} + [\partial_x(\frac{\Phi(w)}{2(1+\varepsilon Q)^2}) + \frac{\partial_x \Phi(w)}{2(1+\varepsilon Q)^2}]\xi_x, \\ b_2^4 &= b_2^3, \\ b^4 &= b^3, \end{aligned}$$

for some P_{11}^4, P_1^{41}, P_1^{42}, and $P_{11}^4 \ge C_1$ for some constant $C_1 > 0$ independent of ε and w. Furthermore, $\sum |b_{ij}^4|_{C^{12}} + |b_i^4|_{C^{12}} + |b^4|_{C^{12}} \le C_2$, for some C_2 independent of ε and w.

Proof. Using the chain rule we find that $b^4_{12} = b^3_{12}\xi_x + b^3_{22}\xi_y$. Therefore, we seek a smooth function $\xi(x, y)$ such that

$$b^4_{12} = b^3_{12}\xi_x + b^3_{22}\xi_y = 0 \quad \text{in} \quad X_1, \quad \xi(x, 0) = x, \tag{1.4}$$

where X_1 will be defined below. Since $b^3_{22} \equiv 1$, the line $y = 0$ will be non-characteristic for (1.4). Then by the theory of first order partial differential equations, (1.4) is reduced to the following system of first order ODE:

$$\begin{aligned} \dot{x} &= b^3_{12}, \; x(0) = s, \; -\mu_1 x_0 \le s \le \mu_1 x_0, \\ \dot{y} &= 1, \quad y(0) = 0, \\ \dot{\xi} &= 0, \quad \xi(0) = s, \end{aligned}$$

where $x = x(t)$, $y = y(t)$, $\xi(t) = \xi(x(t), y(t))$ and $\dot{x}$, $\dot{y}$, $\dot{\xi}$ are derivatives with respect to t.

Choose $\mu_1 > 1$. We first show that the characteristic curves, given parametrically by $(x, y) = (x(t), t)$, exist globally for $-\mu_1 y_0 \le t \le \mu_1 y_0$. We apply lemma 1.1 with $T_1 = 2\mu_1 x_0$, and $T_2 = \mu_1 y_0$, to the initial-value problem $\dot{x} = b^3_{12}$, $x(0) = s$. Let T be as in lemma 1.1. Since $|w|_{C^{16}} \le C$, we have

$$T = \sup_{X_1} |b^3_{12}| \le \varepsilon C_3,$$

for some C_3 independent of ε. Then for ε small, $T \le \frac{2x_0}{y_0}$, implying that

$$\min(T_2, T_1/T) = \mu_1 y_0.$$

Then lemma 1.1 gives the desired global existence.

Let X_1 be the domain with boundary consisting of the two lines $y = \pm\mu_1 y_0$, and the two characteristics passing through $\pm\mu_1 x_0$. Then the mapping (ξ, η) takes ∂X_1 onto $\partial \mu_1 X$. We now show that the map $\rho : \mu_1 X \to X_1$ given by $(s, t) \mapsto (x(s, t), y(s, t)) = (x(s, t), t)$, is a diffeomorphism. It will then follow that the map $(x, y) \mapsto (\xi(x, y), \eta(x, y)) = (s(x, y), y) = \rho^{-1}(x, y)$ is a diffeomorphism of X_1 onto $\mu_1 X$. To show that ρ is 1-1, suppose that $\rho(s_1, t_1) = \rho(s_2, t_2)$. Then $t_1 = t_2$ and $x(s_1, t_1) = x(s_2, t_2)$, which implies that $s_1 = s_2$ by uniqueness for the initial-value problem for ordinary differential equations. To show that ρ is onto, take an arbitrary point $(x_1, y_1) \in X_1$, then we will show that there exists $s \in [-\mu_1 x_0, \mu_1 x_0]$ such that $\rho(s, y_1) = (x(s, y_1), y_1) = (x_1, y_1)$. Since the map $x(s, y_1) : [-\mu_1 x_0, \mu_1 x_0] \to [x(-\mu_1 x_0, y_1), x(\mu_1 x_0, y_1)]$ is continuous, and $x(-\mu_1 x_0, y_1) \le x_1 \le x(\mu_1 x_0, y_1)$ by definition of X_1, the intermediate value theorem guarantees that there exists $s \in [-\mu_1 x_0, \mu_1 x_0]$ with $x(s, y_1) = x_1$. Therefore, ρ has a well-defined inverse $\rho^{-1} : X_1 \to \mu_1 X$.

To show that ρ^{-1} is smooth it is sufficient, by the inverse function theorem, to show that the Jacobian of ρ does not vanish at each point of $\mu_1 X$. Since

$$D\rho = \begin{pmatrix} x_s & x_t \\ 0 & 1 \end{pmatrix},$$

this is equivalent to showing that x_s does not vanish in $\mu_1 X$. Differentiate the equation for x with respect to s to obtain, $\frac{d}{dt}(x_s) = (b^3_{12})_x x_s$, $x_s(0) = 1$. Then by the mean value theorem

$$|x_s(s,t) - 1| = |x_s(s,t) - x_s(s,0)| \leq \mu_1 y_0 \sup_{X_1} |(b^3_{12})_x| \sup_{\mu_1 X} |x_s|$$

for all $(s,t) \in \mu_1 X$. Thus, since $|w|_{C^{16}} \leq C$,

$$1 - \varepsilon \mu_1 y_0 C_4 \sup_{\mu_1 X} |x_s| \leq x_s(s,t) \leq \varepsilon \mu_1 y_0 C_4 \sup_{\mu_1 X} |x_s| + 1$$

for all $(s,t) \in \mu_1 X$. Hence for ε sufficiently small, $x_s(s,t) > 0$ in $\mu_1 X$. We have now shown that ρ is a diffeomorphism. Moreover, by lemma 1.1 and the inverse function theorem $\rho, \rho^{-1} \in C^r$.

We now calculate b^4_{11} and b^4_1. We have

$$b^4_{11} = \frac{(\varepsilon w_{xy} + \varepsilon^{2n} H^n P_{12})^2 + \varepsilon^{2(n+1)} H^{n+1} P}{(1+\varepsilon Q)^2} \xi_x^2 - \frac{2(\varepsilon w_{xy} + \varepsilon^{2n} H^n P_{12})}{1+\varepsilon Q} \xi_x \xi_y + \xi_y^2. \quad (1.5)$$

Since $\xi_y = -b^3_{12} \xi_x$, plugging into (1.5) we obtain

$$b^4_{11} = \frac{\varepsilon^{2(n+1)} H^{n+1} P \xi_x^2}{(1+\varepsilon Q)^2} := \varepsilon^{2(n+1)} H^{n+1} P^4_{11}.$$

To show that $P^4_{11} \geq C_1$, we now estimate ξ_x. By differentiating (1.4) with respect to x, we obtain

$$b^3_{12}(\xi_x)_x + (\xi_x)_y = -(b^3_{12})_x \xi_x, \quad \xi_x(x,0) = 1.$$

As above let $(x(t), y(t))$ be the parameterization for an arbitrary characteristic, then $\xi_x(t) = \xi_x(x(t), y(t))$ satisfies $\dot{\xi_x} = -(b^3_{12})_x \xi_x$, $\xi_x(0) = 1$. By the mean value theorem

$$|\xi_x(t) - 1| = |\xi_x(t) - \xi_x(0)| \leq \mu_1 y_0 \sup_{X_1} |(b^3_{12})_x| \sup_{X_1} |\xi_x|.$$

Therefore,

$$1 - \varepsilon \mu_1 y_0 C_5 \sup_{X_1} |\xi_x| \leq \xi_x(t) \leq \varepsilon \mu_1 y_0 C_5 \sup_{X_1} |\xi_x| + 1. \quad (1.6)$$

Thus, for ε small $\xi_x \geq C_6 > 0$, showing that $P^4_{11} \geq C_1$ for some $C_1 > 0$ independent of ε and w.

We now calculate b^4_1. We have

$$b^4_1 = b^3_{11} \xi_{xx} + 2 b^3_{12} \xi_{xy} + b^3_{22} \xi_{yy} + b^3_1 \xi_x + b^3_2 \xi_y. \quad (1.7)$$

From (1.4) we obtain

$$\xi_{xy} = -(b^3_{12})_x \xi_x - b^3_{12} \xi_{xx}, \quad \xi_{yy} = -(b^3_{12})_y \xi_x - b^3_{12} \xi_{xy}. \quad (1.8)$$

Plugging into (1.7) produces

$$\begin{aligned}
b_1^4 &= \frac{\varepsilon^{2(n+1)}H^{n+1}P}{(1+\varepsilon Q)^2}\xi_{xx} + b_1^3\xi_x + b_2^3\xi_y \\
&\quad +[\partial_y(\frac{\varepsilon w_{xy}+\varepsilon^{2n}H^nP_{12}}{1+\varepsilon Q}) - \frac{1}{2}\partial_x(\frac{\varepsilon w_{xy}+\varepsilon^{2n}H^nP_{12}}{1+\varepsilon Q})^2]\xi_x \\
&= \varepsilon^{2n}H^nQ_1 + n\varepsilon^{2n}H^{n-1}Q_2 \\
&\quad +[\partial_y(\frac{\varepsilon w_{xy}}{1+\varepsilon Q}) - \frac{1}{2}\partial_x(\frac{\varepsilon w_{xy}}{1+\varepsilon Q})^2]\xi_x,
\end{aligned} \tag{1.9}$$

for some Q_1, Q_2. We now calculate the last term of (1.9). From (0.6) we have

$$\frac{-\varepsilon^2 w_{xy}^2}{(1+\varepsilon Q)^2} = \frac{-\varepsilon w_{yy}(1+\varepsilon Q)+\varepsilon^{2n}H^nQ_3+\Phi(w)}{(1+\varepsilon Q)^2}, \tag{1.10}$$

for some Q_3. Then plugging (1.10) into (1.9), we obtain

$$\begin{aligned}
&\partial_y(\frac{\varepsilon w_{xy}}{1+\varepsilon Q}) - \frac{1}{2}\partial_x(\frac{\varepsilon w_{xy}}{1+\varepsilon Q})^2 \\
=\;& \partial_y(\frac{\varepsilon w_{xy}}{1+\varepsilon Q}) - \frac{1}{2}\partial_x(\frac{\varepsilon w_{yy}}{1+\varepsilon Q}) \\
&+\varepsilon^{2n}H^nQ_4 + n\varepsilon^{2n}H^{n-1}Q_5 + \partial_x[\frac{\Phi(w)}{2(1+\varepsilon Q)^2}] \\
=\;& \frac{\varepsilon/2 w_{xyy}(1+\varepsilon w_{xx}) - \varepsilon^2 w_{xy}w_{xxy} + \varepsilon^2/2 w_{yy}w_{xxx}}{(1+\varepsilon Q)^2} \\
&+\varepsilon^{2n}H^nQ_6 + n\varepsilon^{2n}H^{n-1}Q_7 + \partial_x[\frac{\Phi(w)}{2(1+\varepsilon Q)^2}] \\
=\;& \frac{\partial_x}{2(1+\varepsilon Q)^2}[\varepsilon w_{yy}(1+\varepsilon w_{xx}) - \varepsilon^2 w_{xy}^2] \\
&+\varepsilon^{2n}H^nQ_6 + n\varepsilon^{2n}H^{n-1}Q_7 + \partial_x[\frac{\Phi(w)}{2(1+\varepsilon Q)^2}] \\
=\;& \frac{\partial_x\Phi(w)}{2(1+\varepsilon Q)^2} + \partial_x[\frac{\Phi(w)}{2(1+\varepsilon Q)^2}] \\
&+\varepsilon^{2n}H^nQ_8 + n\varepsilon^{2n}H^{n-1}Q_9,
\end{aligned}$$

for some $Q_4,\ldots,Q_9$. It follows from (1.9), that b_1^4 has the desired form.

To complete the proof of lemma 1.2, we now show that $\sum|b_{ij}^4|_{C^{12}} + |b_i^4|_{C^{12}} + |b^4|_{C^{12}} \leq C_2$, for some constant C_2 independent of ε and w. In view of the fact that $|w|_{C^{16}} \leq C$, this will be accomplished by showing that $|\xi|_{C^{14}} \leq C_7$ for some C_7 independent of ε and w. By (1.6) we find that

$$\sup_{X_1}|\xi_x| \leq \frac{1}{1-\varepsilon C_5\mu_1 y_0} := C_8.$$

It follows from (1.4) that

$$\sup_{X_1} |\xi_y| \leq C_9,$$

where C_9 is independent of ε and w.

We now estimate ξ_{xx}. Differentiate (1.4) two times with respect to x to obtain

$$b^3_{12}(\xi_{xx})_x + (\xi_{xx})_y = -2(b^3_{12})_x \xi_{xx} - (b^3_{12})_{xx}\xi_x, \quad \xi_{xx}(x,0) = 0.$$

Then the same procedure that yielded (1.6), produces

$$\sup_{X_1} |\xi_{xx}| \leq \varepsilon\mu_1 y_0 C_{10} \sup_{X_1} |\xi_{xx}| + \varepsilon\mu_1 y_0 C_{11} C_8,$$

implying that

$$\sup_{X_1} |\xi_{xx}| \leq \frac{\varepsilon\mu_1 y_0 C_{11} C_8}{1 - \varepsilon\mu_1 y_0 C_{10}} := C_{12}.$$

Furthermore, in light of (1.8), we can use the estimates for ξ_x and ξ_{xx} to estimate ξ_{xy}, and then subsequently ξ_{yy}. Clearly, we can continue this procedure to yield $|\xi|_{C^{14}} \leq C_7$. q.e.d.

We now continue defining the sequence of linear operators $L_i(w)$. To simplify the coefficient of ∂_ξ in $L_4(w)$, we remove the portion of b^4_1 involving $\Phi(w)$ and define

$$\begin{aligned} L_5(w) &:= L_4(w) - [\partial_x(\frac{\Phi(w)}{2(1+\varepsilon Q)^2}) + \frac{\partial_x \Phi(w)}{2(1+\varepsilon Q)^2}]\xi_x \partial_\xi \\ &= \sum_{i,j} b^5_{ij}\partial_{x_i x_j} + \sum_i b^5_i \partial_{x_i} + b^5, \end{aligned}$$

where $x_1 = \xi$, $x_2 = \eta$.

To bring $L_5(w)$ into the canonical form (0.4), we shall need one more change of variables.

Lemma 1.3. *For ε sufficiently small, there exists a C^r diffeomorphism*

$$\alpha = \alpha(\xi,\eta), \ \beta = H(\xi,\eta),$$

of a domain $X_2 \subset \mu_1 X$ onto $\mu_2 X$, $1 < \mu_2 < \mu_1$, such that $\mu_3 X$ properly contains the image of $\rho^{-1}(X)$ (where ρ^{-1} is the diffeomorphism given by lemma 1.2), *for some μ_3, $1 < \mu_3 < \mu_2$. In the new variables (α,β), $L_5(w)$ is denoted by $L_6(w)$ and is given by*

$$L_6(w) = \sum_{i,j} b^6_{ij}\partial_{x_i x_j} + \sum_i b^6_i \partial_{x_i} + b^6,$$

where $x_1 = \alpha$, $x_2 = \beta$, *and*

$$\begin{aligned} b^6_{11} &= \varepsilon^{2(n+1)}\beta^{n+1}P^6_{11}, \\ b^6_{12} = b^6_{21} &\equiv 0, \\ b^6_{22} &= P^6_{22}, \\ b^6_1 &= \varepsilon^{2n}\beta^n P^{61}_1 + n\varepsilon^{2n}\beta^{n-1}P^{62}_1, \\ b^6_2 &= \varepsilon P^{61}_2 + n\varepsilon^{2n}\beta^{n-1}P^{62}_2, \\ b^6 &= \varepsilon^{2n}\beta^n P^6_3, \end{aligned}$$

for some P^6_{11}, P^6_{22}, P^{61}_1, P^{62}_1, P^{61}_2, P^{62}_2, P^6_3, *such that* $P^6_{11}, P^6_{22} \geq C_{13}$ *for some constant* $C_{13} > 0$ *independent of* ε *and* w. *Furthermore,* $\sum |b^6_{ij}|_{C^{12}} + |b^6_i|_{C^{12}} + |b^6|_{C^{12}} \leq C_{14}$, *for some* C_{14} *independent of* ε *and* w.

Proof. Using the chain rule we find that $b^6_{12} = b^5_{11}\beta_\xi\alpha_\xi + b^5_{22}\beta_\eta\alpha_\eta$. Therefore, we seek a smooth function $\alpha(\xi,\eta)$ such that

$$b^6_{12} = b^5_{11}\beta_\xi\alpha_\xi + b^5_{22}\beta_\eta\alpha_\eta = 0 \quad \text{in} \quad X_2, \quad \alpha(\xi,0) = \xi, \tag{1.11}$$

where X_2 will be defined below. By our original assumption on H made in the introduction, $H_y \geq C_{15}$ for some $C_{15} > 0$ independent of ε. Therefore,

$$H_\eta = H_x\frac{\partial x}{\partial \eta} + H_y\frac{\partial y}{\partial \eta} = -H_x\frac{\xi_y}{\xi_x} + H_y \geq \varepsilon C_{16} + C_{15} \geq C_{17} > 0,$$

for some C_{16}, C_{17} independent of ε. Since $b^5_{22} \equiv 1$, it follows that the line $\eta = 0$ is noncharacteristic for (1.11). Therefore, the methods used in the proof of lemma 1.2 show that the desired function $\alpha(\xi,\eta)$ exists.

We now define X_2. Since $H_\eta \geq C_{17} > 0$, we may choose $\mu_1 > \mu_2 > 1$ such that the curves $H(\xi,\eta) = \pm\mu_2 y_0$ are properly contained in the strips $\{(\xi,\eta) \mid y_0 \leq \eta \leq \mu_1 y_0\}$, $\{(\xi,\eta) \mid -y_0 \geq \eta \geq -\mu_1 y_0\}$. Then define $X_2 \subset \mu_1 X$ to be the domain in the ξ,η plane bounded by the curves $H(\xi,\eta) = \pm\mu_2 y_0$ and the characteristic curves of (1.11) passing through the points $(\pm\mu_2 x_0, 0)$. Then the methods of the proof of lemma 1.2 show that the mapping $\tau : (\xi,\eta) \mapsto (\alpha(\xi,\eta),\beta(\xi,\eta))$ is a C^r diffeomorphism from X_2 onto $\mu_2 X$. Furthermore, since $\rho^{-1}(X) \subset X_2$, if μ_3 is chosen large then $\tau(\rho^{-1}(X)) \subset \mu_3 X$.

We now compute the coefficients b^6_{ij}, b^6_i, b^6. We have

$$\begin{aligned} b^6_{11} &= \varepsilon^{2(n+1)}\beta^{n+1}P^4_{11}\alpha^2_\xi + \alpha^2_\eta \\ &= \varepsilon^{2(n+1)}\beta^{n+1}P^4_{11}\alpha^2_\xi + \varepsilon^{4(n+1)}\beta^{2(n+1)}(P^4_{11})^2\frac{\beta^2_\xi}{\beta^2_\eta}\alpha^2_\xi \\ &= \varepsilon^{2(n+1)}\beta^{n+1}[P^4_{11} + \varepsilon^{2(n+1)}\beta^{n+1}(P^4_{11})^2\frac{\beta^2_\xi}{\beta^2_\eta}]\alpha^2_\xi \\ &:= \varepsilon^{2(n+1)}\beta^{n+1}P^6_{11}. \end{aligned}$$

As in the proof of lemma 1.2, $\alpha_\xi \geq C_{18}$ for some $C_{18} > 0$ independent of ε and w. Thus, if ε is sufficiently small the properties of P_{11}^4 imply that $P_{11}^6 \geq C_{13}$ for some $C_{13} > 0$ independent of ε and w. Next we calculate b_{22}^6:

$$b_{22}^6 = \varepsilon^{2(n+1)}\beta^{n+1}P_{11}^4\beta_\xi^2 + \beta_\eta^2 := P_{22}^6.$$

Since $H_\eta \geq C_{17}$, if ε is sufficiently small then $P_{22}^6 \geq C_{13}$. Furthermore, by (1.11)

$$\begin{aligned} b_1^6 &= b_{11}^5\alpha_{\xi\xi} + \alpha_{\eta\eta} + b_1^5\alpha_\xi + b_2^5\alpha_\eta \\ &= b_{11}^5\alpha_{\xi\xi} - \partial_\eta\left(\frac{\varepsilon^{2(n+1)}\beta^{n+1}P_{11}^4\beta_\xi\alpha_\xi}{\beta_\eta}\right) + b_1^5\alpha_\xi + b_2^5\alpha_\eta \\ &:= \varepsilon^{2n}\beta^n P_1^{61} + n\varepsilon^{2n}\beta^{n-1}P_1^{62}. \end{aligned}$$

Lastly, since $\beta_\eta = H_x(\frac{-\xi_y}{\xi_x}) + H_y = O(\varepsilon) + H_y$, we have

$$\beta_{\eta\eta} = O(\varepsilon) + H_{yy} = O(\varepsilon) + \varepsilon^2\widetilde{H}_{vv} = O(\varepsilon).$$

Thus,

$$\begin{aligned} b_2^6 &= b_{11}^5\beta_{\xi\xi} + \beta_{\eta\eta} + b_1^5\beta_\xi + b_2^5\beta_\eta \\ &:= \varepsilon P_2^{61} + n\varepsilon^{2n}\beta^{n-1}P_2^{62}. \end{aligned}$$

We complete the proof by noting that the methods of the proof of lemma 1.2 show that $\sum |b_{ij}^6|_{C^{12}} + |b_i^6|_{C^{12}} + |b^6|_{C^{12}} \leq C_{14}$, for some C_{14} independent of ε and w. q.e.d.

To obtain the canonical form (0.4), we define

$$\begin{aligned} L_7(w) &:= \frac{1}{b_{22}^6}L_6(w) \\ &= \sum_{i,j} b_{ij}^7\partial_{x_ix_j} + \sum_i b_i^7\partial_{x_i} + b^7, \end{aligned}$$

where $x_1 = \alpha$, $x_2 = \beta$, and

$$\begin{aligned} b_{11}^7 &= \varepsilon^{2(n+1)}\beta^{n+1}P_{11}^7, \\ b_{12}^7 = b_{21}^7 &\equiv 0, \\ b_{22}^7 &\equiv 1, \\ b_1^7 &= \varepsilon^{2n}\beta^n P_1^{71} + n\varepsilon^{2n}\beta^{n-1}P_1^{72}, \\ b_2^7 &= \varepsilon P_2^{71} + n\varepsilon^{2n}\beta^{n-1}P_2^{72}, \\ b^7 &= \varepsilon^{2n}\beta^n P_3^7, \end{aligned}$$

for some P_{11}^7, P_1^{71}, P_1^{72}, P_2^{71}, P_2^{72}, P_3^7, such that $P_{11}^7 \geq C_{19}$ for some constant $C_{19} > 0$ independent of ε and w. In the following section, we shall study the existence and regularity theory for the operator $L_7(w)$.

2. Linear Theory

In this section we study the existence and regularity theory for the operator L_7. More precisely, we will first extend the coefficients of L_7 onto the entire plane in a manner that facilitates an a priori estimate, and then prove the existence of weak solutions having regularity in the α-direction. It will then be shown that these weak solutions are also regular in the β-direction via a boot-strap argument.

For simplicity of notation, put $x = \alpha$, $y = \beta$, and $\overline{L} = L_7(w)$. Then

$$\begin{aligned} \overline{L} &= \varepsilon^{2(n+1)} y^{n+1} B_1 \partial_{xx} + \partial_{yy} + (\varepsilon^{2n} y^n B_2 + n\varepsilon^{2n} y^{n-1} B_3)\partial_x \\ &\quad + (\varepsilon B_4 + n\varepsilon^{2n} y^{n-1} B_5)\partial_y + \varepsilon^{2n} y^n B_6 \\ &:= \overline{A}\partial_{xx} + \partial_{yy} + \overline{D}\partial_x + \overline{E}\partial_y + \overline{F}, \end{aligned}$$

for some $B_1, \ldots, B_6 \in C^r$ such that $B_1 \geq M$ and $|B_i|_{C^{12}} \leq M'$, for some constants $M, M' > 0$ independent of ε and w. By lemma 1.3 $\overline{A}, \overline{D}, \overline{E}$, and $\overline{F}$ are defined in the rectangle $\mu_2 X$. We will modify these coefficients on $\mathbb{R}^2 - \mu_2 X$, so that they will be defined and of class C^r on the entire plane.

Choose values $y_1, \ldots, y_6$ such that $0 < y_1 < \cdots < y_6$ and $y_1 = \mu_3 y_0$, $y_6 = \mu_2 y_0$. Let $\delta, M_1 > 0$ be constants, where δ will be chosen small. Fix a nonnegative cut-off function $\phi \in C^\infty(\mathbb{R})$ such that

$$\phi(y) = \begin{cases} 1 & \text{if } |y| \leq y_5, \\ 0 & \text{if } |y| \geq y_6. \end{cases}$$

Furthermore, define functions $\psi_1, \psi_2, \psi_3 \in C^\infty(\mathbb{R})$ with properties:

i) $\psi_1(y) = \begin{cases} 0 & \text{if } |y| \leq y_2, \\ -1 & \text{if } y \leq -y_3, \\ 1 & \text{if } y \geq y_3, \end{cases}$

ii) $\psi_1 \leq 0$ if $y \leq 0$, $\psi_1 \geq 0$ if $y \geq 0$, and $\psi_1' \geq 0$,

iii) $\psi_2(y) = \begin{cases} 0 & \text{if } y \geq -y_5, \\ -\delta y - \delta(\frac{y_5 + y_6}{2}) & \text{if } y \leq -y_6, \end{cases}$

iv) $\psi_2 \geq 0$, and $-\delta \leq \psi_2' \leq 0$,

v) $\psi_3(y) = \begin{cases} 0 & \text{if } |y| \leq y_3, \\ M_1 & \text{if } y \leq -y_4, \\ -M_1 & \text{if } y \geq y_4, \end{cases}$

vi) $\psi_3 \geq 0$ if $y \leq 0$, $\psi_3 \leq 0$ if $y \geq 0$, and $\psi_3' \leq 0$.

Now define smooth extensions of $\overline{A}, \overline{D}, \overline{E}$, and $\overline{F}$ to the entire plane by

$$\begin{aligned} A &= \psi_1(y) + \phi(x)\phi(y)\overline{A}, \\ D &= \phi(x)\phi(y)\overline{D}, \\ E &= \psi_2(y) + \phi(x)\phi(y)\overline{E}, \\ F &= \psi_3(y) + \phi(x)\phi(y)\overline{F}, \end{aligned}$$

and set

$$L = A\partial_{xx} + \partial_{yy} + D\partial_x + E\partial_y + F.$$

Before making estimates for L, we must define the function spaces that will be utilized. For $m, l \in \mathbb{Z}_{\geq 0}$, let

$$C^{(m,\,l)}(\mathbb{R}^2) = \{u : \mathbb{R}^2 \to \mathbb{R} \mid \partial_x^s \partial_y^t u \in C(\mathbb{R}^2),\ s \leq m,\ t \leq l\},$$

and

$$C_c^{(m,\,l)}(\mathbb{R}^2) = \{u \in C^{(m,\,l)}(\mathbb{R}^2) \mid u \text{ has compact support}\}.$$

Let $\theta > 0$ be a small parameter, and define the norm

$$\| u \|_{(m,\,l)} = \Big(\sum_{s \leq m,\ t \leq l} \theta^s \| \partial_x^s \partial_y^t u \|_{L^2(\mathbb{R}^2)}^2 \Big)^{1/2}.$$

Then define $H_\theta^{(m,\,l)}(\mathbb{R}^2)$ to be the closure of $C_c^{(m,\,l)}(\mathbb{R}^2)$ in the norm $\| \cdot \|_{(m,\,l)}$. Furthermore, let $H^m(\mathbb{R}^2)$ be the Sobolev space with square integrable derivatives up to and including order m, with norm $\| \cdot \|_m$. Lastly, denote the $L^2(\mathbb{R}^2)$ inner product and norm by $(\cdot, \cdot)$ and $\| \cdot \|$ respectively.

We are now ready to establish a basic estimate for the operator L on $\mathbb{R}^2$. This estimate will be used to establish a more general estimate, which will in turn be used as the foundation for the proof of the existence of weak solutions.

Lemma 2.1. *If ε is sufficiently small, then there exists a constant $C_1 > 0$ independent of ε, and functions $a(y), b(y), \gamma(y) \in C^\infty(\mathbb{R})$ where $\gamma = O(1)$ as $y \to \infty$, and $\gamma = O(|y|)$ as $y \to -\infty$ such that*

$$(au + bu_y, Lu) \geq C_1(\| \gamma u_y \|^2 + \| u \|^2), \text{ for all } u \in C_c^\infty(\mathbb{R}^2).$$

Proof. We first define the functions a and b. Let $M_2, M_3, M_4 > 0$ be constants satisfying $M_3 < M_2$ and $\frac{1}{2}M_4 - M_2 \geq 1$. Then choose $a, b \in C^\infty(\mathbb{R})$ and M_2, M_3, M_4 such that:

$$i)\ \ a(y) = \begin{cases} y^2 - M_2 & \text{if } |y| \leq y_5, \\ -M_3 & \text{if } |y| \geq y_6, \end{cases}$$

ii) $a \le -M_3$, $a^{'} \le 0$ if $y \le 0$, $a^{'} \ge 0$ if $y \ge 0$, and $a^{''} \ge -\delta$,

iii) $b(y) = \begin{cases} 1 & \text{if } y \ge 0, \\ -M_4 y + 1 & \text{if } y \le -y_2, \end{cases}$

iv) $b \ge 1$, and $b^{'} \le 0$.

Now let $u \in C_c^\infty(\mathbb{R}^2)$, and integrate by parts to obtain

$$(au + bu_y, Lu) = \int\int_{\mathbb{R}^2} I_1 u_x^2 + 2I_2 u_x u_y + I_3 u_y^2 + I_4 u^2,$$

where

$$\begin{aligned} I_1 &= (\frac{1}{2}b^{'} - a)A + \frac{1}{2}bA_y, \\ I_2 &= -\frac{1}{2}bA_x + \frac{1}{2}bD, \\ I_3 &= -a - \frac{1}{2}b^{'} + bE, \\ I_4 &= \frac{1}{2}aA_{xx} + \frac{1}{2}a^{''} - \frac{1}{2}aD_x - \frac{1}{2}(aE)_y - (\frac{1}{2}b^{'} - a)F - \frac{1}{2}bF_y. \end{aligned}$$

We now estimate I_1. If $|y| \le y_3$ then

$$\begin{aligned} I_1 &\ge [(M_2 - y^2)\varepsilon^{2(n+1)}y^{n+1}B_1 + \frac{(n+1)}{2}\varepsilon^{2(n+1)}y^n B_1 + \frac{1}{2}\varepsilon^{2(n+1)}y^{n+1}b\partial_y B_1]\phi(x) \\ &= \varepsilon^{2(n+1)}y^n[(M_2 - y^2)yB_1 + \frac{(n+1)}{2}B_1 + \frac{1}{2}yb\partial_y B_1]\phi(x) \\ &\ge \varepsilon^{2(n+1)}C_2 y^n \phi(x) \ge 0, \end{aligned}$$

for some constants $C_2 > 0$, if y_3 is chosen sufficiently small. Moreover, if $|y| \ge y_3$ we have

$$I_1 \ge O(\varepsilon^{2(n+1)}) + \begin{cases} M_3 & \text{if } y \ge 0 \\ \frac{1}{2}M_4 - M_2 & \text{if } y < 0 \end{cases} \ge C_3,$$

for some $C_3 > 0$, if ε is small.

To estimate I_3, we observe that for $|y| \le y_6$,

$$I_3 \ge M_3 + O(\varepsilon).$$

Furthermore, if $|y| \ge y_6$ then

$$I_3 \ge M_3 + \begin{cases} 0 & \text{if } y \ge 0, \\ \delta M_4 y^2 & \text{if } y < 0. \end{cases}$$

Hence, $I_3 \geq \gamma^2(y)$ for some $\gamma \in C^\infty(\mathbb{R})$ such that $\gamma = O(1)$ as $y \to \infty$, and $\gamma = O(|y|)$ as $y \to -\infty$.

Next we show that

$$\int\int_{\mathbb{R}^2} I_1 u_x^2 + 2I_2 u_x u_y + I_3 u_y^2 \geq C_4 \parallel \gamma u_y \parallel^2,$$

for some $C_4 > 0$. From our estimates on I_1 and I_3, this will follow if $I_1 I_3 - 2I_2^2 \geq 0$. A calculation shows that when $|y| \leq y_6$, we have

$$\begin{aligned} I_1 I_3 - 2I_2^2 &\geq \varepsilon^{2(n+1)} C_5 y^n \phi(x) + O(n\varepsilon^{2n} y^{n-1}\phi(x) + \varepsilon^{2n} y^n |\phi^{'}(x)|)^2 \\ &= \varepsilon^{2(n+1)} y^n [C_5 + \varepsilon^{2n-2} O(n^2 y^{n-2}\phi(x) + y^n |\phi^{'}(x)|^2 \phi^{-1}(x) \\ &\quad + n y^{n-1} |\phi^{'}(x)|)]\phi(x) \\ &\geq 0, \end{aligned}$$

for some $C_5 > 0$ independent of ε, if ε is sufficiently small. Moreover, if $|y| \geq y_6$ then

$$I_1 I_3 - 2I_2^2 = I_1 I_3 > 0,$$

from which we obtain the desired conclusion.

Lastly, we estimate I_4. In the strip $|y| \leq y_4$, we obtain

$$I_4 \geq 1 + O(\varepsilon).$$

Furthermore, if $|y| \geq y_4$ then

$$I_4 \geq \begin{cases} M_1 M_3 + O(\varepsilon + \delta) & \text{if } y \geq 0, \\ M_1(\frac{1}{2}M_4 - M_2) + O(\varepsilon + \delta) & \text{if } y < 0. \end{cases}$$

Therefore, $I_4 \geq C_6$ for some $C_6 > 0$ independent of ε. q.e.d.

Having established the basic estimate, our goal shall now be to establish a more general estimate that involves derivatives of higher order in the x-direction. Let $< \cdot, \cdot >_m$ denote the inner product on $H_\theta^{(m,0)}(\mathbb{R}^2)$, that is,

$$< u, v >_m = \int\int_{\mathbb{R}^2} \sum_{s=0}^{m} \theta^s \partial_x^s u \partial_x^s v, \text{ for all } u, v \in H_\theta^{(m,0)}(\mathbb{R}^2).$$

Theorem 2.1. *If $\varepsilon = \varepsilon(m)$ is sufficiently small, then for each $m \leq r - 2$, there exist constants $\theta(m) > 0$ and $C_m > 0$, both depending on $|A|_{C^{m+2}(\mathbb{R}^2)}, |D|_{C^{m+2}(\mathbb{R}^2)}$, $|E|_{C^{m+2}(\mathbb{R}^2)}$, and $|F|_{C^{m+2}(\mathbb{R}^2)}$, such that for all $\theta \leq \theta(m)$*

$$< au + bu_y, Lu >_m \geq C_m(\parallel u \parallel^2_{(m,0)} + \sum_{s=0}^{m} \theta^s \parallel \gamma \partial_x^s u_y \parallel^2), \text{ for all } u \in C_c^\infty(\mathbb{R}^2).$$

Proof. We shall prove the estimate by induction on m. The case $m=0$ is given by lemma 2.1. Let $m \geq 1$, and assume that the estimate holds for all integers less than m.

Let $u \in C_c^\infty(\mathbb{R}^2)$ and set $w = \partial_x^m u$, then

$$\begin{aligned} & < au + bu_y, Lu >_m \\ = \; & < au + bu_y, Lu >_{m-1} + \theta^m (aw + bw_y, L_m w) \\ & + \theta^m \Big(a\partial_x^m u + b\partial_x^m u_y, \sum_{i=0}^{m-1} \partial_x^i (E_x \partial_x^{m-1-i} u_y + \partial_x F_{m-1-i} \partial_x^{m-1-i} u)\Big), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} L_m &= A\partial_{xx} + \partial_{yy} + D_m \partial_x + E\partial_y + F_m, \\ D_m &= D + mA_x, \quad F_m = F + mD_x + \frac{m(m-1)}{2} A_{xx}. \end{aligned}$$

We now estimate each term on the right-hand side of (2.1). By the induction assumption,

$$< au + bu_y, Lu >_{m-1} \geq C_{m-1}(\| u \|^2_{(m-1,0)} + \sum_{s=0}^{m-1} \theta^s \| \gamma \partial_x^s u_y \|^2). \tag{2.2}$$

In addition, since D_x, A_x, A_{xx} have compact support and both $D_m = O(mn\varepsilon^{2n} y^{n-1})$, and $mD_x + \frac{m(m-1)}{2} A_{xx} = O(m^2 n \varepsilon^{2n})$ near the origin, if $\varepsilon = \varepsilon(m)$ is sufficiently small then the coefficients of L_m have the same properties as those of L so that lemma 2.1 applies to yield,

$$\theta^m (aw + bw_y, L_m w) \geq \theta^m C_1 (\| \gamma w_y \|^2 + \| w \|^2). \tag{2.3}$$

Furthermore, integrating by parts produces

$$\begin{aligned} & \Big(a\partial_x^m u + b\partial_x^m u_y, \sum_{i=0}^{m-1} \partial_x^i (E_x \partial_x^{m-1-i} u_y + \partial_x F_{m-1-i} \partial_x^{m-1-i} u)\Big) \\ = \; & \int\int_{\mathbb{R}^2} [e_{m-1} (\partial_x^{m-1} u)^2 + e_{m-2} (\partial_x^{m-2} u)^2 + \cdots + e_0 u^2 \\ & + f_{m-1} (\partial_x^{m-1} u_y)^2 + f_{m-2} (\partial_x^{m-2} u_y)^2 + \cdots + f_0 u_y^2 \\ & + g_{m-1} \partial_x^m u \partial_x^{m-1} u_y + g_{m-2} \partial_x^{m-1} u \partial_x^{m-2} u_y + \cdots + g_0 u_x u_y], \end{aligned} \tag{2.4}$$

for some functions e_i, f_i, g_i depending on the derivatives of A, D, E and F up to and including order $m+2$.

Observe that the power of θ in the third term on the right of (2.1), is sufficiently large to guarantee that the right-hand side of (2.4) may be absorbed into the combined right-hand sides of (2.2) and (2.3), for all $\theta < \theta(m)$ if $\theta(m)$ is sufficiently small. Thus, we obtain

$$< au + bu_y, Lu >_m \geq C_m(\| u \|^2_{(m,0)} + \sum_{s=0}^{m} \theta^s \| \gamma \partial_x^s u_y \|^2),$$

completing the proof by induction. q.e.d.

Let $f \in L^2(\mathbb{R}^2)$, and consider the equation

$$Lu = f. \tag{2.5}$$

A function $u \in L^2(\mathbb{R}^2)$ is said to be a weak solution of (2.5) if

$$(u, L^*v) = (f, v), \text{ for all } v \in C_c^\infty(\mathbb{R}^2),$$

where L^* is the formal adjoint of L. The estimate of theorem 2.1 shall serve as the basis for establishing the existence of weak solutions via the method of Galerkin approximation. That is, we shall construct certain finite-dimensional approximations of (2.5), and then pass to the limit to obtain a solution.

Let $\{\phi_l\}_{l=1}^\infty$ be a basis of $H_\theta^{2m+2}(\mathbb{R})$ that is orthonormal in $H_\theta^m(\mathbb{R})$. Such a sequence may be constructed by applying the Gram-Schmidt process to a basis of $H_\theta^{2m+2}(\mathbb{R})$. Choose a positive integer N. We seek an approximate solution, u^N, of equation (2.5) in the form

$$u^N(x,y) = \sum_{l=1}^{N} d_l^N(y)\phi_l(x),$$

where the functions d_l^N are to be determined from the relations

$$\int_{\mathbb{R}} \sum_{s=0}^{m} \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s L u^N dx = \int_{\mathbb{R}} \sum_{s=0}^{m} \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s f dx, \quad l = 1, \ldots, N. \tag{2.6}$$

The following lemma will establish the existence of the d_l^N.

Lemma 2.2. *Suppose that $\varepsilon = \varepsilon(m)$ and $\theta(m)$ are sufficiently small, and $f \in H_\theta^{(m,0)}(\mathbb{R}^2)$, $m \leq r - 2$. Then there exist functions $d_l^N \in H^2(\mathbb{R})$, $l = 1, \ldots, N$, satisfying (2.6) in the $L^2(\mathbb{R})$-sense.*

Proof. Choose ε and θ so small that theorem 2.1 is valid. Since $\{\phi_l\}_{l=1}^\infty$ is an orthonormal set in $H_\theta^m(\mathbb{R})$, (2.6) becomes

$$\begin{aligned}
&(d_l^N)'' + \sum_{i=1}^{N} \sum_{s=0}^{m} (\int_{\mathbb{R}} \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s (E\phi_i) dx)(d_i^N)' \\
&+ \sum_{i=1}^{N} \sum_{s=0}^{m} (\int_{\mathbb{R}} \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s (A\phi_i'') + \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s (D\phi_i') + \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s (F\phi_i) dx) d_i^N \\
&= \int_{\mathbb{R}} \sum_{s=0}^{m} \theta^s \frac{d^s\phi_l}{dx^s} \partial_x^s f dx, \quad l = 1, \ldots, N.
\end{aligned} \tag{2.7}$$

By the theory of ordinary differential equations, it is sufficient to prove uniqueness to obtain existence of a solution to system (2.7).

We now establish the uniqueness of solutions to (2.7) in the space $H^2(\mathbb{R})$. Multiply (2.6) by $a(y)d_l^N(y) + b(y)(d_l^N)'(y)$, sum over l from 1 to N, and then integrate with respect to y over $\mathbb{R}$ to obtain

$$< au^N + bu_y^N, Lu^N >_m = < au^N + bu_y^N, f >_m .$$

It now follows from theorem 2.1 that

$$C_m(\| u^N \|_{(m,0)}^2 + \sum_{s=0}^{m} \theta^s \| \gamma \partial_x^s u_y^N \|^2) \leq < au^N + bu_y^N, f >_m, \tag{2.8}$$

for some constant $C_m > 0$ independent of N. Again using the orthonormal properties of $\{\phi_l\}_{l=1}^{\infty}$, we find

$$\sum_{l=1}^{N}(\| d_l^N \|_{\mathbb{R}}^2 + \| \gamma (d_l^N)' \|_{\mathbb{R}}^2) = \| u^N \|_{(m,0)}^2 + \sum_{s=0}^{m} \theta^s \| \gamma \partial_x^s u_y^N \|^2 . \tag{2.9}$$

Uniqueness for solutions of (2.6) in the space of functions for which the left-hand side of (2.9) is finite, now follows from (2.8) and (2.9). Thus, existence of a solution in this space is guaranteed; furthermore, since we can solve for $(d_l^N)''$ in (2.7), it follows that this solution is in $H^2(\mathbb{R})$. q.e.d.

Before proving the existence of a weak solution to equation (2.5), we will need one more lemma.

Lemma 2.3. *Let $v \in C_c^\infty(\mathbb{R}^2)$. Then there exists a unique solution, $\widehat{v} \in H^{(\infty,0)}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$, of the equation*

$$(-\theta)^m \partial_x^{2m} \widehat{v} + (-\theta)^{m-1} \partial_x^{2(m-1)} \widehat{v} + \cdots + \widehat{v} = v. \tag{2.10}$$

Proof. By the Riesz representation theorem, there exists a unique $\widehat{v} \in H^{(m,0)}(\mathbb{R}^2)$, such that

$$< \widehat{v}, w >_m = (v, w), \text{ for all } w \in C_c^\infty(\mathbb{R}^2). \tag{2.11}$$

Thus, $\widehat{v}$ is a weak solution of (2.10), and by the theory of ordinary differential equations with parameter, we have $\widehat{v} \in C^\infty(\mathbb{R}^2)$.

We now show that $\widehat{v} \in H^{(\infty,0)}(\mathbb{R}^2)$. It follows from (2.11) and the result of Friedrichs [2] on the identity of weak and strong solutions, that there exists a sequence $\{\widehat{v}^k\}_{k=1}^{\infty} \subset C_c^\infty(\mathbb{R}^2)$ such that $\widehat{v}^k \to \widehat{v}$ in $H^{(m,0)}(\mathbb{R}^2)$, and

$$(-\theta)^m \partial_x^{2m} \widehat{v}^k + \cdots + (-\theta)^{\frac{m_0+2}{2}} \partial_x^{m_0+2} \widehat{v}^k \to v - (-\theta)^{\frac{m_0}{2}} \partial_x^{m_0} \widehat{v} - \cdots - \widehat{v} \quad \text{in} \quad L^2(\mathbb{R}^2),$$

where $m_0 = m$ if m is even, and $m_0 = m-1$ if m is odd. Therefore,

$$\begin{aligned}\iint_{\mathbb{R}^2} v^2 = & \iint_{\mathbb{R}^2} [(-\theta)^m \partial_x^{2m}\widehat{v} + \cdots + (-\theta)^{\frac{m_0+2}{2}} \partial_x^{m_0+2}\widehat{v}]^2 \\ & +2[(-\theta)^m \partial_x^{2m}\widehat{v} + \cdots + (-\theta)^{\frac{m_0+2}{2}} \partial_x^{m_0+2}\widehat{v}][(-\theta)^{\frac{m_0}{2}} \partial_x^{m_0}\widehat{v} + \cdots + \widehat{v}] \\ & +[(-\theta)^{\frac{m_0}{2}} \partial_x^{m_0}\widehat{v} + \cdots + \widehat{v}]^2 \\ \geq & \lim_{k\to\infty} \iint_{\mathbb{R}^2} 2[(-\theta)^m \partial_x^{2m}\widehat{v}^k + \cdots + (-\theta)^{\frac{m_0+2}{2}} \partial_x^{m_0+2}\widehat{v}^k][(-\theta)^{\frac{m_0}{2}} \partial_x^{m_0}\widehat{v}^k + \cdots + \widehat{v}^k] \\ & +[(-\theta)^{\frac{m_0}{2}} \partial_x^{m_0}\widehat{v}^k + \cdots + \widehat{v}^k]^2.\end{aligned}$$

Integrating by parts yields,

$$\iint_{\mathbb{R}^2} v^2 \geq \lim_{k\to\infty} \iint_{\mathbb{R}^2} \theta^{m+1}(\partial_x^{m+1}\widehat{v}^k)^2 + \cdots + (\widehat{v}^k)^2,$$

if $m > 1$. Since bounded sets in Hilbert spaces are weakly compact, $\widehat{v}^{k_l} \rightharpoonup \overline{v}$ weakly in $H^{(m+1,0)}(\mathbb{R}^2)$, for some $\overline{v} \in H^{(m+1,0)}(\mathbb{R}^2)$, where $\{\widehat{v}^{k_l}\}_{l=1}^{\infty}$ is a subsequence of $\{\widehat{v}^k\}$. For simplicity, we denote $\widehat{v}^{k_l}$ by $\widehat{v}^k$.

We now show that $\widehat{v} \equiv \overline{v}$. By the Riesz representation theorem, there exists $w \in H^{(m+1,0)}(\mathbb{R}^2)$ such that

$$< w, z >_{m+1} = < \widehat{v} - \overline{v}, z >_m, \text{ for all } z \in H^{(m+1,0)}(\mathbb{R}^2).$$

In particular, setting $z = \widehat{v}^k - \overline{v}$ we have

$$\lim_{k\to\infty} < w, \widehat{v}^k - \overline{v} >_{m+1} = \lim_{k\to\infty} < \widehat{v} - \overline{v}, \widehat{v}^k - \overline{v} >_m = \| \widehat{v} - \overline{v} \|_{(m,0)}^2 . \tag{2.12}$$

Furthermore, since $\widehat{v}^k \rightharpoonup \overline{v}$ we have

$$\lim_{k\to\infty} < w, \widehat{v}^k - \overline{v} >_{m+1} = 0. \tag{2.13}$$

Combining (2.12) and (2.13) we obtain $\widehat{v} \equiv \overline{v}$ in $H^{(m,0)}(\mathbb{R}^2)$, implying that $\widehat{v} \in H^{(m+1,0)}(\mathbb{R}^2)$. Recall that we assumed that $m > 1$; however, if $m = 1$ we still obtain $\widehat{v} \in H^{(m+1,0)}(\mathbb{R}^2)$ by solving for $\partial_{xx}\widehat{v}$ in (2.10). A boot-strap argument can now be used to show that $\widehat{v} \in H^{(\infty,0)}(\mathbb{R}^2)$. q.e.d.

We are now ready to establish the existence of a weak solution of equation (2.5), having regularity in the x-direction.

Theorem 2.2. *If $\varepsilon = \varepsilon(m)$ and $\theta(m)$ are sufficiently small, then for every $f \in H_\theta^{(m,0)}(\mathbb{R}^2)$, $m \leq r-2$, there exists a unique weak solution $u \in H_\theta^{(m,1)}(\mathbb{R}^2)$ of* (2.5).

Proof. For each $N \in \mathbb{Z}_{>0}$, let $u^N \in H_\theta^{(m,2)}(\mathbb{R}^2)$ be given by lemma 2.2. Then applying Cauchy's inequality ($pq \leq \kappa p^2 + \frac{1}{4\kappa}q^2$, $\kappa > 0$) to the right-hand side of (2.8), we obtain

$$\| u^N \|_{(m,1)} \leq C_m' \| f \|_{(m,0)}, \tag{2.14}$$

where C_m' is independent of N. Since bounded sets in Hilbert spaces are weakly compact, there exists a subsequence $\{u^{N_i}\}_{i=1}^\infty$ such that $u^{N_i} \rightharpoonup u$ in $H_\theta^{(m,1)}(\mathbb{R}^2)$, for some $u \in H_\theta^{(m,1)}(\mathbb{R}^2)$.

We now show that u is a weak solution of (2.5). Let $v \in C_c^\infty(\mathbb{R}^2)$ and let $\widehat{v} \in H^{(\infty,0)}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ be the solution of

$$(-\theta)^m \partial_x^{2m} \widehat{v} + (-\theta)^{m-1} \partial_x^{2(m-1)} \widehat{v} + \cdots + \widehat{v} = v,$$

given by lemma 2.3. Since $\{\phi_l(x)\}_{l=1}^\infty$ forms a basis in $H_\theta^{2m+2}(\mathbb{R})$, we can find $e_l^{N_*}(y) \in H^\infty(\mathbb{R})$ such that $v^{N_*} := \sum_{l=1}^{N_*} e_l^{N_*}(y)\phi_l(x) \to \widehat{v}$ in $H_\theta^{(2m+2,2)}(\mathbb{R}^2)$ as $N_* \to \infty$. Then multiply (2.6) by $e_l^{N_*}$, sum over l from 1 to N_*, and integrate with respect to y over $\mathbb{R}$ to obtain

$$< v^{N_*}, Lu^{N_i} >_m = < v^{N_*}, f >_m .$$

Integrating by parts and letting $N_i \to \infty$ produces,

$$(u, L^*(v^{N_*} + \cdots + (-\theta)^m \partial_x^{2m} v^{N_*})) = (f, v^{N_*} + \cdots + (-\theta)^m \partial_x^{2m} v^{N_*}).$$

Furthermore, by letting $N_* \to \infty$ we obtain

$$(u, L^* v) = (f, v).$$

Uniqueness of the weak solution follows from (2.14). q.e.d.

We now prove regularity in the y-direction for the weak solution given by theorem 2.2, in the case that $f \in H^m(\mathbb{R}^2)$. The following standard lemma concerning difference quotients will be needed.

Lemma 2.4. *Let $w \in L^2(\mathbb{R}^2)$ have compact support, and define*

$$w^h = \frac{1}{h}(w(x, y + h) - w(x, y)).$$

If $\| w^h \| \leq C_8$ where C_8 is independent of h, then $w \in H^{(0,1)}(V)$ for any compact $V \subset \mathbb{R}^2$. Furthermore, if $w \in H^{(0,1)}(\mathbb{R}^2)$ then $\| w^h \| \leq C_9 \| w_y \|$, for some C_9 independent of h.

Theorem 2.3. *Suppose that the hypotheses of theorem 2.2 are fulfilled and that $f \in H^m(\mathbb{R}^2)$, then $u \in H^m(\mu_2 X)$.*

Proof. If $m = 0, 1$, then the desired conclusion follows directly from theorem 2.2, so assume that $m \geq 2$. Let $\zeta \in C^\infty(\mathbb{R}^2)$ be a cut-off function such that

$$\zeta(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \mu_2 X, \\ 0 & \text{if } (x, y) \in (\mu_2 + 1)X. \end{cases}$$

Let $u \in H_\theta^{(m,1)}(\mathbb{R}^2)$ be the weak solution of (2.5) given by theorem 2.2. Set $w = \zeta u$, then since u is a weak solution of (2.5) we obtain

$$[w,v] := \int\int_{\mathbb{R}^2} w_y v_y - E w_y v - F w v = \int\int_{\mathbb{R}^2} -\widetilde{f} v, \text{ for all } v \in C_c^\infty(\mathbb{R}^2),$$

where $\widetilde{f} = \zeta f - A\zeta u_{xx} + \zeta_{yy} u + 2\zeta_y u_y - D\zeta u_x + E\zeta_y u$.

Using lemma 2.4 and the fact that $\widetilde{f} \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned} |[w^h, v]| &\leq |[w, v^{-h}]| + C_{10} \parallel v \parallel_{(0,1)} \\ &= |\int\int_{\mathbb{R}^2} \widetilde{f} v^{-h}| + C_{10} \parallel v \parallel_{(0,1)} \\ &\leq C_{11} \parallel v \parallel_{(0,1)}, \end{aligned} \tag{2.15}$$

for some constants C_{10}, C_{11} independent of h. Furthermore, integrating by parts yields

$$C_{12} \parallel v \parallel_{(0,1)}^2 \leq |[v,v]| + C_{13} \parallel v \parallel . \tag{2.16}$$

The estimates (2.15) and (2.16) also hold if $v = w^h$. Therefore

$$\begin{aligned} C_{12} \parallel w^h \parallel_{(0,1)}^2 &\leq C_{11} \parallel w^h \parallel_{(0,1)} + C_{13} \parallel w^h \parallel \\ &\leq C_{11} \parallel w^h \parallel_{(0,1)} + C_{14}, \end{aligned}$$

for some constant C_{14} independent of h. It follows that $\parallel w^h \parallel_{(0,1)} \leq C_{15}$ independent of h. Hence, by lemma 2.4 $w \in H^{(0,2)}(V)$ for any compact $V \subset \mathbb{R}^2$. Since $w \equiv u$ in $\mu_2 X$, we have $u \in H^{(0,2)}(\mu_2 X)$.

By differentiating $Lu = f$ with respect to x, $s = 1, \ldots, m-2$ times, we obtain

$$L_s z = \partial_x^s f - \sum_{i=0}^{s-1} \partial_x^i (E_x \partial_x^{s-1-i} u_y + \partial_x F_{s-1-i} \partial_x^{s-1-i} u), \tag{2.17}$$

where $z = \partial_x^s u$ and L_s, F_s were defined in (2.1). We may then apply the above procedure to equation (2.17) and obtain $\partial_x^s u \in H^{(0,2)}(\mu_2 X)$, $s = 1, \ldots, m-2$.

Lastly, denote the right-hand side of (2.17) by f_s, then the following equation holds in $L^2(\mu_2 X)$,

$$z_{yy} = f_s - A z_{xx} - (D + sA_x) z_x - E z_y - (F + sD_x + \frac{s(s-1)}{2} A_{xx}) z. \tag{2.18}$$

Since the right-hand side of (2.18) is in $H^{(0,1)}(\mu_2 X)$, it follows that $z_{yy} \in H^{(0,1)}(\mu_2 X)$. Then by differentiating (2.18) with respect to y, we may apply a boot-strap argument to obtain $u \in H^m(\mu_2 X)$. q.e.d.

3. The Moser Estimate

Having established the existence of regular solutions to a small perturbation of the linearized equation for (0.6), we intend to apply a Nash-Moser type iteration procedure in the following section, to obtain a smooth solution of (0.6) in X. In the current section, we shall make preparations for the Nash-Moser procedure by establishing a certain a priori estimate. This estimate, referred to as the Moser estimate, will establish the dependence of the solution u of (2.5), on the coefficients of the linearization as well as on the right-hand side, f. If the linearization is evaluated at some function $w \in C^{\infty}(\mu_2\overline{X})$, then the Moser estimate is of the form

$$\| u \|_{H^m} \leq C_m(\| f \|_{H^m} + \| w \|_{H^{m+m_1}} \| f \|_{H^2}), \tag{3.1}$$

for some constants C_m and m_1 independent of ε and w.

Estimate (3.1) will first be established in the coordinates (α, β), which we have been denoting by (x, y) for convenience, and later converted into the original coordinates (x, y) of the introduction. We will need the Gagliardo-Nirenberg estimates contained in the following lemma.

Lemma 3.1. *Let $u, v \in C^k(\overline{\Omega})$.*

i) If σ and ϱ are multi-indices such that $|\sigma| + |\varrho| = k$, then there exist constants $\mathcal{M}_1$ and $\mathcal{M}_2$ depending on k, such that

$$\| \partial^{\sigma} u \partial^{\varrho} v \|_{L^2(\Omega)} \leq \mathcal{M}_1(|u|_{L^{\infty}(\Omega)} \| v \|_{H^k(\Omega)} + \| u \|_{H^k(\Omega)} |v|_{L^{\infty}(\Omega)}),$$

and

$$|\partial^{\sigma} u \partial^{\varrho} v|_{C^0(\overline{\Omega})} \leq \mathcal{M}_2(|u|_{C^0(\overline{\Omega})}|v|_{C^k(\overline{\Omega})} + |u|_{C^k(\overline{\Omega})}|v|_{C^0(\overline{\Omega})}).$$

ii) If $\sigma_1, \ldots, \sigma_l$ are multi-indices such that $|\sigma_1| + \cdots + |\sigma_l| = k$, then there exists a constant $\mathcal{M}_3$ depending on l and k, such that

$$\| \partial^{\sigma_1} u_1 \cdots \partial^{\sigma_l} u_l \|_{L^2(\Omega)} \leq \mathcal{M}_3 \sum_{j=1}^{l} (|u_1|_{L^{\infty}(\Omega)} \cdots \widehat{|u_j|}_{L^{\infty}(\Omega)} \cdots |u_l|_{L^{\infty}(\Omega)}) \| u_j \|_{H^k(\Omega)},$$

where $\widehat{|u_j|}_{L^{\infty}(\Omega)}$ indicates the absence of $|u_j|_{L^{\infty}(\Omega)}$.

iii) Let $B \subset \mathbb{R}^N$ be compact and contain the origin, and let $G \in C^{\infty}(B)$. If $u \in H^{k+2}(\Omega, B)$ and $\| u \|_{H^2(\Omega)} \leq \mathcal{C}$ for some fixed $\mathcal{C}$, then there exist constants $\mathcal{M}, \mathcal{M}_k > 0$ such that

$$\| G \circ u \|_{H^k(\Omega)} \leq \mathcal{M} + \mathcal{M}_k \| u \|_{H^{k+2}(\Omega)},$$

where $\mathcal{M} = \mathrm{Vol}(\Omega)|G(0)|$.

Proof. These estimates are standard consequences of the interpolation inequalities, and may be found in, for instance, [16]. q.e.d.

Estimate (3.1) will follow by induction from the next two propositions. The first shall establish an estimate for the x-derivatives only, while the second deals with all remaining derivatives.

Proposition 3.1. *Suppose that the linearization, L_1, is evaluated at some function $w \in C^\infty(\mathbb{R}^2)$ with $|w|_{C^{16}} \leq C_1$, as in (1.1). Let $f \in H^m(\mathbb{R}^2)$ and $u \in H^{(m,1)}(\mathbb{R}^2) \cap H^m(\mu_2 X)$, $m \leq r-7$, be the solution of (2.5). If $\varepsilon = \varepsilon(m)$ is sufficiently small, then*

$$\| \partial_x^m u \| + \| \partial_x^m u_y \| \leq C_m(\| f \|_m + \| u \|_{H^{m-1}(\mu_2 X)} + \| w \|_{H^{m+7}(\mu_2 X)} \| f \|_{H^2(\mu_2 X)}),$$

for some constant C_m independent of ε and w.

Proof. We proceed by induction on m. The case $m = 0$ is given by lemma 2.1. Now assume that the estimate holds for all positive integers less than m. Differentiate $L(w)u = f$ m-times with respect to x and put $v = \partial_x^m u$, then

$$L_m v = \partial_x^m f - \sum_{i=0}^{m-1} \partial_x^i (E_x \partial_x^{m-1-i} u_y + \partial_x F_{m-1-i} \partial_x^{m-1-i} u) := f_m,$$

where L_m and F_m were defined in (2.1). If $\varepsilon = \varepsilon(m)$ is sufficiently small , we can apply lemma 2.1 to obtain

$$\| \partial_x^m u \| + \| \partial_x^m u_y \| \leq M \| f_m \| . \tag{3.2}$$

We now estimate each term of f_m. Let $\| \cdot \|_{m,\, \mu_2 X}$ denote $\| \cdot \|_{H^m(\mu_2 X)}$, and $| \cdot |_\infty$ denote $| \cdot |_{L^\infty(\mu_2 X)}$. A calculation shows that

$$\sum_{i=0}^{m-1} \partial_x^i (E_x \partial_x^{m-1-i} u_y) = m E_x \partial_x^{m-1} u_y + \sum_{i=1}^{m-1} \sum_{j=1}^{i} \begin{pmatrix} i \\ j \end{pmatrix} \partial_x^{j+1} E \partial_x^{m-1-j} u_y.$$

Then using lemma 3.1 (i) and (iii), and recalling that E_x vanishes on $\mathbb{R}^2 - \mu_2 X$, we obtain

$$\begin{aligned} \| \sum_{i=0}^{m-1} \partial_x^i (E_x \partial_x^{m-1-i} u_y) \| \quad &\leq \quad M_1 \| \partial_x^{m-1} u_y \| \\ &+ \quad M_2 (|\partial_x^2 E|_\infty \| u \|_{m-1,\, \mu_2 X} + \| \partial_x^2 E \|_{m-1,\, \mu_2 X} |u|_\infty) \\ &\leq \quad M_1 \| \partial_x^{m-1} u_y \| \\ &+ \quad M_3 (|E|_{C^2(\mu_2 \overline{X})} \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+6,\, \mu_2 X} \| u \|_{2,\, \mu_2 X}). \end{aligned}$$

Using the fact that $|E|_{C^2(\mu_2 \overline{X})} \leq C_{14}'$ (lemma 1.3), and the induction assumption, we have

$$\| \sum_{i=0}^{m-1} \partial_x^i (E_x \partial_x^{m-1-i} u_y) \| \tag{3.3}$$

$$\leq C'_{m-1}(\| f \|_{m-1} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+6,\, \mu_2 X} \| u \|_{2,\, \mu_2 X}).$$

In a similar manner, we may estimate

$$\| \sum_{i=0}^{m-1} \partial_x^i (\partial_x F_{m-1-i} \partial_x^{m-1-i} u) \| \tag{3.4}$$

$$\leq C''_{m-1}(\| f \|_{m-1} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| u \|_{2,\, \mu_2 X}).$$

Furthermore, the methods used above can be made to show that $\| u \|_{2,\, \mu_2 X} \leq M_4 \| f \|_{2,\, \mu_2 X}$. Then (3.3) and (3.4) yield

$$\| \partial_x^m u \| + \| \partial_x^m u_y \| \leq C_m(\| f \|_m + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| f \|_{2,\, \mu_2 X}),$$

completing the proof by induction. q.e.d.

We now estimate the remaining derivatives.

Proposition 3.2. *Let u, w, f, ε, and m be as in proposition 3.1. Then*

$$\| \partial_x^s \partial_y^t u \|_{\mu_2 X} \leq C_m(\| f \|_{m,\, \mu_2 X} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| f \|_{2,\, \mu_2 X}),$$

for $0 \leq s \leq m-t$, $0 \leq t \leq m$, where C_m is independent of ε and w.

Proof. The cases $t = 0, 1$ are given by proposition 3.1. We will proceed by induction on t. Assume that the desired estimate holds for $0 \leq s \leq m-t$, $0 \leq t \leq k-1$, $0 \leq k \leq m$.

Solving for u_{yy} in the equation $L(w)u = f$, we obtain

$$u_{yy} = f - Au_{xx} - Du_x - Eu_y - Fu := \overline{f}. \tag{3.5}$$

Differentiate (3.5) with respect to $\partial_x^s \partial_y^{k-2}$ where $0 \leq s \leq m-k$, then

$$\partial_x^s \partial_y^k u = \partial_x^s \partial_y^{k-2} \overline{f}. \tag{3.6}$$

We now estimate each term on the right-hand side of (3.6). Using lemma 3.1 (i) and (iii), we have

$$\begin{aligned} & \| \partial_x^s \partial_y^{k-2}(Au_{xx}) \|_{\mu_2 X} \\ \leq\ & M_5(\| \partial_x^{s+2} \partial_y^{k-2} u \|_{\mu_2 X} + \sum_{\substack{p \leq s,\ q \leq k-2 \\ (p,q) \neq (0,0)}} \| \partial_x^p \partial_y^q A \partial_x^{s-p} \partial_y^{k-2-q} u_{xx} \|_{\mu_2 X}) \\ \leq\ & M'_5(\| \partial_x^{s+2} \partial_y^{k-2} u \|_{\mu_2 X} + |A|_{C^1(\mu_2 \overline{X})} \| u \|_{m-1,\, \mu_2 X} + \| A \|_{m,\, \mu_2 X} |u|_\infty) \\ \leq\ & M''_5(\| \partial_x^{s+2} \partial_y^{k-2} u \|_{\mu_2 X} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+4,\, \mu_2 X} \| f \|_{2, \mu_2 X}). \end{aligned}$$

Furthermore, since $s \leq m-k$ the induction assumption implies that

$$\| \partial_x^{s+2} \partial_y^{k-2} u \|_{\mu_2 X} \leq M_6(\| f \|_{m,\, \mu_2 X} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| f \|_{2,\, \mu_2 X}).$$

Thus,

$$\| \partial_x^s \partial_y^{k-2}(Au_{xx}) \|_{\mu_2 X} \leq M_7(\| f \|_{m,\, \mu_2 X} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| f \|_{2,\, \mu_2 X}).$$

The remaining terms on the right-hand side of (3.6) may be estimated in a similar manner. Therefore,

$$\| \partial_x^s \partial_y^k u \|_{\mu_2 X} \leq M_8(\| f \|_{m,\, \mu_2 X} + \| u \|_{m-1,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| f \|_{2,\, \mu_2 X}),$$

for $0 \leq s \leq m-k$. The proof is now complete by induction. q.e.d.

By combining the previous two propositions, we obtain the following Moser estimate.

Theorem 3.1. *Let u, w, f, ε, and m be as in proposition* 3.2. *Then*

$$\| u \|_{m,\, \mu_2 X} \leq C_m(\| f \|_{m,\, \mu_2 X} + \| w \|_{m+7,\, \mu_2 X} \| f \|_{2,\, \mu_2 X}),$$

where C_m is independent of ε and w.

Proof. This follows by induction on m, using proposition 3.2. q.e.d.

The Moser estimate of theorem 3.1 is in terms of the variables (α, β) of lemma 1.3. Since the Nash-Moser iteration procedure of the following section will be carried out in the original variables, (x, y), of the introduction, we will now obtain an analogous Moser estimate in these original coordinates. Let $\| \cdot \|_{m,\, \Omega}$, $\| \cdot \|'_{m,\, \Omega}$, and $\| \cdot \|''_{m,\, \Omega}$ denote the $H^m(\Omega)$ norm with respect to the variables (x, y), (ξ, η), and (α, β) respectively; a similar notation will be used for the $C^m(\overline{\Omega})$ norms. The following estimates will be needed in transforming the estimate of theorem 3.1 into the original variables.

Lemma 3.2. *If $\varepsilon = \varepsilon(m)$ is sufficiently small, then*

$$\| \xi_x \|_{m,\, X_1} \leq C_m(1 + \| w \|_{m+7,\, X_1}), \ \text{and} \ \| \alpha_\xi \|'_{m,\, X_2} \leq C'_m(1 + \| w \|'_{m+7,\, X_2}),$$

where C_m and C'_m are independent of ε and w, and X_1, X_2 were defined in lemmas 1.2 *and* 1.3.

Proof. We shall only prove the first estimate, since a similar argument yields the second. The estimate will be proven by induction on m. From the proof of lemma 1.2 we have,

$$|\xi_x|_{C^0(\overline{X}_1)} \leq M_9,$$

which gives the case $m = 0$. Now assume that the following estimate holds,

$$|\xi_x|_{C^{m-1}(\overline{X}_1)} \leq C_{m-1} |b_{12}^3|_{C^m(\overline{X}_1)}.$$

We will first estimate the x-derivatives. Differentiate the equation,

$$b_{12}^3(\xi_x)_x + (\xi_x)_y = -(b_{12}^3)_x \xi_x, \tag{3.7}$$

m-times with respect to x to obtain

$$b_{12}^3(\partial_x^m \xi_x)_x + (\partial_x^m \xi_x)_y = -\partial_x^m[(b_{12}^3)_x \xi_x] - \sum_{i=0}^{m-1} \partial_x^i[(b_{12}^3)_x \partial_x^{m-i}\xi_x] := g.$$

Then estimating $\partial_x^m \xi_x$ along the characteristics of (3.7) as in the proof of lemma 1.2, we find

$$|\partial_x^m \xi_x|_{C^0(\overline{X}_1)} \leq \mu_1 y_0 |g|_{C^0(\overline{X}_1)}.$$

Using the second half of lemma 3.1 (i) in the same way that the first half was used in proposition 3.1, and recalling that $|b_{12}^3|_{C^2(\overline{X}_1)} \leq \varepsilon M_{10}$, produces

$$\begin{aligned} |g|_{C^0(\overline{X}_1)} \;\leq\; & (m+1)\varepsilon M_{10}|\partial_x^m \xi_x|_{C^0(\overline{X}_1)} \\ & + M_{10}'(|(b_{12}^3)_{xx}|_{C^0(\overline{X}_1)}|\xi_x|_{C^{m-1}(\overline{X}_1)} + |(b_{12}^3)_{xx}|_{C^{m-1}(\overline{X}_1)}|\xi_x|_{C^0(\overline{X}_1)}). \end{aligned}$$

Therefore, if ε is small enough to guarantee that $(m+1)\mu_1 y_0 \varepsilon M_{10} < \frac{1}{2}$, we can bring $(m+1)\mu_1 y_0 \varepsilon M_{10}|\partial_x^m \xi_x|_{C^0(\overline{X}_1)}$ to the left-hand side:

$$|\partial_x^m \xi_x|_{C^0(\overline{X}_1)} \leq M_{11}(|\xi_x|_{C^{m-1}(\overline{X}_1)} + |b_{12}^3|_{C^{m+1}(\overline{X}_1)}). \tag{3.8}$$

By solving for $(\xi_x)_y$ in equation (3.7), and differentiating the result with respect to $\partial_x^s \partial_y^{t-1}$, $0 \leq s \leq m-t$, $0 \leq t \leq m$, we can use the techniques of proposition 3.2, combined with lemma 3.1 (i), to obtain

$$|\partial_x^s \partial_y^t \xi_x|_{C^0(\overline{X}_1)} \leq M_{12}(|\xi_x|_{C^{m-1}(\overline{X}_1)} + |b_{12}^3|_{C^{m+1}(\overline{X}_1)}). \tag{3.9}$$

By the induction assumption on m, (3.9) implies that

$$|\xi_x|_{C^m(\overline{X}_1)} \leq M_{13}|b_{12}^3|_{C^{m+1}(\overline{X}_1)}.$$

Then the Sobolev embedding theorem gives

$$\| \xi_x \|_{m,\, X_1} \leq M_{14} \| b_{12}^3 \|_{m+3,\, X_1}.$$

Thus, by lemma 3.1 (iii) we have

$$\| \xi_x \|_{m,\, X_1} \leq M_{15}(1 + \| w \|_{m+7,\, X_1}).$$

q.e.d.

Theorem 3.2. *Let u, w, and f be as in theorem* 3.1, *and $m \leq r - 25$. If $\varepsilon = \varepsilon(m)$ is sufficiently small, then*

$$\| u \|_{m,\, X} \leq C_m(\| f \|_{m,\, X_1} + \| w \|_{m+25,\, X_1} \| f \|_{2,\, X_1}),$$

where C_m is independent of ε and w.

Proof. We first prove an analogue of the desired estimate in terms of the variables (ξ,η). Observe that

$$\xi_\alpha = \frac{1}{\alpha_\xi}\left(\frac{\beta_\eta^2}{\beta_\eta^2+\beta_\xi^2 b_{12}^5}\right) \geq M_{16} \tag{3.10}$$

for some $M_{16} > 0$, if ε is sufficiently small. Let $G(b_{12}^5) = \beta_\eta^2/(\beta_\eta^2+\beta_\xi^2 b_{12}^5)$, and $s = m-t$, $0 \leq t \leq m$. A calculation shows that

$$\| \partial_\xi^s \partial_\eta^t u \|_{X_2}^{'} \leq M_{17} \sum_{k=0}^{m}\sum_{i=0}^{k} \| R_{ik}\partial_\alpha^{k-i}\partial_\beta^i u \|_{\mu_2 X}^{''},$$

where the R_{ik} are polynomials in the variables $\nabla_{\alpha,\beta}^{\sigma_1}\xi_\alpha$, $\nabla_{\alpha,\beta}^{\sigma_2}\xi_\alpha^{-1}$, $\nabla_{\alpha,\beta}^{\sigma_3}b_{12}^5$, $\nabla_{\alpha,\beta}^{\sigma_4}G(b_{12}^5)$, $\nabla_{\xi,\eta}^{\sigma_5+1}\beta$, such that $|\sigma_j| \leq m-k$, $1 \leq j \leq 5$, and $\sum_\nu |\sigma_\nu| \leq m-k$, where $\sum_\nu |\sigma_\nu|$ represents the sum over all σ_j appearing in an arbitrary term of R_{ik}. Then using lemma 3.1 (ii) and (iii), we find that

$$\begin{aligned}
\| \partial_\xi^s \partial_\eta^t u \|_{X_2}^{'} &\leq M_{18}[\| u \|_{m,\,\mu_2 X}^{''} \\
&\quad +(\| \xi_\alpha \|_{m,\,\mu_2 X}^{''} + \| \xi_\alpha^{-1} \|_{m,\,\mu_2 X}^{''} + \| b_{12}^5 \|_{m+2,\,\mu_2 X}^{''})|u|_\infty] \quad (3.11)\\
&\leq M_{18}^{'}[\| u \|_{m,\,\mu_2 X}^{''} +(\| \xi_\alpha \|_{m+2,\,\mu_2 X}^{''} + \| w \|_{m+6,\,\mu_2 X}^{''})|u|_\infty].
\end{aligned}$$

Similarly,

$$\| \partial_\alpha^s \partial_\beta^t u \|_{\mu_2 X}^{''} \leq M_{19}[\| u \|_{m,\,X_2}^{'} +(\| \alpha_\xi \|_{m+2,\,X_2}^{'} + \| w \|_{m+6,\,X_2}^{'})|u|_\infty]. \tag{3.12}$$

Then by theorem 3.1 and the Sobolev lemma, we have

$$\begin{aligned}
\| \partial_\xi^s \partial_\eta^t u \|_{X_2}^{'} &\leq M_{20}(\| f \|_{m,\,\mu_2 X}^{''} + \| w \|_{m+7,\,\mu_2 X}^{''}\| f \|_{2,\,\mu_2 X}^{''}) \quad (3.13)\\
&\quad +M_{20}^{'}(\| \xi_\alpha \|_{m+2,\,\mu_2 X}^{''} + \| w \|_{m+6,\,\mu_2 X}^{''}) \| f \|_{2,\,X_2}^{'}.
\end{aligned}$$

We now estimate the terms on the right-hand side of (3.13). Using lemma 3.1 (i), (iii), lemma 3.2, (3.12), and (3.10) we have

$$\begin{aligned}
\| \xi_\alpha \|_{m+2,\,\mu_2 X}^{''} &\leq M_{21}[\| \xi_\alpha \|_{m+2,\,X_2}^{'} +(\| \alpha_\xi \|_{m+4,\,X_2}^{'} + \| w \|_{m+8,\,X_2}^{'})|\xi_\alpha|_\infty]\\
&\leq M_{22}[\| \alpha_\xi^{-1}G(b_{12}^5) \|_{m+2,\,X_2}^{'} + \| \alpha_\xi \|_{m+4,\,X_2}^{'} + \| w \|_{m+8,\,X_2}^{'}]\\
&\leq M_{23}[|G(b_{12}^5)|_\infty \| \alpha_\xi^{-1} \|_{m+2,\,X_2}^{'} + \| G(b_{12}^5) \|_{m+2,\,X_2}^{'} |\alpha_\xi^{-1}|_\infty\\
&\quad + \| \alpha_\xi \|_{m+4,\,X_2}^{'} + \| w \|_{m+8,\,X_2}^{'}]\\
&\leq M_{24}[\| \alpha_\xi \|_{m+4,\,X_2}^{'} + \| w \|_{m+8,\,X_2}^{'}]\\
&\leq M_{25}[1+ \| w \|_{m+11,\,X_2}^{'}].
\end{aligned}$$

Furthermore, by (3.12), lemma 3.2, and the Sobolev lemma

$$\begin{aligned}
\| f \|_{m,\,\mu_2 X}^{''} &\leq M_{26}[\| f \|_{m,\,X_2}^{'} +(\| \alpha_\xi \|_{m+2,\,X_2}^{'} + \| w \|_{m+6,\,X_2}^{'}) \| f \|_{2,\,X_2}^{'}]\\
&\leq M_{26}^{'}[\| f \|_{m,\,X_2}^{'} + \| w \|_{m+9,\,X_2}^{'}\| f \|_{2,\,X_2}^{'}].
\end{aligned}$$

Also, the same method yields

$$\begin{aligned}\| w \|''_{m+7,\,\mu_2 X} &\leq M_{27}(\| w \|'_{m+7,\,X_2} + \| w \|'_{m+16,\,X_2}\| w \|'_{2,\,X_2})\\ &\leq M'_{27} \| w \|'_{m+16,\,X_2}\,.\end{aligned}$$

Therefore, from (3.13) and the above estimates we obtain

$$\| u \|'_{m,\,X_2} \leq M_{28}(\| f \|'_{m,\,X_2} + \| w \|'_{m+16,\,X_2}\| f \|'_{2,\,X_2}). \tag{3.14}$$

We can now apply the same procedure to obtain the following analogue of (3.14) in terms of the original variables (x, y),

$$\| u \|_{m,\,X} \leq M_{29}(\| f \|_{m,\,X_1} + \| w \|_{m+25,\,X_1}\| f \|_{2,\,X_1}).$$

q.e.d.

4. The Nash-Moser Procedure

In this section we will carry out a Nash-Moser type iteration procedure to obtain a solution of

$$\Phi(w) = 0 \text{ in } X. \tag{4.1}$$

Instead of solving the linearized equation at each iteration, we shall solve a small perturbation of the modified linearized equation $L_7(v)u = f$, and then estimate the error at each step. However, the theory of sections §2 and §3 requires that v and f be defined on the whole plane. Therefore, we will need the following extension theorem.

Theorem 4.1 [15]. *Let Ω be a bounded convex domain in $\mathbb{R}^2$, with Lipschitz smooth boundary. Then there exists a linear operator $T_\Omega : L^2(\Omega) \to L^2(\mathbb{R}^2)$ such that:*

i) $T_\Omega(g)|_\Omega = g$,

ii) $T_\Omega : H^m(\Omega) \to H^m(\mathbb{R}^2)$ continuously for each $m \in \mathbb{Z}_{\geq 0}$.

As with all Nash-Moser iteration schemes we will need smoothing operators, which we now construct. Fix $\widehat{\chi} \in C_c^\infty(\mathbb{R}^2)$ such that $\widehat{\chi} \equiv 1$ inside X. Let $\chi(x) = \int\int_{\mathbb{R}^2} \widehat{\chi}(\eta) e^{2\pi i \eta \cdot x} d\eta$ be the inverse Fourier transform of $\widehat{\chi}$. Then χ is a Schwartz function and satisfies $\int\int_{\mathbb{R}^2} \chi(x)dx \equiv 1$, $\int\int_{\mathbb{R}^2} x^\beta \chi(x)dx = 0$ for any multi-index β, $\beta \neq 0$. If $g \in L^2(\mathbb{R}^2)$ and $\mu \geq 1$, we define smoothing operators $S'_\mu : L^2(\mathbb{R}^2) \to H^\infty(\mathbb{R}^2)$ by

$$(S'_\mu g)(x) = \mu^2 \int\int_{\mathbb{R}^2} \chi(\mu(x-y))g(y)dy.$$

Then we have (see [14]),

Lemma 4.1. *Let* $l, m \in \mathbb{Z}_{\geq 0}$ *and* $g \in H^l(\mathbb{R}^2)$, *then*
i) $\| S'_\mu g \|_{H^m(\mathbb{R}^2)} \leq C_{l,m} \| g \|_{H^l(\mathbb{R}^2)}$, $m \leq l$,
ii) $\| S'_\mu g \|_{H^m(\mathbb{R}^2)} \leq C_{l,m}\mu^{m-l} \| g \|_{H^l(\mathbb{R}^2)}$, $l \leq m$,
iii) $\| g - S'_\mu g \|_{H^m(\mathbb{R}^2)} \leq C_{l,m}\mu^{m-l} \| g \|_{H^l(\mathbb{R}^2)}$, $m \leq l$.

Furthermore, we obtain smoothing operators on X, $S_\mu : L^2(X) \to H^\infty(X)$, by setting $S_\mu g = (S'_\mu T g)|_X$, where T is the extension operator given by theorem 4.1 with $\Omega = X$. Moreover, it is clear that the corresponding results of lemma 4.1 hold for S_μ.

We now set up the underlying iterative procedure. Let $\mu_k = \mu^k$, $S'_k = S'_{\mu_k}$, $S_k = S_{\mu_k}$, and $w_0 = 0$. Suppose that functions $w_0, w_1, \ldots, w_k$ have been defined on X, and put $v_j = S'_j T w_j$, $0 \leq j \leq k$. Let $L(v_k)$ denote the linearization of (4.1) evaluated at v_k, and let $L_8(v_k)$ be a small perturbation (on X) of $L_7(v_k)$ to be given below, where $L_7(v_k)$ is as in section §1. Then define $w_{k+1} = w_k + u_k$ where u_k is the solution, restricted to X, of

$$L_8(v_k)u_k = f_k, \tag{4.2}$$

given by theorem 2.2 (see lemma 4.2 below), and where f_k will be specified below.

Let $Q_k(w_k, u_k)$ denote the quadratic error in the Taylor expansion of Φ at w_k. Then using the definition of L_7 we have

$$\begin{aligned}\Phi(w_{k+1}) &= \Phi(w_k) + L(w_k)u_k + Q_k(w_k, u_k) \qquad (4.3)\\ &= \Phi(w_k) + \varepsilon(1 + \varepsilon(w_k)_{xx} + \varepsilon^{2n} H^n P_{11}(w_k))(P^6_{22}(w_k)L_7(w_k)u_k + D_k(w_k)\partial_x u_k)\\ &\quad + A_k(w_k)\partial_{xx}u_k + Q_k(w_k, u_k)\\ &= \Phi(w_k) + \varepsilon(1 + \varepsilon(v_k)_{xx} + \varepsilon^{2n} H^n P_{11}(v_k))|_X P^6_{22}(v_k|_X)L_8(v_k|_X)u_k + e_k,\end{aligned}$$

where

$$\begin{aligned} e_k \;=\; & \varepsilon(P_k(w_k)L_8(w_k) - P_k(v_k|_X)L_8(v_k|_X))u_k + A_k(w_k)\partial_{xx}u_k + Q_k(w_k, u_k)\\ & -\varepsilon\overline{P}_k(w_k)(P^6_{22}(w_k)\overline{A}_k\partial_{\alpha\alpha}u_k - (S_k D_k(w_k))\partial_x u_k),\end{aligned}$$

$$P_k(w_k) = (1 + \varepsilon(w_k)_{xx} + \varepsilon^{2n} H^n P_{11}(w_k))P^6_{22}(w_k),$$

$$\overline{P}_k(w_k) = 1 + \varepsilon(w_k)_{xx} + \varepsilon^{2n} H^n P_{11}(w_k),$$

$$A_k(w_k) = \varepsilon\overline{P}_k^{-1}(w_k)\Phi(w_k), \qquad \overline{A}_k = \varepsilon^n \mu_k^{-4}\beta\phi(\alpha)\phi(\beta) + \psi_1(\beta),$$

$$D_k(w_k) = \frac{1}{2}\partial_x[\overline{P}_k^{-2}(w_k)\Phi(w_k)] + \frac{1}{2}\overline{P}_k^{-2}(w_k)\partial_x\Phi(w_k),$$

$$L_8(w_k)u_k = L_7(w_k)u_k + \overline{A}_k\partial_{\alpha\alpha}u_k + \phi(\alpha)\phi(\beta)T[(P^6_{22}(w_k))^{-1}(I - S_k)D_k(w_k)]\partial_x u_k,$$

the functions ϕ and ψ_1 are as in section §2, (α, β) are the coordinates of lemma 1.3; note also also that we use $\phi|_X \equiv 1$ and $T(\cdot)|_X = I$ in (4.3).

We now define f_k. In order to solve (4.2) with the theory of section §2, we require f_k to be defined on all of $\mathbb{R}^2$. Furthermore, we need the right-hand side of (4.3) to tend to zero sufficiently fast, to make up for the error incurred at each step by solving (4.2) instead of solving the unmodified linearized equation. Therefore we set $E_0 = 0$, $E_k = \sum_{i=0}^{k-1} e_i$, and define

$$f_0 = -T[(\varepsilon P_0(v_0))^{-1} S_0 \Phi(w_0)],$$

$$f_k = T[(\varepsilon P_k(v_k))^{-1}(S_{k-1}E_{k-1} - S_k E_k + (S_{k-1} - S_k)\Phi(w_0))].$$

It follows that

$$\begin{aligned} \Phi(w_{k+1}) &= \Phi(w_0) + \sum_{i=0}^{k} \varepsilon P_i(v_i|_X)(f_i|_X) + E_k + e_k \\ &= (I - S_k)\Phi(w_0) + (I - S_k)E_k + e_k. \end{aligned} \tag{4.4}$$

In what follows, we will show that the right-hand side of (4.4) tends to zero sufficiently fast to guarantee the convergence of $\{w_k\}_{k=0}^{\infty}$ to a solution of (4.1).

Let b be a positive number that will be chosen as large as possible, set $\delta = \varepsilon^{n-1}$, and $\mu = \varepsilon^{\frac{1-n}{b+1}}$. Furthermore, let $m_* \in \mathbb{Z}_{\geq 0}$ be such that $\Phi(w_0) \in H^{m_*}(X)$. For convenience we will denote the $H^m(X)$ and $H^m(\mathbb{R}^2)$ norms by $\| \cdot \|_m$ and $\| \cdot \|_{m,\, \mathbb{R}^2}$, respectively. The convergence of $\{w_k\}_{k=0}^{\infty}$ will follow from the following eight statements, valid for $0 \leq m \leq m_* - 25$ unless specified otherwise, which shall be proven by induction on j, for some constants C_1, C_2, C_3, and C_4 independent of j, ε, and μ, but dependent on m.

I_j: $\| u_{j-1} \|_m \leq \delta \mu_{j-1}^{m-b}$,

II_j: $\| w_j \|_m \leq \begin{cases} C_1\delta & \text{if } m - b \leq -1/2, \\ C_1\delta\mu_j^{m-b} & \text{if } m - b \geq 1/2, \end{cases}$

III_j: $\| w_j \|_{18} \leq C_1\delta, \quad \| v_j \|_{18,\, \mathbb{R}^2} \leq C_3\delta$,

IV_j: $\| w_j - v_j \|_m \leq C_2\delta\mu_j^{m-b}$,

V_j: $\| v_j \|_{m,\, \mathbb{R}^2} \leq \begin{cases} C_3\delta & \text{if } m - b \leq -1/2, \\ C_3\delta\mu_j^{m-b} & \text{if } m - b \geq 1/2, \end{cases} \quad 0 \leq m < \infty$,

VI_j: $\| e_{j-1} \|_m \leq \varepsilon\delta^2\mu_{j-1}^{m-b}$, $0 \leq m \leq m_* - 30$,

VII_j: $\| f_j \|_{m,\, \mathbb{R}^2} \leq C_4\delta^2(1 + \mu^{b-m})\mu_j^{m-b}$, $0 \leq m \leq m_*$,

VIII_j: $\| \Phi(w_j) \|_m \leq \delta\mu_j^{m-b}$, $0 \leq m \leq m_* - 30$.

Assume that the above eight statements hold for $j = 0, \dots, k$. Before showing the induction step we will need the following preliminary lemma which allows us to study equation (4.2).

Lemma 4.2. *If ε is sufficiently small, then the theory of sections §2 and §3 applies to the operators $L_8(v_k)$ and $L_8(v_0)$.*

Proof. We first show that lemma 2.1 holds for $L_8(v_k)$. Extend the coefficients of $L_7(v_k)$ to the entire $\alpha\beta$-plane and denote them by A_k, D_k, E_k, F_k as in section §2. Write

$$L_8(v_k) = \widetilde{A}_k\partial_{\alpha\alpha} + \partial_{\beta\beta} + \widetilde{D}_k\partial_\alpha + \widetilde{E}_k\partial_\beta + \widetilde{F}_k,$$

let I_i, $i = 1, 2, 3, 4$, be as in the proof of lemma 2.1, and let $\widetilde{I}_i$ be analogous to I_i with A_k, D_k, E_k, F_k replaced by $\widetilde{A}_k$, $\widetilde{D}_k$, $\widetilde{E}_k$, $\widetilde{F}_k$. Then a calculation shows that

$$\widetilde{I}_1 \geq I_1 + \begin{cases} \varepsilon\delta\mu_k^{-4}\phi(\alpha)(\frac{1}{2} + O(|\beta|)) & \text{if } |\beta| \leq y_3, \\ C + O(\varepsilon) & \text{if } |\beta| \geq y_3, \end{cases}$$

for some constant $C > 0$ independent of ε and k, where y_3 is as in the proof of lemma 2.1. Furthermore, using the definition of Φ, lemma 3.1 (iii), and III_k, we have

$$\begin{aligned} |(I - S_k)D_k(v_k)|_{C^0(X)} &\leq C \parallel (I - S_k)D_k(v_k) \parallel_2 \\ &\leq C\mu_k^{-5} \parallel D_k(v_k) \parallel_7 \\ &\leq C\mu_k^{-5}(\varepsilon \parallel v_k \parallel_{12} + \varepsilon^{2n}) \\ &\leq C\varepsilon\delta\mu_k^{-5} \end{aligned}$$

since $\Phi(0) = O(\varepsilon^{2n})$. It follows that

$$\widetilde{I}_3 \geq I_3 + O(\varepsilon\delta\mu_k^{-5}\phi(\alpha)), \qquad \widetilde{I}_4 = I_4 + O(\varepsilon),$$

$$\widetilde{I}_2 = I_2 + O(\varepsilon\delta\mu_k^{-4}|\phi'(\alpha)| + \varepsilon\delta\mu_k^{-5}\phi(\alpha)),$$

from which we also find

$$\widetilde{I}_1\widetilde{I}_3 - 2\widetilde{I}_2^2 \geq I_1I_3 - 2I_2^2 + \varepsilon\delta\mu_k^{-4}\phi(\alpha)(C + O(\mu_k^{-1} + \varepsilon)) \geq 0,$$

if ε is sufficiently small. We then conclude that lemma 2.1 holds for $L_8(v_k)$. Similarly, the proofs of the remaining results of sections §2 and §3 need only slight modifications to show that they also hold for $L_8(v_k)$. Lastly, the same method applies to $L_8(v_0)$ if we note that

$$|(I - S_0)D_0(v_0)|_{C^0(X)} \leq C\varepsilon^{2n}.$$

q.e.d.

The next four propositions will show that the above eight statements hold for $j = k + 1$. The case $j = 0$ will be proven shortly there after.

Proposition 4.1. *If $27 \leq b \leq m_* - 26$, $0 \leq m \leq m_* - 25$, and ε is sufficiently small, then I_{k+1}, II_{k+1}, III_{k+1}, IV_{k+1}, and V_{k+1} hold.*

Proof. I_{k+1}: First note that by III_k,

$$|v_k|_{C^{16}(\mathbb{R}^2)} \leq C \parallel v_k \parallel_{18,\, \mathbb{R}^2} \leq C'.$$

Therefore, we may apply lemma 4.2 and the theory of section §2 to obtain the solution u_k of (4.2). We require $m \leq m_* - 25$ so that the hypotheses of theorem 3.2 are fulfilled. If $m + 25 - b \geq 1/2$ then using theorem 3.2, V_k, VII_k, and $b \geq 27$, we have

$$\begin{aligned} \parallel u_k \parallel_m &\leq C_m(\parallel f_k \parallel_{m,\, \mathbb{R}^2} + \parallel v_k \parallel_{m+25,\, \mathbb{R}^2} \parallel f_k \parallel_{2,\, \mathbb{R}^2}) \\ &\leq C_m(C_4\delta^2(1+\mu^{b-m})\mu_k^{m-b} + C_3C_4\delta^3(1+\mu^{b-2})\mu_k^{m+25-b}\mu_k^{2-b}) \\ &\leq \delta\mu_k^{m-b}, \end{aligned}$$

if ε is sufficiently small, since $\delta\mu^{b-m} = \varepsilon^{(n-1)(1-\frac{b-m}{b+1})} \leq \varepsilon^{\frac{1}{b+1}}$. If $m + 25 - b \leq -1/2$ and $m \geq 2$, then using $\parallel v_k \parallel_{m+25,\, \mathbb{R}^2} \leq C_3\delta$ in the estimate above gives the desired result. Furthermore, if $0 \leq m < 2$ then the methods of theorem 3.2 show that $\parallel u_k \parallel_m \leq M \parallel f_k \parallel_{m,\, \mathbb{R}^2}$; in which case VII_k gives the desired result.

II_{k+1}: Since $w_{k+1} = \sum_{i=0}^{k} u_i$, we have

$$\parallel w_{k+1} \parallel_m \leq \sum_{i=0}^{k} \parallel u_i \parallel_m \leq \delta \sum_{i=0}^{k} \mu_i^{m-b}.$$

Hence, if $m - b \leq -1/2$

$$\parallel w_{k+1} \parallel_m \leq \delta \sum_{i=0}^{\infty} (\mu^i)^{-1/2} \leq \delta \sum_{i=0}^{\infty} (2^i)^{-1/2} := C_1\delta,$$

and if $m - b \geq 1/2$,

$$\parallel w_{k+1} \parallel_m \leq \delta\mu_{k+1}^{m-b} \sum_{i=0}^{k} (\frac{\mu_i}{\mu_{k+1}})^{m-b} \leq \delta\mu_{k+1}^{m-b} \sum_{i=0}^{\infty} (\mu^{-i})^{1/2} \leq C_1\delta\mu_{k+1}^{m-b}.$$

III_{k+1}: Since $b \geq 27$ we have $18 - b \leq -1/2$. Therefore II_{k+1} and V_{k+1} (proven below) imply that

$$\parallel w_{k+1} \parallel_{18} \leq C_1\delta \quad \text{and} \quad \parallel v_{k+1} \parallel_{18,\, \mathbb{R}^2} \leq C_3\delta.$$

IV_{k+1}: Since $b \leq m_* - 26$ we have $m_* - 25 - b \geq 1/2$. Therefore lemma 4.1 and II_{k+1} yield,

$$\begin{aligned} \parallel w_{k+1} - v_{k+1} \parallel_m &= \parallel (I - S_{k+1})w_{k+1} \parallel_m \\ &\leq C_m\mu_{k+1}^{m-(m_*-25)} \parallel w_{k+1} \parallel_{m_*-25} \\ &\leq C_m\mu_{k+1}^{m-(m_*-25)} C_1\delta\mu_{k+1}^{m_*-25-b} \\ &:= C_2\delta\mu_{k+1}^{m-b}. \end{aligned}$$

V_{k+1}: From lemma 4.1 and $b \le m_* - 26$ we have for all $m \ge 0$,

$$\| v_{k+1} \|_{m,\,\mathbb{R}^2} = \| S'_{k+1} T w_{k+1} \|_{m,\,\mathbb{R}^2} \le C'_m \| T \| \begin{cases} \| w_{k+1} \|_{b-1} & \text{if } m-b \le -1/2, \\ \mu_{k+1}^{m-b-1} \| w_{k+1} \|_{b+1} & \text{if } m-b \ge 1/2. \end{cases}$$

V_{k+1} now follows from II_{k+1}. q.e.d.

Write $e_k = e'_k + e''_k + e'''_k$, where

$$\begin{aligned} e'_k &= \varepsilon(P_k(w_k)L_8(w_k) - P_k(v_k|_X)L_8(v_k|_X))u_k, \\ e''_k &= -\varepsilon \overline{P}_k(w_k)(P^6_{22}(w_k)\overline{A}_k\partial_{\alpha\alpha}u_k - (S_kD_k(w_k))\partial_x u_k) + A_k(w_k)\partial_{xx}u_k, \\ e'''_k &= Q_k(w_k, u_k). \end{aligned}$$

Proposition 4.2. *If the hypotheses of proposition* 4.1 *hold and* $0 \le m \le m_* - 30$, *then* VI_{k+1} *holds.*

Proof. We will estimate e'_k, e''_k, and e'''_k separately. Denote

$$(P_k(w_k)L_8(w_k) - P_k(v_k|_X)L_8(v_k|_X))u_k = \sum_{i,j} d_{ij}(u_k)_{x_ix_j} + \sum_i d_i(u_k)_{x_i} + du_k,$$

then lemma 3.1 (i) and (iii), I_k, and IV_k show that

$$\begin{aligned} \| e'_k \|_m &\le \varepsilon C_{m,1}[(\sum_{i,j} \| d_{ij} \|_m + \sum_i \| d_i \|_m + \| d \|_m) \| u_k \|_4 \\ &\quad + (\sum_{i,j} \| d_{ij} \|_2 + \sum_i \| d_i \|_2 + \| d \|_2) \| u_k \|_{m+2}] \\ &\le \varepsilon C_{m,2}(\| w_k - v_k \|_{m+5} \| u_k \|_4 + \| w_k - v_k \|_7 \| u_k \|_{m+2}) \\ &\le C_{m,3}\varepsilon\delta^2\mu_k^{9-b}\mu_k^{m-b} \\ &\le \frac{\varepsilon}{3}\delta^2\mu_k^{m-b} \end{aligned}$$

if ε is sufficiently small, since $\mu_k^{9-b} \le \mu^{9-b} = \varepsilon^{(9-b)(\frac{1-n}{b+1})} \le \varepsilon^{18/28}$. Note that we have also used $m \le m_* - 30$, which allows us to apply IV_k.

We now estimate e''_k. By lemma 3.1 (i) and (iii), I_k, II_k, and $VIII_k$,

$$\begin{aligned} \| A_k\partial_{xx}u_k \|_m &\le C_{m,4}(\| \partial_{xx}u_k \|_2 \| A_k \|_m + \| \partial_{xx}u_k \|_m \| A_k \|_2) \\ &\le \varepsilon C_{m,5}[\| u_k \|_4 ((1 + \| w_k \|_6) \| \Phi(w_k) \|_m + \| w_k \|_{m+4} \| \Phi(w_k) \|_2) \\ &\quad + \| u_k \|_{m+2} \| \Phi(w_k) \|_2] \\ &\le \varepsilon C_{m,6}[\delta\mu_k^{4-b}(\delta\mu_k^{m-b} + \delta^2\mu_k^{m+4-b}\mu_k^{2-b}) + \delta^2\mu_k^{m+2-b}\mu_k^{2-b}] \\ &\le \varepsilon C_{m,7}\mu_k^{10-b}\delta^2\mu_k^{m-b} \\ &\le \frac{\varepsilon}{9}\delta^2\mu_k^{m-b}, \end{aligned}$$

if ε is sufficiently small and $m+4-b \geq 1/2$. If $m+4-b \leq -1/2$ then we may use the estimate $\| w_k \|_{m+4} \leq C_1\delta$ to obtain the same outcome. Furthermore, the same methods combined with lemma 4.1 show that

$$\begin{aligned}
\| \varepsilon\overline{P}_k(w_k)(S_kD_k)\partial_x u_k \|_m &\leq \varepsilon C_{m,8}(\| \partial_x u_k \|_2 \| \overline{P}_k(S_kD_k) \|_m + \| \partial_x u_k \|_m \| \overline{P}_k(S_kD_k) \|_2) \\
&\leq \varepsilon C_{m,9}[\| u_k \|_3 \ (\mu_k \| \overline{P}_k \|_2 \| D_k \|_{m-1} + \| \overline{P}_k \|_m \| D_k \|_2) \\
&\quad + \| u_k \|_{m+1} \| D_k \|_2] \\
&\leq \varepsilon C_{m,10}[\| u_k \|_3 \ (\mu_k \| \Phi(w_k) \|_m + \mu_k(1+ \| w_k \|_{m+4}) \| \Phi(w_k) \|_3) \\
&\quad + \| u_k \|_{m+1} \| \Phi(w_k) \|_3] \\
&\leq \varepsilon C_{m,11}[\delta\mu_k^{3-b}(\delta\mu_k^{m+1-b} + \delta^2\mu_k^{m+5-b}\mu_k^{3-b}) + \delta^2\mu_k^{m+1-b}\mu_k^{3-b}] \\
&\leq \varepsilon C_{m,12}\mu_k^{11-b}\delta^2\mu_k^{m-b} \\
&\leq \frac{\varepsilon}{9}\delta^2\mu_k^{m-b}.
\end{aligned}$$

Similarly, since $\psi_1(\beta) \equiv 0$ in X it follows that

$$\begin{aligned}
\| \varepsilon P_k(w_k)\overline{A}_k\partial_{\alpha\alpha}u_k \|_m &\leq \varepsilon^2\delta\mu_k^{-4}C_{m,13}(\| u_k \|_4 \| w_k \|_{m+4} + \| u_k \|_{m+2} \ (1+ \| w_k \|_6)) \\
&\leq \varepsilon^2\delta\mu_k^{-4}C_{m,14}(\delta^2\mu_k^{4-b}\mu_k^{m+4-b} + \delta\mu_k^{m+2-b}) \\
&\leq \frac{\varepsilon}{9}\delta^2\mu_k^{m-b}.
\end{aligned}$$

Therefore

$$\| e_k^{''} \| \leq \frac{\varepsilon}{3}\delta^2\mu_k^{m-b}.$$

We now estimate $e_k^{'''}$. We have

$$e_k^{'''} = Q_k(w_k, u_k) = \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}\Phi(w_k + tu_k)dt.$$

Apply lemma 3.1 (i) and (ii), as well as the Sobolev lemma to obtain

$$\begin{aligned}
\| e_k^{'''} \|_m &\leq \int_0^1 \sum_{|\sigma|,|\gamma|\leq 2} \| \nabla_{\overline{\sigma}\overline{\gamma}}\Phi(w_k + tu_k)\partial^\sigma u_k\partial^\gamma u_k \|_m \, dt \\
&\leq \int_0^1 \sum_{|\sigma|,|\gamma|\leq 2} C_{m,15}(|\nabla_{\overline{\sigma}\overline{\gamma}}\Phi(w_k + tu_k)|_\infty \| \partial^\sigma u_k\partial^\gamma u_k \|_m \\
&\quad + \| \nabla_{\overline{\sigma}\overline{\gamma}}\Phi(w_k + tu_k) \|_m \, |\partial^\sigma u_k\partial^\gamma u_k|_\infty)dt \\
&\leq \int_0^1 C_{m,16}(\| \nabla^2\Phi(w_k + tu_k) \|_2 \| u_k \|_4 \| u_k \|_{m+2} \\
&\quad + \| \nabla^2\Phi(w_k + tu_k) \|_m \| u_k \|_4^2)dt,
\end{aligned}$$

where $\overline{\sigma} = \partial^\sigma(w_k + tu_k)$ and $\overline{\gamma} = \partial^\gamma(w_k + tu_k)$. The notation $\nabla^2\Phi$ represents the collection of second partial derivatives with respect to the variables $\overline{\sigma}$, $\overline{\gamma}$, so by (0.6)

$\nabla^2\Phi = O(\varepsilon^2)$. Therefore using lemma 3.1 (iii), I_k, and II_k, we have

$$\begin{aligned}
\| e_k''' \|_m &\leq \varepsilon^2 C_{m,17}[(1+ \| w_k \|_6 + \| u_k \|_6) \| u_k \|_4 \| u_k \|_{m+2} \\
&\quad +(1+ \| w_k \|_{m+4} + \| u_k \|_{m+4}) \| u_k \|_4^2] \\
&\leq \varepsilon^2 C_{m,18}[\delta^2 \mu_k^{4-b}\mu_k^{m+2-b} + \delta^2 \mu_k^{2(4-b)} + \delta^3 \mu_k^{m+4-b}\mu_k^{2(4-b)}] \\
&\leq \frac{\varepsilon}{3}\delta^2\mu_k^{m-b}
\end{aligned}$$

if ε is sufficiently small, since $b \geq 27$. Combining the estimates of e_k', e_k'', and e_k''' yields the desired result. q.e.d.

Assume that $b \leq m_* - 31$, then $E_k \in H^{b+1}(X)$ by theorem 2.3. The following estimate of E_k will be utilized in the next proposition:

$$\| E_k \|_{b+1} \leq \sum_{i=0}^{k-1} \| e_i \|_{b+1} \leq \varepsilon\delta^2 \sum_{i=0}^{k-1} \mu_i \leq \varepsilon(\sum_{i=0}^{\infty} \mu_i^{-1})\delta^2\mu^k \leq \varepsilon(\sum_{i=0}^{\infty} 2^{-i})\delta^2\mu_k. \qquad (4.5)$$

Proposition 4.3. *If the hypotheses of proposition* 4.2 *hold and* $b \leq m_* - 31$, *then* VII_{k+1} *holds for all* $0 \leq m \leq m_*$.

Proof. By lemma 3.1 (iii),

$$\begin{aligned}
\| f_{k+1} \|_{m,\, \mathbb{R}^2} \leq \varepsilon^{-1} \| T \| C_{m,19}(& \| S_k E_k - S_{k+1}E_{k+1} + (S_k - S_{k+1})\Phi(w_0) \|_m \\
&+ \| v_{k+1} \|_{m+4} \| S_k E_k - S_{k+1}E_{k+1} + (S_k - S_{k+1})\Phi(w_0) \|_2).
\end{aligned} \qquad (4.6)$$

Furthermore using (4.5) and the estimate $\| \Phi(w_0) \|_{b+1} \leq C_b\varepsilon^{2n}$, we obtain for all $m \geq b+1$,

$$\begin{aligned}
&\| S_k E_k - S_{k+1}E_{k+1} + (S_k - S_{k+1})\Phi(w_0) \|_m \\
&\leq C_{m,20}(\mu_k^{m-b-1} \| E_k \|_{b+1} + \mu_{k+1}^{m-b-1} \| E_{k+1} \|_{b+1} + (\mu_k^{m-b-1} + \mu_{k+1}^{m-b-1}) \| \Phi(w_0) \|_{b+1}) \\
&\leq C_{m,21}\varepsilon\delta^2(1+\mu^{b-m})\mu_{k+1}^{m-b}.
\end{aligned} \qquad (4.7)$$

If $m < b+1$, then applying similar methods along with VI_{k+1} to

$$\begin{aligned}
&\| S_k E_k - S_{k+1}E_{k+1} + (S_k - S_{k+1})\Phi(w_0) \|_m \\
\leq\ &\| (I - S_k)E_k \|_m + \| (I - S_{k+1})E_k \|_m + \| S_{k+1}e_k \|_m \\
&+ \| (I - S_k)\Phi(w_0) \|_m + \| (I - S_{k+1})\Phi(w_0) \|_m,
\end{aligned}$$

yields the same estimate found in (4.7). Therefore plugging into (4.6) produces

$$\begin{aligned}
\| f_{k+1} \|_{m,\, \mathbb{R}^2} &\leq C_{m,22}[\delta^2(1+\mu^{b-m})\mu_{k+1}^{m-b} + \delta^3(1+\mu^{b-2})\mu_{k+1}^{m+6-2b}] \\
&\leq C_{m,23}\delta^2(1+\mu^{b-m})\mu_{k+1}^{m-b},
\end{aligned}$$

if $m+4-b \geq 1/2$. If $m+4-b \leq -1/2$ and $m \geq 2$, then using $\| v_{k+1} \|_{m+4} \leq C_3\delta$ in the estimate above gives the desired result. Moreover if $0 \leq m < 2$, then in place of (4.6) we use the estimate

$$\| f_{k+1} \|_{m,\,\mathbb{R}^2} \leq \varepsilon^{-1} \| T \| C_{m,24} \| S_k E_k - S_{k+1}E_{k+1} + (S_k - S_{k+1})\Phi(w_0) \|_m$$

combined with the above method to obtain the desired result. Lastly if $m+4-b=0$, then replace $\| v_{k+1} \|_{m+4}$ in (4.6) by $\| v_{k+1} \|_{m+5}$ and follow the above method. q.e.d.

Proposition 4.4. *If the hypotheses of proposition* 4.3 *hold and* $b = m_* - 31$, *then* VIII_{k+1} *holds for* $0 \leq m \leq m_* - 30$.

Proof. By (4.4), VI_{k+1}, and $m \leq b+1 = m_* - 30$, we have

$$\begin{aligned} \| \Phi(w_{k+1}) \|_m &\leq \| (I-S_k)\Phi(w_0) \|_m + \| (I-S_k)E_k \|_m + \| e_k \|_m \\ &\leq C_{m,25}(\mu_k^{m-b-1} \| \Phi(w_0) \|_{b+1} + \mu_k^{m-b-1} \| E_k \|_{b+1} + \varepsilon\delta^2\mu_k^{m-b}). \end{aligned}$$

Applying the estimate (4.5), $\| \Phi(w_0) \|_{b+1} \leq C_b\varepsilon^{2n} \leq \delta^2$, and $\delta\mu^{b-m} \leq \varepsilon^{\frac{1}{b+1}}$ produces

$$\| \Phi(w_{k+1}) \|_m \leq C_{m,26}(\delta^2\mu^{b-m} + \varepsilon\delta^2\mu^{b-m})\mu_{k+1}^{m-b} \leq \delta\mu_{k+1}^{m-b},$$

if ε is sufficiently small. q.e.d.

To complete the proof by induction we will now prove the case $k=0$. Since $w_0 = 0$, II_0, III_0, IV_0, and V_0 are trivial. Furthermore since $\| \Phi(w_0) \|_m \leq \varepsilon\delta^2$ if $\varepsilon = \varepsilon(m)$ is sufficiently small and $m \leq m_*$, VII_0 and VIII_0 hold. In addition, by lemma 4.2 we can apply theorem 3.2 to obtain

$$\| u_0 \|_m \leq C_m \| f_0 \|_{m,\,\mathbb{R}^2} \leq C_m'\delta^2 \leq \delta$$

if δ is small, so that I_1 is valid. Lastly, the proof of proposition 4.2 now shows that VI_1 is valid. This completes the proof by induction.

In view of the hypotheses of propositions 4.1-4.4, we require $m_* \geq 58$ and choose $b = m_* - 31$. The following corollaries will complete the proof of theorem 0.3.

Corollary 4.1. $w_k \to w$ *in* $H^{m_*-32}(X)$.

Proof. For $0 \leq m \leq m_* - 32$ and $i > j$, I_k implies that

$$\| w_i - w_j \|_m \leq \sum_{k=j}^{i-1} \| u_k \|_m \leq \delta \sum_{k=j}^{i-1} \mu_k^{m-b} \leq \delta \sum_{k=j}^{i-1} \mu^{-k}.$$

Hence, $\{w_k\}$ is Cauchy in $H^m(X)$ for all $0 \leq m \leq m_* - 32$. q.e.d.

Corollary 4.2. $\Phi(w_k) \to 0$ *in* $C^0(X)$.

Proof. By the Sobolev lemma and VIII_k,

$$|\Phi(w_k)|_{C^0(X)} \leq C \| \Phi(w_k) \|_2 \leq C\delta\mu_k^{2-b}.$$

The desired conclusion follows since $b = m_* - 31 \geq 27$. q.e.d.

Let r, K, a_{ij}, and f be as in theorem 0.3. If $K, a_{ij}, f \in C^r$, $r \geq 58$, then there exists a C^{r-34} solution of (4.1).

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