HEEGAARD FLOER INVARIANTS AND CABLING

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ABSTRACT

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Jennifer Hom

Paul Melvin, Advisor

A natural question in knot theory is to ask how certain properties of a knot behave under satellite operations. We will focus on the satellite operation of cabling, and on Heegaard Floer-theoretic properties. In particular, we will give a formula for the Ozsváth-Szabó concordance invariant $\tau$ of iterated cables of a knot $K$ in terms of the cabling parameters, $\tau(K)$, and a new concordance invariant, $\varepsilon(K)$. We show that, in many cases, $\varepsilon$ gives better bounds on the 4-ball genus of a knot that $\tau$ alone, and discuss further applications of $\varepsilon$. We will also completely classify when the iterated cable of a knot admits a positive $L$-space surgery.

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# Chapter 1

# Introduction

In this thesis, we address the question of how certain properties of knots behave under the satellite operation of cabling. The $(p, q)$-cable of a knot $K$, denoted $K_{p,q}$, is the satellite knot with pattern the $(p, q)$-torus knot $T_{p,q}$ (where $p$ indicates the longitudinal winding and $q$ indicates the meridional winding) and companion $K$. We will assume throughout that $p > 1$. (This assumption does not cause any loss of generality, since $K_{-p,-q} = rK_{p,q}$, where $rK_{p,q}$ denotes $K_{p,q}$ with the opposite orientation, and since $K_{1,q} = K$.)
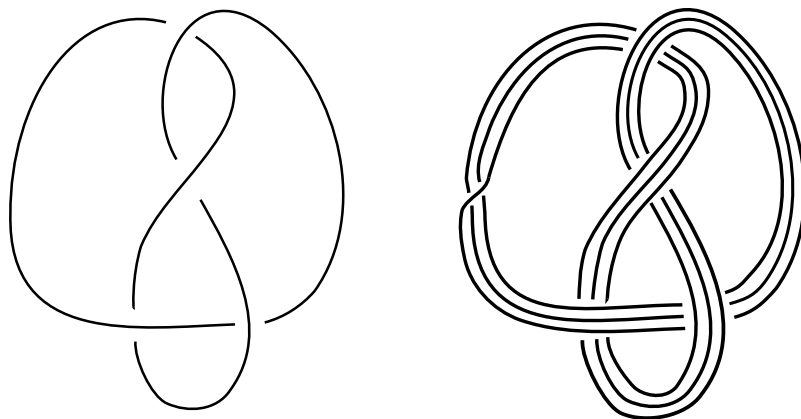


Figure 1.1: The figure 8 knot, and its $(3, 1)$-cable.

It is well-known that the Alexander polynomial of the $(p, q)$-cable of a knot $K$ is completely determined by $p$, $q$, and the Alexander polynomial of $K$, $\Delta_K(t)$, in the following manner:

$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t),$$

where

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

In this thesis, we focus on the behavior of the Ozsváth-Szabó concordance invariant $\tau$ under cabling.

Two knots $K_0, K_1 \subset S^3$ are called *concordant*, denoted $K_0 \sim K_1$, if there exists a smooth, properly embedded cylinder in $S^3 \times [0, 1]$ such that one end of the cylinder is $K_0 \times \{0\}$ and the other is $K_1 \times \{1\}$. This gives us an equivalence relation on the set of knots. A knot $K$ is called *slice* if $K$ is concordant to the unknot. The set $\{K\}/\sim$ forms the *concordance group* $\mathcal{C}$, where the operation is induced by connected sum. The class of slice knots is the identity element, and the inverse of $[K]$ is $[-K]$, where $-K$ denotes the reverse of the mirror image of $K$. If we loosen the conditions and only require that the cylinder be locally flat, rather than smooth, we obtain the *topological concordance group*.

To a knot $K \subset S^3$, Ozsváth and Szabó [11], and independently Rasmussen [20], associate a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex $CFK^\infty(K)$, whose doubly filtered chain homotopy type is an invariant of $K$. Looking at just one of the filtrations (i.e., taking the degree zero summand of the associated graded object with respect to the other filtration) yields the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(K)$, and associated to this chain complex is the $\mathbb{Z}$-valued smooth concordance invariant $\tau(K)$; see [9].

In [9], Ozsváth and Szabó show that the concordance invariant $\tau$ has the following properties:

1. $\tau : \mathcal{C} \to \mathbb{Z}$ is a surjective homomorphism.

2. $|\tau(K)| \leq g_4(K)$, where $g_4(K)$ denotes the smooth 4-ball genus of $K$.

3. $|\tau(T_{p,q})| = g(T_{p,q})$, where $g(K)$ denotes the Seifert genus of $K$.

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We completely describe the behavior of $\tau$ under cabling, generalizing work of Hedden [2], Van Cott [21], and Petkova [19]. As one might expect, $\tau(K_{p,q})$ is closely related to $\tau(T_{p,q})$. However, it depends on strictly more than just $\tau(K)$, $p$, and $q$; we also need to know the value of $\varepsilon(K)$, a new $\{-1, 0, 1\}$-valued concordance invariant associated to the knot Floer complex $CFK^{\infty}(K)$.

**Theorem 1.** *The behavior of $\tau(K_{p,q})$ is completely determined by $p$, $q$, $\tau(K)$, and $\varepsilon(K)$. In particular,*

1. *If $\varepsilon(K) = 0$, then $\tau(K) = 0$ and $\tau(K_{p,q}) = \tau(T_{p,q}) = \begin{cases} \frac{(p-1)(q+1)}{2} & \text{if } q < 0 \\ \frac{(p-1)(q-1)}{2} & \text{if } q > 0. \end{cases}$*

2. *If $\varepsilon(K) \neq 0$, then*

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q - \varepsilon(K))}{2}.$$

One consequence of Theorems 1 and 2 is that the only additional concordance information about $K$ coming from $\tau$ of iterated cables of $K$ is the invariant $\varepsilon$. Conversely, knowing $\tau$ of just two cables of $K$, one positive and one negative, is sufficient to determine $\varepsilon(K)$. In particular, knowing information about the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(K_{p,q})$, namely $\tau(K_{p,q})$, can tell us information about the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex $CFK^{\infty}(K)$, i.e., $\varepsilon(K)$.

Since $\tau(K_{p,q})$ depends on both $\tau(K)$ and $\varepsilon(K)$, we would also like to know the behavior of $\varepsilon$ under cabling in order to compute $\tau$ of iterated cables.

**Theorem 2.** *The invariant $\varepsilon$ behaves in the following manner under cabling:*

1. *If $\varepsilon(K) = 0$, then $\varepsilon(K_{p,q}) = \varepsilon(T_{p,q}) = \begin{cases} -1 & \text{if } q < -1 \\ 0 & \text{if } |q| = 1 \\ 1 & \text{if } q > 1. \end{cases}$*

2. *If $\varepsilon(K) \neq 0$, then $\varepsilon(K_{p,q}) = \varepsilon(K)$ for all $p$ and $q$.*

We see that $\tau(K_{p,q})$ depends on strictly more than just $\tau(K)$, so it is natural to ask if there exist knots $K$ and $K'$ with $\tau(K) = \tau(K')$ but $\tau(K_{p,q}) \neq \tau(K'_{p,q})$. We answer this question in the affirmative:

**Corollary 3.** *For any integer n, there exists knots $K$ and $K'$ with $\tau(K) = \tau(K') = n$, such that $\tau(K_{p,q}) \neq \tau(K'_{p,q})$, for all $p$ and $q$, $p \neq 1$.*

Recall that the absolute value of $\tau(K)$ gives a lower bound on the 4-ball genus of a knot; that is, $g_4(K) \geq |\tau(K)|$. By looking at both $\tau$ and $\varepsilon$, we can give stronger bounds in many cases. The following corollary was suggested to me by Livingston:

**Corollary 4** (Livingston). *If $\varepsilon(K) \neq \operatorname{sgn} \tau(K)$, then $g_4(K) \geq |\tau(K)| + 1$.*

If two knots are concordant, it follows that their cables are also concordant. Thus, it follows from Therorem 1 that $\varepsilon$ is a concordance invariant. We also prove the following properties about $\varepsilon$:

- If $K$ is slice, then $\varepsilon(K) = 0$.

- $\varepsilon(-K) = -\varepsilon(K)$.

- If $\varepsilon(K) = 0$, then $\tau(K) = 0$.

- There exist knots $K$ with $\tau(K) = 0$ but $\varepsilon(K) \neq 0$; that is, $\varepsilon(K)$ is *strictly stronger* than $\tau(K)$ at obstructing sliceness.

- Let $g(K)$ denote the genus of $K$. If $|\tau(K)| = g(K)$, then $\varepsilon(K) = \operatorname{sgn} \tau(K)$.

- If $K$ is *homologically thin* (meaning $\widehat{HFK}(K)$ is supported on a single diagonal with respect to its bigrading), then $\varepsilon(K) = \operatorname{sgn} \tau(K)$.

- If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K \# K') = \varepsilon(K) = \varepsilon(K')$. If $\varepsilon(K) = 0$, then $\varepsilon(K \# K') = \varepsilon(K')$.

Moreover, the invariant $\varepsilon$ can be used to define a new concordance homomorphism, as discussed in [4].

In the latter part of this thesis, we will consider a special kind of 3-manifold, called an $L$-space, and knots on which some positive integral surgery yields an $L$-space. Such a knot is called an *L-space knot.*

Let $g(K)$ denote the Seifert genus of $K$. In Theorem 1.10 of [2], Hedden proves that if $K$ is an $L$-space knot and $q/p \geq 2g(K) - 1$, then $K_{p,q}$ is an $L$-space knot. We will prove the converse:

**Theorem 5.** *The $(p, q)$-cable of a knot $K \subset S^3$ is an $L$-space knot if and only if $K$ is an $L$-space knot and $q/p \geq 2g(K) - 1$.*

Thus, we see that whether or not the cable of $K$ is an $L$-space knot depends only on the cabling parameters, and whether or not $K$ is an $L$-space knot. The proof of this theorem relies on classical low-dimensional techniques, as well as various properties of Ozsváth-Szabó invariants.

# Chapter 2

# A quick trip through bordered

# Heegaard Floer homology

We begin with a few algebraic preliminaries, before proceding to a brief overview of bordered Heegaard Floer homology and knot Floer homology.

## 2.1 Algebraic preliminaries

For the reader unfamiliar with the algebraic structures involved in bordered Heegaard Floer homology, such as $\mathcal{A}_\infty$-modules, the Type $D$ structures of [7], and the "box" tensor product, we recount the definitions below. For a more detailed description, we refer the reader to [7, Section 2].

Let $\mathcal{A}$ be a unital graded algebra over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with an orthogonal basis $\{\iota_i\}$ for the subalgebra of idempotents, $\mathcal{I} \subset \mathcal{A}$, such that $\sum \iota_i = 1 \in \mathcal{A}$. In what follows, all of the tensor products are over $\mathcal{I}$. We suppress grading shifts for ease of exposition.

A *(right unital)* $\mathcal{A}_\infty$-*module* is an $\mathbb{F}$-vector space $M$ equipped with a right $\mathcal{I}$-action such that

$$M = \bigoplus_i M\iota_i,$$

and a family of maps

$$m_i : M \otimes \mathcal{A}^{\otimes i-1} \to M$$

satisfying the $\mathcal{A}_\infty$ conditions

$$0 = \sum_{i=1}^{n-1} m_{n-i+1}(m_i(x \otimes a_1 \otimes \ldots \otimes a_{i-1}) \otimes \ldots \otimes a_{n-1})$$
$$+ \sum_{i=1}^{n-2} m_{n-1}(x \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_{n-1})$$

and the unital conditions

$$m_2(x, 1) = x$$

$$m_i(x, \ldots, 1, \ldots) = 0, \quad i > 2.$$

We say that $M$ is *bounded* if there exists an integer $n$ such that $m_i = 0$ for all $i > n$.

A *Type D structure over* $\mathcal{A}$ is an $\mathbb{F}$-vector space $N$ equipped with a left $\mathcal{I}$-action such that

$$N = \bigoplus_i \iota_i N,$$

and a map

$$\delta_1 : N \to \mathcal{A} \otimes N$$

satisfying the Type $D$ condition

$$(\mu \otimes \mathbb{I}_N) \circ (\mathbb{I}_\mathcal{A} \otimes \delta_1) \circ \delta_1 = 0,$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication on $\mathcal{A}$.

On the Type $D$ structure $N$, we define maps

$$\delta_k : N \to \mathcal{A}^{\otimes k} \otimes N$$

inductively by

$$\delta_0 = \mathbb{I}_N$$

$$\delta_i = (\mathbb{I}_{\mathcal{A}}^{\otimes i-1} \otimes \delta_1) \circ \delta_{i-1}.$$

We say that $N$ is *bounded* if there exists an integer $n$ such that $\delta_i = 0$ for all $i > n$.

The *box tensor product* $M \boxtimes N$ is the $\mathbb{F}$-vector space

$$M \otimes_{\mathcal{I}} N,$$

endowed with the differential

$$\partial^{\boxtimes}(x \otimes y) = \sum_{k=0}^{\infty}(m_{k+1} \otimes \mathbb{I}_N)(x \otimes \delta_k(y)).$$

If at least one of $M$ or $N$ is bounded, then the above sum is guaranteed to be finite.

The above definitions can be suitably modified if one would like to work over a *differential graded algebra* instead of merely a graded algebra; see [7, Section 2] or [6, Section 2.1].

## 2.2 Bordered Heegaard Floer homology

We assume the reader is familiar with Heegaard Floer homology for closed 3-manifolds, and with the filtration induced on this invariant by a knot $K$ in the 3-manifold. See, for example, the expository overview [15]. We begin with an overview of the invariants associated to 3-manifolds with parameterized boundary, as defined by Lipshitz, Ozsváth and Thurston in [7]. Let $Y$ be a closed 3-manifold. Decompose $Y$ along a closed surface $F$ into pieces $Y_1$ and $Y_2$ such that $\partial Y_1 = -\partial Y_2 = F$. In particular, this gives us an orientation preserving diffeomorphism from $F$ to $\partial Y_1$, and an orientation reversing diffeomorphism from $F$ to $\partial Y_2$. A 3-manifold with a diffeomorphism (up to isotopy) from a standard surface to its boundary is called a *bordered* 3-manifold, and we call this isotopy class of diffeomorphisms a *marking* of the boundary. To the closed surface $F$, we associate a differential graded algebra $\mathcal{A}(F)$. To $Y_1$, we associate the invariant,

$\widehat{CFA}(Y_1)$, which will be a right $\mathcal{A}_\infty$-module over the algebra $\mathcal{A}(F)$, while to $Y_2$, we associate the invariant, $\widehat{CFD}(Y_2)$, which will be a Type $D$ structure. To a knot $K$ in $Y_1$, we may associate either $\widehat{CFA}(Y_1, K)$, a filtered $\mathcal{A}_\infty$-module, or $CFA^-(Y_1, K)$, an $\mathcal{A}_\infty$-module over the ground ring $\mathbb{F}[U]$.

The pairing theorems of [7, Theorems 1.3 and 10.12] state that there exists a quasi-isomorphism between $\widehat{CF}(Y)$ and the box tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$:

$$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2).$$

We may also consider the case where we have a knot $K_1 \subset Y_1$, in which case we have the following quasi-isomorphism of $\mathbb{Z}$-filtered chain complexes:

$$\widehat{CFK}(Y, K) \simeq \widehat{CFA}(Y_1, K_1) \boxtimes \widehat{CFD}(Y_2),$$

and the following quasi-isomorphism of $\mathbb{F}[U]$-modules:

$$CFK^-(Y, K) \simeq CFA^-(Y_1, K_1) \boxtimes \widehat{CFD}(Y_2).$$

Note that the information contained in the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(Y, K)$ is equivalent to that in the $\mathbb{F}[U]$-module $CFK^-(Y, K)$. Similar pairing theorems hold when we have a knot $K_2 \subset Y_2$.

In this thesis, we will use these tools to study cabling. Thus, we will restrict ourselves to the case where $F$ is a torus. To use the bordered Heegaard Floer package to study the $(p, pn+1)$-cable of a knot $K$, we will let $Y_1$ be a solid torus equipped with a $(p, 1)$-torus knot, and let $Y_2$ be the knot complement $S^3 -$ nbd $K$ with framing $n$; that is, the marking specifies a meridian of the knot and a $n$-framed longitude.

We will now describe the algebra $\mathcal{A}(F)$, the modules $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$, and the box tensor product, all in the case of $F = T^2$. When $F$ is a torus, $\mathcal{A}(F)$ is merely a graded algebra, while when $g(F) \geq 2$, it is a *differential* graded algebra. At the end of this section, we note the modifications needed in the more general case.

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To specify the identification of $T^2$ with $\partial Y_1$ and $-\partial Y_2$, we need to identify a meridian and a longitude of the torus. One way to do this is to specify a handle-decomposition for the surface; that is, a disk with two 1-handles attached such that the resulting boundary is connected and can be capped off with a disk. For technical reasons, we also place a basepoint somewhere along the boundary of the disk.

Schematically, we can represent this information by a *pointed matched circle*, which we think of as the boundary of the disk with markings at the feet of the 1-handles. In this case, the pointed matched circle $\mathcal{Z}$ consists of a circle with five marked points: $a_1$, $a_2$, $a_3$, $a_4$, and $z$, in that order as we traverse the circle in the clockwise direction, where the arc $\alpha_1^a$ has endpoints at $a_1$ and $a_3$, and the arc $\alpha_2^a$ has endpoints at $a_2$ and $a_4$.
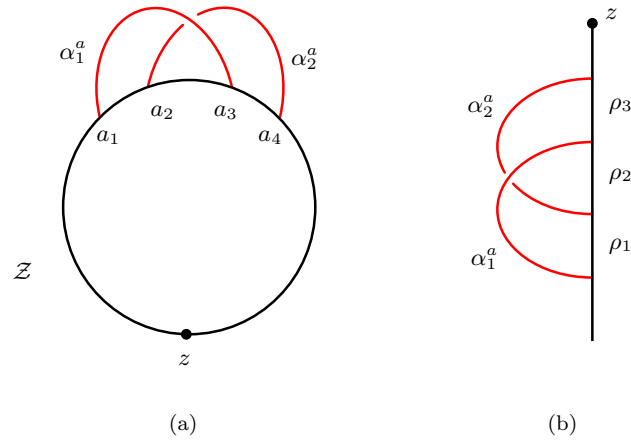


Figure 2.1: Above left, the pointed matched circle for the surface $T^2$. Above right, the same pointed matched circle cut open at $z$.

To the surface $T^2$ parametrized by the pointed matched circle $\mathcal{Z}$, we associate a graded algebra, $\mathcal{A}(T^2)$. The algebra $\mathcal{A}(T^2)$ is generated over $\mathbb{F}$ by the two idempotents

$$\iota_1 \quad \text{and} \quad \iota_2,$$

and the six "Reeb" elements

$$\rho_1, \ \rho_2, \ \rho_3, \ \rho_{12}, \ \rho_{23}, \ \rho_{123}.$$

These algebra elements may be understood pictorially, as in Figure 2.2, where multiplication is understood to correspond to concatenation.
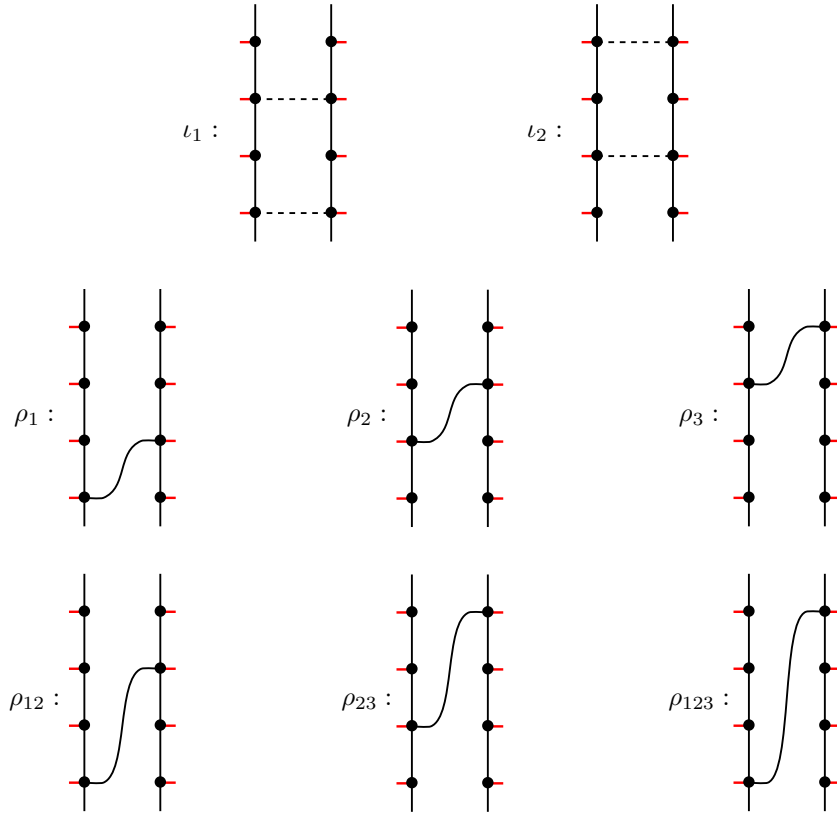


Figure 2.2: The idempotents and algebra elements.

The idempotents correspond to $\alpha_1^a$ and $\alpha_2^a$, respectively. We will often need to consider the *ring of idempotents*,

$$\mathcal{I} = \mathbb{F}\langle \iota_1 \rangle \oplus \mathbb{F}\langle \iota_2 \rangle.$$

We have the following compatibility conditions with the idempotents:

$$\rho_1 = \iota_1 \rho_1 = \rho_1 \iota_2 \qquad \rho_2 = \iota_2 \rho_2 = \rho_2 \iota_1 \qquad \rho_3 = \iota_1 \rho_3 = \rho_3 \iota_2$$

$$\rho_{12} = \iota_1 \rho_{12} = \rho_{12} \iota_1 \qquad \rho_{23} = \iota_2 \rho_{23} = \rho_{23} \iota_2 \qquad \rho_{123} = \iota_1 \rho_{123} = \rho_{123} \iota_2,$$

and the following non-zero products:

$$\rho_1 \rho_2 = \rho_{12} \qquad \rho_2 \rho_3 = \rho_{23} \qquad \rho_1 \rho_{23} = \rho_{12} \rho_3 = \rho_{123}.$$

We will let $\rho_1$ refer to the arc of $\mathcal{Z} - \{z\}$ between $a_1$ and $a_2$, $\rho_2$ the arc between $a_2$ and $a_3$, and $\rho_3$ the arc between $a_3$ and $a_4$. Similarly, $\rho_{12}$, $\rho_{23}$ and $\rho_{123}$ will refer to the appropriate concatenations.

This completes the description of the algebra $\mathcal{A}(T^2)$. See Section 3 of [7] for the full description of the algebra in the general case.

A *bordered Heegaard diagram* for a 3-manifold $Y$ with $\partial Y = T^2$ is a tuple $(\overline{\Sigma}, \boldsymbol{\alpha}^c, \boldsymbol{\alpha}^a, \boldsymbol{\beta}, z)$ consisting of the following:

- a compact, oriented surface $\overline{\Sigma}$ of genus $g$ with a single boundary, $\partial\overline{\Sigma}$

- a $(g-1)$-tuple of pairwise disjoint circles $\boldsymbol{\alpha}^c = (\alpha_1^c, \ldots, \alpha_{g-1}^c)$ in the interior of $\overline{\Sigma}$

- a pair of disjoint arcs $\boldsymbol{\alpha}^a = (\alpha_1^a, \alpha_2^a)$ in $\overline{\Sigma} \backslash \boldsymbol{\alpha}^c$ with endpoints on $\partial\overline{\Sigma}$

- a $g$-tuple of pairwise disjoint circles $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)$ in the interior of $\overline{\Sigma}$

- a basepoint $z$ on $\mathcal{Z} \backslash \partial\boldsymbol{\alpha}^a$

Let $\boldsymbol{\alpha}$ denote $\boldsymbol{\alpha}^c \cup \boldsymbol{\alpha}^a$. We further require that all intersections be transverse and that $\overline{\Sigma} \backslash \boldsymbol{\alpha}$ and $\overline{\Sigma} \backslash \boldsymbol{\beta}$ are connected. The data $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\}$ describes a pointed matched circle.

For bordered Heegaard diagrams, there are two types of periodic domains, and hence two notions of admissibility. Consider closed domains $\mathcal{P}$ in $\overline{\Sigma}$ whose interiors consist of linear combinations of connected components in $\overline{\Sigma} \backslash (\boldsymbol{\alpha}^a, \boldsymbol{\alpha}^c, \boldsymbol{\beta})$. We call $\mathcal{P}$ a *periodic domain* if $\partial\mathcal{P}$ consists of a collection of $\alpha$-arcs, $\alpha$-circles, $\beta$-circles, and arcs in $\partial\overline{\Sigma}$, with $z \notin \partial\mathcal{P}$. We call $\mathcal{P}$ a *provincial periodic domain* if $\partial\mathcal{P}$ consists of a collection of full $\alpha$-circles and $\beta$-circles, with $z \notin \partial\mathcal{P}$. Notice that this implies that $\mathcal{P}$ is not adjacent to $\partial\overline{\Sigma}$.

We say a bordered Heegaard diagram is *provincially admissible* if every provincial periodic domain has both positive and negative multiplicities. A bordered Heegaard diagram is *admissible* if every periodic domain has both positive and negative multiplicities. Provincial admissibility is sufficient for the bordered invariants to be well-defined, and admissibility is sufficient for the bordered invariants to be bounded. We will return to this point in more detail later in this section.

To construct a 3-manifold with parameterized boundary, $Y$, from the data $(\overline{\Sigma}, \boldsymbol{\alpha}^c, \boldsymbol{\alpha}^a, \boldsymbol{\beta}, z)$ , we attach 2-handles to $\overline{\Sigma} \times [0,1]$ along $\boldsymbol{\alpha}^c \times \{0\}$ and $\boldsymbol{\beta} \times \{1\}$. The parametrization of the boundary of $Y$ is given by the identification of $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\} \times \{0\}$ with the pointed matched circle $\mathcal{Z}$.

Let $\mathcal{H}$ be a bordered Heegaard diagram for $Y$. We will now describe the invariants $\widehat{CFD}(\mathcal{H})$ and $\widehat{CFA}(\mathcal{H})$. The set of generators, $\mathfrak{S}(\mathcal{H})$, are unordered $g$-tuples of intersection points of $\alpha$- and $\beta$-curves such that

- each $\beta$-circle is occupied exactly once

- each $\alpha$-circle is occupied exactly once

- each $\alpha$-arc is occupied at most once.

In the case we are considering, where $\partial Y = T^2$, notice that these conditions imply that exactly one of the $\alpha$-arcs is occupied.

We now define the Type $D$ structure $\widehat{CFD}(\mathcal{H})$. We identify $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\}$ with $-\mathcal{Z}$. Let $\widehat{CFD}(\mathcal{H})$, or simply $\widehat{CFD}$, denote the $\mathbb{F}$-vector space generated by $\mathfrak{S}(\mathcal{H})$, with the left $\mathcal{I}$-action on $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ defined to be

$$
\iota_1 \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does } not \text{ occupy the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases}
$$

$$
\iota_2 \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does } not \text{ occupy the arc } \alpha_2^a \\ 0 & \text{otherwise.} \end{cases}
$$

We will define maps

$$
\delta_1 : \widehat{CFD} \to \mathcal{A} \otimes \widehat{CFD}
$$

by counting certain pseudo-holomorphic curves. Note that the above tensor product is over the ring of idempotents, $\mathcal{I}$, as are the rest of the tensor products in this section. Let $\Sigma$ denote Int $\overline{\Sigma}$. Define a *decorated source* $S^\triangleright$ to be a topological type of smooth surface $S$ with boundary and a finite number of boundary punctures endowed with

- a labeling of each puncture by one of $-$, $+$, or $e$

- a labeling of each $e$ puncture of $S$ by a Reeb chord $\rho$.

Consider the 4-manifold $\Sigma \times [0,1] \times \mathbb{R}$, with the following projection maps:

$$\pi_\Sigma : \Sigma \times [0,1] \times \mathbb{R} \to \Sigma$$

$$\pi_D : \Sigma \times [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$$

$$\pi_I : \Sigma \times [0,1] \times \mathbb{R} \to [0,1]$$

$$\pi_\mathbb{R} : \Sigma \times [0,1] \times \mathbb{R} \to \mathbb{R}.$$

Let $\Sigma_{\bar{e}}$ denote $\Sigma$ with its puncture filled in. Similarly, let $S_{\bar{e}}$ denote $S$ with its $e$ punctures filled in.
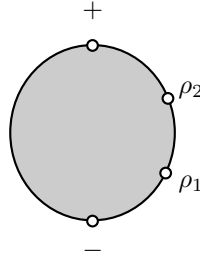


Figure 2.3: A decorated source, $S^\triangleright$.

We are interested in proper maps

$$u : (S, \partial S) \to \left(\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})\right)$$

such that

- The map $\pi_D \circ u$ is a $g$-fold branched cover.

- At each $-$-puncture $q$ of $S$, $\lim_{z \to q}(\pi_\mathbb{R} \circ u)(z) = -\infty$.

- At each $+$-puncture $q$ of $S$, $\lim_{z \to q}(\pi_\mathbb{R} \circ u)(z) = +\infty$.

- At each $e$ puncture $q$ of $S$, $\lim_{z \to q}(\pi_\Sigma \circ u)(z)$ is the Reeb chord $\rho$ labeling $q$.

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- $\pi_\Sigma \circ u$ does not cover the region of $\Sigma$ adjacent to $z$.

- The map $u$ extends to a proper map $u_{\overline{e}} : S_{\overline{e}} \to \Sigma_{\overline{e}} \times [0,1] \times \mathbb{R}$.

- For each $t \in \mathbb{R}$ and each $i = 1, \ldots, g$, there is exactly one point in $u^{-1}(\beta_i \times \{0\} \times \{t\})$. Similarly, for each $t \in \mathbb{R}$ and each $i = 1, \ldots, g-1$, there is exactly one point in $u^{-1}(\alpha_i^c \times \{0\} \times \{t\})$.

We also require the map $u$ to be $J$-holomorphic and of finite energy in the appropriate sense. See Section 5 of [7].

The map $\pi_\mathbb{R} \circ u_{\overline{e}}$ gives an ordering on the $e$ punctures, and their respective labels; this is induced by the $\mathbb{R}$-coordinate of their images. We denote the resulting sequence of Reeb chords by $\overrightarrow{\rho} = (\rho_{i_1}, \ldots, \rho_{i_n})$.

We let $\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})\big)$ denote a certain reduced moduli space. Roughly, this moduli space consists of curves from a decorated source $S^\triangleright$ with asymptotics corresponding to $\overrightarrow{\rho}$ and in the homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, where $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homology classes of curves connecting $\mathbf{x}$ to $\mathbf{y}$. The index $\mathrm{ind}(B, \overrightarrow{\rho})$ is equal to the expected dimension of $\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})\big)$ plus one.

The map $\delta_1$ is defined as:

$$\delta_1(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\overrightarrow{\rho}| \ \mathrm{ind}(B, \overrightarrow{\rho})=1\}}} \#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})\big) \rho_{i_1} \cdot \ldots \cdot \rho_{i_n} \otimes \mathbf{y},$$

where $\#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})\big)$ is the number of points, modulo two, in the zero-dimensional moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})$. Provincial admissibility implies that the sum is well-defined.

As in the theory for closed 3-manifolds, there is a combinatorial formula to compute the index of a map, in terms of the Euler measure and local multiplicities of the associated domain on $\overline{\Sigma}$ (that is, the image of $\pi_\Sigma \circ u$), and the manner in which the domain abuts $\partial\overline{\Sigma}$. See Definition 5.46 of [7].
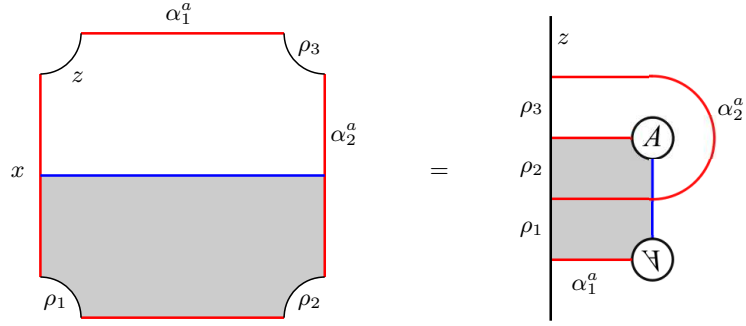
(a) Two different ways of viewing the Heegaard diagram $\mathcal{H}_0$. The shaded region contributes to the map $\delta_1(x) = \rho_{12}x$.
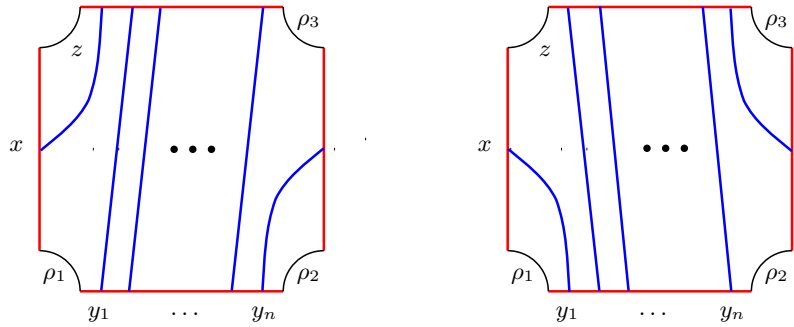


(b) The Heegaard diagram $\mathcal{H}_n$.



(c) The Heegaard diagram $\mathcal{H}_{-n}$.

Figure 2.4: Three bordered Heegaard diagrams for the solid torus, labeled to compute $\widehat{CFD}$, with different parametrizations of the boundary. In 2.4(a), we show two equivalent ways of drawing the Heegaard surface: first, as a square with opposite sides identified, and second, as a disk with a 1-handle attached.

**Example 2.1.** *The invariant* $\widehat{CFD}(\mathcal{H}_0)$ *associated to the bordered Heegaard diagram in Figure 2.4(a) has a single generator $x$ in the idempotent $\iota_1$ and the map*

$$\delta_1 : x \mapsto \rho_{12}x.$$

**Example 2.2.** *The invariant* $\widehat{CFD}(\mathcal{H}_n)$ *associated to the bordered Heegaard diagram in Figure 2.4(b) has one generator $x$ in the idempotent $\iota_1$, $n$ generators $y_1, \ldots, y_n$ in the idempotent $\iota_2$ and*

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the maps

$$\delta_1 : x \mapsto \rho_{123}y_1$$

$$\delta_1 : y_i \mapsto \rho_{23}y_{i+1}, \quad 1 \le i \le n-1$$

$$\delta_1 : y_n \mapsto \rho_2 x.$$

**Example 2.3.** *The invariant* $\widehat{CFD}(\mathcal{H}_{-n})$ *associated to the bordered Heegaard diagram in Figure 2.4(b) has one generator* $x$ *in the idempotent* $\iota_1$, $n$ *generators* $y_1, \dots, y_n$ *in the idempotent* $\iota_2$ *and the maps*

$$\delta_1 : x \mapsto \rho_1 y_1$$

$$\delta_1 : y_i \mapsto \rho_{23}y_{i-1}, \quad 2 \le i \le n$$

$$\delta_1 : x \mapsto \rho_3 y_n.$$

To complete the definition of the Type $D$ structure on $\widehat{CFD}$, we define the maps

$$\delta_k : \widehat{CFD} \to (\mathcal{A}^{\otimes k} \otimes \widehat{CFD})$$

inductively by

$$\delta_0 = \mathbb{I}_{\widehat{CFD}}$$

$$\delta_i = (\mathbb{I}_{\mathcal{A}}^{\otimes i-1} \otimes \delta_1) \circ \delta_{i-1}.$$

Recall that $\widehat{CFD}$ is *bounded* if there exists an integer $N$ such that $\delta_i = 0$ for all $i > N$. Lemma 6.5 of [7] tells us that if $\mathcal{H}$ is admissible, then $\widehat{CFD}(\mathcal{H})$ is bounded.

We now define $\widehat{CFA}$. We identify $\{\partial\overline{\Sigma}, \ z, \ \partial\alpha_1^a, \ \partial\alpha_2^a\}$ with $\mathcal{Z}$. As an $\mathbb{F}$-vector space, $\widehat{CFA}$ is generated by $\mathfrak{S}(\mathcal{H})$, with the right $\mathcal{I}$-action defined to be

$$\mathbf{x}\iota_1 = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ occupies the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{x}\iota_2 = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ occupies the arc } \alpha_2^a \\ 0 & \text{otherwise.} \end{cases}$$

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The $\mathcal{A}_\infty$-structure on $\widehat{CFA}$ is defined by counting certain pseudoholomorphic curves, giving maps

$$m_{j+1} : \widehat{CFA} \otimes \mathcal{A}^{\otimes j} \to \widehat{CFA},$$

defined to be

$$m_{j+1}(\mathbf{x}, \rho_{i_1}, \ldots, \rho_{i_j}) = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(B, \overrightarrow{\rho})=1}} \#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})\big)\mathbf{y}$$

$$m_2(\mathbf{x}, 1) = \mathbf{x}$$

$$m_{j+1}(\mathbf{x}, \ldots, 1, \ldots) = 0, \quad j > 1.$$

As in the case of $\widehat{CFD}$, provincial admissibility of the Heegaard diagram guarantees that the above sum is well-defined. Recall that $\widehat{CFA}$ is *bounded* if there exists an integer $N$ such that $m_j = 0$ for all $j > N$. If $\mathcal{H}$ is admissible, then $\widehat{CFA}$ is bounded.

**Example 2.4.** *The invariant $\widehat{CFA}(\mathcal{H})$ associated to the bordered Heegaard diagram in Figure 2.5(a) has a single generator $b$ in the idempotent $\iota_2$, with the algebra relations*

$$m_{3+i}(b, \rho_2, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) = b, \quad i \geq 0.$$

**Example 2.5.** *The invariant $\widehat{CFA}(\mathcal{H}')$ associated to the bordered Heegaard diagram in Figure 2.5(b) has two generators $a_1$ and $a_2$ in the idempotent $\iota_1$ and a single generator $b$ in the idempotent $\iota_2$, with the algebra relations*
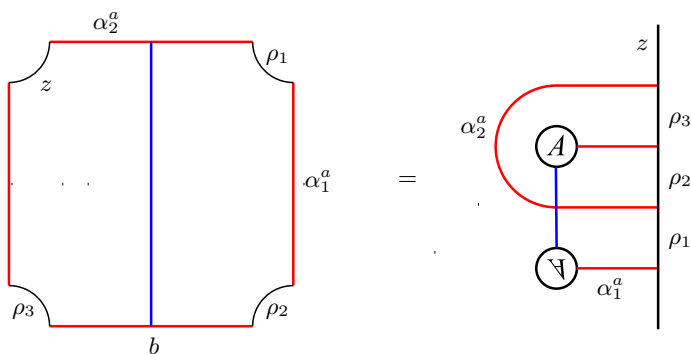
$$m_1(a_1) = a_2$$

$$m_2(a_1, \rho_{12}) = a_2$$

$$m_2(a_1, \rho_1) = b$$

$$m_2(b, \rho_2) = a_2$$

The tensor product $\widehat{CFA} \boxtimes \widehat{CFD}$ is the $\mathbb{F}$-vector space

$$\widehat{CFA} \otimes_{\mathcal{I}} \widehat{CFD},$$

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(a) The Heegaard diagram $\mathcal{H}$.
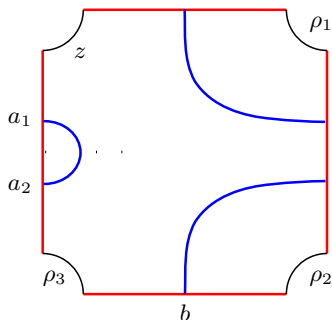


(b) The admissible Heegaard diagram $\mathcal{H}'$.

Figure 2.5: A bordered Heegaard diagram for the framed solid torus, labeled to compute the invariant $\widehat{CFA}$. In 2.5(a), we show the Heegaard surface first as a square with opposite sides identified, and second as disk with a 1-handle attached. In 2.5(b), we have isotoped the $\beta$-circle so that the diagram is admissible.

equipped with the differential

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_{\widehat{CFD}})(\mathbf{x} \otimes \delta_k(\mathbf{y})).$$

If at least one of $\widehat{CFA}$ and $\widehat{CFD}$ is bounded, then the above sum is guaranteed to be finite.

The following description of the tensor product in terms of a basis for $\mathcal{A}$ is often useful for

calculations. Define $\rho_\emptyset$ to be $\iota_1 + \iota_2$. Then we can rewrite $\delta_1$ as

$$\delta_1 = \sum_i \rho_i \otimes D_i,$$

where the sum is taken over $i \in \{\emptyset, 1, 2, 3, 12, 23, 123\}$, and the $D_i$ are coefficient maps

$$D_i : \widehat{CFD} \to \widehat{CFD}.$$

The tensor product $\widehat{CFA} \boxtimes \widehat{CFD}$ is still the $\mathbb{F}$-vector space

$$\widehat{CFA} \otimes_{\mathcal{I}} \widehat{CFD},$$

with the differential now given by

$$\partial^\boxtimes(\mathbf{x} \otimes \mathbf{y}) = \sum m_{k+1}(\mathbf{x}, \rho_{i_1}, \ldots, \rho_{i_k}) D_{i_k} \circ \ldots \circ D_{i_1}(\mathbf{y}),$$

where the sum is taken over all $k$-element sequences $i_1, \ldots, i_k$ (including the empty sequence when $k = 0$) of elements in $\{\emptyset, 1, 2, 3, 12, 23, 123\}$.

**Example 2.6.** *Using Examples 2.2 and 2.5 above, we compute the tensor product $\widehat{CFA}(\mathcal{H}') \boxtimes \widehat{CFD}(\mathcal{H}_n) \simeq \widehat{CF}(L_{n,1})$. The generators in the tensor product are*

$$a_1 x, \quad a_2 x, \quad b y_1, \quad \ldots, \quad b y_n,$$

*with the differential*

$$\partial(a_1 x) = a_2 x.$$

**Example 2.7.** *Using Examples 2.1 and 2.5 above, we compute the tensor product $\widehat{CFA}(\mathcal{H}') \boxtimes \widehat{CFD}(\mathcal{H}_0) \simeq \widehat{CF}(S^1 \times S^2)$. Note that the only nontrivial coefficient map on $\widehat{CFD}(\mathcal{H}_0)$ is*

$$D_{12}(x) = x.$$

*The generators in the tensor product are $a_1 x$ and $a_2 x$, with trivial differential since*

$$\partial(a_1 x) = m_1(a_1) \otimes x + m_2(a_1, \rho_{12}) \otimes D_{12}(x) = a_2 x + a_2 x = 0 \quad \text{and} \quad \partial(a_2 x) = 0.$$

We conclude this section by highlighting a few of the differences that occur when $\partial Y$ has genus $\geq 2$.

A bordered Heegaard diagram for a 3-manifold $Y$ with parameterized boundary $F$ of genus $k$ consists of a punctured surface $\overline{\Sigma}$ of genus $g$, a $g$-tuple of $\beta$-circles, a $(g-k)$-tuple of $\alpha$-circles, a $2k$-tuple of $\alpha$-arcs, and a basepoint $z$ on $\partial\overline{\Sigma}$, such that $\overline{\Sigma}\backslash\boldsymbol{\alpha}$ and $\overline{\Sigma}\backslash\boldsymbol{\beta}$ are connected (where $\boldsymbol{\alpha}$ denotes the collection of $\alpha$-arcs and -circles, and $\boldsymbol{\beta}$ denotes the collection of $\beta$-circles). Notice that when $g(F) \geq 2$, there is a not a unique parametrization (i.e., handle-decomposition) of the diffeomorphism type of $F$, hence there is not a unique algebra associated to the diffeomorphism type of the surface $F$. In other words, for different pointed matched circle describing a surface of genus $g(F)$, we get a different algebra.

Furthermore, while in the case of torus boundary, $\mathcal{A}(T^2)$ is a simply a graded algebra, the algebra associated to a parameterized surface of genus 2 or higher is a differential graded algebra. See Section 3 of [7] for details.

In the case of torus boundary, each generator $\mathbf{x}$ occupies exactly one $\alpha$-arc, hence for a map $u$, we have at most one Reeb chord occurring at any given time (i.e., the $\mathbb{R}$-coordinate of its image). Thus, the map $u$ allows us to consider a sequence of Reeb chords. In the general case, it is possible for multiple Reeb chords to occur at the same time, so $u$ induces a sequence of *sets* of Reeb chords instead. See Section 5 of [7] for a complete description, or Section 5 of Zarev [22] for a description of the analogous construction in the bordered sutured case.

## 2.3   The knot Floer complex

We now review some basic facts about the various flavors of the knot Floer complex, defined by Ozsváth and Szabó in [11] and independently by Rasmussen in [20]. For an expository overview of these invariants, we again refer the reader to [15]. We specify a knot $K \subset S^3$ by a doubly pointed Heegaard diagram, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, where $w$ and $z$ are each basepoints in the complement of

the $\alpha$- and $\beta$-circles. The chain complex $CFK^-(K)$ is freely generated over $\mathbb{F}[U]$ by the set of $g$-tuples of intersection points between the $\alpha$- and $\beta$-circles, where each $\alpha$- and each $\beta$-circle are used exactly once, and $g$ is the genus of the surface $\Sigma$. The differential is defined as

$$\partial \mathbf{x} := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(\phi) = 1}} \#\widehat{\mathcal{M}}(\phi) \, U^{n_w(\phi)} \cdot \mathbf{y}.$$

This complex has a homological $\mathbb{Z}$-grading, called the *Maslov grading M*, as well as a $\mathbb{Z}$-filtration, called the *Alexander filtration A*. The relative Maslov and Alexander gradings are defined as

$$M(\mathbf{x}) - M(\mathbf{y}) = \mathrm{ind}(\phi) - 2n_w(\phi) \qquad \text{and} \qquad A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi),$$

for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. The differential, $\partial$, decreases the Maslov grading by one, and respects the Alexander filtration; that is,

$$M(\partial \mathbf{x}) = M(\mathbf{x}) - 1 \qquad \text{and} \qquad A(\partial \mathbf{x}) \leq A(\mathbf{x}).$$

Multiplication by $U$ shifts the Maslov grading and respects the Alexander filtration as follows:

$$M(U \cdot \mathbf{x}) = M(\mathbf{x}) - 2 \qquad \text{and} \qquad A(U \cdot \mathbf{x}) = A(\mathbf{x}) - 1.$$

Setting $U = 0$, we obtain the filtered chain complex $\widehat{CFK}(K) = CFK^-(K)/(U = 0)$. The total homology of $\widehat{CFK}(K)$ is isomorphic to $\widehat{HF}(S^3) \cong \mathbb{F}$. The normalization for the Maslov grading is chosen so that the generator for $\widehat{HF}(S^3)$ lies in Maslov grading zero. We denote the homology of the associated graded object of $\widehat{CFK}(K)$ by

$$\widehat{HFK}(K) = \bigoplus_s \widehat{HFK}(K, s),$$

where $s$ indicates the Alexander grading induced by the filtration. Similarly, we denote the homology of the associated graded object of $CFK^-(K)$ by

$$HFK^-(K) = \bigoplus_s HFK^-(K, s).$$

We normalize the Alexander grading so that

$$\min\{s \mid \widehat{HFK}(K,s) \neq 0\} = -\max\{s \mid \widehat{HFK}(K,s) \neq 0\}.$$

Equivalently, we can define the absolute Alexander grading of a generator $\mathbf{x}$ to be

$$A(\mathbf{x}) = \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle,$$

where $\widehat{F}$ is a Seifert surface for $K$ capped off in the 0-surgery, $\underline{\mathfrak{s}}(\mathbf{x}) \in \mathrm{Spin}^c(S_0^3(K))$ denotes the Spin$^c$ structure over $S_0^3(K)$ associated to the generator $\mathbf{x}$ by the basepoints $w$ and $z$, and $c_1(\underline{\mathfrak{s}}(\mathbf{x}))$ is the relative first Chern class of $\underline{\mathfrak{s}}(\mathbf{x})$; see [11, Section 3].

At times, we will consider the closely related complex

$$CFK^\infty(K) := CFK^-(K) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}],$$

which is naturally a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex, with one filtration induced by the Alexander filtration and the other by $-(U\text{-exponent})$. It is often convenient to view $CFK^\infty(K)$ and $CFK^-(K)$ graphically in the $(i,j)$-plane, suppressing the homological grading from the picture, where the $i$-coordinate corresponds to $-(U\text{-exponent})$, and the $j$-coordinate corresponds to the Alexander grading. An element of the form $U^i \cdot x$ is plotted at the coordinate $(-i, A(U^i \cdot x))$, or equivalently, $(-i, A(x) - i)$. In particular, the complex $CFK^-(K)$ is contained in the part of the $(i,j)$-plane with $i \leq 0$, and a generator $x$ of $CFK^-(K)$ has coordinates $(0, A(x))$.

We may denote the differential by arrows which will necessarily point non-strictly downwards and to the left. We say that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. Given $S \subset \mathbb{Z} \oplus \mathbb{Z}$, we let $C\{S\}$ denote the set of elements in $CFK^\infty(K)$ whose $(i,j)$-coordinates are in $S$. If $S$ has the property that $(i,j) \in S$ implies that $(i',j') \in S$ for all $(i',j') \leq (i,j)$, then $C\{S\}$ inherits the structure of a subcomplex. Similarly, for appropriate $S$, $C\{S\}$ may inherit the structure of a quotient complex, or of a subquotient complex. For example, $\widehat{CFK}(K)$ is the subquotient complex $C\{i = 0\}$.

The integer-valued smooth concordance invariant $\tau(K)$ is defined in [9] to be

$$\tau(K) := \min\{s \mid \iota : C\{i = 0, j \leq s\} \to C\{i = 0\} \text{ induces a non-trivial map on homology}\},$$

where $\iota$ is the natural inclusion of chain complexes. Alternatively, $\tau(K)$ may be defined in terms of the $U$-action on $HFK^-(K)$, as in [18, Appendix A]:

$$\tau(K) = -\max\{s \mid \exists\, [\mathbf{x}] \in HFK^-(K, s) \text{ such that } \forall\, d \geq 0,\ U^d[\mathbf{x}] \neq 0\}.$$

Recall that the complex $CFK^\infty(K)$ is doubly filtered, by the Alexander filtration, and by powers of $U$. Taking the degree $0$ part of the associated graded object with respect to the Alexander filtration, we define the *horizontal complex*,

$$C^{\mathrm{horz}} := C\{j = 0\},$$

equipped with a differential, $\partial^{\mathrm{horz}}$. Graphically, this can be viewed as the subquotient complex of $CFK^\infty(K)$ consisting of elements with $j$-coordinate equal to zero, with the induced differential consisting of horizontal arrows pointing non-strictly to the left. The horizontal complex inherits the structure of a $\mathbb{Z}$-filtered chain complex, with the filtration induced by $-(U$-exponent$)$. Similarly, we may consider the degree $0$ part of the associated graded object with respect to the filtration by powers of $U$, and define the *vertical complex*,

$$C^{\mathrm{vert}} := C\{i = 0\},$$

equipped with a differential, $\partial^{\mathrm{vert}}$. Note that this is equivalent to $CFK^-(K)/(U \cdot CFK^-(K))$. In the vertical complex, the induced differential may be graphically depicted as vertical arrows pointing non-strictly downwards. The vertical complex inherits the structure of a $\mathbb{Z}$-filtered chain complex, with the filtration induced by the Alexander filtration.

Symmetry properties of $CFK^\infty(K)$ from [11, Section 3.5] show that both $C^{\mathrm{horz}}$ and $C^{\mathrm{vert}}$ are filtered chain homotopy equivalent to $\widehat{CFK}(K)$. (In fact, if we ignore grading and filtration shifts, any row or column is filtered chain homotopic to $\widehat{CFK}(K)$.) More generally, $CFK^\infty(K)$ is filtered chain homotopic to the complex obtained by reversing the roles of $i$ and $j$. The filtered chain homotopy type of $\widehat{CFK}(K)$, $CFK^-(K)$ and $CFK^\infty(K)$ are all invariants of the knot $K$.

The chain complex $CFK^-(K)$ is called *reduced* if the differential $\partial$ strictly drops either the

Alexander filtration or the filtration by powers of $U$. Graphically, this means that each arrow points strictly downwards or to the left (or both). A filtered chain complex is always filtered chain homotopic to a reduced complex, i.e., it is filtered chain homotopic to the $E_1$ page of its associated spectral sequence.

A basis $\{x_i\}$ for a filtered chain complex $(C, \partial)$ is called a *filtered basis* if the set $\{x_i \mid x_i \in C_S\}$ is a basis for $C_S$ for all filtered subcomplexes $C_S \subset C$. Two filtered bases can be related by a filtered change of basis. For example, given a filtered basis $\{x_i\}$, replacing $x_j$ with $x_j + x_k$, where the filtration level of $x_k$ is less than or equal to that of $x_j$, is a filtered change of basis. More generally, we may consider a doubly filtered chain complex with two doubly filtered bases, related by a doubly filtered change of basis.

We say a filtered basis $\{x_i\}$ over $\mathbb{F}[U]$ for the reduced complex $CFK^-(K)$ is *vertically simplified* if for each basis element $x_i$, exactly one of the following holds:

- $x_i$ is in the image of $\partial^{\mathrm{vert}}$ and there exists a unique basis element $x_{i-1}$ such that $\partial^{\mathrm{vert}} x_{i-1} = x_i$.

- $x_i$ is in the kernel, but not the image, of $\partial^{\mathrm{vert}}$.

- $x_i$ is not in the kernel of $\partial^{\mathrm{vert}}$, and $\partial^{\mathrm{vert}} x_i = x_{i+1}$.

(In the statements above, we are considering the basis that $\{x_i\}$ naturally induces on $C^{\mathrm{vert}}$; that is, $\{x_i \bmod(U \cdot CFK^-(K))\}$. For ease of exposition, we suppress this from the notation.) When $\partial^{\mathrm{vert}} x_i = x_{i+1}$, we say that there is a vertical arrow from $x_i$ to $x_{i+1}$, and the length of this arrow is $A(x_i) - A(x_{i+1})$. Notice that upon taking homology, the differential $\partial^{\mathrm{vert}}$ cancels basis elements in pairs. Since $H_*(C^{\mathrm{vert}}) \cong \mathbb{F}$, there is a distinguished element, which after reordering we denote $x_0$, with the property that it has no incoming or outgoing vertical arrows.

Similarly, we define what it means for a filtered basis $\{x_i\}$ over $\mathbb{F}[U]$ for the reduced complex $CFK^-(K)$ to be horizontally simplified. Notice that $\{U^{m_i} x_i\}$, where $m_i = A(x_i)$, naturally induces a basis on $C^{\mathrm{horz}}$. We say the basis $\{x_i\}$ is *horizontally simplified* if for each basis element

$x_i$, exactly one of the following holds:

- $U^{m_i}x_i$ is in the image of $\partial^{\mathrm{horz}}$ and there exists a unique basis element $x_{i-1}$ such that

  $$\partial^{\mathrm{horz}}U^{m_{i-1}}x_{i-1} = U^{m_i}x_i.$$

- $U^{m_i}x_i$ is in the kernel, but not the image, of $\partial^{\mathrm{horz}}$.

- $U^{m_i}x_i$ is not in the kernel of $\partial^{\mathrm{horz}}$, and $\partial^{\mathrm{horz}}U^{m_i}x_i = U^{m_{i+1}}x_{i+1}$.

When $\partial^{\mathrm{horz}}U^{m_i}x_i = U^{m_{i+1}}x_{i+1}$, we say that there is a horizontal arrow from $x_i$ to $x_{i+1}$, and the length of this arrow is $A(x_i) - A(x_{i+1})$. Notice that upon taking homology, the differential $\partial^{\mathrm{horz}}$ cancels basis elements in pairs. Since $H_*(C^{\mathrm{horz}}) \cong \mathbb{F}$, there is a distinguished element, which after reordering we denote $x_0$, with the property that it has no incoming or outgoing horizontal arrows.

The following technical fact, proven at the end of this section, will be of use to us:

**Lemma 2.8.** $CFK^-(K)$ *is* $\mathbb{Z} \oplus \mathbb{Z}$-*filtered,* $\mathbb{Z}$-*graded homotopy equivalent to a chain complex* $C$ *that is reduced. Moreover, one can find a vertically simplified basis over* $\mathbb{F}[U]$ *for* $C$, *or, if one would rather, a horizontally simplified basis over* $\mathbb{F}[U]$ *for* $C$.

## 2.4 From the knot Floer complex to the bordered invariant

Theorems 10.17 and 11.7 of [7] give an algorithm for computing $\widehat{CFD}(Y)$ for a framed knot complement $Y = S^3 - \mathrm{nbd}\, K$ from $CFK^-(K)$. More precisely, we frame the knot complement by letting $\alpha_1^a$ correspond to an $n$-framed longitude, and $\alpha_2^a$ to a meridian. We recount the algorithm from $CFK^-$ to $\widehat{CFD}$ here.

Let $\{x_i\}$ be a vertically simplified basis. Then $\{x_i\}$ is a basis for $\iota_1\widehat{CFD}(Y)$. To each arrow of length $\ell$ from $x_i$ to $x_{i+1}$ we introduce a string of basis elements $y_1^i, \ldots, y_\ell^i$ for $\iota_2\widehat{CFD}(Y)$ and differentials

$$x_i \xrightarrow{D_1} y_1^i \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} y_k^i \xleftarrow{D_{23}} y_{k+1}^i \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} y_\ell^i \xleftarrow{D_{123}} x_{i+1}.$$

Note the directions of the arrows. Similarly, let $\{x'_i\}$ be a horizontally simplified basis. Then $\{x'_i\}$ is also a basis for $\iota_1\widehat{CFD}(Y)$. To each arrow of length $\ell$ from $x'_i$ to $x'_{i+1}$ we introduce a string of basis elements $w^i_1, \ldots, w^i_\ell$ for $\iota_2\widehat{CFD}(Y)$ and differentials

$$x'_i \xrightarrow{D_3} w^i_1 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} w^i_k \xrightarrow{D_{23}} w^i_{k+1} \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} w^i_\ell \xrightarrow{D_2} x'_{i+1}.$$

Finally, there is the *unstable chain*, consisting of generators $z_1, \ldots, z_m$ connecting $x_0$ and $x'_0$. The form of the unstable chain depends on the framing $n$ relative to $2\tau(K)$. When $n < 2\tau(K)$, we introduce a string of basis elements $z_1, \ldots, z_m$ for $\iota_2\widehat{CFD}(Y)$, where $m = 2\tau(K) - n$, and differentials

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x'_0.$$

When $n = 2\tau(K)$, the unstable chain has the form

$$x_0 \xrightarrow{D_{12}} x'_0.$$

Lastly, when $n > 2\tau(K)$, the unstable chain has the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x'_0,$$

where $m = n - 2\tau(K)$.

**Remark 2.9.** It is often possible to find a basis for $CFK^-$ that is simultaneously vertically and horizontally simplified. It is an open question whether or not there always exists such a basis.

**Remark 2.10.** The examples in Figure 2.4 can be viewed as framed complements of the unknot, verifying the above algorithm in the special case where $K$ is the unknot.

We conclude this section with the proof of Lemma 2.8.

*Proof of Lemma 2.8.* We need to show that we can find a basis over $\mathbb{F}[U]$ for $CFK^-(K)$ that is vertically simplified. What follows is essentially the well-known "cancellation lemma" for chain complexes in the filtered setting.

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Let $\{x_i\}$ be a filtered basis (over $\mathbb{F}$) for $C^{\text{vert}} = C/U \cdot C$. For the remainder of the proof, we will let $\partial$ denote the differential on $C^{\text{vert}}$. Consider the set

$$B_n = \{x_i \mid A(\partial x_i) = A(x_i) - n\}.$$

We will prove the lemma by induction. Note that $B_{-1} = \emptyset$, since the differential $\partial$ respects the Alexander filtration. We say that $B_n$ is *simplified with respect to the basis* $\{x_i\}$ if $\{B_n, \partial B_n\}$ is a direct summand of $C^{\text{vert}}$ such that $\{x_i \mid x_i \in B_n\} \cup \{\partial x_i \mid x_i \in B_n\}$ form a simplified basis for $\{B_n, \partial B_n\}$.

Assume that $B_0, B_1 \ldots, B_{n-1}$ are simplified with respect to $\{x_i\}$. We will find a change of basis from $\{x_i\}$ to $\{x_i'\}$ so that $B_n$ is simplified as well. If $\partial x_i = \sum c_j x_j$, then define

$$\partial_j x_i := c_j.$$

For $x_j \in B_n$, we would like to perform a change of basis such that $x_j'$ and $\partial x_j'$ are elements in the new basis and form a direct summand. We begin by noticing that

$$x_j \in B_n \text{ implies } \exists\, k \text{ such that } \partial_k x_j = 1 \text{ and } A(x_k) = A(x_j) - n.$$

We now choose a new filtered basis $\{x_i'\}$ as follows:

$$x_j' = x_j$$

$$x_k' = \partial x_j$$

$$x_\ell' = x_\ell + (\partial_k x_\ell) x_j, \quad \ell \neq j, k.$$

This is a filtered change of basis. Indeed, we have that $A(\partial x_j) = A(x_k)$ by construction. Whenever $\partial_k x_\ell \neq 0$, we have that $A(x_\ell) \geq A(x_k) + n$, by the assumption that $B_0, \ldots, B_{n-1}$ are simplified with respect to $\{x_i\}$; then $A(x_\ell) \geq A(x_j)$, since $A(x_k) = A(x_j) - n$.

Now if $\partial x_i' = \sum c_j' x_j'$, then similarly define $\partial_j' x_i' = c_j'$, and we notice that

$$\partial_j' x_\ell' = 0 \quad \forall\, \ell$$

$$\partial_k' x_\ell' = 0 \quad \forall\, \ell \neq j.$$

Indeed, that first equation follows from the facts that

$$\partial x_j' = x_k'$$

$$\partial_k' x_m' = 0 \ \forall \ m \neq j$$

$$\partial^2 = 0.$$

The second equation is true by construction of the basis $\{x_i'\}$. Notice that in this process, we have left $B_0, \ldots, B_{n-1}$ unchanged, and that we have not increased the size of $B_n$.

Now $\{x_j', x_k'\}$ splits as a direct summand as desired. Iterating this process, we can continue to change bases until $B_n$ is simplified with respect to our new basis. By induction, we can construct a simplified basis for all of $C^{\mathrm{vert}}$. This basis is also a basis for $C$.

Similarly, we may find a simplified basis for $C^{\mathrm{horz}}$. This, too, will be a basis for $C$. $\qquad\square$

(a)                                        (b)

(c)                                        (d)

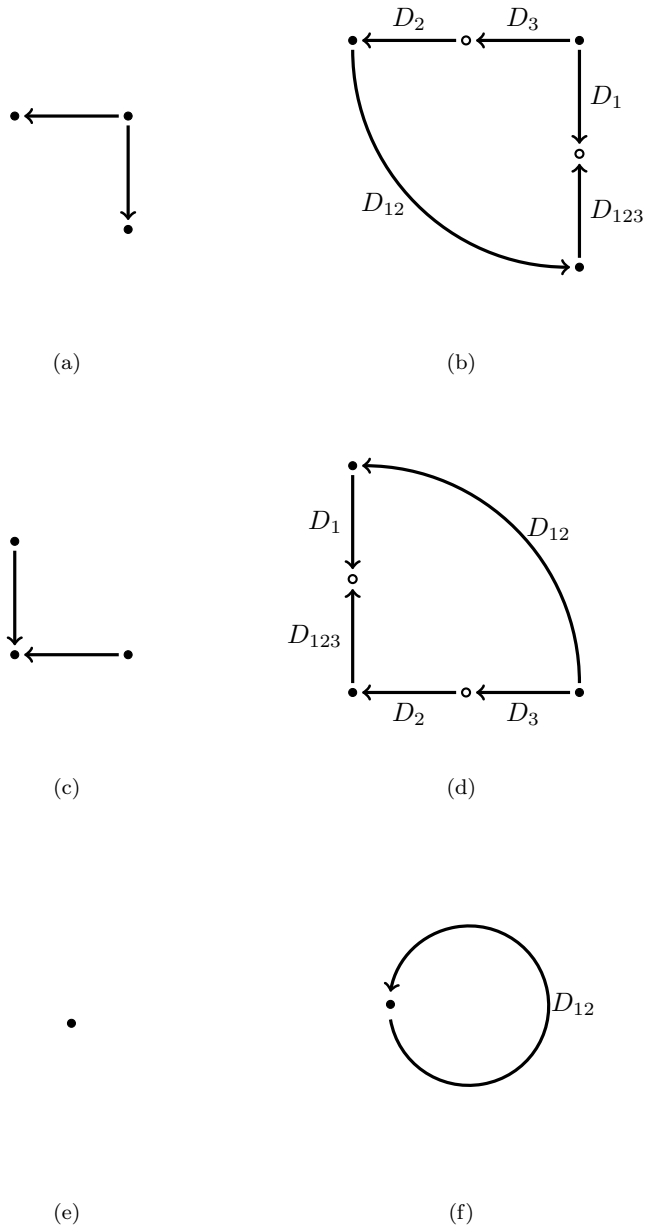(e)                                        (f)

Figure 2.6: $CFK^\infty$ and $\widehat{CFD}$ for different knots and their framed complements, respectively. (To be precise, $CFK^\infty(K)$ is the above diagram tensored with $\mathbb{F}[U, U^{-1}]$.) Top, $K$ is the right-handed trefoil, and the framing on the complement is $2\tau(K) = 2$. Middle, $K$ is the left-handed trefoil, and the framing on the complement is $2\tau(K) = -2$. Bottom, $K$ is the unknot, and the framing is $2\tau(K) = 0$.

# Chapter 3

# Definition and properties of $\varepsilon(K)$

We will begin by defining the invariant $\varepsilon(K)$, in terms of $\tau(K)$ and the invariant $\nu(K)$, defined by

Ozsváth and Szabó in [17, Definition 9.1]. Let $(A_s, \partial_s)$ and $(A_s', \partial_s')$ be the following sub-quotient

complexes of $CFK^\infty(K)$:

$$A_s = C\{\max(i, j - s) = 0\}$$

$$A_s' = C\{\min(i, j - s) = 0\},$$

with induced differentials $\partial_s$ and $\partial_s'$, respectively. We consider both $A_s$ and $A_s'$ as complexes

over $\mathbb{F}$. Notice that if $\{x_i\}$ is a basis for $CFK^-(K)$, then $\{U^{n_i} x_i\}$ is a basis for $A_s$, where

$n_i = \max(0, A(x_i) - s)$. Similarly, $\{U^{m_i} x_i\}$ will be a basis for $A_s'$, where $m_i = \min(0, A(x_i) - s)$.

We have a map $v_s : A_s \to \widehat{CF}(S^3)$ defined as the following composition:

$$A_s \to C\{i = 0, j \le s\} \to C\{i = 0\} \simeq \widehat{CF}(S^3)$$

where the first map is projection and the second is inclusion. Similarly, we have a map $v_s'$ :

$\widehat{CF}(S^3) \to A_s'$ defined as the following composition:

$$\widehat{CF}(S^3) \simeq C\{i = 0\} \to C\{i = 0, j \ge s\} \to A_s'$$

where the first map is projection and the second is inclusion. These definitions have geometric significance. Recall from [11, Section 4] that for $N \in \mathbb{N}$ sufficiently large,

$$A_s \simeq \widehat{CF}(S_N^3(K), [s])$$

where $S_N^3(K)$ denotes $N$-surgery along $K$, $|s| \leq N/2$, and $[s]$ denotes the Spin$^c$ structure $\mathfrak{s}_s$ that extends over the corresponding 2-handle cobordism with the property that

$$\langle c_1(\mathfrak{s}_s), [\widehat{F}] \rangle + N = 2s,$$

where $\widehat{F}$ denotes the capped off Seifert surface in the 4-manifold. The map

$$v_s : \widehat{CF}(S_N^3(K), [s]) \to \widehat{CF}(S^3)$$

is induced by the cobordism from $S_N^3(K)$ to $S^3$ endowed with the Spin$^c$ structure above. Similarly,

$$A_s' \simeq \widehat{CF}(S_{-N}^3(K), [s]),$$

and

$$v_s' : \widehat{CF}(S^3) \to \widehat{CF}(S_{-N}^3(K), [s])$$

corresponds to the 2-handle cobordism from $S^3$ to $S_{-N}^3(K)$.

**Definition 3.1.** *Define $\nu(K)$ by*

$$\nu(K) = \min\{s \in \mathbb{Z} \mid v_s : A_s \to \widehat{CF}(S^3) \text{ induces a non-trivial map on homology}\}.$$

*Similarly, define $\nu'(K)$ by*

$$\nu'(K) = \max\{s \in \mathbb{Z} \mid v_s' : \widehat{CF}(S^3) \to A_s' \text{ induces a non-trivial map on homology}\}.$$

For ease of notation, we will often write $\tau$ for $\tau(K)$ when the meaning is clear from context. Recall from [9, Proposition 3.1] that $\nu'(K)$ is equal to either $\tau - 1$ or $\tau$. The idea is that if $s > \tau$, then $v_s'$ induces the trivial map on homology, because quotienting by $C\{i = 0, j < s\}$ gives the zero map, and if $s < \tau$, then $v_s'$ induces a non-trivial map on homology, because a generator $x$ for $\widehat{HF}(S^3)$

must be supported in the $(i, j)$-coordinate $(0, \tau)$, thus $x$ is not a boundary in $A'_s$. Thus, we should focus on the map $v'_\tau$. In particular,

- $\nu'(K) = \tau - 1$ if and only if $v'_\tau$ is trivial on homology

- $\nu'(K) = \tau$ if and only if $v'_\tau$ is non-trivial on homology.

A similar argument shows that $\nu(K)$ is equal to either $\tau(K)$ or $\tau(K) + 1$; in particular,

- $\nu(K) = \tau$ if and only if $v_\tau$ is non-trivial on homology

- $\nu(K) = \tau + 1$ if and only if $v_\tau$ is trivial on homology.

We now proceed to show that $v_\tau$ and $v'_\tau$ cannot both be trivial on homology. Roughly, the idea is that $v'_\tau$ is trivial on homology when the class $[x]$ generating $\widehat{HF}(S^3) \cong H_*(C\{i = 0\})$ is in the image of the horizontal differential. Thus, $[x]$ must also be in the kernel of the horizontal differential, implying that $v_\tau$ is non-trivial. Similarly, $v_\tau$ is trivial on homology when $[x]$ is not in the kernel of the horizontal differential, hence $[x]$ is not in the image of the horizontal differential, implying that $v'_\tau$ is non-trivial.

The following lemma, and its proof, make this precise:

**Lemma 3.2.** *If $\nu'(K) = \tau(K) - 1$, then $\nu(K) = \tau(K)$, and there exists a horizontally simplified basis $\{x_i\}$ for $CFK^-(K)$ such that, after possible reordering, there is a pair of basis elements, $x_1$ and $x_2$, with the property that:*

*1. $x_2$ is the distinguished element of some vertically simplified basis.*

*2. $\partial^{\mathrm{horz}} x_1 = x_2$.*

*Similarly, if $\nu(K) = \tau(K) + 1$, then $\nu'(K) = \tau(K)$, and there exists a horizontally simplified basis $\{x_i\}$ for $CFK^-(K)$ such that, after possible reordering, there is a pair of basis elements, $x_1$ and $x_2$, with the property that:*

*1. $x_1$ is the distinguished element of some vertically simplified basis.*

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2. $\partial^{\mathrm{horz}} x_1 = x_2$.

*Proof.* Let $x_V$ be the distinguished element of a vertically simplified basis, and let $\partial_s^{\mathrm{horz}}$ be the differential on $C\{j = s\} \simeq C^{\mathrm{horz}}$.

Since $\nu'(K) = \tau - 1$, for any element $x$ that generates $H_*(C\{i = 0\})$, we have that $[x] = 0 \in H_*(A'_\tau)$. Thus, there exists $x' \in A'_\tau$ such that $\partial'_\tau x' = x_V$. Moreover, we may choose $x'$ to be the element of minimal $i$-filtration such that $\partial'_\tau x' = x_V$; this will be convenient to us later. (Note that the complex $A'_s$ inherits a natural $i$-filtration as a subquotient complex of $CFK^\infty$.)

We can write $x'$ as the sum of chains $x_{i=0}$ and $x_{i>0}$, where $x_{i=0} \in C\{i = 0, j > \tau\}$ and $x_{i>0} \in C\{i > 0, j = \tau\}$. Furthermore, $x_{i=0}$ can be taken to be a sum of vertically simplified basis elements that are not in the kernel of the vertical differential. Hence,

- $x_V + \partial^{\mathrm{vert}} x_{i=0}$ generates $H_*(C\{i = 0\})$

- $\partial'_s x_{i>0} = x_V + \partial^{\mathrm{vert}} x_{i=0}$.

We notice that $[\partial_\tau^{\mathrm{horz}} x_{i>0}]$ is non-zero in both $H_*(A_\tau)$ and $H_*(C\{i = 0\})$. Therefore, $\nu(K) = \tau(K)$.

We now need to find an appropriate horizontally simplified basis. Replace $x_V$ with $\partial_\tau^{\mathrm{horz}} x_{i>0}$. This is a filtered change of basis and this new basis element is still the distinguished element of a vertically simplified basis, since elements in $\partial_\tau^{\mathrm{horz}} x_{i>0} + x_V$ either have $i$-coordinate $< 0$, or have $i$-coordinate equal to zero and are in the image of $\partial^{\mathrm{vert}}$.

Now apply the algorithm in Lemma 2.8, splitting off the arrow of length, say $n$, from $x_{i>0}$ to the new $x_V$ first when simplifying $B_n$. This will yield a horizontally simplified basis with the desired property.

The proof in the case that $\nu(K) = \tau(K) + 1$ follows similarly. $\qquad\square$

When both $v_\tau$ and $v'_\tau$ are non-trivial on homology, the class $[x]$ generating $\widehat{HF}(S^3)$, which we identify with $H_*(C\{i = 0\})$, is in the kernel of the horizontal differential but not in the image. We make this more precise in the following lemma:

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**Lemma 3.3.** *If $\nu(K) = \nu'(K)$, then there exists a vertically simplified basis $\{x_i\}$ for $CFK^-(K)$ such that the distinguished element, $x_0$, is also the distinguished element of a horizontally simplified basis for $CFK^-(K)$.*

*Proof.* Note that $\nu(K) = \nu'(K)$ implies that both are equal to $\tau(K)$. Let $\{x_i\}$ be a vertically simplified basis $\{x_i\}$ for $CFK^-(K)$ with distinguished element $x_0$. We have the following series of implications:

1. $\nu(K) = \tau(K)$ implies there exists a chain $x_H \in A_\tau$ such that $v_{\tau*}([x_H]) = [x_0]$.

2. Hence, $[x_H] \neq 0 \in H_*(A_\tau)$, and so $\partial_\tau x_H = 0$.

3. $\partial_\tau^{\text{horz}} x_H = 0$ as well, since $\partial_\tau^{\text{horz}} x_H = \partial_\tau x_H / C\{i = 0, j < \tau\}$.

Notice that since $v_{\tau*}([x_H]) = [x_0]$, it follows that $x_H$ must be equal to $x_0$ plus possibly some basis elements in the image of $\partial^{\text{vert}}$ and some elements in $C\{i < 0, j = \tau\}$. Hence, we may replace our distinguished vertical element $x_0$ with $x_H$.

With $x_H$ is as above, $\nu'(K) = \tau(K)$ implies that $v'_{\tau*}([x_H]) \neq 0 \in H_*(A'_\tau)$. Thus, $x_H \notin \text{Im } \partial'_\tau$. Since $x_H \notin \text{Im } \partial'_\tau$, $x_H$ is not homologous in $C\{j = \tau\} \simeq C^{\text{horz}}$ to anything of strictly lower filtration level. Moreover, $x_H \notin \text{Im } \partial_\tau^{\text{horz}}$. Therefore, there exists a horizontally simplified basis for $CFK^-(K)$ with distinguished element $x_H$, which is also the distinguished element of a vertically simplified basis. $\qquad\square$

We see that there are three different possibilities for the values of the pair $(\nu(K), \nu'(K))$: $(\tau(K), \tau(K) - 1)$, $(\tau(K), \tau(K))$ or $(\tau(K) + 1, \tau(K))$. This motivates the following definition:

**Definition 3.4.** *Define $\varepsilon(K)$ to be*

$$\varepsilon(K) := 2\tau(K) - \nu(K) - \nu'(K).$$

*In particular, $\varepsilon(K)$ can take on the values $-1$, $0$, or $1$.*

**Remark 3.5.** By various symmetry properties of $CFK^\infty(K)$ (see [11, Section 3.5]), we may equivalently define $\varepsilon(K)$ as

$$\varepsilon(K) = \big(\tau(K) - \nu(K)\big) - \big(\tau(\overline{K}) - \nu(\overline{K})\big),$$

where $\overline{K}$ denotes the mirror of $K$.



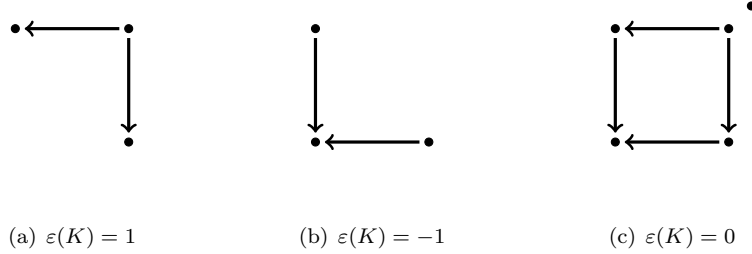(a) $\varepsilon(K) = 1$   (b) $\varepsilon(K) = -1$   (c) $\varepsilon(K) = 0$

Figure 3.1: $CFK^\infty(K)$, for different knots $K$. Above left, $K$ is the right-handed trefoil, which has $\varepsilon(K) = 1$, and the unique generator with no incoming or outgoing vertical arrows lies at the head of a horizontal arrow. Center, $K$ is the left-handed trefoil, which has $\varepsilon(K) = -1$, and the unique generator with no incoming or outgoing vertical arrows lies at the tail of a horizontal arrow. Right, $K$ is the figure 8 knot, which has $\varepsilon(K) = 0$, and the unique generator with no incoming or outgoing vertical arrows also has no incoming or outgoing horizontal arrows.

See Figure 3.1 for examples of knots $K$ with different values of $\varepsilon(K)$. Recall that a knot $K$ is called *homologically thin* if $\widehat{HFK}(K)$ is supported on a single diagonal in the plane whose axes correspond to Alexander and Maslov gradings of the groups.

**Proposition 3.6.** *The following are properties of $\varepsilon(K)$:*

*1. If $K$ is smoothly slice, then $\varepsilon(K) = 0$.*

*2. If $\varepsilon(K) = 0$, then $\tau(K) = 0$.*

*3. $\varepsilon(\overline{K}) = -\varepsilon(K)$.*

4. If $|\tau(K)| = g(K)$, then $\varepsilon(K) = \text{sgn } \tau(K)$.

5. If $K$ is homologically thin, then $\varepsilon(K) = \text{sgn } \tau(K)$.

6. (a) If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K\#K') = \varepsilon(K) = \varepsilon(K')$.

   (b) If $\varepsilon(K) = 0$, then $\varepsilon(K\#K') = \varepsilon(K')$.

*Proof of (1).* If $K$ is slice, then the surgery correction terms defined in [8] vanish (i.e, agree with those of the unknot), and the maps

$$\widehat{HF}(S^3_N(K), [0]) \to \widehat{HF}(S^3) \quad \text{and} \quad \widehat{HF}(S^3) \to \widehat{HF}(S^3_{-N}(K), [0])$$

are non-trivial. Indeed, the surgery corrections terms can be defined in terms of the maps

$$HF^+(S^3) \to HF^+(S^3_{-N}(K), [s])$$

and we have the commutative diagram

$$
\begin{array}{ccc}
\widehat{HF}(S^3) & \xrightarrow{\;v'_s\;} & \widehat{HF}(S^3_{-N}(K), [s]) \\
\iota \downarrow & & \iota_{-N,s} \downarrow \\
HF^+(S^3) & \xrightarrow{\;u'_s\;} & HF^+(S^3_{-N}(K), [s]).
\end{array}
$$

Let $N \gg 0$. If the surgery corrections terms vanish and $s = 0$, then $u'_s$ is an injection and so the composition $\iota \circ u'_s$ is non-trivial. By commutativity of the diagram, it follows that $v'_s$ must be non-trivial. A similar diagram in the case of large positive surgery shows that $v_s$ must be non-trivial as well. Thus, $\nu(K)$ and $\nu'(K)$ both equal $\tau(K) = 0$, and so $\varepsilon(K) = 0$. $\square$

*Proof of (2).* If $\varepsilon(K) = 0$, then by Lemma 3.3, there exists an element $x_0$ that is the distinguished element of both a vertically and horizontally simplified basis. If $A(x_0)$ is the Alexander grading of $x_0$ viewed in $C^{\text{vert}} \simeq \widehat{CFK}(K)$, then $-A(x_0)$ is the Alexander grading of $x_0$ viewed in $C^{\text{horz}} \simeq \widehat{CFK}(K)$. This implies that $\tau(K) = -\tau(K)$, hence $\tau(K) = 0$. $\square$

*Proof of (3).* The symmetry properties of $CFK^\infty(K)$ imply that $\tau(\overline{K}) = -\tau(K)$ and $\nu(\overline{K}) = -\nu'(K)$. Hence, $\varepsilon(\overline{K}) = -\varepsilon(K)$. $\square$

*Proof of (4).* Without loss of generality, we can consider the case $\tau(K) > 0$. By the adjunction inequality for knot Floer homology [11, Theorem 5.1],

$$H_*(C\{i < 0, j = g(K)\}) = 0.$$

Hence by considering the short exact sequence

$$0 \to C\{i < 0, j = g(K)\} \to A_{g(K)} \to C\{i = 0, j \leq g(K)\} \to 0$$

and the fact that the inclusion $C\{i = 0, j \leq g(K)\} \hookrightarrow C\{i = 0\}$ is a quasi-isomorphism (again, by the adjunction inequality), we see that the map $v_{g(K)} : A_{g(K)} \to C\{i = 0\}$ induces an isomorphism on homology. Thus $\nu(K) = \tau(K)$, so $\varepsilon(K)$ is equal to 0 or 1. Since $\tau(K) > 0$, it follows that $\varepsilon(K)$ is not equal to zero, and hence is equal to 1. $\square$

*Proof of (5).* In [19, Lemma 5], Petkova constructs model complexes for $CFK^-(K)$ of homologically thin knots, from which the values of $\tau(K)$ and $\varepsilon(K)$ follow readily. $\square$

*Proof of (6).* Recall from [11, Theorem 7.1] that $CFK^-(K \# K') \simeq CFK^-(K) \otimes_{\mathbb{F}[U]} CFK^-(K')$. We first consider the case where $\varepsilon(K) = \varepsilon(K') = 1$. Then by Lemma 3.2, there exists a horizontally simplified basis $\{x_i\}$ for $CFK^-(K)$ such that

1. $x_2$ is the distinguished element of a vertically simplified basis

2. $\partial^{\mathrm{horz}} x_1 = U^m x_2$,

and similarly, a horizontally simplified basis $\{y_i\}$ for $CFK^-(K')$ with $y_2$ the distinguished element of a vertically simplified basis, and $\partial^{\mathrm{horz}} y_1 = U^n y_2$. Note that $A(x_2) = \tau(K)$ and $A(y_2) = \tau(K')$. Then $A(x_2 \otimes y_2) = \tau(K) + \tau(K') = \tau(K \# K')$, and $[x_2 \otimes y_2] \neq 0 \in H_*(C^{\mathrm{vert}}(K \# K'))$. Recall the map

$$v'_s : C^{\mathrm{vert}} \to A'_s,$$

and notice that $[x_2 \otimes y_2] = 0 \in H_*(A'_{\tau(K \# K')}(K \# K'))$ since $\partial'_{\tau(K \# K')}(U^{-m} x_1 \otimes y_2) = x_2 \otimes y_2$. Hence $\nu'(K \# K') = \tau(K \# K') - 1$, implying that $\varepsilon(K \# K') = 1$. The case where $\varepsilon(K) = \varepsilon(K') = -1$ follows similarly.

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Finally, if $\varepsilon(K) = 0$, then by Lemma 3.3, there exists a basis $\{x_i\}$ for $CFK^-(K)$ such that the element $x_0$ is the distinguished element of both a horizontally simplified basis and a vertically simplified basis. Then to determine $\nu(K \# K')$ and $\nu'(K \# K')$, it is sufficient to consider just $\{x_0\} \otimes CFK^-(K')$, in which case, $\varepsilon(K \# K') = \varepsilon(K')$. $\qquad\square$

# Chapter 4

# Computation of $\tau$ for $(p, pn + 1)$-cables

We will first consider $(p, pn + 1)$-cables, whose Heegaard diagrams are easier to work with, and prove the following version of Theorem 1:

**Theorem 4.1.** $\tau(K_{p,pn+1})$ *behaves in one of three ways. If* $\varepsilon(K) = 1$*, then*

$$\tau(K_{p,pn+1}) = p\tau(K) + \frac{pn(p-1)}{2}.$$

*If* $\varepsilon(K) = -1$*, then*

$$\tau(K_{p,pn+1}) = p\tau(K) + \frac{pn(p-1)}{2} + p - 1.$$

*Finally, if* $\varepsilon(K) = 0$*, then*

$$\tau(K_{p,pn+1}) = \begin{cases} \frac{pn(p-1)}{2} + p - 1 & \text{if } n < 0 \\ \frac{pn(p-1)}{2} & \text{if } n \geq 0. \end{cases}$$

The proof will proceed as follows. We will determine that only a certain small piece of the Type $D$ bordered invariant associated to the framed knot complement is necessary to determine a suitable generator for $\widehat{HF}(S^3)$. The form of this piece of $\widehat{CFD}$ depends only on the framing parameter

relative to $2\tau(K)$, and on $\varepsilon(K)$. We will then determine the absolute Alexander grading of this generator in terms of combinatorial data associated to the Heegaard diagrams for the pattern and companion knots.

## 4.1 The case $\varepsilon(K) = 1$

We first consider the case $\varepsilon(K) = 1$. By Lemma 3.2 and the symmetry properties of $CFK^\infty(K)$, we can find a *vertically* simplified basis $\{x_i\}$ over $\mathbb{F}[U]$ for $CFK^-(K)$ with the following properties, after possible reordering:

1. $x_2$ is the distinguished element of a *horizontally* simplified basis.

2. $\partial^{\text{vert}} x_1 = x_2$.

Let $Y_{K,n}$ be the 3-manifold $S^3 - \text{nbd } K$ with framing $n$. We will use Theorems 10.17 and 11.7 of [7] to compute $\widehat{CFD}(Y_{K,n})$ from $CFK^\infty(K)$. Consider the basis $\{x_i\}$ as above. Then if $n < 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ (consisting of the unstable chain and an additional generator $y$ from a vertical chain) of the form

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_2 \xrightarrow{D_{123}} y, \tag{4.1}$$

where $m = 2\tau(K) - n$. If $n = 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{12}} x_2 \xrightarrow{D_{123}} y. \tag{4.2}$$

Finally, if $n > 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x_2 \xrightarrow{D_{123}} y, \tag{4.3}$$

where $m = n - 2\tau(K)$. The generators $x_0$ and $x_2$ are in the idempotent $\iota_1$, while the generators $z_1, \ldots, z_m$ and $y$ are in the idempotent $\iota_2$.

Let $\widehat{CFA}(p, 1)$ be the bordered invariant associated to the pattern knot in the solid torus that winds $p$ times longitudinally and once meridionally. See Figure 4.1. We will use the pairing

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theorem for bordered Heegaard Floer homology of [7, Theorem 10.12] to compute $\tau(K_{p,pn+1})$ by studying $\widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n})$, which is filtered quasi-isomorphic to $\widehat{CFK}(K_{p,pn+1})$.
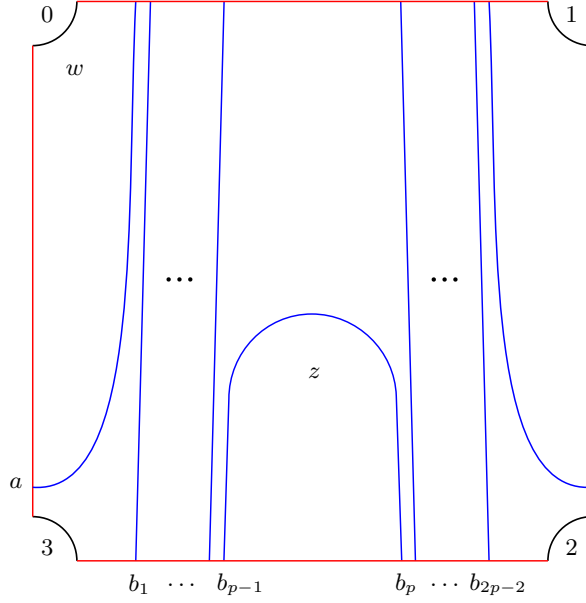


Figure 4.1: A genus one bordered Heegaard diagram $\mathcal{H}(p,1)$ for the $(p,1)$-cable in the solid torus.

**Remark 4.2.** We remark here on the basepoint conventions used henceforth in this thesis. We prefer to work with filtered chain complexes, rather than $\mathbb{F}[U]$-modules, and thus compute the filtered chain complex $\widehat{CFK}$, rather than the $\mathbb{F}[U]$-module $CFK^-$. This convention requires that we interchange the roles of $w$ and $z$, i.e., we now place the basepoint $w$ on $\partial\overline{\Sigma}$.

Each algebra relation $m_i$ in $\widehat{CFA}(p,1)$ contributes a relative filtration shift, denoted $\Delta_A$, which is equal to the number of times that the domain inducing $m_i$ crosses the basepoint $z$. (Each domain must miss the basepoint $w$ completely.) This relative filtration shift naturally extends to the tensor product.

Since switching the roles of $w$ and $z$ induces a chain homotopy equivalence on $CFK^\infty(K)$, it does not change the homotopy type of $\widehat{CFD}$ of the framed knot complement.

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We see that we have the following algebra relations on $\widehat{CFA}(p,1)$, where $\Delta_A$ records the relative filtration shift, i.e., the number of times that the associated domain crosses the basepoint $z$:

$$m_{3+i}(a,\rho_3,\overbrace{\rho_{23},\ldots,\rho_{23}}^{i},\rho_2) = a, \qquad\qquad \Delta_A = pi+p, \qquad i \geq 0$$

$$m_{4+i+j}(a,\rho_3,\overbrace{\rho_{23},\ldots,\rho_{23}}^{i},\rho_2,\overbrace{\rho_{12},\ldots,\rho_{12}}^{j},\rho_1) = b_{j+1}, \quad \Delta_A = pi+j+1, \quad 0 \leq j \leq p-2$$

$$i \geq 0$$

$$m_{2+j}(a,\overbrace{\rho_{12},\ldots,\rho_{12}}^{j},\rho_1) = b_{2p-j-2}, \qquad\qquad \Delta_A = 0, \qquad 0 \leq j \leq p-2$$

$$m_1(b_j) = b_{2p-j-1}, \qquad\qquad \Delta_A = p-j, \qquad 1 \leq j \leq p-1$$

$$m_{3+i}(b_j,\rho_2,\overbrace{\rho_{12},\ldots,\rho_{12}}^{i},\rho_1) = b_{j+i+1}, \qquad \Delta_A = i+1, \qquad 1 \leq j \leq p-2$$

$$0 \leq i \leq p-j-2$$

$$m_{3+i}(b_j,\rho_2,\overbrace{\rho_{12},\ldots,\rho_{12}}^{i},\rho_1) = b_{j-i-1}, \qquad \Delta_A = 0, \qquad p+1 \leq j \leq 2p-2$$

$$0 \leq i \leq j-p-1$$

The generator $a$ is in the idempotent $\iota_1$ and the generators $b_1,\ldots,b_{2p-2}$ are in the idempotent $\iota_2$. By the pairing theorem for bordered Floer homology [7, Theorem 10.12], we have the filtered quasi-isomorphism

$$\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n}).$$

The following lemma identifies the generator of $\widehat{HF}(S^3)$ in the tensor product:

**Lemma 4.3.** *When $\varepsilon(K) = 1$, the element in the tensor product*

$$\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n})$$

*that survives to generator $\widehat{HF}(S^3)$ is $ax_2$, regardless of $n$, the framing on $Y_{K,n}$.*

*Proof.* When we tensor $\widehat{CFA}(p,1)$ with the portion of $\widehat{CFD}(Y_{K,n})$ in Equation 4.1, we see that $ax_2$ has no incoming or outgoing differentials in the tensor product. This can be seen by noticing that $a$ has no $m_1$ algebra relations, nor any algebra relations beginning with $\rho_{123}$, nor any algebra

relations of the form $m_{2+i}(a, \rho_3, \rho_{23}, \ldots, \rho_{23})$. Hence, $ax_2$ represents a generator for $\widehat{HF}(S^3)$ of minimal Alexander grading. Similarly, we see that tensoring $\widehat{CFA}(p, 1)$ with either of the pieces of $\widehat{CFD}(Y_{K,n})$ in Equations 4.2 or 4.3 also gives us $ax_2$ as the generator for $\widehat{HF}(S^3)$. $\square$

We now need to compute the absolute Alexander grading of the generator $ax_2$. Recall that one way to define the absolute Alexander grading is

$$A(\mathbf{x}) = \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle.$$

Also recall that $\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle$ can be computed in terms of combinatorial data associated to the Heegaard diagram for $S^3$ compatible with the knot $K$. More precisely, replace the $\alpha$-circle representing a meridian of $K$ with a 0-framed longitude $\lambda$. We refer to this local region of the Heegaard diagram as the winding region. Then we have the following formula [11, Equation 9]:

$$\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle = \chi(\mathcal{P}) + 2n_{\mathbf{x}}(\mathcal{P}),$$

where $\mathcal{P}$ is a periodic domain representing $[\widehat{F}]$, $\chi(\mathcal{P})$ is the Euler measure of $\mathcal{P}$ and $n_{\mathbf{x}}(\mathcal{P})$ is the local multiplicity of $\mathbf{x}'$ at $\mathcal{P}$, where $\mathbf{x}'$ is obtained from $\mathbf{x}$ by moving the support of $\mathbf{x}$ on the meridian to the longitude, as in Figure 4.2. We will use this formula to compute the Alexander grading of $ax_2$.

**Lemma 4.4.** *The Alexander grading of $ax_2$ is*

$$A(ax_2) = p\tau(K) + \tfrac{pn(p-1)}{2}.$$

*Proof.* We will construct a domain $\mathcal{P}$ that may be decomposed into a domain $\mathcal{P}_A$ on $\mathcal{H}(p, 1)$ and a domain $\mathcal{P}_D$ on $\mathcal{H}(Y_{K,n})$, whose multiplicities agree in the four regions surrounding the puncture on each surface. Then

$$\langle c_1(\underline{\mathfrak{s}}(ax_2)), [\widehat{F}]\rangle = \chi(\mathcal{P}_A) + 2n_a(\mathcal{P}_A) + \chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D)$$

since Euler measure and local multiplicity are both additive under disjoint union.

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We first consider the domain $\mathcal{P}_D$ in $\mathcal{H}(Y_{K,n})$. Recall that $x_2$ is the preferred element of a horizontally simplified basis, and it corresponds to some linear combination of generators in the diagram $\mathcal{H}(Y_{K,n})$. Choose an element in that linear combination of maximal Alexander grading. For ease of notation, we will also denote this generator by $x_2$. Our conventions in this thesis for the base points in $\mathcal{H}(Y_{K,n})$ are the opposite of those in [7]; that is, we have switched the roles of $w$ and $z$. (This was done so that we could compute the tensor product as a filtered chain complex, rather than as a $\mathbb{F}[U]$-module.) With our conventions, $x_2$, the preferred element of a horizontally simplified basis, will have Alexander grading $\tau(K)$ in $\widehat{CFK}(K)$. However, we are interested in the quantity $\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D)$, which we claim is equal to

$$\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D) = 2\tau(K) - \tfrac{n}{2} - \tfrac{1}{2}.$$

This can be seen from the fact that the domain $\mathcal{P}'$ (representing a Seifert surface for $K$) used to compute the Alexander grading of $x_2$ in $\widehat{CFK}(K)$ has multiplicities in the winding region as shown in Figure 4.2. Winding the longitude (that is, changing the framing) does not change the quantity $\chi(\mathcal{P}') + 2n_{x_2}(\mathcal{P}') = 2\tau(K)$. However, $\mathcal{P}_D$ will differ from $\mathcal{P}'$ by the removal of a small disk around the intersection of the longitude and the meridian, which implies that $\chi(\mathcal{P}_D) = \chi(\mathcal{P}') + \tfrac{n}{2} + \tfrac{1}{2}$. See Figure 4.3. We have also moved the support of $x_2$ in the winding region from the intersection of the longitude with a $\beta$ circle (denoted $x_2'$ in Figure 4.2) to the unique intersection of the meridian with the same $\beta$ circle (denoted $x_2$ in Figure 4.2), which implies that $n_{x_2}(\mathcal{P}_D) = n_{x_2}(\mathcal{P}') - \tfrac{n}{2} - \tfrac{1}{2}$. Hence, $\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D)$ has the value claimed above.

We now consider the domain $\mathcal{P}_A$ in $\mathcal{H}(p, 1)$. First, we stablilize the diagram to obtain a curve, $\beta_2$, that represents the meridian of the knot sitting in the solid torus. We replace the generator $a$ with the generator $a$ union the unique intersection of $\beta_2$ with an $\alpha$ circle; for ease of notation, we also denote this generator by $a$. We then add a closed curve, $\lambda$, to $\mathcal{H}(p, 1)$, such that $\lambda$ represents a 0-framed longitude for the knot $K_{p,pn+1}$ in $S^3$. See Figure 4.4. Note that $\lambda$, which is contained entirely in $\mathcal{H}(p, 1)$, will depend on the framing parameter $n$ of the knot complement $Y_{K,n}$. We
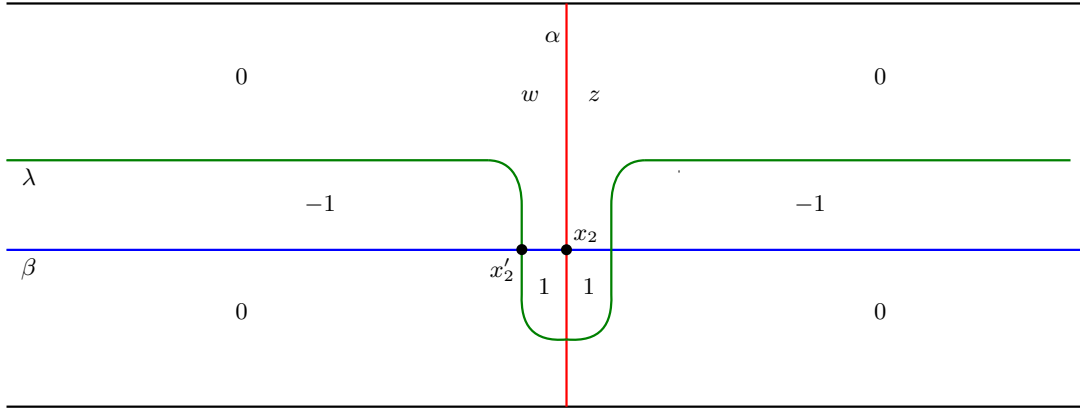
Figure 4.2: Winding region for a knot complement. The numbers indicate the multiplicities of $\mathcal{P}'$.
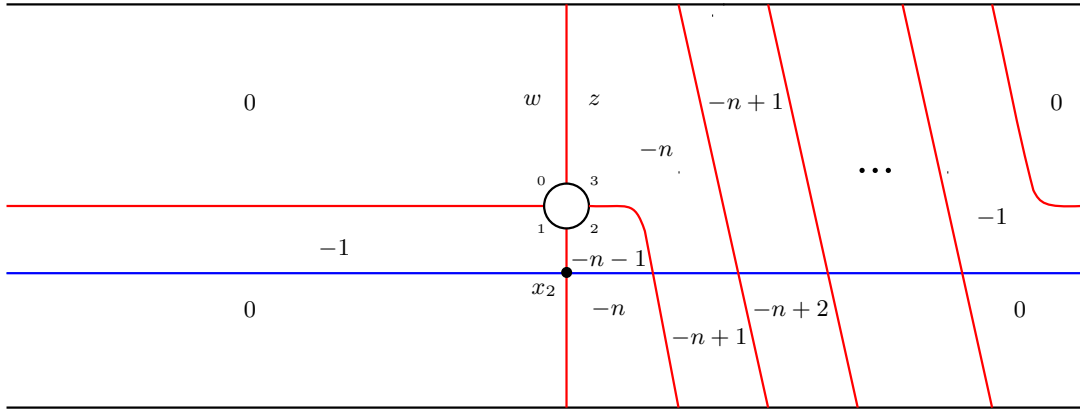


Figure 4.3: Winding region for a bordered knot complement with the multiplicities of $\mathcal{P}_D$ shown.

require $\partial \mathcal{P}_A$ to contain $\lambda$ exactly once. Furthermore, we require the multiplicities of $\mathcal{P}_A$ in the regions 0, 1, 2 and 3 surrounding the puncture to be $0$, $-p$, $-pn - p$ and $-pn$, respectively, in order to coincide with $p$ (the winding number) times the multiplicities in the corresponding regions in $\mathcal{H}(Y_{K,n})$.

First consider the domain $\mathcal{P}_\mu$ shown in Figure 4.5. $\mathcal{P}_\mu$ has zero multiplicity in the regions 0 and 1 near the puncture, and multiplicity $-1$ in the regions 2 and 3. Furthermore, $\partial \mathcal{P}_\mu$ contains
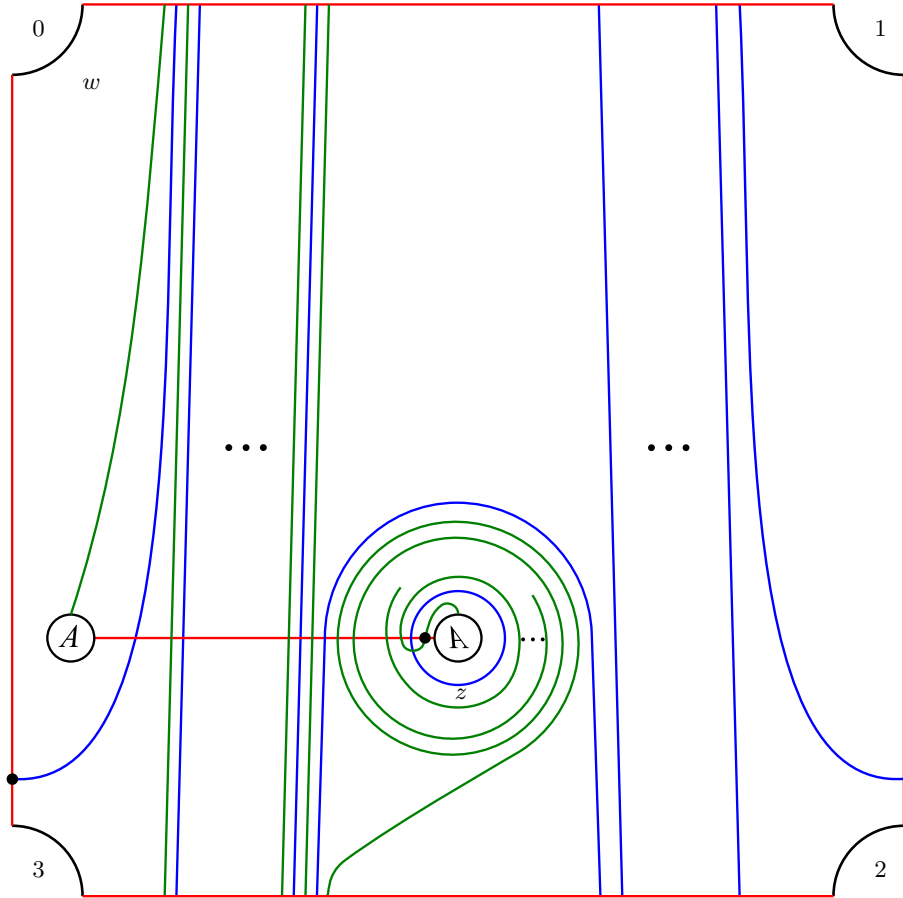
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Figure 4.4: Stabilized bordered Heegaard diagram $\mathcal{H}(p,1)$ for the $(p,1)$-cable in the solid torus with the longitude $\lambda$ shown in green. The pair of black dots indicate the generator $a$ (with its support on the meridian moved to the longitude).

$\beta_2$ with multiplicity $p$ (for an appropriate orientation of $\beta_2$). We see that $\chi(\mathcal{P}_\mu) = p + \frac{1}{2}$, and $n_a(\mathcal{P}_\mu) = -\frac{1}{2}$. Next, consider the domain $\mathcal{P}_\lambda$ shown in Figure 4.6. $\mathcal{P}_\lambda$ has zero multiplicity in regions 0 and 3, and multiplicity $-p$ in the regions 1 and 2. $\partial\mathcal{P}_\lambda$ contains the curve $\beta_2$ with multiplicity $-p^2 n$. We also have that $\chi(\mathcal{P}_\lambda) = \frac{3p}{2}$ and $n_a(\mathcal{P}_\lambda) = -\frac{p}{2}$. Let

$$\mathcal{P}_A = pn \cdot \mathcal{P}_\mu + \mathcal{P}_\lambda.$$

Notice that $\mathcal{P}_A$ has the desired multiplicities in the regions surrounding the puncture, and $\partial\mathcal{P}_A$

contains the longitude for the pattern knot exactly once. We have that $\chi(\mathcal{P}_A) = p^2 n + \frac{pn+3p}{2}$ and $n_a(\mathcal{P}_A) = \frac{-pn-p}{2}$.
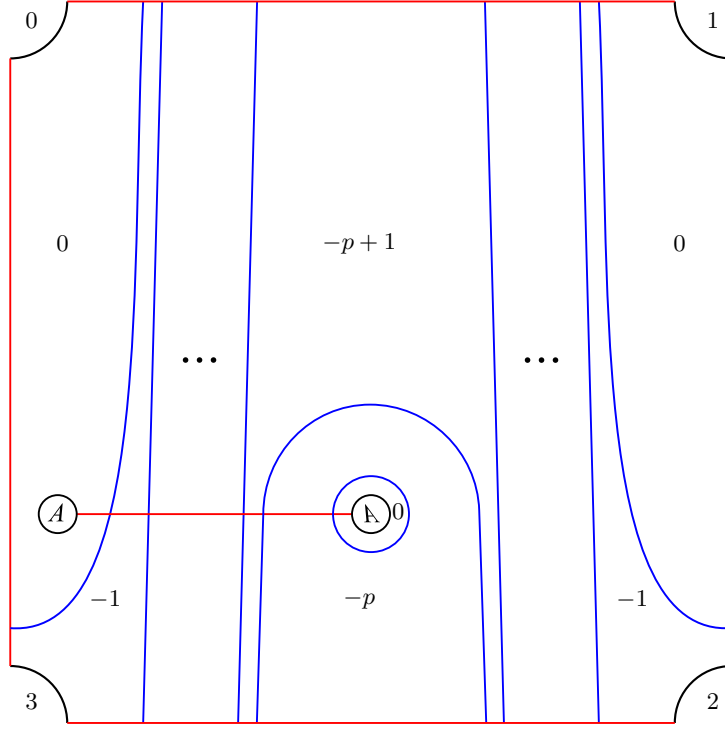


Figure 4.5: The periodic domain $\mathcal{P}_\mu$ in $\mathcal{H}(p, 1)$.

The union of $\mathcal{P}_A$ and $p \cdot \mathcal{P}_D$ represents a Seifert surface for the cable knot $K_{p,pn+1}$, and we find that the Alexander grading of $ax_2$ is

$$A(ax_2) = \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(ax_2)), [\widehat{F}]\rangle$$

$$= \tfrac{1}{2}\big(\chi(\mathcal{P}_A) + 2n_a(\mathcal{P}_A) + p\chi(\mathcal{P}_D) + 2pn_{x_2}(\mathcal{P}_D)\big)$$

$$= \tfrac{1}{2}\big(p^2 n + \tfrac{pn+3p}{2} - pn - p + 2p\tau(K) - \tfrac{pn+p}{2}\big)$$

$$= p\tau(K) + \tfrac{pn(p-1)}{2}.$$

$\square$

Combining this lemma with Lemma 4.3 yields the result that when $\varepsilon(K) = 1$,

$$\tau(K_{p,pn+1}) = p\tau(K) + \tfrac{pn(p-1)}{2}.$$

## 4.2 The case $\varepsilon(K) = -1$

We now consider the case $\varepsilon(K) = -1$, proceeding exactly as in the case $\varepsilon(K) = 1$ above, with the appropriate modifications.

By Lemma 3.2 and the symmetry properties of $CFK^\infty(K)$, we can find a *vertically* simplified basis $\{x_i\}$ over $\mathbb{F}[U]$ for $CFK^-(K)$ with the following properties, after possible reordering:

1. $x_1$ is the distinguished element of a *horizontally* simplified basis.

2. $\partial^{\mathrm{vert}} x_1 = x_2$.

We again let $Y_{K,n}$ be the 3-manifold $S^3 - \mathrm{nbd}\, K$ with framing $n$, and consider the basis $\{x_i\}$ as above. Now, if $n < 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ (consisting of the unstable chain and an additional generator $y$ from a vertical chain) of the form

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_1 \xrightarrow{D_1} y, \tag{4.4}$$

where $m = 2\tau(K) - n$. If $n = 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{12}} x_1 \xrightarrow{D_1} y. \tag{4.5}$$

Finally, if $n > 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x_1 \xrightarrow{D_1} y, \tag{4.6}$$

where $m = n - 2\tau(K)$. In all of the cases above, $y$ has an incoming arrow labeled either $D_{23}$ or $D_{123}$, depending on the exact form of $CFK^\infty(K)$.

**Lemma 4.5.** *When $\varepsilon(K) = -1$, the element in the tensor product*

$$\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n})$$

*that survives to generator $\widehat{HF}(S^3)$ is $b_1 y + ax_2$, regardless of $n$, the framing on $Y_{K,n}$.*

*Proof.* The proof of this lemma follows identically to the proof of Lemma 4.3. For example, tensoring $\widehat{CFA}(p,1)$ with the piece of $\widehat{CFD}(Y_{K,n})$ in Equation 4.4, we see that $\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n})$ has a direct summand consisting of the three generators $ax_1$, $b_1 y$ and $b_{2p-2}y$ with a filtration-preserving differential $\partial(ax_1) = b_{2p-2}y$ and a differential $\partial(b_1 y) = b_{2p-2}y$ that drops filtration level by $p-1$. There are no other differentials in this summand, since $y$ has an incoming arrow labeled either $D_{23}$ or $D_{123}$, neither of which can tensor non-trivially with any of the algebra relations in $\widehat{CFA}(p,1)$. Thus, $b_1 y + ax_1$ generates $\widehat{HF}(S^3)$. The other cases follow similarly. $\square$

The Alexander grading of $ax_1$ is $p\tau(K) + \frac{pn(p-1)}{2}$, by Lemma 4.4, where now $x_1$, rather than $x_2$, is the distinguished element of a vertically simplified basis. By examining the grading shifts of the differentials in the subcomplex of $\widehat{CFK}(K_{p,pn+1})$ above, we see immediately that the Alexander grading of $b_1 y$ is $p\tau(K) + \frac{pn(p-1)}{2} + p - 1$, as desired.

## 4.3   The case $\varepsilon(K) = 0$

The values of $\tau(K_{p,pn+1})$ in the case $\varepsilon(K) = 0$ can be computed by considering the model calculation where $K$ is the unknot. When $\varepsilon(K) = 0$, $\widehat{CFD}(Y_{K,n})$ has a direct summand that is isomorphic to $\widehat{CFD}(Y_{U,n})$, where $U$ denotes the unknot. The tensor product splits along direct summands, so $\widehat{CFK}(K_{p,pn+1})$ has a direct summand that is filtered quasi-isomorphic to $\widehat{CFK}(T_{p,pn+1})$, where $T_{p,pn+1}$ is the $(p, pn+1)$-torus knot; that is, the $(p, pn+1)$-cable of the unknot.

We remark that when $n \geq 2\tau(K)$ and $\varepsilon(K) = 0$, $\widehat{CFD}(Y_{K,n})$ is not bounded. However, by [7, Proposition 4.16], there exists an admissible diagram, and hence bounded $\widehat{CFA}$, for the $(p,1)$-torus knot in $S^1 \times D^2$, in which case the tensor products above will be well-defined.
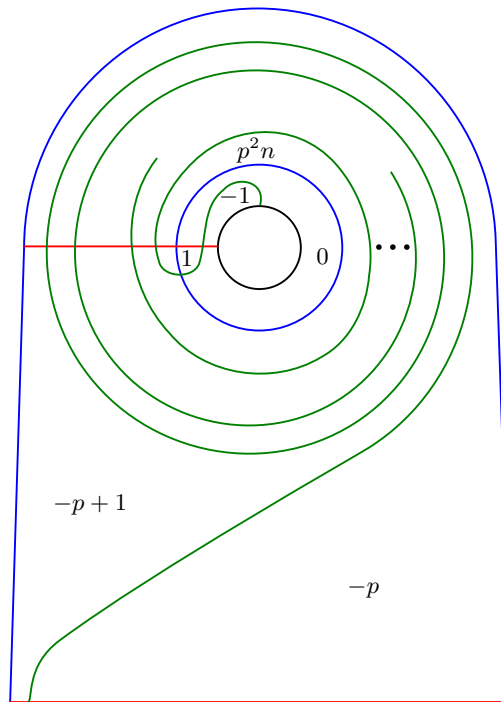
Hence, when $\varepsilon(K) = 0$, the results of [9] computing $\tau$ of torus knots tell us that

$$\tau(K_{p,pn+1}) = \begin{cases} \frac{pn(p-1)}{2} + p - 1 & \text{if } n < 0 \\ \frac{pn(p-1)}{2} & \text{if } n \geq 0. \end{cases}$$

This completes the proof of Theorem 4.1.
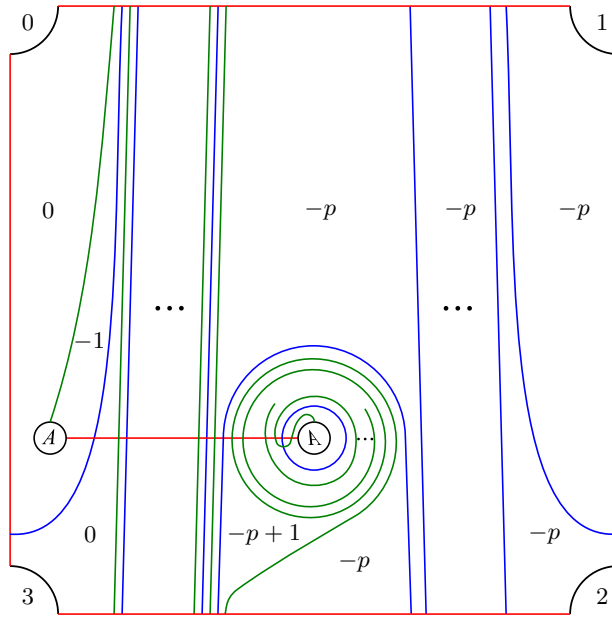
(a)

(b)

Figure 4.6: The periodic domain $\mathcal{P}_\lambda$ in $\mathcal{H}(p, 1)$, with detail in (b).

# Chapter 5

# Computation of $\tau$ for general $(p, q)$-cables

We will now extend our results for $(p, pn + 1)$-cables to general $(p, q)$-cables. That is, we would like to prove the following restatement of Theorem 1:

**Theorem 5.1.** $\tau(K_{p,q})$ *behaves in one of three ways. If* $\varepsilon(K) = 1$, *then*

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}.$$

*If* $\varepsilon(K) = -1$, *then*

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}.$$

*Finally, if* $\varepsilon(K) = 0$, *then*

$$\tau(K_{p,q}) = \begin{cases} \frac{(p-1)(q+1)}{2} & \text{if } q < 0 \\ \frac{(p-1)(q-1)}{2} & \text{if } q > 0. \end{cases}$$

This could be done by considering patterns for $(p, r)$-cables, for all $0 < r < p$ with $r$ relatively prime to $p$. However, Van Cott's results from [21] eliminate the need to consider these more complicated patterns. We summarize her results below.

We expect the behavior of $\tau(K_{p,q})$ to be somehow related to $\tau(T_{p,q})$. Recall that as a function of $q$, $\tau(K_{p,q})$ is linear of slope $\frac{p-1}{2}$ for fixed $p$ and $q > 0$. This motivates the following definition from [21]:

**Definition 5.2.** *Fix an integer $p$ and a knot $K \subset S^3$. For all integers $q$ relatively prime to $p$, define $h(q)$ to be*

$$h(q) = \tau(K_{p,q}) - \frac{(p-1)q}{2}.$$

Van Cott proves the following theorem:

**Theorem 5.3** ([21]). *The function $h(q)$ is a non-increasing $\frac{1}{2} \cdot \mathbb{Z}$-valued function which is bounded below. In particular, we have*

$$-(p-1) \leq h(q) - h(r) \leq 0$$

*for all $q > r$, where both $q$ and $r$ are relatively prime to $p$.*

She then extends Hedden's work on $(p, pn+1)$-cables in [2] to general $(p,q)$-cables:

**Theorem 5.4** ([21]). *Let $K \subset S^3$ be a non-trivial knot. Then the following inequality holds for all pairs of relatively prime integers $p$ and $q$:*

$$p\tau(K) + \frac{(p-1)(q-1)}{2} \leq \tau(K_{p,q}) \leq p\tau(K) + \frac{(p-1)(q-1)}{2} + p - 1.$$

*When $K$ satisfies $\tau(K) = g(K)$, we have $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}$, whereas when $\tau(K) = -g(K)$, we have $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2} + p - 1$.*

The same argument used to prove the final two statements in the above theorem can be used to extend our Theorem 4.1 for $(p, pn+1)$-cables to general $(p,q)$-cables. For completeness, we repeat the argument here.

Let $\varepsilon(K) = 1$. By Theorem 4.1, we know that $\tau(K_{p,pn+1}) = p\tau(K) + \frac{pn(p-1)}{2}$. Our goal is to prove the analogous statement for general $(p,q)$-cables; that is, $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}$. We

see that

$$h(pn + 1) = \tau(K_{p,pn+1}) - \frac{(p-1)(pn+1)}{2}$$

$$= p\tau(K) - \frac{p-1}{2},$$

for all $n$. Since the function $h$ is non-increasing, it follows that

$$h(q) = p\tau(K) - \frac{p-1}{2}$$

for all $q$. Hence

$$\tau(K_{p,q}) = h(q) + \frac{(p-1)q}{2}$$

$$= p\tau(K) + \frac{(p-1)(q-1)}{2},$$

as desired. A similar argument shows that in the case $\varepsilon(K) = -1$,

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}.$$

We are left with the case $\varepsilon(K) = 0$. Let $\widehat{CFA}(p, q)$ denote the bordered invariant associated to a bordered Heegaard diagram compatible with the $(p, q)$-torus knot in $S^1 \times D^2$. (Such a diagram exists by [7, Section 10.4], and can be made admissible by [7, Proposition 4.16].) We again consider the tensor product of $\widehat{CFA}(p, q)$ with $\widehat{CFD}(Y_{K,0})$, i.e., $Y_{K,0} = S^3 - $ nbd $K$ with the zero framing. Since $\varepsilon(K) = 0$, the tensor product $\widehat{CFA}(p, q) \boxtimes \widehat{CFD}(Y_{K,0})$ contains a summand that is filtered quasi-isomorphic to $\widehat{CFK}(T_{p,q})$. Therefore, $\tau(K_{p,q})$ agrees with $\tau(T_{p,q})$, and by [9], we have

$$\tau(K_{p,q}) = \begin{cases} \frac{(p-1)(q+1)}{2} & \text{if } q < 0 \\ \\ \frac{(p-1)(q-1)}{2} & \text{if } q > 0. \end{cases}$$

This completes the proof of Theorem 5.1, and hence also the proof of Theorem 1.

# Chapter 6

# Computation of $\varepsilon(K_{p,pn+1})$ when

# $\varepsilon(K) = 1$

In this chapter, we will compute $\varepsilon(K_{p,pn+1})$ in the case where $\varepsilon(K) = 1$. We will do this by computing $\tau(K_{p,pn+1;2,-1})$, since Theorem 1 tells us that if $\tau(K_{p,pn+1;2,-1}) = 2\tau(K_{p,pn+1}) - 1$, then $\varepsilon(K_{p,pn+1}) = 1$.

**Proposition 6.1.** *If $\varepsilon(K)=1$, then $\varepsilon(K_{p,pn+1}) = 1$.*

We consider the pattern knot $T_{p,1;2,2m+1} \subset S^1 \times D^2$. See Figure 6.1 and denote the associated bordered invariant $\widehat{CFA}(p, 1; 2, 2m + 1)$. Letting $Y$ be $S^3 - \mathrm{nbd}\ K$ with framing $n$, we then have

$$\widehat{CFK}(K_{p,pn+1;2,2p^2n+2m+1}) \simeq \widehat{CFA}(p, 1; 2, 2m + 1) \boxtimes \widehat{CFD}(Y).$$

Thus, we need to consider the case when $m = -p^2n - 1$.

We will proceed as in Chapter 4, by computing a portion of $\widehat{CFA}(p, 1; 2, 2m + 1)$ that is sufficient to determine a generator for $\widehat{HF}(S^3)$, and then determining the Alexander grading of that generator. The remainder of this chapter consists of those detailed computations.

If $\varepsilon(K) = 1$, then by Lemma 3.2, we can find a vertically simplified basis $\{x_i\}$ over $\mathbb{F}[U]$ for
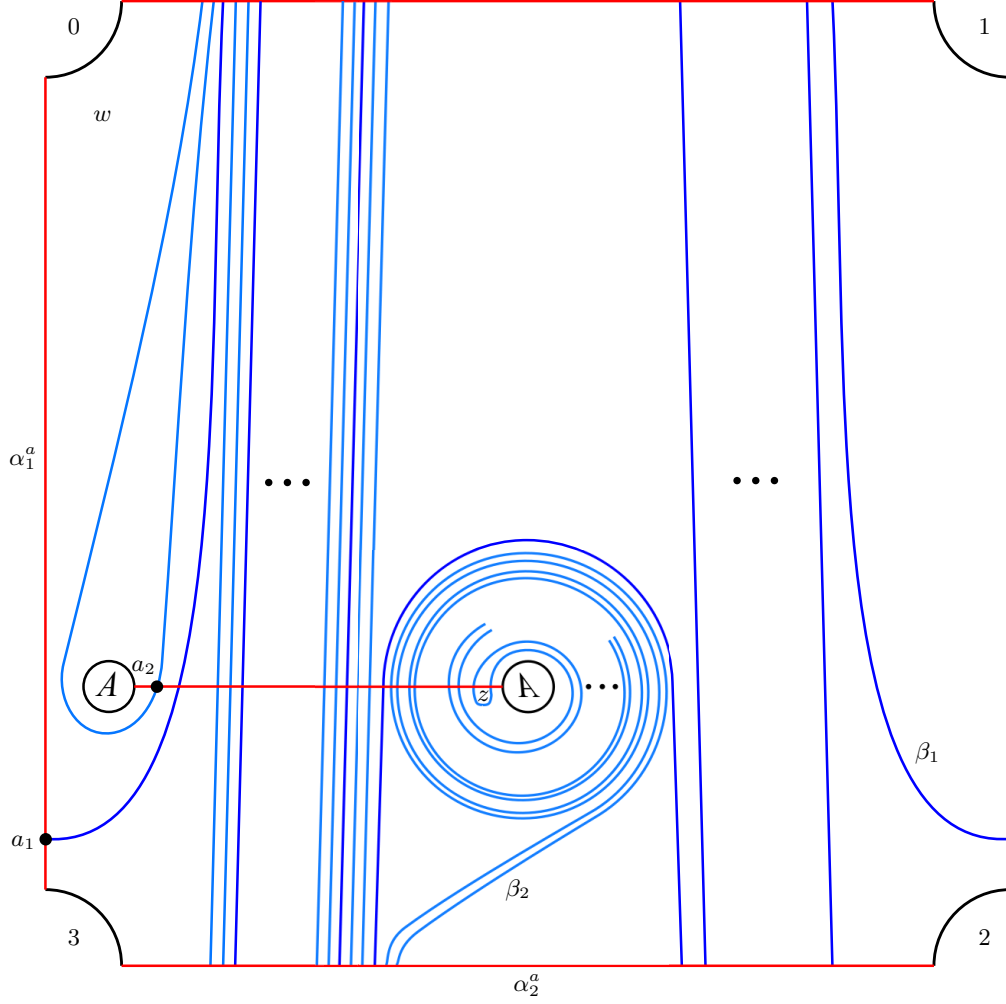
Figure 6.1: Bordered Heegaard diagram for the $(p, 1; 2, 2m + 1)$-iterated torus knot in the solid torus. The light blue circle, $\beta_2$, winds $p + m$ times.

$CFK^-(K)$ with the following properties, after possible reordering:

1. $x_2$ is the distinguished element of a horizontally simplified basis.

2. $\partial x_1 = x_2$.

Let $a = (a_1, a_2)$ in Figure 6.1. We claim that $ax_2$ will be a generator for $\widehat{HF}(S^3)$ in the tensor product $\widehat{CFK}(K_{p,pn+1;2,-1}) \simeq \widehat{CFA}(p, 1; 2, -2p^2n - 1) \boxtimes \widehat{CFD}(Y)$.

Letting $Y_{K,n}$ be the 3-manifold $S^3 - $ nbd $K$ with framing $n$, and considering the basis $\{x_i\}$

above, we again have the following pieces of $\widehat{CFD}(Y_{K,n})$:

- If $n < 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_2 \xrightarrow{D_{123}} y,$$

  where $m = 2\tau(K) - n$.

- If $n = 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{12}} x_2 \xrightarrow{D_{123}} y.$$

- Finally, if $n > 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x_2 \xrightarrow{D_{123}} y,$$

  where $m = n - 2\tau(K)$.

Recall that the generators $x_0$ and $x_2$ are in the idempotent $\iota_1$, while the generators $z_1, \ldots, z_m$ and $y$ are in the idempotent $\iota_2$. In all of the above cases, there is either an arrow labeled $D_{23}$ leaving $y$, or an arrow labeled $D_1$ entering $y$.

Let us now consider $\widehat{CFA}(p, 1; 2, 2m + 1)$. In particular, we would like to compute enough of $\widehat{CFA}$ to show that the generator $ax_2$ survives to generate $\widehat{HF}(S^3)$, so we look for algebra relations coming from domains entering or leaving $a = (a_1, a_2)$. We say that a domain from $a = (a_1, a_2)$ to $b = (b_1, b_2)$ *fixes* $a_1$ if one of $b_1$ or $b_2$ is equal to $a_1$. If a domain does not fix $a_1$, then we say that the domain *moves* $a_1$.

We first notice that no domains *from* $a$ that fix $a_2$ contribute to arrows leaving $ax_2$ in the complex $\widehat{CFA} \boxtimes \widehat{CFD}$. Nor do any domains *to* $a$ that fix $a_2$ contribute to arrows entering $ax_2$ in $\widehat{CFA} \boxtimes \widehat{CFD}$. Both of these statements follow from the computation in Section 4.1.

In light of the above observation, we must consider domains that move $a_2$. There are no domains to $a$ that move $a_2$. This follows from the fact that there are only 3 distinct regions in $\overline{\Sigma} \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ adjacent to $a_2$, and that the basepoint $w$, which we must miss, is in one of them.

We now consider domains from $a$ that move $a_2$. We claim that none of these domains will contribute to arrows leaving $ax_2$ in the complex $\widehat{CFA} \boxtimes \widehat{CFD}$. By inspection, there are no domains contributing to an algebra relation of the form

$$m_1(a).$$

Furthermore, there are no algebra relations in $\widehat{CFA}$ of the form

$$m_{2+i}(a, \rho_3, \overbrace{\rho_{23}, \ldots, \rho_{23}}^{i}), \quad i \geq 0$$

$$m_{2+i}(a, \rho_{123}, \overbrace{\rho_{23}, \ldots, \rho_{23}}^{i}), \quad i \geq 0.$$

We can exclude relations of the from $m_{2+i}(a, \rho_3, \rho_{23}, \ldots, \rho_{23})$ by attempting to find the boundary of the corresponding domain, or by using the techniques of [6, Section 2.3].

To exclude relations of the form $m_{2+i}(a, \rho_{123}, \rho_{23}, \ldots, \rho_{23})$, we will use $\mathcal{A}_\infty$-relations (Section 2.1) to reach a contradiction. For simplicity, we will consider the relation $m_2(a, \rho_{123})$; the other cases follow in an identical manner. We have the following $\mathcal{A}_\infty$-relation:

$$0 = m_3(m_1(a), \rho_{12}, \rho_3) + m_2(m_2(a, \rho_{12}), \rho_3) + m_2(a, \rho_{12} \cdot \rho_3) + m_1(m_3(a, \rho_{12}, \rho_3))$$

The first term above is zero, since $m_1(a) = 0$. Moreover, by attempting to find the boundary of the corresponding domain, we can also conclude that $m_2(a, \rho_{12})$ and $m_3(a, \rho_{12}, \rho_3)$ are equal to zero. Hence, $m_2(a, \rho_{123}) = 0$ as well.

To compute the Alexander grading of $ax_2$, we again use the formula

$$A(ax_2) = \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(ax_2)), [\widehat{F}]\rangle$$

$$= \tfrac{1}{2}\big(\chi(\mathcal{P}) + 2n_{ax_2}(\mathcal{P})\big),$$

where now $\mathcal{P} = \mathcal{P}_A + 2p\mathcal{P}_D$, and

$$\chi(\mathcal{P}_A) + 2n_a(\mathcal{P}_A) = 2p^2 n - pn + p - 2$$

$$\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D) = 2\tau(K) - \tfrac{n}{2} - \tfrac{1}{2}.$$

(To compute these quantities, we again find it convenient to decompose the domain $\mathcal{P}_A$ as $\mathcal{P}_\lambda +$ $2pn\mathcal{P}_\mu$, with

$$\chi(\mathcal{P}_\lambda) = 3p - 4p^2 n$$

$$n_a(\mathcal{P}_\lambda) = -p + p^2 n - 1$$

$$\chi(\mathcal{P}_\mu) = 3p + \tfrac{1}{2}$$

$$n_a(\mathcal{P}_\mu) = -\tfrac{p}{2} - \tfrac{1}{2}.)$$

This implies that

$$A(ax_2) = 2p\tau(K) + p^2 n - pn - 1$$

$$= 2\tau(K_{p,pn+1}) - 1,$$

and so $\varepsilon(K_{p,pn+1}) = 1$, completing the proof of Proposition 6.1.

# Chapter 7

# Computation of $\varepsilon$ for $(p, q)$-cables

In the previous chapter, we proved that if $\varepsilon(K) = 1$, then $\varepsilon(K_{p,pn+1}) = 1$. The goal of this chapter is to prove Theorem 2; that is, to describe the behavior of $\varepsilon$ under cabling, for all values of $\varepsilon$ and for all $p$ and $q$.

What follows is a straightforward modification of Van Cott's work in [21]. Fix a knot $K$ and integers $p$ and $m$, and define the function

$$H(q) := \tau(K_{p,q;2,m}) - (p - 1)q$$

for all $q$ relatively prime to $p$.

**Proposition 7.1.** *The function $H$ is non-increasing; that is,*

$$H(q) - H(r) \leq 0$$

*for all $q > r$, where both $q$ and $r$ are relatively prime to $p$.*

*Proof.* Recall our convention that $p > 1$. Let $q$ and $r$ be integers relatively prime to $p$ with $q > r$. Consider the connect sum

$$K_{p,q;2,m} \# - (K_{p,r;2,m}).$$

Notice that $-(K_{p,r;2,m}) = (-K)_{p,-r;2,-m}$. Let $k$ be the smallest positive integer such that $q-r-k$ is relatively prime to $p$. (Note that $k$ may be equal to zero, and that $q-r-k > 0$.) We leave it as an exercise for the reader to show that by performing $2p + 2k(p-1)$ band moves, we can obtain the knot

$$(K\# - K)_{p,q-r-k;2,-1},$$

which is concordant to the iterated torus knot $T_{p,q-r-k;2,-1}$ since $K\# - K$ is slice. Thus, we have a genus $p + k(p-1)$ cobordism between $K_{p,q;2,m}\# - K_{p,r;2,m}$ and $T_{p,q-r-k;2,-1}$. Since $|\tau|$ is a lower-bound on the 4-ball genus, we have

$$|\tau(K_{p,q;2,m}\# - K_{p,r;2,m}\# - T_{p,q-r-k;2,-1})| \le p + k(p-1)$$

$$|\tau(K_{p,q;2,m}) - \tau(K_{p,r;2,m}) - ((p-1)(q-r-k-1)-1)| \le p + k(p-1)$$

$$|H(q) - H(r) - (p-1)(-k-1) + 1| \le p + k(p-1)$$

$$H(q) - H(r) \le 0,$$

completing the proof of the proposition. $\square$

For $K$ with $\varepsilon(K) = 1$, we have that

$$H(pn+1) = \tau(K_{p,pn+1;2,m}) - (p-1)(pn+1)$$

$$= 2p\tau(K) + (p-1)pn + \tfrac{m-1}{2} - (p-1)(pn+1)$$

$$= 2p\tau(K) + \tfrac{m-1}{2} - (p-1)$$

for all $n$. But since the function $H$ is non-increasing, this implies that $H(q) = 2p\tau(K) + \tfrac{m-1}{2} - (p-1)$ for all $q$ relatively prime to $p$. Hence,

$$\tau(K_{p,q;2,m}) = 2p\tau(K) + \tfrac{m-1}{2} - (p-1) + (p-1)q$$

$$= 2\big(p\tau(K) + \tfrac{(p-1)(q-1)}{2}\big) + \tfrac{m-1}{2}$$

$$= 2\tau(K_{p,q}) + \tfrac{m-1}{2}$$

and so $\varepsilon(K_{p,q}) = 1$. Thus, we have shown that if $\varepsilon(K) = 1$, then $\varepsilon(K_{p,q}) = 1$ for all $p$ and $q$.

Since $\varepsilon(-K) = -\varepsilon(K)$ and $(-K)_{p,q} = -K_{p,-q}$, we have that if $\varepsilon(K) = -1$, then $\varepsilon(K_{p,q}) = -\varepsilon(-K_{p,q}) = -\varepsilon((-K)_{p,-q}) = -1$; that is, if $\varepsilon(K) = -1$, then $\varepsilon(K_{p,q}) = -1$.

For the case $\varepsilon(K) = 0$, we again appeal to a model calculation; that is, if $\varepsilon(K) = 0$, then $\tau$ of any iterated cable agrees with $\tau$ of the underlying torus knot, implying that if $\varepsilon(K) = 0$, then

$$\varepsilon(K_{p,q}) = \varepsilon(T_{p,q}).$$

Thus, we have completely described the behavior of $\varepsilon$ under cabling.

# Chapter 8

# Proof of Corollaries 3 and 4

In this chapter, we prove the two corollaries from the introduction.

*Proof of Corollary 3.* By Theorem 1, it is sufficient to find knots $K_n^+$ and $K_n^-$ with $\tau(K_n^\pm) = n$ and $\varepsilon(K_n^\pm) = \pm 1$.

For the right-handed trefoil, which we will denote $R$, we have that $\tau(R) = \varepsilon(R) = 1$, and for the left-handed trefoil, we have that $L$, $\tau(L) = \varepsilon(L) = -1$. Hence, by Theorems 1 and 2:

$$\tau(R_{2,2m+1}) = 2 + m$$

$$\varepsilon(R_{2,2m+1}) = 1$$

$$\tau(L_{2,2m+1}) = 3 + m$$

$$\varepsilon(L_{2,2m+1}) = -1,$$

and so by taking a cable of a right- or left-handed trefoil, we can construct knots with arbitrary $\tau$, and with $\varepsilon$ equal to our choice of $\pm 1$. (Note that this is one way to construct a knot $K$ with $\tau(K) = 0$ but $\varepsilon(K) \neq 0$.) More precisely, let $K_n^+ = R_{2,2n-3}$ and let $K_n^- = L_{2,2n-5}$, and so $\tau(K_n^\pm) = n$ and $\varepsilon(K_n^\pm) = \pm 1$. This completes the proof of Corollary 3. $\qquad\square$

*Proof of Corollary 4.* This corollary was suggested to me by Livingston. We would like to prove

that if $\varepsilon(K) \neq \operatorname{sgn} \tau(K)$, then $g_4(K) \geq |\tau(K)| + 1$. Recall that

- If $\varepsilon(K) = 0$, then $\tau(K) = 0$.

- $\tau(\overline{K}) = -\tau(K)$.

- $\varepsilon(\overline{K}) = -\varepsilon(K)$.

Hence, without loss of generality, we may assume that $\tau(K) \geq 0$ and that $\varepsilon(K) = -1$, in which case $\tau(K_{2,1}) = 2\tau(K) + 1$.

We can construct a slice surface for $K_{2,1}$ by taking two copies of the slice surface for $K$ and connecting them with a single twisted strip, hence

$$g_4(K_{2,1}) \leq 2g_4(K).$$

We also have that $|\tau(K_{2,1})| \leq g_4(K)$, or

$$2\tau(K) + 1 \leq g_4(K_{2,1}),$$

so upon combining these two inequalities, we get

$$\tau(K) + \tfrac{1}{2} \leq g_4(K).$$

But $\tau(K)$ and $g_4(K)$ are both integers, hence

$$\tau(K) + 1 \leq g_4(K),$$

concluding the proof of Corollary 4 when $\tau(K) \geq 0$. The case $\tau(K) < 0$ follows by taking mirrors. $\square$

# Chapter 9

# Cabling and $L$-space Knots

In this chapter, we focus our attention on a class of 3-manifolds with particularly simple Heegaard Floer homology. For a rational homology sphere $Y$, Proposition 5.1 of [12] tells us that

$$\mathrm{rk}\ \widehat{HF}(Y) \geq |H_1(Y, \mathbb{Z})|.$$

An $L$-space is a rational homology sphere $Y$ for which the above bound is sharp. The name comes from the fact that lens spaces are $L$-spaces, which can be seen by examining the Heegaard Floer complex associated to a standard genus one Heegaard decomposition of a lens space.

We call a knot $K \subset S^3$ an $L$-space knot if there exists $n \in \mathbb{Z}$, $n > 0$, such that $n$ surgery on $K$ yields an $L$-space. We will denote the resulting 3-manifold by $S^3_n(K)$. Torus knots are a convenient source of $L$-space knots, since $pq \pm 1$ surgery on the $(p, q)$-torus knot yields a lens space. It was proved in [14, Theorem 1.2] that if a knot $K$ is an $L$-space knot, then the knot Floer complex associated to $K$ has a particularly simple form that can be deduced from the Alexander polynomial of $K$, $\Delta_K(t)$. Thus, knowing that a knot $K$ admits a lens space (or $L$-space) surgery yields a remarkable amount of information about the Heegaard Floer invariants associated to both the knot $K$, and manifolds obtained by Dehn surgery on $K$. In particular, [14, Theorem 1.2] combined with [17, Theorem 1.1] allows one to compute the Heegaard Floer invariants of any

Dehn surgery on an $L$-space knot $K$ from the Alexander polynomial of $K$.

Let $g(K)$ denote the Seifert genus of $K$. In [2, Theorem 1.10], Hedden proves that if $K$ is an $L$-space knot and $q/p \geq 2g(K) - 1$, then $K_{p,q}$ is an $L$-space knot. We will prove the converse:

**Theorem 9.1.** *The $(p,q)$-cable of a knot $K \subset S^3$ is an $L$-space knot if and only if $K$ is an $L$-space knot and $q/p \geq 2g(K) - 1$.*

It was already known that if $K_{p,q}$ is an $L$-space knot, then $q > 0$ and $\tau(K) = g(K)$ [21, Corollary 6]. We prove our theorem by methods similar to those used in [2, Theorem 1.10]. An interesting question to consider is whether there are other satellite constructions that also yield $L$-space knots.

An $L$-space $Y$ can be thought of as rational homology sphere with the "smallest" possible Heegaard Floer invariants, i.e. rk $\widehat{HF}(Y) = |H_1(Y, \mathbb{Z})|$. In a similar spirit, an $L$-space knot $K$ can be thought of as a knot with the "smallest" possible knot Floer invariants. For example, since

$$\Delta_K(t) = \sum_{m,s}(-1)^m \text{ rk } \widehat{HFK}_m(K, s)t^s,$$

so we see immediately that the total rank of $\widehat{HFK}(K)$ is bounded below by the sum of the absolute value of the coefficients of the Alexander polynomial of $K$, $\Delta_K(t)$. A necessary, but not sufficient, condition for a knot $K$ to be an $L$-space knot is for this bound to be sharp; see [14, Theorem 1.2] for the complete statement. The spirit of our proof is that when either $K$ is not an $L$-space knot, or $q/p < 2g(K) - 1$, the knot Floer invariants of $K_{p,q}$ are not "small" enough for $K_{p,q}$ to be an $L$-space knot. We will determine this by looking at the rank of $\widehat{HF}(S^3_{pq}(K_{p,q}))$.

*Proof.* Recall that $\tau(K)$ is the integer-valued concordance invariant defined by Ozsváth and Szabó in [9]. Let $\mathcal{P}$ denote the set of all knots $K$ for which $g(K) = \tau(K)$. We begin by assembling the following collection of facts.

1. If $K$ is an $L$-space knot, then $K \in \mathcal{P}$. This follows from [14, Theorem 1.2] combined with the fact that knot Floer homology detects genus [10, Theorem 1.2]

2. Let

$$s_K = \sum_{s \in \mathbb{Z}} (\text{rk } H_*(\widehat{A}_s(K)) - 1),$$

where $\widehat{A}_s(K)$ is the sub-quotient complex of $CFK^\infty(K)$ defined in [16, Section 4.3]. We may think of $CFK^\infty(K)$ as generated over $\mathbb{F}[U, U^{-1}]$ by $\widehat{CFK}(K)$, in which case rk $\widehat{A}_s(K) = $ rk $\widehat{CFK}(K)$ for all $s$. Recall that rk $\widehat{CFK}(K)$ is always odd, since the graded Euler characteristic of $\widehat{CFK}(K)$ is the Alexander polynomial of $K$. Therefore, rk $H_*(\widehat{A}_s(K))$ is odd, hence greater than or equal to 1, and so $s_K$ is always non-negative. Let

$$t_K^{a/b} = 2 \ \max(0, (2g(K) - 1)b - a),$$

for a pair of relatively prime integers $a$ and $b$, $b > 0$. Notice that

$$t_K^{a/b} = 0 \text{ if and only if } a/b \geq 2g(K) - 1.$$

For $K \in \mathcal{P}$ and $a$, $b$ as above,

$$\text{rk } \widehat{HF}(S_{a/b}^3(K)) = a + bs_K + t_K^{a/b}.$$

This is a special case of Proposition 9.5 of [17]. In particular, the term $\nu(K)$ appearing in Proposition 9.5 is bounded below by $\tau(K)$ [9, Proposition 3.1] and above by $g(K)$ [11, Theorem 5.1], so $K \in \mathcal{P}$ implies $\nu(K) = g(K)$. We notice that

$$K \text{ admits a positive } L\text{-space surgery if and only if } s_K = 0.$$

Indeed, if $s_K = 0$, then $p$ surgery on $K$ yields an $L$-space, for any integer $p \geq 2g(K) - 1$. Conversely, if $K$ is an $L$-space knot, then there exists some integer $p > 0$ such that $p$ surgery on $K$ is an $L$-space, in which case $s_K$, which is always non-negative, must be 0.

3. Recall our convention that $p$, $q$ are relatively prime integers, with $p > 1$. If $K_{p,q} \in \mathcal{P}$, then $K \in \mathcal{P}$, and if $K \in \mathcal{P}$, then $\tau(K_{p,q}) = p\tau(K) + \frac{1}{2}(p - 1)(q - 1)$. These facts are Corollaries

4 and 3, respectively, in [21]. Therefore, if $K_{p,q} \in \mathcal{P}$, we have

$$(2g(K) - 1)p - q = (2\tau(K) - 1)p - q$$

$$= 2\big(p\tau(K) + \tfrac{(p-1)(q-1)}{2}\big) - 1 - pq$$

$$= 2\tau(K_{p,q}) - 1 - pq$$

$$= 2g(K_{p,q}) - 1 - pq,$$

or equivalently,

$$\text{if } K_{p,q} \in \mathcal{P}, \text{ then } t_K^{q/p} = t_{K_{p,q}}^{pq}.$$

4. It is well-known that $pq$ surgery on $K_{p,q}$ is the manifold $L(p,q)\#S_{q/p}^3(K)$ (see [2, Proof of Theorem 1.10] for a nice proof of this fact). We also have from [13, Proposition 6.1] that

$$\text{rk } \widehat{HF}(Y_1\#Y_2) = \text{rk } \widehat{HF}(Y_1) \cdot \text{rk } \widehat{HF}(Y_2).$$

Then

$$\text{rk } \widehat{HF}(S_{pq}^3(K_{p,q})) = \text{rk } \widehat{HF}(L(p,q)) \cdot \text{rk } \widehat{HF}(S_{q/p}^3(K))$$

$$= p \cdot \text{rk } \widehat{HF}(S_{q/p}^3(K)).$$

With these facts in place, we are ready to prove the theorem. Assume $K_{p,q}$ is an $L$-space knot. Then by (1) and (3), $K_{p,q} \in \mathcal{P}$ and $t_{K_{p,q}}^{pq} = t_K^{q/p}$, and by (2),

$$\text{rk } \widehat{HF}(S_{pq}^3(K_{p,q})) = pq + s_{K_{p,q}} + t_{K_{p,q}}^{pq} \quad \text{and} \quad \text{rk } \widehat{HF}(S_{q/p}^3(K)) = q + ps_K + t_K^{q/p}.$$

Then by (4), $\text{rk } \widehat{HF}(S_{pq}^3(K_{p,q})) = p \cdot \text{rk } \widehat{HF}(S_{q/p}^3(K))$, and $s_{K_{p,q}} = 0$, since $K_{p,q}$ is an $L$-space knot. So we find that

$$p^2 s_K + (p-1)t_K^{q/p} = 0.$$

Therefore, since $p > 1$, we have that $s_K$ and $t_K^{q/p}$ must both be zero, or equivalently, $K$ is an $L$-space knot and $q/p \geq 2g(K) - 1$. This completes the proof of the theorem. $\qquad\square$

**Remark 9.2.** The contents of this chapter appears in [5].

# Chapter 10

# Future directions

The invariant $\varepsilon$ defined in this thesis has applications beyond determining $\tau$ of iterated cables. These applications will be discussed in depth in [4]; we outline the main ideas below.

The goal is to use $\varepsilon$ to define a new concordance homomorphism that is strong enough to detect linear independence in $\mathcal{C}$. We will turn the monoid of chain complexes $CFK^\infty(K)$ (under tensor product) into a group, which we will denote $\mathcal{F}$, in much the same way that the monoid of knots (under connected sum) can be made into the group $\mathcal{C}$ by quotienting by slice knots.

**Definition 10.1.** *Let $CFK^\infty(K)^*$ denote the dual of $CFK^\infty(K)$; that is,*

$$CFK^\infty(K)^* = \operatorname{Hom}_{\mathbb{F}[U,U^{-1}]}(CFK^\infty(K), \mathbb{F}).$$

*Define the group $\mathcal{F}$ to be*

$$\mathcal{F} := \big(\{CFK^\infty(K) \mid K \subset S^3\}, \otimes\big)/\sim$$

*where*

$$CFK^\infty(K_1) \sim CFK^\infty(K_2) \iff \varepsilon(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*) = 0.$$

**Theorem 10.2** ([4])**.** *The map*

$$\mathcal{C} \to \mathcal{F},$$

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*sending a class in $\mathcal{C}$ represented by $K$ to the class in $\mathcal{F}$ represented by $CFK^\infty(K)$ is a group homomorphism.*

This group $\mathcal{F}$ has the advantage that it can be studied from an algebraic perspective. In particular, $\mathcal{F}$ is totally ordered, with an additional well-defined notion of "$\ll$". Moreover, we can use the relation $\ll$ to define a filtration on $\mathcal{F}$. We can also use spectral sequences to define a second, independent filtration.

The rich structure on $\mathcal{F}$ is powerful enough to detect linear independence in $\mathcal{C}$. Let $T_{p,q}$ denote the $(p,q)$-torus knot, $K_{p,q}$ the $(p,q)$-cable of $K$, and $D$ the (positive, untwisted) Whitehead double of the right-handed trefoil.

**Theorem 10.3** ([4]). *The topologically slice knots*

$$D_{p,p+1} \# - T_{p,p+1}, \quad p \geq 1$$

*are independent in the smooth concordance group; that is, they freely generate a subgroup of infinite rank.*

The first examples of an infinite family of smoothly independent, topologically slice knots was given by Endo [1]. His examples consist of certain pretzel knots. More recently, Hedden and Kirk [3] showed that an infinite family of (untwisted) Whitehead doubles of certain torus knots are smoothly independent. The structure of $\mathcal{F}$ shows that our examples (when $p \geq 2$) are smoothly independent from both of these earlier families.

Let $P(K)$ denote the satellite of $K$ with pattern $P$; that is, $P$ is a knot in $S^1 \times D^2$, which we then glue into the (zero framed) knot complement $S^3 - \operatorname{nbd} K$. Recall that the map $P(-) : \mathcal{C} \to \mathcal{C}$ given by

$$[K] \mapsto [P(K)]$$

is well-defined, by "following" the concordance along the satellite.

We obtain a similar well-defined map on $\mathcal{F}$:

**Proposition 10.4** ([4]). *The map* $P(-) : \mathcal{F} \to \mathcal{F}$ *given by*

$$[CFK^\infty(K)] \mapsto [CFK^\infty\big(P(K)\big)]$$

*is well-defined.*

By composing $P$ with $\tau$, we obtain a new concordance invariant

$$\tau_P(K) := \tau\big(P(K)\big),$$

since $K_1 \sim K_2$ implies that $P(K_1) \sim P(K_2)$. Let $\{\tau_P(K)\}_P$ denote the collection of $\tau_P$ for all patterns $P$ in $S^1 \times D^2$. In the following theorem, we see that the information contained in $\{\tau_P(K)\}_P$ is exactly the information contained in $[CFK^\infty(K)]$.

**Theorem 10.5** ([4]). $[CFK^\infty(K_1)] = [CFK^\infty(K_2)]$ *if and only if* $\{\tau_P(K_1)\}_P = \{\tau_P(K_2)\}_P$.

Does the map $P(-) : \mathcal{C} \to \mathcal{C}$ always take linearly independent collections of knots to linearly independent collections of knots? To address this question, we have the following theorem:

**Theorem 10.6** ([4]). *For each* $n \in \mathbb{N}$, *there exists a collection of linearly independent knots*

$$\{K_i\}_{i=1}^n$$

*and a pattern* $P$ *such that*

$$\{P(K_i)\}_{i=1}^n$$

*are independent in* $\mathcal{C}$.

We hope that the applications of $\varepsilon$, both in this thesis and in future work, convince the reader of the utility and power of this new concordance invariant. Our goal is to be able to give a complete algebraic description of the group $\mathcal{F}$, helping to shed light on the structure of the smooth concordance group.

# Bibliography

[1] H. Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology Appl. **63** (1995), no. 3, 257–262.

[2] M. Hedden, *On knot Floer homology and cabling II*, Int. Math. Res. Not. **12** (2009), 2248–2274.

[3] M. Hedden and P. Kirk, *Instantons, concordance and Whitehead doubling*, preprint (2010), arXiv:1009.5361v2.

[4] J. Hom, *The knot Floer complex and the smooth concordance group*, preprint (2011), in preparation.

[5] ———, *A note on cabling and L-space surgeries*, Algebr. Geo. Topol. **11** (2011), no. 1, 219–223.

[6] A. Levine, *Knot doubling operators and bordered Heegaard Floer homology*, preprint (2010), arXiv:1008.3349v1.

[7] R. Lipshitz, P. S. Ozsváth, and D. Thurston, *Bordered Heegaard Floer homology: Invariance and pairing*, preprint (2008), arXiv:0810.0687v3.

[8] P. S. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), 179–261.

[9] _____, *Knot Floer homology and the four-ball genus*, Geom. Topol. **7** (2003), 615–639.

[10] _____, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004).

[11] _____, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116.

[12] _____, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), no. 3, 1159–1245.

[13] _____, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158.

[14] _____, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300.

[15] _____, *Heegaard diagrams and Floer homology*, preprint (2006), arXiv:0602232v1.

[16] _____, *Knot Floer homology and integer surgeries*, Algebr. Geo. Topol. **8** (2008), no. 1, 101–153.

[17] _____, *Knot Floer homology and rational surgeries*, Algebr. Geo. Topol. **11** (2011), no. 1, 1–68.

[18] P. S. Ozsváth, Z. Szabó, and D. Thurston, *Legendrian knots, transverse knots and combinatorial Floer homology*, preprint (2003), arXiv:0303017v2.

[19] I. Petkova, *Cables of thin knots and bordered Heegaard Floer homology*, preprint (2009), arXiv:0911.2679v1.

[20] J. Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003, arXiv:0306378v1.

[21] C. Van Cott, *Ozsváth-Szabó and Rasmussen invariants of cable knots*, Algebr. Geo. Topol. **10** (2010), no. 2, 825–836.

[22] R. Zarev, *Bordered Floer homology for sutured manifolds*, preprint (2009), arXiv:0908.1106v2.