ON THE HOPF CONJECTURES WITH SYMMETRY

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ABSTRACT

ON THE HOPF CONJECTURES WITH SYMMETRY

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We prove results related to the classical Hopf conjectures about positively curved manifolds under the assumption of a logarithmic symmetry rank bound. The main new tool is the action of the Steenrod algebra on cohomology. Along the way, we formulate a certain notion of periodicity in cohomology and obtain a generalization in this context of Adem's theorem on singly generated cohomology rings.

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Chapter 1

Introduction

1.1 Summary of results

Positively curved spaces have been of interest since the beginning of global Riemannian geometry. Unfortunately, there are few known examples (see [39] for a survey and [25, 9, 15] for recent examples) and few topological obstructions to any given manifold admitting a positively curved metric. In fact, all known simply connected examples in dimensions larger than 24 are spheres and projective spaces, and all known obstructions to positive curvature for simply connected manifolds are already obstructions to nonnegative curvature.

One famous conjectured obstruction to positive curvature was made by H. Hopf in the 1930s. It states that even-dimensional manifolds admitting positive sectional curvature have positive Euler characteristic. This conjecture holds in dimensions two and four by the theorems of Gauss-Bonnet or Bonnet-Myers (see [4, 7]), but it remains open in higher dimensions.

In the 1990s, Karsten Grove proposed a research program to address our lack of knowledge in this subject. The idea is to study positively curved metrics with large isometry groups. This approach has proven to be quite fruitful (see [37, 13] for surveys). Our main result falls into this category:

Theorem A. *Let M^n be a closed Riemannian manifold with positive sectional curvature and $n \equiv 0 \bmod 4$. If M admits an effective, isometric T^r-action with $r \geq 2\log_2 n$, then $\chi(M) > 0$.*

Previous results showed that $\chi(M^n) > 0$ under the assumption of a linear bound on r. For example, a positively curved n-manifold with an isometric T^r-action has positive Euler characterisic if n is even and $r \geq n/8$ or if $n \equiv 0 \bmod 4$ and $r \geq n/10$ (see Table 1.1 on page 8 for a summary).

The assumption of the existence of an isometric T^r-action is equivalent to assuming the rank of the isometry group $\mathrm{Isom}(M)$ is at least r. This measure of symmetry is called the symmetry rank. Other measures include the symmetry degree and the cohomogeneity, which are defined as the dimensions of $\mathrm{Isom}(M)$ and $M/\mathrm{Isom}(M)$, respectively. Low cohomogenity implies large symmetry degree, which in turn implies large symmetry rank. Hence Theorem A easily implies similar results when the assumption is replaced by the assumption of large symmetry degree or small cohomogeneity (see Corollary 7.1 for a statement and Table 1.2 on page 9 for a comparison with previous results).

Another well known conjecture of Hopf is that $S^2 \times S^2$ admits no metric of positive sectional curvature. More generally, one might ask whether any nontrivial product manifold admits a metric with positive curvature. It is easy to see that, if M and N admit nonnegative curvature metrics, then $M \times N$ with the product metric has nonnegative curvaure. This generalized conjecture is open.

Another way to generalize this conjecture of Hopf is to observe that $S^2 \times S^2$ has the structure of a rank two symmetric space. The rank one symmetric spaces are S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathrm{Ca}P^2$, and all of these admit positive sectional curvature. Higher rank symmetric spaces include $SU(n)/SO(n)$, $Sp(n)/U(n)$, the Grassmann manifolds, and products of these and the other irreducible symmetric spaces. Not one symmetric space of rank greater than one is known to admit a positively curved metric. Moreover, no such space is known not to. It has been conjectured that the only simply connected, compact symmetric spaces which admit positively curved metrics are the rank one spaces. Our second main result provides evidence for this conjecture under the assumption of symmetry:

Theorem B. *Suppose M^n has the rational cohomology of a simply connected, compact symmetric space N. If M admits a positively curved metric invariant under a T^r-action with $r \geq 2\log_2 n + 8$, then N is a product of spheres times either a rank one symmetric space or a rank p Grassmannian $SO(p+q)/SO(p) \times SO(q)$ with $p \in \{2, 3\}$.*

In the irreducible case, product manifolds are excluded, so the only possibilities are that N has rank one or that N is a rank two or rank three real Grassmannian.

Aside from more general classification results which assume much larger symmetry rank, the author is unaware of previous work on this specific question. In addition to weakening the symmetry rank assumption, the author would be particularly interested in methods which would exclude products and the rank two and rank three Grassmannians from the conclusion. The obstruction (see Theorem P in Chapter 5) we prove in order to obtain Theorem B is at the level of rational cohomology in low degrees. Since taking products with spheres does not affect cohomology in low degrees, and since the Grassmannians $SO(2+q)/SO(2) \times SO(q)$ and $SO(3+q)/SO(3) \times SO(q)$ have the same rational cohomology ring in low degrees as the complex and quaternionic projective spaces, respectively, our methods cannot exclude them.

1.2 Grove's research program

The precursor to Grove's research program was the 1990 thesis of Kleiner. The main result is contained in [19]: If M^4 is a simply connected closed manifold, and if M admits a positively curved metric invariant under a circle action on M, then M is homeomorphic to S^4 or $\mathbb{C}P^2$.

An immediate application to Hopf's conjecture on $S^2 \times S^2$ is that the isometry group of any supposed positively curved metric on $S^2 \times S^2$ must be finite.

A flurry of work followed [19]. As explained in [21], it follows from earlier work ([11, 12, 24]) that the conclusion can be improved to a diffeomorphism classification. Moreover, it has recently been anounced that Grove and Wilking have improved the conclusion to an equivariant diffeomorphism classification. In [22, 31], there are similar classifications under the more general assumption of nonnegative sectional curvature. And in [38] and [18], it was shown that $\chi(M) \leq 7$ or $\chi(M) \leq 5$, respectively, if a large enough $\mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ acts effectively by isometries.

Moving beyond dimension four, Grove proposed the study of positively curved metrics which admit large isometry groups. To put it another way, in order to study manifolds with positive curvature, one might assume something about the symmetry of the metric and try to conclude something about the topology of the manifold. The proposal was purposefully vague so as to allow flexibility. The work on positively curved 4-manifolds with symmetry illlustrates an example of this.

The first higher dimensional result in this direction is due to Grove and Searle (see [14]): If M^n is a closed, positively curved manifold which admits an isometric, effective r-torus action, then $r \leq \left\lfloor \frac{n+1}{2} \right\rfloor$ with equality only if M is diffeomorphic to S^n, $\mathbb{C}P^{n/2}$, or a lens space $S^n/\mathbb{Z}_m$. In particular, closed n-manifolds with postive curvature and symmetry rank at least $n/2$ are classified up to diffeomorphism.

In dimension five, the Grove-Searle result implies that the presence of an isometric T^3-action on a positively curved, simply connected closed manifold M^5 implies that M is diffeomorphic to S^5. In [29], Rong showed that the same conclusion holds if only a T^2 is assumed to act. Dimensions 6 and 7 contain lots of examples, so no classification is complete, unless maximal symmetry rank is assumed. In dimensions 8 and 9, Fang and Rong (see [10]) prove that a 3-torus or a 4-torus action, respectively, implies M is homeomorphic to the 8- or 9-sphere, $\mathbb{C}P^4$, or $\mathbb{H}P^2$.

In the early 2000s, a postdoc at Penn named Burkhard Wilking gave new life to the Grove program. Most significantly, he developed a tool which he called the connectedness lemma. I will upgrade the status of the result to a theorem, as we will use it frequently in this thesis. See Theorem 2.9 for a statement. Wilking used the connectedness theorem to prove the following classfication theorem for positively curved manifolds with large isometry group:

Theorem (Wilking, [35, 36])**.** *Let M^n be a simply connected closed manifold with positive sectional curvature.*

1. *If the symmetry rank is at least $\frac{n}{4} + 1$ and $n \geq 10$, then M is homotopy equivalent to a rank one symmetric space.*
2. *If the symmetry degree is at least $2n - 6$, then M is tangentially homotopy equivalent to a rank one symmetric space or isometric to a homogeneous space which admits positive sectional curvature.*
3. *If the cohomogeneity k satisfies $1 \leq k < \sqrt{n/18} - 1$, then M is tangentially homotopy equivalent to a rank one symmetric space.*

We make a few remarks. First, in dimensions greater than 6000, Wilking obtained a classification assuming the symmetry rank is at least $\frac{n}{6}+1$. In dimensions $n \equiv 0, 1 \bmod 4$, M^n has the integral cohomology of S^n, $\mathbb{C}P^{n/2}$, or $\mathbb{H}P^{n/4}$. In dimensions $n \equiv 2, 3 \bmod 4$, the conclusion is that, for all primes p, $H^*(M; \mathbb{Z}_p)$ is that of S^n, $\mathbb{C}P^{n/2}$, $S^2 \times \mathbb{H}P^{(n-2)/4}$, or $S^3 \times \mathbb{H}P^{(n-3)/4}$. Note that the integral cohomology of M may not agree with any of these spaces. However, for all n, the conclusion implies that the rational cohomology is that of a rank one symmetric space, $S^2 \times \mathbb{H}P^m$, or $S^3 \times \mathbb{H}P^m$. One might compare this to Theorem P in Chapter 5, which implies the following: Given $n \geq c \geq 2$ and a positively curved, simply connected closed manifold M^n, if $b_3(M) \leq 1$ and the symmetry rank is at least $2\log_2 n + \frac{c}{2}$, then the rational cohomology up to degree c is that of S^n, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $S^2 \times \mathbb{H}P^{(n-2)/4}$, or $S^3 \times \mathbb{H}P^{(n-3)/4}$.

Second, we remark on positively curved homogenous spaces and their generalizations. A biquotient is the quotient by a free Lie group action on a homogeneous space. These spaces contain nearly all examples of positively curved manifolds. A second generalization comes from observing that the cohomogeneity of a Riemannian homogeneous space is zero. Since cohomogeneity zero spaces with positive curvature are classified, one then asks about cohomogeneity one spaces with positive curvature. In even dimensions, only the rank one symmetric spaces occur (see [33]). In odd dimensions, a partial classification was done in [16]. The bad news is that the classification is incomplete. The good news however is that recently one of the spaces which came out of the classification was proven to have a metric with positive sectional curvature (see [9, 15]). Examples are so difficult to find that this has been one of the great achievements of the Grove program.

We conclude this section with a remark about Hopf's conjecture that $\chi(M) > 0$ for all even-dimensional, positively curved Riemannian manifolds M. Many of the results we have discussed imply that the Hopf conjecture in the presence of symmetry. Tables 1.1 and 1.2 together provide a timeline of results on this conjecture in

Year	Symmetry rank r	Dimension restriction	Source
1994	$r \geq \frac{n}{2}$	–	[14]
2001	$r \geq \frac{n}{4} - 1$	–	[27]
2003	$r \geq \frac{n}{6} + 1$	$n \geq 6000$	[35]
2005	$r \geq \frac{n-4}{8}$	$n \neq 12$	[30]
2008	$r \geq \frac{n-2}{10}$	$n \equiv_4 0$ and $n > 20$	[32]
	$r \geq \frac{n+4}{12}$	$n \equiv_{20} 0, 4, 12$ and $n > 20$	
2012	$r \geq 2\log_2 n$	$n \equiv_4 0$	Theorem A

Table 1.1: Timeline of the symmetry rank assumptions which imply that an even-dimensional, positively curved n-manifold has $\chi > 0$.

the presence of symmetry. The general trend is toward weaker symmetry assumptions and therefore stronger results, however the main exceptions are the result of [27] and those involving divisibility assumptions on the dimension. We remark that

Year	Cohomogeneity k is small, or symmetry degree d is large	Dimension restriction	Source
1972	$k = 0$	–	[34]
1998	$k \leq 1$	–	[26]
2001	$k \leq 5$	–	[27]
2006	$k < \sqrt{n/18} - 1$ or $d \geq 2n - 6$	–	[36]
2012	$k \leq n - (4\log_2 n)^2$ or $d \geq (4\log_2 n)^2$	$n \equiv 0 \bmod 4$	Corollary 7.1

Table 1.2: Timeline of other symmetry assumptions which imply that an even-dimensional, positively curved n-manifold has $\chi > 0$.

many of the entries in the table are trivial consequences of much stronger theorems.

1.3 Methods

A key tool is Wilking's connectedness theorem, which relates the topology of a closed, positively curved manifold with that of its totally geodesic submanifolds of small codimension. Since fixed-point sets of isometries are totally geodesic, this becomes a powerful tool in the presence of symmetry.

Part of the utility of the connectedness theorem is to allow proofs by induction over the dimension of the manifold. Another important implication is a certain periodicity in cohomology. By using the action of the Steenrod algebra on cohomology, we refine this periodicity in some cases. For example, we prove:

Theorem (Periodicity Theorem)**.** *Let N^n be a closed, simply connected, positively curved manifold which contains a pair of totally geodesic, transversely intersecting submanifolds of codimensions $k_1 \leq k_2$. If $k_1 + 3k_2 \leq n$, then $H^*(N;\mathbb{Q})$ is* $\gcd(4, k_1, k_2)$*-periodic.*

It follows from [35] that, under these assumptions, $H^*(N;\mathbb{Q})$ is $\gcd(k_1, k_2)$-periodic. For a closed, orientable n-manifold N and a coefficient ring R, we say that $H^*(N;R)$ is k-periodic if there exists $x \in H^k(N;R)$ such that the map $H^i(N;R) \to H^{i+k}(N;R)$ induced by multiplication by x is surjective for $0 \leq i < n-k$ and injective for $0 < i \leq n-k$.

To illustrate the strength of the conclusion of Theorem 1.3, we observe the following:

- If $\gcd(4, k_1, k_2) = 1$, then N is a rational homology sphere.
- If $\gcd(4, k_1, k_2) = 2$, then N has the rational cohomology of S^n or $\mathbb{C}P^{n/2}$.
- If $\gcd(4, k_1, k_2) = 4$ and $n \not\equiv 2 \bmod 4$, then N has the rational cohomology ring of S^n, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, or $S^3 \times \mathbb{H}P^{(n-3)/4}$.

When $\gcd(4, k_1, k_2) = 4$ and $n \equiv 2 \bmod 4$, the rational cohomology rings of S^n, $\mathbb{C}P^{n/2}$, $S^2 \times \mathbb{H}P^{(n-2)/4}$ and

$$M^6 = (S^2 \times S^4)\#(S^3 \times S^3)\# \cdots \#(S^3 \times S^3)$$

are 4-periodic, but we do not know whether other examples exist in dimensions greater than six. This uncertainty is what prevents us from proving Theorem A in all even dimensions (see Chapter 7).

The main step in the proof of Theorem 1.3 is the following topological result:

Theorem C. *If M^n is a closed, simply connected manifold such that $H^*(M;\mathbb{Z})$ is k-periodic with $3k \leq n$, then $H^*(M;\mathbb{Q})$ is $\gcd(4,k)$-periodic.*

To prove Theorem C, we note that the assumption implies the same periodicity with coefficients in $\mathbb{Z}_p$. We then use the action of the Steenrod algebra for $p = 2$ and $p = 3$ to improve these periodicity statements with coefficients in $\mathbb{Z}_p$. When combined, this information implies $\gcd(4,k)$-periodicity with coefficients in $\mathbb{Q}$. See Theorem D for more general periodicity statements with coefficients in $\mathbb{Z}_p$, which together can be viewed as a generalization of Adem's theorem on singly generated cohomology rings (see [2]).

With the periodicity theorem in hand, we briefly explain some of the tools that go into the proofs of Theorems A and B.

First, the starting point for the proof of Theorem A is a theorem of Lefschetz which states that the Euler characteristic satisfies $\chi(M) = \chi(M^T)$, where M^T is the fixed-point set of the torus action. Since M is even-dimensional with positive curvature, M^T is nonempty by a theorem of Berger. Writing $\chi(M^T) = \sum \chi(F)$ where the sum runs over components F of M^T, we see that it suffices to show $\chi(F) > 0$ for all F. In fact, we prove that F has vanishing odd Betti numbers.

An important tool is a theorem of Conner, which states that, if P is a manifold on which T acts, then $b_{odd}(P^T) \leq b_{odd}(P)$, where b_{odd} denotes the sum of the odd Betti numbers. The strategy is to find a submanifold P on which a subtorus $T' \subseteq T$ acts such that $b_{odd}(P) = 0$ and such that F is a component of $P^{T'}$.

In order to find such a submanifold P, we investigate the web of fixed-point sets of $H \subseteq T$, where H ranges over subgroups of involutions. These fixed-point sets are totally geodesic submanifolds on which T acts, so, under the right conditions, we can induct over dimension. In addition, studying fixed-point sets of involutions has the added advantage that we can easily control the intersection data by studying the isotropy representation at a fixed-point of T.

In order to apply the periodicity theorem, we must find a transverse intersection in the web of fixed-point sets of subgroups of involutions. To strip away complication while preserving the required codimension, symmetry, and intersection data, we define an abstract graph Γ where the vertices correspond to involutions whose fixed-point sets satisfy certain codimension and symmetry conditions. An edge exists between two involutions if the intersection of the corresponding fixed-point sets is not transverse. We then break up the proof into several parts, corresponding to the structure of this graph.

For Theorem B, we first prove Theorem P, which we state and prove in Chapter 5. Theorem P implies that a closed, simply connected, positively curved n-manifold which admits an isometric T^r-action with $r \geq 2\log_2 n + \frac{c}{2}$ contains a c-connected inclusion $P \hookrightarrow M$ such that P has 4-periodic rational cohomology ring. In particular, the Betti numbers of M satisfy $1 \geq b_4$ and $b_i = b_{i+4}$ for $0 < i < c - 4$. Since symmetric spaces are classified and their rational cohomology is known, this easily implies Theorem B. Chapter 6 contains the proof of Theorem B using Theorem P.

In addition to the techniques explained above to prove Theorem A, we use, following [35], the theory of error-correcting codes to prove the existence of fixed-point sets of small codimension. In particular, we use the Griesmer bound for linear codes to find submanifolds which are simply connected and whose inclusions into M are highly connected.

We summarize in Table 1.3 how our two main tools – the action of the Steenrod algebra on cohomology, the connectedness theorem, and the Griesmer bound – fit into the logical structure of this thesis.

We close this chapter with a short description of the individual chapters. In Chapter 2, we prove Theorem C and use it to prove the periodicity theorem. In Chapter 3, we prove Theorem A. In Chapter 4, we state the Griesmer bound from the theory of error-correcting codes and use it to prove the existence of fixed-point sets of small codimension. In Chapter 5, we state and prove Theorem P, and we explain how it implies Theorem A. In Chapter 6, we study the topological obstructions imposed by Theorem P and prove Theorem B. Finally, in Chapter 7, we derive a few immediate corollaries of the above theorem, and we state a periodicity conjecture which would imply that Theorem A holds in all even dimensions.

Steenrod operations	$\Longrightarrow$	Theorem D		
		$\Downarrow$		
		Theorem C		
		$\Downarrow$		
Connectedness Theorem	$\Longrightarrow$	Periodicity Theorem	$\Longrightarrow$	Theorem A
	$\searrow$	$\Downarrow$		
Griesmer bound	$\Longrightarrow$	Theorem P		
		$\Downarrow$		
		Theorem B		

Table 1.3: Logical structure of this thesis

Chapter 2

Cohomological consequences of the connectedness theorem

In this chapter, we define a notion of periodicity in cohomology, discuss its basic properties, and explain how it arises in the context of positively curved Riemannian manifolds. We will then study the action of the Steenrod algebra on spaces with periodic cohomology and prove a generalization of Adem's theorem on singly generated cohomology rings. Finally, we will combine our main topological result with the connectedness theorem to prove the periodicity theorem.

2.1 Periodic cohomology and Theorem D

The following is the definition of periodicity for reference:

Definition 2.1. For a topological space M, a ring R, and an integer c, we say that $x \in H^k(M;R)$ induces periodicity in $H^*(M;R)$ up to degree c if the maps $H^i(M;R) \to H^{i+k}(M;R)$ given by multiplication by x are surjective for $0 \leq i < c-k$ and injective for $0 < i \leq c-k$.

When such an element $x \in H^k(M;R)$ exists, we say that $H^*(M;R)$ is k-periodic up to degree c. If in addition M is a closed, orientable manifold and $c = \dim(M)$, then we say that $H^*(M;R)$ is k-periodic.

We remark that $H^*(M;R)$ is trivially k-periodic up to degree c when $k \geq c$. By a slight abuse of notation, we also say that $H^*(M;R)$ is k-periodic if $2k \leq c$ and $H^i(M;R) = 0$ for $0 < i < c$. One thinks of 0 as the element inducing periodicity. This convention simplifies the discussion.

To motivate the discussion in this section, we give an example of how periodicity in this sense arises in the study of positively curved manifolds. Suppose M^n is a simply connected, closed, positively curved manifold which contains a pair of totally geodesic submanifolds $N_1^{n-k_1}$ and $N_2^{n-k_2}$ with $k_1 \leq k_2$. If the pair intersects transversely, then the connectedness theorem implies $H^*(N_2;\mathbb{Z})$ is k_1-periodic.

Observe that the definition implies $\dim_R H^{ik}(M;R) \leq 1$ for $ik < c$ when R is a field and M is connected. In particular, if $x \in H^k(M;R)$ induces periodicity up to degree c, then x^i generates $H^{ik}(M;R)$ for $ik < c$. If, in fact, $H^j(M;R) = 0$ for $0 < j < c$ with $j \not\equiv 0 \bmod k$, then $H^*(M;R)$ is isomorphic to the singly generated cohomology ring $R[x]$ up to degree c. In [2], J. Adem proved the following:

Theorem (Adem's theorem)**.** *Let M be a topological space, and let p be a prime. Assume $H^*(M;\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p[x]$ or $\mathbb{Z}_p[x]/x^{q+1}$ with $q \geq p$. Let k denote the degree of x.*

1. *If $p = 2$, then k is a power of 2.*

2. *If $p > 2$, then $k = 2\lambda p^r$ for some $r \geq 0$ and $\lambda | p-1$.*

As a corollary, if $H^*(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z}[x]$ or $\mathbb{Z}[x]/x^{q+1}$ with $q \geq 3$, then the degree of x is 2 or 4, corresponding to the case where M is complex or quaternionic projective space.

The main step in the proof of the periodicity theorem is the following generalization of Adem's theorem:

Theorem D. *Let M be a topological space, and let p be a prime. Assume $x \in H^k(M;\mathbb{Z}_p)$ is nonzero, induces periodicity up to degree pk, and that $k = \deg(x)$ is minimal among all such elements x.*

1. *If $p = 2$, then k is a power of 2.*

2. *If $p > 2$, then $k = 2\lambda p^r$ for some $r \geq 0$ and $\lambda | p-1$.*

Before starting the proof, we prove the following general lemma about periodicity:

Lemma 2.2. *Let R be a field. If $x \in H^k(M;R)$ is a nonzero element inducing periodicity up to degree c with $2k \leq c$, and if $x^r = yz$ for some $1 \leq r \leq c/k$ with $\deg(y) \not\equiv 0 \bmod k$, then y also induces periodicity.*

In particular, if $x = yz$ *with* $0 < \deg(y) < k$*, then* y *induces periodicity.*

The way in which we will use this lemma is to take an element x of minimal degree that induces periodicity, and to conclude that the only factorizations of x^r are those of the form $(ax^s)(bx^t)$ where $a, b \in R$ are multiplicative inverses and $r = s + t$.

Proof. Use periodicity to write $y = y'y''$ and $z = z'z''$ where y'' and z'' are powers of x, $0 < \deg(y') < k$, and $0 < \deg(z') < k$. Since x generates $H^k(M; R)$, it follows that $y'z' = ax$ for some multiple $a \in R$. If $a = 0$, then $x^r = (ax)y''z'' = 0$, a contradiction to periodicity and the assumption that $x \neq 0$. Supposing therefore that $a \neq 0$, we may multiply by a^{-1} to assume without loss of generality that $x = y'z'$. Since y'' is a multiple of x, we note that it suffices to show that y' induces periodicity up to degree c. Let $k' = \deg(y')$.

Since multiplying by x is injective from $H^i(M; \mathbb{Z}_2) \to H^{i+k}(M; \mathbb{Z}_2)$, and since this map factors as multiplication by y' followed by multiplication by z', it follows that multiplication by y' from $H^i(M; \mathbb{Z}_2) \to H^{i+k'}(M; \mathbb{Z}_2)$ is injective for $0 < i \leq c - k$. In addition, to see that multiplication by y' is injective from $H^i(M; \mathbb{Z}_2) \to H^{i+k'}(M; \mathbb{Z}_2)$ for $c - k < i \leq c - k'$, consider that multiplying by y' and then by x from

$$H^{i-k}(M; \mathbb{Z}_2) \to H^{i-k+k'}(M; \mathbb{Z}_2) \to H^{i+k'}(M; \mathbb{Z}_2)$$

is the same as multiplying by x and then by y' from

$$H^{i-k}(M;\mathbb{Z}_2) \to H^i(M;\mathbb{Z}_2) \to H^{i+k'}(M;\mathbb{Z}_2).$$

Since the first composition is injective, and since the first map in the second is an isomorphism, we conclude that multiplication by y' is injective from $H^i(M;\mathbb{Z}_2) \to H^{i+k'}(M;\mathbb{Z}_2)$ for all $0 < i \leq c - k'$. The proof that multiplication by y' is surjective in all required degrees is similar. □

We proceed to the proof of the first part of Theorem D. The key tool in the proof is the existence of Steenrod squares, so we review some of their properties now. The Steenrod squares are group homomorphisms

$$Sq^i : H^*(M;\mathbb{Z}_2) \to H^*(M;\mathbb{Z}_2)$$

which exist for all $i \geq 0$ and satisfy the following properties:

1. If $x \in H^j(M;\mathbb{Z}_2)$, then $Sq^i(x) \in H^{i+j}(M;\mathbb{Z}_2)$, and
   - if $i = 0$, then $Sq^i(x) = x$,
   - if $i = j$, then $Sq^i(x) = x^2$, and
   - if $i > j$, then $Sq^i(x) = 0$.
2. (Cartan formula) If $x, y \in H^*(M;\mathbb{Z}_2)$, then $Sq^i(xy) = \sum_{0 \leq j \leq i} Sq^j(x) Sq^{i-j}(y)$.
3. (Adem relations) For $a < 2b$, one has the following relation among composi-

tions of Steenrod squares:

$$Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j.$$

A consequence of the Adem relations is the following: If k is not a power of two, there exists a relation of the form $Sq^k = \sum_{0<i<k} a_i Sq^i Sq^{k-i}$ for some constants a_i. Indeed, if $k = 2^c + d$ for integers c and $d \equiv 0 \bmod 2^{c+1}$, then we can use the Adem relation with $(a, b) = (2^c, d)$.

The first application of the Steenrod squares in the presence of periodicity is to show the following:

Lemma 2.3. *Suppose $x \in H^k(M; \mathbb{Z}_2)$ is nonzero and induces periodicity up to degree c with $2k \leq c$. If $x = Sq^i(y)$ for some $i > 0$, then x factors as a product of elements of degree less than k.*

Combined with Lemma 2.2, we conclude that if $i > 0$ and if $x = Sq^i(y) \in H^k(M; \mathbb{Z}_2)$ is nonzero and induces periodicity up to degree c with $2k \leq c$, then there is another nonzero element x' inducing periodicity up to degree c with $0 < \deg(x') < k$.

Proof. Let $i > 0$ be maximal such that $x = Sq^i(y)$ for some cohomology element y. Using the Cartan relation, we compute x^2 as follows:

$$x^2 = Sq^i(y)^2 = Sq^{2i}(y^2) - \sum_{j \neq i} Sq^j(y) Sq^{2i-j}(y).$$

Now $Sq^j(y)$ and $Sq^{2i-j}(y)$ commute, so the sum over $j \neq i$ is twice the sum over $j < i$. Hence $x^2 = Sq^{2i}(y^2)$.

Next, $Sq^i(y) = x \neq 0$ implies $i \leq \deg(y)$. Moreover, $i = \deg(y)$ implies that x factors as y^2. Suppose then that $i < \deg(y)$. Since $k = i + \deg(y) < \deg(y^2)$, it follows from the surjectivity assumption of periodicity that $y^2 = xy'$ for some y' with $0 < \deg(y') < \deg(y)$. Using periodicity again, observe that $Sq^j(x)$ for $0 \leq j < k$ can be factored as xx_j for some $x_j \in H^j(M;\mathbb{Z}_2)$. Applying the Cartan formula again, we have

$$x^2 = Sq^{2i}(xy') = \sum_{j \leq 2i} xx_j Sq^{2i-j}(y').$$

The injectivity assumption of periodicity implies we may cancel an x and conclude $x = x_j Sq^{2i-j}(y')$ for some $j \leq 2i$. Because i was chosen to be maximal, we must have $j > 0$, that is, we must have that x factors as a product of elements of degree less than k. $\square$

We proceed to the proof of the first part of Theorem D:

Proof of Theorem D when $p = 2$. Suppose $x \in H^*(M;\mathbb{Z}_2)$ is nonzero, induces periodicity up to degree c with $2k \leq c$, and has minimal degree among all such elements. Assume k is not a power of 2. We will show that x factors nontrivially or that $x = Sq^i(y)$ for some $i > 0$, which contradicts Lemmas 2.2 and 2.3.

The first step is to evaluate the Adem relation $Sq^k = \sum_{0<i<k} a_i Sq^i Sq^{k-i}$ on x. Using the factorization $Sq^j(x) = xx_j$ as above, together with the Cartan formula,

we obtain

$$x^2 = Sq^k(x) = \sum_{0<i<k} a_i Sq^i(xx_{k-i}) = \sum_{0<i<k} a_i \sum_{0\leq j\leq i} xx_j Sq^{i-j}(x_{k-i}).$$

Using the injectivity assumption of periodicity, we can cancel an x to conclude that

$$x = \sum_{0<i<k} \sum_{0\leq j\leq i} a_i x_j Sq^{i-j}(x_{k-i}).$$

Now periodicity and our assumption that x is nonzero imply that $H^k(M;\mathbb{Z}_2)\mathbb{Z}_2$ and is generated by x. It follows that $x = x_j Sq^{i-j}(x_{k-i})$ for some $0 < i < k$ and $0 \leq j \leq i$. If $j > 0$, we have proven a nontrivial factorization of x, and if $j = 0$, we have proven that $x = Sq^i(x_{k-i})$ for some $i > 0$. As explained at the beginning of the proof, this is a contradiction. □

We now proceed to the proof of the second part of Theorem D, which we restate here for easy reference:

Proposition. *Let p be an odd prime. Suppose $x \in H^l(M;\mathbb{Z}_p)$ is nonzero and induces periodicity in $H^*(M;\mathbb{Z}_p)$ up to degree c with $pl \leq c$. If x has minimal degree among all such elements, then $l = 2\lambda p^r$ for some $r \geq 0$ and $\lambda \mid p-1$.*

The proof uses Steenrod powers. These are group homomorphisms

$$P^i : H^*(M;\mathbb{Z}_p) \to H^*(M;\mathbb{Z}_p)$$

for $i \geq 0$ that satisfy the following properties:

1. If $x \in H^j(M;\mathbb{Z}_p)$, then $P^i(x) \in H^{j+2i(p-1)}(M;\mathbb{Z}_p)$, and

- if $i = 0$, then $P^i(x) = x$,
- if $2i = j$, then $P^i(x) = x^p$, and
- if $2i > j$, then $P^i(x) = 0$.

2. (Cartan formula) For $x, y \in H^*(M; \mathbb{Z}_p)$, $P^i(xy) = \sum_{0 \le j \le i} P^j(x) P^{i-j}(y)$.

3. (Adem relations) For $a < pb$,

$$P^a P^b = \sum_{j \le a/p} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j. \tag{2.1.1}$$

Despite the similarity of the statements of Theorem D when $p = 2$ and $p > 2$, the proof in the odd prime case is more involved. We proceed with a sequence of steps.

We first study the structure of the Adem relations to obtain a specific relation in the $\mathbb{Z}_p$-algebra $\mathcal{A}$ generated by $\{P^i\}_{i \ge 0}$ modulo the Adem relations. This lemma does not use periodicity.

Lemma 2.4. *Let $k = \lambda p^a + \mu$ where $0 < \lambda < p$ and $\mu \equiv 0 \bmod p^{a+1}$. For all $1 \le m \le \lambda$, there exist $Q_i \in \mathcal{A}$ such that $P^k = P^{mp^a} \circ Q_a + \sum_{i<a} P^{p^i} \circ Q_i$.*

Proof. We induct over k. For $k = 1$, the result is trivial. Suppose the result holds for all $k' < k$. Write $k = \lambda p^a + \mu$ where $0 < \lambda < p$ and $\mu \equiv 0 \bmod p^{a+1}$, and let $1 \le m \le \lambda$. If $\mu = 0$ and $m = \lambda$, then $P^k = P^{mp^a}$ is already of the desired form. If not, then $mp^a < p(k - mp^a)$ and we have the Adem relation (see Equation 2.1.1)

$$c_0 P^k = P^{mp^a} P^{k-mp^a} - \sum_{0 < j \le mp^{a-1}} c_j P^{k-j} P^j.$$

For $0 < j \le mp^{a-1}$, $k-j$ is less than k and not congruent to 0 modulo p^a. Hence the induction hypothesis implies that each P^{k-j} term is of the form $\sum_{i<a} P^{p^i} Q_i$. It therefore suffices to prove that $c_0 \not\equiv 0 \bmod p$.

For this, we use the following elementary fact: If $x = \sum_{i \ge 0} x_i p^i$ and $y = \sum_{i \ge 0} y_i p^i$ are base p expansions (where p is prime), then the modulo p binomial coefficients satisfy $\binom{x}{y} \equiv \prod \binom{x_i}{y_i} \bmod p$. Hence we have that

$$\pm c_0 = \binom{(p-1)(k-mp^a)-1}{mp^a} \equiv \binom{(p-1)(\lambda-m)p^a - 1}{mp^a} \equiv \binom{p-(\lambda-m)-1}{m},$$

which is not congruent to 0 modulo p since $0 \le m \le p-(\lambda-m)-1 < p$. This completes the proof. □

To simplify the remainder of the proof, we assume throughout the rest of the section the following:

Assumption 2.5. Assume that $x \in H^l(M;\mathbb{Z}_p)$ is nonzero and that x induces periodicity up to degree c with $pl \le c$. Also assume that $l = \deg(x)$ is minimal among all such elements.

In particular, Lemma 2.2 implies that the only factorizations of ax^r with $a \neq 0$ and $r \le p$ are of the form $(a'x^{r'})(a''x^{r''})$ with $a', a'' \in \mathbb{Z}_p$ and $r = r' + r''$. The next step is to prove an analogue of Lemma 2.3:

Lemma 2.6. *No nontrivial multiple of x is of the form $P^i(y)$ with $i > 0$.*

Proof. Without loss of generality, we may assume x itself is equal to $P^i(y)$ for some $i > 0$. Set $d = \deg(y)$. Our first task is to write some power of x as $P^{i_1}(y_1)$ with

$0 < \deg(y_1) < d$. Because $x \neq 0$, we have $2i \leq d$, which implies $\deg(y^p) \geq l$. Let r be the integer such that $l + d > \deg(y^r) \geq l$. Lemma 2.2 implies a strict inequality here. Using periodicity, write $y^r = xy_1$ with $0 < d_1 < d$.

We next calculate the r-th power of both sides of the equation $x = P^i(y)$ using the Cartan relation:

$$x^r = P^{ri}(y^r) - \sum c_{j_1,\ldots,j_r} P^{j_1}(y) \cdots P^{j_r}(y)$$

where the $c_{j_1,\ldots,j_r}$ are constants and the sum runs over $j_1 \geq \cdots \geq j_r$ with $j_1 + \ldots + j_r = ri$ and $(j_1, \ldots, j_r) \neq (i, \ldots, i)$. Observe that $j_1 > i$, so $P^{j_1}(y) = xz_{j_1}$ for some $z_{j_1} \in H^{2(j_1-i)(p-1)}(M; \mathbb{Z}_p)$. Using $y^r = xy_1$, the first term on the right-hand side becomes

$$P^{ri}(y^r) = P^{ri}(xy_1) = \sum_{k'+k=ri} P^{k'}(x)P^k(y_1) = x \sum_{k'+k=ri} x_{k'} P^k(y_1)$$

for some $x_{k'} \in H^{2k'(p-1)}(M; \mathbb{Z}_p)$. Combining these calculations, and using periodicity to cancel the x, we obtain

$$x^{r-1} = \sum_{k'+k=ri} x_{k'} P^k(y_1) + \sum c_{j_1,\ldots,j_r} z_{j_1} P^{j_2}(y) \cdots P^{j_r}(y).$$

Now $r - 1 < p$, so periodicity implies that x^{r-1} generates $H^{(r-1)l}(M; \mathbb{Z}_p)$. By Lemma 2.2 therefore, every term of the form $z_{j_1} P^{j_2}(y) \cdots P^{j_r}(y)$ vanishes since $0 < \deg(P^{j_r}(y)) < \deg(P^i(y)) = l$. Similarly, all terms of the form $x_{k'} P^k(y_1)$ vanish unless $P^k(y_1)$ is a power of x. Hence some power of x is of the form $P^{i_1}(y_1)$, as claimed.

We now show that, given an expression $x^{r_j} = P^{i_j}(y_j)$ for some $j \geq 1$ with $0 < \deg(y_j) < d$, there exists another expression $x^{r_{j+1}} = P^{i_{j+1}}(y_{j+1})$ with $0 < \deg(y_{j+1}) < d$. Moreover, it will be apparent that $l + \deg(y_{j+1}) = \deg(y_j) + m_j d$ for some integer m_j. First, among all such expressions $x^{r_j} = P^{i_j}(y_j)$, fix y_j and take r_j (or, equivalently, i_j) to be minimal. Next, note that $P^{i_j}(y_j) = x^{r_j} \neq 0$ implies $p \deg(y_j) \geq r_j l$, which together with $pd \geq l$ implies

$$p \deg(y_j y^{p-r_j}) = p \deg(y_j) + (p - r_j)pd \geq pl.$$

Hence we can choose an integer $m_j \leq p - r_j$ satisfying $l \leq \deg(y_j) + m_j d < l + d$. Once again, Lemma 2.2 implies both inequalities are strict. Using periodicity, we can write $y_j y^{m_j} = x y_{j+1}$ with $0 < \deg(y_{j+1}) < d$ and $l + \deg(y_{j+1}) = \deg(y_j) + m_j d$. We now calculate

$$x^{r_j + m_j} = P^{i_j}(y_j) P^i(y)^{m_j} = P^{i_j + m_j i}(y_j y^{m_j}) - \sum P^{k_0}(y_j) P^{k_1}(y) \cdots P^{k_{m_j}}(y)$$

where the sum runs over $(k_0, \ldots, k_{m_j}) \neq (i_j, i, \ldots, i)$ with $k_0 + \ldots + k_{m_j} = i_j + m_j i$. As when we calculated x^r above, we are able to factor an x from each term on the right-hand side and use periodicity to cancel it. Using that $x^{r_j + m_j - 1}$ is a generator and Lemma 2.2, together with the assumption that r_j is minimal, we conclude that x is a nonzero multiple of $P^{i_{j+1}}(y_{j+1})$, as claimed.

We therefore have a sequence of cohomology elements (y_j) with $0 < \deg(y_j) < d$ and $l + \deg(y_{j+1}) = \deg(y_j) + m_j d$ for some integer m_j for all $j \geq 1$. This cannot be. Indeed, adding the equations $l + \deg(y_1) = rd$ and $l + \deg(y_{j+1}) = \deg(y_j) + m_j d$

for $1 \leq j \leq d-1$ yields

$$ld + \deg(y_d) = (r + m_1 + \ldots + m_{d-1})d,$$

which implies that $\deg(y_d)$ is divisible by d. But $0 < \deg(y_d) < d$, so this is a contradiction. □

Lemma 2.6 easily implies the following:

Lemma 2.7. *No nontrivial multiple of x^r with $1 \leq r \leq p$ is of the form $P^i(y)$ with $0 < i < \frac{l}{2(p-1)}$.*

Proof. Indeed, the bound on i implies $\deg(y) = ml - 2i(p-1) > (m-1)l$, so periodicity implies $y = x^{m-1}z$ for some z with $0 < \deg(z) < l$. Applying the Cartan formula and periodicity, we obtain $x^m = P^j(x^{m-1})P^{j'}(z)$ for some $j + j' = i$. By Lemma 2.2, $P^{j'}(z)$ is a power of x. But since $\deg(P^j(x^{m-1})) \geq (m-1)l$, we must have $x = P^{j'}(z)$. Since $\deg(z) < l$, we have a contradiction to Lemma 2.6. □

At this point, we combine what we have established so far. Recall that we are assuming $x \in H^l(M;\mathbb{Z}_p)$ is nonzero and induces periodicity up to degree $2l \leq c$ and that x has minimal degree among all such elements. Observe that $p > 2$ implies $x^3 \neq 0$. Hence $l = 2k$ for some k.

Lemma 2.8. *Suppose $l = 2k$ and $k = \lambda p^a + \mu$ for some $0 < \lambda < p$ and $\mu \equiv 0 \bmod p^{a+1}$. For all $1 \leq m \leq \lambda$, there exists $r < p$, $0 < j \leq mp^a$, and $z \in H^{2p^a(r\lambda - m(p-1))}(M;\mathbb{Z}_p)$ such that $x^r = P^j(z)$.*

Moreover, $j \equiv 0 \bmod p^a$ *and* $0 \le \deg(z) < l$ *with* $\deg(z) = 0$ *only if* $rk = (p-1)mp^a$.

Proof. Let l, k, and m be as in the assumption. Evaluating the expression in Lemma 2.4 on x yields

$$x^p = P^k(x) = P^{mp^a}(Q_a(x)) + \sum_{i<a} P^{p^i}(Q_i(x)).$$

Using periodicity, we can write $Q_i(x) = xz_i$ for $i < a$, $Q_a(x) = x^{p-r}z$ for some $1 \le r < p$ such that $0 \le \deg(z) < l$, $P^j(x) = xy_i$ for all j, and $P^{mp^a-j}(x^{p-r}) = x^{p-r}w_j$ for all j.

Using this notation and the Cartan formula, we have

$$x^p = x^{p-r} \sum_{j \le mp^a} w_j P^j(z) + x \sum_{i<a} \sum_{j \le i} y_j P^{p^i-j}(z_i).$$

Using periodicity again, we obtain

$$x^{p-1} = x^{p-r-1} \sum_{j \le mp^a} w_j P^j(z) + \sum_{i<a} \sum_{j \le i} y_j P^{p^i-j}(z_i).$$

Periodicity implies x^{p-1} is a (nonzero) generator of $H^{(p-1)l}(M; \mathbb{Z}_p)$, hence a nontrivial multiple of x^{p-1} is $x^{p-r-1}w_j P^j(z)$ for some $j \le mp^a$ or $y_j P^{p^i-j}(z_i)$ for some $j \le i < a$. In the second case, we have a contradiction to Lemma 2.2 or Lemma 2.7 since $p^i - j \le p^{a-1} < l/2(p-1)$. Similarly, we have a contradiction to Lemma 2.2 in the first case unless w_j is a power of x. Moreover, $\deg(w_j) = 2(p-1)(mp^a - j)$ implies $j = ip^a$ for some $0 \le i \le m$.

Using periodicity to cancel powers of x, we conclude that $x^r = P^j(z)$ for some r, j, and z as in the conclusion of the lemma. □

Using this result, we are in a position to prove Theorem D for $p > 2$, that is, we are ready to show that $l = 2\lambda p^r$ for some $r \geq 0$ and some $\lambda | p-1$:

Proof of Theorem D when $p > 2$. Suppose $x \in H^l(M;\mathbb{Z}_p)$ is nonzero, induces periodicity up to degree c with $pl \leq c$, and has minimal degree among all such elements.

Since $p > 2$, we have $l = 2k$ for some k. Write $k = \lambda p^a + \mu$ where $0 < \lambda < p$ and $\mu \equiv 0 \bmod p^{a+1}$. Let $g = \gcd(\lambda, p-1)$. Our task is to show that $\mu = 0$ and that $g = \lambda$. We prove this by contradiction using three cases.

Suppose first that $\mu > 0$. Take $m = 1$ in Lemma 2.8. Then we have $x^r = P^{p^a}(z)$ with $\deg(z) > 0$ since

$$rk \geq k \geq \mu \geq p^{a+1} > (p-1)mp^a.$$

Because

$$p^a = (p^a + p^{a+1})/(p+1) < l/2(p-1),$$

we have a contradiction to Lemma 2.7.

Second, suppose that $\mu = 0$ and $1 = g < \lambda$. Choose $1 \leq m < \lambda$ such that $m(p-1) \equiv -1 \mod \lambda$. Lemma 2.8 implies the existence of $r < p$, $0 < j \leq mp^a$ with $j \equiv 0 \bmod p^a$, and $z \in H^{2p^a(r\lambda - m(p-1))}(M;\mathbb{Z}_p)$ with $0 \leq \deg(z) < l$ such that $x^r = P^j(z)$. Our choice of m and the conditions on $\deg(z)$ imply $\deg(z) = 2p^a$. In addition, $x^r \neq 0$ implies $j \leq p^a$, so the conditions on j imply $j = p^a$. Putting these facts together implies $x^r = P^j(z) = z^p$. Because $0 < \deg(z) < l$, this contradicts Lemma 2.2.

Finally, suppose that $\mu = 0$ and $1 < g < \lambda$. Taking $m = 1$ yields $x^r = P^{p^a}(z)$ with $\deg(z) > 0$ since $g < \lambda$ implies

$$(p-1)mp^a = (p-1)p^a \neq r\lambda p^a = rk.$$

Raising both sides to the (λ/g)-th power, we obtain

$$x^{r\lambda/g} = P^{\lambda p^a/g}(z^{\lambda/g}) - \sum P^{i_1}(z)\cdots P^{i_{\lambda/g}}(z)$$

where the sum runs over $i_1 + \ldots + i_{\lambda/g} = \lambda p^a/g$ with $(i_1, \ldots, i_{\lambda/g}) \neq (p^a, \ldots, p^a)$. Observe that $\deg(z^{\lambda/g})$ is a multiple of $l = \deg(x)$ while $0 < \deg(z) < l$, so Lemma 2.2 implies $z^{\lambda/g} = 0$. Now $g \geq 2$ and $rl - 2(p-1)p^a = \deg(z) < l$ implies $r\lambda/g < p$, so $x^{r\lambda/g}$ is nonzero and generates $H^{lr\lambda/g}(M;\mathbb{Z}_p)$. This implies $x^{r\lambda/g}$ is a nontrivial multiple of $P^{i_1}(z)\cdots P^{i_{\lambda/g}}(z)$ for some $(i_1, \ldots, i_{\lambda/g})$. Using Lemma 2.2 again, we conclude that each $P^{i_j}(z)$ is a power of x. But the degrees of x and z implies that this is only the case if $i_j \geq p^a$ for all j. Since there is no such term in the sum, we obtain a contradiction. □

2.2 From Theorem D to the periodicity theorem

In this section, we use Theorem D to prove Theorem C in the introduction. We then pull together Theorem C and the connectedness theorem to prove the periodicity theorem.

We recall the setting of Theorem C: We are given a closed, simply connected manifold M^n and an element $x \in H^k(M;\mathbb{Z})$ inducing periodicity with $3k \leq n$. Our goal is to show that $H^*(M;\mathbb{Q})$ is $\gcd(4,k)$-periodic. We proceed to the proof, the summary of which is included in Table 2.1.

$H^*(M;\mathbb{Z})$ is k-periodic	$\Longrightarrow$	$H^*(M;\mathbb{Z}_p)$ is k-periodic	$\Longrightarrow$	$H^*(M;\mathbb{Z}_p)$ is $2(p-1)p^r$-periodic with $p \in \{2,3\}$
	$\searrow$		$\swarrow$	
		$H^*(M;\mathbb{Q})$ is $2(p-1)p^r$-periodic with $p \in \{2,3\}$		
		$\Downarrow$		
		$H^*(M;\mathbb{Q})$ is $\gcd(2^r, 4 \cdot 3^s)$-periodic		

Table 2.1: Proof sketch for Theorem C.

Proof of Theorem C. Note that if x is a torsion element, then periodicity implies that M is a rational homology sphere. Since $H^*(M;\mathbb{Q})$ is then trivially $\gcd(4,k)$-periodic, we may assume x is not a torsion element. By periodicity, $H^k(M;\mathbb{Z}) \cong \mathbb{Z}$ and is generated by x.

Consider the nonzero image $x_2 \in H^k(M;\mathbb{Z}_2)$ of x under the reduction homomorphism $H^k(M;\mathbb{Z}) \to H^k(M;\mathbb{Z}_2)$. It follows by the Bockstein sequence

$$\cdots \xrightarrow{\cdot 2} H^i(M;\mathbb{Z}) \xrightarrow{\rho} H^i(M;\mathbb{Z}_2) \longrightarrow H^{i+1}(M;\mathbb{Z}) \longrightarrow \cdots$$

and the five lemma that x_2 induces periodicity in $H^*(M;\mathbb{Z}_2)$. Denote by $y \in$

$H^l(M;\mathbb{Z}_2)$ the element of minimal degree which induces periodicity. Theorem D says l is a power of 2.

We claim that l divides k. If not, there exists a cohomology element y' with $0 < \deg(y') < l$ such that $y^m = x_2 y'$ for some integer m. By Lemma 2.2, it follows that y' also induces periodicity, a contradiction to the minimality of l. We have then that $l \mid k$. Moreover, periodicity implies $y^{k/l} = x_2$.

Next we show that y comes from an integral element $\tilde{y} \in H^l(M;\mathbb{Z})$ such that the map $H^i(M;\mathbb{Z}) \to H^{i+l}(M;\mathbb{Z})$ induced by multiplication by $\tilde{y}$ has finite kernel for all $0 < i < n$. Let $\rho : H^l(M;\mathbb{Z}) \to H^l(M;\mathbb{Z}_2)$ be the map induced by reduction modulo 2. Consider first that (via multiplication by y) $0 = H^1(M;\mathbb{Z}_2) \cong H^{1+l}(M;\mathbb{Z}_2)$, which implies $b_{l+1}(M) \leq b_{l+1}(M;\mathbb{Z}_2) = 0$. Next consider the following portion

$$H^l(M;\mathbb{Z}) \to H^l(M;\mathbb{Z}_2) \to H^{l+1}(M;\mathbb{Z}) \to H^{l+1}(M;\mathbb{Z}) \to H^{l+1}(M;\mathbb{Z}_2)$$

of the Bockstein sequence. We see that that $H^{l+1}(M;\mathbb{Z}) \to H^{l+1}(M;\mathbb{Z})$ is a surjection and hence an isomorphism since $H^{l+1}(M;\mathbb{Z})$ is finite. Using exactness again, we conclude $\rho : H^l(M;\mathbb{Z}) \to H^l(M;\mathbb{Z}_2)$ is surjective, so that we can choose some $\tilde{y} \in H^l(M;\mathbb{Z})$ with $\rho(\tilde{y}) = y$. Now $H^k(M;\mathbb{Z})$ is generated by x, so $\tilde{y}^{k/l} = mx$ for some $m \in \mathbb{Z}$. Applying ρ to both sides yields

$$mx_2 = \rho(\tilde{y}^{k/l}) = y^{k/l} = x_2 \neq 0,$$

hence $m \neq 0$. This proves that multiplication by $\tilde{y}$ has finite kernel.

Moving to rational coefficients, we conclude that $\bar{y} \in H^l(M;\mathbb{Q})$, the image of $\tilde{y}$

under the coefficient map $H^l(M;\mathbb{Z}) \to H^l(M;\mathbb{Q})$, induces periodicity in $H^*(M;\mathbb{Q})$.

A completely analogous argument using Theorem D with $p = 3$ shows that $H^*(M;\mathbb{Q})$ is m-periodic with $m = 4 \cdot 3^s$. Taking m to be minimal, it follows again that $m \mid k$. At this point it is clear that the Betti numbers of M are $\gcd(k,l,m)$-periodic and hence $\gcd(4,k)$-periodic.

To conclude that $H^*(M;\mathbb{Q})$ is $\gcd(4,k)$-periodic, consider the set D of all positive integers d such that $H^*(M;\mathbb{Q})$ has an element in degree d which induces periodicity. Clearly $k,l,m \in D$. We claim that $d_1, d_2 \in D$ with $d_1 > d_2$ implies $d_1 - d_2 \in D$. Indeed suppose $z_1 \in H^{d_1}(M;\mathbb{Q})$ and $z_2 \in H^{d_2}(M;\mathbb{Q})$ induce periodicity in $H^*(M;\mathbb{Q})$. Since z_2 induces periodicity, there exists $z_3 \in H^{d_1-d_2}(M;\mathbb{Q})$ such that $z_1 = z_2 z_3$. Since z_1 induces periodicity, Lemma 2.2 implies that z_3 does as well. Since the difference of any two elements in D lies in D, it follows that $\gcd(k,l,m)$, and hence $\gcd(4,k)$, also lies in D. □

We now prove the periodicity theorem using Theorem C and the connectedness theorem. We recall the statement of the connectedness theorem:

Theorem 2.9 (Connectedness Theorem, [35])**.** *Suppose M^n is a closed Riemannian manifold with positive sectional curvature.*

1. *If N^{n-k} is connected and totally geodesic in M, then $N \hookrightarrow M$ is $(n-2k+1)$-connected.*

2. *If $N_1^{n-k_1}$ and $N_2^{n-k_2}$ are totally geodesic with $k_1 \leq k_2$, then $N_1 \cap N_2 \hookrightarrow N_2$ is*

$(n-k_1-k_2)$-connected.

Recall an inclusion $N \hookrightarrow M$ is called h-connected if $\pi_i(M,N)=0$ for all $i \leq h$. It follows from the relative Hurewicz theorem that the induced map $H_i(N;\mathbb{Z}) \to H_i(M;\mathbb{Z})$ is an isomorphism for $i<h$ and a surjection for $i=h$. The following is a topological consequence of highly connected inclusions of closed, orientable manifolds (see [35]):

Theorem 2.10. *Let M^n and N^{n-k} be closed, orientable manifolds. If $N \hookrightarrow M$ is $(n-k-l)$-connected with $n-k-2l>0$, then there exists $e \in H^k(M;\mathbb{Z})$ such that the maps $H^i(M;\mathbb{Z}) \to H^{i+k}(M;\mathbb{Z})$ given by $x \mapsto ex$ are surjective for $l \leq i < n-k-l$ and injective for $l < i \leq n-k-l$.*

Combining these results with Theorem C, we will prove in this section the following slightly stronger version of the periodicity theorem.

Theorem 2.11. *Let N^n be a closed, simply connected Riemannian manifold with positive sectional curvature. Let $N_1^{n-k_1}$ and $N_2^{n-k_2}$ be totally geodesic, transversely intersecting submanifolds with $k_1 \leq k_2$.*

1. *If $k_1+3k_2 \leq n$, then the rational cohomology rings of N, N_1, N_2, and $N_1 \cap N_2$ are $\gcd(4,k_1,k_2)$-periodic.*

2. *If $2k_1+2k_2 \leq n$, then the rational cohomology rings of N, N_1, N_2, and $N_1 \cap N_2$ are $\gcd(4,k_1)$-periodic.*

3. If $3k_1 + k_2 \leq n$ and if N_2 is simply connected, then the rational cohomology rings of N_2 and $N_1 \cap N_2$ are $\gcd(4, k_1)$*-periodic.*

We make two remarks. First, all three codimension assumptions imply that N_1 is simply connected and that $N_1 \cap N_2$ is simply connected if N_2 is. This follows by the connectedness theorem since the bounds on the codimensions imply that the inclusions $N_1 \hookrightarrow N$ and $N_1 \cap N_2 \hookrightarrow N_2$ induce isomorphisms of fundamental groups. Similarly the first two assumptions imply that N_2 is simply connected, but the third condition does not.

Second, in the proof of Theorem A, we will use the following consequence of Theorem 2.11:

Corollary 2.12. *Let N^n be a closed, positively curved manifold with $n \equiv 0 \bmod 4$. Let $N_1^{n-k_1}$ and $N_2^{n-k_2}$ be totally geodesic, transversely intersecting submanifolds with $2k_1 + 2k_2 \leq n$. Then* $\mathrm{b}_{\mathrm{odd}}(N) = \sum b_{1+2i}(N) = 0$.

Proof of Corollary 2.12. Let $\pi : \tilde{N} \to N$ denote the universal Riemannian covering. The submanifolds $\pi^{-1}(N_i) \subseteq N$ are transversely intersecting, totally geodesic, $(n - k_i)$-dimensional submanifolds of the closed, simply connected, positively curved manifold $\tilde{N}$. Since $2k_1 + 2k_2 \leq n$, Theorem 2.11 implies $H^*(\tilde{N}; \mathbb{Q})$ is 4-periodic.

Observe that 4-periodicity and Poincaré duality imply $\mathrm{b}_{\mathrm{odd}}(\tilde{N}) = 0$ since $\tilde{N}$ is simply connected and $n \equiv 0 \bmod 4$. Recall now that the transfer theorem implies that $H^*(N; \mathbb{Q})$ is isomorphic to $H^*(\tilde{N}; \mathbb{Q})^{\pi_1(N)}$, the subring of invariant elements under the action of $\pi_1(N)$ on $H^*(\tilde{N};\mathbb{Q})$. Since $\mathrm{b}_{\mathrm{odd}}(\tilde{N}) = 0$, it follows that $\mathrm{b}_{\mathrm{odd}}(N) = 0$. $\square$

We proceed now to the proof of Theorem 2.11. Recall that we have a closed, simply connected Riemannian manifold N^n with positive sectional curvature. We also have totally geodesic, transversely intersecting submanifolds $N_1^{n-k_1}$ and $N_2^{n-k_2}$ with $k_1 \leq k_2$. As discussed above, we may assume that N_1, N_2, and $N_1 \cap N_2$ are simply connected and therefore orientable.

Observe that we have $3k_1 + k_2 \leq n$ in all three cases. By the corollary to the connectedness theorem, $H^*(N_2;\mathbb{Z})$ is k_1-periodic since the intersection is transverse. The bound on the codimensions implies $3k_1 \leq \dim(N_2)$, hence Theorem C implies $H^*(N_2;\mathbb{Q})$ is g-periodic, where $g = \gcd(4, k_1)$. Since $N_1 \cap N_2 \hookrightarrow N_2$ is $\dim(N_1 \cap N_2)$-connected, $H^*(N_1 \cap N_2;\mathbb{Q})$ is g-periodic as well. This concludes the proof of the third statement.

Assume now that $2k_1 + 2k_2 \leq n$. We claim that $H^*(N;\mathbb{Q})$ is g-periodic. Let $j : N_2 \hookrightarrow N$ be the inclusion map, and let $x' \in H^g(N_2;\mathbb{Q})$ be an element inducing periodicity in $H^*(N_2;\mathbb{Q})$. Because $n - 2k_2 \geq 2k_1 \geq g$, the connectedness theorem implies $j^* : H^g(N;\mathbb{Q}) \to H^g(N_2;\mathbb{Q})$ is an isomorphism. Let $x \in H^g(N;\mathbb{Q})$ satisfy $j^*(x) = x'$. By the first part of the connectedness theorem, the inclusion $N_1 \hookrightarrow N$ is $n - k_1 - (k_1 - 1)$ connected, so there exists $e_1 \in H^{k_1}(N;\mathbb{Z})$ such that the maps $H^i(N;\mathbb{Z}) \to H^{i+k_1}(N;\mathbb{Z})$ given by $y \mapsto e_1 y$ are isomorphisms for $k_1 \leq i \leq n - 2k_1$. Denote by $\overline{e}_1$ the image of e_1 under the natural map $H^{k_1}(N;\mathbb{Z}) \to H^{k_1}(N;\mathbb{Q})$, and note that $\overline{e}_1$ satisfies the corresponding property with rational coefficients.

Note that $\overline{e}_1$ is some nonzero multiple of $x^{k_1/g}$. This follows since $(x')^{k_1/g}$ generates $H^{k_1}(N_2;\mathbb{Q})$, and since j is $n-2k_2+1 \geq 2k_1+1$ connected. Replacing $\overline{e}_1$ by any nonzero multiple preserves the multiplicative property of $\overline{e}_1$, so suppose without loss of generality that $\overline{e}_1 = x^{k_1/g}$. We prove in three cases the claim that multiplication by x induces isomorphisms $H^i(N;\mathbb{Q}) \to H^{i+g}(N;\mathbb{Q})$ for all $0 < i < n-g$:

- For $0 < i < 2k_1 - g$, the claim follows since $N_2 \hookrightarrow N$ is $n-2k_2$ connected and $n - 2k_2 + 1 \geq 2k_1 + 1$. For $n - 2k_1 < i < n - g$, one uses the cap product isomorphisms given by x together with Poincaré duality to conclude the claim.

- For $2k_1 < i < n - 2k_1 - g$, one chooses $l \geq 1$ such that $k_1 < i - lk_1 \leq 2k_1$ and uses the fact that multiplication by $\overline{e}_1$ induces isomorphisms in some middle degrees and that $\overline{e}_1$ and x commute.

- For $2k_1 - g \leq i \leq 2k_1$ or $n - 2k_1 - g \leq i \leq n - 2k_1$, one factors multiplication by $\overline{e}_1$ as multiplication by $x^{k_1/g-1}$ followed by multiplication by x and uses the previous two cases.

Hence x induces g-periodicity in N, as claimed.

Next let $g' = g$ if $k_1 + 3k_2 > n$ and $g' = \gcd(4, k_1, k_2)$ if $k_1 + 3k_2 \leq n$. Our proof will be complete once we show that N, N_1, N_2, and $N_1 \cap N_2$ are g'-periodic. First, we claim that N is g'-periodic.

If $k_1 + 3k_2 > n$, then $H^*(N;\mathbb{Q})$ is already g'-periodic. Suppose then that $k_1 + 3k_2 \leq n$. By the corollary to the connectedness theorem, there exists $e_2 \in H^{k_2}(N;\mathbb{Q})$ such that the maps $H^i(N;\mathbb{Q}) \to H^{i+k_2}(N;\mathbb{Q})$ induced by multiplication by e_2 are isomorphisms for $k_2 \leq i \leq n - 2k_2$. Given that x and e_2 commute, we conclude that e_2 induces periodicity in $H^*(N;\mathbb{Q})$. Indeed, suppose $0 < i < k_2$. Choose $j \geq 0$ with $k_2 \leq i + jg \leq n - 2k_2$. Observe that $H^i(N;\mathbb{Q}) \to H^{i+k_2}(N;\mathbb{Q})$, induced by multiplication by e_2, composed with the isomorphism $H^{i+k_2}(N;\mathbb{Q}) \to H^{i+k_2+jg}(N;\mathbb{Q})$, induced by multiplication by x^j, is the same as the composition of isomorphisms

$$H^i(N;\mathbb{Q}) \to H^{i+jg}(N;\mathbb{Q}) \to H^{i+jg+k_2}(N;\mathbb{Q})$$

induced by multiplying in the other order. It follows that multiplication by e_2 induces isomorphisms $H^i(N;\mathbb{Q}) \to H^{i+k_2}(N;\mathbb{Q})$ for $0 < i < k_2$. Checking the other required periodicity conditions requires similar arguments. Hence we have that $H^*(N;\mathbb{Q})$ is g'-periodic.

Using this periodicity, we now conclude that the rational cohomology rings of N_1, N_2, and $N_1 \cap N_2$ are g'-periodic.

First, since $4k_1 \leq 2k_1 + 2k_2 \leq n$, $N_1 \hookrightarrow N$ induces isomorphisms on cohomology up to half of the dimension of N_2. Using Poincaré duality, it follows from the fact that N is rationally g'-periodic that N_1 is too.

Second, observe that $N_2 \hookrightarrow N$ is $n - 2k_2 + 1 \geq 2k_1 + 1$ periodic. Hence $H^*(N_2;\mathbb{Q})$ is both g-periodic and g'-periodic up to degree $2k_1$ (which is at least twice g). Since

N_2 is rationally g-periodic, it follows that N_2 is rationally g'-periodic by arguments similar to those above.

Finally, $N_1 \cap N_2 \hookrightarrow N_2$ is $\dim(N_1 \cap N_2)$-connected, so $N_1 \cap N_2$ is clearly g'-periodic as well. This concludes the proof of the Theorem 2.11.

Chapter 3

Proof of Theorem A

Before we begin, we state three well known theorems for easy reference:

Theorem 3.1 (Berger)**.** *If T is a torus acting by isometries on an compact, even-dimensional, positively curved manifold M, then the fixed-point set M^T is nonempty.*

Theorem 3.2 (Lefschetz)**.** *If T is a torus acting on a manifold M, then the Euler characteristic satisfies $\chi(M) = \chi(M^T)$.*

Theorem 3.3 (Conner)**.** *If T is a torus acting on a manifold P, then the sum of the odd Betti numbers satisfies $\mathrm{b}_{\mathrm{odd}}(P^T) \leq \mathrm{b}_{\mathrm{odd}}(P)$.*

We also pause to make a definition:

Definition 3.4. Let T be a torus acting effectively on a manifold M. If the T-action restricts to an action on a submanifold $N \subseteq M$, let $\dim\ker(T|_N)$ denote the dimension of the kernel of the induced action on N.

Observe that if $\dim\ker(T|_N) = d$, then one can find a codimension d subtorus $T' \subseteq T$ whose Lie algebra is complementary to the kernel of the induced T-action on N. It follows that T' acts almost effectively on N.

We recall the setup of Theorem A. We are given a closed, positively curved Riemannian manifold M with $\dim(M) \equiv 0 \bmod 4$, and we have an effective, isometric action by a torus T with $\dim(T) \geq 2\log_2(\dim M)$. By Theorems 3.1 and 3.2, M^T is nonempty and $\chi(M) = \chi(M^T)$, hence it suffices to show that $\mathrm{b}_{\mathrm{odd}}(F) = 0$ for all components F of M^T.

Fix a component F of M^T. Our goal is to find a submanifold P with $F \subseteq P \subseteq M$ and $\mathrm{b}_{\mathrm{odd}}(P) = 0$ such that T acts on P. It would follow that F is a component of P^T, and hence Theorem 3.3 would imply $\mathrm{b}_{\mathrm{odd}}(F) = 0$.

In order to find such a submanifold P, the first step is to set up a sort of induction argument. To do this, we look at our situation from the point of view of F. We consider all closed, totally geodesic submanifolds N^n with $F \subseteq N^n \subseteq M$ such that

$$(*) \quad \begin{cases} n \equiv 0 \bmod 4, \text{ and there exists a subtorus } T' \subseteq T \text{ acting} \\ \text{almost effectively on } N \text{ with } \dim(T') \geq 2\log_2(n). \end{cases}$$

Clearly M satisfies property $(*)$ by the assumption in Theorem A, so the collection of submanifolds N satisfying $(*)$ is nonempty. As it will simplify later arguments, we complete the induction setup by choosing a submanifold N^n with minimal n satisfying $F \subseteq N \subseteq M$ and property $(*)$.

We make a few remarks before continuing with the proof. First, observe that F is a component of the fixed-point set $N^{T'}$. This follows since $F \subseteq N \subseteq M$ and T' acts almost effectively on N. Since our goal is to show $\mathrm{b}_{\mathrm{odd}}(F) = 0$, and since we will do this by finding a submanifold $F \subseteq P \subseteq N$ on which T' acts with $\mathrm{b}_{\mathrm{odd}}(P) = 0$, we may forget about M and T and instead focus on N and T'.

Second, since we are focused only on the action of T' on N, we may divide T' by its discrete ineffective kernel to assume without loss of generality that T' acts effectively on N. Third, it will be convenient to adopt the following notation (recall that F is fixed):

Definition 3.5.

1. For a submanifold $N' \subseteq N$, let $\mathrm{cod}(N')$ denote the codimension of N' in N.

2. For a subgroup $H \subseteq T'$, let $F(H)$ denote the component of the fixed-point set N^H of H which contains F. If H is generated by $\sigma \in T'$, we will write $F(\sigma)$ for $F(H)$.

Finally, the following lemma is a consequence of our choice of N. It is one of the two places where the logarithmic bound appears. We will refer to it frequently.

Lemma 3.6. *For a non-trivial subgroup $H \subseteq T'$ with $\dim F(H) \equiv 0 \bmod 4$, we have $\dim F(H) > n/2^{d/2}$ where $d = \dim \ker T'|_{F(H)}$.*

Proof. Indeed, if $\dim F(H) \leq n/2^{d/2}$, then the remark following Definition 3.4 implies the existence of a codimension d subtorus $T'' \subseteq T'$ acting almost effectively on $F(H)$ with

$$\dim(T'') = \dim(T') - d \geq 2\log_2(n) - d \geq 2\log_2(\dim F(H)).$$

Hence $F(H)$ satisfies property $(*)$. But since the action of T' on N is effective, $\dim F(H) < n$, a contradiction to our choice of N. □

We now proceed with the second part of the proof, in which we study the array of intersections of fixed-point sets of involutions in T'. The strategy is to find $F \subseteq P \subseteq N$ such that $\dim(P) \equiv 0 \bmod 4$ and such that P contains a pair of transversely intersecting submanifolds. This takes work and is the heart of the proof. Once we find this transverse intersection, we will apply Lemma 3.6 to show that the two codimensions are small enough so that the periodicity theorem applies. Corollary 2.12 will then imply $\mathrm{b}_{\mathrm{odd}}(P) = 0$, as required.

To organize the required intersection, codimension, and symmetry data, we define an abstract graph which simplifies the picture while retaining this information:

Definition 3.7. Define a graph Γ by declaring the following

- An involution $\sigma \in T'$ is in Γ if $\operatorname{cod} F(\sigma) \equiv_4 0$ and $\dim\ker\left(T'|_{F(\sigma)}\right) \leq 1$, and
- An edge exists between distinct $\sigma, \tau \in \Gamma$ if $F(\sigma) \cap F(\tau)$ is not transverse.

We are ready to prove the existence of a submanifold $F \subseteq P \subseteq N$ on which T' acts with $\mathrm{b}_{\mathrm{odd}}(P) = 0$. As we will see, the P we choose will be N itself or $F(H)$

for some $H \subseteq T'$, so we will only need to show that $\mathrm{b}_{\mathrm{odd}}(P) = 0$. We separate the proof into five cases, according to the structure of Γ.

Lemma 3.8 (Case 1)**.** *Let $r = \dim(T')$. If Γ does not contain $r-1$ algebraically independent involutions, then $\mathrm{b}_{\mathrm{odd}}(N) = 0$.*

Proof. Let $0 \leq j \leq r-2$ be maximal such that there exist $\iota_1, \ldots, \iota_j \in \Gamma$ generating a $\mathbb{Z}_2^j$. We wish to show that $\mathrm{b}_{\mathrm{odd}}(N) = 0$.

Consider the isotropy $T' \hookrightarrow SO(T_x N)$ for some $x \in F$. Choose a basis of the tangent space so that the image of the $\mathbb{Z}_2^r \subseteq T'$ lies in a copy of $\mathbb{Z}_2^{n/2} \subseteq T^{n/2} \subseteq SO(T_x N)$. Let $\mathbb{Z}_2^{r-1}$ denote a subspace of the kernel of the composition $\mathbb{Z}_2^r \to \mathbb{Z}_2^{n/2} \to \mathbb{Z}_2$, where the last map is given by $(\tau_1, \ldots, \tau_{n/2}) \mapsto \sum \tau_i$. It follows that every $\sigma \in \mathbb{Z}_2^{r-1}$ has $\operatorname{cod} F(\sigma) \equiv 0 \bmod 4$.

Since $j \leq r-2$, there exists $\iota_{j+1} \in \mathbb{Z}_2^{r-1} \setminus \langle \iota_1, \ldots, \iota_j \rangle$. Choosing ι_{j+1} to have minimial $\operatorname{cod} F(\iota_{j+1})$ ensures that $\dim \ker \left(T'|_{F(\iota_{j+1})}\right) \leq 2$. Moreover, because j is maximal, we cannot have $\iota_{j+1} \in \Gamma$. Hence $\dim \ker \left(T'|_{F(\iota_{j+1})}\right) = 2$. This implies the existence of an involution $\iota \in T'$ such that $F(\iota_{j+1}) \subseteq F(\iota) \subseteq M$ with all inclusion strict. It follows that $F(\iota_{j+1})$ is the transverse intersection of $F(\iota)$ and $F(\iota\iota_{j+1})$. Moreover, Lemma 3.6 implies $\dim F(\iota_{j+1}) > \frac{n}{2}$, which implies

$$2 \operatorname{cod} F(\iota) + 2 \operatorname{cod} F(\iota\iota_1) = 2 \operatorname{cod} F(\iota_{j+1}) < n.$$

By Corollary 2.12, $\mathrm{b}_{\mathrm{odd}}(N) = 0$. $\square$

Lemma 3.9 (Case 2)**.** *If* $\dim \ker \left(T'|_{F(\langle\sigma,\tau\rangle)}\right) \geq 3$ *for some distinct* $\sigma, \tau \in \Gamma$*, then* $\mathrm{b}_{\mathrm{odd}}(F(\tau)) = 0$*.*

Proof. Let $H = \langle\sigma, \tau\rangle$. Since $\dim \ker \left(T'|_{F(H)}\right) \geq 3$ and $\dim \ker \left(T'|_{F(\tau)}\right) \leq 1$, there exists a 2-torus that acts almost effectively on $F(\tau)$ and fixes $F(H)$. Restricting our attention to the action on $F(\tau)$, we may divide by the kernel of this action to conclude that a 2-torus acts effectively on $F(\tau)$ and fixes $F(H)$. This implies the existence of an involution ι such that $F(H) \subseteq F(\iota) \subseteq F(\tau)$ with all inclusions strict. Since $F(H)$ is the F-component of the fixed-point set of the σ-action on $F(\tau)$, it follows that $F(H)$ is the transverse intersection inside $F(\tau)$ of $F\left(\iota|_{F(\tau)}\right)$ and $F\left(\iota\sigma|_{F(\tau)}\right)$.

Lemma 3.6 implies $\dim F(\tau) \geq n/\sqrt{2} > 2n/3$ and similarly for $\dim F(\sigma)$. Hence

$$\mathrm{cod}_{F(\tau)} F(\iota|_{F(\tau)}) + \mathrm{cod}_{F(\tau)} F(\iota\sigma|_{F(\tau)}) = \mathrm{cod}_{F(\tau)} F(\sigma|_{F(\tau)}) \leq \mathrm{cod}\, F(\sigma) < \frac{1}{2} \dim F(\tau).$$

Corollary 2.12, together with the observation

$$\dim F(\tau) = n - \mathrm{cod}\, F(\tau) \equiv 0 \bmod 4,$$

therefore implies $\mathrm{b}_{\mathrm{odd}}(F(\tau)) = 0$. □

Lemma 3.10 (Case 3)**.** *If there exist distinct* $\sigma, \tau \in \Gamma$ *with no edge connecting them, then* $\mathrm{b}_{\mathrm{odd}}(F(\tau)) = 0$*.*

Proof. Let $H = \langle\sigma, \tau\rangle$. We may assume that $\dim \ker \left(T'|_{F(H)}\right) \leq 2$ by the proof of Case 2. By Lemma 3.6, therefore, $\dim F(H) > n/2$. The assumption that no edge exists between σ and τ means that $F(\sigma) \cap F(\tau)$ is transverse. Since

$$2 \operatorname{cod} F(\sigma) + 2 \operatorname{cod} F(\tau) = 2 \operatorname{cod} F(H) < n,$$

Corollary 2.12 implies $\mathrm{b}_{\mathrm{odd}}(F(\tau)) = 0$. □

Lemma 3.11 (Case 4)**.** *If there exist distinct $\sigma, \tau \in \Gamma$ such that $\sigma\tau \notin \Gamma$, then $\mathrm{b}_{\mathrm{odd}}(F(\tau)) = 0$ or $\mathrm{b}_{\mathrm{odd}}(N) = 0$.*

Proof. It follows from the isotropy representation that

$$\operatorname{cod} F(\sigma\tau) \equiv \operatorname{cod} F(\sigma) + \operatorname{cod} F(\tau)$$

modulo 4, so $\sigma\tau \notin \Gamma$ implies $\dim\ker \left(T'|_{F(\sigma\tau)}\right) \geq 2$. On the other hand, the fact that $F(H) \subseteq F(\sigma\tau)$ implies

$$\dim\ker \left(T'|_{F(\sigma\tau)}\right) \leq \dim\ker \left(T'|_{F(H)}\right),$$

and the proof in Case 2 implies that we may assume $\dim\ker T'|_{F(H)} \leq 2$, hence we have $\dim\ker \left(T'|_{F(\sigma\tau)}\right) = 2$.

This implies the existence of an involution $\rho \in T'$ satsifying $F(\sigma\tau) \subseteq F(\rho) \subseteq M$ with all inclusions strict, which in turn implies $F(\sigma\tau)$ is the transverse intersection in M of $F(\rho)$ and $F(\rho\sigma\tau)$. Additionally $\dim\ker \left(T'|_{F(\sigma\tau)}\right) = 2$ implies $\dim F(\sigma\tau) > n/2$ by Lemma 3.6. Hence

$$2 \operatorname{cod} F(\rho) + 2 \operatorname{cod} F(\rho\sigma\tau) = 2 \operatorname{cod} F(\sigma\tau) < n,$$

so the periodicity theorem implies $\mathrm{b}_{\mathrm{odd}}(N) = 0$. □

We pause before considering the last case. By the proof of Cases 3 and 4, we may assume that Γ is a complete graph and that the set of vertices in Γ is closed under multiplication. Adding the proof of Case 1, we may assume, in fact, that Γ is a complete graph on $\mathbb{Z}_2^m$ for some $m \geq \dim(T') - 1$. The last case considers this possibility.

We make a minor modification to our definition of $F(\rho)$. If $\rho \in T'$ and $H \subseteq T'$, then ρ acts on $F(H)$. Let $F(\rho|_{F(H)})$ denote the F-component of the fixed-point set of the ρ-action on $F(H)$. Observe that $F(\rho|_{F(H)}) = F(H')$ where H' is the subgroup generated by ρ and H.

Lemma 3.12 (Case 5)**.** *Suppose* Γ *is a complete graph on* $\mathbb{Z}_2^m$ *with* $m \geq \dim(T') - 1$. *There exists* $H \subseteq T'$ *such that* $\mathrm{b}_{\mathrm{odd}}(F(H)) = 0$.

Proof. Set $l = \left\lfloor \frac{m+1}{2} \right\rfloor$. Choose subgroups

$$\mathbb{Z}_2^m \supseteq \mathbb{Z}_2^{m-1} \supseteq \cdots \supseteq \mathbb{Z}_2^{m-(l-1)}$$

and

$$\rho_i \in \mathbb{Z}_2^{m-(i-1)} \setminus \langle \rho_1, \ldots, \rho_{i-1} \rangle$$

for $1 \leq i \leq l$ according to the following procedure:

- Choose $\rho_1 \in \mathbb{Z}_2^m$ such that $k_1 = \operatorname{cod} F(\rho_1)$ is maximal.

- Given $\mathbb{Z}_2^m \supseteq \cdots \supseteq \mathbb{Z}_2^{m-(i-1)}$ and $\rho_j \in \mathbb{Z}_2^{m-(j-1)}$ for $1 \leq j \leq i$, choose $\mathbb{Z}_2^{m-i} \subseteq \mathbb{Z}_2^{m-(i-1)}$ such that every $\rho \in \mathbb{Z}_2^{m-i}$ satisfies

$$\operatorname{cod}\left(F(\rho|_{R_i}) \hookrightarrow R_i\right) \equiv 0 \bmod 4$$

where $R_i = F(\langle \rho_1, \ldots, F(\rho_i)\rangle)$, and then choose $\rho_{i+1} \in \mathbb{Z}_2^{m-i}$ such that

$$\mathrm{cod}\left(F\left(\rho_{i+1}|_{R_i}\right) \hookrightarrow R_i\right) = k_{i+1}$$

is maximal.

We claim that our choices imply

1. $\dim(R_h) \equiv 0 \bmod 4$ for all h,
2. $k_h \geq 2k_{h+1}$ for all h, and
3. $k_l = 0$.

The first point follows by observing that $\dim(R_h) = n - (k_1 + \ldots + k_h)$ by definition and that $k_i \equiv 0 \bmod 4$ for all i by our choices.

To prove the second claim, fix $h \geq 1$. Observe that $\rho_h \in \mathbb{Z}_2^{m-(h-1)}$ and $\rho_{h+1} \in \mathbb{Z}_2^{m-h} \subseteq \mathbb{Z}_2^{m-(h-1)}$, so $\rho_h\rho_{h+1} \in \mathbb{Z}_2^{m-(h-1)}$ as well. By maximality then, we have

$$\begin{aligned} k_h &\geq \mathrm{cod}\left(F\left(\rho_{h+1}|R_{h-1}\right) \hookrightarrow R_{h-1}\right) = k_{h+1} + a, \text{ and} \\ k_h &\geq \mathrm{cod}\left(F\left(\rho_h\rho_{h+1}|R_{h-1}\right) \hookrightarrow R_{h-1}\right) = k_{h+1} + (k_h - a) \end{aligned}$$

where $a = \mathrm{cod}\left(R_{h+1} \hookrightarrow F\left(\rho_h\rho_{h+1}\right)|_{R_{h-1}}\right)$. Adding these inequalities shows that $k_h \geq 2k_{h+1}$.

Finally, the third claim follows from the second claim together with the estimate

$$l = \left\lfloor \frac{m+1}{2} \right\rfloor \geq \left\lfloor \frac{\dim(T')}{2} \right\rfloor > \log_2(n) - 1$$

and the fact that $k_1 < n/\sqrt{2}$ by Lemma 3.6.

We now use these facts to find a transverse intersection. Let $0 < j \leq l$ be the smallest index such that $k_j = 0$. For $1 \leq i \leq j-1$, let l_i be the number of (-1)s in the image of ρ_j in $SO(T_x R_{i-1} \cap \nu_x R_i)$. Geometrically, l_i is the codimension of

$$F(\rho_i|_{R_{i-1}}) \cap F(\rho_j|_{R_{i-1}}) \hookrightarrow F(\rho_i \rho_j|_{R_{i-1}}).$$

By replacing ρ_j by $\rho_{j-1}\rho_j$ if necessary, we can ensure that $l_{j-1} \leq \frac{k_{j-1}}{2}$. Observe that this may change l_i for $i < j-1$. Next, replace ρ_j by $\rho_{j-2}\rho_j$ if necessary to ensure that $l_{j-2} \leq \frac{k_{j-2}}{2}$. Observe again that the l_i may have changed for $i < j-2$, but that l_{j-1} does not. Continuing in this way, we may replace ρ_j by $\rho\rho_j$ for some $\rho \in \langle \rho_1, \ldots, \rho_{j-1} \rangle$ to ensure that $l_i \leq \frac{k_i}{2}$ for all $i < j$.

Now some of the $l_{j-1}, l_{j-2}, \ldots$ may be zero, but they cannot all be zero because the action of T' is effective and $\rho_j \notin \langle \rho_1, \ldots, \rho_{j-1} \rangle$. Let $1 \leq i \leq j-1$ denote the largest index where $l_i > 0$. Observe that $l_i \leq \frac{k_i}{2}$ implies $l_i > 0$ and $k_i - l_i > 0$.

Consider now the transverse intersection of $F(\rho_j|_{R_{i-1}})$ and $F(\rho_j \rho_i|_{R_{i-1}})$ inside R_{i-1}. The intersection is R_i (by choice of i), and the codimensions are l_i and $k_i - l_i$. We wish to apply Corollary 2.12 to this intersection to conclude $\mathrm{b}_{\mathrm{odd}}(R_{i-1}) = 0$.

First, observe that $k_1 \leq n - \frac{n}{\sqrt{2}} < n/2$ by Lemma 3.6. Also recall that $k_h \geq 2k_{h+1}$ for all h. Hence

$$2l_i + 2(k_i - l_i) = 2k_i \leq \frac{n}{2^{i-1}} = n - \sum_{h=1}^{i-1} \frac{n}{2^h} \leq n - \sum_{h=1}^{i-1} k_h = \dim(R_{i-1}).$$

Moreover $\dim R_{i-1} \equiv 0 \bmod 4$, so Corollary 2.12 implies $\mathrm{b}_{\mathrm{odd}}(R_{i-1}) = 0$. Since

$R_{i-1} = F(H)$ where $H = \langle \rho_1, \ldots, \rho_{i-1} \rangle$, this concludes the proof in this case. $\square$

We have shown in all five cases the existence of a submanifold $F \subseteq P \subseteq N$ on which T' acts such that $\mathrm{b}_{\mathrm{odd}}(P) = 0$. As explained at the beginning of the proof, Conner's theorem then implies $\mathrm{b}_{\mathrm{odd}}(F) = 0$, as required.

Chapter 4

Griesmer's bound from the theory of error-correcting codes

In this chapter, we consider a Riemannian manifold M^n and an isometric T^r-action on M fixing a point $x \in M$. We assume r is bounded below by a logarithmic function of n. We use the Griesmer bound from the theory of error-correcting codes to prove an estimate on the codimensions of fixed-point sets of involutions in T^r. This estimate will be used in the next section.

Proposition 4.1. *Let $n \geq 4$, and assume T^r acts effectively by isometries on N^n with fixed point x. Let c be a nonnegative integer.*

1. *If $r > \log_2 n + \frac{c}{2} + 1.5$, then there exists an involution $\sigma \in T^r$ satisfying $\operatorname{cod} F(\sigma) \equiv 0 \bmod 4$ and $\operatorname{cod} F(\sigma) \leq \frac{n-c-1}{2}$.*

2. Suppose $\sigma \in T^r$ *has been chosen as in the previous part such that* $\operatorname{cod} F(\sigma)$ *is minimal. If* $r > \log_2 n + \frac{c}{2} + 2.5$*, then there exists an involution* $\tau \in T^r$ *satisfying* $F(\tau) \not\subseteq F(\sigma)$*,* $\operatorname{cod} F(\tau) \equiv 0 \bmod 4$*,* $\operatorname{cod} F(\sigma) \cap F(\tau) \equiv 0 \bmod 4$*, and* $\operatorname{cod} F(\tau) \leq \frac{n-c-1}{2}$.

Note that, by the connectedness theorem, a totally geodesic inclusion $N^{n-k} \hookrightarrow M^n$ with $k \leq \frac{n-c-1}{2}$ is $(c+2)$-connected. In particular, $\pi_1(N) \cong \pi_1(M)$, so N is simply connected if M is.

We proceed to the proof of Proposition 4.1. Choose a basis of T_xM such that the image of $\mathbb{Z}_2^r \subseteq T^r$ under the isotropy representation $T^r \hookrightarrow SO(T_xM)$ lies in a copy of $\mathbb{Z}_2^m \subseteq T^m \subseteq SO(T_xM)$ where $m = \lfloor \frac{n}{2} \rfloor$. Endow $\mathbb{Z}_2^m$ with a $\mathbb{Z}_2$-vector space structure, and consider now the linear embedding $\iota : \mathbb{Z}_2^r \hookrightarrow \mathbb{Z}_2^m$.

Consider the first statement. The bound on r implies $r \geq 2$. Consider the composition of $\mathbb{Z}_2^r \to \mathbb{Z}_2^m \to \mathbb{Z}_2$, where the last map takes $\tau = (\tau_1, \ldots, \tau_m)$ to $\sum \tau_i \bmod 2$. Clearly there exists $\mathbb{Z}_2^{r-1}$ inside the kernel, and for each $\sigma \in \mathbb{Z}_2^{r-1}$, $\operatorname{cod} F(\sigma)$ is twice the even weight of $\iota(\sigma) = \tau$. Hence it suffices to prove the existence of $\sigma \in \mathbb{Z}_2^{r-1}$ with $\operatorname{cod} F(\sigma) \leq \frac{n-c-1}{2}$.

Equivalently, it suffices to prove that the weight of the $\sigma \in \mathbb{Z}_2^{r-1}$ whose image in $\mathbb{Z}_2^m$ has minimal (positive) Hamming weight is at most $\frac{n-c-1}{4}$. If this is not the case, then the image of that σ has weight at least $\frac{n-c}{4}$. We now apply the Griesmer bound from the theory of error-correcting codes:

Theorem 4.2. *(Griesmer bound) If* $\mathbb{Z}_2^u \hookrightarrow \mathbb{Z}_2^m$ *is an injective linear map such that* *every element in the image has weight at least w, then*

$$m \geq \sum_{i=0}^{u-1} \left\lceil \frac{w}{2^i} \right\rceil .$$

This bound implies

$$\frac{n}{2} \geq \left\lfloor \frac{n}{2} \right\rfloor \geq \sum_{i=0}^{r-2} \left\lceil \frac{n-c}{2^{i+2}} \right\rceil \geq \sum_{i=0}^{r-3-\lceil \frac{c}{2} \rceil} \frac{n-c}{2^{i+2}} + \sum_{i=r-2-\lceil \frac{c}{2} \rceil}^{r-2} 1.$$

Observe that the lower bound on r implies $r-3-\lceil \frac{c}{2} \rceil \geq 0$, so the second inequality is justified. Computing the geometric sum, canceling like terms, and rearranging yields

$$2^{r-\lceil \frac{c}{2} \rceil -1} \leq n-c \leq n,$$

a contradiction to the assumed lower bound on r.

We now prove the second statement of Proposition 4.1. First, observe that the lower bound on r implies $r \geq 4$. Let $\sigma \in \mathbb{Z}_2^r$ be as in the statement. We define three linear maps $\mathbb{Z}_2^r \to \mathbb{Z}_2$. For the first, fix a component i such that the i-th component of $\iota(\sigma)$ is 1 (which corresponds to a normal direction of $F(\sigma)$), and define the map $\mathbb{Z}_2^r \to \mathbb{Z}_2$ as the projection onto the i-th component of $\iota(\tau)$. For the second map, assign $\tau \in \mathbb{Z}_2^r$ to the sum of the components of $\iota(\tau)$ (as we did previously to choose σ). And for the third map, let I be the subset of indices i where the i-th component of $\iota(\sigma)$ is 0, and define the map by assigning $\tau \in \mathbb{Z}_2^r$ to the sum over $i \in I$ of the i-th componets of $\iota(\tau)$. The intersection of the kernels of these three maps contains a $\mathbb{Z}_2^{r-3}$, and every $\tau \in \mathbb{Z}_2^{r-3}$ satisfies $F(\tau) \not\subseteq F(\sigma)$, $\operatorname{cod} F(\tau) \equiv_4 0$, and $\operatorname{cod} F(\tau) \cap F(\sigma) \equiv_4 0$. Moreover, the image of every $\tau \in \mathbb{Z}_2^{r-3}$ has a 0 in the

i-component of $\mathbb{Z}_2^m$, hence we have an (injective) linear code $\mathbb{Z}_2^{r-3} \to \mathbb{Z}_2^{m-1}$ where twice the weight of the image of $\tau \in \mathbb{Z}_2^{r-3}$ is equal to $\operatorname{cod} F(\tau)$.

It therefore suffices to prove that the element in $\mathbb{Z}_2^{r-3}$ whose image has minimal weight has weight at most $\frac{n-c-1}{2}$. The Griesmer bound implies

$$\frac{n}{2} - 1 \geq \sum_{i=0}^{r-4} \left\lceil \frac{n-c}{2^{i+2}} \right\rceil .$$

Splitting the sum on the right-hand side into the first $r - 4 - \left\lceil \frac{c}{2} \right\rceil$ terms plus the last $\left\lceil \frac{c}{2} \right\rceil + 1$ terms and estimating as in the previous case yields a contradiction to the lower bound on r.

Chapter 5

Theorem P

In this chapter, we state and prove Theorem P. The "P" is for periodicity. The assumptions involve a positively curved manifold with symmetry rank depending on a constant c, and the conclusion is that the first c rational cohomology groups of M are 4-periodic. The following chapter will show that Theorem P easily implies Theorem B as well.

We proceed to the statement of Theorem P. For an integer m, let $\delta(m)$ be 0 if m is even and 1 if m is odd.

Theorem P. *Let $m \geq c \geq 0$. Assume M^m is a closed, simply connected, positively curved manifold with an effective, isometric T^s-action satisfying $s \geq 2\log_2 m + \frac{c}{2} - \delta(m)$. If F is a component of the fixed-point set of T^s, then there exists $H \subseteq T^s$ with F-component P such that $P \hookrightarrow M$ is c-connected and $H^*(P;\mathbb{Q})$ is 4-periodic. Moreover, P may be chosen to satisfy $\dim P \equiv m \bmod 4$ and $\dim P \geq c$.*

Using Theorem 3.1, an immediate corollary is the following:

Corollary 5.1. *Let $m \geq c \geq 0$, and let M^m be a closed, simply connected, positively curved manifold with an effective, isometric T^s-action. If $s \geq 2\log_2 m + \frac{c}{2}$, then there exists a c-connected inclusion $P \hookrightarrow M$ such that $H^*(P;\mathbb{Q})$ is 4-periodic.*

We now set up the proof of Theorem P. To simplify various statements in the proof, we make the following definitions. Suppose M^m is a closed, simply connected, positively curved manifold, and suppose T is a torus acting almost effectively by isometries on M with fixed-point component F. Finally, define δ to be 0 if m is even and 1 if m is odd, and suppose that c is a nonnegative integer.

For an element or subgroup H of T, let $F(H)$ denote the F-component of M^H. Since T is abelian and connected, T acts on $F(H)$. In general, if T acts almost effectively on S and restricts to an action on $N \subseteq S$, let $\mathrm{dk}_S(N)$ denote the dimension of the kernel of the action on N. Recall that we called this quantity $\dim\ker(T|_N)$ in Chapter 3. Equivalently, if T' is a maximal subtorus of T acting almost effectively on N, then

$$\mathrm{dk}_S(N) = \dim T - \dim T'.$$

For example, if T itself acts almost effectively on N, then $\mathrm{dk}_S(N) = 0$, and if (at least) a codimension one subtorus acts almost effectively, then $\mathrm{dk}_S(N) \leq 1$.

Next, we will write $\mathrm{con}_M(N) \geq c$ when $N \hookrightarrow M$ is at least c-connected. Define the following collection of totally geodesic, simply connected submanifolds of

M whose dimensions are at least c and congruent to m modulo four and whose inclusions into M are c-connected:

$$\mathcal{C} = \{F(H) \mid \operatorname{cod}_M F(H) \equiv_4 0,\ \dim F(H) \geq c,\ \text{and}\ \operatorname{con}_M F(H) \geq c+2\}.$$

We now define three properties that M might satisfy. The first with $S = M$ is the assumption of Theorem P, the last is the conclusion, and the intermediate property is one which occurs in the course of the proof that $\mathcal{S}$ implies $\mathcal{P}$. With the notation above, we define the following properties:

Property $\mathcal{S}$: There exists $S \in \mathcal{C}$ and $T' \subseteq T$ acting almost effectively on S with $\dim(T') \geq 2\log_2(\dim S) + \frac{c}{2} - \delta$.

Property $\mathcal{I}$: Property $\mathcal{S}$ holds, and there exist $N \in \mathcal{C}$ and an involution $\sigma \in T'$ with $\operatorname{cod}_N F(\sigma|_N) \equiv_4 0$ and $0 < \operatorname{cod}_N F(\sigma|_N) < \dim(S)/(3 \cdot 2^{\mathrm{dk}_S(N)})$.

Property $\mathcal{P}$: There exists $P \in \mathcal{C}$ such that $H^*(P;\mathbb{Q})$ is 4-periodic.

Theorem P can now be stated as "$\mathcal{S}$ with $\dim(S) = m$ implies $\mathcal{P}$." The basic strategy is to use double induction as follows. Given that $\mathcal{S}$ holds for some S, we consider the situation with $\dim(S)$ minimal. The first step is to show that this implies $\mathcal{I}$ for some $\dim(N) \leq \dim(S)$. Now take N with minimal dimension such that $\mathcal{I}$ holds. We then show that $\mathcal{P}$ holds by using our choice of S and N.

With the setup complete, we proceed to the proof of Theorem P. Roughly speaking, the first step is to show that $\mathcal{S}$ implies $\mathcal{I}$:

Lemma 5.2. *If $S \in \mathcal{C}$ has minimal dimension such that Property $\mathcal{S}$ holds, then Property $\mathcal{P}$ holds or $\mathcal{I}$ holds for some N with* $\dim(N) \leq \dim(S)$.

Proof. First, by dividing T' by the kernel of its action on S, we may assume that T' acts effectively. If $\dim(S) \leq 5$, then S is trivially 4-periodic. Hence $\mathcal{P}$ holds with $P = S$.

Suppose therefore that $\dim(S) \geq 6$. Using Property $\mathcal{S}$, we conclude

$$\begin{aligned}\dim(T') &\geq 2\log_2(\dim S) + \frac{c}{2} - \delta \\ &\geq \log_2(\dim S) + \frac{c}{2} + \log_2(6) - 1 \\ &> \log_2(\dim S) + \frac{c}{2} + 1.5.\end{aligned}$$

By Proposition 4.1, there exists $\sigma \in T'$ with $\mathrm{cod}_S\, F(\sigma|_S) \leq \frac{1}{2}(\dim S - c - 1)$ and $\mathrm{cod}_S\, F(\sigma|_S) \equiv 0 \bmod 4$. Choose the σ satisfying these properties which has the minimal codimension in S. Since $\mathrm{cod}\, S \equiv 0 \bmod 4$, we have $\mathrm{cod}\, F(\sigma|_S) \equiv 0 \bmod 4$. In addition, the connectedness theorem and our assumption on S implies $F(\sigma|_S) \hookrightarrow S \hookrightarrow M$ is $(c+2)$-connected. Finally, $\mathrm{cod}_S\, F(\sigma|_S) \leq (\dim S - c - 1)/2$ and $\dim S \geq c$ imply $\dim F(\sigma|_S) \geq (\dim S + c + 1)/2 \geq c$. Hence $F(\sigma|_S) \in \mathcal{C}$.

We claim that $\mathrm{dk}_S\, F(\sigma|_S) \leq 2$. If not, then N is a fixed by a 3-torus acting effectively on S. Since one of the involutions, say σ', inside the subgroup of involutions has $\mathrm{cod}\, F(\sigma'|_S) \equiv 0 \bmod 4$ and $\sigma' \notin \langle \sigma \rangle$, we obtain a contradiction to the minimality of $\mathrm{cod}\, F(\sigma|_S)$. Hence we have $\mathrm{dk}_S\, F(\sigma|_S) \leq 2$.

We next claim that $\mathrm{dk}_S\, F(\sigma|_S) = 2$ implies $\mathcal{P}$. Indeed, suppose for a moment that a 2-torus acting effectively on S fixes $F(\sigma|_S)$. Then there exists an involution σ' with $F(\sigma|_S) \subseteq F(\sigma'|_S) \subseteq S$ with all inclusions strict. It follows that $F(\sigma|_S)$ is the transverse intersection of $F(\sigma'|_S)$ and $F(\sigma\sigma'|_S)$. If $\dim F(\sigma|_S) \leq \frac{1}{2}\dim(S)$, then $\mathcal{S}$ holds for $F(\sigma|_S)$ in place of S, a contradiction to the minimality of $\dim(S)$. On the other hand, $\dim F(\sigma|_S) \geq \frac{1}{2}\dim(S)$ implies

$$2\,\mathrm{cod}_S\, F(\sigma'|_S) + 2\,\mathrm{cod}_S\, F(\sigma\sigma'|_S) \leq \dim(S),$$

which implies $\mathcal{P}$ holds with $P = S$ by the periodicity theorem.

This leaves us with the case $\mathrm{dk}_S\, F(\sigma|_S) \leq 1$. Let T'' denote a codimension 1 subtorus of T' that acts almost effectively on $F(\sigma|_S)$. By minimality of $\dim(S)$, we have $\dim(T'') < 2\log_2\left(\dim F(\sigma|_S)\right) + \frac{c}{2} - \delta$. Since $\dim(T'') \geq \dim(T') - 1$ and $\dim(T') \geq 2\log_2(\dim S) + \frac{c}{2} - \delta$, this implies $2\log_2(\dim F(\sigma|_S)) > 2\log_2(\dim S) - 1$, or

$$\mathrm{cod}\, F(\sigma|_S) = \dim S - \dim F(\sigma|_S) < \left(1 - \frac{1}{\sqrt{2}}\right)\dim S < \frac{1}{3}\dim S.$$

Taking $N = S$ and $i = 0$, we see that $\mathcal{I}$ holds with $\dim(N) \leq \dim(S)$. $\square$

We now show the second part of the proof, which is roughly that $\mathcal{I}$ implies $\mathcal{P}$. Table 5.1 displays a summary of the notation used in the proof of Lemma 5.3. All submanifolds shown are in $\mathcal{C}$, that is, each is the F-component of the fixed-point set of some subgroup of T, and each has dimension both divisible by four and at least c, and the inclusion of each in M is c-connected. The codimension k of $F(\sigma) \hookrightarrow N$ satisfies $0 < k < \frac{\dim S}{3\cdot 2^j}$.

$$
\begin{array}{ccccc}
 & & M \curvearrowleft & T & \text{where } \dim(T) \geq 2\log_2(\dim M) + \tfrac{c}{2} - \delta \\
 & & | & & \\
 & & S \curvearrowleft & T' & \text{where } \dim(T') \geq 2\log_2(\dim S) + \tfrac{c}{2} - \delta \\
 & & | & & \\
 & & N \curvearrowleft & T'' & \text{where } \dim(T'') \geq \dim(T') - j \\
 & \diagup & & \diagdown & \\
F(\sigma) & & & F(\tau) & \\
 & \diagdown & & \diagup & \\
 & & F(\langle \sigma, \tau \rangle) & &
\end{array}
$$

Table 5.1: Summary of notation in the proof of Lemma 5.3.

Lemma 5.3. *Assume* $\dim(S)$ *is minimal such that* $\mathcal{S}$ *holds and that* $\dim(N)$ *is minimal such that* $\mathcal{I}$ *holds. Then* $\mathcal{P}$ *holds.*

As we already established, Lemmas 5.2 and 5.3 imply Theorem P. We spend the rest of the chapter proving Lemma 5.3.

Let T'' denote a subtorus of dimension $\dim(T'') = \dim(T') - \mathrm{dk}_S(N)$ acting almost effectively on N. After dividing T'' by the kernel of this action if necessary, we assume that T'' acts effectively. Observe that the image of σ (which we also denote by σ) in the quotient still acts effectively. Without loss of generality, we may assume $F(\sigma|_N)$ has minimal codimension among those with codimension divisible by four.

The assumption in $\mathcal{S}$ that $\dim(T') \geq 2\log_2(\dim S) + \frac{c}{2} - \delta$ and the assumption in $\mathcal{I}$ that $4 \leq \operatorname{cod}_N F(\sigma|_N) < \dim(S)/(2^{\mathrm{dk}_S(N)+1.5})$ imply

$$\dim(T'') = \dim(T') - \mathrm{dk}_S(N) > \log_2(\dim N) + \frac{c}{2} + 2.5.$$

Hence Proposition 4.1 implies the existence of $\tau \in T'$ with $F(\tau|_N) \not\subseteq F(\sigma|_N)$, $\operatorname{cod}_N F(\tau|_N) \equiv_4 0$, $\operatorname{cod}_N(F(\sigma|_N) \cap F(\tau|_N)) \equiv_4 0$, and

$$0 < \operatorname{cod}_N F(\tau|_N) \leq \frac{1}{2}(\dim(N) - c - 1).$$

Choose such a τ with minimal $\operatorname{cod}_N F(\tau|_N)$.

Claim. *$F(\tau|_N)$, $F(\sigma\tau|_N)$, and $F(\sigma|_N) \cap F(\tau|_N)$ are in $\mathcal{C}$.*

Proof. First, our choice of τ implies all three codimensions are congruent to zero modulo four. Second, all three dimensions are at least c since $F(\sigma|_N) \cap F(\tau|_N)$ has dimension at least

$$\dim(N) - \operatorname{cod}_N F(\sigma|_N) - \operatorname{cod}_N F(\sigma|_N) \geq \dim N - 2\left(\frac{\dim N - c - 1}{2}\right) \geq c.$$

Third, this same estimate shows by the connectedness theorem that

$$F(\sigma|_N) \cap F(\tau|_N) \hookrightarrow F(\tau|_N) \hookrightarrow N$$

is c-connected. Since N is c-connected in M, the inclusions of $F(\sigma|_N) \cap F(\tau|_N)$ and $F(\tau|_N)$ into M are c-connected. Finally, the connectedness theorem implies

$F\,(\sigma|_N) \cap F(\tau|_N) \hookrightarrow F(\sigma\tau|_N)$ is t-connected with

$$\begin{aligned} t &\geq \dim F\,(\sigma\tau|_N) - 2\,\mathrm{cod}_{F(\sigma\tau|_N)}(F\,(\sigma|_N) \cap F(\tau|_N)) + 1 \\ &= \dim N - \mathrm{cod}_N\, F(\sigma|_N) - \mathrm{cod}_N\, F(\tau) + 1 \\ &\geq c + 1. \end{aligned}$$

Hence $F(\sigma\tau|_N) \hookrightarrow M$ is c-connected by seeing the map on homotopy as that which goes via $F(\sigma|_N) \cap F(\tau|_N)$, $F(\tau|_N)$, N, and finally M. □

As in the previous lemma, we use the the minimality of $\mathrm{cod}_N\, F(\tau|_N)$ to prove an upper bound for $\mathrm{dk}_N\, F(\tau|_N)$:

Claim. $\mathrm{dk}_N\, F(\tau|_N) \leq 3$.

Proof. Indeed, suppose a 4-torus T^4 acting effectively on N fixes $F(\tau|_N)$. Choose a $\mathbb{Z}_2^2 \subseteq T^4$ with the property that every $\tau' \in \mathbb{Z}_2^2$ satisfies $\mathrm{cod}_N\, F(\tau'|_N) \equiv_4 0$ and $\mathrm{cod}_N\,(F(\sigma|_N) \cap F(\tau'|_N)) \equiv_4 0$, then choose $\tau' \in \mathbb{Z}_2^2 \setminus \langle\tau\rangle$.

It follows that $F(\tau'|_N)$ lies strictly between $F(\tau|_N)$ and N. Since $F(\tau|_N) \not\subseteq F(\sigma|_N)$, we have $F(\tau'|_N) \not\subseteq F(\sigma|_N)$ as well. This contradicts to the minimality of $\mathrm{cod}_N\, F(\tau|_N)$. □

The remainder of the proof now follows a sequence of claims. In each, we claim that we may assume something. The idea in each case is to use the periodicity theorem and the minimality of $\dim(S)$ to show that $\mathcal{P}$ holds if the claim does not. At the end of the sequence of claims, we conclude the proof by showing that, if

$\mathcal{P}$ does not hold, then the combination of assumptions from the claims implies a contradiction to the minimality of $\dim(N)$.

The first claim improves upon the previous one:

Claim. *We may assume* $\mathrm{dk}_N F(\tau|_N) \leq 2$.

Proof. Indeed, suppose $\mathrm{dk}_N F(\tau|_N) = 3$. We show that this implies $\mathcal{P}$.

There exists $\tau' \in T''$ such that $F(\tau'|_N) \not\subseteq F(\sigma|_N)$, $\mathrm{cod}_N F(\tau|_N) \equiv_4 0$, and $F(\tau|_N) \subseteq F(\tau'|_N) \subseteq N$ with all inclusions strict. Choose such a τ' with minimal $\mathrm{cod}_N F(\tau'|_N)$. Choose a basis of the isotropy representation $T' \hookrightarrow SO(T_pN)$ so that the actions of σ, τ, and τ' are given by

$$\begin{aligned} \sigma &= \mathrm{diag}(-I, -I, -I, \quad I, \quad I, \quad I), \\ \tau &= \mathrm{diag}(-I, -I, \quad I, -I, -I, \quad I), \\ \tau' &= \mathrm{diag}(-I, \quad I, \quad I, -I, \quad I, \quad I), \end{aligned}$$

where the blocks are of size y, $x-y$, $k-x$, z, $l-x-z$, and $\dim(F(\sigma|_N) \cap F(\tau|_N))$, where $k = \mathrm{cod}_N F(\sigma|_N)$ and $l = \mathrm{cod}_N F(\tau|_N)$. Note that we can replace τ' by $\tau\tau'$ if necessary to ensure that $2y \leq x$. In addition, note that maximality of $F(\tau|_N)$ implies $2x \leq k$, $y > 0$, and $z > 0$.

Observe that $F\left(\tau'|_{F(\sigma\tau|_N)}\right)$ and $F\left(\tau\tau'|_{F(\sigma\tau|_N)}\right)$ intersect transversely in $F\left(\sigma\tau|_N\right)$ and have (positive) codimensions x and $x-y$, respectively. If

$$2y + 2(x-y) \leq \dim F(\sigma\tau|_N),$$

then the periodicity theorem implies $\mathcal{P}$ with $P = F(\sigma\tau|_N)$. We claim the other case leads to a contradiction. Indeed, if

$$2x > \dim F(\sigma\tau|_N) = \dim F(\tau|_N) - k + 2x,$$

then the assumption in $\mathcal{I}$ implies

$$\dim F(\tau|_N) < \frac{\dim S}{2^{\operatorname{dk}_S N + 1.5}} \leq \frac{\dim S}{2^{(\operatorname{dk}_S N+3)/2}} = \frac{\dim S}{2^{(\operatorname{dk}_S F(\tau|_N))/2}}.$$

Using the assumption in $\mathcal{S}$ on the size of $\dim(T')$, we conclude from this that $\mathcal{S}$ also holds for $F(\tau|_N)$, a contradiction to the minimality of $\dim S$. □

We will soon show that, in fact, we may assume $\operatorname{dk}_N F(\tau|_N) \leq 1$, but we must first show the following:

Claim. *We may assume* $\dim F(\tau|_N) \geq 2k$.

Proof. We assume $\dim F(\tau|_N) < 2k$ and show that $\mathcal{P}$ holds. Suppose first that $\operatorname{dk}_S N \geq 1$ or $\operatorname{dk}_N F(\tau|_N) \leq 1$. This together with the previous claim implies

$$\operatorname{dk}_N F(\tau|_N) \leq \operatorname{dk}_S N + 1.$$

Combining this with the equality

$$\operatorname{dk}_S F(\tau|_N) = \operatorname{dk}_S N + \operatorname{dk}_N F(\tau|_N),$$

we have

$$\operatorname{dk}_S F(\tau|_N) \leq 2 \operatorname{dk}_S N + 1.$$

Using this together with the assumption in $\mathcal{I}$, we have

$$\dim F(\tau|_N) \leq 2k \leq \frac{\dim S}{2^{\mathrm{dk}_S N + 0.5}} \leq \frac{\dim S}{2^{(\mathrm{dk}_S F(\tau|_N))/2}},$$

which by the assumption in $\mathcal{S}$ implies that $\mathcal{S}$ holds with S replaced by $F(\tau|_N)$, a contradiction to the minimality of $\dim S$. Assume therefore that $\mathrm{dk}_S N = 0$ and $\mathrm{dk}_N F(\tau|_N) = 2$.

By minimality of $\dim S$, we have $N = S$, which implies

$$\mathrm{dk}_S F(\tau|_N) = \mathrm{dk}_N F(\tau|_N) = 2.$$

Using minimality of $\dim S$ again, it follows that $\dim F(\tau|_N) > \frac{1}{2}\dim(N)$. Now $\mathrm{dk}_N F(\tau|_N) = 2$ implies that an involution τ' exists so that $F(\tau|_N) \subseteq F(\tau'|_N) \subseteq N$ with all inclusions strict. Since $\dim F(\tau|_N) > \frac{1}{2}\dim N$, it follows that

$$2\,\mathrm{cod}_N F(\tau'|_N) + 2\,\mathrm{cod}_N F(\tau\tau'|_N) = 2\,\mathrm{cod}_N F(\tau|_N) \leq n,$$

so we conclude from the periodicity theorem that $\mathcal{P}$ holds. □

As mentioned before the previous claim, our goal is to show the following:

Claim. *We may assume* $\mathrm{dk}_N F(\tau|_N) \leq 1$.

Proof. As in the previous proofs, our goal is to show that $\mathcal{P}$ holds if the statement in the claim does not. Suppose therefore that $\mathrm{dk}_N F(\tau|_N) = 2$.

We can choose $\tau' \in T''$ satisfying $F(\sigma|_N) \not\subseteq F(\tau'|_N)$ and $F(\tau|_N) \subseteq F(\tau'|_N) \subseteq N$ with all inclusions strict. We reuse the notation for the images of σ, τ, and τ'

under the isotropy representation as above. Observe that, by replacing τ' by $\tau\tau'$ if necessary, we may assume $y \leq x - y$.

Using the assumption that $\dim F(\tau|_N) \geq 2k$, we apply the periodicity theorem in each of three cases to conclude $\mathcal{P}$. The relevant transverse intersections in the three cases are the following:

- If $y > 0$, we consider the intersection of $F(\tau'|_{F(\sigma\tau|_N)})$ and $F(\tau\tau'|_{F(\sigma\tau)})$ inside $F(\sigma\tau|_N)$, and we conclude $\mathcal{P}$ with $P = F(\sigma\tau|_N)$.

- If $y = 0$ and $x > 0$, we consider the intersection of $F(\sigma|_{F(\sigma\tau\tau'|_N)})$ and $F(\tau'|_{F(\sigma\tau\tau'|_N)})$ inside $F(\sigma\tau\tau'|_N)$, and we conclude $\mathcal{P}$ for $P = F(\sigma\tau|_N) = F(\tau'|_{F(\sigma\tau\tau'|_N)})$.

- If $y = 0$ and $x = 0$, we first replace τ' by $\tau\tau'$ if necessary to suppose that $z \leq l - z$. We then consider the intersection of $F(\sigma|_N)$ and $F(\tau'|_N)$ inside N, and we conclude $\mathcal{P}$ for $P = N$.

□

To summarize, we now have $\mathrm{dk}_N\, F(\tau|_N) \leq 1$ and $\dim F(\tau|_N) \geq 2k$. We must prove three more claims in the same manner before concluding the theorem.

Claim. *We may assume that* $\mathrm{dk}_N\,(F(\sigma|_N) \cap F(\tau|_N)) \leq 2$.

Proof. If this is not the case, then $\mathrm{dk}_{F(\tau|_N)}\,(F(\sigma|_N) \cap F(\tau|_N)) \geq 2$. Hence there exists an involution $\sigma' \in T'$ such that $F(\sigma'|_{F(\tau|_N)})$ lies strictly between $F(\sigma|_N) \cap$

$F(\tau|_N) = F(\sigma|_{F(\tau|_N)})$ and $F(\tau|_N)$. Because $F(\sigma'|_{F(\tau|_N)})$ and $F(\sigma\sigma'|_{F(\tau|_N})$ intersect transversely in $F(\tau|_N)$ with

$$2\,\mathrm{cod}_{F(\tau|_N)} F(\sigma'|_{F(\tau|_N)}) + 2\,\mathrm{cod}_{F(\tau|_N)} F(\sigma\sigma'|_{F(\tau|_N)}) \leq 2k \leq \dim F(\tau|_N),$$

the periodicity theorem implies $\mathcal{P}$ with $P = F(\tau|_N)$. □

Claim. *We may assume that* $F\,(\sigma|_N) \cap F(\tau|_N)$ *is not transverse.*

Proof. Suppose instead that $F(\sigma|_N) \cap F(\tau|_N)$ is transverse. If $\mathrm{dk}_S(N) = 0$, then the minimality of $\dim(S)$ implies $S = N$ and $\dim F(\sigma|_N) \cap F(\tau|_N) \geq \frac{1}{2}\dim(N)$. Hence

$$2\,\mathrm{cod}\, F(\sigma|_N) + 2\,\mathrm{cod}\, F(\tau|_N) = 2\,\mathrm{cod}_N\, F(\sigma|_N) \cap F(\tau|_N) \leq \dim(N),$$

which by the periodicity theorem implies $\mathcal{P}$ with, for example, $P = F(\tau|_N)$. And if $\mathrm{dk}_S(N) \geq 1$, then minimality of $\dim(S)$ implies

$$\dim(F(\sigma|_N) \cap F(\tau|_N)) \geq \frac{\dim S}{2^{0.5\,\mathrm{dk}_S(F(\sigma|_N)\cap F(\tau|_N))}} \geq \frac{\dim S}{2^{0.5(\mathrm{dk}_S N+2)}} \geq \frac{\dim S}{2^{\mathrm{dk}_S N+0.5}} \geq 2k,$$

where the last inequality follows from the assumptions in $\mathcal{I}$. Hence

$$3k + l \leq \dim\left(F(\sigma|_N) \cap F(\tau|_N)\right) + k + l \leq \dim(N),$$

so the periodicity theorem again implies $\mathcal{P}$ with $P = F(\tau|_N)$. □

The final claim is the following:

Claim. *We may assume that* $\mathrm{dk}_N\, F(\sigma\tau|_N) \leq 1$.

Proof. Suppose instead that $\mathrm{dk}_N F(\sigma\tau|_N) \geq 2$. Choose and involution $\iota \in T'$ such that $F(\iota|_N)$ lies strictly between $F(\sigma\tau|_N)$ and N. Choose a basis for the isotropy representation $T' \to SO(T_x N)$ so that σ, τ, ι are represented as follows:

$$\begin{aligned}
\sigma &= \mathrm{diag}(-I, -I, -I, \quad I, \quad I, \quad I),\\
\tau &= \mathrm{diag}(-I, \quad I, \quad I, -I, -I, \quad I),\\
\iota &= \mathrm{diag}(\quad I, -I, \quad I, -I, \quad I, \quad I),
\end{aligned}$$

where the first five blocks have sizes x, a, $k-x-a$, b, $l-x-b$, and $\dim(F(\sigma|_N) \cap F(\tau|_N))$, where again $k = \mathrm{cod}_N F(\sigma|_N)$ and $l = \mathrm{cod}_N F(\tau|_N)$. Moreover, we may replace ι by $\sigma\tau\iota$ if necessary to ensure that $a \leq l - x - a$. We prove Property $\mathcal{P}$ in each of two cases:

1. Suppose $a > 0$. Then $F(\iota|_{F(\tau|_N)})$ and $F(\sigma\iota|_{F(\tau|_N)})$ intersect transversely in $F(\tau|_N)$ with codimensions a and $k-x-a$, respectively. Since $\dim F(\tau|_N) \geq 2k$ by our claim above,

$$2a + 2(k - x - a) \leq 2k \leq \dim F(\tau|N).$$

   Hence the periodicity theorem implies $\mathcal{P}$ with $P = F(\tau|_N)$.

2. Suppose $a = 0$. Then $F(\sigma|_{F(\sigma\tau\iota|_N)})$ and $F(\iota|_{F(\sigma\tau\iota|_N)})$ intersect transversely in $F(\sigma\tau\iota|_N)$ with codimensions x and b, respectively. Observe that the minimality of $\dim S$, the assumption on k in $\mathcal{I}$, and the assumption $\mathrm{dk}_N F(\sigma\tau|_N) \leq 2$

imply that $\dim F(\sigma\tau|_N) \geq \frac{3}{2}k$. Hence

$$3x + b \leq \frac{3}{2}k + b \leq \dim F(\sigma\tau|_N) + b = \dim F(\sigma\tau\iota|_N)$$

and

$$2x + 2b \leq 3x + b \leq \dim F(\sigma\tau\iota|_N)$$

in the case that $b \leq x$. By the periodicity theorem, therefore, $\mathcal{P}$ holds with $P = F(\iota|_{F(\sigma\tau\iota|_N)}) = F(\sigma\tau|_N)$.

□

We now pull together what we have proven. We claim that Property $\mathcal{I}$ holds with N replaced by $F(\sigma\tau|_N)$. This will contradict the assumption that $\dim(N)$ is minimal, and hence prove the lemma.

First, set $S' = S$, $N' = F(\sigma\tau|_N)$, and $\sigma' = \sigma|_{N'}$. Clearly $\mathcal{S}$ holds with S' since it holds for S. In addition, we have already shown that $N' \in \mathcal{C}$. Next, our choice of τ ensured that $\operatorname{cod}_{N'} F(\sigma'|_{N'})$ is positive and divisible by four. We just have to show that $\operatorname{cod}_{N'} F(\sigma'|_{N'}) < \dim(S')/(3 \cdot 2^{\operatorname{dk}_{S'} N'})$. To see this, note that

$$\operatorname{dk}_{S'}(N') = \operatorname{dk}_S(N) + \operatorname{dk}_N(N') \leq \operatorname{dk}_S(N) + 1$$

and that the maximal choices of $F(\sigma|_N)$ and $F(\tau|_N)$ imply

$$\operatorname{cod}_{N'} F(\sigma'|_{N'}) \leq \frac{1}{2} \operatorname{cod}_N F(\sigma|_N).$$

The desired bound now follows from the assumed bound on $\operatorname{cod}_N F(\sigma|_N)$ in Property $\mathcal{I}$. This completes the proof of Theorem P.

Chapter 6

Proof of Theorem B

The proof of Theorem B contains three steps. The first step classifies 1-connected, compact, irreducible symmetric spaces whose rational cohomology is 4-periodic up to degree 16. The second step is a lemma about product manifolds whose rational cohomology is 4-periodic up to degree 16. The final step combines these lemmas to classify 1-connected, compact symmetric spaces whose rational cohomology is 4-periodic up to degree 16. From Theorem P, these results immediately imply Theorem B.

The relevant lemma concerning 1-connected, compact, irreducible symmetric spaces is the following:

Lemma 6.1. *Assume M^n is a 1-connected, compact, irreducible symmetric space. Let $c \geq 16$ and $n \geq 16$ be integers. If $H^*(M;\mathbb{Q})$ is 4-periodic up to degree c, then M is one of the following: S^n, $\mathbb{C}P^q$, $\mathbb{H}P^{q'}$, $SO(2+q)/SO(2) \times SO(q)$, or*

$SO(3+q')/SO(3) \times SO(q')$ *where* $q = n/2$ *and* $q' = n/3$.

The only facts about 4-periodicity up to degree 16 we use in the proof are the following: $b_i = b_{i+4}$ for $0 < i < 12$, $b_4 \leq 1$, and $b_4 = 0$ only if $b_i = 0$ for all $0 < i < 16$. Observe that, in particular, if $b_4(M) = 1$, then $b_{12} = 1$ and hence $\dim(M) \geq 12$. Here and throughout the section, b_i denotes the i-th Betti number of M.

The following is an easy corollary of the proof. We will use it as well:

Lemma 6.2. *Assume* M^n *is a 1-connected, compact, irreducible symmetric space. If* $H^i(M;\mathbb{Q}) = 0$ *for* $3 < i < 16$*, then* M *is* S^2 *or* S^3.

Proof of Lemma 6.1. We use Cartan's classification of simply connected, irreducible compact symmetric spaces. We also keep Cartan's notation. See [17] for a reference.

One possibility is that M is a simple Lie group. It is known that M has the rational cohomology ring of a product of spheres $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_s}$ for some $s \geq 1$ where the n_i are odd. In fact, these sphere dimensions are known and are listed in Table 6.1 on page 72 (see [23] for a reference). Since M is simply connected, we may assume

$$3 = n_1 \leq n_2 \leq \cdots \leq n_s.$$

Since $b_3(M) \neq 0$, we have $b_4(M) = 1$ by our comments above. But this cannot be since the n_i are odd, so there are no simple Lie groups with 4-periodic rational cohomology up to degree 16.

G	$n_1, n_2, \ldots, n_s$
$Sp(n)$	$3, 7, \ldots, 4n-1$
$Spin(2n+1)$	$3, 7, \ldots, 4n-1$
$Spin(2n)$	$3, 7, \ldots, 4n-5, 2n-1$
$U(n)$	$1, 3, \ldots, 2n-1$
$SU(n)$	$3, 5, \ldots, 2n-1$
G_2	$3, 11$
F_4	$3, 11, 15, 23$
E_6	$3, 9, 11, 15, 17, 23$
E_7	$3, 11, 15, 19, 23, 27, 35$
E_8	$3, 15, 23, 27, 35, 39, 47, 59$

Table 6.1: Dimensions of spheres

We now consider the irreducible spaces which are not Lie groups. We have that $M = G/H$ for some compact Lie groups G and H where G is simple. The possible pairs (G, H) that occur fall into one of seven classical families and 12 exceptional examples. We calculate the first 15 Betti numbers in each case, then compare the results to the requirement that they be 4-periodic as described above. We summarize the results in Tables 6.2 and 6.3.

To explain our calculations, we first consider $M = G/H$ with $\text{rank}(G) = \text{rank}(H)$. Let $S^{n_1} \times \cdots \times S^{n_s}$ and $S^{m_1} \times \cdots \times S^{m_s}$ denote the rational homotopy

G/H	$P_{G/H}(t) - 1$ if $rk(G) = rk(H)$	Reference if not	Obstruction
$SU(n)/SO(n), n \geq 5$	$-$	[5]	$b_5 > 0$
$SU(2n)/Sp(n), n \geq 3$	$-$	[5]	$b_5 > 0$
$SO(p+q)/SO(p) \times SO(q)$	$-$	[28]	$1 < b_4$
$SU(p+q)/S(U(p) \times U(q))$	$2t^4 + \dots$	$-$	$1 < b_4$
$Sp(n)/U(n), n \geq 3$	$t^2 + t^4 + 2t^6 + \dots$	$-$	$b_2 < b_6$
$Sp(p+q)/Sp(p) \times Sp(q)$	$t^4 + 2t^8 + \dots$	$-$	$b_4 < b_8$
$SO(2n)/U(n), n \geq 4$	$t^2 + t^4 + 2t^6 + \dots$	$-$	$b_2 < b_6$

Table 6.2: Classical irreducible simply connected compact symmetric spaces not listed in Lemma 6.1

types of G and H, respectively. Then one has the following formula for the Poincaré polynomial of M (see [5]):

$$P_M(t) = \sum_{i \geq 0} b_i(M) t^i = \frac{(1 - t^{n_1+1}) \cdots (1 - t^{n_s+1})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_s+1})}.$$

For each simple Lie group G, the dimensions of the spheres are listed in Table 6.1. When $\text{rank}(G) = \text{rank}(H)$, we compute the Poincaré polynomial of M and list the relevant terms in Tables 6.2 and 6.3.

In the case where $\text{rank}(G) \neq \text{rank}(H)$, we simply cite a source where the cohomology is calculated. The tables give the pair (G, H) realizing the space, the first few terms of the Poincaré polynomial if $\text{rank}(G) = \text{rank}(H)$, and the the relevant

G/H	$P_{G/H}(t)-1$ if $rk(G)=rk(H)$	Reference if not	Obstruction
$E_6/Sp(4)$	$-$	[20]	$0 < b_9$
E_6/F_4	$-$	[3]	$0 < b_9$
$E_6/SU(6)\times SU(2)$	$t^4+t^6+2t^8$	$-$	$b_4 < b_8$
$E_6/SO(10)\times SO(2)$	$t^4+2t^8+\dots$	$-$	$b_4 < b_8$
$E_7/SU(8)$	$t^8+\dots$	$-$	$b_4 < b_8$
$E_7/SO(12)\times SU(2)$	$t^4+2t^8+\dots$	$-$	$b_4 < b_8$
$E_7/E_6\times SO(2)$	$t^4+t^8+2t^{12}+\dots$	$-$	$b_8 < b_{12}$
$E_8/SO(16)$	$t^8+\dots$	$-$	$b_4 < b_8$
$E_8/E_7\times SU(2)$	$t^4+t^8+2t^{12}+\dots$	$-$	$b_8 < b_{12}$
$F_4/Sp(3)\times SU(2)$	$t^4+2t^8+\dots$	$-$	$b_4 < b_8$
$F_4/Spin(9)$	$t^8+\dots$	$-$	$b_4 < b_8$
$G_2/SO(4)$	t^4+t^8	$-$	$b_8 > b_{12}$

Table 6.3: Exceptional irreducible simply connected compact symmetric spaces not listed in Lemma 6.1

Betti number inequalities that show M is not rationally 4-periodic up to degree 16.

In Table 6.2, we exclude spaces with dimension less than 12. We also exclude the rank one Grassmannians and the rank two and rank three real Grassmannians, as these are the spaces that appear in the conclusion of Theorem B. □

With the first step complete, we now prove the following lemma about general products $M = M' \times M''$ whose rational cohomology ring is 4-periodic up to degree c. For simplicity we denote the Betti numbers of M, M', and M'' by b_i, b'_i, and b''_i, respectively.

Lemma 6.3. *Suppose $b_1 = 0$ and that $H^*(M;\mathbb{Q})$ is 4-periodic up to degree c with $c \geq 9$. Suppose that $M = M' \times M''$ with $b'_4 \geq b''_4$. Then either M is rationally $(c-1)$-connected or $b'_4 = 1$ and the following hold:*

1. *$H^*(M';\mathbb{Q})$ is 4-periodic up to degree c,*

2. *$b''_i = 0$ for $4 \leq i < c$, and*

3. *if $b'_2 > 0$ or $b'_3 > 0$, then $b''_2 = b''_3 = 0$.*

Proof of lemma. Let $x \in H^4(M;\mathbb{Q})$ be an element inducing periodicity. If $x = 0$, then $c \geq 8$ implies M is rationally $(c-1)$-connected. Assume therefore that $b_4(M) = 1$ (i.e., that $x \neq 0$).

We first claim that $b'_4 = 1$. Suppose instead that $0 = b'_4 = b''_4$. Then Künneth theorem implies $1 = b_4 = b'_2 b''_2$, and hence $b'_2 = b''_2 = 1$. Using periodicity and the

Künneth theorem again, we have

$$0 = b_1 = b_5 = b'_5 + b''_5 + b'_3 + b''_3,$$

and hence that all four terms on the right-hand side are zero. Similarly, now, we have

$$2 = b'_2 + b''_2 = b_2 = b_6 = b'_6 + b''_6.$$

Finally, we obtain

$$1 = b_4 = b_8 \geq b'_2 b''_6 + b'_6 b''_2 = b'_6 + b''_6 \geq 2,$$

a contradiction. Assume therefore that $b'_4 = 1$ and $b'_2 b''_2 = 0$.

Let $p : M \to M'$ be the projection map. It follows from this and the Künneth theorem that

$$H^4(M') \cong H^4(M') \otimes H^0(M'') \hookrightarrow \bigoplus_{i+j=4} H^i(M') \otimes H^j(M'') \xrightarrow{\times} H^4(M)$$

is an isomorphism. Choose $\bar{x} \in H^4(M'; \mathbb{Q})$ with $p^*(\bar{x}) = x$. We claim that $\bar{x}$ induces periodicity in $H^*(M')$ up to degree c.

First, $b'_4 = 1$ implies that multiplication by $\bar{x}$ induces a surjection $H^0(M') \to H^4(M')$. Second, consider the commutative diagram

$$\begin{array}{ccccc} H^i(M') & \hookrightarrow & \bigoplus_{i'+i''=i} H^{i'}(M') \otimes H^{i''}(M'') & \xrightarrow{\times} & H^i(M) \\ \downarrow & & \downarrow & & \downarrow \\ H^{i+4}(M') & \hookrightarrow & \bigoplus_{i'+i''=i+4} H^{i'}(M') \otimes H^{i''}(M'') & \xrightarrow{\times} & H^{i+4}(M) \end{array}$$

where the vertical arrows from left to right are given by multiplication by $\bar{x}$, $\bar{x} \otimes 1$, and x, respectively. Because mutliplication by x is injective for $0 < i \leq c-4$, it follows that multiplication by $\bar{x}$ is injective in these degrees as well. It therefore suffices to check that multiplication by $\bar{x}$ is surjective for $0 < i < c-4$. We accomplish this by a dimension counting argument. Specifically, we claim $b'_i = b'_{i+4}$ for $0 < i < c-4$. Indeed, for all $0 \leq i < c-4$, we have from periodicity, the Künneth theorem, and injectivity of multiplication by $\bar{x}$ the following estimate:

$$\sum_{j=0}^{i} b'_{i-j} b''_j = b_i = b_{i+4} \geq b''_{i+4} + \sum_{j=0}^{i} b'_{i+4-j} b''_j \geq b''_{i+4} + \sum_{j=0}^{i} b'_{i-j} b''_j.$$

Equality must hold everywhere, proving $b'_i = b'_{i+4}$ and $b''_{i+4} = 0$ for all $0 \leq i < c-4$. This completes the proof of the first part, as well as the second part, of the lemma.

Finally, suppose $b'_2 + b'_3 > 0$. Then

$$b'_4 + (b'_2 + b'_3) b''_2 \leq b_4 + b_5 = 1 + b_1 = b'_4$$

implies $b''_2 = 0$, and

$$b'_6 + (b'_2 + b'_3) b''_3 \leq b_5 + b_6 = b_2 = b'_2 = b'_6$$

implies $b''_3 = 0$.

□

Although we do not pursue the topic here, there is a similar result for rationally 4-periodic manifolds which are decomposable in the sense of being a connected sum of two other manifolds. In particular, consider an n-manifold constructed by taking products and connected sums of a collection $N_1, \ldots, N_t$ of irreducible symmetric spaces. If such a space admits a positively curved metric invariant under an r-torus action with $r \geq 2\log_2(n) + 8$, then at most one of the N_i is not a sphere. See also the discussion following Corollary 7.2 for an application to Cheeger manifolds.

Finally, we come to the proof of Theorem B. In fact, we prove the following stronger theorem:

Theorem 6.4. *Suppose M^n has the rational cohomology ring of a simply connected, compact symmetric space N. Let $c \geq 16$, and assume M has a metric with positive sectional curvature and symmetry rank at least $2\log_2 n + \frac{c}{2}$. Then there exists a (possibly trivial) product S of spheres, each of dimension at least c, such that one of the following holds:*

1. *$N = S$,*
2. *$N = S \times R$ with $R \in \{\mathbb{C}P^q, SO(2+q)/SO(2) \times SO(q)\}$, or*
3. *$N = S \times R \times Q$ with $R \in \{\mathbb{H}P^q, SO(3+q)/SO(3) \times SO(q)\}$ and $Q \in \{*, S^2, S^3\}$.*

As mentioned in the introduction, one might compare this result to Theorem 6 of [35]. There the topological assumption is that M is an arbitrary closed, simply connected manifold with $n \geq 6000$, the metric assumption is that a positively curved metric exists which is invariant under an r-torus with $r \geq \frac{n}{6} + 1$, and the conclusion implies, in particular, that M has the rational cohomology ring of S^n, $\mathbb{C}P^{n/2}$, or $\mathbb{H}P^q \times Q$ with $Q \in \{*, S^2, S^3\}$.

Using this conclusion as a reference point, our result allows the complex or quaternionic projective space to be replaced by a rank 2 or rank 3 Grassmannian, as well as possibly multiplying by a product of spheres of dimension at least c. It would be interesting to exclude these additional possibilities.

Observe that the Lie group E_8 has the rational cohomology of a product of spheres in dimensions $3, 15, \ldots$. It follows that the rational cohomology of $E_8 \times \mathbb{H}P^3$ is 4-periodic up to degree 15, so we must take $c \geq 16$ in this theorem for our proof to work.

Proof. Let N^n be as in the theorem. Assuming $n > 0$, the maximal symmetry rank result in [14] implies $n \geq 16$.

By the corollary to Theorem P, we conclude that N is rationally 4-periodic up to degree c. Write $N = N_1 \times \cdots \times N_t$ where the N_i are irreducible symmetric spaces and $b_4(N_1) \geq b_4(N_i)$ for all i. By Lemma 6.3, $b_4(N_1) = 0$ implies N is rationally $(c-1)$-connected and hence is a product of spheres since $c \geq 16$. Assume therefore that $b_4(N_1) = b_4(N) = 1$.

By the same lemma, N_1 is 4-periodic up to degree c. Observe that periodicity implies $n \geq 16$ since $x^4 \neq 0$ where $x \in H^4(N_1; \mathbb{Q})$ is an element inducing periodicity up to degree 16. By Lemma 6.1, we have that N_1 is $\mathbb{C}P^m$, $SO(2+q)/SO(2) \times SO(q)$, $\mathbb{H}P^m$, or $SO(3+q)/SO(3) \times SO(q)$. We also have that N_i for $i > 1$ has $b_j(N_i) = 0$

for all $4 \leq j < c$. If N is $\mathbb{C}P^q$ or $SO(2+q)/SO(2)\times SO(q)$, then $b_2(N_1)+b_3(N_1) > 0$, which by Lemmas 6.3 and 6.2 implies N_i is a sphere for all $i > 1$. This completes the proof in this case. Suppose therefore that N_1 is $\mathbb{H}P^q$ or $SO(3+q)/SO(3)\times SO(q)$. If $b_2(N_i)+b_3(N_i) = 0$ for all $i > 0$, then once again we have that each N_i with $i > 1$ is a sphere. Suppose therefore that some N_i, say with $i = 2$, has $b_2(N_2) + b_3(N_2) > 0$. Taking $M' = N_1 \times N_2$ and $M'' = N_3 \times \cdots \times N_t$ in the lemma implies N_i is a sphere for all $i > 2$. Finally, N_2 is an irreducible symmetric space with $b_2(N_2)+b_3(N_2) > 0$ and $b_i(N_2) = 0$ for all $4 \leq i < 16$. Lemma 6.2 implies that N_2 is a 2-sphere or a 3-sphere, completing the proof. $\square$

Chapter 7

Corollaries and Conjectures

Our first point of discussion regards general Lie group actions. By examining the list of simple Lie groups, one easily shows that $(2\,\mathrm{rank}(G))^2 \geq \dim(G)$ for all compact, 1-connected, simple Lie groups. The inequality persists for all compact Lie groups. In addition, $\dim(M^n/G) \leq n - d$ clearly implies $\dim(G) \geq d$. Hence, letting $I(M)$ denote the isometry group of M, we have the following corollary:

Corollary 7.1. *Let M^n be a closed Riemannian manifold with positive sectional curvature and $n \equiv 0 \bmod 4$. If $\dim I(M) \geq (4\log_2 n)^2$ or $\dim M/I(M) \leq n - (4\log_2 n)^2$, then $\chi(M) > 0$.*

We remark that in [27] it was shown that $\chi(M^{2n}) > 0$ if $\dim M/I(M) < 6$, and in [36] it was shown that $\chi(M^{2n}) > 0$ if $\dim M/I(M) \leq \sqrt{n}/3 - 1$ or $\dim I(M) \geq 4n - 6$.

There are analogous corollaries to Theorems B and P. For example, we may take $c = 4$ in Theorem P and to obtain the following:

Corollary 7.2. *Let M^n be a closed Riemannian manifold with positive sectional curvature. If the symmetry rank is at least $2\log_2 n + 2$, if the symmetry degree is at least $(4\log_2 n + 4)^2$, or if the cohomogeneity is at most $n - (4\log_2 n + 4)^2$, then $b_4(M) \leq 1$.*

Taking $c = 6$ instead allows us to conclude $b_5 = 0$ and $b_2 \geq b_6$, and taking $c = 8$ allows us to conclude in addition that $b_2 = b_6$, $b_3 = b_7$, and $b_8 \leq b_4 \leq 1$. This immediately applies to Cheeger manifolds, which are connected sums of rank one symmetric spaces. It was shown in [6] that each of these manifolds has a metric of nonnegative curvature. However, this corollary shows that no Cheeger manifold of dimension n admits a positively curved metric which admits an isometric T^r-action with $r \geq 2\log_2 n + 4$.

As another application of Theorem P, we obtain another proof of Theorem A. The proof begins in the same way as the proof in Chapter 3, so the goal is to show $\mathrm{b}_{\mathrm{odd}}(F) = 0$ for every component F inside M^T. As before, this follows from Conner's theorem once we prove the existence of a submanifold $P \subseteq M$ with $\mathrm{b}_{\mathrm{odd}}(P) = 0$ such that F is a component of P^T. Setting $c = 0$ in Theorem P and assuming $n \equiv 0 \bmod 4$, we obtain such a submanifold P. We omit the details.

Another observation is a curious one involving the rational category in the sense of Lusternik and Schnirelmann. Let M^n be a simply connected, n-dimensional CW complex. The rational category $\mathrm{cat}_0(M)$ can be defined as the category of the rationalization $M_{\mathbb{Q}}$ of M. (See [8] for definitions.) If M is an n-dimensional CW complex that is rationally $(c-1)$-connected, the rational category is bounded above by n/c. On the other hand, $\mathrm{cat}_0(M)$ is at least the cup length of M, where the cup length is the maximal integer k such that there exist positive degree elements $x_i \in H^*(M;\mathbb{Q})$ with $x_1 \cdots x_k \neq 0$.

Now, let M^n be as in the statement of Theorem P. The conclusion implies $H^*(M;\mathbb{Q})$ is 4-periodic up to degree c. Let $x \in H^4(M;\mathbb{Q})$ denote the element inducing this periodicity. It follows that M is rationally $(c-1)$-connected (if $x = 0$) or that $x^{c/4} \neq 0$. According to the bounds mentioned above, we have either $\mathrm{cat}_0(M) \leq n/c$ or $\mathrm{cat}_0(M) \geq c/4$.

Corollary 7.3. *Assume M^n is a closed, simply connected Riemannian manifold with positive sectional curvature. Let c be a nonnegative integer. Assume an r-torus acts effectively by isometries with $r \geq 2\log_2 n + \frac{c}{2}$. Then $\mathrm{cat}_0(M) \leq n/c$ or $\mathrm{cat}_0(M) \geq c/4$.*

In particular, if $c > 2\sqrt{n}$, then there exists a gap in the possible values of $\mathrm{cat}_0(M)$.

Observe that if $n \equiv 1 \bmod 4$, then the submanifold P from Theorem P is rationally 4-periodic and has $\dim(P) \equiv n \equiv 1 \bmod 4$. Hence Poincaré duality implies that P is a rational homology sphere, and c-connectedness implies that M is rationally $(c-1)$-connected. We state this as a corollary:

Corollary 7.4. *Let $n \equiv 1 \bmod 4$, and assume M^n is a closed, simply connected Rie-*

mannian manifold with positive sectional curvature. Let c be a nonnegative integer. If the symmetry rank is at least $2\log_2 n + \frac{c}{2}$, then M is rationally $(c-1)$-connected.

In particular, if the symmetry rank is at least $\frac{n}{2C} + 2\log_2 n$, then the rational category of M is at most C.

To conclude, we state a conjecture which would improve the conclusion of the periodicity theorem. Recall that the periodicity theorem rested on Theorem D, which we referred to as generalizations of Adem's theorem on singly generated cohomology rings. The conclusion of Adem's theorem was improved by Adams after he developed the theory of secondary cohomology operations. The result is the following:

Theorem (Adams, [1])**.** *Let p be a prime, and let M be a topological space. Assume $H^*(M;\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p[x]$ or $\mathbb{Z}_p[x]/x^{q+1}$ with $p \leq q$.*

1. *If $p = 2$, then $k \in \{1, 2, 4, 8\}$. Moreover, $k = 8$ only occurs when $q = 2$.*
2. *If $p > 2$, then $k = 2\lambda$ for some $\lambda | p - 1$.*

Recall that singly generated cohomology rings are periodic in the sense of this paper. The corresponding strengthening in our case would be the following:

Conjecture. *Let p be a prime, and let M be a topological space. Assume $x \in H^k(M;\mathbb{Z}_p)$ is nonzero and induces periodicity up to degree pk, and suppose x has minimal degree among all such elements.*

1. *If* $p = 2$, *then* $k \in \{1, 2, 4, 8\}$. *Moreover, if* x *induces periodicity up to degree* $3k$, *then* $k \neq 8$.

2. *If* $p > 2$, *then* $k = 2\lambda$ *for some* $\lambda | p - 1$.

We first note that, regarding the first statement, $S^{k-1} \times S^k$ is k-periodic but not k'-periodic for any $k' < k$, and $S^7 \times \mathrm{Ca}P^2$ is 8-periodic but not 4-periodic. Hence one must assume periodicity up to degree $2k$, respectively $3k$.

Second, we wish to outline how a proof of this conjecture would imply that Theorem A holds in all even dimensions. First, one would use the conjecture to improve Theorem D when $p = 2$ to prove the following: If M is a simply connected, closed manifold such that $H^*(M^n; \mathbb{Z}_2)$ is k-periodic with $3k \leq n$, then M has the $\mathbb{Z}_2$-cohomology ring of S^n, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{H}P^{(n-3)/4} \times S^3$, or $\mathbb{H}P^{(n-2)/4} \times S^2$. Indeed, a proof of the $\mathbb{Z}_2$-periodicity conjecture combined with Poincaré duality implies this when $n \not\equiv 2 \bmod 4$.

Suppose then that $n \equiv 2 \bmod 4$. We may assume without loss of generality that $H^4(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and that the generator x has minimal degree among all elements inducing periodicity. It follows that $Sq^1(H^3(M; \mathbb{Z}_2)) = 0$, $Sq^1(H^7(M; \mathbb{Z}_2)) = 0$, and $Sq^2(H^2(M; \mathbb{Z}_2)) = 0$.

By periodicity and Poincaré duality, $H^2(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Let $z_2 \in H^2(M; \mathbb{Z}_2)$ be a generator. If $H^3(M; \mathbb{Z}_2) = 0$, it follows that $H^*(M; \mathbb{Z}_2) \cong H^*(S^2 \times \mathbb{H}P^{(n-2)/4}; \mathbb{Z}_2)$. To see that this is the case, suppose there exists a nonzero $u \in H^3(M; \mathbb{Z}_2)$. Using Poincaré duality and periodicity again, we conclude the existence of a relation

$uv = xz$ for some $v \in H^3(M;\mathbb{Z}_2)$. One can now use the Cartan formula to prove that $Sq^4(uv) = 0$ and $Sq^4(xz) = x^2z \neq 0$, which is a contradiction.

Given this, the basic outline of our proof of Theorem A implies the result without the assumption that the dimension is divisible by four. In fact, the proof simplifies since one does not have to keep track of the divisibility of the codimensions. The optimal bound, as far as the proof is concerned, would be $r \geq \log_2(n) - 2$.

Bibliography

[1] J.F. Adams, *On the non-existence of elements of Hopf invariant one*, The Annals of Mathematics **72** (1960), no. 1, 20–104.

[2] J. Adem, *The iteration of the Steenrod squares in algebraic topology*, Proceedings of the National Academy of Sciences **38** (1952), 720–726.

[3] S. Arakai and Y. Shikata, *Cohomology mod 2 of the compact exceptional group E_8*, Proc. Japan Acad. **37** (1961), no. 10, 619–622.

[4] R.L. Bishop and S.I. Goldberg, *Some implications of the generalized gauss-bonnet theorem*, Transactions of the American Mathematical Society **112** (1964), no. 3, 508–535.

[5] A. Borel, *Sur la cohomologie des espaces fibres principaux et des espaces homogenes de groupes de Lie compacts*, The Annals of Mathematics **57** (1953), no. 1, 115–207.

[6] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, Journal of Differential Geometry **8** (1973), 623–628.

[7] S.S. Chern, *On the curvature and characteristic classes of a Riemannian manifold*, Abh. Math. Sem. Univ. Hamburg **20** (1956), 117–126.

[8] O. Cornea, G. Lupton, J. Oprea, and D. Tanré, *Lusternik-Schnirelmann category*, American Mathematical Society, 2003.

[9] O. Dearricott, *A 7-manifold with positive curvature*, Duke Mathematical Journal **158** (2011), no. 2, 307–346.

[10] F. Fang and X. Rong, *Homeomorphism classification of positively curved manifolds with almost maximal symmetry rank*, Mathematische Annalen **332** (2005), 81–101.

[11] R. Fintushel, *Circle actions on simply connected* 4*-manifolds*, Transactions of the American Mathematical Society **230** (1977), 147–171.

[12] ———, *Classification of circle actions on* 4*-manifolds*, Transactions of the American Mathematical Society **242** (1978), 377–390.

[13] K. Grove, *Developments around positive sectional curvature*, Surveys in Differential Geometry, Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry (2009), 117–133.

[14] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry-rank*, Journal of Pure and Applied Algebra **91** (1994), no. 1, 137–142.

[15] K. Grove, L. Verdiani, and W. Ziller, *An exotic T_1S^4 with positive curvature*, Geometric and Functional Analysis **21** (2011), no. 3, 499–524.

[16] K. Grove, B. Wilking, and W. Ziller, *Positively curved cohomogeneity one manifolds and 3-sasakian geometry*, Journal of Differential Geometry **78** (2008), no. 1, 33–111.

[17] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, American Mathematical Society, 2001.

[18] A. Hicks, *Group actions and the topology of nonnegatively curved 4-manifolds*, Illinois Journal of Mathematics **41** (1997), 421–437.

[19] W.Y. Hsiang and B. Kleiner, *On the topology of positively curved 4-manifolds with symmetry*, Differential Geometry **30** (1989), 615–621.

[20] K. Ishitoya, *Cohomology of the symmetric space EI*, Proc. Japan Acad. Ser. A Math. Sci. **53** (1977), no. 2, 56–60.

[21] J.H. Kim, *On positively curved 4-manifolds with s^1-symmetry*, Preprint: arXiv:0810.1232v1 [math.DG] (2008).

[22] B. Kleiner, *Riemannian four-manifolds with nonnegative curvature and continuous symmetry*, Dissertation, University of California, Berkeley (1990).

[23] M. Mimura and H. Toda, *Topology of Lie Groups, I and II*, American Mathematical Society, 1978.

[24] P.S. Pao, *Nonlinear circle actions on the* 4*-sphere and twisting spun knots*, Topology **17** (1978), 291–296.

[25] P. Petersen and F. Wilhelm, *An exotic sphere with positive curvature*, Preprint: arXiv:0805.0812v3 [math.DG].

[26] F. Podesta and L. Verdiani, *Totally geodesic orbits of isometries*, Annals of Global Analysis and Geometry **16** (1998), no. 5, 399–412.

[27] T. Püttmann and C. Searle, *The Hopf conjecture for manifolds with low cohomogeneity or high symmetry rank*, Proceedings of the American Mathematical Society **130** (2001), no. 1, 163–166.

[28] V. Ramani and P. Sankaran, *On degrees of maps between Grassmannians*, Proceedings Mathematical Sciences **107** (1997), no. 1, 13–19.

[29] X. Rong, *Positively curved manifolds with almost maximal symmetry rank*, Geometriae Dedicata **95** (2002), no. 1, 157–182.

[30] X. Rong and X. Su, *The Hopf conjecture for manifolds with abelian group actions*, Communications in Contemporary Mathematics **7** (2005), 121–136.

[31] C. Searle and D. Yang, *On the topology of nonnegatively curved simply connected 4-manifolds with continuous symmetry*, Duke Mathematical Journal **74** (1994), 547 – 556.

[32] X. Su and Y. Wang, *The Hopf conjecture for positively curved manifolds with discrete abelian group actions*, Differential Geometry and its Applications **26** (2008), no. 3, 313–322.

[33] L. Verdiani, *Cohomogeneity one manifolds of even dimension with strictly positive curvature*, Journal of Differential Geometry **68** (2004), no. 1, 31–72.

[34] N.R. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Annals of Mathematics **96** (1972), no. 2, 277–295.

[35] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta mathematica **191** (2003), no. 2, 259–297.

[36] ______, *Positively curved manifolds with symmetry*, Annals of Mathematics **163** (2006), 607–668.

[37] ______, *Nonnegatively and Positively Curved Manifolds*, Surveys in Differential Geometry, Vol. XI. Metric and Comparison Geometry (2007), 25–62.

[38] D. Yang, *On the topology of nonnegatively curved simply connected 4-manifolds with discrete symmetry*, Duke Mathematical Journal **74** (1994), 531–545.

[39] W. Ziller, *Examples of Riemannian manifolds with non-negative sectional curvature*, Surveys in Differential Geometry, Vol. XI. Metric and Comparison Geometry (2007), 63–102.