ASYMPTOTIC INVARIANTS OF HADAMARD MANIFOLDS

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ABSTRACT

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We study various asymptotic invariants of manifolds of nonpositive curvature. First, we study the filling invariants at infinity div_k for Hadamard manifolds defined by Noel Brady and Benson Farb. Among other results, we give a positive answer to the question they posed: can these invariants be used to detect the rank of a symmetric space of noncompact type?

Second, we study the asymptotic cones of the universal covers of 4-dimensional closed nonpositively curved real analytic manifolds. We show that the existence of nonstandard components in the Tits boundary, discovered by Christoph Hummel and Victor Schroeder, depends only on the quasi-isometry type of the fundamental group.

Contents

Chapter 1
Introduction

1.1 Historical Background

A class of groups is called quasi-isometry rigid if any finitely generated group which is quasi-isometric[1] (with the word metric) to a group in this class is weakly commensurable to a member of the same class. The first example was given by Gromov [21], where he showed that a finitely generated group has polynomial growth if and only if it is virtually nilpotent, i.e. has nilpotent subgroup of finite index.

Gromov's polynomial growth theorem inspired the more general problem of understanding to what extent the asymptotic geometry of a finitely generated group determines the algebraic structure of the group (see for example [6, 22, 23]). Many interesting algebraic properties have already been characterized by corresponding geometric invariants.

[1]A quasi-isometry between two metric spaces is a map $\Phi\colon X \longrightarrow Y$ such that, for two constants $K, C > 0$ the function satisfies the following two conditions. For every $x_1, x_2 \in X$, $K^{-1} d_X(x_1, x_2) - C \leq d_Y\big(\Phi(x_1), \Phi(x_2)\big) \leq K d_X(x_1, x_2) + C$ and the C-neighborhood of $\Phi(X)$ is all of Y.

A Riemannian manifold is called a Hadamard manifold if it is complete, simply connected, and of nonpositive sectional curvature. CAT(0) spaces are generalizations of Hadamard manifolds. They are geodesic metric spaces which satisfy Aleksandrov's nonpositively curved condition that geodesic triangles are not fatter than corresponding triangles in the Euclidean plane. A group is called a CAT(0) group if it acts discretely, cocompactly, and isometrically on a CAT(0) space.

Given a CAT(0) space X, the ideal boundary, introduced in its full generality by Eberlein and O'Neil [15], is the set of equivalence classes of asymptotic geodesic rays. Two rays are asymptotic if they lie within a bounded distance from each other. The geometric boundary $\partial_\infty X$ is the ideal boundary equipped with the cone topology, i.e. the topology of uniform convergence on compact sets. One also associates to X the Tits boundary $\partial_T X$, which is the ideal boundary equipped with the the Tits metric. The Tits boundary reflects part of the asymptotic geometry of the space. Both boundaries are well-known constructions with many applications in the literature (see for example [2, 5, 31]).

For one to be able to distinguish the quasi-isometry types of groups, quasi-isometry invariants are needed. If a group G acts isometrically, cocompactly, and discretely on a CAT(0) space, it is a natural question to ask whether the ideal boundary of X is a quasi-isometry invariant of the group. In the case of CAT(-1) spaces, or more generally Gromov hyperbolic spaces, the geometric boundary behaves well with respect to quasi-isometries. More precisely, a quasi-isometry between two spaces induces a homeomorphism between the geometric boundaries. This is a consequence of the Morse Lemma, which says that every quasi-geodesic[2] lies within a bounded distance from a unique geodesic. As a consequence, the geometric boundary is a quasi-isometry invariant of the group itself.

The Morse Lemma fails in general for CAT(0) spaces. An easy example is a logarithmic spiral in the Euclidean plane which is a quasi-geodesic but does not lie within a bounded distance of any straight line. For CAT(0) spaces, the ideal boundary is *no longer* a quasi-isometry invariant.

Croke and Kleiner [11] gave the first example of two quasi-isometric spaces with non-homeomorphic geometric boundaries. They constructed a pair of compact piecewise Euclidean 2-complexes with nonpositive curvature which are homeomorphic but whose universal covers have non-homeomorphic geometric boundaries.

For an n-dimensional manifold of nonpositive curvature, the geometric boundary is always homeomorphic to S^{n-1}. However by studying graph manifolds of nonpositive curvature, Buyalo [10] and Croke and Kleiner [12] independently showed that neither the geometric nor the Tits boundary is equivariantly determined by the quasi-isometry type of the space.

The failure of the ideal boundary to be a quasi-isometry invariant for a general CAT(0) group, rises the following natural questions. If X and Y are two quasi-isometric CAT(0) spaces with cocompact isometry groups, What properties of the geometric boundary are quasi-isometry invariants? And what properties of the Tits boundary are quasi-isometry invariants? Another natural question is when can one guarantee that any quasi-isometry

[2] A quasi-flat of dimension n is a quasi-isometrically embedded n-dimensional Euclidean space. A quasi-geodesic is a 1-dimensional quasi-flat.

maps flats to within a bounded distance of flats? Answers to these questions can also be viewed as partial information towards quasi-isometry rigidity.

1.2 Overview of Main Results

In this work we study two different topics. First in Chapter 3, we study the filling invariants at infinity div_k for Hadamard manifolds defined by Brady and Farb [7]. Second in Chapter 4, we study the large scale geometry of 4-dimensional closed nonpositively curved real analytic manifolds.

Description of Results in Chapter 3. Gersten [19] studied the divergence of geodesics in CAT(0) spaces, and gave an example of a finite CAT(0) 2-complex whose universal cover possesses two geodesic rays which diverge quadratically and such that no pair of geodesics diverges faster than quadratically. Adopting a definition of divergence which is a quasi-isometry invariant, Gersten introduced a new invariant for geodesic metric spaces, which we refer to as div_0. Gersten [18] used the div_0 invariant to distinguish the quasi isometry type of graph manifolds among all closed Haken 3-manifolds, as the only manifolds whose div_0 has quadratic polynomial growth.

Using the same idea introduced by Gersten [19] to get a quasi-isometry invariant, Brady and Farb [7] introduced a family of new quasi-isometry invariants $\text{div}_k(X)$ for $1 \leq k \leq n-2$, for a Hadamard manifold X^n. These invariants were meant to be a finer measure of the spread of geodesics in X.

The precise definition will be given in Section 3.1, but roughly the definition of $\text{div}_k(X)$

is as follows: Find the minimum volume of a $(k+1)$-ball which is needed to fill a k-sphere which lies on the $(n-1)$-dimensional distance sphere $S_r(x_0)$ in X. The filling is required to lie outside the n-dimensional open ball $B_r^\circ(x_0)$. The $\mathrm{div}_k(X)$ invariant measures the asymptotic behavior of the volume of the filling as $r \to \infty$ when the volume of the k-sphere grows polynomially in r.

After computing some of these invariants for certain Hadamard manifolds, Brady and Farb posed the following two questions.

Question 1.2.1. Can the $\mathrm{div}_k(X)$ invariants be used to detect the rank of a noncompact symmetric space X?

Question 1.2.2. What symmetric spaces can be distinguished by the invariants div_k?

We study the div_k invariants for various Hadamard manifolds including symmetric spaces of noncompact type. The first result we obtain is the following theorem.

Theorem 1.2.3. *If X is an n-dimensional symmetric space of nonpositive curvature and of rank l, then $\mathrm{div}_k(X)$ has a polynomial growth of degree at most $k+1$ for every $k \geq l$.*

Brady and Farb [7] showed that $\mathrm{div}_{k-1}(X)$ has exponential growth when $X = H^{m_1} \times \cdots \times H^{m_k}$ is a product of k hyperbolic spaces. The idea was to show that there are quasi-isometric embeddings of $H^{(m_1+\cdots+m_k)-k+1}$ in X and use that to show an exponential growth of the filling of a $(k-1)$-sphere which lies in a k-flat, and therefore proving that $\mathrm{div}_{k-1}(X)$ is indeed exponential. For more details, see Section 4 in [7].

The same idea was taken further by Leuzinger [30] to generalize the above result to any rank k symmetric space X of nonpositive curvature by showing the existence of an embedded submanifold $Y \subset X$ of dimension $n-k+1$ which is quasi-isometric to a Riemannian manifold with strictly negative sectional curvature, and which intersects a maximal flat in a geodesic.

Combining Theorem 1.2.3 with Leuzinger's theorem mentioned above, we obtain the following corollary which gives a positive answer to Question 1.2.1.

Corollary 1.2.4. *The rank of a nonpositively curved symmetric space* X *can be detected using the* $\mathrm{div}_k(X)$ *invariants.*

Brady and Farb [7] suspected that div_1 has exponential growth for the symmetric space $SL_n(\mathbb{R})/SO_n(\mathbb{R})$. For $n = 3$, this follows from Leuzinger's theorem [30] since the rank of $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ is 2. We show that the same result does not hold anymore for $n > 3$. More generally we prove,

Theorem 1.2.5. *If* X *is a symmetric space of nonpositive curvature and rank* $k \geq 3$, *then* $\mathrm{div}_1(X)$ *has a quadratic polynomial growth.*

After studying the case of symmetric spaces, we turn our focus to the class of Hadamard manifolds of pinched negative curvature. We show,

Theorem 1.2.6. *If* X *is a Hadamard manifold with sectional curvature* $-b^2 \leq K \leq -a^2 < 0$, *then* $\mathrm{div}_k(X)$ *has a polynomial growth of degree at most* k *for every* $k \geq 1$.

A natural question at this point would be whether the same result holds if we weaken the assumption on the manifold X to be a rank 1 instead of being negatively curved. We give a negative answer to this question. We give an example of a nonpositively curved graph manifold of rank 1 whose div_1 has exponential growth.

Finally we used the techniques we developed in [24] to give a different and shorter proof of Leuzinger's theorem which we mentioned above.

Description of Results in Chapter 4. Nonpositively curved closed 4-dimensional real analytic manifolds form an interesting class of manifolds. Many examples of these manifolds have been constructed by Schroeder [34, 35] and later by Abresch and Schroeder [1]. Schroeder [36] studied the structure of flats in these manifolds. While the Tits alternative of their fundamental groups was studied by Xiangdong Xie [39].

Given a Hadamard space X, the Tits boundary $\partial_T X = \big(X(\infty), Td\big)$ is a metric space which reflects part of the asymptotic geometry of the space. The ideal boundary of a Hadamard space in general is *not* a quasi-isometry invariant.

By investigating the Tits boundary of graph manifolds, Croke and Kleiner [12] discovered nontrivial connected components of the Tits boundary which are not subsets of unions of ideal boundaries of flats. These components are intervals of length less than π. Hummel and Schroeder [26] discovered similar components, which they called *nonstandard components,* in the Tits boundary of certain 4-dimensional nonpositively curved real analytic manifolds. It follows from [36] that the existence of these nonstandard components depends on the fundamental group of the 4-dimensional real analytic manifold.

We analyze, using some ideas of Kapovich and Leeb [27], the asymptotic cones of the universal cover of these 4-dimensional nonpositively curved analytic manifolds, and show that the existence of nonstandard components depends only on the quasi-isometry type of the fundamental group of the manifold.

Theorem 1.2.7. *Let X_1 and X_2 be the universal covers of two closed 4-dimensional nonpositively curved real analytic manifolds. If the fundamental groups $\pi_1(X_1)$ and $\pi_1(X_2)$ are quasi-isometric, then $\partial_T X_1$ contains a nonstandard component if and only if $\partial_T X_2$ does.*

1.3 Outline of Subsequent Chapters

In Chapter 2 we collect all the background information which we need later on. In Section 2.1 we give a brief introduction to spaces of nonpositive curvature and state some of their basic properties. In Section 2.2 we concentrate on symmetric spaces of noncompact type. We discuss their basic properties including Cartan decomposition, Weyl chambers, and the Tits buildings at infinity. In Section 2.3 we give a short introduction to ultrafilters, ultralimits, and asymptotic cones of metric spaces.

We start our study of the filling invariants at infinity in Chapter 3. In section 3.1 we give the precise definition of these invariants. In section 3.2 the proof of Theorem 1.2.3 in the simple case of rank 1 symmetric spaces is given where the basic idea is illustrated. In section 3.3 we prove Theorem 1.2.3 for higher rank symmetric spaces. The proof of Theorem 1.2.5 is give in section 3.4. In section 3.5 we prove Theorem 1.2.6. In section 3.6 we give the graph manifold example mentioned above to show that Theorem 1.2.6 does not hold if we weaken the assumption on the manifold X to be rank 1 instead of being negatively curved. In Section 3.7 we give a different and shorter proof of Leuzinger's theorem, which was proved in [30].

Our study of the large scale geometry of the 4-dimensional closed nonpositively curved real analytic manifolds is done in Chapter 4. In Section 4.1 we recall the properties of 4-dimensional real analytic manifolds which we need through the rest of that chapter. In Section 4.2 we show that for certain 4-dimensional real analytic manifolds all triangles are thin relative to a maximal higher rank submanifold, which is the key to understand their asymptotic cones. We use that to describe all flats in the asymptotic cones of these manifolds, which is done in Section 4.3. Finally, in Section 4.4 we finish by giving the proof of Theorem 1.2.7.

Chapter 2
Background

2.1 Spaces of Nonpositive Curvature

Hadamard Manifolds. The sign of the sectional curvature of a Riemannian manifold describes locally how fast geodesics spread apart compared with Euclidean space. If the sectional curvature K is everywhere nonnegative ($K \geq 0$), then two geodesics starting from the same point tend to come together compared with two geodesics in Euclidean space with the same angle. The situation is the opposite if the curvature is nonpositive ($K \leq 0$). In such a case geodesics diverge faster than in Euclidean space.

A *Hadamard manifold* is a simply connected complete Riemannian manifold of nonpositive sectional curvature. Since geodesics spread in a Hadamard manifold X at least as fast as in Euclidean space, X has no conjugate points. By the Cartan-Hadamard Theorem X^n is diffeomorphic to $\mathbb{R}^n$. In fact the exponential map $\exp_p$ at any point $p \in X$ is a diffeomorphism.

Given two points $p, q \in X$, there exists a unique geodesic connecting them. We denote by $\overline{pq}$, $\overrightarrow{pq}$ respectively the geodesic segment connecting p to q, and the geodesic ray staring at p and passing through q.

Triangles in a Hadamard manifold X are slimmer than triangles in Euclidean space. This means if Δ is any geodesic triangle in X and we choose a triangle $\overline{\Delta}$ in the Euclidean plane with the same edge lengths then distances between points on Δ are less than or equal to the distances between corresponding points on $\overline{\Delta}$. In fact, a simply connected manifold which satisfies this condition for every triangle is a Hadamard manifold. This observation allows us to extend the definition of spaces of nonpositive curvature to a larger class of singular spaces, CAT(0) spaces, which are not manifolds by requiring that triangles in the singular space to be slimmer than triangles in Euclidean space.

CAT (0) Spaces. In this section we recall the definition of spaces of curvature bounded from above with emphasis on spaces of nonpositive curvature. We will only consider complete metric spaces. Let κ be a real number, we denote by M^2_κ the unique simply connected surface of constant curvature κ. We denote by D_κ the diameter of M^2_κ, which is ∞ if $\kappa \leq 0$ and $\pi/\sqrt{\kappa}$ if $\kappa > 0$.

Given a metric space (X, d), a curve in X is a continuous map $\gamma\colon [a, b] \longrightarrow X$. The length of γ is defined in the usual way.

$$L(\gamma) = \sup \sum_{i=1}^{k} d\big(\gamma(t_{i-1}), \gamma(t_i)\big), \tag{2.1.1}$$

where the supremum is taken over all partitions $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$ of $[a, b]$. The curve γ is called a geodesic segment if it is parameterized by arc length and

$L(\gamma) = d\big(\gamma(a), \gamma(b)\big)$.

Definition 2.1.1. A metric space (X, d) is called a *geodesic metric space* if there is a geodesic segment connecting every two points.

A triangle in a geodesic metric space (X, d) consists of three geodesic segments σ_1, σ_2, σ_3, whose end points match in the usual way. These geodesic segments are called the edges of the triangle. A triangle $\overline{\Delta}\big(\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}\big)$ in a model space M^2_κ is called a *comparison triangle* for Δ if $L(\sigma_i) = L\big(\overline{\sigma_i}\big)$ for $i = 1, 2, 3$.

A comparison triangle exists (and unique up to congruence) in the model space M^2_κ if the sides of Δ satisfy the following two conditions:

$$L(\sigma_i) + L(\sigma_j) \geq L(\sigma_k) \quad \text{for } i \neq j \neq k, \tag{2.1.2a}$$

and

$$L(\sigma_1) + L(\sigma_2) + L(\sigma_3) < 2D_\kappa. \tag{2.1.2b}$$

Definition 2.1.2. A geodesic triangle Δ in a geodesic metric space X is said to satisfy the CAT(κ) inequality, if it has a comparison triangle $\overline{\Delta}$ in the model space M^2_κ and distances between points on Δ are less than or equal to the distances between corresponding points on $\overline{\Delta}$.

Definition 2.1.3. A geodesic metric space (X, d) is a CAT(κ) space if all geodesic triangles of perimeter less than $2D_\kappa$ satisfy the CAT(κ) inequality.

Let X be a CAT(κ) space. Now we define angles between two geodesics segments in X. Let $\epsilon > 0$ and let $\sigma_1, \sigma_2 \colon [0, \epsilon] \longrightarrow X$ be two unit speed geodesics with $\sigma_1(0) = \sigma_2(0) =$

p. For $s, t \in [0, \epsilon]$, let Δ_{st} be the triangle spanned by $\sigma_1|_{[0,s]}$ and $\sigma_2|_{[0,t]}$ respectively. Let $\theta(s,t)$ be the angle at $\overline{p}$ of the comparison triangle $\overline{\Delta}_{st}$ in M^2_κ. The angle $\theta(s,t)$ is monotonically decreasing function in s and t and therefore

$$\angle_p(\sigma_1, \sigma_2) = \lim_{s,t \to 0} \theta(s,t) \qquad (2.1.3)$$

exists and is called the *angle* between σ_1 and σ_2.

We now focus our attention on the case $\kappa = 0$. Following the terminology in the manifold case, we call a CAT(0) space a *Hadamard space*. We state some known facts about Hadamard spaces. Hadamard spaces are contractible and between every two points there exists a unique geodesic segment connecting them. The most important property of Hadamard spaces is that distance functions are convex.

Proposition 2.1.4 ([3] Proposition I.5.4). *Let* I *be an interval, and let* $\sigma_1, \sigma_2 \colon I \longrightarrow X$ *be two geodesics in a Hadamard space* X*. Then* $d\big(\sigma_1(t), \sigma_2(t)\big)$ *is convex in* t.

Proposition 2.1.5 ([3] Proposition I.5.2). *Let* X *be a Hadamard space and let* Δ *be a triangle in* X *with edges of length* a, b, *and* c *and angles* α, β, *and* γ *at the opposite vertices respectively. Then*

1. $\alpha + \beta + \gamma \leq \pi$.
2. *First Cosine Inequality:* $c^2 \geq a^2 + b^2 - 2ab\cos\gamma$.
3. *Second Cosine Inequality:* $c \leq b\cos\alpha + a\cos\beta$.

In each case, equality holds if and only if Δ *is flat, that is,* Δ *bounds a convex region in* X *isometric to the triangular region bounded by the comparison triangle in the flat plane.*

Two unit speed geodesics $\sigma_1, \sigma_2 \colon [0, \infty) \longrightarrow X$ are said to be *parallel* if $d(\sigma_1(t), \sigma_2(t))$ is uniformly bounded.

Definition 2.1.6. Given a geodesic σ in a Hadamard space X. The *parallel set* of σ, denoted by P_σ, is the set consists of the union of all geodesics which are parallel to σ. The parallel set P_σ splits isometrically as $Q \times \mathbb{R}$, where the $\mathbb{R}$-factor corresponds to the geodesic σ and $Q \subset X$ is closed and convex.

Lemma 2.1.7 ([4] Lemma 2.4). *If X is a real analytic Hadamard manifold, σ is a geodesic, and $P_\sigma = Q \times \mathbb{R}$ is its parallel set, then Q is a complete submanifold of X without boundary.*

Theorem 2.1.8 (Flat Strip). *Let σ_1 and σ_2 be two parallel geodesics in a Hadamard space X. If the Hausdorff distance $Hd(\sigma_1, \sigma_2) = a$, then there is an isometric and convex embedding $\phi \colon \mathbb{R} \times [0, a] \longrightarrow X$ such that $\sigma_1(\mathbb{R}) = \phi(\mathbb{R} \times \{0\})$ and $\sigma_2(\mathbb{R}) = \phi(\mathbb{R} \times \{a\})$.*

Projection onto Convex Subsets. If X is a Hadamard space, a subset $W \subset X$ is called convex if for every two points $p, q \in W$, $\overline{pq} \subset W$. If W is also closed then projection onto W resembles the situation in the Euclidean plane. More precisely, for every point $p \in X$ there is a unique point $\pi_W(p) \in W$ of minimal distance to p. Moreover, the projection map π_W is distance nonincreasing.

Ideal Boundary. To simplify the discussion in this part we restrict ourselves to metric spaces which are locally compact. The Hyperbolic plane (considered as the unit disc) can be compactified by adding the unit sphere. Isometries extends to homeomorphisms of the compactified space. The ideal boundary $X(\infty)$ for a Hadamard space was introduced in its full generality by Eberlein and O'Neil [15], and it is an analogous compactification for arbitrary Hadamard space. Recall that we assume that X is a locally compact Hadamard space.

Definition 2.1.9. Given two unit speed geodesic rays $\sigma_1, \sigma_2\colon [0,\infty) \longrightarrow X$, we say that σ_1 and σ_2 are asymptotic if $d\big(\sigma_1(t), \sigma_2(t)\big)$ is uniformly bounded.

Being asymptotic is an equivalence relation on the set of all geodesic rays. The set of equivalence classes is denoted by $X(\infty)$ and called the ideal boundary of the Hadamard space X. If $\xi \in X(\infty)$ and σ belongs to ξ, we write $\sigma(\infty) = \xi$.

By convexity of the distance function, if σ_1 and σ_2 are two asymptotic geodesic rays and $\sigma_1(0) = \sigma_2(0)$, then $\sigma_1 = \sigma_2$. This shows that for any base point $p \in X$ and any $\xi \in X(\infty)$ there is at most one geodesic ray starting at p in this class. In fact, it is not hard to see that there is a geodesic ray starting at p representing this class. One way to see that is to take any geodesic ray σ_2 such that $\sigma_2(\infty) = \xi$. and take the following sequence of geodesic rays $\overrightarrow{p\sigma_2(n)}$. Since X is locally compact, we can show that this sequence has a subsequence which converges to a geodesic ray starting at p and asymptotic to σ_2.

There are two different topologies which one can put on the ideal boundary $X(\infty)$ of X. The first topology is the *cone topology*. Let $\overline{X} = X \cup X(\infty)$. The cone topology on $\overline{X}$ is the topology generated by the open sets of X together with the sets

$$U(x,\xi,R,\epsilon) = \left\{z \in \overline{X} \mid z \notin B(x,R) \text{ and } d\big(\overrightarrow{xz}(R), \overrightarrow{x\xi}(R)\big) < \epsilon\right\}, \tag{2.1.4}$$

where $x \in X$, $\xi \in X(\infty)$, $R, \epsilon > 0$, and $\overrightarrow{x\xi}$ represents the geodesic ray starting at x in the class ξ. It is possible to show that this collection of sets forms a basis for a topology (see [3], Lemma II.2.2).

If X is a Hadamard manifold, then $\left(\overline{X}, X(\infty)\right)$ with the cone topology is homeomorphic to (D^n, S^{n-1}), where n is the dimension of the manifold. $X(\infty)$ is not homeomorphic to a sphere for arbitrary Hadamard space. For example, if X is the Cayley graph of the free group on two generators, then the ideal boundary is a Cantor set. The ideal boundary of a Hadamard space with the cone topology is called the *geometric boundary* and denoted by $\partial_\infty X$.

The second topology one can put on the ideal boundary $X(\infty)$ of a Hadamard space X is the topology induced by the *Tits metric* which we define now.

Definition 2.1.10 (Tits Angle). Given two points $\xi, \eta \in X(\infty)$, the *Tits angle* between them is $\angle(\xi, \eta) = \sup_{p \in X} \angle_p(\xi, \eta)$.

For any metric space (Y, d), the *interior metric* d_i of (Y, d) is defined as follows.

$$d_i(y_1, y_2) = \inf\{L(\gamma) \mid \gamma \text{ is a curve connecting } y_1 \text{ to } y_2\}. \tag{2.1.5}$$

The *Tits metric* on the ideal boundary $X(\infty)$ of a Hadamard space X is the interior metric of $\left(X(\infty), \angle\right)$. The ideal boundary equipped with the Tits metric is called the *Tits boundary* and denoted by $\partial_T X$.

Example 2.1.11. If $X = \mathbb{R}^n$ and $\xi, \eta \in X(\infty)$, then for every point $p \in X$, $\angle_p(\xi, \eta) = \angle(\xi, \eta)$. Therefore $\partial_T X$ is isometric to the round sphere.

Example 2.1.12. Let X be n-dimensional manifold of strictly negative curvature. Every two points $\xi, \eta \in X(\infty)$ are connected with a geodesic inside X. By taking p on that geodesic, we immediately get that $\angle(\xi,\eta) = \angle_p(\xi,\eta) = \pi$. Therefore the Tits distance between each two different points in $X(\infty)$ is infinite.

Example 2.1.13. If X is a symmetric space of nonpositive curvature, then $\partial_T X$ is a spherical building. The apartments are the ideal boundaries of flats of maximal dimension in X.

2.2 Symmetric Spaces

The purpose of this section is to give a very short and brief introduction to symmetric spaces with emphasis on symmetric spaces of noncompact type. For a standard reference on symmetric spaces of nonpositive curvature and proofs of most of the facts in this section we refer the reader to [14].

Definition 2.2.1. A symmetric space is a simply connected Riemannian manifold X with the property that the geodesic reflection at every point is an isometry of X. More precisely, for any $x_0 \in X$ there is an isometry S_{x_0} of X such that $S_{x_0}(x_0) = x_0$ and $(dS_{x_0})_{x_0} = -I$.

Proposition 2.2.2. *If X is a simply connected Riemannian manifold then the following conditions are equivalent.*

1. *X is a symmetric space.*

2. *The curvature tensor is parallel, i.e. $\nabla R = 0$.*

3. *If $X(t), Y(t), Z(t)$ are parallel vector fields along a geodesic $\gamma(t)$, then the vector field $R\big(X(t), Y(t)\big)Z(t)$ is parallel along $\gamma(t)$.*

If X is a symmetric space of nonpositive sectional curvature and is not a Riemannian product of an Euclidean space $\mathbb{R}^k$, $k \geq 1$, with another Riemannian manifold, then X is said to be a symmetric space of *noncompact type.*

Every symmetric space is a homogenous manifold, i.e. the isometry group $\mathrm{Isom}(X)$ acts transitively on X. We denote the connected component of $\mathrm{Isom}(X)$ that contains the identity map by G. If we fix a base point $x_0 \in X$, then the closed subgroup $K = \{g \in G \mid g(x_0) = x_0\}$ is called the *isotropy group* at x_0.

Cartan Decompositions. From now on we assume that X is a symmetric space of noncompact type and therefore G is a real semisimple Lie group. Fix a base point $x_0 \in X$ and recall that K denotes the isotropy group at x_0. We define an involution $\sigma_{x_0} \colon G \longrightarrow G$ by

$$\sigma_{x_0}(g) = S_{x_0} \circ g \circ S_{x_0}, \tag{2.2.1}$$

where S_{x_0} is the geodesic reflection at x_0. The involution σ_{x_0} induces the following involution on the Lie algebra $\mathfrak{g}$ of G,

$$\theta_{x_0} = d\sigma_{x_0} \colon \mathfrak{g} \longrightarrow \mathfrak{g}. \tag{2.2.2}$$

Let

$$\mathfrak{k} = \{A \in \mathfrak{g} \mid \theta_{x_0} A = A\} \quad \text{and} \quad \mathfrak{p} = \{A \in \mathfrak{g} \mid \theta_{x_0} A = -A\}. \tag{2.2.3}$$

Since $\theta_{x_0}^2 = I$, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The fact that θ_{x_0} preserves the Lie brackets implies that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \tag{2.2.4}$$

The *Killing form* of the Lie algebra $\mathfrak{g}$ is the symmetric bilinear form $\mathcal{B}\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{B}(A, B) = \operatorname{trace}(\operatorname{ad} A \circ \operatorname{ad} B). \tag{2.2.5}$$

Because $\mathfrak{g}$ is semisimple, $\mathcal{B}$ is nondegenerate on $\mathfrak{g}$. Moreover $\mathcal{B}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$ and with respect to $\mathcal{B}$, $\mathfrak{g}$ is an orthogonal direct sum of $\mathfrak{k}$ and $\mathfrak{p}$. For any two vectors $A, B \in \mathfrak{g}$, we set $\phi_{x_0}(A, B) = -\mathcal{B}(A, \theta_{x_0} B)$. ϕ_{x_0} is a positive definite bilinear form on $\mathfrak{g}$ and called the *canonical inner product* on $\mathfrak{g}$. If we extend this metric by left translation, the natural projection $\pi\colon G \longrightarrow X$ becomes Riemannian submersion and this metric up to a scale is the original metric on X.

Curvature Tensor of X. The curvature tensor on X has the following form at the point x_0.

$$R(A, B)C = -\operatorname{ad}[A, B](C) = -[[A, B], C], \tag{2.2.6}$$

for all $A, B, C \in \mathfrak{p}$.

Root Space Decomposition. A *flat* in a Hadamard manifold X is a complete totally geodesic Euclidean submanifold. For symmetric spaces we also insist that it is of maximal dimension. Let F be a flat which contains the base point x_0. The tangent space to F at x_0 corresponds to a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$. All maximal Abelian subalgebras of $\mathfrak{p}$ are conjugate which implies that for any two flats F and F' containing x_0, there is an isometry $\phi \in K$ such that $\phi F = F'$.

For every element $H \in \mathfrak{a}$ the map $\operatorname{ad} H\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is symmetric with respect to the canonical inner product ϕ_{x_0} on $\mathfrak{g}$. Since $\mathfrak{a}$ is Abelian, the different elements $\operatorname{ad} H$, where $H \in \mathfrak{a}$, commute and therefore can be diagonalized simultaneously. This yields the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda \tag{2.2.7}$$

where $\mathfrak{g}_\lambda = \{B \in \mathfrak{g} \mid \forall H \in \mathfrak{a}, [H, B] = \lambda(H)B\}$, $\lambda \in \operatorname{Hom}(\mathfrak{a}, \mathbb{R})$, and $\Lambda = \{\lambda \in \operatorname{Hom}(\mathfrak{a}, \mathbb{R}) \mid \mathfrak{g}_\lambda \neq 0\}$. The elements of Λ are called the *roots.*

A geodesic $\gamma\colon \mathbb{R} \longrightarrow F$ with $\gamma(0) = x_0$ is called a *regular* geodesic if it is not contained in any maximal flat other than F, otherwise it is called a *singular* geodesic. A vector of $\mathfrak{a}$ is called a *singular vector* if it is tangent to a singular geodesic otherwise it is called a *regular geodesic.* A vector $H \in \mathfrak{a}$ is singular if and only if $\lambda(H) = 0$ for some $\lambda \in \Lambda$. Therefore the set of singular vectors in $\mathfrak{a}$ consists of a finite number of hyperplanes which are called *singular hyperplanes.*

Weyl Chambers. The singular hyperplanes in $\mathfrak{a}$ divide the set of regular vectors into finitely many components. Each component is called a *Weyl Chamber* of $\mathfrak{a}$. Fix a Weyl Chamber $\mathfrak{a}^+$. This choice corresponds to the choice of a set of positive roots $\Lambda^+ \subset \Lambda$, where

$$\Lambda^+ = \{\lambda \in \Lambda \mid \lambda(H) > 0 \text{ for every } H \in \mathfrak{a}^+\}. \tag{2.2.8}$$

Given $\lambda \in \Lambda^+$, the kernel of λ is a singular hyperplane and is called a *wall* of the Weyl chamber.

The intersection of a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ with the unit sphere $S_{x_0}X \subset T_{x_0}X$ is called a Weyl chamber of the *first type*, while the image of $\mathfrak{a}^+$ under the exponential map $\exp_{x_0}$ is called a Weyl chamber of the *second type*. The ideal boundary of a Weyl chamber of the second type is a subset of $X(\infty)$ and is called a Weyl Chamber *at infinity*. The intersection of a number of walls of a Weyl chamber with the sphere at infinity is called a face of the corresponding Weyl chamber at infinity. It can be shown that the set of all Weyl chambers at infinity and their faces form a partition of the sphere at infinity. A *closed* Weyl chamber is the union of a Weyl chamber and its faces.

It is reasonable to expect that the set of Weyl chambers and their faces depends on the base point x_0. Surprisingly that is not the case. The isotropy group K acts transitively on the set of Weyl chamber at infinity and the fundamental domain of this action is a closed Weyl chamber.

Algebraic Centroid of a Weyl Chamber. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subspace and let Λ be the set of corresponding roots. Let $A \in \mathfrak{a}$ be a regular vector and $\mathscr{C}_A$ be the unique Weyl chamber contains A. Let Λ^+ is the set of positive roots determined by $\mathscr{C}_A$. The *algebraic centroid* of $\mathscr{C}_A$ is the vector ν defined by

$$\nu = \sum_{\lambda \in \Lambda^+} n_\lambda, \quad \text{where } n_\lambda = H_\lambda / \|H_\lambda\|. \tag{2.2.9}$$

H_λ is the dual vector of λ with respect to the canonical inner product. The isotropy group K at x_0 acts transitively on the set of algebraic centroids of all Weyl chambers.

Weyl Group. Fix a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and let $A = \exp(\mathfrak{a})$. $F = Ax_0$ is a maximal flat in X. Let M be the centralizer of A in K. M is the subgroup of G which leaves F fixed pointwise. Let M' be the normalizer of A in K. M' is the subgroup of G which fixes x_0 and leaves F invariant. M is a normal subgroup of M' and the *Weyl group* is the group M'/M. One can show that the Weyl group is a finite group which acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$.

Tits Building at Infinity.

Definition 2.2.3. Let S be a finite set. A *Coxeter matrix* $M = (m_{ss'})$ is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ss'} = 1$ if $s = s'$ and at least 2 otherwise.

Definition 2.2.4. Given a Coxeter matrix M, define a group W with presentation,

$$W = \langle S \mid (ss')^{m_{ss'}} = 1 \text{ for every } (s, s') \in S \times S\rangle. \tag{2.2.10}$$

W is called a *Coxeter group.*

To any Coxeter group W, one can associate a simplicial complex Σ_W which is called the *Coxeter complex* of W. In the case of a finite reflection group, which is the only case we are interested in here, the Coxeter complex has a simple description. We assume that W is a finite group generated by reflections along hyperplanes in some Euclidean space $\mathbb{R}^n$. Let $M = S^{n-1}$ be the unit sphere. For each reflection $r \in W$, i.e. a generator or a conjugate of a generator, r fixes a hyperplane which is called a *Wall.* The Walls divide M into simplices. The resulting simplicial complex Σ_W is the Coxeter complex for W. The top dimensional simplices of Σ_W are called chambers. One can shows that W acts simply transitively on the set of chambers.

We now give one of the possible definitions of Buildings and refer the reader to [9] for details.

Definition 2.2.5. A building is a simplicial complex Δ together with a collection of subcomplexes $\mathcal{A}$, called *apartments*, which satisfies the following three conditions.

1. Each apartment is isomorphic to a Coxeter complex.

2. Any two simplices of Δ are contained in a common apartment.

3. If A_1 and A_2 are two apartments which share a chamber, then there is an isomorphism $\phi\colon A_1 \longrightarrow A_2$ which fixes $A_1 \cap A_2$.

A building is called a *thick* building if every codimension 1 simplex is contained in at least three chambers. It follows from the definition that each apartment is isomorphic to the same Coxeter complex Σ_W. We say that a building Δ is *spherical* if the realization of Σ_W is homeomorphic to a sphere.

One can show that the collection of all Weyl chambers at infinity with their faces with the obvious face relation of a symmetric space of noncompact type is a spherical building and it is called the *Tits building at infinity* of X. The system of apartments are the collection of the ideal boundaries of all maximal dimensional flats in X. It is worth noting that the Tits distance Td at $X(\infty)$ determines the Tits building of X and vice versa. See [4] or [14] for the proof of this fact.

2.3 Asymptotic Cones

The concept of the asymptotic cone was introduced by van den Dries and Wilkie [38] and by Gromov [23]. It is a general construction which can be applied to any metric space. It encodes part of the coarse geometry of the space, and it has been used by several authors to study the large scale geometry of certain spaces and to distinguish the quasi-isometry type of these spaces.

Kleiner and Leeb [29] used the asymptotic cone to prove the Margulis' conjecture which says that if X and X' are irreducible symmetric spaces of noncompact type, each non-flat and of rank at least 2, and if $\Phi \colon X \to X'$ is a quasi-isometry, then the metric on X may be rescaled, multiplying by a constant, so that Φ is a bounded distance from some isometry. Eskin and Farb [17] also gave a different and effective proof of the theorem of Kleiner and Leeb.

Kapovich and Leeb [27, 28] also used the asymptotic cone to study the quasi-isometry types of Haken 3-manifolds. By studying the topological type of the asymptotic cones of different 3-manifolds, Kapovich and Leeb [27] showed that the existence of a Seifert component in a Haken manifold is a quasi-isometry invariant of its fundamental group. Later they extend this result to show that quasi-isometries preserve the geometric decomposition of the universal covers of Haken manifolds [28].

Ultrafilters. Let $\mathbb{N}$ be the set of natural numbers. A filter on $\mathbb{N}$ is a nonempty family ω of subsets of $\mathbb{N}$ which satisfies the following properties.

1. $\emptyset \notin \omega$.

2. If $A \in \omega$ and $A \subset B$, then $B \in \omega$.

3. If $A_1, \ldots, A_k \in \omega$, then $A_1 \cap \cdots \cap A_k \in \omega$.

Any filter ω defines a probability measure on $\mathbb{N}$, where a subset $A \subset \mathbb{N}$ has measure 1 if and only if $A \in \omega$, otherwise it has measure 0.

An *ultrafilter* is a maximal filter. The maximality condition easily implies that if $\mathbb{N}$ is written as a finite union of disjoint subsets, $\mathbb{N} = A_1 \sqcup \cdots \sqcup A_k$, then the ultrafilter contains exactly one of those subsets.

Definition 2.3.1. We call an ultrafilter ω a *nonprincipal ultrafilter* if and only if it does not contain finite sets.

The interesting ultrafilters are the nonprincipal ones. They exist by Zorn's Lemma, and they can not be described explicitly. Given a compact space Y and an ultrafilter ω on $\mathbb{N}$, we assign to each sequence $x \colon \mathbb{N} \longrightarrow Y$ a unique limit in Y. The limit $\omega\text{-}\lim x(n) = x_0$ is the unique point $x_0 \in Y$ such that for every open neighborhood U of x_0, $x^{-1}(U) \in \omega$. If $Y = \mathbb{R} \cup \{\infty\}$ is the one point compactification of $\mathbb{R}$, then every sequence of real numbers converges to a unique limit (possibly ∞). In general the limit depends on the choice of the ultrafilter ω.

Ultralimits of Metric Spaces. Let $\left(X_n, d_n\right)_{n\in\mathbb{N}}$ be a sequence of metric spaces. Given a nonprincipal ultrafilter ω on $\mathbb{N}$, the ultralimit of this sequences is the metric space

$$X_\omega = \omega\text{-}\lim X_n, \tag{2.3.1}$$

which is defined as follows. Let $X^* = \prod_{\mathbb{N}} X_n$. Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ be two points in X^*. We define the distance between x and y as follows.

$$d_\omega(x, y) = \omega\text{-}\lim d_n(x_n, y_n). \tag{2.3.2}$$

In general d_ω is a pseudometric. The quotient space obtained by identifying points with zero d_ω-distance is called the *ultralimit* of the sequence $(X_n, d_n)_{n\in\mathbb{N}}$ and is denoted by (X_ω, d_ω).

The definition of ultralimits extends the notion of Gromov-Hausdorff convergence.

Proposition 2.3.2 ([27] Proposition 3.2). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces converging in the Gromov-Hausdorff topology to a compact metric space X. Then $X_\omega \cong X$ for all nonprincipal ultrafilters ω.*

If the diameters of $(X_n, d_n)_{n\in\mathbb{N}}$ have no upper bound, the ultralimit X_ω generically decomposes into uncountable number of components. If we pick a base point $x_n^0 \in X_n$ for every $n \in \mathbb{N}$, the sequence $(x_n^0)_{n\in\mathbb{N}}$ represents a base point $x_0 \in X_\omega$. The *based ultralimit* (X_ω, x_0) is the set of points of finite distance from the base point x_0.

$$X_\omega^0 = \{x \in X_\omega \mid d_\omega(x, x_0) < \infty\}. \tag{2.3.3}$$

Asymptotic Cones of a Metric Spaces. Let (X, d) be a metric space. Fix a non-principle ultrafilter ω on $\mathbb{N}$, a base points $x_0 \in X$, and a sequence of rescaling factors λ_n such that $\omega\text{-}\lim \lambda_n = \infty$. The *asymptotic cone* $\mathrm{Cone}_\omega(X)$ of X is the based ultralimit of $((X, \lambda_n^{-1} \cdot d), x_0)$. The asymptotic cone $\mathrm{Cone}_\omega(X)$ is independent of which base point $x_0 \in X$ we choose.

An important property of asymptotic cones is that they are *complete* metric spaces, the proof can be found in [8] or [38]. The asymptotic cone construction respects products, i.e. for any two metric spaces X and Y and any ultrafilter ω, $\mathrm{Cone}_\omega(X \times Y) = \mathrm{Cone}_\omega(X) \times \mathrm{Cone}_\omega(Y)$. This easily follows from the definition. For any nonprinciple ultrafilter ω, the asymptotic cone of any Euclidean space $\mathbb{R}^n$ is again $\mathbb{R}^n$.

The following proposition summarizes some of the basic properties of the asymptotic cone.

Proposition 2.3.3 ([29] Proposition 2.4.6).

1. *If* X *is a geodesic metric spaces, then* $\mathrm{Cone}_\omega(X)$ *is a geodesic metric space.*
2. *If* X *is a Hadamard space, then* $\mathrm{Cone}_\omega(X)$ *is a Hadamard space.*
3. *If* X *is a CAT*(κ) *space for* $\kappa < 0$. *then* $\mathrm{Cone}_\omega(X)$ *is a metric tree.*
4. *If the orbits of* $\mathrm{Isom}(X)$ *are at bounded Hausdorff distance from* X, *then* $\mathrm{Cone}_\omega(X)$ *is a homogenous metric space.*
5. *A* (K, C) *quasi-isometry of metric spaces* $f \colon X \longrightarrow Y$ *induces a* K *bi-Lipschitz homeomorphism* $\mathrm{Cone}_\omega(f) \colon \mathrm{Cone}_\omega(X) \longrightarrow \mathrm{Cone}_\omega(Y)$.

The map $\mathrm{Cone}_\omega(f)$ is defined as follows: Let $x \in \mathrm{Cone}_\omega(X)$ and let $(x_n)_{n\in\mathbb{N}}$ be any sequence which represents it. The point $\mathrm{Cone}_\omega(f)(x)$ is represented by $(f(x_n))_{n\in\mathbb{N}}$. The map $\mathrm{Cone}_\omega(f)$ is well defined and it is bi-Lipschitz homeomorphism if f is a quasi-isometry.

We finish this section with the following remark. Suppose that X admits a cocompact discrete isometric action by a group G. The dependance of the asymptotic cone of X on the choice of the ultrafilter was an open question until recently. Thomas and Velickovic [37] constructed a finitely generated (but not finitely presented) group which has two nonhomeomorphic asymptotic cones. The finitely presented case is still an open question.

Chapter 3

Filling Invariants at Infinity

3.1 Definitions

Let X^n be an n-dimensional Hadamard manifold. Recall that we denote the ideal boundary of X by $X(\infty)$, for any two different points $p, q \in X$, $\overline{pq}$, $\overrightarrow{pq}$ denote respectively the geodesic segment connecting p to q, and the geodesic ray staring at p and passing through q, and by $\overrightarrow{pq}(\infty)$ we denote the limit point in $X(\infty)$ of the ray $\overrightarrow{pq}$.

Let $S_r(x_0)$, $B_r(x_0)$, and $B_r^\circ(x_0)$ denote respectively the distance sphere, the distance ball, and the open distance ball of radius r and center x_0. Let S^k and B^{k+1} denote respectively the unit sphere and the unit ball in $\mathbb{R}^{k+1}$. Let $C_r(x_0) = X \setminus B_r^\circ(x_0)$. Projection along geodesics of $C_r(x_0)$ onto the sphere $S_r(x_0)$ is a deformation retract, which decreases distances, since the ball $B_r(x_0)$ is convex and the manifold is nonpositively curved. Any continuous map $f\colon S^k \longrightarrow S_r(x_0)$ admits a continuous extension, "filling", $\hat{f}\colon B^{k+1} \longrightarrow C_r(x_0)$, and the extension could be chosen to be Lipschitz if f is Lipschitz.

Lipschitz maps are differential almost everywhere. Let $|D_x(f)|$ denote the Jacobian of f at x. The k-volume of f and the $(k+1)$-volume of $\hat{f}$ are defined as follows,

$$\mathrm{vol}_k(f) = \int_{S^k} |D_x f| \,, \tag{3.1.1}$$

$$\mathrm{vol}_{k+1}(\hat{f}) = \int_{B^{k+1}} \left|D_x \hat{f}\right| . \tag{3.1.2}$$

Let $0 < A$ and $0 < \rho \leq 1$ be given. A Lipschitz map $f\colon S^k \longrightarrow S_r(x_0)$ is called A-*admissible* if $\mathrm{vol}_k(f) \leq Ar^k$ and a Lipschitz filling $\hat{f}$ is called ρ-*admissible* if $\hat{f}(B^{k+1}) \subset C_{\rho r}(x_0)$. Let

$$\delta^k_{\rho, A} = \sup_f \inf_{\hat{f}} \mathrm{vol}_{k+1}(\hat{f}), \tag{3.1.3}$$

where the supremum is taken over all A-admissible maps and the infimum is taken over all ρ-admissible fillings.

Definition 3.1.1. The invariant $\mathrm{div}_k(X)$ is the two parameter family of functions

$$\mathrm{div}_k(X) = \{\delta^k_{\rho, A} \mid 0 < \rho \leq 1 \text{ and } 0 < A\}.$$

Fix an integer $k \geq 0$, for any two functions $f, g\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we write $f \preceq_k g$ if there exist two positive constants a, b and a polynomial $p_{k+1}(x)$ of degree at most $k+1$ with a positive leading coefficient such that $f(x) \leq a g(bx) + p_{k+1}(x)$. Now we write $f \sim_k g$ if $f \preceq_k g$ and $g \preceq_k f$. This defines an equivalence relation.

We say that $\mathrm{div}_k \preceq \mathrm{div}'_k$ if there exist $0 < \rho_0, \rho'_0 \leq 1$ and $A_0, A'_0 > 0$ such that for every $\rho < \rho_0$ and $A > A_0$ there exist $\rho' < \rho'_0$ and $A' > A'_0$ such that $\delta^k_{\rho, A} \preceq_k \delta'^k_{\rho, A}$. We define $\mathrm{div}_k \sim \mathrm{div}'_k$ if $\mathrm{div}_k \preceq \mathrm{div}'_k$ and $\mathrm{div}'_k \preceq \mathrm{div}_k$. This is an equivalence relation and under this identification div_k is a quasi-isometry invariant (see [7] for details).

Remark 3.1.2. As a quasi-isometry invariant, a polynomial growth rate of div_k is only defined up to $k+1$. And the reader should view the k polynomial growth rate of div_k in Theorem 1.2.6 accordingly.

The polynomial bound on $\mathrm{vol}_k(f)$ in the definition above is essential to prevent the possibility of constructing exponential k-volume maps requiring exponential volume fillings, and therefore making the filling invariants always exponential. On the other hand, allowing the filling to be "slightly" inside the ball, i.e. inside $C_{\rho r}(x_0)$ for some fixed $0 < \rho$ is needed since continuous quasi-isometries map spheres to distorted spheres.

Remark 3.1.3. Through this chapter we will use the following cone construction. Given an n-dimensional Hadamard manifold X and a Lipschitz map $f\colon S^k \longrightarrow X$, if we cone f from a point $x_0 \in X$ we obtain a Lipschitz extension $\hat{f}\colon D^{k+1} \longrightarrow X$. Using comparison with Euclidean space, it is clear that

$$\mathrm{vol}_{k+1}(\hat{f}) \leq \mathrm{vol}_k(f) \sup_{p \in S^k} d(x_0, f(p)). \tag{3.1.4}$$

If X has sectional curvature $K \leq -a^2 < 0$, then there exists a constant $C = C(a, n)$ which does not depend on x_0 nor f such that $\mathrm{vol}_{k+1}(\hat{f}) \leq C\,\mathrm{vol}_k(f)$. If $a = 1$ this constant can be chosen to be 1.

3.2 Rank One Symmetric Spaces

The $\mathrm{div}_0(X)$ of a Hadamard manifold X is the same as the "rate of divergence" of geodesics. Therefore for rank 1 symmetric spaces it is exponential since they have pinched negative curvature.

In this section we calculate div_k where $k \geq 1$ for any rank one symmetric space. First we start with a lemma which will be needed in the proof of Theorem 3.2.3.

Lemma 3.2.1. *Given a Lipschitz function* $f\colon S^k \longrightarrow \mathbb{R}^n$, *where* $1 \leq k < n$, *there exists a constant* $c = c(n)$ *such that for any ball* $B_r(u_0)$, *we can find a map* $g\colon S^k \longrightarrow \mathbb{R}^n$ *which satisfies the following conditions:*

1. f *is homotopic to* g.
2. $\operatorname{vol}_k(g) \leq c \operatorname{vol}_k(f)$.
3. $g(S^k) \cap B_r^\circ(u_0) = \emptyset$.
4. f *and* g *agree outside* $B_r(u_0)$.
5. *The* $(k+1)$*-volume of the homotopy is bounded by* $cr \operatorname{vol}_k(f)$.

Proof. The proof follows the proof of Theorem 10.3.3 in [16]. The idea is to project the part inside the ball $B_r(u_0)$ to the sphere $S_r(u_0)$. We will project from a point within the ball $B_{r/2}(u_0)$, but since projecting from the wrong center might increase the volume by a huge factor, we average over all possible projections and prove that the average is under control, and therefore we have plenty of centers from which we can project and still have the volume of the new function under control.

Let ω_{n-1} be the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. Let $|D_x f|$ represent the Jacobian of f at the point x. Let π_u be the projection map from the point $u \in B_{r/2}(u_0)$ to the sphere $S_r(u_0)$.

$$\operatorname{vol}_k(\pi_u \circ f) \leq \int_{f^{-1}(B_r(u_0))} \frac{|D_x f|\,(2r)^k}{\|f(x) - u\|^k \cos\theta}\, dx + \operatorname{vol}_k(f), \qquad (3.2.1)$$

where θ is the angle between the ray connecting u to $f(x)$ and the outward normal vector to the sphere at the point of intersection of the ray and the sphere $S_r(u_0)$. If $u \in B_{r/2}(u_0)$ then it is easy to see that $\theta \leq \pi/6$ and therefore $1/\cos\theta \leq 2/\sqrt{3} \leq 2$.

By integrating (3.2.1) over the ball $B_{r/2}(u_0)$ we get,

$$\begin{aligned}
\int_{B_{r/2}(u_0)} \mathrm{vol}_k(\pi_u \circ f)\, du &\leq \int_{B_{r/2}(u_0)} \int_{f^{-1}(B_r(u_0))} \frac{|D_x f|\,(2r)^k}{\|f(x)-u\|^k \cos\theta}\, dx\, du \\
&\qquad + \mathrm{vol}(B_{r/2}(u_0))\ \mathrm{vol}_k(f) \\
&\leq \int_{f^{-1}(B_r(u_0))} |D_x f| \int_{B_{r/2}(u_0)} \frac{2\,(2r)^k}{\|f(x)-u\|^k}\, du\, dx \\
&\qquad + \mathrm{vol}(B_{r/2}(u_0))\ \mathrm{vol}_k(f) \\
&\leq 2^{k+1} r^k \int_{f^{-1}(B_r(u_0))} |D_x f| \int_{B_{3r/2}(f(x))} \frac{1}{\|u\|^k}\, du\, dx \\
&\qquad + \mathrm{vol}(B_{r/2}(u_0))\ \mathrm{vol}_k(f) \\
&= 2^{k+1} r^k \int_{f^{-1}(B_r(u_0))} |D_x f| \int_{S^{n-1}} \int_0^{3r/2} \frac{1}{t^k} t^{n-1}\, dt\, d\mu\, dx \\
&\qquad + \mathrm{vol}(B_{r/2}(u_0))\ \mathrm{vol}_k(f) \\
&= \frac{2^{k+1} 3^{n-k} r^n \omega_{n-1}}{2^{n-k}(n-k)} \int_{f^{-1}(B_r(u_0))} |D_x f|\, dx \\
&\qquad + \mathrm{vol}(B_{r/2}(u_0))\ \mathrm{vol}_k(f) \\
&\leq \left[\frac{2^{2k+1} 3^{n-k} n}{n-k} + 1\right] \mathrm{vol}(B_{r/2}(u_0))\, \mathrm{vol}_k(f) \\
&\leq \left[2^{2n-1} 3^{n-1} n + 1\right] \mathrm{vol}(B_{r/2}(u_0))\, \mathrm{vol}_k(f).
\end{aligned} \tag{3.2.2}$$

The $(k+1)$-volume of the homotopy is bounded by $2r[2^{2n-1}3^{n-1}n+1]\,\mathrm{vol}_k(f)$. We take $c = 2[2^{2n-1}3^{n-1}n+1]$, which now satisfies all the requirements of the lemma. □

Using the standard notation, we denote by sn_k respectively cs_k the solution to the differential equation $x''(t) + kx(t) = 0$ with the initial conditions $x(0) = 0$ and $x'(0) = 1$ respectively $x(0) = 1$ and $x'(0) = 0$. We also set $\mathrm{ct}_k = \mathrm{cs}_k / \mathrm{sn}_k$.

Now we use Lemma 3.2.1 to prove a similar result for any Riemannian manifold with bounded geometry which we will call the "Deformation Lemma".

Lemma 3.2.2 (Deformation Lemma). *Given a Riemannian manifold M, with sectional curvature* $L \leq K \leq H$ *and injectivity radius* $\mathrm{inj}(M) \geq \epsilon$*, then there exist constants* $\delta = \delta(L, K, \epsilon) > 0$ *and* $c = c(n, L, K, \epsilon)$*, which do not depend on* M *such that for any Lipschitz map* $f\colon S^k \longrightarrow M$ *and any ball* $B_r(p) \subset M$ *with radius* $r \leq \delta$*, we can find a map* $g\colon S^k \longrightarrow M$ *which satisfies the following conditions:*

1. f *is homotopic to* g.
2. $\mathrm{vol}_k(g) \leq c\ \mathrm{vol}_k(f)$.
3. $g(S^k) \cap B_r^\circ(p) = \emptyset$.
4. f *and* g *agree outside* $B_r(p)$.
5. *The* $(k+1)$*-volume of the homotopy is bounded by* $c\ \mathrm{vol}_k(f)$.

Proof. Take $\delta = \min\{\,\epsilon/2, \pi/2\sqrt{H}\,\}$. By comparison with the spaces of constant curvature L and H, we have the following bounds within $B_\delta(p)$.

$$\left\|D\exp_p^{-1}\right\| \leq A = \max\{1, \delta/\,\mathrm{sn}_H(\delta)\}, \tag{3.2.3}$$

$$\left\|D\exp_p\right\| \leq B = \max\{1, \mathrm{sn}_L(\delta)/\delta\}. \tag{3.2.4}$$

See Theorem 2.3 and Corollary 2.4 in chapter 6 of [32] for details but the reader should be aware that the estimate for $\|D\exp_p^{-1}\|$ is stated incorrectly there. Note that A and B only depend on L, K and ϵ but not on p nor M.

Now we use $\exp_p^{-1}$ to lift the map f locally, i.e. in $B_\delta(p)$, to T_pM then use Lemma 3.2.1 to deform this lifted map to a map that lands outside the ball $B_\delta(0) \subset T_pM$ and then project back using $\exp_p$ to the manifold M. The k-volume of the new map and the $(k+1)$-volume of the homotopy will be controlled because of the bounds on $\|D\exp_p\|$ and $\|D\exp_p^{-1}\|$ given by equations (3.2.3) and (3.2.4) and the estimates given in Lemma 3.2.1. This finishes the proof of the lemma. □

Theorem 3.2.3. *If X is a rank 1 symmetric space of noncompact type, then $\operatorname{div}_k(X)$ has a polynomial growth of degree at most k for every $k \geq 1$.*

Proof. Let x_0 be any point in X, $S_r(x_0)$ the distance sphere of radius r centered at x_0, and $d_{S_r(x_0)}$ the Riemannian distance function on the sphere.

We will show that there exists $0 < \rho < 1$ such that there is a filling of any admissible k-sphere on $S_r(x_0)$ outside $B^\circ_{\rho r}(x_0)$ which grows polynomially of degree at most k as $r \to \infty$.

Let $\pi_r^R \colon S_R(x_0) \longrightarrow S_r(x_0)$ be the radial projection from $S_R(x_0)$ to $S_r(x_0)$, and let $\lambda_r^R \colon S_r(x_0) \longrightarrow S_R(x_0)$ be its inverse. Assume that the metric on X is normalized such that the sectional curvature is bounded between -4 and -1. By comparison with the spaces of constant curvatures -1 and -4, it is easy to see that the map π_r^R decreases distance by at least a factor of $\sinh R/\sinh r$, and λ_r^R increases distance at most by a factor of

$\sinh 2R / \sinh 2r$.

Fix $A > 0$ and let $f\colon S^k \longrightarrow S_r(x_0)$ be a Lipschitz map such that $\mathrm{vol}_k(f) \leq Ar^k$. Then $\mathrm{vol}_k(\pi^r_{\rho r} \circ f) \leq Ar^k \left(\sinh \rho r / \sinh r\right)^k$.

Since horospheres are Lie groups with left invariant metrics, the curvature is bounded above and below and there is a lower bound on the injectivity radius. This puts a uniform bound on the curvature as well as the injectivity radius of distance spheres $S_r(x_0)$ as long as r is big enough. Because of that and by using the Deformation Lemma, we could deform $\pi^r_{\rho r} \circ f$ on $S_{\rho r}(x_0)$ to a new function g which misses a ball, on the sphere $S_{\rho r}(x_0)$, of radius δ and has k-volume $\leq c\,\mathrm{vol}_k(\pi^r_{\rho r} \circ f)$ where δ and c are the constants given by the Deformation Lemma. And such that the homotopy between $\pi^r_{\rho r} \circ f$ and g has $(k+1)$-volume $\leq c\,\mathrm{vol}_k(\pi^r_{\rho r} \circ f)$.

Let $p \in S_{\rho r}(x_0)$ be the center of this ball and let q be its antipodal point. We will cone g from q, inside the sphere, and then project the cone from x_0 back to the sphere $S_{\rho r}(x_0)$. Because the curvature is less than -1 the $(k+1)$-volume of the cone is smaller than $\mathrm{vol}_k(g) \leq c\,\mathrm{vol}_k(\pi^r_{\rho r} \circ f)$ (see Remark 3.1.3).

To estimate the volume of the projection of the cone, we need to find a lower bound on the distance from x_0 to the image of the cone. Let $x \in S_{\rho r}(x_0)$ such that $d_{S_{\rho r}(x_0)}(x, p) = \delta$. Let x' be the intersection point of $S_{2\rho r}(q)$ and the ray $\overrightarrow{qx}$. For large values of r, x and x' are close, and therefore we have $d_{S_{2\rho r}(q)}(p, x') \geq \delta/2$. By comparison to the space of constant curvature -4 we see that $\theta = \angle_q(p, x) \geq \delta/(2\sinh 4\rho r)$. Now by comparison with Euclidian space we have $d(x_0, \overrightarrow{qx}) \geq \rho r \sin\theta \geq \rho r \sin(\delta/(2\sinh 4\rho r))$.

So the cone from q misses a ball of radius $\rho r \sin(\delta/(2\sinh 4\rho r))$ around x_0. Projecting the cone which lies inside $S_{\rho r}(x_0)$ from x_0 to $S_{\rho r}(x_0)$ gives us a filling for g of volume $\leq cAr^k (\sinh \rho r/\sinh r)^k \left(\frac{\sinh 2\rho r}{\sinh(2\rho r \sin(\delta/2\sinh 4\rho r))}\right)^{k+1}$.

Notice that $\sinh t$ behaves like $e^t/2$ as $t \to \infty$ and like t as $t \to 0$, while $\sin t$ behaves like t as $t \to 0$. Because of that the above estimate of the filling of g behaves like $cAr^k (e^{\rho r}/e^r)^k \left(\frac{e^{2\rho r}/2}{2\delta\rho r/e^{4\rho r}}\right)^{k+1} = \frac{cA}{(4\delta\rho)^{k+1} r} e^{((7k+6)\rho - k)r}$. Clearly we can choose ρ small enough to make $(7k+6)\rho - k$ negative. Therefore there exists $r_0 > 0$ such that for all $r \geq r_0$ the $(k+1)$-volume of the filling of g will be smaller than 1. Now the filling of the original map f consists of three parts.

1. The radial projection of f to the sphere $S_{\rho r}(x_0)$, which has $(k+1)$-volume $\leq Ar^k$.

2. The homotopy used to deform the map $\pi^r_{\rho r} \circ f$ to the new map g, which has $(k+1)$-volume $\leq c \operatorname{vol}_k(\pi^r_{\rho r} \circ f) \leq c \operatorname{vol}_k(f) \leq cAr^k$.

3. The filling of g, which has $(k+1)$-volume < 1.

This filling of f is Lipschitz and lies outside $B^\circ_{\rho r}(x_0)$, and is therefore ρ-admissible, and it has $(k+1)$-volume smaller than $A(c+1)r^k + 1$. This finishes the proof of the theorem. □

3.3 Higher Rank Symmetric Spaces

In this section we give the proof for Theorem 1.2.3. The idea is similar to the one used in the proof of Theorem 3.2.3, namely project the Lipschitz map you wish to fill, to a smaller sphere to make the k-volume of the projection small and then fill the projection on the smaller sphere.

For the rank one symmetric space case, the projection decreases distances exponentially in all direction, which is no longer true for the higher rank case. Nevertheless, we still have $n-l$ directions in which the projection, to smaller spheres, decreases distance by an exponential factor. Hence, the projection will decrease the volume of a Lipschitz map f by an exponential factor as long as the dimension of the domain of f is bigger than $l-1$, to include at least one of the exponentially decreasing directions. See Lemma 3.3.1 for details.

We start by recalling some basic facts about symmetric spaces which we need. See Section 2.2 for more details. For a general reference of symmetric spaces of nonpositive curvature and for the proofs of the facts used in this section see chapter 2 in [14].

Let $X = G/K$ be a symmetric space of nonpositive curvature of dimension n and rank l. Fix a point $x_0 \in X$ and let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition, where $\mathfrak{k}$ is the Lie algebra of the isotropy group K at x_0. Fix a maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and let F be the flat determined by $\mathfrak{a}$. We identify $\mathfrak{p}$ with $T_{x_0}X$ in the usual way. For each regular vector $v \in \mathfrak{a}$, let $R_v \colon v^\perp \longrightarrow v^\perp$ denote the curvature tensor where $R_v(w) = R(w, v)v$. Let $\lambda_1(v), \dots, \lambda_{n-1}(v)$ be the eigenvalues of R_v with corresponding eigenvectors $E_1, \dots, E_{n-1}$ such that $E_1, \dots, E_{l-1}$ are tangent to the l-flat F. Therefore $\lambda_1(v) = \cdots = \lambda_{l-1}(v) = 0$ while $\lambda_j(v) < 0$ for $j \geq l$. Since for every $v_1, v_2 \in \mathfrak{a}$, R_{v_1} and R_{v_2} commute and therefore can be simultaneously diagonalized, $E_l, \dots, E_{n-1} \in (T_{x_0}F)^\perp$ can be chosen not to depend on the choice of v and the $\lambda_j(v)$ are continuous functions in v.

Let $E_j(t)$ be the parallel translates along the geodesic $\gamma_v(t)$ with $\gamma_v(0) = x_0$ and $\gamma_v'(0) = v$. Then $J_j(t) = f_j(t)E_j(t)$ determine an orthogonal basis for the space of Jacobi fields along $\gamma_v(t)$ which vanishes at x_0 and are orthogonal to $\gamma_v(t)$, where $f_j(t) = t$ if $\lambda_j(v) = 0$ (i.e. $1 \le j \le l-1$) and $f_j(t) = 1/\sqrt{-\lambda_j(v)} \sinh \sqrt{-\lambda_j(v)}t$ if $\lambda_j(v) < 0$ (i.e. $l \le j \le n-1$).

Let $\mathcal{C}$ be the collection of all Weyl chambers of the first type (i.e. Weyl chambers defined on $S_{x_0}X$), where $S_{x_0}X \subset T_{x_0}X$ is the sphere of unit vectors in the tangent space $T_{x_0}X$. We will often identify $S_{x_0}X$ with $S_r(x_0)$ in an obvious way. Let $\mathscr{C}(v)$ be the Weyl chamber containing v for any regular vector v. Let $v_{\mathscr{C}}$ be the algebraic centroid of the Weyl chamber $\mathscr{C}$. Let $\mathcal{B}$ be the set of the algebraic centroids of all Weyl chambers. By the identification of $S_{x_0}X$ and $S_r(x_0)$, we will think of $\mathcal{B}$ as a set of points on $S_r(x_0)$, a set of vectors of length r in $T_{x_0}X$ and a set of points in $X(\infty)$. The meaning will be clear from the context. Recall that the subgroup of isometries K fixing x_0 acts transitively on $\mathcal{B}$.

Assume $v \in \mathcal{B}$ belongs to the l-flat F. Let $\mathscr{C}(v)$ be the unique Weyl chamber containing v. Since v is a regular vector then $\lambda_l(v), \dots, \lambda_{n-1}(v)$ are negative. By the continuity of λ_j, there exist $\epsilon > 0$ and $\delta > 0$ such that for every $w \in N_\epsilon(v)$, $\lambda_j(w) \le -\delta^2$ for every $j \ge l$, where $N_\epsilon(v) = \{w \in \mathscr{C}(v) \mid \angle_{x_0}(v,w) = \angle(v,w) \le \epsilon\}$. We choose ϵ small enough such that $N_\epsilon(v)$ is contained in the interior of $\mathscr{C}(v)$ and away from the boundary of $\mathscr{C}(v)$. By the transitive action of K on $\mathcal{B}$ we have such neighborhood around each element of $\mathcal{B}$. Let $\mathcal{N}_\epsilon$ be the union of these neighborhoods. Recall that any two Weyl chambers at infinity are contained in the boundary at infinity of a flat in X. Note that if $v \in \mathcal{B}$ and v' is the centroid of an opposite Weyl chamber to $\mathscr{C}(v)$ then $\angle(v, v') = \pi$. And therefore, for any $v \in \mathcal{B}$ and any $w \notin \mathcal{N}_\epsilon$, we have $\angle(v, w) < \pi - \epsilon$.

From the above discussion we have immediately the following lemma.

Lemma 3.3.1. *Let* $0 < \rho \leq 1$, $k \geq l$ *and* $\pi^r_{\rho r}\colon S_r(x_0) \longrightarrow S_{\rho r}(x_0)$ *be the projection map. If* $f\colon S^k \longrightarrow S_r(x_0)$ *is a Lipschitz map then* $\mathrm{vol}_k(\pi^r_{\rho r} \circ f \cap \mathcal{N}_\epsilon) \leq \mathrm{vol}_k(f \cap \mathcal{N}_\epsilon)\,(\sinh \rho\delta r/ \sinh \delta r) \leq \mathrm{vol}_k(f)\,(\sinh \rho\delta r/ \sinh \delta r)$.

Before starting the proof of Theorem 1.2.3 we need the following lemma.

Lemma 3.3.2. *Let* X *be a Hadamard manifold, and let* $x_0 \in X$. *For every* $\epsilon > 0$ *there exists* $\eta > 0$ *such that for any two points* p_1, $p_2 \in S_r(x_0)$ *with* $\angle(\overrightarrow{x_0p_1}(\infty), \overrightarrow{x_0p_2}(\infty)) \leq \pi - \epsilon$ *we have* $d(x_0, \overline{p_1p_2}) \geq \eta r$.

Proof. Let $\gamma_i = \overrightarrow{x_0p_i}(t)$ and $z_i = \gamma_i(\infty)$ for $i = 1, 2$. Let $\alpha_i(t) = \angle_{\gamma_i(t)}(x_0, \gamma_j(t))$ for $j \neq i$.

Note that $\pi - (\alpha_1(t) + \alpha_2(t))$ is an increasing function of t which converges to $\angle(z_1, z_2)$ as $t \to \infty$. Therefore $\epsilon \leq \alpha_1(r) + \alpha_2(r)$. Without any loss of generality we may assume that $\epsilon/2 \leq \alpha_1(r)$. Using the first law of cosines we have $r^2 + d(p_1, p_2)^2 - 2rd(p_1, p_2)\cos(\epsilon/2) \leq r^2$. Therefore $d(p_1, p_2) \leq 2r\cos(\epsilon/2)$. Let m be the closest point on $\overline{p_1p_2}$ to x_0, and without loss of generality assume that $d(m, p_1) \leq d(m, p_2)$. By taking $\eta = 1 - \cos(\epsilon/2)$, we have

$$\begin{aligned} d(x_0, \overline{p_1 p_2}) &= d(x_0, m) \\ &\geq d(x_0, p_1) - d(p_1, m) \\ &\geq r - r\cos(\epsilon/2) \\ &= \eta r, \end{aligned} \tag{3.3.1}$$

which finishes the proof of the lemma. □

Proof of Theorem 1.2.3. Let $-\lambda^2$ be a lower bound on the sectional curvature of X. Fix $A > 0$ and let $f\colon S^k \longrightarrow S_r(x_0)$ be a Lipschitz map such that $\mathrm{vol}_k(f) \leq Ar^k$. By Lemma 3.3.1, $\mathrm{vol}_k(\pi^r_{\rho r} \circ f \cap \mathcal{N}_\epsilon) \leq \mathrm{vol}_k(f)\,(\sinh \rho\delta r/\sinh \delta r)$.

Fix $q \in S_{\rho r}(x_0)$ to be the algebraic centroid of a Weyl chamber, and let $p \in S_{\rho r}(x_0)$ be its antipodal point. Since curvature and injectivity radii of large spheres are again controlled, using the Deformation Lemma as in the proof of Theorem 3.2.3, we can assume that $\pi^r_{\rho r} \circ f$ misses a ball $B_\mu(p)$ on $S_{\rho r}(x_0)$ and μ does not depends on r as long as r is large enough.

Now cone the image of $\pi^r_{\rho r} \circ f$ from q inside the sphere $S_{\rho r}(x_0)$. Let C_f be the image of the cone. Let C_1 be the part of the cone coming from $\pi^r_{\rho r} \circ f \cap \mathcal{N}_\epsilon$ and $C_2 = C_f \setminus C_1$. By Lemma 3.3.1, $\mathrm{vol}_{k+1}(C_1) \leq 2\rho A r^{k+1}(\sinh \rho\delta r/\sinh \delta r) \leq 2Ar^{k+1}(\sinh \rho\delta r/\sinh \delta r)$ and $\mathrm{vol}_{k+1}(C_2) \leq 2\rho A r^{k+1} \leq 2Ar^{k+1}$. Taking η to be the constant given by Lemma 3.3.2, we see that $C_2 \cap B^\circ_{\eta\rho r}(x_0) = \emptyset$.

By comparison with the space of constant curvature $-\lambda^2$ and arguing as in the proof of Theorem 3.2.3, it is easy to see that C_1 misses a ball of radius $\rho r \sin(\mu/(2\sinh 2\lambda\rho r))$

around x_0. Now we project from x_0 the part of C_1 in $B_{\eta\rho r}(x_0)$ to the sphere $S_{\eta\rho r}(x_0)$ to obtain a filling g of $\pi^r_{\rho r} \circ f$ lying outside $B^\circ_{\eta\rho r}(x_0)$. Now

$$\begin{aligned} \mathrm{vol}_{k+1}(g) \leq\ & \mathrm{vol}_{k+1}(C_2) + \mathrm{vol}_{k+1}(C_1)\left(\frac{\sinh \lambda\eta\rho r}{\sinh(\lambda\rho r \sin(\mu/(2\sinh 2\lambda\rho r)))}\right)^{k+1} \\ & \leq 2Ar^{k+1}\left[1 + \left(\frac{\sinh \rho\delta r}{\sinh \delta r}\right)\left(\frac{\sinh \lambda\eta\rho r}{\sinh(\lambda\rho r \sin(\mu/(2\sinh 2\lambda\rho r)))}\right)^{k+1}\right]. \end{aligned} \tag{3.3.2}$$

This estimate behaves like $2Ar^{k+1}[1+e^{(\rho(\delta+\lambda(k+1)(2+\eta))-\delta)r}/(2\lambda\rho\mu)^{k+1}r^{k+1}]$ as $r \to \infty$. Clearly we can choose ρ small enough to make the exponent $\rho(\delta + \lambda(k+1)(2+\eta)) - \delta$ negative and therefore the $(k+1)$-volume of the $\eta\rho$-admissible filling of $\pi^r_{\rho r} \circ f$ will be bounded by $4Ar^{k+1}$ and the (k+1)-volume of the $\eta\rho$-admissible filling of f is bounded by $5Ar^{k+1}$. This finishes the proof of the theorem where $\eta\rho$ is taken in place of ρ. □

3.4 First Filling Invariant for Symmetric Spaces

In this section we prove Theorem 1.2.5. The proof consists of two steps. The first step is to deform the closed curve we wish to fill to a new curve which is continuous when viewed as a curve at infinity with respect to the Tits metric. The length of the new curve and the area of the deformation will have to be under control. The proof is valid for any symmetric space of higher rank. We will establish this in Lemma 3.4.1.

In the second step we will use the assumption that the rank is bigger than or equal 3, to fill the new curve with a disk with controlled area. This step will be done in Proposition 3.4.2.

We denote by $\lambda^\infty_r : S_r(x_0) \longrightarrow (X(\infty), Td)$ the map obtained by sending a point $x \in$

$S_r(x_0)$ to the point $\overrightarrow{x_0x}(\infty) \in X(\infty)$. Notice that this map is almost never continuous.

Lemma 3.4.1. *Let* X *be a symmetric space of noncompact type and rank* $k \geq 2$. *There exist two constants* $c > 0$ *and* $0 < \rho \leq 1$ *which depend only on* X *such that, for every Lipschitz curve* $f\colon S^1 \longrightarrow S_r(x_0)$, *there exists a new curve* $g\colon S^1 \longrightarrow S_r(x_0)$ *which satisfies the following conditions:*

1. $\lambda_r^\infty \circ g$ *is continuous with respect to the Tits metric.*
2. *There exists a homotopy between* f *and* g *which lies outside* $B^\circ_{\rho r}(x_0)$.
3. *The area of the homotopy is bounded above by* $cr \operatorname{vol}_1(f)$.
4. $\operatorname{vol}_1(g) \leq c \operatorname{vol}_1(f)$.

Proof. Let $\delta = \min(1, \eta/2)$, where η is the constant given by Lemma 3.3.2 for $\epsilon = \pi/2$. Divide f into pieces each of length δr, the last piece possibly could be shorter than δr. If that is the case we will call it "short" and all the other pieces "long".

Let $c_j\colon [0,1] \longrightarrow S_r(x_0)$ be one of these pieces. Let $\mathscr{C}_i$ be a Weyl chamber containing $c_j(i)$ for $i = 0, 1$. Let $\mathscr{A}$ be an apartment containing $\mathscr{C}_0$ and $\mathscr{C}_1$. Let $\gamma_j\colon [0,1] \longrightarrow S_r(x_0)$ be a geodesic (with respect to the Tits metric) in the apartment $\mathscr{A}$ connecting $c_j(0)$ to $c_j(1)$. The goal is to replace the piece c_j with the geodesic γ_j. The homotopy between them which leaves the end points $c_j(0)$ and $c_j(1)$ fixed will be through geodesics in X connecting $c_j(t)$ to $\gamma_j(t)$ for every $0 \leq t \leq 1$. The length of these geodesics is no longer than $2r$, since they lie inside the ball $B_r(x_0)$.

The new curve g will be formed by replacing each piece c_j with the curve γ_j. Notice that the length of each γ_j is no bigger than πr. If $\mathrm{vol}_1(f) < \delta r$, i.e. we have no "long" pieces then g is just a point and the statement trivially follows. If $\mathrm{vol}_1(f) \geq \delta r$, i.e. we have at least one long piece then it is not hard to see that $\mathrm{vol}_1(g) \leq 2\pi \,\mathrm{vol}_1(f)/\delta$.

Since X is nonpositively curved, the area of the homotopy between c_j and γ_j is no bigger than $2\pi r^2$. And the area of the homotopy between f and g is no bigger than $cr\,\mathrm{vol}_1(f)$, where $c = 4\pi/\delta$.

To finish the proof we need to show that the homotopy lies outside $B^\circ_{\rho r}(x_0)$. So we need to show that the distance between x_0 and the geodesic, in X, connecting $c_j(t)$ and $\gamma_j(t)$ is at least ρr for every $0 \leq t \leq 1$. Assume that $t \leq 1/2$, the other case is similar. Notice that $\angle(c_j(0), \gamma_j(t)) \leq \pi/2$. Applying Lemma 3.3.2, we get that $d_X(x_0, \overline{c_j(0)\gamma_j(t)}) \geq \eta r$, where η is the constant in Lemma 3.3.2.

Recall that $d_X(c_j(0), c_j(t)) \leq \delta r$. For every point s on the geodesic $\overline{c_j(t)\gamma_j(t)}$ there exists a point s' on the geodesic $\overline{\gamma_j(t)c_j(0)}$ such that $d_X(s, s') \leq \delta r$. Therefore $d_X(x_0, s) \geq (\eta - \delta)r \geq \eta r/2$. By taking $\rho = \eta/2$, the image of the homotopy lies outside $B^\circ_{\rho r}(x_0)$. This finishes the proof of the lemma. □

Now we proceed to the second step of the proof. We prove a more general result.

Proposition 3.4.2. *Let Δ be a spherical building with a re-scaled metric such that each apartment is isometric to $S^{n-1}(r)$, the round sphere of radius r. There exists a constant $c > 0$ which depends only on Δ but not r such that for any Lipschitz function $g\colon S^k \longrightarrow \Delta$, where $k < n-1$, we can extend g to a new function $\hat{g}\colon B^{k+1} \longrightarrow \Delta$ such that $\mathrm{vol}_{k+1}(\hat{g}) \leq cr\,\mathrm{vol}_k(g)$.*

Proof. Fix a point p to be the center of a Weyl chamber $\mathscr{C}$. Let $\mathcal{A}$ be the collection of all opposite Weyl chambers to $\mathscr{C}$, and let $\mathcal{B}$ be the collection of all antipodal points to p. Let $\epsilon > 0$ be the largest positive number such for any $q \in \mathcal{B}$ the ball $B_{\epsilon r}(q)$ is contained in the interior of the Weyl chamber containing q. Notice that ϵ only depends on Δ.

The proof of Lemma 3.2.1 can be easily modified to deform the function g to miss the ball $B_{\epsilon r}(q)$ for every $q \in \mathcal{B}$. Now we cone the new deformed function from the point p to obtain the desired extension. □

Proof of Theorem 1.2.5. The proof is immediate. We deform the function f we wish to fill to a new function g using Lemma 3.4.1. We fill g with a disc by invoking Proposition 3.4.2, for $k = 1$ where Δ is the Tits building on the sphere $S_r(x_0)$. □

Remark 3.4.3. We expect a similar result to Theorem 1.2.5 to hold for a larger class than Symmetric spaces. One candidate is the class of Hadamard manifolds whose boundary at infinity is simply connected with respect to the Tits metric.

3.5 Riemannian Manifolds of Pinched Negative Curvature

In this section we give the proof of Theorem 1.2.6. All the steps of the proof of Theorem 3.2.3 carry over automatically to our new setting, except the uniform bounds on the curvature and the injectivity radius of distance spheres $S_r(x_0)$. By uniform we mean independent of r for large values of r. These bounds will be established in Lemma 3.5.3 and Lemma 3.5.4 below. The proofs are straightforward and we include them for completeness. First we recall the following proposition concerning estimates on Jacobi fields.

Proposition 3.5.1 ([3] Proposition IV.2.5). *Let* $\gamma\colon \mathbb{R} \longrightarrow M$ *be a unit speed geodesic and suppose that the sectional curvature of* M *along* γ *is bounded from below by a constant* λ*. If* J *is a Jacobi field along* γ *with* $J(0) = 0$, $J'(0) \perp \gamma'(0)$, *and* $\|J'(0)\| = 1$, *then*

$$\|J(t)\| \leq sn_\lambda(t) \text{ and } \|J'(t)\| \leq ct_\lambda(t)\,\|J(t)\|\,,$$

if there is no pair of conjugate points along $\gamma|[0,t]$.

Notice that the estimate on $\|J'(t)\|$ is still valid without requiring that $\|J'(0)\| = 1$. From the proposition we have the following immediate corollary.

Corollary 3.5.2. *Let* X *be a Hadamard manifold, with sectional curvature* $-b^2 \leq K \leq -a^2 < 0$, *if* γ *is a unit speed geodesic,* $J(t)$ *is a Jacobi field along* γ *with* $J(0) = 0$, $J(t) \perp \gamma'(t)$, *and* $\|J(r)\| = 1$ *then* $\|J'(r)\| \leq ct_{-b^2}(r)$, *and therefore the upper bound goes uniformly, independent of* γ, *to* b *as* $r \to \infty$.

We use the estimate on the derivative of Jacobi fields to put an estimate on the second fundamental form of distance spheres, and hence bounds on the sectional curvature.

Lemma 3.5.3. *Let* X *be a Hadamard manifold, with sectional curvature* $-b^2 \leq K \leq -a^2 < 0$, *and* $x_0 \in X$*. There exists a number* $H = H(a,b) > 0$ *such that the absolute value of the sectional curvature of* $S_r(x_0)$ *is bounded by* H *for every* $r \geq 1$.

Proof. The lemma is immediate using the Gauss formula for the sectional curvature of hypersurfaces, namely

$$K(Y,Z) - \overline{K}(Y,Z) = \langle B(Y,Y), B(Z,Z)\rangle - \|B(Y,Z)\|^2\,. \tag{3.5.1}$$

Where Y and Z are orthogonal unit vectors tangent to the hypersurface.

Notice that for $S_r(x_0)$ we have $\|B(Y,Z)\| \leq \|J'_Y(r)\|$, with $J(0) = 0$ and $J_Y(r) = Y$, for every two unit vectors Y and Z tangent to the sphere. But $\|J'_Y(r)\|$ is bounded by b for large values of r by Corollary 3.5.2. □

In the next lemma we establish a lower bound on the injectivity radius of distance spheres.

Lemma 3.5.4. *Let* X *be a Hadamard manifold, with sectional curvature* $-b^2 \leq K \leq -a^2 < 0$*, and* $x_0 \in X$*. There exists a number* $\delta > 0$ *such that* $\operatorname{inj}(S_r(x_0)) \geq \delta$ *for every* $r \geq 1$*.*

Proof. Using Lemma 3.5.3 we have a uniform upper bound H on the sectional curvature of distance spheres $S_r(x_0)$ for $r \geq 1$ and therefore a lower bound $\pi/\sqrt{H}$ on the conjugate radius.

Our plan is to choose δ small and prove that if $\operatorname{inj}(S_r(x_0)) \geq \delta$ then $\operatorname{inj}(S_{r+s}(x_0)) \geq \delta$ for all $0 \leq s \leq 1$. Using the lower bound $-b^2$ on the sectional curvature of X and by comparison with the space of constant curvature $-b^2$, there exists a constant $0 < B = B(b) < 1$ independent of r and $0 \leq s \leq 1$ such that $\pi_r^{r+s}\colon S_{r+s}(x_0) \longrightarrow S_r(x_0)$ satisfies the following inequality

$$B\, d(x,y) \leq d(\pi_r^{r+s}(x), \pi_r^{r+s}(y)) \leq d(x,y), \quad \forall\, x,y \in S_{r+s}(x_0). \tag{3.5.2}$$

Assume $\delta < \min\{\operatorname{inj}(S_1(x_0)), B\pi/2\sqrt{H}\}$. Let us assume that $\operatorname{inj}(S_r(x_0)) \geq \delta$ but $\operatorname{inj}(S_{r+s}(x_0)) < \delta$. Since $\delta \leq \pi/2\sqrt{H}$, there exists two points p, $q \in S_{r+s}(x_0)$ and two minimizing geodesic (with respect to the induced Riemannian metric on $S_{r+s}(x_0)$)

γ_1, γ_2 connecting p to q. Moreover $d(p,q) < \delta$. By Klingenberg's Lemma, see [13], any homotopy from γ_1 to γ_2 with the end points fixed must contain a curve which goes outside $B^\circ_{\pi/\sqrt{H}}(p)$. Therefore these two curves are not homotopic inside the ball $B^\circ_{\delta/B}(p)$. Using (3.5.2) we have,

$$B^\circ_\delta(\pi_r^{r+s}(p)) \subseteq \pi_r^{r+s}(B^\circ_{\delta/B}(p)). \tag{3.5.3}$$

Notice that $\pi_r^{r+s} \circ \gamma_i$ are contained in $B^\circ_\delta(\pi_r^{r+s}(p))$, but not homotopic to each other within that ball, which is a topological ball since $\mathrm{inj}(S_r(x_0)) \geq \delta$ and this is a contradiction. This finishes the proof of the lemma. □

3.6 The Graph Manifold Example

In this section we give the example mentioned in the introduction showing that Theorem 1.2.6 is false if we relaxed the condition on the manifold from being negatively curved to being merely rank 1.

Our example will be a graph manifold. Graph manifolds of nonpositive curvature form an interesting class of 3-dimensional manifolds for various reasons. They are the easiest nontrivial examples of rank 1 manifolds whose fundamental group is not hyperbolic. They are rank 1, yet still have a lot of 0-curvature. In fact every tangent vector $v \in T_pX$ is contained in a 2-plane $\sigma \subset T_pX$ with curvature $K(\sigma) = 0$.

They were used by Gromov in [20] to give examples of open manifolds with curvature $-a^2 \leq K \leq 0$ and finite volume which have infinite topological type, contrary to the case of pinched negative curvature. The compact ones have been extensively studied by Schroeder in [33].

Since we are mainly interested in giving a counter example, and for the sake of clarity, we will consider the simplest possible graph manifold. Even though the same idea works for a large class of graph manifolds.

We start by giving a description of the manifold. Let W_1 and W_2 be two tori with one disk removed from each one of them. Let $B_i = W_i \times S^1$. Each B_i is called a block. The boundary of each block is diffeomorphic to $S^1 \times S^1$. The manifold M is obtained by gluing the two blocks B_1 and B_2 along the boundary after interchanging the S^1-factors.

Since the invariants div_k do not depend on the metric, we choose a metric which is convenient to work with. Take the flat torus, corresponding to the lattice $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. Let β_1 and β_2 be the unique closed geodesics of length 1. We picture the torus as the unit square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$ with the boundary identified. Remove a small disk in the middle and pull the boundary up, such that the metric is rotationally symmetric around the z-axis. It is easy to see we can obtain a metric on W_i with curvature $-b^2 \leq K \leq 0$, and make it product near the boundary which is a closed geodesic. Rescale the metric on W_i to make the curvature $-1 \leq K \leq 0$. The closed curves β_1 and β_2 are still closed geodesics, and generate the fundamental group of the punctured torus. Take the metric on each block to be the product of this metric with a circle of length equal to the length of the boundary of W_i. Gluing the two metrics on the two blocks together gives a smooth metric on M with curvature $-1 \leq K \leq 0$.

Let X be the universal covering space of M, and $\pi\colon X \longrightarrow M$ be the covering map. We show that $\mathrm{div}_1(X)$ is exponential.

Let Y be a connected component of $\pi^{-1}(B_1)$. Since B_1 is a convex subset of M, it is easy to see that $Y = Z \times \mathbb{R}$ is the universal covering space of B_1, where Z is the universal covering space of W_1, and the restriction of π to $Z \times \{0\}$ is the covering map onto W_1. We will identify $Z \times \{0\}$ with Z. Clearly $\pi_1(B_1) = \pi_1(W_1) \times \mathbb{Z}$ and $\pi_1(W_1) = \mathbb{Z} * \mathbb{Z}$ is a free group generated by the closed geodesics β_1 and β_2. Moreover the universal covering space Z is a thickening of the Cayley graph of $Z * Z$, and it retracts to it.

Let w_0 be the unique intersection point of β_1 and β_2. Take w_0 to be the base point of the fundamental group of W_1. Let a_1, $a_2 \in \pi_1(W_1, w_0)$ represent the elements corresponding to β_1 and β_2.

Let $\psi \colon Z \longrightarrow W_1$ be the covering map which is the restriction to $Z = Z \times \{0\}$ of the covering map $\pi \colon X \longrightarrow M$. Fix $p_0 \in Z$ such that $\psi(p_0) = w_0$. We denote the deck transformation corresponding to any element $s \in \pi_1(W_1, w_0)$ by ϕ_s.

Let γ_i be the lift of β_i to a geodesic in Z starting at p_0. Notice that ϕ_{a_i} is a translation along the geodesic γ_i. Take $F = \gamma_1 \times \mathbb{R}$, which is a totally geodesic submanifold isometric to $\mathbb{R}^2$. And let $x_0 = (p_0, 0) \in Z \times \mathbb{R}$, where $p_0 = \gamma_1(0) = \gamma_2(0)$. Let $S_F(r) = S_r(x_0) \cap F$, be the 1-sphere in F with center x_0 and radius r, $A_Z(r) = S_r(x_0) \cap Z$ and $B_Z(r) = B_r(x_0) \cap Z$.

Let $f_r \colon S^1 \longrightarrow S_F(r)$ be the canonical diffeomorphism with constant velocity. Notice that f_r is a Lipschitz map and $\mathrm{vol}_1(f_r) = 2\pi r$ and therefore an admissible map.

Our goal is to show that for every fixed $0 < \rho \leq 1$ the infimum over all ρ-admissible fillings of f_r grows exponentially as $r \to \infty$. We show this first for $\rho = 1$ and then the general case will follow easily from that.

Notice that a smallest filling of f_r outside $B_r^\circ(x_0)$ would lie on the sphere $S_r(x_0)$, so we will only look at those fillings which lie on the sphere. $S_r(x_0)$ is a 2-dimensional sphere and f_r is a simple closed curve on it, therefore it divides the sphere into two halves H_1S and H_2S. Any filling of f_r has to cover one of these two halves. So it is enough to show that $\mathrm{vol}_2(H_iS)$ grows exponentially with r for $i = 1, 2$.

We will estimate the area of each half from below by estimating the area of the part which lies inside Y. The geodesic γ_1 divides Z into two halves H_1 and H_2. We concentrate on one of them, say H_1. Let $b(r) = \mathrm{vol}_2(B_Z(r) \cap H_1)$, which is an increasing function.

The portion of the half sphere H_1S inside Y has area bigger than the area of $B_Z(r) \cap H_1$. That is easy to see since vertical projection of that part will cover $B_Z(r) \cap H_1$, and the projection from $Y = Z \times \mathbb{R}$ onto Z decreases distance since the metric is a product. So it is enough to show that $b(r) = \mathrm{vol}_2(B_Z(r) \cap H_1)$ grows exponentially.

It is easy to see that $\mathrm{vol}_2(B_Z(r))$ grows exponentially with r. One way to see it is to consider the covering map $\psi \colon Z \longrightarrow W_1$, and look at all lifts, under deck transformations, of a small ball around $w_0 \in W_1$. The number of these disjoint lifted balls which are contained in $B_Z(r)$ increases exponentially because the fundamental group is free and therefore has exponential growth.

We need to show that the number of these lifted balls in each half H_1 or H_2 increases exponentially. We show that for H_1.

Without loss of generality we assume that $\phi_{a_2}(p_0) \in H_1$. We claim that under the deck transformations corresponding to the subset $S = \{sa_2 \mid sa_2 \text{ is a reduced word}\} \subset \pi_1(W_1, w_0)$, the image of p_0 is in H_1. Recall that Z is a thickening of the the Cayley graph of $Z * Z$ and the action of the deck transformation corresponding to the action of the free group on its Cayley graph. Now the statement follows since the Cayley graph is a tree. But the number of elements in S of length less than or equal m increases exponentially as $m \to \infty$. This finishes the proof of the claim. Therefore $\mathrm{vol}_2(H_1 S) \geq b(r)$ grows exponentially and that finishes the proof for $\rho = 1$.

Now we turn to the general case where $0 < \rho \leq 1$. We showed that there exists $0 < \epsilon$ such that for large values of r, $e^{\epsilon r} \leq \inf(\mathrm{vol}_2(\hat{f}_r))$, where the infimum is taken over all 1-admissible fillings, i.e. the fillings which lie in $C_r(x_0) = X \setminus B_r^\circ(x_0)$. Fix $0 < \rho \leq 1$ and let $\pi^r_{\rho r} \colon S_r(x_0) \longrightarrow S_{\rho r}(x_0)$, be the projection map. Let g be any ρ-admissible filling of f_r. Clearly $\pi^r_{\rho r} \circ g$ is a 1-admissible filling of $\pi^r_{\rho r} \circ f_r = f_{\rho r}$. Therefore $\mathrm{vol}_2(g) \geq \mathrm{vol}_2(\pi^r_{\rho r} \circ g) \geq e^{\epsilon \rho r}$, as long as r is big enough. And this finishes the proof.

Remark 3.6.1. Gersten [18] studied the growth rate of the div_0 invariant for a large class of 3-manifolds including graph manifolds. He showed in Theorem 5 [18] that a closed Haken 3-manifold is a graph manifold if and only if the div_0 invariant has a quadratic growth. The author would like to thank Bruce Kleiner for bringing this paper to his attention.

3.7 Leuzinger's Theorem

In this section we give independent and shorter proof of the following theorem.

Theorem 3.7.1 (Leuzinger [30]). *If* X *is a rank* k *symmetric space of nonpositive curvature, then* $\mathrm{div}_{k-1}(X)$ *has exponential growth.*

Proof. We use the same notation as in section 3.3. Fix a maximal k-flat F passing through x_0. Fix a Weyl chamber $\mathscr{C}$ in F. Let $v \in \mathscr{C}$ be the algebraic centroid of $\mathscr{C}$. We identify $X(\infty)$ and $S_r(x_0)$. Since the set of regular points is an open subset of $S_r(x_0)$ with the cone topology, we can find $0 < \eta$ such that the set $S = \{w \in S_r(x_0) \mid \angle_{x_0}(v, w) < \eta\}$ does not contain any singular point. Moreover by choosing η small enough we may assume that $S \subset \mathcal{N}_\epsilon$. To see this it is enough to show a small neighborhood (with respect to the cone topology) of v is contained in $\mathcal{N}_\epsilon$. Take any sequence $\{v_n\}$ which converges to v in the cone topology. We may assume that v_n is a regular point for each n since the set of regular points is open. Any open Weyl chamber is a fundamental domain of the action of K on the set of regular points in $X(\infty)$, see Proposition 2.17.24 in [14]. Note that the algebraic centroid of a Weyl chamber is mapped to the algebraic centroid of another Weyl chamber under the action of K. Therefore for large values of n, v_n would be close to the algebraic centroid of the unique Weyl chamber containing v_n, and therefore contained in $\mathcal{N}_\epsilon$.

Let $f_r \colon S^{k-1} \longrightarrow S_r(x_0) \cap F$ be a diffeomorphism. We will show that any filling of f_r grows exponentially with $r \to \infty$. Assume not, then for each $n \in \mathbb{N}$ there is a filling $\hat{f}_n$ for f_n such that the $\mathrm{vol}_k(\hat{f}_n)$ grows sub-exponentially. Let $\pi_1^n \colon S_n(x_0) \longrightarrow S_1(x_0)$ be the projection map. By Lemma 3.3.1 $\mathrm{vol}_k(\pi_1^n \circ \hat{f}_n \cap \mathcal{N}_\epsilon) \leq \mathrm{vol}_k(\hat{f}_n)(\sinh \delta / \sinh \delta n)$, which goes to zero as $n \to \infty$. Let ϕ be the projection to the flat F. And g_n be the projection of $\pi_1^n \circ \hat{f}_n$ to F. The image of g_n has to cover the closed unit ball in F.

For every $w \notin \mathcal{N}_\epsilon$, $\angle_{x_0}(v, w) \geq \eta$ and therefore $d(\phi(w), v) \geq 2a = 1 - \cos\eta$. Let $A = B_a(v) \cap B_1(x_0)$ be the part of the k-ball in the flat F centered at v with radius a which lies inside the unit closed k-ball in F. Now it is easy to see that the only part of g_n which will lie inside A would be coming from the portion inside $\mathcal{N}_\epsilon$. The k-volume of that part goes to zero as $n \to \infty$, nevertheless it has to cover A which is a contradiction. Therefore the filling of f_r grows exponentially. $\square$

Chapter 4

Four Dimensional Analytic Manifolds

4.1 Background

Recall that we denote the ideal boundary $X(\infty)$ of a Hadamard manifold X equipped with the cone topology by $\partial_\infty X$, and denote the ideal boundary with the Tits metric by $\partial_T X$.

Through out this chapter, with the exception of Proposition 4.2.3, X will denote the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold and we denote by Γ its fundamental group.

For any unit vector v, we write $v(\pm\infty) \in X(\infty)$ to denote the end points of the unique geodesic c_v with $c_v'(0) = v$. The parallel set of c_v is denoted by P_v. This set consists of the union of all geodesics which are parallel to c_v. By analyticity, P_v is a complete totally geodesic submanifold without boundary. Moreover P_v splits isometrically as $Q \times \mathbb{R}$, where the $\mathbb{R}$-factor corresponds to the geodesic c_v. See Lemma 2.1.7 for more details.

Definition 4.1.1. A connected submanifold F is said to be a *higher rank submanifold* of X if it is a totally geodesic submanifold with the property that every geodesic c in F has a parallel c' in F such that $c \neq c'$. We say F is a *maximal higher rank submanifold* if it is not properly contained in any other higher rank submanifold.

Given a unit vector v, P_v is a higher rank submanifold unless $\mathrm{rank}(v) = 1$. Schroeder [36] gave a complete description of the higher rank submanifolds in X. If X is irreducible, then there are exactly three types, $\mathbb{R}^2$, $\mathbb{R}^3$, and $Q \times \mathbb{R}$, where Q is a 2-dimensional visibility manifold[1]. Moreover all the maximal higher rank submanifolds are closed under the action of the fundamental group, i.e. for any maximal higher rank submanifold F, $\Gamma_F = \{\phi \in \Gamma \mid \phi F = F\}$ operates with compact quotient on F.

We denote by $\mathcal{V}$ the set of all maximal higher rank submanifolds and by $\mathcal{W}$ the set of all maximal higher rank submanifolds of the form $Q \times \mathbb{R}$, where Q is a 2-dimensional visibility manifold.

Schroeder [36] proved that modulo Γ there are only finite number of maximal higher rank submanifolds. He also gave a description of the possible intersections of the maximal higher rank submanifolds. We now recall them. Given two different maximal higher rank submanifolds F_1 and F_2.

1. If $F_1 \approx \mathbb{R}^3$, then $F_1 \cap F_2 = \emptyset$.

2. If $F_1 \approx \mathbb{R}^2$ and $F_1 \cap F_2 \neq \emptyset$, then $F_2 \approx \mathbb{R}^2$ and $F_1 \cap F_2$ is a single point.

[1] A Hadamard manifold X is a visibility manifold if for every two points $\xi, \eta \in X(\infty)$ there exists a geodesic σ in X such that $\sigma(\infty) = \xi$ and $\sigma(-\infty) = \eta$.

3. If $F_1 \in \mathcal{W}$ and $F_1 \cap F_2 \neq \emptyset$, then $F_2 \in \mathcal{W}$ and $F_1 \cap F_2$ is a 2-flat.

For any maximal higher rank submanifold $F \in \mathcal{V}$, $\partial_T F$ is a connected subset of $\partial_T X$. If $F \notin \mathcal{W}$, then $\partial_T F$ is isometric either to S^1 or S^2. If $F \in \mathcal{W}$, then $\partial_T F$ is a graph with two vertices and uncountable number of edges, where the length of each edge is π. The union of any two edges is a closed geodesic in $\partial_T F$ and it is the ideal boundary of a 2-flat in F. The two vertices are called singular points and they are precisely the end points of any singular geodesic, i.e. a geodesic of the form $\{q\} \times \mathbb{R} \subset Q \times \mathbb{R} = F$.

Schroeder [36] also described the possible intersections of the ideal boundaries of the maximal higher rank submanifolds. The only possible intersection is if $F_1, F_2 \in \mathcal{W}$ and $F_1 \cap F_2 = K$, where K is a 2-flat. In such a case $\partial_T F_1 \cap \partial_T F_2 = \partial_T K = S^1$.

Definition 4.1.2. A connected component of $\partial_T X$ is called *standard* if it contains a boundary point of a flat and *nonstandard* if it is not a single point and not standard.

The main result of Hummel and Schroeder in [26] was to show that $\partial_T X$ contains a nonstandard components precisely when there exist two maximal higher rank submanifolds $F_1, F_2 \in \mathcal{W}$ which intersect in a 2-flat, and that all the nonstandard components are intervals of length less than π.

4.2 Fat Triangles

Through out this section, with the exception of Proposition 4.2.3, X will denote the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold which has no nonstandard component in $\partial_T X$.

If X contains any higher rank submanifold, then it is not Gromov δ-hyperbolic space for any $\delta \geq 0$. Easy examples of triangles which are not δ-thin are triangles which lie in a flat in a higher rank submanifold. The goal of this section is to show that all "fat" triangles have to lie near a maximal higher rank submanifold, see Theorem 4.2.2. We begin with the following definition.

Definition 4.2.1. Let $\delta > 0$ and let F be a maximal higher rank submanifold in X, a triangle $\Delta(p, q, r)$ is called δ-thin relative to F, if every side is contained in a δ-neighborhood of the union of the other two sides and F.

This definition resembles the definition of thin triangles relative to flats used by Hruska in [25].

Theorem 4.2.2. *There exists some constant $\delta > 0$ such that any triangle in X is δ-thin relative to some maximal higher rank submanifold.*

Most of the proofs in this section depends on the following observation.

Proposition 4.2.3. *Let X^n be an n-dimensional Hadamard manifold, not necessarily analytic. Let $p \in X$ and let $\overline{x_n y_n}$ be a sequence of geodesic segments such that x_n, y_n converge respectively to $\xi_x, \xi_y \in X(\infty)$. If $d\left(p, \overline{x_n y_n}\right)$ goes to infinity then $Td\left(\xi_x, \xi_y\right) \leq \pi$.*

Proof. The proof is easy and resembles the proof of Lemma 2.1 in [2]. Assume that the proposition is not true. If $Td\left(\xi_x, \xi_y\right) > \pi$ then there exists a geodesic σ in X such that $\sigma(\infty) = \xi_x$ and $\sigma(-\infty) = \xi_y$. Notice that for any two points $p, q \in X$ the metrics $\angle_p$ and $\angle_q$ define the same topology on $\overline{X} = X \cup \partial_\infty X$. So without loss of generality and to simplify the notation, we assume that $p = \sigma(0)$.

Let σ_n be the complete geodesic extending $\overline{x_n y_n}$ parameterized such that $\sigma_n(0)$ is the closest point to $\sigma(0)$. It is easy to see that for large values of n, x_n and y_n have to be on opposite sides of $\sigma_n(0)$. Therefore $d\big(\sigma_n(0), p\big)$ goes to infinity. Fix $R > 0$ and let q_n be the point on $\overline{p\sigma_n(0)}$ such that $d(p, q_n) = R$. By passing to a subsequence we assume that q_n converges to q and that the rays $\overrightarrow{q_n x_n}$ and $\overrightarrow{q_n y_n}$ converge respectively to two rays starting at q and asymptotic to $\overrightarrow{p\xi_x}$ and $\overrightarrow{p\xi_y}$. In fact these two rays form a complete geodesic, see [2] for details. Therefore σ is contained in a flat strip of width R. Since R is arbitrary number, the geodesic σ is contained in flat strips of arbitrary width. By a standard compactness argument, it is easy to show that a subsequence of the flat strips will converge to a flat half plane, which implies that $Td\big(\xi_x, \xi_y\big) = \pi$, and this is a contradiction. □

For any convex subset K of a Hadamard manifold X, we denote by π_K the projection map from X to K.

Proposition 4.2.4. *There exists a constant $\delta > 0$ which depends only on the analytic 4-manifold X, such that for any maximal higher rank submanifold F, any $q \in F$, and any $p \notin F$, $d\big(\pi_F(p), \overline{pq}\,\big) \leq \delta$. In particular the triangle $\Delta\big(p, q, \pi_F(p)\big)$ is δ-thin.*

Proof. Modulo the action of the fundamental group Γ, there are only a finite number of maximal higher rank submanifolds. If the proposition is false, then there exists a maximal higher rank submanifold F and a sequence of triangles $\Delta\big(p_n, q_n, \pi_F(p_n)\big)$ such that $d\big(\pi_F(p_n), \overline{p_n q_n}\,\big) \geq n$. Since F is closed, we could assume that $\pi_F(p_n)$ is contained in a compact subset of F. By passing to a subsequence we assume that p_n, q_n, and $\pi_F(p_n)$ converge respectively to $\xi_p \in \partial_\infty X$, $\xi_q \in \partial_\infty F$, and $c \in F$. By Proposition 4.2.3, $\xi_p \in \partial_\infty F$, which is a contradiction. This finishes the proof. □

Lemma 4.2.5. *There exists a constant $D_1 > 0$, which depends only on the space X such that if F is a maximal higher rank submanifold and $\overline{pq}$ is a geodesic segment, then either $d\big(\pi_F(p), \pi_F(q)\big) \leq D_1$ or $d\big(\overline{pq}, F\big) \leq D_1$.*

Proof. Assume that the statement is false, then there is a maximal higher rank submanifold F such that the following is true: for every $n \in \mathbb{N}$ there exists a geodesic segment $\overline{p_n q_n}$ such that $d\big(\pi_F(p_n), \pi_F(q_n)\big) \geq n$ and $d\big(\overline{p_n q_n}, F\big) \geq n$. Since F is closed, we could assume that $\pi_F(p_n)$ is contained in a compact subset of F. By passing to a subsequence we assume that $\pi_F(p_n)$, $\pi_F(q_n)$, p_n, and q_n converge respectively to $c \in F$, $\eta \in \partial_\infty F$, and ξ_p, $\xi_q \in \partial_\infty X$. It is not hard to see that since F is a totally geodesic submanifold, $\xi_p \notin \partial F$. Since $d\big(c, \overline{\pi_F(q_n) q_n}\big) \geq n-1$ for large values of n and $d\big(c, \overline{p_n q_n}\big) \geq n$, using Proposition 4.2.3 we see that ξ_p is path connected to $\eta \in \partial_\infty F$, and therefore $\xi_p \in \partial F$ which is a contradiction. This finishes the proof of the lemma. □

Corollary 4.2.6. *There exists a constant $D_2 > 0$ such that if F is a maximal higher rank submanifold and $\overline{pq}$ is a geodesic segment such $d\big(\pi_F(p), \pi_F(q)\big) > D_2$ then there are two points $p', q' \in \overline{pq}$ such that $d\big(p', \pi_F(p)\big) \leq D_2$ and $d\big(q', \pi_F(q)\big) \leq D_2$. In particular $\overline{pq}$ runs within D_2 distance from the path consisting of the geodesic segments $\overline{p\pi_F(p)}$, $\overline{\pi_F(p)\pi_F(q)}$, and $\overline{\pi_F(q)q}$.*

Proof. Let $D_2 = D_1 + \delta$, where D_1 is the constant in Lemma 4.2.5 and δ is the constant in Proposition 4.2.4. Let $r \in \overline{pq}$ be a point which is closest to F. By Lemma 4.2.5,

$d(r, \pi_F(r)) \le D_1$ and therefore the two geodesic segments $\overline{pr}$ and $\overline{p\pi_F(r)}$ are at most D_1 apart. By Proposition 4.2.4, there exists a point on $\overline{p\pi_F(r)}$ which is δ close to $\pi_F(p)$. Therefore there is a point on $\overline{pr}$ which is D_2 close to $\pi_F(p)$. Similarly there exists a point on $\overline{rq}$ which is D_2 close to $\pi_F(q)$ and the corollary follows. □

Now we start the proof of the main result in this section.

Proof of Theorem 4.2.2. Assume that the statement is false, then there exists a sequence of triangles $\Delta(p_n, q_n, r_n)$ which are not n-thin relative to any maximal higher rank submanifold. In particular they are not n-thin triangles. Therefore there exists a point c_n on $\overline{p_n q_n}$ such that $d(c_n, \overline{p_n r_n}) \ge n$ and $d(c_n, \overline{q_n r_n}) \ge n$. Since Γ acts cocompactly on X, we assume that c_n are contained in a compact subset. By passing to a subsequence we could assume that c_n converges to a point $c \in X$. Notice that p_n, q_n, and r_n diverge to infinity. By passing further to a subsequence we could assume that p_n, q_n, and r_n converge respectively to ξ_p, ξ_q, and ξ_r in $\partial_\infty X$. By Proposition 4.2.3, $Td(\xi_p, \xi_r) \le \pi$ and $Td(\xi_r, \xi_q) \le \pi$ and therefore they belong to a connected component of the Tits boundary. Clearly $\xi_p \ne \xi_q$, therefore there exists a maximal higher rank submanifold F such that $\xi_p, \xi_q, \xi_r \in \partial_T F$.

We claim that c belongs to F. If not, then the geodesic segments $\overline{p_n q_n}$ would converge to a geodesic σ passing through c and parallel to F. Let σ' be a geodesic in F which is parallel to σ. By the analyticity of X, σ and σ' are contained in a 2-flat. If $F \notin \mathcal{W}$, i.e. isometric to $\mathbb{R}^2$ or $\mathbb{R}^3$ then the parallel set $P_{\sigma'}$ of σ' properly contains F which contradicts the maximality of F. If $F \in \mathcal{W}$, then σ' can not be a singular geodesics in F, otherwise the parallel set $P_{\sigma'}$ is 4-dimensional and X would be reducible. So, we assume that σ' is not a singular geodesic in F. In this case P_σ is 3-dimensional and therefore has to be of the form $Q \times \mathbb{R}$ and therefore in $\mathcal{W}$. But by the assumption on X that can not happen. Otherwise $\partial_T X$ would have a nonstandard component.

Let $p'_n = \pi_F(p_n)$, $q'_n = \pi_F(q_n)$, and $r'_n = \pi_F(r_n)$. Without loss of generality assume that $d(q'_n, r'_n) \geq d(p'_n, r'_n)$, we might need to pass to a subsequence to guarantee that for every n. Notice that $d(p'_n, q'_n)$ goes to infinity and therefore $d(q'_n, r'_n)$ goes to infinity. We need to show that $d(r'_n, p'_n)$ goes to infinity as well. If not then by passing to a subsequence we could assume that $d(p'_n, r'_n) \leq C$, for some constant C. By Corollary 4.2.6, there exist four points $t_n, t'_n \in \overline{p_n q_n}$ and $s_n, s'_n \in \overline{q_n r_n}$ such that $d(t_n, p'_n)$, $d(t'_n, q'_n)$, $d(s_n, r'_n)$, and $d(s'_n, q'_n)$ are smaller than or equal D_2. Therefore $d(t_n, s_n) \leq 2D_2 + C$. Therefore the two geodesic segments $\overline{q_n s_n}$ and $\overline{q_n t_n}$ are at most $2D_2 + C$ apart. But this contradicts that $d(c, \overline{q_n r_n})$ goes to infinity. Therefore $d(p'_n, r'_n)$ goes to infinity as well. Again by Corollary 4.2.6, there are two point $l_n, l'_n \in \overline{p_n r_n}$ such that $d(l_n, p'_n)$ and $d(l'_n, r'_n)$ are smaller than or equal D_2. Now it is easy to see that all the triangles $\Delta(p_n, q_n, r_n)$ are D_2-thin relative to F, contradicting the choice of these triangles. This finishes the proof. □

Remark 4.2.7. The proof of Theorem 4.2.2 shows that the "fat" part of any triangle is close to a maximal higher rank submanifold.

4.3 The Asymptotic Cone of X

In this section we analyze the asymptotic cone of the Hadamard manifold X. As in Section 4.2, we still assume that X is the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold without nonstandard components in $\partial_T X$.

See Section 2.3 for the definition of the asymptotic cone and for review of some of its basic properties. Fix a non-principle ultrafilter ω on $\mathbb{N}$ and a base point $x_0 \in X$. Let the sequence of rescaling factors $\lambda_n = n$.

Recall the following two well known facts about the asymptotic cone. For any non-principle ultrafilter ω, $\mathrm{Cone}_\omega(\mathbb{R}^n) = \mathbb{R}^n$, and if X is a quasi-homogeneous Gromov hyperbolic space with uncountable number of ideal boundary points, then $\mathrm{Cone}_\omega(X)$ is an $\mathbb{R}$-tree with uncountable branching.

If X is a Hadamard manifold, then $\mathrm{Cone}_\omega(X)$ is a Hadamard space. Any sequence of flats in X gives rise to a flat in the ultralimit of $\big((X, n^{-1} \cdot d), x_0\big)$. If the distance between that flat and the base point in the asymptotic cone is finite, then the flat is in $\mathrm{Cone}_\omega(X)$. The goal of this section is to show that these are the only flats which appear in $\mathrm{Cone}_\omega(X)$.

Our analysis is similar to the analysis done by Kapovich and Leeb in [27]. And we will use some of their results. We will often mention the results without proofs and refer the reader to their paper for the proofs.

Given a maximal higher rank submanifold $F \in \mathcal{V}$, if $F \approx \mathbb{R}^2$, $\mathbb{R}^3$, or $Q \times \mathbb{R}$, the sequence $F_n = F$ has a limit $\mathbb{R}^2$, $\mathbb{R}^3$, or $T \times \mathbb{R}$, where T is an $\mathbb{R}$-tree, in $\mathrm{Cone}_\omega(X)$.

The group $\Gamma^* = \prod_{\mathbb{N}} \Gamma$ acts by isometries on $\mathrm{Cone}_\omega(X)$. Recall that up to the action of Γ there is only a finite number of maximal higher rank submanifolds, we denote them by $K_1, \dots, K_r$. Any sequence F_n of maximal higher rank submanifolds in $\mathcal{V}$ gives rise to a partition $A_1 \sqcup \dots \sqcup A_r$ of $\mathbb{N}$ as follows: $n \in A_s$ if $F_n = \phi(K_s)$ for some isometry $\phi \in \Gamma$. By basic properties of ultrafilters, there is exactly one subset $A_s \in \omega$. Therefore the sequence F_n and the subsequence corresponding to A_s give rise to the same limit in $\mathrm{Cone}_\omega(X)$. Therefore without loss of generality, we could assume that for a fixed K_s, $F_n = \phi_n(K_s)$, where $\phi_n \in \Gamma$. Since Γ^* acts by isometries on $\mathrm{Cone}_\omega(X)$, it is easy to see that the limit of F_n is $\phi^*(\omega\text{-}\lim K_s)$ where $\phi^* = (\phi_1, \phi_2, \dots) \in \Gamma^*$.

We denote by $\mathcal{F}$ the limits in $\mathrm{Cone}_\omega(X)$ of all sequences of maximal higher rank submanifolds of X. The above discussion shows the following,

Proposition 4.3.1. *Every element in $\mathcal{F}$ is isometric to $\mathbb{R}^2$, $\mathbb{R}^3$, or $T \times \mathbb{R}$, where T is an $\mathbb{R}$-tree.*

Definition 4.3.2. Let X be a CAT(0) space. A triangle $\Delta(p_1, p_2, p_3)$ in X is called an open triangle if p_1, p_2, p_3 are different points, and $\overline{p_i p_j} \cap \overline{p_i p_k} = \{p_i\}$, where $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$.

The following lemma is a rephrase of Proposition 4.3 in [27].

Lemma 4.3.3. *The asymptotic cone $\mathrm{Cone}_\omega(X)$ satisfies the following properties:*

1. *Every open triangle is contained in some $F \in \mathcal{F}$.*

2. *Any two different elements $F, F' \in \mathcal{F}$ have at most one point in common.*

Proof. The proof of the first part is identical to the proof of the first part in Proposition 4.3 in [27], and the proof carries over to our sitting.

Now we give the proof of the second part. Notice that every $F \in \mathcal{F}$ is a convex subset of $\mathrm{Cone}_\omega(X)$. Assume that F and F' intersect in two different points x and y. Therefore $\overline{xy} \subset F \cap F'$. Let F_n be a sequence of maximal higher rank submanifolds such that $\omega\text{-}\lim F_n = F$.

Let $z \in F'$ be any point such that the triangle $\Delta(x, y, z)$ is an open triangle. The goal is to show that $z \in F$. Choose a sequence of triangles $\Delta_n = \Delta(x_n, y_n, z_n)$ in X such that they converge to $\Delta(x, y, z)$ in the asymptotic cone. We can choose $x_n, y_n \in F_n$. Let $z'_n = \pi_{F_n}(z_n)$ be the projection of z_n to F_n.

We claim that the two sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ represent the same point z in $\mathrm{Cone}_\omega(X)$, which implies that $z \in \omega\text{-}\lim F_n = F$. To see that, assume that $(z'_n)_{n\in\mathbb{N}}$ converges to $z' \in \mathrm{Cone}_\omega(X)$ and $z \neq z'$. Using Proposition 4.2.4, for every $n \in \mathbb{N}$, we can choose two points $s_n \in \overline{z_n x_n}$ and $t_n \in \overline{z_n y_n}$ within distance at most δ from z'_n, where δ is the constant given in Proposition 4.2.4. Clearly, $(z'_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, and $(t_n)_{n\in\mathbb{N}}$ represent the same point $z' \in \mathrm{Cone}_\omega(X)$ which implies that $\{z\} \subsetneqq \overline{zz'} \subset \overline{zx} \cap \overline{zy}$. This is a contradiction since we assumed that $\Delta(x, y, z)$ is an open triangle.

Notice that the set of points in F' where the triangle $\Delta(x, y, z)$ is open is an open dense subset of F', for the three different possibilities of F' given by Proposition 4.3.1. By a continuity argument we see that $F' \subset F$. Similarly we show that $F \subset F'$ which finishes the proof. □

Remark 4.3.4. The proof of Lemma 4.3.3 rules out the possibility that the limit of a sequence of maximal higher rank submanifolds which are isometric to $\mathbb{R}^2$ is contained in the limit of a sequence of maximal higher rank submanifolds of higher dimensions.

For every element $F \in \mathcal{F}$, we denote by $\pi_F\colon \mathrm{Cone}_\omega(X) \longrightarrow F$ the projection map which is well defined and distance non-increasing since F is a convex subset of $\mathrm{Cone}_\omega(X)$ which is CAT(0) space. Now we state Lemma 4.4 and Lemma 4.5 from [27] in our new sitting where 2-flats are replaced by maximal higher rank submanifolds.

Lemma 4.3.5. *Let* $\gamma\colon I \longrightarrow \mathrm{Cone}_\omega(X)$ *be a curve in the complement of* F*. Then* $\pi_F \circ \gamma$ *is constant.*

Lemma 4.3.6. *Every embedded closed curve* $\gamma \subset \mathrm{Cone}_\omega(X)$ *is contained in some element* $F \in \mathcal{F}$*.*

The proofs of these two lemmas in [27] carry over to our new setting. The only ingredient used there was that the limits of 2-flats in the asymptotic cones of the universal covers of certain Haken manifolds, satisfy the two properties in Lemma 4.3.3. The proofs carry over to our new sitting after replacing 2-flats by maximal higher rank submanifolds and using Lemma 4.3.3.

Lemma 4.3.3 easily implies that the only flats in $\mathrm{Cone}_\omega(X)$ are the ones which are subsets of elements of $\mathcal{F}$. While Lemma 4.3.6 shows that every embedded disk of dimension at least 2 is contained in an element of $\mathcal{F}$.

4.4 Proof of Main Theorem

In this section we give the proof of Theorem 1.2.7. We assume that $\partial_T X_1$ has a nonstandard component and $\partial_T X_2$ does not have any nonstandard components. If $f\colon X_1 \longrightarrow X_2$ is a quasi-isometry, then it induces a map $\mathrm{Cone}_\omega(f)\colon \mathrm{Cone}_\omega(X_1) \longrightarrow \mathrm{Cone}_\omega(X_2)$ which is bi-Lipschitz homeomorphism. Since $\partial_T X_1$ contains a nonstandard component, there exist two maximal higher rank submanifolds $H_1 = Q_1 \times \mathbb{R}$ and $H_2 = Q_2 \times \mathbb{R}$ which intersect in a 2-flat, which we denote by K. In the asymptotic cone $Q_1 \times \mathbb{R}$ and $Q_2 \times \mathbb{R}$ give rise to two convex subsets $W_1 = T_1 \times \mathbb{R}$ and $W_2 = T_2 \times \mathbb{R}$, where T_1 and T_2 are $\mathbb{R}$-trees. The $\mathbb{R}$-factors of W_1 and W_2, which we call the singular directions are different. The 2-flat K gives rise to a 2-flat, which we denote by L, in the asymptotic cone $\mathrm{Cone}_\omega X_1$. Clearly $W_1 \cap W_2 \supseteqq L$. In the following proposition we show that the intersection of W_1 and W_2 is precisely L.

Proposition 4.4.1. *Using the above notation,* $W_1 \cap W_2 = L$.

Proof. Let $z \in W_1 \cap W_2$. Choose two sequences $(x_n)_{n\in\mathbb{N}} \subset H_1$ and $(y_n)_{n\in\mathbb{N}} \subset H_2$ which represent the point z in $\mathrm{Cone}_\omega(X_1)$. This implies that $\omega\text{-}\lim d(x_n, y_n)/n = 0$. Since H_1 and H_2 are orthogonal to each other at the intersection (see Lemma 3.4 in [36]), therefore $\pi_{H_1}(y_n) \in K = H_1 \cap H_2$. Since $d\big(\pi_{H_1}(y_n), y_n\big) \leq d(x_n, y_n)$, it easy to see that $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, and $\big(\pi_{H_1}(y_n)\big)_{n\in\mathbb{N}}$ represent the same point in $\mathrm{Cone}_\omega(X_1)$. This shows that $z \in L$, which finishes the proof. □

Since $\partial_T X_2$ does not contain any nonstandard components, Section 4.3 describe all the flats in $\mathrm{Cone}_\omega(X_2)$. We are finally ready to start the proof of the main theorem.

Proof of Theorem 1.2.7. Assume that $f\colon X_1 \longrightarrow X_2$ is a quasi-isometry. The induced map $\mathrm{Cone}_\omega(f)\colon \mathrm{Cone}_\omega(X_1) \longrightarrow \mathrm{Cone}_\omega(X_2)$ is a bi-Lipschitz homeomorphism. Assume that $T_i \times \mathbb{R} \subset \mathrm{Cone}_\omega(X_1)$, for $i = 1, 2$, where T_i is an $\mathbb{R}$-tree such that $T_1 \times \mathbb{R} \cap T_2 \times \mathbb{R} = \mathbb{R}^2$ as mentioned above. By Lemma 4.3.6 the image of each flat in $T_i \times \mathbb{R}$, for $i = 1, 2$, has to be contained inside some element $F \in \mathcal{F}$. For any two 2-flats in $T_i \times \mathbb{R}$, there exists a third flat which intersects both of them in half planes. That shows that the images of $T_i \times \mathbb{R}$, for

$i = 1, 2$, have to be contained in some $F \in \mathcal{F}$. We need to show this is not possible for the three different types of F, namely $\mathbb{R}^2$, $\mathbb{R}^3$, and $T \times \mathbb{R}$.

Case 1: $F = \mathbb{R}^2$

This case is easy, a bi-Lipschitz embedding of any 2-flat in $T_1 \times \mathbb{R}$ has to be onto F, contradicting that $\mathrm{Cone}_\omega(f)$ is a bijection.

Case 2: $F = \mathbb{R}^3$

Since $T_i = \mathrm{Cone}_\omega(Q_i)$ where Q_i are visibility 2-manifolds (with cocompact action) and therefore hyperbolic, then T_i branches uncountably many times at each point. Now it is not hard to see that an $\mathbb{R}$-tree which branches uncountably many times at each point can not be bi-Lipschitz embedded in $\mathbb{R}^3$. One easy way to see that is to fix a base point p in the tree, and take a sequence of points p_i such that $d(p, p_i) = 1$ and $d(p_i, p_j) = 2$, for $i \neq j$. But $\mathrm{Cone}_\omega(f)(p_i)$ is contained in a compact subset of $\mathbb{R}^3$ and therefore by passing to a subsequence it will converge in $\mathbb{R}^3$, which contradicts that $\mathrm{Cone}_\omega(f)$ is bi-Lipschitz homeomorphism.

Case 3: $F = T \times \mathbb{R}$

First we recall Lemma 2.14 in [27] which states that if T is a metric tree then the image of any bi-Lipschitz embedding $\pi \colon \mathbb{R}^2 \longrightarrow T \times \mathbb{R}$, is a flat in $T \times \mathbb{R}$. Notice that any two different flats in $F = T \times \mathbb{R}$ either do not intersect, intersect in a line, a strip, or a half plane.

Denote by α_1, α_2 the unique geodesics in T_1, T_2 respectively which satisfies that $\alpha_1 \times \mathbb{R} = \alpha_2 \times \mathbb{R}$ is the unique 2-flat which is the intersection of $T_1 \times \mathbb{R}$ and $T_2 \times \mathbb{R}$. Parameterize α_1 and α_2 such $p = \alpha_1(0) = \alpha_2(0)$ is the unique intersection point. Choose a geodesic β_1 in T_1 such that, β_1 follows α_1 until they reach the point p and then branches at p. Choose a geodesic β_2 in T_2 such the intersection of α_2 and β_2 is the unique point p. Consider the two flats $\beta_1 \times \mathbb{R} \subset T_1 \times \mathbb{R}$ and $\beta_2 \times \mathbb{R} \subset T_2 \times \mathbb{R}$. It is not hard to see that these two planes intersect in a half line. But by Lemma 2.14 in [27] mentioned above, the image of these two planes in F are two flats. This is a contradiction since a half line is not bi-Lipschitz homeomorphic to either a line, a strip or a half plane. $\square$

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