ON TYPE II_0 E_0-SEMIGROUPS INDUCED BY q-PURE MAPS ON $M_n(\mathbb{C})$

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ABSTRACT

ON TYPE II_0 E_0-SEMIGROUPS INDUCED BY q-PURE MAPS ON $M_n(\mathbb{C})$

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Using a dilation theorem of Bhat, Powers has shown that every non-trivial spatial E_0-semigroup can be obtained from a CP-flow acting on $B(K \otimes L^2(0, \infty))$, where K is a separable Hilbert space. In this thesis, we use boundary weight doubles (ϕ, ν) to define natural boundary weight maps which then induce CP-flows over K for $1 < dim(K) < \infty$. Developing a comparison theory for the E_0-semigroups arising from certain boundary weight doubles, we obtain examples of type II_0 E_0-semigroups and classify them up to cocycle conjugacy. We study the unital q-pure maps acting on $M_n(\mathbb{C})$, classifying all such maps which are invertible or have rank one. We find that the rank one unital q-pure maps arise from faithful states on $M_n(\mathbb{C})$ and, through a boundary weight construction using particular unbounded weights ν on $B(L^2(0, \infty))$, yield uncountably many mutually non-cocycle conjugate type II_0 E_0-semigroups for each $1 < n \in \mathbb{N}$. We ask whether every unital q-pure map acting on $M_n(\mathbb{C})$ has rank one or is invertible, and show that this is the case when $n = 2$. In conclusion, we discuss future problems involving the examination and generalization of these results.

Contents

Chapter 1

Introduction

1.1 Outline

Let H be a separable Hilbert space. A result of Wigner in [Wi] shows that every weakly continuous one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ of $B(H)$ is implemented by a strongly continuous unitary group $\{U_t\}_{t\in\mathbb{R}}$ in that $\alpha_t(A) = U_t A U_t^*$ for all $A \in B(H)$ and $t \in \mathbb{R}$. This leads us to pursue the more general task of classifying all semigroups of $*$-endomorphisms of $B(H)$. We will study one such class of semigroups, called E_0-semigroups, which arise naturally in the study of quantum mechanics.

Definition 1. *We say a family $\{\alpha_t\}_{t\geq 0}$ of $*$-endomorphisms of $B(H)$ is an* E_0-semigroup *if:*

1. *$\alpha_{s+t} = \alpha_s \circ \alpha_t$ for all $s, t \geq 0$, and $\alpha_0(A) = A$ for all $A \in B(H)$.*

2. *For each $f, g \in H$ and $A \in B(H)$, the inner product $(f, \alpha_t(A)g)$ is continuous in t.*

3. *$\alpha_t(I) = I$ for all $t \geq 0$ (in other words, α is* unital*).*

We have two different notions of what it means for two E_0-semigroups to be the same, namely conjugacy and cocycle conjugacy, the latter of which arises from Alain Connes' definition of outer conjugacy.

Definition 2. *Let α and β be E_0-semigroups on $B(H_1)$ and $B(H_2)$, respectively. We say that α and β are* conjugate *if there is a $*$-isomorphism θ from $B(H_1)$ onto $B(H_2)$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$ for all $t \geq 0$. We say that α and β are* cocycle conjugate *if α is conjugate to β', where β' is an E_0-semigroup on $B(H_2)$ satisfying the following condition: For some strongly continuous family of unitaries $U = \{U(t) : t \geq 0\}$ acting on H_2 and satisfying $U(t+s) = U(t)\beta_t(U(s))$ for all $s, t \geq 0$, we have $\beta'_t(A) = U_t \beta_t(A) U_t^*$ for all $A \in B(H_2)$ and $t \geq 0$. Such a semigroup of unitaries is called a* unitary cocycle *for β, and the semigroup $\beta' = \{\beta'_t\}$ is called a* cocycle perturbation *of β.*

E_0-semigroups are divided into three types based upon the existence, and structure of, their units. More specifically, let α be an E_0-semigroup on $B(H)$. A *unit* for α is a strongly continuous semigroup of bounded operators $U = \{U(t) : t \geq 0\}$ such that $\alpha_t(A)U(t) = U(t)A$ for all $A \in B(H)$. Let $\mathcal{U}_\alpha$ be the set of all units for α. We say α is *spatial* if $\mathcal{U}_\alpha \neq \emptyset$, while we say that α is *completely spatial* if, for each $t \geq 0$, the closed linear span of the set $\{U_1(t_1) \cdots U_n(t_n)f : f \in H, t_i \geq 0 \text{ and } U_i \in \mathcal{U}_\alpha \ \forall\ i, \sum t_i = t\}$ is H. An E_0-semigroup α is said to be of type I if it is completely spatial, type II if it is spatial but not completely spatial, and type III if it is not spatial. If α is of type I or II, we may further assign a numerical index $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to α, in which case we say α is of type I_n or II_n. Arveson has shown in [A5] that the type I E_0-semigroups are entirely classified by their index: the type I_0 E_0-semigroups are semigroups of $*$-automorphisms, while for $n \in \mathbb{N} \cup \{\infty\}$, every type I_n E_0-semigroup is cocycle conjugate to the CAR flow of index n.

However, at the present time, we do not have such a classification for those of type II or III. The first type II and type III examples were constructed by Powers in [P4] and [P5]. Tsirelson, who contructed uncountably many mutually non-cocycle conjugate type III E_0-semigroups in [T1], exhibited uncountably many mutually non-cocycle conjugate E_0-semigroups of types II [T2] through Arveson's theory of product systems. A dilation theorem of Bhat in [Bh] shows that every CP-flow α can be dilated to an E_0-semigroup, and that there is a minimal dilation α^d of α which is unique up to conjugacy. Using Bhat's result, Powers has recently proved in [P1] that every spatial E_0-semigroup can be obtained from the boundary weight map of a CP-flow over a separable Hilbert space K. He has also constructed spatial E_0-semigroups using boundary weights for the case where $dim(K) = 1$ in [P2].

Our goal is to use boundary weight maps to induce CP-flows over K for $1 < dim(K) < \infty$ and to classify their minimal dilations to E_0-semigroups up to cocycle conjugacy. To do so, we define a natural boundary weight map $\rho \to \omega(\rho)$ using a unital completely positive map ϕ and a boundary weight ν on $B(L^2(0, \infty))$. The necessary and sufficient condition that this map induce a CP-flow α is that ϕ satisfies a generalization of the definition of q-positive from [P1] (see Definition 6 and Proposition 1), in which case we say that α is the CP-flow induced by the boundary weight double (ϕ, ν). We develop a comparison theory for boundary weight doubles (ϕ, ν) and (ψ, ν) in the case that ν is a normalized unbounded boundary weight on $B(L^2(0, \infty))$ of the form $\nu(B) = (f, Bf)$, finding that the doubles induce cocycle conjugate E_0-semigroups if and only if there is a hypermaximal q-corner from ϕ to ψ (see Definition 10 and Proposition 7).

The problem of determining hypermaximal q-corners from ϕ to ψ becomes much easier if we focus on a particular class of q-positive maps, called the q-pure maps, which have the least possible q-subordinates (see Definition 9). Given a q-positive map ϕ acting on $M_n(\mathbb{C})$ and a unitary $U \in M_n(\mathbb{C})$, we can form a new map ϕ_U by $\phi_U(A) = U^*\phi(UAU^*)U$. We describe the order isomorphism between the q-subordinates of ϕ and those of ϕ_U, which in turn leads to the existence of a hypermaximal q-corner from ϕ to ϕ_U if ϕ is q-pure (Proposition 3). With this result in mind, we turn our attention to the task of classifying the q-pure maps. In Proposition 4, we show that the rank one unital q-pure maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are precisely the maps $\phi(A) = \rho(A)I$ for faithful states ρ on $M_n(\mathbb{C})$. That these maps give us an enormous class of mutually non cocycle conjugate E_0-semigroups in one of our main results (Theorem 8). Furthermore, for $n > 1$, none of the E_0-semigroups constructed from boundary weight doubles satisfying the conditions of Theorem 5 are cocycle conjugate to any of the E_0-semigroups obtained from one-dimensional weights by Powers in [P2] (see Corollary 2).

However, for unital q-pure maps that are invertible rather than rank one, the opposite holds. These maps are best understood through their (conditionally negative) inverses. In Theorem 9, we find a necessary and sufficient condition for an invertible unital map ϕ on $M_n(\mathbb{C})$ to be q-pure. However, if ν is a normalized unbounded boundary weight of the form $\nu(B) = (f, Bf)$, then the E_0-semigroup induced by the boundary weight double (ϕ, ν) is entirely determined by ν. This E_0-semigroup is the one induced by ν in the sense of [P2]. In conclusion, we are led to ask if all unital q-pure maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ either have rank one or are invertible. We find that this is the case when $n = 2$. In fact, there are no unital q-positive maps ϕ on $M_2(\mathbb{C})$ with rank three.

1.2 Background

The development of quantum mechanics during the twentieth century brought the study of operator algebras to prominence. Classical scalar-valued and vector-valued observables became operators acting on a Hilbert space, and deterministic solutions gave way to probability and expected values. The Schrödinger and Heisenberg pictures of quantum mechanics, each looking at the evolution of time through a different lens, were unified in a mathematical setting. Many other fundamental contributions to the field not only enlightened scientists with regard to the quantum picture, but also challenged human intuition and brought the philosophy behind the apparent indeterminism of quantum mechanics to the forefront. While it is not within the author's expertise to discuss the fine points of these scientific and philosophical matters, they are the motivation behind a significant portion of operator theory.

We will briefly discuss this motivation in the vein of the introduction of [BR1], which the interested reader should consult for a broader and more detailed treatment. In rough terms, a quantum system is represented by several mathematical constructs. The framework begins with the Hilbert space $L^2(\mathbb{R}^n)$ for some $n \in \mathbb{N}$ which depends on the number of particles in the system and the dimension of the encompassing space. We denote the inner product on $L^2(\mathbb{R}^n)$ through the symbol (,) which is conjugate-linear in its first coordinate and linear in its second. The observables of the system are identified with self-adjoint linear operators, not necessarily bounded, acting on $L^2(\mathbb{R}^n)$. We represent the initial state of the system through a positive linear functional $\rho \in L^2(\mathbb{R}^n)^*$ of the form $\rho(A) = (f, Af)$ for some $f \in L^2(\mathbb{R}^n)$ with $||f|| = (\int_{\mathbb{R}^n} |f(x)|^2 dx)^{1/2} = 1$. A functional ρ of this form is called a *pure state.*[1]

The evolution of time is implemented by a one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ in the following manner: The value of the observable A at time t is equal to $(f, U_t A U_t^* f)$. In the Schrödinger picture, the observables are fixed (as operators) while the function f varies. The functions $f_t := \{U_t^* f\}$ are the solutions to Schrödinger's equation for the system's Hamiltonian. In the Heisenberg picture, f stays fixed while the observables vary with time through the operation $A \to U_t A U_t^*$. The two pictures are the same, since

$$(f, U_t A U_t^* f) = (U_t^* f, A U_t^* f).$$

Stone's Theorem asserts that, for some self-adjoint B acting on $L^2(\mathbb{R}^n)$, these unitaries satisfy $U_t = e^{itB}$ for all t (as it turns out, $\hbar B$ is the Hamiltonian of the system), and that the family $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous in t, meaning that for each $g \in L^2(\mathbb{R}^n)$ and $s \in \mathbb{R}$ we have $U_t g \to U_s g$ as $t \to s$. Furthermore, if B is bounded, then $\{U_t\}_{t\in\mathbb{R}}$ is norm continuous in t; that is, for every $s \in \mathbb{R}$ we have $||U_t - U_s|| \to 0$ as $t \to s$. Wigner's aforementioned result shows that every weakly-continuous one-parameter family of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ of $B(H)$ is implemented by a strongly continuous unitary group. If we replace $*$-automorphism (and group) with $*$-endomorphism (and semigroup), we get E_0-semigroups. The study of E_0-semigroups acting on $B(H)$ began as a problem that Robert Powers thought would take a delightful afternoon to solve. Needless to say, it has evolved into a complex subject with fruitful examples whose partial classification, and division into types, have mirrored that of von Neumann algebra factors.

We should first recall our two notions of equivalence for E_0-semigroups. It is obvious that

[1] This setting is simplified. A general quantum state ρ has the form $\rho(A) = \sum_i (f_i, Af_i)$ for mutually orthogonal vectors $\{f_i\}$ with $\sum_i ||f_i||^2 = 1$. As far as the results of this paper are concerned, the least pure states of all (the *faithful states*) will be the most fruitful.

conjugacy is an equivalence relation, but it is not clear that cocycle conjugacy is an equivalence relation as well. To this end, symmetry and transitivity are non-trivial. For symmetry, suppose E_0-semigroups α and β acting on $B(H_1)$ and $B(H_2)$, respectively, are cocycle conjugate. Then for some onto $*$-isomorphism $\theta : B(H_1) \to B(H_2)$ and cocycle perturbation $\beta'_t(A) = U_t\beta_t(A)U_t^*$ of β, we have $\alpha_t = \theta^{-1} \circ \beta'_t \circ \theta$ for all $t \geq 0$. We can check that $\{\theta^{-1}(U_t^*)\}_{t\geq 0}$ is a unitary cocycle for α, and that the cocycle perturbation $\alpha' = \{\alpha'_t\}_{t\geq 0}$ of α defined by $\alpha'_t(A) = \theta^{-1}(U_t^*)\alpha_t(A)\theta^{-1}(U_t)$ satisfies $\beta_t = \theta \circ \alpha'_t \circ \theta^{-1}$ for all $t \geq 0$. Therefore, β is cocycle conjugate to α, and symmetry follows.

For transitivity, suppose that α is cocycle conjugate to β and β is cocycle conjugate to γ, so $\alpha_t = \theta_1^{-1} \circ \beta'_t \circ \theta_1$ and $\beta_t = \theta_2^{-1} \circ \gamma'_t \circ \theta_2$ for some onto $*$-isomorphisms θ_1 and θ_2 and cocycle perturbations $\beta'_t(B) = U_t\beta_t(B)U_t^*$ (all $t \geq 0$) and $\gamma'_t(A) = V_t\gamma_t(A)V_t^*$ (all $t \geq 0$) of β and γ, respectively. Let $\theta_3 = \theta_2 \circ \theta_1$. A straightforward computation yields that the family of unitaries $\{Z_t = \theta_3^{-1}(V_t^*)\theta_1^{-1}(U_t^*)\}_{t\geq 0}$ is a unitary cocycle for α. Defining a cocycle perturbation $\alpha' = \{\alpha'_t\}_{t\geq 0}$ by $\alpha'_t = Z_t\alpha_t(A)Z_t^*$, we find that $\gamma_t = \theta_3 \circ \alpha'_t \circ \theta_3^{-1}$ for all $t \geq 0$, so α is cocycle conjugate to γ, proving transitivity.

1.2.1 The CAR and CCR flows

The CCR and CAR flows, which arise from second quantization of bosons and fermions, respectively, were the first (non-trivial) examples of E_0-semigroups. We should note that nothing in this section is original: we are outlining Arveson's comprehensive treatment of both from [A2], as well as providing some details from [BR2]. We shamelessly borrow notation from both sources, which are fully responsible for any and all insight in this section (except for whatever mistakes you can find, which are, of course, mine).

We start with the Hilbert space $H = K \otimes L^2(0, \infty)$, where K is a separable Hilbert space. We identify H with $L^2((0, \infty); K)$, the space of K-valued measurable functions on $(0, \infty)$ which are square integrable. We denote by $\{U_t\}_{t\geq 0}$ the right shift semigroup on H, so for $f \in H$ we have

$(U_t f)(x) = 0$ if $x \leq t$ and $(U_t f)(x) = f(x-t)$ if $x > t$.

For every $n \in \mathbb{N}$, let $H^{\otimes n}$ be the n-fold tensor product of H with itself (setting $H^{\otimes 0} = \mathbb{C}$), and let $F(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}$. Given $z \in H$, we define an operator $C(z)$ on $F(H)$ by letting $C(z)(1) = z$ and

$$C(z)(z_1 \otimes \cdots \otimes z_n) = \sqrt{n+1}\ (z \otimes z_1 \cdots \otimes z_n)$$

for all simple tensors, then extending linearly. Note that $C(z)$ is densely defined and unbounded if $z \neq 0$, and its adjoint satisfies

$$C(z)^*(z_1 \otimes \cdots \otimes z_n) = \sqrt{n}\ (z, z_1)\ z_2 \otimes \cdots \otimes z_n$$

for all simple tensors and $n \in \mathbb{N}$, along with $C(z)^*(1) = 0$.

1.2.2 The CAR flow

We obtain the antisymmetrization operator P_- on each $H^{\otimes n}$ ($n \in \mathbb{N}$) by defining

$$P_-(z_1 \otimes \cdots \otimes z_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sgn(\sigma)} z_{\sigma_1} \otimes \cdots \otimes z_{\sigma_n}$$

for all $z_1, \ldots, z_n \in H$ and extending linearly. Note that P_- merely projects $H^{\otimes n}$ onto the space of antisymmetric tensors of $H^{\otimes n}$, and the space $H^{\wedge n} := P_-(H^{\otimes n})$ is a Hilbert space under the inner product inherited from $H^{\otimes n}$. We can describe the elements of $H^{\wedge n}$ in terms of wedge products. More specifically, $H^{\wedge n}$ is the closed linear span of elements of the form

$$z_1 \wedge \cdots \wedge z_n = \sqrt{n!}\ P_-(z_1 \otimes \cdots \otimes z_n).$$

We now form the antisymmetric Fock space

$$F_-(H) = \bigoplus_{n=0}^{\infty} H^{\wedge n},$$

where $H^0 = \mathbb{C}$. This serves as the state space for quantum systems with $dim(K)$ non-identical fermions.

Given $z \in H$, let $c(z) \in B(F_-(H))$ be the (unique) operator that satisfies the following for all simple tensors and $n \in \mathbb{N}$:

$$c(z)(P_-(z_1 \otimes \cdots \otimes z_n)) = P_- C(z) P_-(z_1 \otimes \cdots \otimes z_n).$$

We call $c(z)$ the creation operator corresponding to z, and we say that $C^*(c(H))$ is the CAR algebra over H. Since

$$P_-(z_{\sigma_1} \otimes \cdots \otimes z_{\sigma_n}) = (-1)^{sgn(\sigma)}\ P_-(z_1 \otimes \cdots \otimes z_n)$$

for all $\sigma \in \mathfrak{G}_n$, we actually have $c(z)(P_-(z_1 \otimes \cdots \otimes z_n)) = P_- C(z)(z_1 \otimes \cdots \otimes z_n)$. Indeed,

$$\begin{aligned}
c(z)(P_-(z_1 \otimes \cdots \otimes z_n)) &= P_- C(z) P_-(z_1 \otimes \cdots \otimes z_n) \\
&= \frac{\sqrt{n+1}}{n!} P_-\Big(z \otimes (\sum_{\sigma \in \mathfrak{G}_n} (-1)^{sgn(\sigma)} z_{\sigma_1} \otimes \cdots \otimes z_{\sigma_n})\Big) \\
&= \frac{\sqrt{n+1}}{n!} P_-\Big(\sum_{\sigma \in \mathfrak{G}_n} z \otimes z_{\sigma_1} \otimes \cdots \otimes z_{\sigma_n}\Big) \\
&= \frac{\sqrt{n+1}}{n!}\Big(\sum_{\sigma \in \mathfrak{G}_n} ((-1)^{sgn(\sigma)})^2 P_-(z \otimes z_1 \otimes \cdots \otimes z_n)\Big) \\
&= \frac{\sqrt{n+1}}{n!} \sum_{\sigma \in \mathfrak{G}_n} P_-(z \otimes z_1 \otimes \cdots \otimes z_n) \\
&= \sqrt{n+1}\ P_-(z \otimes z_1 \otimes \cdots \otimes z_n) \\
&= P_-\Big((C(z)(z_1 \otimes \cdots \otimes z_n)\Big).
\end{aligned}$$

In terms of wedge products, we have the formula

$$\begin{aligned}
c(z)(z_1 \wedge \cdots \wedge z_n) &= c(z)(\sqrt{n!}\ P_-(z_1 \otimes \cdots \otimes z_n) \\
&= \sqrt{n!}\ c(z)(P_-(z_1 \otimes \cdots \otimes z_n)) \\
&= c(z)\Big(P_- C(z)(z_1 \otimes \cdots \otimes z_n)\Big) \\
&= \sqrt{n!}\sqrt{n+1}\ P_-(z \otimes z_1 \otimes \cdots \otimes z_n) \\
&= \sqrt{(n+1)!}\ P_-(z \otimes z_1 \otimes \cdots \otimes z_n) \\
&= z \wedge z_1 \wedge \cdots \wedge z_n.
\end{aligned}$$

We have obtained the rather simple formula for $c(z)$:

$$c(z)(z_1 \wedge \cdots \wedge z_n) = z \wedge z_1 \cdots \wedge z_n. \tag{1.2.1}$$

Similar calculations show that $c(z)^*$ (the annihilation operator corresponding to z) satisfies

$$\begin{aligned} c(z)^*(z_1 \wedge \cdots \wedge z_n) &= P_- C(z)^*(\sqrt{n!}\ P_-(z_1 \otimes \cdots \otimes z_n)) \\ &= \sum_{k=1}^{n} (-1)^{k+1} (z, z_k)\ z_1 \wedge \cdots \wedge z_{k-1} \wedge z_{k+1} \wedge \cdots \wedge z_n. \end{aligned} \tag{1.2.2}$$

These operators satisfy the canonical anticommutation relations (CARs) for all $z_1, z_2 \in H$:

$$c(z_1)c(z_2) + c(z_2)c(z_1) = 0, \tag{1.2.3}$$

$$c(z_1)^* c(z_2) + c(z_2)c(z_1)^* = (z_1, z_2) I. \tag{1.2.4}$$

To see this, let $z_1, z_2, w_1, \ldots, w_n \in H$ be arbitrary. Relation (1.2.3) follows from (1.2.1) since

$$z_1 \wedge z_2 \wedge w_1 \wedge \cdots \wedge w_n = -z_2 \wedge z_1 \wedge w_1 \wedge \cdots \wedge w_n,$$

while relation (1.2.4) follows from the calculation that the quantity

$$\Big(c(z_1)^* c(z_2) + c(z_2)c(z_1)^*\Big)(w_1 \wedge \cdots \wedge w_n)$$

is equal to the following, where we write $z_2 = w_{n+1}$:

$$\Big(\sum_{k=1}^{n+1} (-1)^{k+1+(-1)^n} (z_1, w_k)\ w_1 \wedge \cdots \wedge w_{k-1} \wedge w_{k+1} \wedge \cdots \wedge w_{n+1}$$

$$+ z_2 \wedge \sum_{k=1}^{n} (-1)^{k+1} (z_1, w_k)\ w_1 \wedge \cdots \wedge w_{k-1} \wedge w_{k+1} \wedge \cdots \wedge w_n\Big)$$

$$= \Big(\sum_{k=1}^{n+1} (-1)^{k+1+(-1)^n} (z_1, w_k)\ w_1 \wedge \cdots \wedge w_{k-1} \wedge w_{k+1} \wedge \cdots \wedge w_{n+1}$$

$$+ \sum_{k=1}^{n} (-1)^{k+1+(-1)^{n-1}} (z_1, w_k)\ w_1 \wedge \cdots \wedge w_{k-1} \wedge w_{k+1} \wedge \cdots \wedge w_n \wedge w_{n+1}\Big)$$

$$= (z_1, w_{n+1})\ w_1 \wedge \cdots \wedge w_n$$

$$= (z_1, z_2)\ w_1 \wedge \cdots \wedge w_n.$$

Having established that our creation operators and their adjoints satisfy the canonical anticommutation relations, we note that for each $t \geq 0$, the map $z \to c(U_t z) : H \to B(F_-(U_t H))$ defines another presentation (in the sense of [BR2], Theorem 5.2.5) of the canonical anticommutation relations. A well-known result asserts that any two presentations of the infinite canonical anticommutation relations on H are $*$-isomorphic, giving us a $*$-isomorphism from $C^*(c(H))$ onto $C^*(c(U_t H))$ which is unique once we specify where to send $\{c(z_j)\}_{j\in\mathbb{N}}$ for an orthonormal basis $\{z_j\}$ of H (see Theorem 5.2.5 of [BR2] or [A1] for a good treatise). Therefore, for each $t \geq 0$, there is a unique $*$-isomorphism $\alpha_t : C^*(c(H)) \to C^*(c(U_t H))$ satisfying

$$\alpha_t(c(z)) = c(U_t z)$$

for all $z \in H$. Since $C^*(c(U_t H))$ is a C^*-subalgebra of $C^*(c(H))$, we may equivalently view α_t as a $*$-endomorphism of $C^*(c(H))$. The family of maps $\alpha = \{\alpha_t\}_{t\geq 0}$ is unital and clearly satisfies the semigroup property $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all nonnegative s and t, and $\alpha_0 = I_{C^*(c(H))}$. Furthermore, from continuity of c and strong continuity of the shift semigroup in t, it follows that α is weakly continuous in t. As it turns out, each α_t can be extended to a $*$-endomorphism of $B(F_-(H))$ while maintaining these properties (see Proposition 2.1.7 and Lemma 2.1.8 of [A2]). We conclude that $\alpha = \{\alpha_t\}_{t\geq 0}$ is an E_0-semigroup, which we call the CAR flow of rank $dim(K)$.

1.2.3 The CCR flow

For the CCR flow of rank $dim(K)$, we start with the space $F(H)$ (where, as before, $H = K \otimes L^2(0, \infty)$) and symmetrize rather than antisymmetrize. Letting P_+ be the (unique) operator defined on $F(H)$ by

$$P_+(z_1 \otimes \cdots \otimes z_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} z_{\sigma_1} \otimes \cdots \otimes z_{\sigma_n}$$

for all $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in H$, we define $H^n := P_+(H^{\otimes n})$ and $H^0 := \mathbb{C}$. Like in the previous section, each H^n is a Hilbert space under the inner product inherited from $H^{\otimes n}$.

Following Arveson's notation, we form the symmetric Fock space $e^H = \bigoplus_{n=0}^{\infty} H^n$. In quantum mechanics, e^H serves as the state space for a system of $dim(K)$ non-identical bosons interacting in a one-dimensional space. We define the exponential map $exp : H \to e^H$ by

$$exp(\xi) = \oplus \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} \xi^{\otimes n}.$$

Next, for each $\xi \in H$ we define a linear map $W(\xi) \in B(e^H)$ by setting

$$W(\xi) \exp(\eta) = e^{-\frac{1}{2}||\xi||^2 - (\xi,\eta)} \exp(\xi + \eta)$$

for all $\eta \in H$ and extending linearly. There is no ambiguity in doing so, since the closed linear span of the set $\{exp(\xi) : \xi \in H\}$ is e^H. In fact, $W(\xi)$ is unitary for each $\xi \in H$. We call $C^*(W(H))$ the CCR algebra over H.

For each $t \geq 0$, there is a $*$-endomorphism α_t of $B(e^H)$ satisfying

$$\alpha_t(W(\xi)) = W(U_t \xi)$$

for every $\xi \in H$ (see [A2]). This property defines α_t uniquely, since the set $\{W(\xi) : \xi \in H\}$ is irreducible in $B(e^H)$. The family $\alpha = \{\alpha_t\}_{t \geq 0}$ trivially satisfies the semigroup property $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all nonnegative s and t, and $\alpha_0 = I_{B(e^H)}$. Furthermore, α is weakly continuous in t since the right shift semigroup is strongly continuous in t and the map $\xi \to W(\xi)$ is strongly continuous on H. We conclude that $\alpha = \{\alpha_t\}_{t \geq 0}$ is an E_0-semigroup on $B(e^H)$. We call α the CCR flow of rank $dim(K)$. What is the relationship between the CAR and CCR flows of rank $n \in \mathbb{N} \cup \{\infty\}$? They are, in fact, conjugate. Arveson has shown in [A5] that, up to cocycle conjugacy, these flows gave all type I E_0-semigroups (aside of semigroups of $*$-automorphisms).

As an aside, we should note that what we are being somewhat dishonest when referring to the above as "the" CCR and CAR algebras over $H = K \otimes L^2(0, \infty)$. Our CCR algebra over H is the CCR algebra obtained from the Fock representation of the canonical commutation relations. The CAR algebra over H can significantly be realized in the big picture as an irreducible representation of the UHF algebra of type $\{2^j\}_{j \in \mathbb{N}}$ (the latter of which is often called "the" CAR algebra). A

UHF (uniformly hyperfinite) algebra is a C^*-algebra $\mathfrak{U}$ which can be written as the norm closure of a union of type I factors $\{\mathfrak{U}_j\}$, where $\mathfrak{U}_j$ is a proper subset of $\mathfrak{U}_{j+1}$ for all j. As each $\mathfrak{U}_j$ is $*$-isomorphic to $M_{n_j}(\mathbb{C})$ for some $n_j \in \mathbb{N}$, we call $\mathfrak{U}$ the UHF algebra of type $\{n_j\}$. Glimm proved a necessary and sufficient condition for two UHF algebras to be $*$-isomorphic in [Gl]. While fruitful in their own right, they have made profound contributions to the theory of von Neumann algebra factors. For example, Powers used product states on UHF algebras to exhibit uncountably many mutually non-isomorphic type III factors (see [P6]). The reader looking for a comprehensive treatment of UHF algebras will find one in chapters 10 and 12 of [KR].

1.2.4 The mighty index

One of the first asked questions in the study of E_0-semigroups was that of equivalence. What is the best notion of when two E_0-semigroups the same? By now, we have likely spoiled all suspense, as we have primarily discussed cocycle conjugacy rather than conjugacy. The *index* of a spatial E_0-semigroup is, perhaps, the most fundamentally useful cocycle conjugacy invariant in the subject of E_0-semigroups.

Let α be a spatial E_0-semigroup acting on $B(H)$. We recall our definition of a unit for α as a strongly continuous one-parameter semigroup of bounded operators $V = \{V_t\}_{t\geq 0}$ that intertwines α in that

$$\alpha_t(A)V_t = V_t A$$

for all $A \in B(H)$ and $t \geq 0$.

We (very briefly) outline Arveson's approach to the index (for the details, and for a thorough treatment of product systems, the reader should consult Chapters 2 and 3 of [A2]). The index of α is the dimension of a particular Hilbert space $H(\mathcal{U}_\alpha)$ defined by its units. More specifically, we start with the covariance function $c : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$, which associates to any two units $U = \{U_t\}$

and $V = \{V_t\}$ the number $c(U,V)$ such that

$$U_t^* V_t = e^{tc(U,V)} I$$

for all $t \geq 0$. If we let $\mathbb{C}_0\mathcal{U}_\alpha$ be the space of functions $f : \mathcal{U}_\alpha \to \mathbb{C}$ which vanish at all but finitely many units and satisfy $\sum_{U \in \mathcal{U}_\alpha} f(U) = 0$, then we may define a semi-definite inner product $\langle\ ,\ \rangle$ on $\mathbb{C}_0\mathcal{U}_\alpha$ by

$$\langle f, g \rangle = \sum_{U,V \in \mathcal{U}_\alpha} c(U,V)\overline{f(U)}g(V).$$

Letting $\mathcal{M} = \{f \in \mathbb{C}_0\mathcal{U}_\alpha : \langle f, f \rangle = 0\}$, we finally construct $H(\mathcal{U}_\alpha)$ as the completion of $\mathbb{C}_0\mathcal{U}_\alpha/\mathcal{M}$. We define the *index* of α to be the dimension of $H(\mathcal{U}_\alpha)$. In later sections, our primary focus will be E_0-semigroups of type II and index 0 (that is, type II_0).

The index satisfies the property

$$Index(\alpha_t \otimes \beta_t) = Index(\alpha_t) + Index(\beta_t),$$

as shown in [A6]. In our proof that cocycle conjugacy is a symmetric property, we established an isomorphism between unitary cocycles for cocycle conjugate E_0-semigroups. A more difficult proof yields a bijection between units of cocycle conjugate E_0-semigroups, and some more work shows that the index of an E_0-semigroup is invariant under cocycle conjugacy.

Given two E_0-semigroups α on $B(H)$ and β on $B(K)$, we ask whether they can be joined together in a suitable way. In essence, can they exist in some larger space on which they together represent the evolution of time in a quantum system? In mathematical terms, this roughly reduces to the following question: Is there a one-parameter group of $*$-automorphisms $\gamma = \{\gamma_t\}_{t \in \mathbb{R}}$ of $B(H \otimes K)$ such that $\gamma_{-t}(A \otimes I_K) = \alpha_t(A) \otimes I_K$ and $\gamma_t(I_H \otimes B) = I_H \otimes \beta_t(B)$ for all $t \geq 0$, $A \in B(H)$, and $B \in B(K)$? If so, we say that α and β are *paired*.

This question has been answered completely in the case that α and β are both type I, as Arveson has shown that α and β are paired if and only if they have the same index. Viewing α and β as the CCR or CAR flows, this fits with our intuition that we should be able to pair such E_0-semigroups together if their systems have the same number of (distinguishable) particles. How do we approach pairing for type II E_0-semigroups? Arveson's theory of product systems gives a necessary and sufficient condition for spatial E_0-semigroups to be paired, but the pairing of explicit type II examples remains mostly unsolved.

1.2.5 Completely positive maps

Among the most significant results in operator algebras is the fact that every C^*-algebra can be represented as a $*$-subalgebra of $B(H)$ for some Hilbert space H. This fact was first proved using the Gelfand-Neumark-Segal construction for states: Given any state ρ acting on a C^*-algebra $\mathfrak{U}$, there exists a $*$-homomorphism π from $\mathfrak{U}$ into $B(H)$, along with a cyclic unit vector f for π, such that for all $A \in \mathfrak{U}$ we have

$$\rho(A) = (f, \pi(A)f).$$

With this result in mind, we might ask ourselves if, in general, a positive linear map $\phi : \mathfrak{U} \to \mathfrak{B}$ between C^*-algebras dilates in a canonical manner to a representation. The answer is yes if the map is *completely* positive.

Definition 3. *Let $\phi : \mathfrak{U} \to \mathfrak{B}$ be a linear map between C^*-algebras. For each $n \in \mathbb{N}$, define $\phi_n : M_n(\mathfrak{U}) \to M_n(\mathfrak{B})$ by*

$$\phi_n \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(A_{1n}) & \cdots & \phi(A_{nn}) \end{pmatrix}.$$

We say that ϕ is completely positive if ϕ_n is positive for all $n \in \mathbb{N}$.

We present Stinespring's Theorem, which allows us to dilate any completely positive map to a $*$-homomorphism (see Theorem 4.1 of [Pa]):

Theorem 1. *If $\mathfrak{U}$ is a unital C^*-algebra and $\phi : \mathfrak{U} \to B(H)$ is a unital completely positive map, then there is a Hilbert space K, a $*$-homomorphism $\pi : \mathfrak{U} \to B(K)$, and an isometry $V : H \to K$*

such that

$$\phi(A) = V^* \pi(A) V$$

for all $A \in \mathfrak{U}$.

Stinespring's Theorem can be used to show that a linear map $\phi : B(H) \to B(H)$ is completely positive if and only if for all $A_1, \dots A_n \in B(H)$, $f_1, \dots, f_n \in H$, and $n = 1, 2, \dots$, we have

$$\sum_{i,j=1}^{n} (f_i, \phi(A_i^* A_j) f_j) \geq 0.$$

From the work of Choi (see [Ch]) and Arveson (see [A4]), we also know that a normal linear map $\phi : B(H_1) \to B(H_2)$ is completely positive if and only if it can be written in the form

$$\phi(A) = \sum_{i=1}^{n} S_i A S_i^*$$

for some maps $S_i : H_1 \to H_2$ which are linearly independent over $\ell_2(\mathbb{N})$ in that $\sum_{i=1}^{r \in \mathbb{N} \cup \infty} z_i S_i = 0$ for a sequence $\{z_i\}_{i=1}^r \in \ell_2(\mathbb{N})$ implies $z_i = 0$ for all i. With these hypotheses satisfied, the number n is unique. We call n the rank of ϕ *as a completely positive map* and note the difference between this notion of rank and the regular notion of rank (as in, the dimension of the image of ϕ). For example, the map ϕ on $M_2(\mathbb{C})$ defined by $\phi(A) = \frac{tr(A)}{2} I$ is a rank one map, but its rank as a completely positive map is four, as

$$\phi(A) = \frac{e_{11} A e_{11} + e_{12} A e_{21} + e_{21} A e_{12} + e_{22} A e_{22}}{2}.$$

We will frequently use the three equivalent formulations of complete positivity we have discussed.

All $*$-homomorphisms are completely positive, as are all positive linear functionals on C^*-algebras. A $*$-homomorphism ϕ is positive since $spec(\phi(A)) \subseteq spec(A)$ for all $A \in \mathfrak{U}$, and complete positivity now follows since ϕ_n is also a $*$-homomorphism for each $n \in \mathbb{N}$. The fact that every positive linear functional on a C^*-algebra $\mathfrak{U}$ is completely positive follows from the more general theorem that any positive map from $\mathfrak{U}$ into a commutative C^*-algebra $\mathfrak{B}$ is completely positive (Stinespring showed that this last statement is also true if we exchange $\mathfrak{U}$ with $\mathfrak{B}$; see Proposition

3.7 and Theorem 3.10 of [Pa]). We will soon meet another class of completely positive maps, the Schur maps associated with positive matrices.

1.2.6 Conditionally negative maps

Having discussed completely positive maps, we turn our attention to a similar notion:

Definition 4. *A self-adjoint linear map* $\psi : B(K) \to B(K)$ *is said to be* conditionally negative *if, whenever* $\sum_{i=1}^{m} A_i f_i = 0$ *for* $A_1, \ldots, A_m \in B(K)$*,* $f_1, \ldots, f_m \in K$*, and* $m \in \mathbb{N}$*, we have* $\sum_{i=1}^{m}(f_i, \psi(A_i^* A_j) f_j) \leq 0$.

Let $n = dim(K) < \infty$. From the literature we know that ψ has the form

$$\psi(A) = sA + YA + AY^* - \sum_{i=1}^{p} \lambda_i S_i A S_i^*,$$

where $s \in \mathbb{R}$, $tr(Y) = 0$, and for all i and j we have $\lambda_i > 0$, $tr(S_i) = 0$ and $tr(S_i^* S_j) = n\delta_{ij}$, where $p \leq n^4$ is independent of the maps S_i. We call p the rank of L *as a conditionally negative map.*

We note that this form for L is unique in the sense that if L is written in the form

$$\psi(A) = tA + ZA + AZ^* - \sum_{i=1}^{p} \mu_i T_i A T_i^*,$$

where $t \in \mathbb{R}$, $tr(Z) = 0$, and for all i and j we have $\mu_i > 0$, $tr(T_i) = 0$, and $tr(T_i^* T_j) = n\delta_{ij}$, then $s = t$, $Z = Y$, and $\sum_{i=1}^{p} \lambda_i S_i A S_i^* = \sum_{i=1}^{p} \mu_i T_i A T_i^*$ for all $A \in B(K)$. Indeed, let $\{v_k\}_{k=1}^{n}$ be any orthonormal basis for K, let $h_k = v_k/\sqrt{n}$ for each k, let $f \in K$ be arbitrary, and for $k = 1, \ldots, n$ define $A_k \in B(K)$ by $A_k = f h_k^*$. Using the trace conditions, we find

$$\begin{aligned}
\sum_{k=1}^{n} \psi(A_k) h_k &= \sum_{k=1}^{n}(h_k, h_k) sf + \sum_{k=1}^{n}(h_k, h_k) Yf + \sum_{k=1}^{n}(h_k, Y^* h_k) Af \\
&\quad - \sum_{k=1}^{n} \Big(\sum_{i=1}^{p} \lambda_i (h_k, S_i^* h_k) S_i f \Big) \\
&= sf + Yf + 0 - \sum_{i=1}^{p} \Big(\sum_{k=1}^{n} \lambda_i (h_k, S_i^* h_k) S_i f \Big) \\
&= sf + Yf - \sum_{i=1}^{p} \lambda_i (0) S_i f = sf + Yf.
\end{aligned}$$

An analogous computation shows that $\sum_{k=1}^{n} \psi(A_k)h_k = tf + Zf$. Since $f \in K$ was arbitrary, we conclude $(t-s)I = Y - Z$. Therefore, $tr((t-s)I) = tr(Y-Z) = 0$, so $t = s$ and $Y = Z$. Consequently, $\sum_{i=1}^{p} \lambda_i S_i A S_i^* = \sum_{i=1}^{p} \mu_i T_i A T_i^*$ for all $A \in B(K)$. This uniqueness will come in handy several times later.

What is the connection between conditional negativity and complete positivity? A result of Evans and Lewis (see [EL]) tells us that if ψ is conditionally negative, then $e^{-\psi}$ is completely positive. This fact will play a vital role in allowing us to establish the connection between unital invertible q-positive maps and conditionally negative maps on $M_n(\mathbb{C})$.

1.2.7 Schur maps

The first time each of us saw matrices, we likely thought that the way to multiply $n \times n$ matrices A and B was to proceed in the obvious manner:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \bullet \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & \cdots & a_{1n}b_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n}b_{1n} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

Our teacher then likely told us how wrong we were to multiply matrices in this way. As the years go by, many of us shudder at the thought of this memory, having long ago failed to consider the possibility that this dazzlingly simple method of matrix multiplication could be mathematically meaningful. Until, that is, the day when we learn that the above is called the Schur product of A with B, and that its impact on mathematics is substantial (for some of us, today *is* that day).

After all, let us consider the map $\phi_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\phi_A(B) = A \bullet B$, the *Schur map* associated with A. This is clearly a linear map. Moreover, if A is a positive matrix, then ϕ_A is completely positive. With this result at hand, we see that Schur maps provide a plethora of relatively uncomplicated completely positive maps. They actually provide, up to change of basis, *all* unital invertible completely positive maps on $M_n(\mathbb{C})$ that enjoy the additional property of being q-pure (see Definition 9 and Theorem 9).

1.2.8 CP-semigroups on matrix algebras

A theorem of Bhat in [Bh] allows us to generate E_0-semigroups from strongly continuous completely positive semigroups of unital maps on $B(H)$, called CP-semigroups. We state a reformulation of Bhat's theorem from [P2]:

Theorem 2. *Suppose α is a unital CP-semigroup of $B(H)$. Then there is an E_0-semigroup α^d of $B(K)$ and an isometry $W : H \to K$ such that*

$$\alpha_t(A) = W^* \alpha_t^d(WAW^*)W$$

and $\alpha_t(WW^*) \geq WW^*$ for all $t > 0$. If the projection $E = WW^*$ is minimal in that the closed linear span of the vectors

$$\alpha_{t_1}^d(EA_1E) \cdots \alpha_{t_n}^d(EA_nE)Ef$$

for $f \in K, A_i \in B(H)$ and $t_i \geq 0$ for all $i = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ is H (in which case we say α is minimal), then α^d is unique up to conjugacy.

Whereas E_0-semigroups are very complicated (we had to toil through the CAR and CCR algebras earlier just to find basic type I examples), CP-semigroups can be incredibly simple. Take, for example, the semigroup of Schur maps $\alpha = \{\alpha_t\}_{t \geq 0}$ on $M_2(\mathbb{C})$ defined by

$$\alpha_t \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & e^{-t}a_{12} \\ e^{-t}a_{21} & a_{22} \end{pmatrix}.$$

Positivity of the matrix $e_{11} + e^{-t}(e_{12} + e_{21}) + e_{22}$ for all $t \geq 0$ implies that the mappings α_t are completely positive, so α is a CP-semigroup. By Bhat's dilation theorem, α dilates minimally to some E_0-semigroup α^d. What can we say about α^d? The answer is that we can tell everything about α^d from the generator ∂ of α, which is the map

$$\partial A = \lim_{t \to 0} \frac{\alpha_t(A) - A}{t}.$$

Powers showed in [P3] that α^d (where α was any CP-semigroup acting on an $n \times n$ matrix algebra) is type I_n, where n is the rank of ∂ as a conditionally negative map (see Definition 4 and the remark following it). At almost exactly the same time, Arveson obtained an even stronger result, as he proved in [A3] that every CP-semigroup α whose generator is bounded minimally dilates to a type I E_0-semigroup.

1.2.9 CP-flows

We have now reached the final section of the background, which, coincidentally, is the framework around which many of our thesis results are built. In perhaps the most herculean effort of his mathematical career, Powers (in [P1]) used Bhat's dilation theorem to concoct a method through which every (non-trivial) spatial E_0-semigroup could be obtained from dilating CP-flows. As before, let K be a separable Hilbert space and let $H = K \otimes L^2(0, \infty)$. We identify H with $L^2((0, \infty); K)$, the space of K-valued measurable functions on $(0, \infty)$ which are square integrable. Under this identification, the inner product on H is

$$(f, g) = \int_0^\infty (f(x), g(x))dx.$$

Let U_t be the right shift on H, so for $f \in H$ we have $(U_t f)(x) = f(x-t)$ for $x > t$ and $(U_t f)(x) = 0$ otherwise. Let $\Lambda : B(K) \to B(H)$ be the map $(\Lambda(A)f)(x) = e^{-x}Af(x)$.

Definition 5. *Assume the notation of the above paragraph. A strongly continuous semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of completely positive maps of $B(H)$ into itself is a CP-flow if $\alpha_t(A)U_t = U_t A$ for all $A \in B(H)$.*

A result of Powers (Theorem 4.0A of [P1]) shows that every spatial E_0-semigroup acting on $B(H)$ (for H a separable Hilbert space) is cocycle conjugate to an E_0-semigroup which is a CP-flow, and that every CP-flow over K arises from a *boundary weight map.* The boundary weight map $\rho \to \omega(\rho)$ of a CP-flow α associates to every $\rho \in B(K)_*$ a weight $\omega(\rho)$ on $B(K \otimes L^2(0, \infty))$

so that the functional $\ell(\rho)$ on $B(K \otimes L^2(0,\infty))$ defined by

$$\ell(\rho)(A) = \omega(\rho)\Big((I - \Lambda(I))^{1/2} A (I - \Lambda(I))^{1/2}\Big)$$

satisfies $\ell(\rho) \in B(K \otimes L^2(0,\infty))_*$.

If $\omega(\rho)(I - \Lambda(I)) = \rho(I)$ for all $\rho \in B(K)_*$, then α is unital.

Given any CP-flow α over K, there is a unique family of completely positive contractions $\pi^\# = \{\pi_t^\# : t > 0\}$, called the *generalized boundary representation* of α. Its relationship to the boundary weight map is as follows. Defining maps $\rho \to \omega_t(\rho)$ by

$$\omega_t(\rho)(A) = \omega(\rho)\Big(U_t U_t^* A U_t U_t^*\Big), \tag{1.2.5}$$

we have $\hat{\pi}_t = \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$. The maps $\{\pi_b^\#\}_{b>0}$ have a σ-strong limit $\pi_0^\#$ as $b \to 0$ on $\bigcup_{t>0} U_t B(H) U_t^*$, called the *normal spine* of α. If α is unital, then the index of α^d as an E_0-semigroup is equal to the rank of $\pi_0^\#$ as a completely positive map (Theorem 4.49 of [P1]). Powers has determined when a boundary weight map $\rho \to \omega(\rho)$ gives rise to a CP-flow (see Theorem 3.3 of [P2]):

Theorem 3. *If $\rho \to \omega(\rho)$ is a completely positive mapping from $B(K)_*$ into weights on $B(H)$ satisfying $\omega(\rho)(I - \Lambda(I)) \leq \rho(I)$ for all positive ρ, and if the maps $\hat{\pi}_t := \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ are completely positive contractions from $B(K)_*$ into $B(H)_*$ for all $t > 0$, then $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow over K.*

The subordinates of a CP-flow are entirely determined by the subordinates of its generalized boundary representation (Theorem 3.4 of [P2]):

Theorem 4. *Let α and β be CP-flows over K with generalized boundary representations $\pi^\# = \{\pi_t^\#\}$ and $\xi^\# = \{\xi_t^\#\}$, respectively. Then β is subordinate to α (that is, $\alpha_t - \beta_t$ is completely positive for all $t \geq 0$, and we write $\alpha \geq \beta$) if and only if $\pi_t^\# - \xi_t^\#$ is completely positive for all $t > 0$.*

Powers investigated boundary weight maps when $dim(K) = 1$ in [P2]. In this case, the boundary weight map is just $c \in \mathbb{C} \to \omega(c) = c\omega(1)$. We may therefore simply view our boundary weight map as a single boundary weight $\omega := \omega(1)$ acting on $B(L^2(0,\infty))$. If ω is unbounded, then it is a non-normal (see page 9 of [MP]) functional defined on the null boundary algebra $\mathcal{U}(L^2(0,\infty)) = \sqrt{I - \Lambda(1)}B(L^2(0,\infty))\sqrt{I-\Lambda(1)}$. Theorem 3.10 of [P2] tells us the following:

Theorem 5. *Suppose ω is a boundary weight. Then ω is of the form*

$$\omega(A) = \sum_{k \in J} (g_k, (I - \Lambda(1))^{-1/2} A (I - \Lambda(1))^{-1/2} g_k)$$

for $A \in \bigcup_{t>0} U_t^* B(L^2(0,\infty))U_t$, where $g_k \in L^2(0,\infty)$ for each $k \in J$ and the g_k are mutually orthogonal with $\sum_{k\in J} ||g_k||^2 \leq 1$.

If ω is a boundary weight on $B(L^2(0,\infty))$, then its associated generalized boundary representation satisfies

$$\pi_t^{\#}(A) = \frac{\omega_t(A)}{1 + \omega_t(\Lambda(1))} \tag{1.2.6}$$

for all $t > 0$. We say ω is *normalized* if $\omega(I - \Lambda(1)) = 1$. An examination of (1.2.6) yields the following: If ω is bounded, then from Theorems 4.27 and 4.49 of [P1] it induces a type $\mathrm{I}_{card(J)}$ E_0-semigroup, while if ω is unbounded, then it induces a type II_0 E_0-semigroup. By Theorem 4, if ν and μ are boundary weights on $B(L^2(0,\infty))$ that give rise to CP-flows α and β, we see from formula (1.2.6) that $\alpha \geq \beta$ if and only if

$$\frac{\nu_t}{1 + \nu_t(\Lambda(1))} - \frac{\mu_t}{1 + \mu_t(\Lambda(1))}$$

is completely positive for all $t > 0$ (we write $\nu \geq_q \mu$). We will use Theorem 4 later to obtain a similar result for finding the subordinates of a CP-flow induced by a particular kind of boundary weight double (ϕ, ν) (see Lemma 2).

We will use unbounded boundary weights on $B(L^2(0,\infty))$, but with slightly different notation. If ν is a normalized unbounded boundary weight on $B(L^2(0,\infty))$ of the form

$$\nu(A) = (g, (I - \Lambda(1))^{-1/2} A (I - \Lambda(1))^{-1/2} g)$$

for $A \in \mathcal{U}(L^2(0,\infty))$, let $f = (I - \Lambda(1))^{-1/2}g$, so $f : (0,\infty) \to \mathbb{C}$ is the function

$$f(x) = \frac{g(x)}{\sqrt{1 - e^{-x}}}.$$

Now $f \in L^2(t,\infty)$ for all $t > 0$, but unboundedness of ν implies that $f \notin L^2(0,\infty)$. Furthermore, we have

$$\begin{aligned} \nu(A) &= (g, (I - \Lambda(1))^{-1/2} A (I - \Lambda(1))^{-1/2} g) \\ &= ((I - \Lambda(1))^{1/2} f, (I - \Lambda(1))^{-1/2} A f) = (f, Af). \end{aligned}$$

for all $A \in \mathcal{U}(L^2(0,\infty))$. Therefore, ν has the form $\nu(A) = (f, Af)$, and $||(I - \Lambda(1))^{1/2} f|| = \nu(I - \Lambda(1)) = 1$. We could also go backwards: Let f be any Lebesgue-measurable defined on $(0,\infty)$ which is in $L^2(t,\infty)$ for all $t > 0$ but satisfies $f \notin L^2(0,\infty)$, with the further property that $||(I - \Lambda(1))^{1/2} f|| = 1$. Then we can define a (normalized) unbounded boundary weight ν on $\mathcal{U}(L^2(0,\infty))$ by $\nu(A) = (f, Af)$. In the notation of [P2], ν is the boundary weight $\nu(A) = (g, (I - \Lambda(1))^{-1/2} A (I - \Lambda(1))^{-1/2} g)$ for the function $g := (I - \Lambda(1))^{1/2} f \in L^2(0,\infty)$.

In summary, the normalized unbounded boundary weights on $B(L^2(0,\infty))$ of the form $\nu(A) = (g, (I - \Lambda(1))^{-1/2} A (I - \Lambda(1))^{-1/2} g)$ are precisely the boundary weights of the form $\nu(A) = (f, Af)$ for Lebesgue-measurable functions f such that $f \in L^2(t,\infty)$ for all $t > 0$, $f \notin L^2(0,\infty)$, and $\int_0^\infty (1 - e^{-x}) |f(x)|^2 dx = 1$.

To define boundary weight maps $\rho \to \omega(\rho)$ for $\rho \in (B(K \otimes L^2(0,\infty))_*$ and $dim(K) > 1$, we will use *boundary weight doubles* (ϕ, ν), where $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital completely positive map and ν is a normalized unbounded boundary weight on $B(L^2(0,\infty))$ of the form $\nu(B) = (f, Bf)$ (see Proposition 1). We find that there is an additional assumption for ϕ which results in a necessary and sufficient condition for $\omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ to give rise to a CP-flow, namely that $\phi(I + t\phi)^{-1}$ be completely positive for all $t > 0$ (we say ϕ is q-positive). This condition is far from redundant, as completely positive maps can have negative eigenvalues. As one example, take the Schur map ϕ

on $M_2(\mathbb{C})$ defined by

$$\phi \left(\begin{array}{cc} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{array} \right) = \left(\begin{array}{cc} a_{11} & -a_{12} \\ \\ -a_{21} & a_{22} \end{array} \right).$$

It is also a good exercise to show that a completely positive map with nonnegative spectrum need not be q-positive either (try some more of your favorite Schur maps).

In [P2], Powers defined corners, q-corners, and hypermaximal q-corners between boundary weights on $B(L^2(0,\infty))$. These serve as the way to test whether such two boundary weights induce cocycle conjugate E_0-semigroups. In Definition 10, we make analogous definitions that allow us to test cocycle conjugacy of Bhat minimal dilations of CP-flows arising from certain boundary weight doubles (ϕ,ν) and (ψ,μ) (see Definition 10).

Chapter 2

Our boundary weight map

2.1 q-positivity and our boundary weight construction

We begin with a definition mentioned in the background:

Definition 6. *Let K be a separable Hilbert space. We say a linear map $\phi : B(K) \to B(K)$ is* $q-$positive *if $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$.*

Henceforth, we naturally identify a finite-dimensional Hilbert space K with $\mathbb{C}^n$ and $B(K \otimes L^2(0,\infty))$ with $M_n(B(L^2(0,\infty))$. Under these identifications, the right shift t units on $K \otimes L^2(0,\infty)$ is the matrix whose ij^{th} entry is $\delta_{ij}V_t$ for V_t the right shift on $L^2(0,\infty)$. The map $\Lambda_{n\times n} : B(K) \to B(K \otimes L^2(0,\infty))$ sends the $n \times n$ matrix $B = (b_{ij})$ to the matrix $\Lambda_{n\times n}(B)$ whose ij^{th} entry is the operator $\Lambda_1 : \mathbb{C} \to B(L^2(0,\infty))$ defined by $\Lambda_1(c)(f)(x) = ce^{-x}f(x)$. Given a boundary weight ν acting on $B(L^2(0,\infty))$, we write $\Omega_{\nu,n\times k}$ as the map that sends an $n \times k$ matrix $A = (A_{ij}) \in M_{n\times k}(B(L^2(0,\infty))$ to the matrix $\Omega_{\nu,n\times k}(A) \in M_{n\times k}(\mathbb{C})$ whose ij^{th} entry is $\nu(A_{ij})$.

We will suppress the integers n and k when they are clear, merely writing the above maps as Ω_ν and Λ. In the proposition and corollary that follow, we show that one can construct a CP-flow using a $q-$positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, a boundary weight ν on $L^2(0,\infty)$ such that

$\nu(I_{L^2(0,\infty)} - \Lambda(1)) = 1$ (we say ν is *normalized*), and the map $\Omega_\nu := \Omega_{\nu,n\times n} : M_n(B(L^2(0,\infty))) \to M_n(\mathbb{C})$. The map Ω_ν is completely positive since ν is positive.

Proposition 1. *Let $H = \mathbb{C}^n \otimes L^2(0,\infty)$. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital completely positive map, and let ν be a normalized unbounded boundary weight on $L^2(0,\infty)$. Define a map ω from $M_n(\mathbb{C})^*$ into weights on $B(H)$ by*

$$\omega(\rho)(A) = \rho(\phi(\Omega_\nu(A))).$$

Then the map $\rho \to \omega(\rho)$ is completely positive. Furthermore, the maps $\hat{\pi}_t := \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ are completely positive contractions of $M_n(\mathbb{C})_*$ into $B(H)_*$ for all $t > 0$ if and only if ϕ is $q-$positive.

Proof: The map $\rho \to \omega(\rho)$ is completely positive since it is the composition of two completely positive maps. Before proving either direction, we prove the equality

$$\pi_t^{\#} = \phi(I + s_t\phi)^{-1} \circ \Omega_{\nu_t}. \tag{2.1.1}$$

Letting $s_t = \nu_t(\Lambda(1))$ and denoting by U_t the right shift on H for every $t > 0$, we claim that $(I + \hat{\Lambda}\omega_t)^{-1} = (I + s_t\hat{\phi})^{-1}$. Indeed, for arbitrary $t > 0$, $B \in M_n(\mathbb{C})$, and $\rho \in M_n(\mathbb{C})^*$, we have

$$\hat{\Lambda}\omega_t(\rho)(B) = \rho\Big(\phi\Big(\Omega_\nu(U_tU_t^*\Lambda(B)U_tU_t^*)\Big)\Big) = \rho\Big(\phi(\Omega_{\nu_t}(\Lambda(B))\Big) = s_t\rho(\phi(B)),$$

hence $\hat{\Lambda}\omega_t = s_t\hat{\phi}$ and $(I + \hat{\Lambda}\omega_t)^{-1} = (I + s_t\hat{\phi})^{-1}$.

For any $t > 0$ and $A \in H$, we have

$$\begin{aligned}
\rho(\pi_t^{\#}(A)) &= \hat{\pi}_t(\rho)(A) = \omega_t(I + \hat{\Lambda}\omega_t)^{-1}(\rho)(A) = \Big((I + \hat{\Lambda}\omega_t)^{-1}(\rho)\Big)(\phi(\Omega_{\nu_t}(A))) \\
&= \Big((I + s_t\hat{\phi})^{-1}(\rho)\Big)(\phi(\Omega_{\nu_t}(A))) = \rho\Big((I + s_t\phi)^{-1}\phi(\Omega_{\nu_t}(A))\Big) \\
&= \rho\Big(\phi(I + s_t\phi)^{-1}(\Omega_{\nu_t}(A))\Big),
\end{aligned}$$

establishing (2.1.1).

Assume the hypotheses of the backward direction and let $t > 0$. Then $\pi_t^{\#}$ is the composition of completely positive maps by (2.1.1) and is thus completely positive. Furthermore, $\pi_t^{\#}$ is a contraction since

$$
\begin{aligned}
||\pi_t^{\#}|| &= ||\pi_t^{\#}(I_H)|| = ||\phi(I+s_t\phi)^{-1}(\Omega_{\nu_t}(I_H))|| \\
&= \Big|\Big|\phi(I+s_t\phi)^{-1}\Big(\nu_t(I_{L^2(0,\infty)})I_K\Big)\Big|\Big| \\
&= \Big|\Big|\frac{\nu_t(I_{L^2(0,\infty)})}{1+s_t}I_K\Big|\Big| = \frac{\nu_t(I_{L^2(0,\infty)})}{1+\nu_t(\Lambda(1))} \leq 1,
\end{aligned}
$$

where the last inequality follows from the fact that

$$\nu_t(I_{L^2(0,\infty)} - \Lambda(1)) \leq \nu(I_{L^2(0,\infty)} - \Lambda(1)) = 1.$$

Now assume the hypotheses of the forward direction. By unboundedness of ν, the (monotonically decreasing) values $\{s_t\}_{t>0}$ form a set equal to either $(0,\infty)$ or $[0,\infty)$. Choose any $t > 0$ such that $s_t > 0$. Let $T \in B(H)$ be the matrix whose ij^{th} entry is $\frac{1}{\nu_t(I)}I_{L^2(0,\infty)}$, and let $\kappa_t : M_n(\mathbb{C}) \to B(H)$ be the map that sends $B = (b_{ij}) \in M_n(\mathbb{C})$ to the matrix $\kappa_t(B) \in B(H)$ whose ij^{th} entry is $\frac{b_{ij}}{\nu_t(I_{L^2(0,\infty)})}I_{L^2(0,\infty)}$. We note that κ_t is the generalized Schur product $B \to B \cdot T$, which is completely positive since T is positive. For all $B \in M_n(\mathbb{C})$ we have

$$\phi(I+s_t\phi)^{-1}(B) = \pi_t^{\#}(\kappa_t(B)),$$

so $\phi(I+s_t\phi)^{-1}$ is the composition of completely positive maps and is thus completely positive. As noted above, the values $\{s_t\}_{t>0}$ span $(0,\infty)$, so ϕ is q-positive.

□

2.1.1 Boundary weight doubles, and type

Corollary 1. *The map $\rho \to \omega(\rho)$ in Proposition 1 is the boundary weight map of a CP-flow α over $\mathbb{C}^n$, and the Bhat minimal dilation α^d of α is a type II_0 E_0-semigroup.*

Proof: The first claim of the corollary follows immediately from Theorem 3 and Proposition 1 since

$$\omega(\rho)(I - \Lambda(I)) = \rho(\phi(I)) = \rho(I)$$

for all $\rho \in M_n(\mathbb{C})^*$. For the second assertion, we note that by Theorem 4.49 of [P1], the index of α^d is equal to the rank of the normal spine $\pi_0^\#$ of α, where $\pi_0^\#$ is the σ-strong limit of the maps $\{\pi_b^\#\}_{b>0}$ on $\bigcup_{t>0} U_t B(H) U_t$. Fix $t > 0$, and let $A \in U_t B(H) U_t^*$. From formula (2.1.1),

$$\pi_b^\#(A)) = \phi(I + \nu_b(\Lambda(1))\phi)^{-1}(\Omega_{\nu_b}(A)).$$

For all $b < t$ we have $||\Omega_{\nu_b}(A)|| = ||\Omega_{\nu_t}(A)|| < \infty$. Since $\nu_b(\Lambda(1)) \to \infty$ as $b \to 0$, we conclude $\lim_{b\to 0} ||\pi_b^\#(A)|| = 0$, hence $\pi_0^\# = 0$ and the index of α is zero. However, α^d is not completely spatial since α is not the CP-flow derived from the zero weight (see Lemma 4.37 and Theorem 4.52 of [P1]), so α^d is of type II_0.

□

Given a $q-$positive $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and a normalized unbounded boundary weight ν acting on $B(L^2(0,\infty))$, we call (ϕ, ν) a *boundary weight double* and note that it gives rise to a unique (up to cocycle conjugacy) type II_0 E_0-semigroup. Indeed, the maps ϕ, ν, and Ω_ν (the last of which depends only on ϕ and ν) define a boundary weight map ω as in Proposition 1 which, by Corollary 1, gives rise to a CP-flow α that minimally dilates to a unique (up to conjugacy) E_0-semigroup α^d.

2.2 Comparison theory for dilated CP-flows

We have a criterion for determining whether two E_0-semigroups induced by CP-flows are cocycle conjugate (see Lemma 3.8 of [P1]):

Theorem 6. *Let α and β be E_0-semigroups on $B(H_1)$ and $B(H_2)$, respectively. Then α and β are cocycle conjugate if and only if there is a collection of linear maps $\gamma = \{\gamma_t : t \geq 0\}$ from $B(H_2, H_1)$ into itself such that the family of maps $\Theta = \{\Theta_t : t \geq 0\}$ defined by*

$$\Theta_t \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := \begin{pmatrix} \alpha_t(A_{11}) & \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) & \beta_t(A_{22}) \end{pmatrix}$$

is an E_0-semigroup of $B(H_1 \oplus H_2)$.

The following definition can be found in [P1]:

Definition 7. *Let α and β be CP-flows over K_1 and K_2. We say that a family of σ-weakly continuous linear maps $\sigma = \{\sigma_t : t \geq 0\}$ from $B(H_2, H_1)$ into itself is a flow corner from α to β if the family of maps $\Theta = \{\Theta_t : t \geq 0\}$ defined by*

$$\Theta_t \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := \begin{pmatrix} \alpha_t(A_{11}) & \sigma_t(A_{12}) \\ \sigma_t^*(A_{21}) & \beta_t(A_{22}) \end{pmatrix}$$

is a CP-flow over $K_1 \oplus K_2$.

If σ is a flow corner from α to β, we consider subordinates Θ' of Θ that are CP-flows of the form

$$\Theta_t \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := \begin{pmatrix} \alpha'_t(A_{11}) & \sigma_t(A_{12}) \\ \sigma_t^*(A_{21}) & \beta'_t(A_{22}) \end{pmatrix}.$$

We call γ a maximal flow corner from α to β if, for every such subordinate Θ' of Θ, we have $\alpha' = \alpha$. We say that γ is a hyper maximal flow corner from α to β if, for every such subordinate Θ' of Θ, we have $\alpha = \alpha'$ and $\beta = \beta'$.

Powers has shown (Theorem 4.56 of [P1]) that if CP-flows α and β both dilate to type II_0 E_0-semigroups, then α^d and β^d are cocycle conjugate if and only if there is a hypermaximal flow corner σ from α to β. We will later use this theorem to determine a necessary and sufficient condition for E_0-semigroups arising from a particular kind of boundary weight double to be cocycle conjugate (see Definition 10 and Lemma 7). Motivated by [P1], we make the following definition:

Definition 8. *Suppose $\alpha : B(H_1) \to B(K_1)$ and $\beta : B(H_2) \to B(K_2)$ are completely positive maps. We say a linear map $\gamma : B(H_2, H_1) \to B(K_2, K_1)$ is a corner from α to β if the matrix*

$$\psi = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}$$

is completely positive.

The following lemma gives us the form of any corner between completely positive maps of finite index. We have been informed that this result (or a nearly identical one) may have already been proved in the literature, in which case we present a proof here for the sake of completeness:

Lemma 1. *Let H_1, H_2, K_1, and K_2 be separable Hilbert spaces. Let $\alpha : B(H_1) \to B(K_1)$ and $\beta : B(H_2) \to B(K_2)$ be completely positive contractions of the form*

$$\alpha(A_{11}) = \sum_{i=1}^{n} S_i A_{11} S_i^*, \beta(A_{22}) = \sum_{j=1}^{p} T_j A_{22} T_j^*,$$

where $n, p \in \mathbb{N}$. Then $\gamma : B(H_2, H_1) \to B(K_2, K_1)$ is a corner from α to β if and only if for all $A_{12} \in B(H_2, H_1)$ we have

$$\gamma(A_{12}) = \sum_{ij} c_{ij} S_i A_{12} T_j^*,$$

where $C = (c_{ij}) \in M_{n\times p}(\mathbb{C})$ is any matrix such that $||C|| \leq 1$.

Proof: For the backward direction, let $C = (c_{ij}) \in M_{n\times p}(\mathbb{C})$ be any contraction, and define a linear map $\gamma : B(H_2, H_1) \to B(K_2, K_1)$ by $\gamma(A) = \sum_{i,j} c_{ij} S_i A T_j^*$. We need to show that the map

$$\Upsilon = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}$$

is completely positive. To prove this, we first assume that $n \geq p$ and note that by Polar Decomposition we may write $C_{n\times p} = V_{n\times p} T_{p\times p}$, where $V_{n\times p}$ is a partial isometry of rank p and T is positive. Unitarily diagonalizing T we see $C_{n\times p} = V_{n\times p} W^*_{p\times p} D_{p\times p} W_{p\times p}$. We may easily add columns to $V_{n\times p} W^*_{p\times p}$ to form a unitary matrix in $M_n(\mathbb{C})$; we call this matrix U^*. Defining $\tilde{D}$ to be the matrix obtained from D by adding $n - p$ rows of zeroes, we see $U^*\tilde{D} = V_{n\times p} W^*_{p\times p} D$, so $C_{n\times p} = U^*\tilde{D} W_{p\times p}$ and

$$U C_{n\times p} W^*_{p\times p} = \tilde{D}.$$

In other words,

$$\sum_{i,j} c_{ij} u_{ki} \overline{w_{\ell j}} = \left\{ \begin{array}{ll} \delta_{k\ell} d_{k\ell} & \text{if } k \leq p \\ 0 & \text{if } k > p \end{array} \right\}.$$

Next, define $\{S_i'\}_{i=1}^n : H_1 \to K_1$ and $\{T_j\}_{j=1}^p : H_2 \to K_2$ by

$$S_i' = \sum_{k=1}^{n} \overline{u_{ik}} S_k, \quad T_j' = \sum_{\ell=1}^{p} \overline{w_{j\ell}} T_j,$$

so $S_i = \sum_{k=1}^n u_{ki} S_k'$ and $T_j = \sum_{\ell=1}^p w_{\ell j} T_j'$ for all i and j.

Since the $n \times n$ matrix U and the $p \times p$ matrix W are unitary, it follows that the maps $\{S_i'\}_{i=1}^n$ are linearly independent, as are the maps $\{T_j'\}_{j=1}^p$. We observe that for any $A_{11} \in B(H_1)$ and $A_{22} \in B(H_2)$,

$$\sum_{i=1}^{n} S_i A_{11} S_i^* = \sum_{i=1}^{n} S_i' A_{11} (S_i')^* \text{ and } \sum_{j=1}^{p} T_j A_{22} T_j^* = \sum_{j=1}^{p} T_j' A_{22} (T_j')^*.$$

Finally, for any $A_{12} \in B(H_2, H_1)$, we find

$$\begin{aligned}
\sum_{i,j} c_{ij} S_i A T_j^* &= \sum_{i,j,k,\ell} c_{ij} u_{ki} \overline{w_{\ell j}} S_k' A (T_\ell')^* = \sum_{k,\ell} \Big(\sum_{i,j} c_{ij} u_{ki} \overline{w_{\ell j}} S_k' A (T_\ell')^* \Big) \\
&= \sum_{(k \le p),\ell} \Big(\sum_{i,j} c_{ij} u_{ki} \overline{w_{\ell j}} S_k' A (T_\ell')^* \Big) + \sum_{(k>p),\ell} \Big(\sum_{i,j} c_{ij} u_{ki} \overline{w_{\ell j}} S_k' A (T_\ell')^* \Big) \\
&= \sum_{k \le p} d_{kk} S_k' A (T_k')^* + 0 = \sum_{k=0}^{p} d_{kk} S_k' A (T_k')^*,
\end{aligned}$$

where $||D|| = ||C||$ since U and W are unitaries in $M_q(\mathbb{C})$ and $M_r(\mathbb{C})$, respectively. We have shown that

$$\begin{pmatrix} \sum_{i=1}^n S_i A_{11} S_i^* & \sum_{i,j} c_{ij} S_i A_{12} T_j^* \\ \sum_{i,j} \overline{c_{ij}} T_j A_{21} S_i^* & \sum_{i=1}^p T_i A_{22} T_i^* \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n S_i' A_{11} (S_i')^* & \sum_{i=1}^p d_{ii} S_i' A_{12} (T_i')^* \\ \sum_{i=1}^p \overline{d_{ii}} T_i' A_{21} (S_i')^* & \sum_{i=1}^p T_i' A_{22} (T_i')^* \end{pmatrix}$$

for all

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in B(H_1 \oplus H_2).$$

For each $i = 1, \ldots, p$, define $Z_i : H_1 \oplus H_2 \to K_1 \oplus K_2$ by

$$Z_i = \begin{pmatrix} d_{ii} S_i & 0 \\ 0 & T_i \end{pmatrix},$$

so

$$\Upsilon(A) = \sum_{i=1}^{p} Z_i A Z_i^* + \sum_{i=1}^{p} \begin{pmatrix} (1-|d_{ii}|^2) S_i A_{11} S_i^* & 0 \\ 0 & 0 \end{pmatrix}.$$

The first sum on the previous line is obviously completely positive, and the second sum is completely positive since $||D|| \leq 1$. We conclude that Υ is completely positive, hence γ is a corner from α to β. The proof in the case that $n < p$ is analogous.

For the forward direction, suppose that γ is a corner from α to β. Define $\Upsilon : B(H_1 \oplus H_2) \to B(K_1 \oplus K_2)$ by

$$\Upsilon = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix},$$

so

$$\Upsilon \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} S_i A_{11} S_i^* & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \sum_{j=1}^{p} T_j A_{22} T_j^* \end{pmatrix}.$$

is completely positive. Therefore, for some $q \in \mathbb{N} \cup \{\infty\}$ and maps $Y_i : H_1 \oplus H_2 \to K_1 \oplus K_2$ for $i = 1, 2, ...$, linearly independent over $\ell_1(\mathbb{N})$, we have

$$\Upsilon(\tilde{A}) = \sum_{i=1}^{q} Y_i \tilde{A} Y_i^*$$

for all $A \in B(H_1 \oplus H_2, K_1 \oplus K_2)$. For $i = 1, 2$, let $E_i \in B(H_1 \oplus H_2)$ be projection onto H_i, and let $F_i \in B(K_1 \oplus K_2)$ be projection onto K_i. Since α and β are contractions we have $\Upsilon(E_1) \leq F_1$ and $\Upsilon(E_2) \leq F_2$, so $Y_i E_j Y_i^* \leq F_j$ for each i and j. It follows that each Y_i, $i = 1, \ldots q$, can be written in the form

$$Y_i = \begin{pmatrix} \tilde{S}_i & 0 \\ 0 & \tilde{T}_i \end{pmatrix}$$

for some $\tilde{S}_i \in B(H_1, K_1)$ and $\tilde{T}_i \in B(H_2, K_2)$.

Since $\alpha(A_{11}) = \sum_{i=1}^{q} \tilde{S}_i A_{11} \tilde{S}_i^* = \sum_{i=1}^{n} S_i A_{11} S_i^*$ for all $A_{11} \in B(H_1)$, we know from the work of Arveson (see [A4]) that the maps $\{S_1', \ldots, S_q'\}$ and $\{S_1, \ldots, S_n\}$ span the same space $\mathcal{E}_\alpha$ in the sense that

$$\mathcal{E}_\alpha = \Big\{ \sum_{i=1}^{q} \lambda_i S_i' : \lambda_i \in \mathbb{C} \Big\} = \Big\{ \sum_{i=1}^{n} u_i S_i : u_i \in \mathbb{C} \Big\}.$$

Similarly, the maps $\{\tilde{T}_j\}_{j=1}^q$ and $\{T_j\}_{j=1}^p$ each span the same space E_β. Therefore, for each $i = 1, \ldots, q$ we have

$$\tilde{S}_i = \sum_{j=1}^n a_{ij} S_j, \quad \tilde{T}_i = \sum_{k=1}^p b_{ik} T_k$$

for some coefficients $\{a_{ij}\}$ and $B = \{b_{ij}\} \in M_{q\times p}(\mathbb{C})$. From linear independence of the maps $\{Y_i\}_{i=1}^q$, it is clear that $q \leq n+p$. Let $A = (a_{ij}) \in M_{q\times n}(\mathbb{C})$ and $B = (b_{ij}) \in M_{q\times p}(\mathbb{C})$. We calculate

$$\begin{aligned}\sum_{i=1}^n S_i A S_i^* &= \sum_{i=1}^q \tilde{S}_i A \tilde{S}_i^* = \sum_{i=1}^q \Big(\sum_{j,k=1}^n a_{ij}\overline{a_{ik}} S_j A S_k^* \Big) \\ &= \sum_{j,k=1}^n \Big(\sum_{i=1}^q a_{ij}\overline{a_{ik}} \Big) S_j A S_k^*.\end{aligned}$$

Let $M \in M_n(\mathbb{C})$ be the matrix with jk^{th} entry $\sum_{i=1}^q a_{ij}\overline{a_{ik}}$. As in the proof of the backward direction, we may "diagonalize" the map $A \to \sum_{i=1}^n S_i A S_i^* = \sum_{j,k=1}^n (\sum_{i=1}^q a_{ij}\overline{a_{ik}}) S_j A S_k^*$ as $A \to \sum_{i=1}^n S_i' A S_i'^* = \sum_{i=1}^n d_{ii} S_i' A S_i'^*$ for some linearly independent maps $\{S_i'\}_{i=1}^n$ and a matrix $D = (d_{ij}) \in M_n(\mathbb{C})$ with $||D|| = ||M||$. Clearly $D = I$, so $||M|| = 1$. But $M = A^T(A^T)^*$, hence $||A|| = 1$. The same argument shows that $||B|| = 1$.

Let

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in B(H_1 \oplus H_2)$$

be arbitrary. Let $C = (c_{jk}) \in M_{n\times p}(\mathbb{C})$ be the matrix $C = (B^*A)^T$, noting that $||C|| \leq 1$. A straightforward computation of $\Upsilon(\tilde{A}) = \sum_{i=1}^q Y_i \tilde{A} Y_i^*$ yields

$$\begin{aligned}\gamma(A_{12}) &= \sum_{i=1}^q S_i' A_{12} T_i'^* = \sum_{i=1}^q \Big(\Big(\sum_{j=1}^n a_{ij} S_j \Big) A_{12} \Big(\sum_{k=1}^p \overline{b_{ik}} T_k^* \Big) \Big) \\ &= \sum_{j,k} \Big(\sum_{i=1}^q a_{ij}\overline{b_{ik}} \Big) S_j A_{12} T_k^* = \sum_{j,k} c_{jk} S_j A_{12} T_k^*,\end{aligned}$$

hence γ is of the form claimed.

□

2.2.1 q-purity

Just as in the general study of various classes of linear operators, it is natural to impose, and examine, an order structure for q-positive maps. If ϕ and ψ are q-positive maps acting on $M_n(\mathbb{C})$, we say that ϕ q-dominates of ψ (and write $\phi \geq_q \psi$) if $\phi(I+t\phi)^{-1} - \psi(I+t\psi)^{-1}$ is completely positive for all $t \geq 0$. We would like to find the q-positive maps with the least complicated structure of q-subordinates. That last statement is not as simple as it seems. We think to define a $q-$positive map ϕ to be "$q-$pure" if $\phi \geq_q \psi \geq_q 0$ implies $\psi = \lambda\phi$ for some $\lambda \in [0,1]$, but there exist $q-$positive maps ϕ such that for every $\lambda \in (0,1)$ we have $\phi \ngeq_q \lambda\phi$. One such example is the Schur map ϕ on $M_2(\mathbb{C})$ given by

$$\phi\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & (\frac{1+i}{2})a_{12} \\ (\frac{1-i}{2})a_{21} & a_{22} \end{pmatrix}.$$

As it turns out, every q-positive map is guaranteed to have a one-parameter family of subordinates of a particular form:

Proposition 2. *Let $\phi \geq_q 0$. For each $s \geq 0$, define $\phi^{(s)} := \phi(I+s\phi)^{-1}$. Then $\phi^{(s)} \geq_q 0$ for all $s \geq 0$. Furthermore, the set $\{\phi^{(s)}\}_{s\geq 0}$ is a monotonically decreasing family of subordinates of ϕ, in the sense that $\phi^{(s_1)} \geq_q \phi^{(s_2)}$ if $s_1 \leq s_2$.*

Proof: For all $s \geq 0$ and $t \geq 0$, we have

$$\begin{aligned} \phi^{(s)}(I+t\phi^{(s)})^{-1} &= \phi(I+s\phi)^{-1}\Big(I+t\phi(I+s\phi)^{-1}\Big)^{-1} \\ &= \phi(I+s\phi)^{-1}\Big[\Big(I+(s+t)\phi\Big)(I+s\phi)^{-1}\Big]^{-1} = \phi(I+(s+t)\phi)^{-1}, \end{aligned}$$

which is completely positive by q-positivity of ϕ. Thus $\phi^{(s)} \geq_q 0$ for all $s \geq 0$.

To prove that $\phi^{(s_1)} \geq_q \phi^{(s_2)}$ if $s_1 \leq s_2$, we let $t \geq 0$ be arbitrary. Writing $t_1 = s_1 + t$ and

$t_2 = s_2 + t$, we observe

$$\begin{aligned}\phi^{(s_1)}(I+t\phi^{(s_1)})^{-1} - \phi^{(s_2)}(I+t\phi^{(s_2)})^{-1} &= \phi(I+t_1\phi)^{-1} - \phi(I+t_2\phi)^{-1} \\ &= (I+t_2\phi)^{-1}\Big((I+t_2\phi)\phi - \phi(I+t_1\phi)\Big)(I+t_1\phi)^{-1} \\ &= (I+t_2\phi)^{-1}\Big((t_2-t_1)\phi^2\Big)(I+t_1\phi)^{-1} \\ &= (t_2-t_1)\Big(\phi(I+t_2\phi)^{-1}\Big)\Big(\phi(I+t_1\phi)^{-1}\Big).\end{aligned}$$

The last line is a non-negative multiple of a composition of completely positive maps and is thus completely positive.

□

We now have the correct notion of what it means to be q-pure:

Definition 9. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital and q-positive. We say that ϕ is* q-pure *if its set of $q-$subordinates is precisely $\{0\} \cup \{\phi^{(s)}\}_{s\geq 0}$.*

2.2.2 Hypermaximality and comparison results

Lemma 2. *Let ν be a normalized unbounded boundary weight on $L^2(0,\infty)$ of the form $\nu(A) = (f, Af)$. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be $q-$positive, and let α be the CP-flow derived from the boundary weight double (ϕ, ν), with boundary representation $\pi = \{\pi_t^{\#}\}_{t>0}$.*

Let β be any CP-flow over $\mathbb{C}^n$, with boundary representation $\xi = \{\xi_t^{\#}\}_{t>0}$ and boundary weight map $\rho \to \eta(\rho)$. Then $\alpha \geq \beta$ if and only if β is derived from the boundary weight double (ψ, ν), where ψ is a $q-$positive map satisfying $\phi \geq_q \psi$.

Proof: As before, for each $t > 0$ we let $s_t = \nu_t(\Lambda(1))$. Assume the hypotheses of the backward direction. Then $\xi_t^{\#} = \psi(I + s_t\psi)^{-1}\Omega_{\nu_t}$, and the direction now follows from Theorem 4 since the line below is completely positive for all $t > 0$:

$$\pi_t^{\#} - \xi_t^{\#} = (\phi(I + s_t\phi)^{-1} - \psi(I + s_t\psi)^{-1})\Omega_{\nu_t}.$$

Now assume the hypotheses of the forward direction. Recall that by construction of ν, the set $\{s_t\}_{t>0}$ is decreasing. If $s_t > 0$ for all $t > 0$ we define $P = \infty$. Otherwise, we define P to be the smallest positive number such that $s_P = 0$. Fix any $t_0 \in (0, P)$. Notationally, write each $g \in H$ in its components as $g(x) = (g_1(x), \ldots, g_n(x))$, and write f_{t_0} for the function $U_{t_0} U_{t_0}^* f \in H$, where U_{t_0} is the right shift t_0 units on H. Under our identifications, $U_{t_0} U_{t_0}^*$ is the diagonal matrix in $M_n(B(L^2(0,\infty)))$ with ii^{th} entries $V_t V_t^*$ for V_t the right shift t_0 units on $L^2(0,\infty)$. Define $S : H \to \mathbb{C}^n$ by

$$Sg = ((f_{t_0}, g_1), \ldots, (f_{t_0}, g_n)),$$

noting that $\Omega_{\nu_{t_0}}(A) = SAS^*$ for all $A \in B(H)$. Since $\phi(I + s_{t_0}\phi)^{-1}$ is completely positive, we have $\phi(I + s_{t_0}\phi)^{-1}(A) = \sum_{i=1}^m R_i A R_i^*$ for some $R_1, \ldots, R_m \in B(H, \mathbb{C}^n)$. Therefore

$$\pi_{t_0}^{\#}(A) = \Big(\phi(I + s_{t_0}\phi)^{-1}\Big)(\Omega_{\nu_{t_0}}(A)) = \sum_{i=1}^m R_i SAS^* R_i^*.$$

The map $\xi_{t_0}^{\#}$ is a subordinate of $\pi_{t_0}^{\#}$, so from Arveson's work in metric operator spaces (see [A4]) we know that $\xi_{t_0}^{\#}$ has the form

$$\xi_{t_0}^{\#}(A) = \sum_{i,j=1}^m c_{ij} R_i SAS^* R_j^*,$$

for some complex numbers $\{c_{ij}\}$. Let L be the map $L(A) = \sum_{i,j} c_{ij} R_i A R_j^*$, noting that $\xi_{t_0}^{\#}(A) = L(SAS^*) = L(\Omega_{\nu_{t_0}}(A))$ for all $A \in B(H)$.

Defining ψ_{t_0} by $\psi_{t_0} = (I - \xi_{t_0}^{\#}\Lambda)^{-1} L$, we find that for arbitrary $A \in B(H)$ and $\acute{A} \in M_n(\mathbb{C})$,

$$\begin{aligned} \eta_{t_0}(\rho)(A) &= \Big(\hat{\xi}_{t_0}(I - \hat{\Lambda}\hat{\xi}_{t_0})^{-1}\Big)(\rho)(A)) = \rho\Big((I - \xi_{t_0}^{\#}\Lambda)^{-1}(\xi_{t_0}^{\#}(A))\Big) \\ &= \rho\Big((I - \xi_{t_0}^{\#}\Lambda)^{-1} L(\Omega_{\nu_{t_0}}(A))\Big) = \rho(\psi_{t_0}(\Omega_{\nu_{t_0}} A)) \end{aligned} \tag{2.2.1}$$

and

$$\hat{\Lambda}\eta_{t_0}(\rho)(\acute{A}) = \eta_{t_0}(\rho)(\Lambda(\acute{A})) = \rho\Big(\psi_{t_0}(\Omega_{\nu_{t_0}}(\Lambda(\acute{A})))\Big) = s_{t_0}\rho(\psi_{t_0}(\acute{A})), \tag{2.2.2}$$

so $\hat{\Lambda}\eta_{t_0} = s_{t_0}\hat{\psi}_{t_0}$.

Using formulas (2.2.1) and (2.2.2) and the fact that $\hat{\xi}_{t_0} = \eta_{t_0}(I + \hat{\Lambda}\tau_{t_0})^{-1}$, we find

$$\begin{aligned}
\rho(\xi_{t_0}^{\#}) &= \hat{\xi}_{t_0}(\rho) = \eta_{t_0}(I + \hat{\Lambda}\eta_{t_0})^{-1}(\rho) = \Big((I + \hat{\Lambda}\eta_{t_0})^{-1}(\rho)\Big)(\psi_{t_0}\Omega_{\nu_{t_0}}) \\
&= \Big((I + s_{t_0}\hat{\psi}_{t_0})^{-1}(\rho)\Big)(\psi_{t_0}\Omega_{\nu_{t_0}}) = \rho\Big((I + s_{t_0}\psi_{t_0})^{-1}\psi_{t_0}\Omega_{\nu_{t_0}}\Big) \\
&= \rho\Big(\psi_{t_0}(I + s_{t_0}\psi_{t_0})^{-1}\Omega_{\nu_{t_0}}\Big)
\end{aligned}$$

for all $\rho \in M_n(\mathbb{C})^*$, hence $\xi_{t_0}^{\#} = \psi_{t_0}(I + s_{t_0}\psi_{t_0})^{-1}\Omega_{\nu_{t_0}}$.

We now show that the maps $\{\psi_t\}_{t>0}$ are constant on the interval $(0, P)$. Let $t \in [t_0, P)$ be arbitrary. For each $\acute{A} = (a_{ij}) \in M_n(\mathbb{C})$, let $A = (\frac{a_{ij}}{\nu_t(I)} V_t V_t^*) \in B(H)$. Let $\rho \in M_n(\mathbb{C})^*$. Straightforward computations using formula (1.2.5) yield $\Omega_{t_0}(A) = \Omega_t(A) = \acute{A}$ and $\eta_{t_0}(\rho)(A) = \eta_t(\rho)(A)$. Combining these equalities gives us

$$\begin{aligned}
\rho(\psi_{t_0}(\acute{A})) &= \rho(\psi_{t_0}\Omega_{\nu_{t_0}}(A))) = \eta_{t_0}(\rho)(A) \\
&= \eta_t(\rho)(A) = \rho(\psi_t\Omega_{\nu_t}(A)) = \rho(\psi_t(\acute{A})).
\end{aligned}$$

Since the above formula holds for every $\acute{A} \in M_n(\mathbb{C})$ and $\rho \in M_n(\mathbb{C})^*$, we have $\psi_{t_0} = \psi_t$. But $t_0 \in (0, P)$ and $t \in [t_0, P)$ were chosen arbitrarily, so the previous sentence shows that $\psi_t = \psi_{t_0}$ for all $t \in (0, P)$.

Letting $\psi = \psi_{t_0}$, we have

$$\xi_t^{\#} = \psi(I + s_t\psi)^{-1}\Omega_{\nu_t} \tag{2.2.3}$$

for all $t \in (0, P)$. Defining κ_t as in the proof of Proposition 1, we observe that $\psi(I + s_t\psi)^{-1} = \xi_t^{\#}\kappa_t$ for all $t \in (0, P)$, where the right hand side is completely positive by hypothesis. Since every $t \in (0, \infty)$ can be written as $t = s_{t'}$ for some $t' \in (0, P)$, it follows that $\psi(I + t\psi)^{-1}$ is completely positive for all $t > 0$. Furthermore, $\psi(I + s_t\psi)^{-1} \to \psi$ in norm as $t \to \infty$, hence $\psi \geq_q 0$. Similarly, we see that since $\pi_t^{\#} - \xi_t^{\#}$ is completely positive for all $t > 0$ by assumption, it follows from our formula

$$\phi(I + s_t\phi)^{-1} - \psi(I + s_t\psi)^{-1} = (\pi_t^{\#} - \xi_t^{\#})\kappa_t$$

that $\phi(I+s_t\phi)^{-1}-\psi(I+s_t\psi)^{-1}$ is completely positive for all $t>0$, and so its norm limit (as $t\to\infty$) $\phi-\psi$ is completely positive. Therefore, $\phi\geq_q\psi$. Finally, since the CP-flow β is entirely determined by its generalized boundary representation ξ (which itself is entirely determined by any sequence $\{\xi^{\#}_{t_n}\}$ with t_n tending to 0), it follows from (2.2.3) that β is derived from the boundary weight double (ψ,ν).

□

In a manner analogous to that used by Powers in [P2] and [P1], we define the terms *q-corner* and *hypermaximal q-corner*:

Definition 10. *Let* $\phi : M_n(\mathbb{C})\to M_n(\mathbb{C})$ *and* $\psi : M_k(\mathbb{C})\to M_k(\mathbb{C})$ *be* $q-$*positive maps. A corner* $\gamma : M_{n\times k}(\mathbb{C})\to M_{n\times k}(\mathbb{C})$ *from* ϕ *to* ψ *is said to be a q-corner from* ϕ *to* ψ *if the map*

$$\Upsilon\begin{pmatrix} A_{n\times n} & B_{n\times k} \\ C_{k\times n} & D_{k\times k}\end{pmatrix}=\begin{pmatrix} \phi(A_{n\times n}) & \gamma(B_{n\times k}) \\ \gamma^*(C_{k\times n}) & \psi(D_{k\times k})\end{pmatrix}$$

is q-positive. A $q-$*corner* γ *is called hypermaximal if, whenever*

$$\Upsilon\geq_q\Upsilon'=\begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi'\end{pmatrix}\geq_q 0,$$

we have $\Upsilon=\Upsilon'$.

Proposition 3. *For any* $q-$*positive* $\phi : M_n(\mathbb{C})\to M_n(\mathbb{C})$ *and unitary* $U\in M_n(\mathbb{C})$*, define a map* ϕ_U *by*

$$\phi_U(A)=U^*\phi(UAU^*)U.$$

1. *The map* ϕ_U *is q-positive, and there is an order isomorphism between* $q-$*positive maps* β *such that* $\phi\geq_q\beta$ *and* $q-$*positive maps* β_U *such that* $\phi_U\geq\beta_U$*. In particular,* ϕ *is* $q-$*pure if and only if* ϕ_U *is* $q-$*pure.*

2. *If* ϕ *is unital and q-pure, then there is a hypermaximal* $q-$*corner from* ϕ *to* ϕ_U*.*

Proof: To prove the first assertion, we define a completely positive map ζ on $M_n(\mathbb{C})$ by $\zeta(A) = U^*AU$, noting that ζ^{-1} is also completely positive. For every $t \geq 0$ and $A \in M_n(\mathbb{C})$, we find that $(I + t\phi_U)^{-1}(A) = U^*(I + t\phi)^{-1}(UAU^*))U$ and

$$\begin{aligned} \phi_U(I + t\phi_U)^{-1}(A) &= U^*\phi\Big(U(U^*(I + t\phi)^{-1}(UAU^*)U)U^*\Big)U \\ &= U^*\phi(I + t\phi)^{-1}(UAU^*)U \\ &= \zeta \circ \phi(I + t\phi)^{-1} \circ \zeta^{-1}(A), \end{aligned} \tag{2.2.4}$$

so $\phi_U \geq_q 0$. Given any $q-$positive map β such that $\phi \geq_q \beta$, define β_U by $\beta_U(A) = U^*\beta(UAU^*)U$. Then β_U is q-positive by (2.2.4), and for each $t \geq 0$ we have

$$\phi_U(I + t\phi_U)^{-1} - \beta_U(I + t\beta_U)^{-1} = \zeta \circ (\phi(I + t\phi)^{-1} - \beta(I + t\beta)^{-1}) \circ \zeta^{-1},$$

hence $\phi_U \geq_q \beta_U$. Of course, since $\phi = (\phi_U)_{U^*}$, the argument just used gives an identical correspondence between $q-$subordinates α of ϕ_U and q-subordinates α_{U^*} of ϕ. Our first assertion now follows.

To prove the second statement, we define $\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\gamma(A) = \phi(AU^*)U$. By Theorem 1, γ is a corner from ϕ to ϕ_U, so the map

$$\Theta \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \phi_U(A_{22}) \end{pmatrix}$$

is completely positive. We calculate $\gamma(I + t\gamma)^{-1}(A) = \phi(I + t\phi)^{-1}(AU^*)U$, so for each $t \geq 0$ we have

$$\Theta(I + t\Theta)^{-1} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \phi(I + t\phi)^{-1}(A_{11}) & \phi(I + t\phi)^{-1}(A_{12}U^*)U \\ U^*\phi(I + t\phi)^{-1}(UA_{21}) & \phi_U(I + t\phi_U)^{-1}(A_{22}) \end{pmatrix}.$$

This shows that $\gamma(I + t\gamma)^{-1}$ is a corner from $\phi(I + t\phi)^{-1}$ to $\phi_U(I + t\phi_U)^{-1}$ for all $t \geq 0$, so γ is a

$q-$corner. Finally, if

$$\Theta' \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \alpha(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \beta(A_{22}) \end{pmatrix}$$

is completely positive and $\Theta - \Theta'$ is completely positive, then since ϕ and ϕ_U are $q-$pure we have $\alpha = \phi(I + t\phi)^{-1}$ for some $t \geq$ and $\beta = \phi_U(I + s\phi_U)^{-1}$ for some $s \geq 0$. Complete positivity of Θ' implies that

$$\Theta' \begin{pmatrix} I & U \\ U^* & I \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t}I & U \\ U^* & \frac{1}{1+s}I \end{pmatrix} \geq 0,$$

so $s = t = 0$ and $\Theta = \Theta'$, hence γ is hypermaximal.

□

We have arrived at the most significant result of the section, which tells us that, under certain conditions, the problem of determining whether two E_0-semigroups induced by boundary weight doubles are cocycle conjugate can be reduced to the much simpler problem of finding hypermaximal q-corners between q-positive maps:

Theorem 7. *Let ν be a normalized unbounded boundary weight on $L^2(0,\infty)$ of the form $\nu(A) = (f, Af)$. Let ϕ and ψ be unital $q-$positive maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively, and induce CP-flows α and β through the boundary weight doubles (ϕ,ν) and (ψ,ν).*

Then α^d and β^d are cocycle conjugate if and only if there is a hypermaximal q-corner from ϕ to ψ.

Proof: For the forward direction, suppose α^d and β^d are cocycle conjugate, so there is a hypermaximal flow corner

$$\Theta = \begin{pmatrix} \alpha & \sigma \\ \sigma^* & \beta \end{pmatrix}$$

from α to β. Let $\Pi = \{\Pi_t^\#\}$ be the boundary representation for Θ and define $s_t = \nu_t(\Lambda(1))$, so for each $t > 0$ there is some $\mathfrak{Z}_t$ such that

$$\Pi_t^{\#} = \begin{pmatrix} \pi_t^{\#} & \mathfrak{z}_t \\ \mathfrak{z}_t^* & \xi_t^{\#} \end{pmatrix} = \begin{pmatrix} \phi(I+s_t\phi)^{-1} \circ \Omega_{\nu_t,n\times n} & \mathfrak{z}_t \\ \mathfrak{z}_t^* & \psi(I+s_t\psi)^{-1} \circ \Omega_{\nu_t,k\times k} \end{pmatrix}.$$

Since each $\mathfrak{z}_t$ is a corner from $\phi(I+s_t\phi)^{-1} \circ \Omega_{\nu_t,n\times n}$ to $\psi(I+s_t\phi)^{-1} \circ \Omega_{\nu_t,k\times k}$, we have $\mathfrak{z}_t = L_t \circ \Omega_{\nu_t,n\times k}$ for some L_t. Define a map B_t for each $t > 0$ by

$$B_t = \begin{pmatrix} \phi(I+s_t\phi)^{-1} & L_t \\ L_t^* & \psi(I+s_t\psi)^{-1} \end{pmatrix}.$$

We observe that $\Pi_t^{\#} = B_t \circ \Omega_{v_t,n+k}$ for all $t > 0$, so an argument analogous to the one given in the proof of Lemma 2 shows that each $\mathfrak{z}_t$ has the form $\mathfrak{z}_t = \gamma_t(I+s_t\gamma_t)^{-1} \circ \Omega_{\nu_t,n\times k}$ and that γ_t does not depend on t. Therefore, for some $\gamma : B(K_2,K_1) \to B(K_2,K_1)$ we have

$$\mathfrak{z}_t = \gamma(I+s_t\gamma)^{-1} \circ \Omega_{\nu_t,n\times k}$$

for all $t > 0$. Letting

$$\vartheta = \begin{pmatrix} \phi & \gamma \\ \gamma^* & \psi \end{pmatrix},$$

we observe for each t that $\vartheta(I+s_t\vartheta)^{-1} = \Pi_t \circ \kappa_{(n+k)\times(n+k),t}$ is the composition of completely positive maps and is thus completely positive, hence $\vartheta \geq_q 0$. Suppose for some map ϑ' we have

$$\vartheta \geq_q \vartheta' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi' \end{pmatrix} \geq_q 0.$$

Let Θ' be the CP-flow induced by θ', so for some CP-flows α' and β' we have

$$\Theta' = \begin{pmatrix} \alpha' & \sigma \\ \sigma^* & \beta' \end{pmatrix}.$$

Since Θ is a hypermaximal flow corner we have $\Theta = \Theta'$, hence $\phi = \phi'$ and $\psi = \psi'$, and we conclude that γ is a hypermaximal $q-$corner.

For the backward direction, let $N = n + k$ and suppose there is a hypermaximal $q-$corner γ from ϕ to ψ, so the map $\Upsilon : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ defined by

$$\Upsilon \begin{pmatrix} A_{n\times n} & B_{n\times k} \\ C_{k\times n} & D_{k\times k} \end{pmatrix} = \begin{pmatrix} \phi(A_{n\times n}) & \gamma(B_{n\times k}) \\ \gamma^*(C_{k\times n}) & \psi(D_{k\times k}) \end{pmatrix}$$

is $q-$positive. By Proposition 1, the map $\rho \to \Xi(\rho)$ from $M_N(\mathbb{C})^*$ into weights on $B(\mathbb{C}^N \otimes L^2(0,\infty))$ defined by

$$\Xi(\rho)(A) = \rho(\Upsilon(\Omega_{\nu_t,N\times N}(A)))$$

is the boundary weight map of a CP-flow ν over $\mathbb{C}^N$, where for some σ we have

$$\nu = \begin{pmatrix} \alpha & \sigma \\ \sigma^* & \beta \end{pmatrix}.$$

Let

$$\nu' = \begin{pmatrix} \alpha' & \sigma \\ \sigma^* & \beta' \end{pmatrix}$$

be any CP-flow such that $\nu \geq \nu'$. Letting $\mathcal{Z}_t = \gamma(I + s_t\gamma)^{-1} \circ \Omega_{\nu_t,n\times k}$, we see the generalized boundary representations $\Pi = \{\Pi_t\}$ and $\Pi' = \{\Pi'_t\}$ for ν and ν' satisfy

$$\Pi_t = \begin{pmatrix} \pi_t & \mathcal{Z}_t \\ \mathcal{Z}_t^* & \xi_t \end{pmatrix} \geq \Pi'_t = \begin{pmatrix} \pi'_t & \mathcal{Z}_t \\ \mathcal{Z}_t^* & \xi'_t \end{pmatrix}$$

for all $t > 0$. Lemma 2 implies that for some ϕ' and ψ' with $\phi \geq_q \phi' \geq_q 0$ and $\psi \geq_q \psi' \geq_q 0$ we have $\pi'_t = \phi'(I+s_t\phi')^{-1} \circ \Omega_{\nu_t,n\times n}$ and $\xi'_t = \psi'(I+s_t\psi')^{-1} \circ \Omega_{\nu_t,k\times k}$. But the maps $\Upsilon'_{s_t} := \Pi'_t \circ \Omega_{\nu_t,N\times N}$ are completely positive, hence γ is a $q-$corner from ϕ' to ψ'. Hypermaximality of γ implies $\phi = \phi'$ and $\psi = \psi'$, thus $\nu = \nu'$. Therefore, σ is a hypermaximal flow corner from α to β, so α^d and β^d are cocycle conjugate by Theorem 4.56 of [P1].

□

Chapter 3

E_0-semigroups obtained from faithful states

3.1 Classification of Unital Rank One $q-$pure Maps

Any unital linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ of rank one is of the form $\phi(A) = \tau(A)I$ for some linear functional τ. If ϕ is positive, then τ is a positive functional and $\tau(I) = 1$, so τ is a state. On the other hand, given any state ρ, the map ϕ defined by $\phi(A) = \rho(A)I$ is unital and completely positive. Furthermore, ϕ is $q-$positive since $\phi(I + t\phi)^{-1} = \frac{1}{1+t}\phi$ for each $t > 0$. The unital rank one $q-$positive maps are therefore precisely the maps $A \to \rho(A)I$ for states ρ.

The goal of the rest of the section is to determine when such maps are $q-$pure, and then to find when there is a hypermaximal $q-$corner between unital rank one q-pure maps ϕ and ψ. In the case that ν is a normalized unbounded boundary weight of the form $\nu(B) = (f, Bf)$, this is equivalent to determining whether (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups by Proposition 7.

We begin with a lemma:

Lemma 3. *Let ρ be a faithful state on $M_n(\mathbb{C})$, and define a q-positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\phi(A) = \rho(A)I$. For any nonzero positive linear functional τ on $M_n(\mathbb{C})$ and nonzero positive operator $C \in M_n(\mathbb{C})$, define $\psi_{\tau,C} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\psi_{\tau,C}(A) = \tau(A)C$.*

Then $\psi_{\tau,C}$ is q-positive, and $\phi \geq_q \psi_{\tau,C}$ if and only if $\psi_{\tau,C} = \lambda\phi$ for some $\lambda \in (0, 1]$.

Proof: We begin by noting that for all $A \in M_n(\mathbb{C})$ and $t \geq 0$ we have $(I + t\psi_{\tau,C})^{-1}(A) = A - \frac{t\tau(A)}{1+t\tau(C)}C$, so

$$\psi_{\tau,C}(I + t\psi_{\tau,C})^{-1}(A) = \frac{\tau(A)}{1 + t\tau(C)}C, \tag{3.1.1}$$

hence $\psi_{\tau,C}$ is $q-$positive. It follows from (3.1.1) that $\phi(I + t\phi)^{-1}(A) = \frac{\rho(A)}{1+t}I$ for all $A \in M_n(\mathbb{C})$.

Assume the hypotheses of the forward direction. Since $\phi \geq_q \psi_{\tau,C}$, we have

$$\frac{\rho(A)I}{1+t} \geq \frac{\tau(A)C}{1 + t\tau(C)} \tag{3.1.2}$$

for all $t \geq 0$ and $A \geq 0$. Note that this is impossible if $\tau(C) = 0$, so assume $\tau(C) \neq 0$. Letting $t \to \infty$ in (3.1.2) yields

$$\rho(A)I \geq \frac{\tau(A)C}{\tau(C)} \tag{3.1.3}$$

for all $A \geq 0$. Setting $A = C$ in (3.1.3), we see $\rho(C)I - C \geq 0$, yet

$$\rho\Big(\rho(C)I - C\Big) = \rho(C) - \rho(C) = 0,$$

hence $C = \rho(C)I$ by faithfulness of ρ.

Rewriting (3.1.3) as $\rho(A)I \geq \frac{\tau(A)}{\tau(\rho(C)I)}\rho(C)I = \frac{\tau(A)}{||\tau||}I$ for all $A \geq 0$, we see that $\rho - \frac{\tau}{||\tau||}$ is a positive linear functional. Therefore,

$$\Big|\Big|\rho - \frac{\tau}{||\tau||}\Big|\Big| = \rho(I) - \frac{\tau(I)}{||\tau||} = 1 - 1 = 0,$$

hence $\tau = ||\tau||\rho$. Setting $t = 0$ and $A = I$ in (3.1.2) gives us $||\tau|| = \tau(I) = \frac{\lambda}{\rho(C)}$ for some $\lambda \in (0, 1]$. Therefore,

$$\psi_{\tau,C}(A) = \tau(A)C = ||\tau||\rho(A)\rho(C)I = \lambda\rho(A)I = \lambda\phi(A)$$

for all $A \in M_n(\mathbb{C})$, proving the forward direction.

The backward direction follows from Proposition 2 since $\lambda\phi = \phi^{(\frac{1-\lambda}{\lambda})}$ for every $\lambda \in (0, 1]$.

□

Let $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $q-$positive contraction such that the map $E_{\psi_t} := t\psi(I + t\psi)^{-1}$ satisfies $||E_{\psi_t}|| < 1$ for all $t > 0$. By compactness of the unit ball of $B(M_n(\mathbb{C}))$, the maps E_{ψ_t} have some norm limit as $t \to \infty$. We claim that the limit is unique. To see this, pick any orthonormal basis with respect to the trace inner product $(A, B) = tr(A^*B)$ of $M_n(\mathbb{C})$, and let M_t be the $n^2 \times n^2$ matrix of E_{ψ_t} with respect to this basis. From the cofactor formula for $(I + t\psi)^{-1}$ we know that the ij^{th} entry of M_t is a rational function $r_{ij}(t)$. Uniqueness of $\lim_{t\to\infty} E_{\psi_t}$ now follows from the fact that each $r_{ij}(t)$ has a unique limit as $t \to \infty$. We call this limit E_ψ. Note that $t\psi = E_{\psi_t}(I - E_{\psi_t})^{-1} = E_{\psi_t} + E_{\psi_t}^2 + \dots$. We claim that E_ψ fixes a positive element T of norm one. Indeed, for each $k \in \mathbb{N}$ and $t > 0$, we find

$$\begin{aligned} t||\psi|| &= t||\psi(I)|| \leq ||E_{\psi_t}(I)|| + \ldots + ||(E_{\psi_t})^{k-1}(I)|| + k\sum_{n=1}^{\infty} ||(E_{\psi_t})^k(I)||^n \\ &< (k-1) + k\sum_{n=1}^{\infty} ||(E_{\psi_t})^k(I)||^n, \end{aligned}$$

so for all k we have

$$||(E_\psi)^k(I)|| = \lim_{t\to\infty} ||(E_{\psi_t})^k(I)|| = 1.$$

Therefore, all elements of the sequence $\{T_k\}_{k\in\mathbb{N}}$ defined by $T_k = (E_\psi)^k(I)$ satisfy $T_k \geq 0$ and $||T_k|| = 1$. Since $T_k - T_{k+1} = (E_\psi)^k(I - T_1) \geq 0$ for all k, the sequence $\{T_k\}_{k\in\mathbb{N}}$ is monotonically decreasing and therefore has a positive norm limit T with $||T|| = 1$. Finally, E_ψ fixes T since $E_\psi(T) = \lim_{k\to\infty} E_\psi^{k+1}(I) = T$. We can say even more about E_ϕ for unital ϕ (see Lemma 8), but for now, the information at hand suffices in showing that a large class of maps is $q-$pure:

Proposition 4. *Let ρ be a state on $M_n(\mathbb{C})$, and define a $q-$positive map ϕ on $M_n(\mathbb{C})$ by $\phi(A) := \rho(A)I$. Then ϕ is $q-$pure if and only if ρ is faithful.*

Proof: For the forward direction, we prove the contrapositive. If ρ is not faithful, then for some $k < n$ and mutually orthogonal vectors $f_1, \ldots, f_k$ with $\sum_{i=1}^k ||f_i||^2 = 1$, we have $\rho(A) = \sum_{i=1}^k (f_i, Af_i)$ for all $A \in M_n(\mathbb{C})$. Let P be the projection onto the k-dimensional subspace of $\mathbb{C}^n$ spanned by the vectors f_i, and define a q-positive map $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\psi(A) = \rho(A)P$. Then for each $t \geq 0$ and $A \in M_n(\mathbb{C})$, we calculate

$$(\phi^{(t)} - \psi^{(t)})(A) = \frac{1}{1+t}(\phi(A) - \psi(A)) = \frac{1}{1+t}\rho(A)(I - P),$$

thus $\phi \geq_q \psi$. Obviously, $\psi \neq \phi^{(s)}$ for any $s \geq 0$, so ϕ is not q-pure.

To prove the backward direction, suppose $\phi \geq_q \psi \geq_q 0$ with $\psi \neq 0$, and form E_ψ and $E_\phi = \phi$. The map $E_{\phi_t} - E_{\psi_t}$ is completely positive for all t, so its limit $\phi - E_\psi$ is completely positive. By the above remarks, we know that E_ψ fixes a positive T with $||T|| = 1$. But $(\phi - E_\psi)(T) = \rho(T)I - T \geq 0$, so $\rho(T) = 1$, hence $T = I$ by faithfulness of ρ.

By complete positivity of $\phi - E_\psi$, we have $||\phi - E_\psi|| = ||\phi(I) - E_\psi(I)|| = 0$, so $\phi = E_\psi$. Therefore,

$$\begin{aligned} 0 &= \lim_{t\to\infty} \Big((\phi - E_{\psi_t})\Big(\frac{I}{t} + \psi\Big)\Big) = \lim_{t\to\infty} \Big(\phi\Big(\frac{I}{t} + \psi\Big) - E_{\psi_t}\Big(\frac{I}{t} + \psi\Big)\Big) \\ &= \lim_{t\to\infty} \Big(\frac{\phi}{t} + \phi\psi - t\psi(I + t\psi)^{-1}\Big(\frac{I}{t} + \psi\Big)\Big) = \lim_{t\to\infty} \frac{\phi}{t} + \phi\psi - \psi \\ &= \phi\psi - \psi. \end{aligned} \tag{3.1.4}$$

Letting τ be the positive linear functional $\tau = \rho \circ \psi$, we conclude from (3.1.4) that $\psi(A) = \rho(\psi(A))I = \tau(A)I$ for all $A \in M_n(\mathbb{C})$. Lemma 3 implies that $\psi = \lambda\phi = \phi^{(\frac{1-\lambda}{\lambda})}$ for some $\lambda \in (0, 1]$.

□

3.2 The Main Theorem

To prove the main result of the section, we need the following:

Lemma 4. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be unital rank one $q-$pure maps. If boundary weight doubles (ϕ, ν) and (ψ, μ) induce cocycle conjugate E_0-semigroups α^d and β^d, then there is a corner γ from ϕ to ψ such that $||\gamma|| = 1$.*

Proof: If α^d and β^d are cocycle conjugate, there is a hypermaximal flow corner σ from α to β with associated CP-flow

$$\Theta = \begin{pmatrix} \alpha & \sigma \\ \sigma^* & \beta \end{pmatrix}.$$

Write the boundary representation Π for Θ as

$$\Pi_t^\# = \begin{pmatrix} \frac{1}{1+\nu_t(\Lambda(1))}\phi & \mathfrak{Z}_t \\ \mathfrak{Z}_t^* & \frac{1}{1+\eta_t(\Lambda(1))}\eta_t \end{pmatrix}$$

for some maps $\{\mathfrak{Z}_t\}_{t>0}$. Let $\rho \to \Xi(\rho)$ be the boundary weight map for Θ, so for some map $\rho_{12} \to \ell(\rho_{12})$ from $M_{n\times k}(\mathbb{C})^*$ to weights on $B(H_2, H_1)$ we have

$$\Xi \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \omega(\rho_{11}) & \ell(\rho_{12}) \\ \ell^*(\rho_{21}) & \eta(\rho_{22}) \end{pmatrix}.$$

For every $A = (A_{ij}) \in \mathcal{U}(H)$ and bounded family of functionals $\{\rho(t) = (\rho_{ij}(t))\}_{t>0}$ in $M_{n+k}(\mathbb{C})^*$, the fact that $\pi_0^\# = \xi_0^\# = 0$ implies

$$\lim_{t\to 0} \omega_t(I + \hat{\Lambda}\omega_t)^{-1}(\rho_{11}(t))(A_{11}) = \lim_{t\to 0} \eta_t(I + \hat{\Lambda}\eta_t)^{-1}(\rho_{22}(t))(A_{22}) = 0,$$

so by complete positivity of the generalized boundary representation, we have

$$\lim_{t\to 0} \ell_t(I + \hat{\Lambda}\ell_t)^{-1}(\rho_{12}(t))(A_{12}) = 0. \tag{3.2.1}$$

We claim that $\rho_{12} \to \ell(\rho_{12})$ is unbounded. If ℓ is bounded, then for each $\rho_{12} \in M_{n\times k}(\mathbb{C})^*$, the family $\rho_{12}(t) := (I + \hat{\Lambda}\ell_t)(\rho_{12})$ is bounded, so by (3.2.1) we have

$$\lim_{t\to 0} \ell_t(\rho_{12})(A_{12}) = 0 \tag{3.2.2}$$

for each $A_{12} \in \bigcup_{t>0} W_t B(H_2, H_1) X_t^*$, where W_t and X_t are the right shift t units on H_1 and H_2, respectively. However, for any $A_{12} \in \bigcup_{t>0} W_t B(H_2, H_1) X_t^*$ we have $A_{12} = W_s B X_s^*$ for some $s > 0$ and $B \in B(H_2, H_1)$, where for all $b < s$ we have

$$\ell(\rho_{12})(A_{12}) = \ell_b(\rho_{12})(A_{12}) = \ell_s(W_s B X_s^*).$$

Therefore, by equation (3.2.2) we have

$$\ell(\rho_{12})(A_{12}) = 0. \tag{3.2.3}$$

Let $A \in B(H_2, H_1)$, $\rho_{12} \in M_{n\times k}(\mathbb{C})^*$, and $t > 0$ be arbitrary. From (3.2.3) we have

$$\ell_t(\rho_{12})(A) = \ell(\rho_{12})(V_t A X_t^*) = 0,$$

hence $\ell_t \equiv 0$ for all $t > 0$. Uniqueness of the generalized boundary representation implies that $\rho_{12} \to \ell(\rho_{12})$ is the zero map. The boundary weight map $\rho \to \Xi'(\rho)$ defined by

$$\Xi' \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \omega(\rho_{11}) & 0 \\ 0 & 0 \end{pmatrix}$$

gives rise to the CP-flow

$$\Theta' = \begin{pmatrix} \alpha & \sigma \\ \sigma^* & \beta', \end{pmatrix}$$

where β' is the CP-flow $\beta'_t(A_{22}) = X_t A_{22} X_t^*$. Trivially, $\Theta \neq \Theta'$ and $\Theta \geq \Theta'$, contradicting hypermaximality of σ. Therefore, ℓ is unbounded.

Since Π_t is a contraction for every $t > 0$ (see Theorem 4.27 of [P1]), so is $\mathfrak{Z}_t$, hence the map $\mathfrak{Z}_t \circ \Lambda : B(K_2, K_1) \to B(K_2, K_1)$ is a contraction for each $t > 0$. A compactness argument shows that $\mathfrak{Z}_{t_n} \circ \Lambda$ has a norm limit γ for some sequence $\{t_n\}$ tending to zero, where $||\gamma|| \leq 1$. From unboundedness of ℓ and the formula $\ell_t = \hat{\mathfrak{Z}}_t(I - \hat{\Lambda}\hat{\mathfrak{Z}}_t)^{-1}$ for all $t > 0$, it follows that $I - \gamma$ is not invertible, so $||\gamma|| \geq 1$, hence $||\gamma|| = 1$. We claim that γ is a corner from ϕ to ψ. Indeed, for the family of completely positive maps $\{R_t\}_{t>0}$ defined by $R_t = \Pi_t \circ \Lambda$, we have

$$\lim_{n\to\infty} R_{t_n} = \lim_{n\to\infty} \begin{pmatrix} \frac{\nu_{t_n}(\Lambda(1))}{1+\nu_{t_n}(\Lambda(1))}\phi & \mathfrak{Z}_{t_n} \circ \Lambda \\ (\mathfrak{Z}_{t_n} \circ \Lambda)^* & \frac{\eta_{t_n}(\Lambda(1))}{1+\eta_{t_n}(\Lambda(1))}\psi. \end{pmatrix} = \begin{pmatrix} \phi & \gamma \\ \gamma^* & \psi \end{pmatrix}.$$

□

Theorem 8. *Let ϕ_1 and ϕ_2 be unital rank one $q-$pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively. Let ν be a normalized unbounded boundary weight on $L^2(0,\infty)$ of the form $\nu(B) = (f, Bf)$.*

Then the boundary weight doubles (ϕ_1, ν) and (ϕ_2, ν) induce cocycle conjugate E_0-semigroups if and only if $n = k$ and $\phi_2 = (\phi_1)_U$ for some unitary $U \in M_n(\mathbb{C})$.

Proof: The backward direction follows immediately from Theorem 7 and Proposition 3. Assume the hypotheses of the forward direction. Since ϕ_1 and ϕ_2 are unital, rank one, and $q-$pure, there exist faithful states ρ_1 on $M_n(\mathbb{C})$ and ρ_2 on $M_k(\mathbb{C})$ such that $\phi_1(M) = \rho_1(M)I_{n\times n}$ and $\phi_2(B) = \rho_2(B)I_{k\times k}$ for all $M \in M_n(\mathbb{C})$, $B \in M_k(\mathbb{C})$. By Lemma 4, there is a corner γ from ϕ_1 to ϕ_2 such that $||\gamma|| = 1$. Therefore, for some $A_0 \in M_{n\times k}(\mathbb{C})$ of norm one and unit vectors $f_0 \in \mathbb{C}^n$ and $g_0 \in \mathbb{C}^k$, we have $|(f_0, \gamma(A_0)g_0)| = 1$. Define $\omega \in (M_{n\times k}(\mathbb{C}))^*$ by $\omega(A) = (f_0, \gamma(A)g_0)$, noting that $||\omega|| = |\omega(A_0)| = 1$. We claim that the map $\tilde{\psi} : M_{n+k}(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$\tilde{\psi}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \rho_1(A_{11}) & \omega(A_{12}) \\ \omega^*(A_{21}) & \rho_2(A_{22}) \end{pmatrix}$$

is completely positive. To see this, let $\{\tilde{F}_i\}_{i=1}^{\ell}$ be arbitrary vectors in $\mathbb{C}^2$, writing each $\tilde{F}_i$ as

$$\tilde{F}_i = \begin{pmatrix} \lambda_{1i} \\ \lambda_{2i} \end{pmatrix}$$

for some complex numbers $\{\lambda_{1i}\}_{i=1}^{\ell}$ and $\{\lambda_{2i}\}_{i=1}^{\ell}$.

Since the map $\psi : M_{n+k}(\mathbb{C}) \to M_{n+k}(\mathbb{C})$ defined by

$$\psi\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \rho_1(A_{11})I & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \rho_2(A_{22})I \end{pmatrix}$$

is completely positive by assumption, we know that for any matrices $A_1, \ldots, A_\ell \in M_{n+k}(\mathbb{C})$ and the vectors

$$F_i = \begin{pmatrix} \lambda_{1i} f_0 \\ \\ \lambda_{2i} g_0 \end{pmatrix} \in \mathbb{C}^{n+k}, \quad i = 1, \ldots, k,$$

we have

$$\sum_{i,j=1}^{\ell} \Big(F_i, \psi(A_i^* A_j) F_j \Big) \geq 0.$$

However, for each i and j we find that

$$\begin{aligned} \Big(F_i, \psi(A_i^* A_j) F_j \Big)_{\mathbb{C}^{n+k}} &= \overline{\lambda_{1i}} \lambda_{1j} \rho_1((A_i^* A_j)_{11}) + \overline{\lambda_{1i}} \lambda_{2j} \omega((A_i^* A_j)_{12}) \\ &\quad + \overline{\lambda_{2i}} \lambda_{j1} \overline{\omega([(A_i^* A_j)_{21}]^*)} + \overline{\lambda_{2i}} \lambda_{2j} \rho_2((A_i^* A_j)_{22}) \\ &= \Big(\tilde{F}_i, \tilde{\psi}(A_i^* A_j) \tilde{F}_j \Big)_{\mathbb{C}^2}. \end{aligned}$$

Therefore, $\sum_{i,j=1}^{\ell} \Big(\tilde{F}_i, \tilde{\psi}(A_i^* A_j) \tilde{F}_j \Big) \geq 0$ for all matrices $A_1, \ldots, A_\ell \in M_{n+k}(\mathbb{C})$ and vectors $\tilde{F}_1, \ldots, \tilde{F}_\ell$ in $\mathbb{C}^2$, that is, $\tilde{\psi} : M_{2n}(\mathbb{C}) \to M_2(\mathbb{C})$ is completely positive. Since ρ_1 and ρ_2 are positive linear functionals (hence completely positive maps), this means that ω is a corner from ρ_1 to ρ_2.

Since ρ_1 and ρ_2 are faithful states, there exist monotonically increasing sequences of strictly positive numbers $\{\lambda_i\}_{i=1}^n$ and $\{\mu_j\}_{j=1}^k$ with $\sum_{i=1}^n \lambda_i^2 = \sum_{j=1}^k \mu_j^2 = 1$, along with orthonormal sets of vectors $\{f_i\}_{i=1}^n$ and $\{g_j\}_{j=1}^k$, such that $\rho_1(M) = \sum_{i=1}^n \lambda_i^2 (f_i, M f_i)$ and $\rho_2(B) = \sum_{j=1}^k \mu_j^2 (g_j, B g_j)$ for all $M \in M_n(\mathbb{C})$, $B \in M_k(\mathbb{C})$. Given $A \in M_{n \times k}(\mathbb{C})$, let $\tilde{A}$ be the matrix whose ji entry is $(f_i, A g_j)$, observing that $||\tilde{A}|| = ||A||$. Let D_λ and D_μ be the diagonal matrices whose ii entries are λ_i and μ_i, respectively, for all i, and let D_{λ^2} and D_{μ^2} be the diagonal matrices whose ii entries are λ_i^2 and μ_i^2, respectively, observing that $D_{\lambda^2} = (D_\lambda)^2$ and $D_{\mu^2} = (D_\mu)^2$.

By Proposition 1, ω has the form

$$\omega(A) = \sum_{i,j} c_{ij} \lambda_i \mu_j (f_i, A g_j) = tr(C D_\mu \tilde{A} D_\lambda) = tr\Big(C D_\mu (D_\lambda \tilde{A}^*)^* \Big)$$

for some $C = (c_{ij}) \in M_{n\times k}(\mathbb{C})$ such that $||C|| \leq 1$. By Cauchy-Schwartz for the inner product $(B, A) = tr(AB^*)$ on $M_{n\times k}(\mathbb{C})$, we have

$$\begin{aligned} 1 &= |\omega(A_0)|^2 = |tr(CD_\mu(D_\lambda \tilde{A_0}^*)^*)|^2 = |(CD_\mu, D_\lambda \tilde{A_0}^*)|^2 \\ &\leq (CD_\mu, CD_\mu)(D_\lambda \tilde{A_0}^*, D_\lambda \tilde{A_0}^*) = tr(D_\mu C^* C D_\mu) tr(D_\lambda \tilde{A_0}^* \tilde{A}_0 D_\lambda) \quad (3.2.4) \\ &\leq tr(D_{\mu^2} I_k) tr(D_{\lambda^2} I_n) \leq 1 * 1 = 1. \quad (3.2.5) \end{aligned}$$

Since equality holds in (3.2.5) and the trace map is faithful, we have $C^*C = I_k$ and $\tilde{A_0}^* \tilde{A}_0 = I_n$. But $C \in M_{n\times k}(\mathbb{C})$ and $\tilde{A_0}^* \in M_{n\times k}(\mathbb{C})$, so $n = k$, hence C and $\tilde{A}_0$ are unitary. Furthermore, since equality holds in (3.2.4), we have $mCD_\mu = D_\lambda \tilde{A_0}^*$ for some $m \in \mathbb{C}$ with $|m| = 1$.

Writing $D_\lambda = mCD_\mu \tilde{A}_0 = (mC\tilde{A}_0)(\tilde{A_0}^* D_\mu \tilde{A}_0)$, we observe that uniqueness of the Polar Decomposition for the invertible matrix D_λ yields

$$D_\lambda = \tilde{A_0}^* D_\mu \tilde{A}_0.$$

Since the ii entries in D_λ and D_μ are listed in increasing order, it follows that $D_\lambda = D_\mu$, hence ρ_2 is of the form $\rho_2(M) = \sum_{i=1}^n \lambda_i^2 (g_i, M g_i)$. Defining a unitary $U \in M_n(\mathbb{C})$ by letting $Uf_i = g_i$ for all i and extending linearly, we observe that

$$\rho_2(M) = \sum_{i=1}^n \lambda_i^2 (U^* f_i, MU^* f_i) = \sum_{i=1}^n \lambda_i^2 (f_i, UMU^* f_i) = \rho_1(UMU^*)$$

for all $M \in M_n(\mathbb{C})$. In other words, $\phi_2 = (\phi_1)_U$.

□

In [P2], Powers constructed E_0-semigroups using boundary weights acting on $B(L^2(0,\infty))$. It is routine to check that in our notation, these are the E_0-semigroups arising from the boundary weight doubles $(\imath_{\mathbb{C}}, \eta)$, where $\imath_{\mathbb{C}}$ is the identity map on $\mathbb{C}$ and η is any boundary weight on $B(L^2(0,\infty))$.

Corollary 2. *Let $n > 1$, and let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital rank one $q-$pure map. Then the boundary weight double (ϕ, ν) induces an E_0-semigroup α^d which is not cocycle conjugate to any E_0-semigroup β^d that arises (in the sense of [P2]) from a boundary weight on $B(L^2(0,\infty))$.*

Proof: By the remarks preceding the corollary, it suffices to show that α^d is not cocycle conjugate to any E_0-semigroup arising from a boundary weight double of the form $(\imath_{\mathbb{C}}, \eta)$. From the proof of Proposition 8, we know that every corner γ from ϕ to $\imath_{\mathbb{C}}$ satisfies $||\gamma|| < 1$ since $n \neq 1$. Lemma 4 now implies α^d and β^d are not cocycle conjugate.

$\square$

Chapter 4

Invertible unital $q-$pure maps

4.1 Conditional negativity of their inverses

Now that we have classified the unital $q-$pure maps on $M_n(\mathbb{C})$ of rank one, we explore the unital q-pure maps ϕ which are invertible. In a stark contrast to the rank one case, we find that for a given normalized unbounded boundary weight $\nu(A) = (f, Af)$ on $L^2(0,\infty)$, the doubles (ϕ,ν) and (ψ,ν) *always* induce cocycle conjugate E_0-semigroups if ϕ and ψ are unital invertible $q-$pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively.

The following proposition gives us a bijective correspondence between invertible unital $q-$positive maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and unital conditionally negative maps $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$:

Proposition 5. *A unital invertible linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive if and only if ϕ^{-1} is conditionally negative.*

Proof: Let $\psi = \phi^{-1}$. The forward direction follows from the fact that for $t \gg 0$ we have

$$t\phi(I + t\phi)^{-1} = t\psi^{-1}(I + t\psi^{-1})^{-1} = t(\psi + tI)^{-1} = \Big(I + \frac{\psi}{t}\Big)^{-1} = I - \frac{\psi}{t} + \frac{\psi}{t}^2 - \dots.$$

For the backward direction, we note that if ψ is conditionally negative, then $e^{-s\psi}$ is completely positive for all $s \geq 0$ (see [EL]). Observing that $\frac{d}{ds}(-e^{-s\psi}) = \psi e^{-s\psi}$, we find that

$$\psi\Big(\int_0^\infty e^{-s\psi} ds\Big) = \int_0^\infty \psi e^{-s\psi} ds = \lim_{t\to\infty} (-e^{-t\psi})]_0^t = I,$$

so $\phi = \int_0^\infty e^{-s\psi} ds$, hence ϕ is completely positive. Furthermore, a quick computation shows that for every $t > 0$,

$$\phi(I + t\phi)^{-1} = (tI + \psi)^{-1} \int_0^\infty e^{-s(tI+\psi)} ds = \int_0^\infty e^{-st} e^{-s\psi} ds,$$

hence $\phi \geq_q 0$.

□

In order to obtain a similar and more specific result for invertible $q-$pure maps, we must find how we can (or even if we can) determine the invertible $q-$subordinates of an invertible $q-$positive map through examination of its inverse. The following proposition tells us what to do in the unital case:

Proposition 6. *Let $\phi_1 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible unital q−positive map, and let $\psi_1 = \phi_1^{-1}$. Suppose $\psi_2 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is conditionally negative and $\psi_2 - \psi_1$ is completely positive. Then ψ_2 is invertible, and $\phi_2 := (\psi_2)^{-1}$ satisfies $\phi_1 \geq_q \phi_2 \geq_q 0$.*

Proof: Assume the hypotheses of the proposition, and let $s > 0$ be arbitrary. Define a function f on $\mathbb{R}$ by $f(t) = e^{-ts\psi_1} e^{(t-1)s\psi_2}$. The equality below is $f(1) - f(0) = \int_0^1 f'(t)dt$:

$$e^{-s\psi_1} - e^{-s\psi_2} = \int_0^1 se^{-ts\psi_1}(\psi_2 - \psi_1)e^{(t-1)s\psi_2} dt.$$

The inside of the integral on the right hand side is the composition of completely positive maps, so $e^{-s\psi_1} - e^{-s\psi_2}$ is completely positive. This implies $e^{-s\psi_1}(I) - e^{-s\psi_2}(I) \geq 0$, so

$$||e^{-s\psi_2}|| = ||e^{-s\psi_2}(I)|| \leq ||e^{-s\psi_1}(I)|| = ||e^{-s}(I)|| = e^{-s}.$$

It follows that $e^{-s\psi_2} \to 0$ as $s \to \infty$ and that $\int_0^\infty e^{-s\psi_2} ds$ converges. An argument analogous to the one given in the proof of Proposition 5 now shows that ψ_2 is invertible and that $\phi_2 := (\psi_2)^{-1}$

is $q-$positive. Furthermore, $\phi_1 \geq_q \phi_2$ since the quantity

$$\phi_1(I+t\phi_1)^{-1} - \phi_2(I+t\phi_2)^{-1} = \int_0^\infty e^{-st}(e^{-s\psi_1} - e^{-s\psi_2})ds$$

is completely positive for every $t \geq 0$.

□

Corollary 3. *Let $\phi_1 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible unital $q-$positive map, and let $\phi_2 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be linear and invertible.*

Then $\phi_1 \geq_q \phi_2 \geq_q 0$ if and only if ϕ_2^{-1} is conditionally negative and $\phi_2^{-1} - \phi_1^{-1}$ is completely positive.

Proof: The backward direction follows from Lemma 6. Assume the hypotheses of the forward direction and let $\psi_1 = \phi_1^{-1}$ and $\psi_2 = \phi_2^{-1}$. For $t \gg 0$ we have

$$\phi_2(I+t\phi_2)^{-1} = \Big(I + \frac{\psi_2}{t}\Big)^{-1} = I - \frac{\psi_2}{t} + \frac{\psi_2}{t}^2 - \dots$$

and

$$\phi_1(I+t\phi_1)^{-1} - \phi_2(I+t\phi_2)^{-1} = \psi_2 - \psi_1 + \Big(\frac{\psi_2^2 - \psi_1^2}{t} - \frac{\psi_2^3 - \psi_1^3}{t^2} + \dots\Big).$$

The first equation shows that ϕ_2^{-1} is conditionally negative, while the second shows that $\phi_2^{-1} - \phi_1^{-1}$ is completely positive.

□

Now that we know how to find all invertible q-subordinates of an invertible unital q-positive map ϕ, we ask if there can be any other q-subordinates of ϕ. We will find that the answer is no (see Proposition 7). Proving this will require the use of some machinery (notably Lemma 7), which we now build.

Definition 11. *For every $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\epsilon \in [0,1]$, we define a map ϕ_ϵ by $\phi_\epsilon = \epsilon I + (1-\epsilon)\phi$.*

If ϕ is $q-$positive, then ϕ_ϵ is invertible for all $\epsilon \in (0,1]$. In the lemmas that follow, we make frequent use of the fact that for all $t \geq 0$ we have

$$t\phi(I+t\phi)^{-1} = I - (I+t\phi)^{-1}. \tag{4.1.1}$$

We present a quick consequence of (4.1.1) for all $a \geq 0$ and $b \geq 0$:

$$a(I+bt\phi)^{-1} = aI - abt\phi(I+bt\phi)^{-1} \tag{4.1.2}$$

Lemma 5. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be completely positive. If $\phi_{\epsilon_k} \geq_q 0$ for some monotonically decreasing sequence $\{\epsilon_k\}$ of positive real numbers tending to 0, then $\phi \geq_q 0$.*

Proof: Assume the hypotheses of the lemma. Let k be arbitrary. Since $\phi_{\epsilon_k} \geq_q 0$, we know that $I - (I + t\phi_{\epsilon_k})^{-1}$ is completely positive for all $t \geq 0$. Noting that

$$I - (I+t\phi_\epsilon)^{-1} = I - \Big((1+t\epsilon I) + (1-\epsilon)t\phi\Big)^{-1} = I - \frac{1}{1+t\epsilon}\Big(I + \frac{t(1-\epsilon)}{1+t\epsilon}\phi\Big)^{-1}$$

and substituting $t' = \frac{t(1-\epsilon_k)}{1+t\epsilon_k}$, we see

$$I - (I+t\phi_\epsilon)^{-1} = I - \frac{1}{1+(\frac{\epsilon_k}{1-\epsilon_k+t'\epsilon_k})t'}(I+t'\phi)^{-1}.$$

Varying t throughout $[0,\infty)$, we find that the above equation is completely positive for all $t' \in [0, \frac{1-\epsilon_k}{\epsilon_k})$. Of course, for any $t' \in [0, \frac{1-\epsilon_k}{\epsilon_k})$, we have $t' \in [0, \frac{1-\epsilon_\ell}{\epsilon_\ell})$ for all $\ell \geq k$ by monotonicity of the sequence $\{\epsilon_n\}$. Therefore, we may repeat the same argument to conclude that for any $t' \in [0, \frac{1-\epsilon_k}{\epsilon_k})$, the map

$$I - \frac{1}{1+(\frac{\epsilon_\ell}{1-\epsilon_\ell+t'\epsilon_\ell})t'}(I+t'\phi)^{-1}$$

is completely positive for all $\ell \geq k$.

Now fix any $t' > 0$, so $t' \in (0, \frac{1-\epsilon_k}{\epsilon_k})$ for some $k \in \mathbb{N}$. A straightforward computation shows that the sequence $\{c_n\}$ defined by $c_n = \frac{\epsilon_n}{1-\epsilon_n+t'\epsilon_n}$ is a monotonically decreasing sequence converging to 0. From the previous paragraph, we know that the map

$$I - \frac{1}{1+c_\ell t'}(I+t'\phi)^{-1}$$

is completely positive for all $\ell \geq k$. Since $c_n \downarrow 0$ it follows that

$$I - (I + t'\phi)^{-1}$$

is completely positive. In other words, $\phi(I+t'\phi)^{-1}$ is completely positive. Since $t' > 0$ was chosen arbitrarily and ϕ is completely positive, the lemma follows.

□

Lemma 6. *If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\phi \geq_q 0$, then $\phi_\epsilon \geq_q 0$ for all $\epsilon \in [0,1)$.*

Proof: Suppose that $\phi \geq_q 0$, and let $\epsilon \in [0,1)$ be arbitrary. For each $t > 0$, we apply formula (4.1.2) to $a = \frac{1}{1+t\epsilon}$ and $b = \frac{t(1-\epsilon)}{1+t\epsilon}$ to find

$$\begin{aligned} I - (I + t\phi_\epsilon)^{-1} &= I - \frac{1}{1+t\epsilon}\Big(I + \frac{t(1-\epsilon)}{1+t\epsilon}\phi\Big)^{-1} \\ &= \Big(1 - \frac{1}{1+t\epsilon}\Big)I + \frac{t(1-\epsilon)}{(1+t\epsilon)^2}\phi\Big(I + \frac{t(1-\epsilon)}{1+t\epsilon}\phi\Big)^{-1}, \end{aligned}$$

where both terms on the last line are completely positive by assumption. Furthermore, ϕ_ϵ is completely positive, hence $\phi_\epsilon \geq_q 0$.

□

Corollary 4. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely positive map. Then $\phi \geq_q 0$ if and only if $\phi_\epsilon \geq_q 0$ for all $\epsilon \in (0,1)$.*

Lemma 7. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be q-positive maps. Then $\phi \geq_q \psi$ if and only if $\phi_\epsilon \geq_q \psi_\epsilon$ for all $\epsilon \in (0,1)$.*

Proof: For any $\epsilon \in (0,1)$ we have $\phi_\epsilon - \psi_\epsilon = \epsilon(\phi - \psi)$, so $\phi - \psi$ is completely positive if and only if $\phi_\epsilon - \psi_\epsilon$ is completely positive for all $\epsilon \in (0,1)$. For all $t' > 0$ we have

$$t'\Big(\phi(I + t'\phi)^{-1} - \psi(I + t'\psi)^{-1}\Big) = (I + t'\psi)^{-1} - (I + t'\phi)^{-1}, \tag{4.1.3}$$

and for all $t > 0$ we have

$$t(\phi_\epsilon(I+t\phi_\epsilon)^{-1} - \psi_\epsilon(I+t\psi_\epsilon)^{-1}) = \Big(I - (I+t\phi_\epsilon)^{-1}\Big) - \Big(I - (I+t\psi_\epsilon)^{-1}\Big)$$
$$= \frac{1}{1+t\epsilon}\Big((I + \frac{t(1-\epsilon)}{1+t\epsilon}\psi)^{-1} - (I + \frac{t(1-\epsilon)}{1+t\epsilon}\phi)^{-1}\Big). \tag{4.1.4}$$

Assume the hypotheses of the forward direction. Showing that $\phi_\epsilon \geq_q \psi_\epsilon$ for all $\epsilon \in (0,1)$ is equivalent to proving that (4.1.4) is completely positive for every for every $t \in (0,\infty)$ and $\epsilon \in (0,1)$. This follows from complete positivity of (4.1.3) since $\frac{t(1-\epsilon)}{1+t\epsilon} \in (0,\infty)$ for every $\epsilon \in (0,1)$ and $t \in (0,\infty)$. Now assume the hypotheses of the backward direction. Any $t' \in (0,\infty)$ can be written $\frac{t(1-\epsilon)}{1+t\epsilon}$ for some $\epsilon \in (0,1)$ and $t \in (0,\infty)$, so complete positivity of (4.1.4) for all such ϵ and t implies that (4.1.3) is completely positive for all $t' > 0$, hence $\phi \geq_q \psi$.

□

We are now in position to prove what is perhaps the most striking result of the section:

Proposition 7. *Let $\xi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible unital q-positive map. If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q−positive and $\xi \geq_q \phi$, then ϕ is either invertible or identically zero.*

Proof: For each $\epsilon \in (0,1)$, form ξ_ϵ and ϕ_ϵ as in Definition 11, and let $\psi_\epsilon := (\phi_\epsilon)^{-1}$, noting that ψ_ϵ is conditionally negative by Corollary 3. We first examine the case when the norms $||\psi_\epsilon||$ remain bounded as $\epsilon \to 0$. More precisely, suppose that for all ϵ sufficiently small we have $||\psi_\epsilon|| < r$ for some $r > 0$. By compactness of the closed unit ball of radius r in $B(M_n(\mathbb{C}))$, there is a decreasing sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ converging to 0 such that $\{\psi_{\epsilon_k}\}_{k\in\mathbb{N}}$ has a (bounded) norm limit ψ as $k \to \infty$. Noting that

$$\phi_{\epsilon_k}\psi_{\epsilon_k} - \phi\psi = (\phi_{\epsilon_k} - \phi)(\psi_{\epsilon_k} - \psi) + \phi(\psi_{\epsilon_k} - \psi) + (\phi_{\epsilon_k} - \phi)\psi$$

and then applying the triangle inequality, we find that

$$\begin{aligned} ||I - \phi\psi|| &= ||\phi_{\epsilon_k}\psi_{\epsilon_k} - \phi\psi|| \\ &\leq ||\phi_{\epsilon_k} - \phi||\ ||\psi_{\epsilon_k} - \psi|| + ||\phi||\ ||\psi_{\epsilon_k} - \psi|| + ||\phi_{\epsilon_k} - \phi||\ ||\psi|| \end{aligned}$$

for all $k \in \mathbb{N}$. But ϕ and ψ are bounded maps while $\psi_{\epsilon_k} \to \psi$ in norm and $\phi_{\epsilon_k} \to \phi$ in norm, so the above equation tends to 0 as $k \to \infty$. We conclude that $\phi\psi = I$. Similarly $\psi\phi = I$, hence ϕ is invertible and $\psi = \phi^{-1}$.

If the first case does not hold, then for some decreasing sequence $\{\epsilon_k\}$ tending to zero, the norms $\{||\psi_{\epsilon_k}||\}_{k\in\mathbb{N}}$ form an unbounded sequence. For each $k \in \mathbb{N}$, we write

$$(\xi_{\epsilon_k})^{-1}(A) = s_k A + Y_k A + AY_k^* - \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^*$$

and

$$\psi_{\epsilon_k}(A) = t_k A + Z_k A + AZ_k^* - \sum_{i=1}^{\ell_k} T_{k_i} A T_{k_i}^*,$$

where $s_k \in \mathbb{R}$, $t_k \in \mathbb{R}$, $tr(Y_k) = tr(Z_k) = 0$, and for all i and j we have $tr(S_{k_i}) = tr(T_{k_i}) = 0$, while for $i \neq j$ we have $tr(S_{k_i}^* S_{k_j}) = tr(T_{k_i}^* T_{k_j}) = 0$.

Since $\xi \geq_q \phi$, we know by Lemma 7 that $\xi_\epsilon \geq_q \phi_\epsilon$ for all $\epsilon \in (0, 1)$, so $\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1}$ is completely positive for all $k \in \mathbb{N}$ by Corollary 3. Therefore, for each k, there exist $p_k \leq n^4$, numbers $\{x_{k_i}\}_{i=1}^{p_k} \in \mathbb{C}$, and maps $\{X_{k_i}\}_{i=1}^{p_k}$ with $tr(X_{k_i}) = 0$, such that for all $A \in M_n(\mathbb{C})$,

$$\begin{aligned}(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A) &= \sum_{i=1}^{p_k} (X_{k_i} + x_{k_i} I) A (X_{k_i} + x_{k_i} I)^* \\ &= \Big(\sum_{i=1}^{p_k} |x_{k_i}|^2\Big) A + \Big(\sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i}\Big) A + A \Big(\sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i}\Big)^* \\ &\quad + \sum_{i=1}^{p_k} X_{k_i} A X_{k_i}^*. \end{aligned} \tag{4.1.5}$$

Simultaneously, for all $A \in M_n(\mathbb{C})$ we have

$$\begin{aligned}(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A) &= (t_k - s_k) A + (Z_k - Y_k) A + A (Z_k - Y_k)^* \\ &\quad + \Big(\sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i} A T_{k_i}^*\Big). \end{aligned} \tag{4.1.6}$$

We claim that

$$\Big|\Big| \sum_{i=1}^{p_k} X_{k_i} A X_{k_i}^* \Big|\Big| \leq \Big|\Big| \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^* \Big|\Big| \tag{4.1.7}$$

for all $k \in \mathbb{N}$. To prove this, we follow the remark below Definition 4. Let $\{v_i\}_{i=1}^n$ be any orthonormal basis for $\mathbb{C}^n$, let $h_i = v_i/\sqrt{n}$ for each i, let $f \in \mathbb{C}^n$ be arbitrary, and define maps A_i for $i = 1, \ldots, n$ by $A_i = fh_i^*$. Using the trace conditions on the maps Y_k, Z_k, $\{T_{k_i}\}$, $\{S_{k_i}\}$, and $\{X_{k_i}\}$, we calculate that

$$\sum_{i=1}^{n}(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A_i)h_i = (t_k - s_k)f + (Z_k - Y_k)f = \Big(\sum_{i=1}^{p_k}|x_{k_i}|^2\Big)f + \Big(\sum_{i=1}^{p_k}\overline{x_{k_i}}X_{k_i}\Big)f.$$

Since f was arbitrary, it follows that

$$\Big(t_k - s_k - \sum_{i=1}^{p_k}|x_{k_i}|^2\Big)I = \Big(\sum_{i=1}^{p_k}\overline{x_{k_i}}X_{k_i}\Big) - (Z_k - Y_k).$$

Taking the trace of both sides yields

$$0 = tr\Big(\Big(\sum_{i=1}^{p_k}\overline{x_{k_i}}X_{k_i}\Big) - (Z_k - Y_k)\Big) = tr\Big(t_k - s_k - \sum_{i=1}^{p_k}|x_{k_i}|^2\Big)I,$$

so $t_k - s_k = \sum_{i=1}^{p_k}|x_{k_i}|^2$ and $Z_k - Y_k = \sum_{i=1}^{p_k}\overline{x_i}X_i$. Formulas (4.1.5) and (4.1.6) now imply that $\sum_{i=1}^{p_k} X_{k_i}AX_{k_i}^* = \Big(\sum_{i=1}^{m_k} S_{k_i}AS_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i}AT_{k_i}^*\Big)$. Therefore, the map $A \to \sum_{i=1}^{m_k} S_{k_i}AS_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i}AT_{k_i}^*$ is completely positive, and

$$\Big|\Big|\sum_{i=1}^{p_k} X_{k_i}X_{k_i}^*\Big|\Big| = \Big|\Big|\sum_{i=1}^{m_k} S_{k_i}S_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i}T_{k_i}^*\Big|\Big| \leq \Big|\Big|\sum_{i=1}^{m_k} S_{k_i}S_{k_i}^*\Big|\Big|,$$

establishing (4.1.7).

We now show that there exists some $M \in \mathbb{N}$ such that

$$||X_{k_i}|| \leq M \tag{4.1.8}$$

for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, p_k\}$. To do this, we first note that since the sequence of invertible maps $\{\xi_{\epsilon_k}\}_{k\in\mathbb{N}}$ converges in norm to the invertible map ξ, the sequence $\{(\xi_{\epsilon_k})^{-1}\}_{k\in\mathbb{N}}$ converges in norm to ξ^{-1}. Therefore, the sequence of bounded linear maps $W_k : A \to \sum_{i=1}^{m_k} S_{k_i}AS_{k_i}^*$ converges to some bounded map W. In particular, the sequence $\{||W_k||\}_{k\in\mathbb{N}}$ is bounded. Choose $M \in \mathbb{N}$ so that

$M^2 \geq n^4 \sup_{k\in\mathbb{N}}\{||W_k||\}$. For every $k \in \mathbb{N}$ and $i \in \{1,\dots,m_k\}$, we have $||S_{k_i}||^2 \leq ||W_k|| \leq M^2/n^4$. Combining this fact with (4.1.7), we find that for every $k \in \mathbb{N}$ and $i \in \{1,\dots,p_k\}$,

$$\begin{aligned} ||X_{k_i}||^2 &= ||X_{k_i}X_{k_i}^*|| \leq ||\sum_{i=1}^{p_k} X_{k_i}X_{k_i}^*|| \leq ||\sum_{i=1}^{m_k} S_{k_i}S_{k_i}^*|| \leq \sum_{i=1}^{m_k} ||S_{k_i}||^2 \\ &\leq n^4 \max\{||S_{k_i}||^2 : i = 1,\dots,m_k\} \leq M^2, \end{aligned}$$

proving (4.1.8).

Finally, we claim that $\lim_{k\to\infty} \sum_{i=1}^{p_k} |x_{k_i}|^2 = \infty$. Indeed, since $||\psi_{\epsilon_k}|| \to \infty$ as $k \to \infty$ while $||(\xi_{\epsilon_k})^{-1}|| \to ||(\xi)^{-1}|| < \infty$, there is a sequence of maps $\{A_{\epsilon_k}\}$ of norm one such that $||(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A_{\epsilon_k})|| \to \infty$ as $k \to \infty$. However, we also have

$$\begin{aligned} ||(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A_{\epsilon_k})|| &= \Big|\Big|\Big(\sum_{i=1}^{p_k} |x_{k_i}|^2\Big)A_{\epsilon_k} + \Big(\sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i}\Big)A_{\epsilon_k} \\ &\quad + A_{\epsilon_k}\Big(\sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i}\Big)^* + \sum_{i=1}^{p_k} X_{k_i}A_{\epsilon_k}X_{k_i}^*\Big|\Big| \\ &\leq \sum_{i=1}^{p_k} |x_{k_i}|^2 + 2M\sum_{i=1}^{p_k} |x_{k_i}| + p_k M^2. \end{aligned}$$

Since the above quantity must go to infinity as $k \to \infty$, and since

$$\Big(\sum_{i=1}^{p_k} |x_{k_i}|\Big)^2 \geq \sum_{i=1}^{p_k} |x_{k_i}|^2 \geq \frac{(\sum_{i=1}^{p_k} |x_{k_i}|)^2}{p_k} \geq \frac{(\sum_{i=1}^{p_k} |x_{k_i}|)^2}{n^4} \tag{4.1.9}$$

for all k, we must have $\sum_{i=1}^{p_k} |x_{k_i}|^2 \to \infty$ and $\sum_{i=1}^{p_k} |x_{k_i}| \to \infty$ as $k \to \infty$.

For each k, let $\lambda_k = \sum_{i=1}^{p_k} |x_{k_i}|$. Let $A \in M_n(\mathbb{C})$ be any matrix such that $||A|| = 1$, and let $C = \sup_{k\in\mathbb{N}} ||(\xi_{\epsilon_k})^{-1}|| < \infty$. Using the reverse triangle inequality and (4.1.9), we find that for each $k \in \mathbb{N}$,

$$\begin{aligned} ||\psi_{\epsilon_k}(A)|| &\geq ||(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A)|| - ||(\xi_{\epsilon_k})^{-1}(A)|| \\ &\geq \frac{\lambda_k^2}{n^4} - 2M\lambda_k - n^4M^2 - C. \end{aligned} \tag{4.1.10}$$

But $\lim_{k\to\infty} \lambda_k = \infty$, so the above equation tends to infinity as $k \to \infty$. Since $\phi_{\epsilon_k} = (\psi_{\epsilon_k})^{-1}$ for all k, we know for $k \gg 0$ that

$$||\phi_{\epsilon_k}|| = \frac{1}{\inf\left\{||\psi_{\epsilon_k}(A)|| : A \in M_n(\mathbb{C}), ||A|| = 1\right\}} \leq \frac{1}{\lambda_k^2/n^4 - 2M\lambda_k - n^4M^2 - C}.$$

Therefore, $\lim_{k\to\infty} ||\phi_{\epsilon_k}|| = 0$. But the sequence $\{\phi_{\epsilon_k}\}_{k=1}^{\infty}$ converges to ϕ in norm, hence $\phi \equiv 0$.

□

4.2 Classification of invertible unital q-pure maps

Proposition 8. *An invertible unital linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is $q-$pure if and only if ϕ^{-1} is of the form*

$$\phi^{-1}(A) = A + YA + AY^*$$

for some $Y = -Y^* \in M_n(\mathbb{C})$ such that $tr(Y) = 0$.

Proof: Let $\psi = \phi^{-1}$. Assume the hypotheses of the forward direction. Write

$$\psi(A) = sA + YA + AY^* - \sum_{i=1}^{k} \lambda_i X_i A X_i^*,$$

where $s \in \mathbb{R}$, $tr(Y) = 0$, and for each i and j we have $\lambda_i \geq 0$, $tr(X_i) = 0$, and $tr(X_i^* X_j) = n\delta_{ij}$.

Defining a conditionally negative map $\psi' : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\psi'(A) = sA + YA + AY^*,$$

we note that ψ' is conditionally negative, and $\psi' - \psi$ is completely positive since $(\psi' - \psi)(A) = \sum_{j=1}^{k} \lambda_j X_j A X_j^*$ for all A. By Lemma 6, it follows that ψ' is invertible and that $\phi' := (\psi')^{-1}$ satisfies $\phi \geq_q \phi' \geq_q 0$.

Since ϕ is $q-$pure, there is some $t_0 \geq 0$ such that $\phi' = \phi^{(t_0)}$, hence

$$\psi' = (\phi')^{-1} = \left(\psi^{-1}(I + t_0\psi^{-1})\right)^{-1} = \left((t_0 I + \psi)^{-1}\right)^{-1} = t_0 I + \psi.$$

Therefore, for all $A \in M_n(\mathbb{C})$ we have

$$\psi'(A) = \psi(A) + \sum_{j=1}^{k} \lambda_j X_j A X_j^* = \psi(A) + t_0 A,$$

so the map $L : A \to \lambda_j X_j A X_j^*$ satisfies $L = t_0 I$. Let $f \in \mathbb{C}^n$ be arbitrary, choose an orthonormal basis $\{v_k\}_{k=1}^n$ of $\mathbb{C}^n$, define $h_k = v_k/\sqrt{n}$ for each k, and form $\{A_k\}_{k=1}^n$ by $A_k = f h_k^*$. The trace conditions for the maps $\{X_j\}$ imply that $\sum_{k=1}^n L(A_k)h_k = 0$. However, since $L = t_0 I$, we must also have $\sum_{k=1}^n L(A_k)h_k = t_0 f$. From arbitrariness of f, we conclude $t_0 = 0$. Therefore, ψ has the form $\psi(A) = sA + YA + AY^*$. Since $\psi(I) = I = sI + Y + Y^*$ and $tr(Y) = 0$, we have $s = 1$ and consequently $Y = -Y^*$.

Now assume the hypotheses of the backward direction. Note that ψ is conditionally negative and unital, hence ϕ is q-positive by Proposition 5. Let Φ be any nonzero $q-$positive map such that $\phi \geq_q \Phi$, so by Corollary 3 and Proposition 7, Φ is invertible and $\Psi := (\Phi)^{-1}$ is a conditionally negative map such that $\Psi - \psi$ is completely positive. Write Ψ in the form

$$\Psi(A) = s'A + ZA + AZ^* - \sum_{i=1}^{m} \mu_i T_i A T_i^*,$$

where $s' \in \mathbb{R}$ and for all i and j, $\mu_i > 0$, $tr(T_i) = 0$, and $tr(T_i^* T_j) = n\delta_{ij}$. Writing $C = Z - Y$, we have

$$(\Psi - \psi)(A) = (s' - s)A + CA + AC^* - \sum_{i=1}^{m} \mu_i T_i A T_i^*.$$

Complete positivity of $\Psi - \psi$ and the trace conditions for the above maps imply that $s' \geq s$, $C = 0$, and $T_i = 0$ for all i. Therefore $\Psi = \psi + (s' - s)I$, so $\Phi = \Psi^{-1} = \phi^{(s'-s)}$. We conclude that ϕ is $q-$pure.

□

Theorem 9. *An invertible unital linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is $q-$pure if and only if for some unitary $U \in M_n(\mathbb{C})$, the map ϕ_U is the Schur map*

$$\phi_U(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+i(\lambda_j - \lambda_k)} e_{jk} & \textit{if } j < k \\ a_{jk}e_{jk} & \textit{if } j = k \\ \frac{a_{jk}}{1-i(\lambda_j - \lambda_k)} e_{jk} & \textit{if } j > k \end{cases}$$

for all $j,k = 1,\dots,n$*, where* $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ *and* $\lambda_1 + \ldots + \lambda_n = 0$.

Assume the hypotheses of the forward direction. By the previous proposition, $\psi := \phi^{-1}$ has the form $\psi(A) = A + \tilde{Y}A + A\tilde{Y}^*$ for some $\tilde{Y} \in M_n(\mathbb{C})$ with $\tilde{Y} = -\tilde{Y}^*$ and $tr(\tilde{Y}) = 0$. Let $B = -i\tilde{Y}$, so $B = B^*$. Defining $Y := \frac{1}{2}I + \tilde{Y} = \frac{1}{2}I + iB$, we find $\psi(A) = YA + AY^*$ for all $A \in M_n(\mathbb{C})$. Since B is self-adjoint, there is some unitary $U \in M_n(\mathbb{C})$ such that U^*BU is a diagonal matrix D. For each $k \in \{1,\dots,n\}$ let $\lambda_k \in \mathbb{R}$ be the kk entry of D. Note that since $tr(B) = 0$ we have $\sum_{k=1}^n \lambda_k = 0$, and that U^*YU is the diagonal matrix M whose kk entry is $\frac{1}{2} + i\lambda_k$. Defining a map ψ_U by $\psi_U(A) = U^*\psi(UAU^*)U$ for all $A \in M_n(\mathbb{C})$, we find that

$$\begin{aligned}\psi_U(A) &= U^*(YUAU^* + UAU^*Y^*)U \\ &= (U^*YU)A + A(U^*YU)^* = MA + AM^*.\end{aligned}$$

A quick calculation shows that this is just the Schur map

$$\psi_U(a_{jk}e_{jk}) = \begin{cases} (1 + i(\lambda_j - \lambda_k))a_{jk}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ (1 - i(\lambda_j - \lambda_k))a_{jk}e_{jk} & \text{if } j > k \end{cases},$$

and so $(\psi_U)^{-1}$ has the form

$$(\psi_U)^{-1}(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1-i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j > k \end{cases}.$$

It is straightforward to verify that $(\psi_U)^{-1}$ is the map $\phi_U(A) = U^*\phi(UAU^*)U$.

Assume the hypotheses of the backward direction. Let T be the diagonal matrix whose kk^{th} entry is λ_k for every $k = 1,\dots,n$. We observe that $tr(T) = 0$ and $T = T^*$. Now let $C = iT$, and let $\tilde{T} = \frac{1}{2}I + C$. We routinely verify that $C = -C^*$ and $tr(C) = 0$, and that $(\phi_U)^{-1}$ satisfies $(\phi_U)^{-1}(A) = \tilde{T}A + A\tilde{T}^* = A + CA + AC^*$ for all $A \in M_n(\mathbb{C})$. Proposition 8 implies that ϕ_U is q-pure, whereby ϕ is q-pure by Proposition 3.

□

4.3 They are more or less the same

After going through all of this effort to find the invertible unital q-pure maps ϕ on $M_n(\mathbb{C})$, we find that, the E_0-semigroup obtained from the boundary weight double (ϕ, ν) (for ν normalized, unbounded and of the form $\nu(B) = (f, Bf)$) is entirely independent of ϕ and is cocycle conjugate to the E_0-semigroup induced by the one-dimensional boundary weight ν. In other words, these q-pure maps give us nothing new:

Theorem 10. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital, invertible, and $q-$pure, and let ν be a normalized unbounded boundary weight on $B(L^2(0,\infty))$ of the form $\nu(B) = (f, Bf)$. Then (ϕ, ν) and $(\imath_{\mathbb{C}}, \nu)$ induce cocycle conjugate E_0-semigroups.*

Proof: By Proposition 3 and Theorems 7 and 9, we may assume that ϕ is the Schur map

$$\phi(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1-i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j > k \end{cases}$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum_{k=1}^n \lambda_k = 0$.

By Theorem 7, it suffices to find a hypermaximal $q-$corner from ϕ to $\imath_{\mathbb{C}}$. For this, define $\gamma : M_{n\times 1}(\mathbb{C}) \to M_{n\times 1}(\mathbb{C})$ by

$$\gamma \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \frac{1}{1+i\lambda_1}b_1 \\ \frac{1}{1+i\lambda_2}b_2 \\ \vdots \\ \frac{1}{1+i\lambda_n}b_n \end{pmatrix}.$$

Now define $\Upsilon : M_{n+1}(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ by

$$\Upsilon\begin{pmatrix} A_{n\times n} & B_{n\times 1} \\ C_{1\times n} & a \end{pmatrix} = \begin{pmatrix} \phi(A_{n\times n}) & \gamma(B_{n\times 1}) \\ \gamma^*(C_{1\times n}) & a \end{pmatrix}.$$

Letting $\lambda_{n+1} = 0$, we observe that Υ is the Schur map on satisfying $j, k = 1, \ldots, n+1$ by

$$\Upsilon(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1-i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j > k \end{cases}$$

for $j, k = 1, \ldots, n+1$ and $A = (a_{jk}) \in M_n(\mathbb{C})$. Since $\sum_{i=1}^{n+1} \lambda_k = \sum_{i=1}^{n} \lambda_k = 0$, it follows from Theorem 9 that Υ is q-positive (in fact, q-pure), hence γ is a $q-$corner from ϕ to $\imath_{\mathbb{C}}$. Now suppose that $\Upsilon \geq_q \Upsilon' \geq_q 0$ for some Υ' of the form

$$\Upsilon'\begin{pmatrix} A_{n\times n} & B_{n\times 1} \\ C_{1\times n} & a \end{pmatrix} = \begin{pmatrix} \phi'(A_{n\times n}) & \gamma(B_{n\times 1}) \\ \gamma^*(C_{1\times n}) & \imath'(a) \end{pmatrix}.$$

Since Υ is $q-$pure and Υ' is not the zero map, we know that $\Upsilon' = \Upsilon^{(t)}$ for some $t \geq 0$, and a quick calculation gives us

$$\Upsilon'\begin{pmatrix} A_{n\times n} & B_{n\times 1} \\ C_{1\times n} & a \end{pmatrix} = \begin{pmatrix} \phi^{(t)}(A_{n\times n}) & \gamma^{(t)}(B_{n\times 1}) \\ (\gamma^*)^{(t)}(C_{1\times n}) & \frac{1}{1+t}(a) \end{pmatrix}.$$

By inspecting the two formulas for Υ' we see $\gamma = \gamma^{(t)}$, hence $t = 0$. Therefore, $\Upsilon' = \Upsilon$, and we conclude the $q-$corner γ is hypermaximal.

$\square$

Chapter 5

Classifying $q-$pure maps on $M_2(\mathbb{C})$

5.1 Revisiting E_ϕ

Having classified both the rank one and invertible unital $q-$pure maps on $M_n(\mathbb{C})$, we approach the broader question of simply finding all $q-$pure maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. In the case $n = 2$, we find that not only are there no unital $q-$pure maps of rank 2, there are not even any unital $q-$positive maps of rank 3. We begin this section by revisiting the map $E_\phi = \lim_{t\to\infty} t\phi(I + t\phi)^{-1}$ for a unital $q-$positive ϕ:

Lemma 8. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital and $q-$positive. Then $rank(\phi) = rank(E_\phi)$, $spec(E_\phi) \subseteq \{0, 1\}$, and if $\phi(A) = \lambda A$ for $\lambda \neq 0$, then $E_\phi(A) = A$. Furthermore, $(E_\phi)^2 = E_\phi$.*

Proof: For the first assertion, we observe for arbitrary $t > 0$ that

$$1 + t = ||I + t\phi|| = \frac{1}{\inf\{||(I + t\phi)^{-1}(A)|| : ||A|| = 1\}},$$

hence $||t(I + t\phi)^{-1}(A)|| \geq \frac{t}{1+t}||A||$ for all $A \in M_n(\mathbb{C})$. Therefore, the image of the unit ball D_1 under $t(I + t\phi)^{-1}$ contains $D_{t/(1+t)}$. Now for all $t \geq 1$ we have $t\phi(I + t\phi)^{-1}(D_1) \supseteq \phi(D_{1/2})$, so $range(\phi) = range(E_\phi)$ and consequently $rank(\phi) = rank(E_\phi)$.

Next, we note that

$$spec(E_\phi) = \left\{ \lim_{t\to\infty} \frac{t\lambda}{1+t\lambda} : \lambda \in spec(\phi) \right\} \subseteq \{0,1\}.$$

Now let λ be any nonzero eigenvalue for ϕ, so for some nonzero $A \in M_n(\mathbb{C})$ we have $\phi(A) = \lambda A$. Then $E_{\phi_t}(A) = \frac{t\lambda}{1+t\lambda}A$, hence $E_\phi(A) = \lim_{t\to\infty} E_{\phi_t}(A) = A$.

For the final claim, recall that $M_n(\mathbb{C})$ is a Hilbert space under the Hilbert-Schmidt inner product $(A, B) = tr(A^*B)$, so we may write ϕ as an upper-triangular matrix $M \in M_{n^2}(\mathbb{C})$ with respect to some orthonormal basis of $M_n(\mathbb{C})$, where the eigenvalues of ϕ are in order of decreasing norm along the diagonal of M. Denoting the Euclidean norm of a matrix by $|| \; ||$ and the Hilbert Schmidt norm by $|| \; ||_{HS}$, we note that $||A|| \leq ||A||_{HS} \leq \sqrt{n}||A||$ for all $A \in M_n(\mathbb{C})$.

The matrix $T = \lim_{t\to\infty} tM(I + tM)^{-1}$ for E_ϕ is upper triangular, where for some $m \in \mathbb{N}$, the first m diagonal entries of T are 1 and the remaining $n^2 - m$ diagonal entries are 0. Since $||E_\phi^k|| = 1$ for all $k \in \mathbb{N}$, we have $||E_\phi^k(A)||_{HS} \leq \sqrt{n}||A|| \leq n||A||_{HS}$ for all $A \in M_n(\mathbb{C})$, so each entry $t_{ij}^{(k)}$ of T^k satisfies $|t_{ij}^{(k)}| \leq n$. However, we readily calculate that $t_{12}^{(k)} = kt_{12}$, which goes to infinity in norm unless $t_{12} = 0$. Now erasing the t_{12} terms in T^k, we find that $t_{13}^{(k)} = kt_{13}$ and so $t_{13} = 0$. Continuing in this way we see that $t_{14} = \cdots = t_{1m} = 0$, that $t_{23} = \cdots t_{2m} = t_{34} \cdots t_{3m} = 0$, and so on. That is to say, we find that $t_{ij} = \delta_{ij}$ whenever i and j are both less than or equal to m.

In a similar manner, we can start at the lower right hand corner of M and observe that when $i \geq m + 1$ and $j \geq m + 1$, the ij entry of $tM(I + tM)^{-1}$ becomes unbounded as $t \to \infty$ unless $m_{ij} = 0$, hence $m_{ij} = 0$ for all such i and j. This implies that $t_{ij} = 0$ when both i and j are greater than or equal to $m + 1$. We conclude that

$$T = \begin{pmatrix} I_{m\times m} & B \\ 0 & 0 \end{pmatrix}$$

for some $m \times (n^2 - m)$ matrix B, hence E_ϕ fixes m linearly independent matrices and $E_\phi = (E_\phi)^2$.

□

5.2 The rank of a unital q-pure map on $M_2(\mathbb{C})$

Proposition 9. *If $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is unital and $q-$positive, then $rank(\phi) \neq 3$.*

Proof: Suppose that $rank(\phi) \geq 2$. By the above remark and Lemma 8, we have $E_\phi(I) = I$ and $E_\phi(A_1) = A_1$ for some A_1 linearly independent from I. Since E_ϕ is completely positive and thus self-adjoint in the sense that $E_\phi(A^*) = E_\phi(A)^*$ for all A, we have $E_\phi(A_1 + A_1^*) = A_1 + A_1^*$ and $E_\phi(i(A_1 - A_1^*)) = i(A_1 - A_1^*)$. A simple calculation shows that the self-adjoint matrices $A_1 + A_1^*$ and $i(A_1 - A_1^*)$ cannot both be multiples of I. We conclude from this that for some $L = L^* \in M_n(\mathbb{C})$ linearly independent from I we have $E_\phi(L) = L$.

Letting U be a unitary matrix such that $U^*LU = D$ for some diagonal matrix D, we note that D is linearly independent from I. Defining $(E_\phi)_U$ as in Proposition 3, we observe that $(E_\phi)_U$ is completely positive, with $(E_\phi)_U(I) = I$ and $(E_\phi)_U(D) = U^*E_\phi(UDU^*)U = U^*LU = D$. It follows easily that $(E_\phi)_U(e_{11}) = e_{11}$ and $(E_\phi)_U(e_{22}) = e_{22}$. We claim that $(E_\phi)_U(e_{12}) = be_{12}$ for some $b \in \mathbb{C}$. Indeed, write

$$(E_\phi)_U(e_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $(E_\phi)_U$ is 2-positive we have

$$0 \leq \begin{pmatrix} (E_\phi)_U(e_{11}) & (E_\phi)_U(e_{12}) \\ (E_\phi)_U(e_{21}) & (E_\phi)_U(e_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 0 & c & d \\ \overline{a} & \overline{c} & 0 & 0 \\ \overline{b} & \overline{d} & 0 & 1 \end{pmatrix}.$$

Positivity of the above matrix implies $a = c = d = 0$, hence $(E_\phi)_U(e_{12}) = be_{12}$, and so $(E_\phi)_U(e_{21}) = \overline{b}e_{12}$. Therefore $(E_\phi)_U$ is merely the Schur mapping

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \begin{pmatrix} a_{11} & ba_{12} \\ \overline{b}a_{21} & a_{22} \end{pmatrix},$$

where $rank((E_\phi)_U)$ is 2 if $b = 0$ and 4 if $b \neq 0$. Since $rank(\phi) = rank(E_\phi) = rank((E_\phi)_U)$, the result follows.

□

Lemma 9. *Let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a unital $q-$positive map of rank 2. Then ϕ has two distinct real nonzero eigenvalues, or the eigenspace for the eigenvalue 1 has dimension 2.*

Proof: Let $M \in M_4(\mathbb{C})$ be an upper-triangular matrix representing ϕ with respect to some orthonormal basis of $M_2(\mathbb{C})$, writing

$$M = \begin{pmatrix} 1 & a & b & c \\ 0 & \lambda_1 & d & f \\ 0 & 0 & \lambda_2 & g \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},$$

where the eigenvalues of ϕ are listed in order of decreasing norm. Note that $\lambda_j \in \mathbb{R}$ for all j. Indeed, since $\phi(A) = \lambda_j A$ if and only if $\phi(A^*) = \overline{\lambda_j} A^*$, we will have $rank(\phi) \geq 3$ if $\lambda_j \notin \mathbb{R}$ for some j. Of course, if $\lambda_1 = 1$ then we must have $a = 0$ since $||\phi^k||_{HS} \leq 2$ for all $k \in \mathbb{N}$, in which case the lemma immediately follows. By the previous two sentences, the lemma holds if any of the λ_j are nonzero. However, if $\lambda_j = 0$ for all j, then

$$M = \begin{pmatrix} 1 & a & b & c \\ 0 & 0 & d & f \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

in which case boundedness of the matrix $T = \lim_{t\to\infty} tM(I+tM)^{-1}$ for E_ϕ implies $d = f = g = 0$, hence by Lemma 8 $rank(T) = rank(\phi) = 1$.

□

Corollary 5. *Let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a unital and $q-$positive map of rank 2. Then for some unitary $U \in M_2(\mathbb{C})$ and states ρ_1 and ρ_2 we have, for all $A \in M_2(\mathbb{C})$,*

$$\phi_U(A) = \rho_1(A)e_{11} + \rho_2(A)e_{22}.$$

Proof: By Lemma 9 we have $\phi(I) = I$ and $\phi(L) = \lambda L$ for some $\lambda \in \mathbb{R}$ and L linearly independent from I. Arguing as we did in the proof of Proposition 9, we may assume $L = L^*$. Unitarily diagonalizing L we have $U^*LU = D$ for some diagonal matrix D, and we easily calculate $\phi_U(I) = I$ and

$$\phi_U(D) = U^*\phi(UDU^*)U = U^*(\lambda L)U = \lambda D.$$

Since D and I are linearly independent, we conclude that $range(\phi) = span\{e_{11}, e_{22}\}$, hence by continuity and complete positivity of ϕ_U it follows that for all $A \in M_2(\mathbb{C})$,

$$\phi_U(A) = \rho_1(A)e_{11} + \rho_2(A)e_{22}$$

for some positive linear functionals ρ_1 and ρ_2. We conclude that ρ_1 and ρ_2 are states since $\phi_U(I) = I$.

□

Proposition 10. *Let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be unital and $q-$pure. Then $rank(\phi) \neq 2$.*

Proof: Suppose $rank(\phi) \leq 2$. By Corollary 5 and Proposition 3, we may assume ϕ is of the form

$$\phi(A) = \rho_1(A)e_{11} + \rho_2(A)e_{22}$$

for some states ρ_1 and ρ_2.

We calculate that $t\phi(I + t\phi)^{-1}(A) = \mu_{1,t}(A) + \mu_{2,t}(A)$ for all $t \geq 0$ and $A \in M_2(\mathbb{C})$, where

$$\mu_{1,t}(A) = \frac{t(1 + t\rho_2(e_{22}))\rho_1(A) - t^2\rho_2(A)\rho_1(e_{22})}{1 + t\Big(\rho_1(e_{11}) + \rho_2(e_{22})\Big) + t^2\Big(\rho_1(e_{11})\rho_2(e_{22}) - \rho_2(e_{11})\rho_1(e_{22})\Big)}$$

and

$$\mu_{2,t}(A) = \frac{t(1+t\rho_1(e_{11}))\rho_2(A) - t^2\rho_1(A)\rho_2(e_{11})}{1+t\Big(\rho_1(e_{11})+\rho_2(e_{22})\Big)+t^2\Big(\rho_1(e_{11})\rho_2(e_{22})-\rho_2(e_{11})\rho_1(e_{22})\Big)}.$$

Let D_t be the denominator of $\mu_{1,t}$ and $\mu_{2,t}$, observing that $D_t \geq 0$ for sufficiently small positive t. Letting $Q = \rho_1(e_{11})\rho_2(e_{22}) - \rho_2(e_{11})\rho_1(e_{22})$, we handle the three possible cases.

First, if $Q < 0$ then for some t_0 we will have $D_{t_0} = 0$, in which case the numerators of μ_{1,t_0} and μ_{2,t_0} must be identically zero, hence $\rho_1 = k\rho_2$ for some $k \in \mathbb{C}$, where $k = 1$ since ρ_1 and ρ_2 are states. Therefore, ϕ is of the form $\phi(A) = \rho_1(A)I$, and $rank(\phi) = 1$.

Second, if $Q = 0$ then for $\mu_{1,t}$ and $\mu_{2,t}$ to remain bounded as $t \to \infty$, the terms $t^2(\rho_1(e_{11})\rho_2(A) - \rho_2(e_{11})\rho_1(A))$ and $t^2(\rho_2(e_{22})\rho_1(A) - \rho_1(e_{22})\rho_2(A))$ must be identically zero, in which case $\rho_1 = k\rho_2$ and, as before, we have $\rho_1 = \rho_2$ and $rank(\phi) = 1$.

Third, if $Q > 0$ then since $t\phi(I + t\phi)^{-1} \to E_\phi$, the linear functionals $\mu_{1,t}$ and $\mu_{2,t}$ have (positive) limits ν_1 and ν_2, respectively. We observe that $\rho_2(e_{22})\rho_1(A) - \rho_1(e_{22})\rho_2(A) = Q\nu_1$ and $\rho_1(e_{11})\rho_2(A) - \rho_2(e_{11})\rho_1(A) = Q\nu_2$. For simplicity of notation, let

$$C = \begin{pmatrix} \rho_1(e_{11}) & \rho_1(e_{22}) \\ \rho_2(e_{11}) & \rho_2(e_{22}) \end{pmatrix}.$$

The entries of C are all nonnegative since ρ_1 and ρ_2 are positive. Furthermore, $c_{11} > 0$ and $c_{22} > 0$ since $det(C) = Q > 0$. Note that

$$\rho_1 = \rho_1(e_{11})\nu_1 + \rho_1(e_{22})\nu_2 = c_{11}\nu_1 + c_{12}\nu_2$$

and

$$\rho_2 = \rho_2(e_{11})\nu_1 + \rho_2(e_{22})\nu_2 = c_{21}\nu_1 + c_{22}\nu_2.$$

This gives us

$$\mu_{1,t} = t\frac{(c_{11}+tQ)\nu_1 + c_{12}\nu_2}{1+t(c_{11}+c_{22})+t^2Q},\ \mu_{2,t} = t\frac{(c_{22}+tQ)\nu_2 + c_{21}\nu_1}{1+t(c_{11}+c_{22})+t^2Q}.$$

Define linear functionals τ_1 and τ_2 by $\tau_1(A) = \frac{Q}{c_{22}}\nu_1$ and $\tau_2(A) = \frac{Q}{c_{11}}\nu_2$, and define $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\Phi(A) = \tau_1(A)e_{11} + \tau_2(A)e_{22}.$$

The map Φ is completely positive since τ_1 and τ_2 are positive linear functionals. One readily calculates that for all $t > 0$ we have

$$t\Phi(I + t\Phi)^{-1}(A) = \omega_{1,t}(A)e_{11} + \omega_{2,t}(A)e_{22},$$

where

$$\omega_{1,t}(A) = \frac{(\frac{tQ}{c_{22}} + \frac{tQ^2}{c_{11}c_{22}})\nu_1}{1 + t(Q/c_{11} + Q/c_{22}) + t^2Q^2/(c_{11}c_{22})}$$

and

$$\omega_{2,t}(A) = \frac{(\frac{tQ}{c_{11}} + \frac{tQ^2}{c_{11}c_{22}})\nu_2}{1 + t(Q/c_{11} + Q/c_{22}) + t^2Q^2/(c_{11}c_{22})}.$$

Since $Q > 0$, it follows that both $\omega_{1,t}$ and $\omega_{2,t}$ are positive linear functionals for every $t > 0$. We conclude that $\Phi \geq_q 0$.

However, we find that $\phi \geq_q \Phi$. Indeed, one (painstakingly) computes that

$$\begin{aligned}
\mu_{1,t} \geq \omega_{1,t} \quad &\iff \quad t\frac{(c_{11} + tQ)\nu_1 + c_{12}\nu_2}{1 + t(c_{11} + c_{22}) + t^2Q} \geq \frac{(\frac{tQ}{c_{22}} + \frac{tQ^2}{c_{11}c_{22}})\nu_1}{1 + t(Q/c_{11} + Q/c_{22}) + t^2Q^2/(c_{11}c_{22})} \\
&\iff \quad \Big(c_{11} - \frac{Q}{c_{22}} + t\Big(1 - \frac{Q^2}{c_{11}c_{22}}\Big)\Big)\nu_1 \\
&+ \quad c_{12}\Big(1 + t(Q/c_{11} + Q/c_{22}) + t^2Q^2/(c_{11}c_{22})\Big)\nu_2 \geq 0.
\end{aligned}$$

Looking at the last line, we note that $c_{11} - \frac{Q}{c_{22}} \geq 0$, $1 - \frac{Q^2}{c_{11}c_{22}} \geq 1 - \frac{Q}{c_{11}c_{22}} \geq 0$, and that the coefficient of ν_2 is nonnegative. Therefore $\mu_{1,t} \geq \omega_{1,t}$ for all $t > 0$. A similar calculation shows that $\mu_{2,t} \geq \omega_{2,t}$ for all $t > 0$, hence $\phi \geq_q \Phi$.

We claim that at least one of c_{12} and c_{21} must be nonzero. Indeed, if $c_{12} = c_{21} = 0$ then $c_{11} = c_{22} = 1$ and $\phi(I + t\phi)^{-1} = \frac{1}{1+t}\phi$, in which case the map $\vartheta(A) = \rho_1(A)e_{11} + \frac{1}{2}\rho_2(A)e_{22}$ satisfies $\phi \geq_q \vartheta \geq_q 0$, yet $\vartheta \neq \phi(I + t\phi)^{-1}$ for any $t \geq 0$, contradicting q-purity of ϕ. Therefore, at least one of c_{12} and c_{21} is nonzero. In this case, since ϕ is $q-$pure we have $\Phi = \phi(I + t\phi)^{-1}$ for some $t \geq 0$. It quickly follows that ν_1 is a multiple of ν_2, which implies that $\rho_1 = k\rho_2$ for some $k \in \mathbb{C}$, and as before we conclude $\rho_1 = \rho_2$ and $rank(\phi) = 1$.

$\square$

The main result of the section now follows immediately from Theorem 8 and Propositions 9 and 10.

Theorem 11. *If* $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ *is unital and* $q-$*pure, then* ϕ *is either invertible or of the form* $\phi(A) = \rho(A)I$ *for some faithful state* ρ*.*

Chapter 6

It is what it is

6.1 Future endeavors

6.1.1 Comparing (ϕ, ν) and (ψ, ν) for q-pure ν

So far, our boundary weight constructions have distinguished the E_0-semigroups arising from specific unbounded boundary weights $\nu(B) = (f, Bf)$ on $B(L^2(0, \infty))$ and unital q-pure maps on $M_n(\mathbb{C})$ which either have rank one or are invertible. This leads us to ask how to approach the problem of comparing boundary weight doubles (ϕ, ν) and (ψ, ν) when ν is *any* q-pure boundary weight on $B(L^2(0, \infty))$. The issue here is that if ν takes a more general form than $\nu(B) = (f, Bf)$, we are not yet quite able to reduce the problem of determining cocycle conjugacy to the much simpler problem of finding hypermaximal q-corners between ϕ and ψ.

6.1.2 Focusing on ν and μ

We can also turn our attention to the unbounded boundary weights ν and μ to see when they alone can restrict cocycle conjugacy of the induced E_0-semigroups. We believe that if ϕ and ψ are unital q-pure maps and ν and μ are normalized unbounded boundary weights on $B(L^2(0, \infty))$ such that (ϕ, ν) and (ψ, μ) induce cocycle conjugate E_0-semigroups, then ν and μ must be connected in the sense of [P2].

6.1.3 Classification of q-pure maps

How can we classify all unital q-pure maps on $M_n(\mathbb{C})$? As previously mentioned, every unital q-pure map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ for $n \leq 2$ is either invertible or has rank one. This seems to be the case for all $n \in \mathbb{N}$. However, the methods we used for the $n \leq 2$ case become unwieldy for $n > 3$. It also seems that every non-invertible q-positive map can be decomposed, in some fashion, into the direct sum of other q-positive maps.

Even if we find all the unital q-pure maps, the problem of finding all unital q-positive maps remains. This class of maps is simply enormous in size and appears unrealistically large to classify. Regardless, it is still reasonable to compare some simple q-positive maps, such as those of the form $A \to \rho(A)I$, where ρ is any state on $M_n(\mathbb{C})$. The problem of determining the existence of a hypermaximal q-corner between unital q-positive maps is much more difficult if the maps are not q-pure. This is due to the fact that a generic unital q-positive map may have a large class of q-subordinates, whereas the non-trivial q-subordinates of a unital q-pure map are easily identifiable and must have norm strictly less than one.

6.2 Some conjectures

Conjecture: *If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is unital and q-pure, then rank$(\phi) = 1$ or ϕ is invertible.*

Conjecture *Let ϕ and ψ be unital rank one q-pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively. Let ν be a normalized unbounded boundary weight on $B(L^2(0,\infty))$ of the form $\nu(B) = (f, Bf)$, and let α^d and β^d be the minimal E_0-semigroups induced by the boundary weight doubles (ϕ, ν) and (ψ, ν), respectively. If α^d and β^d are paired, then $n = k$.*

Conjecture: *Let ϕ and ψ be unital q-pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively, and let ν and μ be normalized unbounded boundary weights on $B(L^2(0,\infty))$. If the boundary weight doubles (ϕ, ν) and (ψ, μ) induce cocycle conjugate E_0-semigroups, then there is a hypermaximal q-corner from ν to μ in the sense of [P2].*

A similar, even stronger conjecture:

Conjecture: *Let ϕ and ψ be unital q-pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively, and let ν and μ be normalized unbounded boundary weights on $B(L^2(0,\infty))$. Then (ϕ, ν) and (ψ, μ) induce cocycle conjugate E_0-semigroups if and only if there exist hypermaximal q-corners γ from ϕ to ψ and η from ν to μ.*

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