Higher Dimensional Class Field Theory: The variety case

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ABSTRACT

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Let k be a finite field, and suppose that the arithmetical variety $X \subset \mathbb{P}^n_k$ is an open subset in projective space. Suppose that $\mathcal{C}_X$ is the Wiesend idèle class group of X, $\pi_1^{ab}(X)$ the abelianised fundamental group, and $\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$ the Wiesend reciprocity map. We use the Artin-Schreier-Witt and Kummer Theory of affine k-algebras to prove a full reciprocity law for X. We find necessary and sufficent conditions for a subgroup $H < \mathcal{C}_X$ to be a norm subgroup: H is a norm subgroup if and only if it is open and its induced covering datum is geometrically bounded. We show that ρ_X is injective and has dense image. We obtain a one-to-one correspondence of open geometrically bounded subgroups of $\mathcal{C}_X$ with open subgroups of $\pi_1^{ab}(X)$. Furthermore, we show that for an étale cover $X'' \longrightarrow X$ with maximal abelian subcover $X' \longrightarrow X$, the reciprocity morphism induces an isomorphism $\mathcal{C}_X/\mathcal{N}\mathcal{C}_{X''} \simeq Gal(X'/X)$.

Contents

Chapter 1

Introduction

Class Field Theory is one of the major achievements in the number theory of the first half of the 20h century. Among other things, Artin reciprocity showed that the unramified extensions of a global field can be described by an abelian object only depending on intrinsic data of the field: the Class Group.

In the language of Grothendieck's algebraic geometry, the theorems of classical global class field theory [12, Ch. VI] can be reformulated as theorems about one-dimensional arithmetical schemes, whose function fields are precisely the global fields. (A summary of results in convenient notation is presented in Section 3.3). The class field theory for such schemes X then turns into the question of describing the unramified abelian covers of the schemes, i.e. describing the fundamental group $\pi_1^{ab}(X)$. It is therefore natural to ask for a generalisation of class field theory to arithmetical schemes of higher dimensions.

Several attempts at a Higher Class Field Theory have already been made, with different generalisations of the class group to higher dimensional schemes: Katz-Lang [4] described the maximal abelian cover of a projective regular arithmetic scheme and Serre [15] gave a description of the abelian covers of schemes over $\mathbb{F}_p$ in terms of generalised Jacobians. Finally, Parshin and Kato, followed by several others, proposed getting a higher dimensional Artin reciprocity map using algebraic K-Theory and cohomology theories. Although promising, these approaches become quite technical, and the heavy machinery involved makes the results very complicated and difficult to apply in concrete situations.

It was G. Wiesend [21] who had the idea to reduce the higher dimensional class field theory to the well developed and known class field theory for arithmetical curves. He defines an "idèle class group" $\mathcal{C}_X$ in terms of the arithmetical curves and closed points contained in X, and gets a canonical homomorphism

$$\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$$

in the hope of establishing properties similar to that of the Artin reciprocity map. Wiesend's work was supplemented by work by M. Kerz and A. Schmidt, where more details of Wiesend's approach were given [5]. Most notably, in the so-called *flat case* of an arithmetical scheme X over $Spec\,\mathbb{Z}$ (c.f. Definition 2.1.2), they were able to prove surjectivity of the canonical homomorphism, and they provided a concrete description of the norm subgroups in $\mathcal{C}_X$.

The focus of this thesis is on the still-open regular variety case, where X is a regular arithmetical variety over some finite field k of characteristic p.

In this case, Kerz and Schmidt have already proved a higher dimensional reciprocity law for the abelianised *tame* fundamental group $\pi_1^{ab,t}(X)$, which classifies all the tame étale abelian covers of X. A generalised Artin map can be defined between

$$\rho_{X,t} : \mathcal{C}_X^t \longrightarrow \pi_1^{ab,t}(X)\,,$$

where $\mathcal{C}_X^t$ denotes the tame class group, as defined by Wiesend [21]. As in the flat case, the proof crucially relies on finiteness theorems for the geometric part of the tame fundamental group [4].

These finiteness results are known to be false for the full fundamental group due to the presence of wild ramification: For any affine variety of dimension ≥ 1 over a finite field, the p-part of the fundamental group is infinitely generated.

Let k be a finite field, and $X \subset \mathbb{P}_k^n$ an open subvariety, then we prove:

Theorem 1.0.1. *Let $X \in \mathbb{P}_k^n$ be an open subvariety, and let*

$$\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$$

be the Wiesend recprocity morphism. Then the following hold:

1) There exists a one-to-one correspondence between open and geometrically bounded subgroups of $\mathcal{C}_X$ and open subgroups $\overline{N}$ of $\pi_1^{ab}(X)$; it is given by $\rho_X^{-1}(\overline{N}) \mapsto \overline{N}$. The reciprocity morphism ρ_X is a continuous injection with dense image in $\pi_1^{ab}(X)$.

2) A subgroup of $\mathcal{C}_X$ is a norm subgroup iff it is open, of finite index and geometrically bounded.

3) If $X'' \longrightarrow X$ is an étale connected cover, $X' \longrightarrow X$ the maximal abelian subcover, then $\mathcal{NC}_{X''} = \mathcal{NC}_{X'}$ and the reciprocity map gives rise to an isomorphism

$$\mathcal{C}_X/\mathcal{NC}_{X''} \xrightarrow{\simeq} Gal(X'/X).$$

This thesis is organised as follows: We begin by introducing notation and reviewing basic facts from the theories of arithmetical schemes, fundamental groups and covering data. We then review Wiesend's definition of the idèle class group and the reciprocity homomorphism, giving the right definitions as to guarantee commutativity of all relevant diagrams. (Unfortunately, in all the published works so far, the given morphisms do not make all the diagrams commute as stated.) We also review the results of classical class field theory as the "base" case of our induction argument in the proof of the Main Theorem 4.3.4. All of Chapter 4 is devoted to assembling and refining the necessary tools for the proof of the Main Theorem 4.3.4, and in the process, the proof of the Main Theorem 4.3.4 is reduced to the Key Lemma 4.4.4. Theorem 1.0.1 is shown as a corollary to the Main Theorem 4.3.4.

The Key Lemma is shown for open subvarieties $X \subset \mathbb{P}^n_k$ in two steps. In Chapter 5, we analyse the behaviour of index-p^m wildly ramified covering data on X to show Part 1). The second part of the Key Lemma, which was already known from the results of Wiesend, Kerz and Schmidt ([21], [5]), is reproven in Chapter 6 without making explicit use of geometric finiteness results.

Chapter 2

Preliminaries

2.1 Basic Facts and Generalities

In this thesis, we shall be concerned with arithmetical schemes X over $Spec\,\mathbb{Z}$:

Definition 2.1.1. X is said to be an *arithmetical scheme over* $\mathbb{Z}$ if it is integral, separated and of finite type over $Spec\,\mathbb{Z}$.

We assume that all schemes are arithmetical, unless otherwise stated, and distinguish two cases:

Definition 2.1.2. If the structural morphism $X \longrightarrow Spec\,\mathbb{Z}$ has open image, X will be called *flat*. If this is not the case, then the image of $X \longrightarrow Spec\,\mathbb{Z}$ is a closed point $p \in Spec\,\mathbb{Z}$, and X is a *variety* over the residue field of the point $k = k(p) = \mathbb{F}_p$.

We now collect some general facts and tools to be used in later chapters:

Fix a field k, and let $K \supset k$ be a field over k. Then let $\deg_{tr} K$ denotes the transcendence degree of a field K. For a scheme X, let $\dim X$ denote the Krull dimension of a scheme X as a topological space (cf. [9, Definition 2.5.1]).

Definition 2.1.3. Let X be an arithmetical scheme with structural morphism $f : X \longrightarrow Spec\,\mathbb{Z}$, and let $\overline{f(X)}$ be the closure of the image of f in $Spec\,\mathbb{Z}$. Denote the function field of X by $K(X)$. Then the *Kronecker dimension* d of X is defined as

$$\dim X = \deg_{tr} K(X) + \dim(\overline{f(X)})\,.$$

Remark 2.1.4. For example, both $Spec\ \mathbb{Q}$ and $Spec\ \mathbb{Z}$ have Kronecker dimension one, while $Spec\ \mathbb{F}_p$ has Kronecker dimension zero. An arithmetical variety of Kronecker dimension one is given by $Spec\ \mathbb{F}_p[t]$.

Definition 2.1.5. A *curve* is an arithmetical scheme of Kronecker dimension one. If X is an arithmetical scheme of arbitrary dimension, a *curve in* X is a closed integral subscheme of Kronecker dimension one.

Under this definition, the curves are precisely those arithmetical schemes whose function field is a global field.

Definition 2.1.6. Let X be any scheme. An *étale cover of* X is a finite étale cover $Y \longrightarrow X$, and a *pro-étale cover* of X is the projective limit of étale covers of X.

Let k be an arbitrary field, and consider a subvariety $X \subset \mathbb{P}^n_k$. If $C \subset X$ is a curve, i.e. a one-dimensional integral closed subscheme of X, then $C \subset \mathbb{P}^n_k$ is quasi-projective. We recall from [9, Section 7.3.2] and [2, Section 1.7] the definitions of the genus and degree of a quasi-projective curve:

Definition 2.1.7. Let C be a geometrically connected projective curve. Then the *arithmetic genus* $g_a(C)$ is defined as $g_a(C) = 1 - \chi_k(\mathcal{O}_X)$, where $\chi_k(\mathcal{O}_X)$ denotes the Euler-Poincaré characteristic of the structural sheaf O_X. Let C be a quasi-projective curve, with regular compactification $\overline{C}$. Then the arithmetic genus is defined by setting $g_a(C) := g_a(\overline{C})$.

Let C be a geometrically connected projective variety over k which is also smooth. Then the arithmetic genus is equal to the geometric genus g_C.

Definition 2.1.8. If k is an arbitrary field, and $X \subset \mathbb{P}_k^n$ is a projective variety of dimension d, the *degree of* X *over* k is defined as the leading coefficient of the Hilbert polynomial $p_X(t)$, multiplied by $d!$. It is denoted by $\deg_k X$.

Fact 2.1.9. Let k be an arbitrary field, $X \subset \mathbb{P}_k^n$ be a subvariety. For every positive integer d, there exists a number $g = g(n,d)$ such that for all curves $C \subset X$ of degree $\leq d$, we have $g_C \leq g$. In particular, this holds for all regular curves $C = \widetilde{C} \subset X$.

Note: The converse of this is not true: If $C \subset \mathbb{P}_k^2$ is the completion of $V(x - y^n) \subset \mathbb{A}_k^2 \subset \mathbb{P}_k^2$, then C has degree n, but $g_C = 0$ as C is rational.

Fact 2.1.10. Let X be any reduced scheme of finite type over the perfect field k. Then there exists a dense open subscheme which is affine and smooth.

Proposition 2.1.11 (Chebotarev Density Theorem)**.** *Let $Y \longrightarrow X$ be a generically Galois cover of connected normal schemes over $\mathbb{Z}$. Let Σ be a subset of $G = G(Y|X)$ that is invariant under conjugation, i.e. $g\Sigma g^{-1} = \Sigma$ for all $g \in G$. Set $S = \{x \in X : Frob_x \in \Sigma\}$. Then the Dirichlet density $\delta(S)$ is defined and equal to $\delta(S) = |\Sigma|/|G|$.*

Proof. See [16]. □

Lemma 2.1.12 (Completely Split Covers)**.** *Let X be a connected, normal scheme of finite type over $\mathbb{Z}$. If $f : Y \longrightarrow X$ is a finite étale cover in which all closed points of X split completely, then this cover is trival. If Y is connected, then f is an isomorphism.*

Proof. Let Y' be the Galois closure of $Y \longrightarrow X$, then a closed point $x \in X$ is completely split in Y if it is completely split in the cover Y', so without loss of generality we may assume that $Y \longrightarrow X$ is Galois with Galois group G. Since $x \in X$ splits completely if and only if $Frob_x = 1$ for all Frobenius elements above x, this follows directly from the Chebotarev Density Theorem 2.1.11. Let $S = \{x : Frob_x = 1\}$, and $\delta(S)$ the Chebotarev density of S. Then

$$1 = \delta(S) = 1/|G|\,,$$

whence it folllows that $|G| = 1$. □

As being completely split is equivalent to saying that the pullback $Y \times x \longrightarrow x$ to every closed point $x \in |X|$ is the trivial cover, we make the following definition:

Definition 2.1.13. An étale cover $f : Y \longrightarrow X$ is called *locally trivial* if it is completely split.

Lemma 2.1.14 (Approximation Lemma)**.** *Let Z be a regular arithmetical curve, X a quasi-projective arithmetical scheme, and let $X \to Z$ be a smooth morphism in of arithmetical schemes. Let $Y \longrightarrow X$ be a finite cover of arithmetical schemes, and let $x_1, \ldots, x_n$ be closed points of X with pairwise different images in Z. Then there exists a curve $C \subset X$ such that the points x_i are contained in the regular locus C^{reg} of C, and such that $C \times Y$ is irreducble.*

Proof. See [5]. □

The following proposition will be essential for dealing with covering data and trivialising morphisms in Chapters 3-5:

Proposition 2.1.15. *Let X be a regular, pure-dimensional, excellent scheme, $X' \subset X$ a dense open subscheme, $Y' \longrightarrow X'$ an étale cover and Y the normalization of X in $k(Y')$. Suppose that for every curve C on X with $C' = C \cap X' \neq \emptyset$, the étale cover $Y' \times \widetilde{C}' \longrightarrow X \times \widetilde{C}'$ extends to an étale cover of $\widetilde{C}$. Then $Y \longrightarrow X$ is étale.*

Proof. See [5, Proposition 2.3] for a proof. □

2.2 Fundamental groups

In this section, we give a brief survey of relevant results in the theory of fundamental groups, all of which are taken from [19, Section 5.5].

Let X be a connected scheme. Then the finite étale covers of X, together with morphisms of schemes over X form a category of X-schemes, which we denote by Et_X.

Now let Ω be an algebraically closed field, and fix $\bar{s} : Spec\,\Omega \longrightarrow X$, a geometric point of X. If $Y \longrightarrow X$ is an element of Et_X, consider the geometric fiber $Y \times_X Spec\,\Omega$, and let $Fib_{\bar{s}}(Y)$ denote its underlying set. Any morphism $Y_1 \longrightarrow Y_2$ in Et_X induces a morphism $Y_1 \times_X Spec\,\Omega \longrightarrow Y_2 \times_X Spec\,\Omega$. Applying the forgetful functor, we get an induced set-theoretic map $Fib_{\bar{s}}(Y_1) \longrightarrow Fib_{\bar{s}}(Y_2)$.

Thus, $Fib_{\bar{s}}(\cdot)$ defines a set-valued functor on the categeroy Et_X, which we call the *fiber functor at the geometric point* $\bar{s}$.

We recall that an automorphism of a functor F is a morphism of $F \longrightarrow F$ which has a two-sided inverse (cf. [20]), and define the fundamental group as follows:

Definition 2.2.1. The *fundamental group* of X with geometric basepoint $\bar{s}$ is the automorphism group of the fiber functor $Fib_{\bar{s}}$ associated to $\bar{s}$, and is denoted by $\pi_1(X, \bar{s})$.

Lemma 2.2.2. *The fundamental group is profinite and acts continuously on* $Fib_{\bar{s}}(X)$.

Proof. See [19, Theorem 5.4.2]. □

Proposition 2.2.3. *The functor $Fib_{\bar{s}}$ induces an equivalence of the category of finite étale covers of X with the category of finite continuous $\pi_1(X, \bar{s})$-Sets. Under this correspondence, connected covers correspond to sets with transitive action, and Galois covers to finite quotients of $\pi_1(X, \bar{s})$*

Proof. Cf. [19, Thm. 5.4.3]). □

Now let $\bar{s}, \bar{s}' : Spec\, \Omega \longrightarrow X$ be two geometric points of X.

Definition 2.2.4. A *path* from $\bar{s}$ to $\bar{s}'$ is an isomorphism of fiber functors $Fib_s \xrightarrow{\simeq} Fib_{s'}$.

Whenever such a path p exists, the fundamental groups are (non-canonically) isomorphic as profinite groups via conjugation by p: $\pi_1(Y, \bar{s}) \simeq \pi_1(Y, \bar{s}')$ [19, 5.5.2].

Proposition 2.2.5. *If X is a connected connected scheme and s are $\bar{s}'$ two geometric points of X, then there always exists a path p from $\bar{s}$ and $\bar{s}'$.*

Proof. See [19, Cor. 5.5.2]. □

Thus, for a connected scheme X, the fundamental group $\pi_1(X)$ is well-defined up to conjugation by an element of itself. In particular, the maximal abelian quotient $\pi_1^{ab}(X)$ does not depend on the choice of geometric basepoint [19, Remark 5.5.3].

Definition 2.2.6. The maximal abelian quotient $\pi_1^{ab}(X)$ is called the *abelianised fundamental group* of X.

Now let X, Y be schemes with geometric points $\bar{s}'$ and $\bar{s}$.

Definition 2.2.7. In the situation above, if $f : Y \longrightarrow X$ is a morphism of schemes such that $f \circ \bar{s} = \bar{s}'$, then the morphism is said to be *compatible* with $\bar{s}$ and $\bar{s}'$.

$$\begin{array}{ccc} Spec\,\Omega & \xrightarrow{\bar{s}} & Y \\ & \searrow^{\bar{s}'} & \downarrow f \\ & & X \end{array}$$

Proposition 2.2.8. *In the situation above, a morphism that is compatible with $\bar{s}$ and $\bar{s}'$ induces a morphism on fundamental groups $\widetilde{f} : \pi_1(Y, \bar{s}) \longrightarrow \pi_1(X, \bar{s}')$.*

Proof. See the remarks following Remark 5.5.3 in [19]. □

If $f : Y \longrightarrow X$ is a finite morphism that is not compatible with the given geometric points, then we can find another geometric point $\bar{s}''$ of Y so that $\bar{s}'$ and $\bar{s}''$ are compatible. Indeed, let x be the image point of $\bar{s}'$ in X. If y is any element of $f^{-1}(x)$, $k(y)$ is a finite extension of $k(x)$ by assumption. Since Ω is algebraically closed, we have $k(y) \subset \Omega$, and may define a geomtric point $\bar{s}''$ of Y by declaring its image to be y.

The fundamental groups of X with respect to the two points are isomorphic via conjugation by some path p, so we obtain a morphism on the fundamental groups induced by f via composition with the isomorphism:

$$\widetilde{f} : \pi_1(X, \bar{s}) \overset{p \cdot p^{-1}}{\simeq} \pi_1(X, \bar{s}'') \longrightarrow \pi_1(Y, \bar{s}')$$

Remark 2.2.9. In the situation above, whether basepoints are already compatible or not, the preimage $\widetilde{f}^{-1}(N)$ of any normal subgroup N of $\pi_1(X, \bar{s}')$ is always well defined. We shall thus drop the geometric points from our notation in later chapters, when dealing with covering data.

Remark 2.2.10. While the fundamental group functor $\pi_1(X, \bar{s})$ is not representable in Et_X, the category of finte étale covers of X, it is pro-representable and thus representable in the larger category of profinite limits of finite étale covers of X. The corresponding universal element is also called a *universal cover* of X, and there is a one-to-one correspondence between universal covers and a system of compatible geometric basepoints for the collection of all pro-étale covers of X.

In particular, fixing a universal cover with geometric basepoint $\widetilde{x}$ amounts to choosing a system of compatible basepoints for every pro-étale cover $Y \longrightarrow X$.

The following proposition summarizes some further properties of fundamental groups:

Proposition 2.2.11. *Let X be a connected scheme. Fix a universal cover $\widetilde{X}$ of X, let $\widetilde{x}$ denote its geometric basepoint, and let $f : Y \longrightarrow X$ be a finite connected étale subcover. Then*

1. *The induced morphism $\widetilde{f} : \pi_1(Y) \longrightarrow \pi_1(X)$ is injective with open image $N_Y := \widetilde{f}(\pi_1(Y))$, and the association $Z \mapsto \widetilde{f}(\pi_1(Z))$ is one-to-one if we restrict to connected étale subcovers Z of $\widetilde{X}$.*

2. *f is a trivial cover if and only if $\widetilde{f}$ is an isomorphism.*

3. *f is Galois if and only if N_Y is normal in $\pi_1(X)$. If this is the case, the Galois*

group $G = Aut(Y/X)$ is isomorphic to the group $\pi_1(X)/N_Y$.

4. *If $g : W \longrightarrow X$ is another étale cover, and let $\overline{w}$, $\overline{y}$ be the geometric basepoints of W, respectively Y that are combaptible with $\widetilde{x}$. Let $Z \subset Y \times_X Y$ be the connected component of $Y \times Z$ containing the geometric basepoint $\overline{y} \times \overline{w}$, then*

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

corresponds to the open subgroup $N_Y \cap N_Z$. If f and g are Galois over X with groups G_f and G_g, then so is $W \longrightarrow X$, and it has Galois group G, where $G < G_f \times G_g$ is a subgroup projecting surjectively onto G_f and G_g.

5. *There exists a minimal étale cover $Z \longrightarrow Y$ such that $Z \longrightarrow X$ is Galois. N_Z is then the smallest normal subgroup of $\pi_1(X)$ contained in N_Y.*

Proof. Apply Proposition 2.2.3 and Propositions 5.5.4-6 of [19]. □

Recall from Definition 2.1.6 that a pro-étale cover $f : Y \longrightarrow X$ is the inverse limit of finite étale covers. Analogously to above, we then have the following proposition:

Proposition 2.2.12. *Let X be a connected scheme. Fix a universal cover $\widetilde{X}$ of X, let $\widetilde{x}$ denote the associated geometric basepoint, and let $f : Y \longrightarrow X$ be a pro-étale subcover. Then*

1. *The induced morphism* $\widetilde{f} : \pi_1(Y) \longrightarrow \pi_1(X)$ *is injective with closed image* $N_Y := \widetilde{f}(\pi_1(Y))$*, and the association* $Z \mapsto \widetilde{f}(\pi_1(Z))$ *is again one-to-one if we restrict to connected étale subcovers* Z *of* $\widetilde{X}$*.*

2. f *is a trivial cover if and only if* $\widetilde{f}$ *is an isomorphism.*

3. f *is Galois if and only if* N_Y *is normal in* $\pi_1(X)$*. If this is the case, the Galois group* $G = Aut(Y/X)$ *is isomorphic to the group* $\pi_1(X)/N_Y$*.*

4. *If* $g : W \longrightarrow X$ *is another pro-étale cover, let* $\overline{w}$*,* $\overline{y}$ *denote the geometric basepoints of* W*, respectively* Y *which are compatible with* $\widetilde{x}$*. Let* $Z \subset W \times_X Y$ *be the irreducible component which contains the geometric basepoint* $\overline{y} \times \overline{w}$*. Then we have a commutative diagam*

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

and the cover $f \times g : Z \longrightarrow X$ *corresponds to the open subgroup* $N_Y \cap N_W$*. If* f *and* g *are Galois over* X *with groups* G_f *and* G_g*, then so is* $f \times g$*, say with Galois group* G*, then* $G < G_f \times G_g$ *is a subgroup projecting surjectively onto* G_f *and* G_g*.*

5. *There exists a minimal étale cover* $Z \longrightarrow Y$ *such that* $Z \longrightarrow X$ *is Galois.* N_Z *is then the smallest normal subgroup of* $\pi_1(X)$ *contained in* N_Y*.*

Notation 2.2.13. Lastly, we establish and summarise some notation for later chapters:

1. We have already introduced the tilde notation for morphisms between fundamental groups which are induced by morphisms of schemes: If $f : Y \xrightarrow{f} X$ is a morphism, then $\widetilde{f} : \pi_1(Y) \longrightarrow \pi_1(X)$ denotes the induced map on the fundamental groups.

2. If X is an arithmetical scheme, then the closed points in an arithmetical scheme X are those points with finite residue field, and the set of closed points is denoted by $|X|$. We let $i_x : x \hookrightarrow X$ denote the inclusion morphism.

3. Recall that a closed integral subscheme $C \hookrightarrow X$ of dimension one is called a curve in X. We denote the normalisation of a curve by $\widetilde{C}$, and note that the normalisation might lie outside the scheme X.

   We let $i_C : \widetilde{C} \longrightarrow C \hookrightarrow X$ denote the composition of the normalisation morphism with the inclusion of the curve in X.

4. Let $i_C : \widetilde{C} \longrightarrow C \hookrightarrow X$ be the composition of the normalisation morphism with the inclusion of a curve into X as defined above, then we denote by $\widetilde{i}_C : \pi_1(\widetilde{C}) \longrightarrow \pi_1(X)$ the induced morphism on the fundamental groups. Similary, for the inclusion $i_x : x \hookrightarrow X$ of a closed point, the induced morphism $\pi_1(x) \longrightarrow \pi_1(X)$ is denoted by $\widetilde{i}_x$.

5. Now let $f : Y \longrightarrow X$ be a morphism of arithmetical schemes. If y is a closed point in Y, then $x = f(y)$ is also a closed point. If $C \subset X$ is a curve, then the closure $\overline{f(C)}$ of the image of C is either a closed point or a curve D. We let

$f|_y = \widetilde{f} \times \widetilde{i}_x$ and $\widetilde{f}|_C = f \times \widetilde{i}_C$ denote the fiber products of f with the maps induced by i_x and i_C, respectively.

Let $D \subset C \times_X Y$ be the irreducble component determined by the fixed universal cover. Then we have a commutative diagram:

$$\begin{array}{ccc} \pi_1(\widetilde{D}) & \xrightarrow{\widetilde{i}_D} & \pi_1(Y) \\ {\scriptstyle \widetilde{f}|_C}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \pi_1(\widetilde{C}) & \xrightarrow{\widetilde{i}_C} & \pi_1(X) \end{array}$$

Similarly, let $y \in x \times_X Y$ be the closed point determined by the univesal cover, then we get the commutative diagram:

$$\begin{array}{ccc} \pi_1(y) & \xrightarrow{\widetilde{i}_y} & \pi_1(Y) \\ {\scriptstyle \widetilde{f}|_x}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \pi_1(x) & \xrightarrow{\widetilde{i}_x} & \pi_1(X) \end{array}$$

2.3 Covering Data

Let X be an arithmetical scheme, and recall from Notation 2.2.13.2 that a curve C of X is an integral, closed, one-dimensional subscheme of X, not necessarily regular. Recall also that $\widetilde{C}$ denotes the normalisation of a curve C.

In this section, we consider collections of open normal subgroups $\mathcal{D} = (N_C, N_x)_{C,x}$, where C and x range over the curves and closed points of X, respectively. For each C, respectively x, $N_C \triangleleft \pi_1(\widetilde{C})$ and $N_x \triangleleft \pi_1(x)$ are taken to be normal subgroups of the fundamental groups of the normalisation of C and the fundamental group of x, respectively. Since the subgroups N_C, N_x are open and normal, they correspond to finite Galois covers of $\widetilde{C}$ and x, respectively.

We notice that whenever C is any curve containing the closed point x, the fibered product gives commutative diagrams

$$\begin{array}{ccc} x \times \widetilde{C} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow{\scriptstyle i_C} \\ x & \xrightarrow{i_x} & X \end{array}$$

in which $x \times \widetilde{C}$ is finite. If x is a regular point of C, then the left vertical morphism is an isomorphism.

Now let $\widetilde{x}$ be a point of $x \times \widetilde{C}$, then we get an induced diagram of fundamental groups

$$\begin{array}{ccc} \pi_1(\widetilde{x}) & \longrightarrow & \pi_1(\widetilde{C}) \\ \downarrow & & \downarrow{\scriptstyle i_{C*}} \\ \pi_1(x) & \xrightarrow{i_{x*}} & \pi_1(X) \end{array} .$$

Definition 2.3.1. A collection $\mathcal{D} = (N_C, N_x)_{C,x}$ of open normal subgroups $N_C < \pi_1(\widetilde{C})$, $N_x < \pi_1(x)$ is a *covering datum* if the N_C and N_x satisfy the following compatibility condition:

(*) For every curve $C \subset X$, $x \in |X|$, and any $\widetilde{x}$ lying above x in $x \times \widetilde{C}$, the preimages of N_C and N_x under the the canonical morphisms above agree as subgroups of $\pi_1(\widetilde{x})$.

Definition 2.3.2. Let $f : Y \longrightarrow X$, and recall the notations for induced morphisms on the fundamental groups set in 2.2.13.1 and 2.2.13.4. If $\mathcal{D}$ is a covering datum on X, then the *pullback of* $\mathcal{D}$ *via* f is the covering datum on Y defined as follows:

1. If y is a closed point in Y, $x = f(y)$, we let $N_y = \widetilde{f}_y^{-1}(N_x)$.

2. For a curve $C \subset Y$, we let $N_C = \widetilde{f}_C^{-1}(N_{\overline{f(C)}})$, where $\overline{f(C)}$ is the closure of the image of C in X, i.e. either a point or a curve.

Definition 2.3.3. Let X be an arithmetical scheme, and let $\mathcal{D}$ be a covering datum on X.

We say that $\mathcal{D}$ is a *covering datum of index* m on X if we have $[\pi_1(\widetilde{C}) : N_C], [\pi_1(x) : N_x] \leq m$ for all points x and curves C in X, and if we have equality for at least one curve or point.

If we do not necessarily have equality, we say instead that $\mathcal{D}$ is of *index bounded by* m, or of *bounded index*.

We say that $\mathcal{D}$ is *a covering datum of cyclic index* l if the associated covers $Y_C^{\mathcal{D}} \longrightarrow C$ and $x^{\mathcal{D}} \longrightarrow x$ all have cyclic Galois groups $\mathbb{Z}/s\mathbb{Z}$ where $s|l$.

This terminology is slightly different from that in [5], for reasons explained in Section 4.3.

Definition 2.3.4. Let X be an arithmetical scheme, and let $\mathcal{D}$,$\mathcal{D}'$ be two covering data on X. Let $C \subset X$ denote a curve, and let $x \in |X|$ denote a closed point of X. Let $f_C^{\mathcal{D}} : Y_C^{\mathcal{D}} \longrightarrow C$, $f_C^{\mathcal{D}'} : Y_C^{\mathcal{D}'} \longrightarrow C$ denote the covers of C defined by $\mathcal{D}$ and $\mathcal{D}'$. Also let $f_x^{\mathcal{D}} : x^{\mathcal{D}} \longrightarrow$, $f_C^{\mathcal{D}'} : x^{\mathcal{D}'} \longrightarrow x$ denote the cover of x defined by $\mathcal{D}$, respectively $\mathcal{D}'$.

We say that $\mathcal{D}$ is a *subdatum* of $\mathcal{D}'$ if $f_C^{\mathcal{D}}$ is a subcover of $f_C^{\mathcal{D}'}$ for all curves $C \subset X$, and if $f_x^{\mathcal{D}}$ is a subcover of $f_x^{\mathcal{D}'}$ for all $x \in |X|$.

Definition 2.3.5. A covering datum $\mathcal{D}$ is called *trivial* if $N_x = \pi_1(x)$ for all closed points $x \in |X|$, and $N(C) = \pi_1(\widetilde{C})$ for all curves $C \subset X$. We say that $\mathcal{D}$ is *weakly trivial* if $N_x = \pi_1(x)$ for all closed points $x \in |X|$.

Definition 2.3.6. Let $\mathcal{D}$ be a covering datum on a scheme X. Given an étale cover $Y = Y_N \longrightarrow X$ corresponding to the open subgroup $N \subset \pi_1(X)$, we say that:

1. f *trivialises* $\mathcal{D}$ if the pullback of $\mathcal{D}$ to Y is the trivial covering datum.

2. f *weakly trivialises* $\mathcal{D}$ if the pullback of $\mathcal{D}$ to Y is weakly trivial.

3. f *weakly realises* the covering datum if $N(x) = N_x$ for all closed points $x \in |X|$.

4. f *realises* $\mathcal{D}$ if $N(x) = N_x$ for all closed points $x \in |X|$ and $N(C) = N_C$ for all cuvers $C \subset X$.

We call f a (weak) trivialisation, respectively realisation, of $\mathcal{D}$.

5. If the pullback $i^*(\mathcal{D})$ to an open subset $U \overset{i}{\hookrightarrow} X$ has one of the above properties, we say that $\mathcal{D}$ has that property *over* U.

Let $f : X_N \longrightarrow X$ be a Galois étale cover corresponding to the open normal subgroup $N \triangleleft \pi_1(X)$, and recall Notations 2.2.13.1 and 2.2.13.4. For every curve $C \subset X$ with normalisation $\widetilde{C}$ and every closed point $x \in |X|$, we define the pullbacks $N(C) := \widetilde{i}_C^{-1}(N)$, $N(x) = \widetilde{i}_x^{-1}(N)$. They correspond to the étale covers $f_C : X_N \times \widetilde{C} \longrightarrow \widetilde{C}$ and $f_x : X_N \times x \longrightarrow x$, respectively. Then for any $\widetilde{x} \in \widetilde{C} \times x$, the diagram

$$\begin{array}{ccc} \pi_1(\widetilde{x}) & \longrightarrow & \pi_1(\widetilde{C}) \\ \downarrow & & \downarrow \\ \pi_1(x) & \longrightarrow & \pi_1(X) \end{array}$$

commutes since it is induced from the commutative diagram

$$\begin{array}{ccc} \widetilde{x} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow \\ x & \longrightarrow & X \end{array}$$

In particular, the pullbacks of $N(C)$ and $N(x)$ must agree in $\pi_1(\widetilde{x})$ for any x and C, i.e. the datum $(N_C, N_x)_{C,x}$ is a covering datum. Thus, the Galois étale cover $f : X_N \longrightarrow X$ induces a covering datum of X, which we shall denote by $\mathcal{D}^N$.

Notation 2.3.7. Given a covering datum $\mathcal{D}$ on X, we let $Y_C^{\mathcal{D}} \longrightarrow \widetilde{C}$, $y \longrightarrow x$ be the étale covers of connected schemes corresponding to $N_C \lhd \pi_1(\widetilde{C})$ and $N_x \lhd \pi_1(x)$.

Then given a morphism $Y \longrightarrow X$, we have diagrams

$$\begin{array}{ccc} Y \times \widetilde{C} \times Y_C^{\mathcal{D}} & \longrightarrow & Y \times \widetilde{C} \\ \downarrow & & \downarrow \\ Y_C^{\mathcal{D}} & \longrightarrow & \widetilde{C} \end{array} \qquad \begin{array}{ccc} Y \times x \times y & \longrightarrow & Y \times x \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \end{array}$$

Notation 2.3.8. Now let Z, Z' be the connected components of $Y \times \widetilde{C}$ and $Y \times \widetilde{C} \times Y_C^{\mathcal{D}}$, respectively, and let z, z' be points in the finite sets $Y \times x$ and $Y \times x \times y$ containing the relevant geometric base points. Then we get the following diagrams of connected schemes:

$$\begin{array}{ccc} Z' & \xrightarrow{q_C} & Z \\ \downarrow{\scriptstyle p_C} & & \downarrow{\scriptstyle f_C} \\ Y_C^{\mathcal{D}} & \xrightarrow{g_C} & \widetilde{C} \end{array} \qquad\qquad \begin{array}{ccc} z' & \xrightarrow{q_x} & z \\ \downarrow{\scriptstyle p_x} & & \downarrow{\scriptstyle f_x} \\ y & \xrightarrow{g_x} & x \end{array}$$

Following the remarks on the correspondence between connected pro-étale covers and closed subgroups of the fundamental group of a scheme X, we note that the covers $Z' \longrightarrow \widetilde{C}$ and $z' \longrightarrow x$ correspond to the subgroups $N_C \cap N(C)$ and $N(x) \cap N_x$ of $\pi_1(\widetilde{C})$ and $\pi_1(x)$, respectively. We also note that we can identify $N(C)$ with $\widetilde{f}_C(\pi_1(Z))$, N_C with $\widetilde{g}_C(\pi_1(Y_C^{\mathcal{D}}))$ and similarly for closed points.

It is now possible to express the properties 2.3.6.1 through 2.3.6.4 of f in relation to $\mathcal{D}$ as inclusions/equalities of subgroups in $\pi_1(\widetilde{C})$ and $\pi_1(x)$, but also as inclusions/equalities of subgroups in $\pi_1(Y_C)$, $\pi_1(Y_C^{\mathcal{D}})$ and $\pi_1(y)$, $\pi_1(Y \times x)$. Full characterisations are given as follows:

Corollary 2.3.9. *Let $\mathcal{D}$ be a covering datum on a scheme X.*

1. *If $f : Y \longrightarrow X$ is a pro-étale cover, then TFAE:*

   (a) *f trivialises $\mathcal{D}$.*

   (b) *$N_x \supseteq N(x)$ in $\pi_1(x)$ for all closed points $x \in |X|$, and $N_C \supseteq N(C)$ in $\pi_1(\widetilde{C})$ for all curves $C \subset X$.*

(c) *We have* $\widetilde{f}_x^{-1}(N_x) = \pi_1(Y \times x)$ *for all closed points* $x \in |X|$ *and* $\widetilde{f}_C^{-1}(N_C) = \pi_1(Y \times \widetilde{C})$ *for all curves* $C \subset X$.

(d) *The covers* p_C *and* p_x *are trivial for any closed points* $x \in |X|$ *and any curve* $C \subset X$.

2. *Similarly, TFAE:*

   (a) f *is a locally trivial cover (cf. Definition 2.1.13).*

   (b) f *weakly trivialises* $\mathcal{D}$.

   (c) $N_x \supset N(x)$ *for all closed points* $x \in |X|$.

   (d) *We have* $\widetilde{f}_x^{-1}(N_x) = \pi_1(y)$ *for all closed points* $x \in |X|$.

   (e) *The covers* p_x *are trivial for all closed points* $x \in |X|$.

3. *For a pro-étale cover* $f : Y \longrightarrow X$, *TFAE:*

   (a) f *realises* $\mathcal{D}$.

   (b) *We have* $\widetilde{f}_x^{-1}(N_x) = \pi_1(Y \times x)$, $\widetilde{g}_x^{-1}(N(x)) = \pi_1(y)$ *for all closed points* $x \in |X|$ *and* $\widetilde{f}_C^{-1}(N_C) = \pi_1(Y \times \widetilde{C})$, $\widetilde{g}_C^{-1}(N(C)) = \pi_1(Y_C^{\mathcal{D}})$ *for all curves* $C \subset X$.

   (c) *The covers* p_C *and* q_C *are trivial for any curve* $C \subset X$ *and the covers* p_x, q_x *are trivial for any closed points* $x \in |X|$.

4. *Analogously to* 3), *TFAE:*

(a) f weakly realises $\mathcal{D}$.

(b) We have $\widetilde{f}_x^{-1}(N_x) = \pi_1(Y \times x)$, $\widetilde{g}_x^{-1}(N(x)) = \pi_1(y)$ for all closed points $x \in |X|$.

(c) The covers p_x, q_x are trivial for any closed points $x \in |X|$.

Lemma 2.3.10. *Let X be an arithmetical scheme, and let $\mathcal{D}$ be a covering datum on X. An étale cover $Y \longrightarrow X$ which weakly trivialises (weakly realises) $\mathcal{D}$ trivialises (realises) the covering datum.*

Proof. We use the notations introduced in 2.3.8, and make repeated use of the equivalence of conditions listed in Cor 2.3.9.2 and 2.3.9.3: f weakly trivialises $\mathcal{D}$ if and only if the morphisms p_x are trivial covers of arithmetical schemes. So now consider the covers p_C. If $\widetilde{x}$ is a point of $\widetilde{C} \times x$, then the morphism $p_{\widetilde{x}}$ induced by base changing from x to $\widetilde{x}$ is again trivial. Thus, p_C is locally trivial. By Lemma 2.1.12, a locally trivial finite cover of arithmetical schemes is trivial, so p_C is a trivial cover for any curve $C \subset X$, as claimed.

Similarly, if f weakly realises $\mathcal{D}$, then we know that all the morphisms p_x and q_x are trivial for all closed points. As above, this implies that p_C and q_C are locally trivial and thus trivial. □

Definition 2.3.11. Let X be an arithmetical scheme, and let $\mathcal{D}$ be a covering datum on X. Then $\mathcal{D}$ is called *effective* if it is realised by a pro-étale cover $Y \longrightarrow X$. If this is the case, the associated cover Y_N is called the *realisation* of the covering datum

D. If the realisation of $\mathcal{D}$ is a finite cover, we say that $\mathcal{D}$ has a *finite realisation.*

Remark 2.3.12. Note that if $\mathcal{D}$ has a finite realisation $f : X_N \longrightarrow X$, then there exists a curve $C \subset X$ such that the canonical morphism $i_C : \widetilde{C} \longrightarrow X$ (cf. Definition 2.2.13) and the induced morphism $\widetilde{i}_C : \pi_1(\widetilde{C}) \longrightarrow \pi_1(X)$ induces an isomorphism

$$\pi_1(\widetilde{C})/N(C) \simeq \pi_1(X)/N\,.$$

In particular, the degree of the realisation f is equal to the index of the covering datum.

Proof. X_N corresponds to the open normal subgroup $N \triangleleft \pi_1(X)$. We define $N(C) := \widetilde{i}_C^{-1}(N)$ for any curve $C \subset X$, and likewise $N(x) := \widetilde{i}_x^{-1}(N)$ for closed points x. Then we have natural inclusions

$$\pi_1(\widetilde{C})/N(C) \hookrightarrow \pi_1(X)/N \text{ and } \pi_1(\widetilde{x})/N(x) \hookrightarrow \pi_1(X)/N$$

for all x and C. As $N(x) = N_x$ and $N(C) = N_C$, this implies that the index of $\mathcal{D}$ is bounded by $\deg(f)$. Now let C be an irreducible curve as guaranteed by Lemma 2.1.14, then $f_C : Y \times \widetilde{C} \longrightarrow \widetilde{C}$ has the same Galois group as f. Therefore

$$\pi_1(\widetilde{C})/N(C) \hookrightarrow \pi_1(X)/N$$

is an isomorphism, and we have $\deg(f) = [\pi_1(\widetilde{C}) : N_C]$. Thus, $\mathcal{D}$ must have index exactly equal to $\deg(f)$, as claimed. □

Proposition 2.3.13. *Let X be a normal, arithmetical scheme and $\mathcal{D} = (N_C, N_x)_{(C,x)}$ a covering datum on X.*

1. *Then $\mathcal{D}$ has at most one finite realisation.*

2. *Let Y_1, Y_2 be two étale covers of X weakly realising $\mathcal{D}$ over an open subset U. Then $Y_1 = Y_2$.*

3. *If there exists an open subscheme $U \overset{i}{\hookrightarrow} X$ such that the pullback $i^*(\mathcal{D})$ can be weakly realised by an étale cover $Y \longrightarrow U$, then $\mathcal{D}$ is effective with finite realisation. The realisation of $\mathcal{D}$ is given by Y', the normalisation of X in $K(Y)$.*

Proof. (cf. [5, Lemma 3.1])

1. Let X_{N_1}, X_{N_2} be two realisations of D, and let $N_i \triangleleft$, $i = 1, 2$ be the corresponding open normal subgroups. Then $N_1 \cap N_2$ correspons to a completely split cover of Y_1, so by Lemma 2.1.12, it is an isomorphim. It follows that $N_2 \subset N_1$, and thus $N_2 = N_1$ by symmetry.

2. Let N_1, N_2 be the open subgroups of $\pi_1(X)$ associated to the two realisations Y_1, Y_2, and let $N_i(x) := \widetilde{i}_x^{-1}(N_i)$ denote their pullbacks to $\pi_1(x)$ for any closed point x. Let $Y = Y_1 \times Y_2$. If $N_1(x) = N_2(x)$ for all $x \in |U|$, then $U \times Y \longrightarrow U \times Y_1$ is completely split, and thus an isomorphism by Lemma 2.1.12. For V an irreducible component of $Y_1 \times U$, this means that $V \times Y_1 \longrightarrow V$ is an isomorphism of connected schemes. In particular, the Galois closure of this cover is just $V \times Y_1$ itself, and is connected. Now if $Y \to Y_1$ were not an isomorphism, then the Galois closure $Y' \to Y_1$ would

have nontrivial Galois group as well. Since $V \times Y_1$ is connected, the Galois groups of the two vertical covers in

$$\begin{array}{ccc} V \times Y' & \hookrightarrow & Y' \\ \downarrow & & \downarrow \\ V & \hookrightarrow & Y_1 \end{array}$$

are isomorphic. Thus this would imply that $V \times Y_1$ has nontrivial Galois groups as well, contrary to assumption.

3. If Y' is the normalisation of X in Y, $Y' \longrightarrow X$ is finite, and we have $Y = Y' \times U$. As $\mathcal{D}$ is a covering datum on all of X, all covers that $i^*(\mathcal{D})$ induces on curves $C' = C \cap U$ of U extend to étale covers Y_C of the full curve C in X. By Proposition 2.1.15, $Y' \longrightarrow X$ is an étale cover; we let $N \lhd \pi_1(X)$ be the corresponding open normal subgroup. Recall the notations set in 2.2.13.5), and let $N(C) := \widetilde{i}_{\widetilde{C}}^{-1}(N)$, $N(x) := \widetilde{i}_x^{-1}(N)$ denote the pullback to $\pi_1(\widetilde{C})$, respectively $\pi_1(x)$.

   Now let C be a curve on X with $C \cap U \neq \emptyset$. Then the preimages $\widetilde{i}_{\widetilde{x}}^{-1}(N(C))$ and $\widetilde{i}_{\widetilde{x}}^{-1}(N_C)$ in $\pi_1(\widetilde{x})$ agree for every point $\widetilde{x}$ of $\widetilde{C}$ lying over U. Applying the argument in 1) to the scheme $\widetilde{C}$, we see that the normal subgroups $N(C)$ and N_C of $\pi_1(\widetilde{C})$ coincide, and so $N(x) = N_x$ for every regular point x of C. By Lemma 2.1.14, every point is contained in the regular locus of a curve meeting U, so we get $N(x) = N_x$ for all closed points $x \in |X|$. By Lemma 2.3.10, we conclude that $Y' = Y_N$ is a realisation of $\mathcal{D}$.

□

Remark 2.3.14. In general, a realisation of a covering datum is automatically finite if the covering datum is of bounded index and tame. It is always étale by Lemma 2.1.15. Since the p-part of the fundamental group is not finitely generated, effective covering data which are note tame have realisations which are not necessarily finite étale covers, but only pro-étale.

Theorem 2.3.15. *Let X be a regular arithmetical scheme, and let $\mathcal{D} = (N_C, N_x)_{C,x}$ be a covering datum on X that is trivialised by a finite cover $f : Y \longrightarrow X$. Then $\mathcal{D}$ is effective with a finite realisation.*

Corollary 2.3.16. *Let X be a regular arithmetical scheme, and let $\mathcal{D} = (N_C, N_x)_{C,x}$ be a covering datum on X. If there exists an open subscheme $U \overset{i}{\hookrightarrow} X$ such that the pullback $i^*(\mathcal{D})$ can be weakly trivialised by a finite cover $Y \longrightarrow U$, then $\mathcal{D}$ is effective with a finite realisation.*

Proof. This follows directly from Theorem 2.3.15 and Lemma 2.3.13.2. □

Proof of Theorem 2.3.15. We first show that it suffices to prove the Theorem under the additional assumption that f is étale.

Claim 2.3.17. If the covering datum $\mathcal{D}$ is trivialised by a finite cover $f : Y \longrightarrow X$, then there also exists a finite étale cover $f' : Y' \longrightarrow X$ trivialising $\mathcal{D}$.

Proof. Let $X' \subset X$ be an open dense subset such that $f|_{X'} : Y'' = X' \times_X Y \longrightarrow X'$ is étale. (Such an X' exists by the purity of the branch locus, cf. [9, Section 8.3, Ex.

2.15].) Then the restriction $\mathcal{D}|_{X'}$ of $\mathcal{D}$ to X' is trivialised by $f|_{X'}$, and thus effective by the results of the previous paragraphs. In particular, there exists a subgroup $N < \pi_1(X')$ giving a weak realisation of $\mathcal{D}$ over the open subscheme X', which must be a full realisation by Proposition 2.3.13. Since the realisation of a covering datum $\mathcal{D}$ trivialises $\mathcal{D}$ by definition, taking Y'' to be the cover corresponding to N proves the claim. □

Returning to the proof of Theorem 2.3.15, by Proposition 2.3.13, it suffices to find a subgroup N of $\pi_1(X)$ which gives a weak realisation over a dense open subset U, i.e. an open normal subgroup N such that $\widetilde{i}_x^{-1}(N) = N_x$ for all $x \in U$. In particular, we can replace X by any open dense subset. Using Lemmas 2.1.10 and 2.3.13, we may thus assume without loss of generality that X is quasi-projective, and that there exists a smooth morphism $X \longrightarrow S$ to a curve S. Replacing $Y \longrightarrow X$ by its Galois hull, we may also assume that $Y \longrightarrow X$ is Galois, say with group G.

By Lemma 2.1.14, there exists a curve $C \subset X$ such that $D = C \times Y$ is irreducible and such that $f_C : D \longrightarrow C$ is again a Galois étale cover with group G. By the Chebotarev Density Theorem 2.1.11, there are infinitely many n-tuples $(x_1, \ldots, x_n)$ of points in C such that G is the union of the conjugacy classes $[Frob_{x_i}]$. Since the regular locus of C is of codimension one, it is finite, so by replacing those x_i which are not contained in the regular locus of C, we may assume that $x_i \in C^{reg}$ for all i.

Since Y trivialises $\mathcal{D}$, we have $\widetilde{f_C}(\pi_1(\widetilde{D})) \leq N_C$ as subgroups of $\pi_1(\widetilde{C})$. Let

$N = \widetilde{i}_C(N_C)$ be the image of N_C in $\pi_1(X)$, then we shall show that N gives a weak realization of $\mathcal{D}$ over an open subset of X. More precisely, if we denote $N(x) = \widetilde{i}_x^{-1}(N)$ for any $x \in |X|$, then we show that $N(x) = N_x$ for all x which $g : X \to \mathbb{Z}$ maps to points which are distinct from the images of $S = \{x_1, \ldots, x_n\}$ under g. The set U of such x is open in X since the set of images of S is closed, and g is continuous in the Zariski topology. For $x \in U$, the Approximation Lemma 2.1.14 yields a curve $C' \subset X$ containing $x, x_1, \ldots x_n$ as regular points, and such that $C' \times Y$ is irreducible.

Lemma 2.3.18. *Let C be a curve in X such that $C \times Y$ is irreducible, let $N = \widetilde{i}_C(N_C)$ be the image of N_C in $\pi_1(X)$, and $N(z) = \widetilde{i}_z(N)$ the pullback to $\pi_1(z)$. Then $N(z) = N_z$ for all regular points z of C'.*

Proof. We first show that if we define $N(C) := \widetilde{i}_C^{-1}(N)$, then $N(C) = N_C$. Indeed, $\widetilde{i}_C$ is easily seen to be injective by applying the second criterion of [19, Corollary 5.5.8] to the canonical morphism $j = \widetilde{i}_C : \widetilde{C} \longrightarrow X$:

First we consider the case where C is normal, i.e. $\widetilde{C}$ is already contained in X, and also assume that X is affine. If j is a closed immersion of affine schemes corresponding to a surjective morphism $B \twoheadrightarrow A \cong B/I$, and $D \longrightarrow Spec\, A$ is a finite étale cover, then D is also affine, say $D = Spec\, E$. We thus have to show that if $E = A[f_1, \ldots, f_n] = A[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$ is a finite étale algebra over A, where the g_i are polynomials with coefficients in A, then E is the tensor product $F \otimes B$ of some finite étale algebra F. But taking $h_i \in B[x_1, \ldots, x_n]$ to be any lifts of the g_i, then we can take $F = B[x_1, \ldots, x_n]/(h_1, \ldots, h_n)$. Clearly, F is finite over B, and it is flat since $Spec\, F \longrightarrow Spec\, B$ is finite and surjective [9, Remark 4.3.11]. Last, it can easily be seen that if F were not étale over B, then E would have to be ramified over B/I as well (e.g. by applying the criterion of [9, Example 4.3.21]).

For the not necessarily affine case of a regular arithmetical scheme X, take $U \subset X$ to be a dense affine open subset. Then the previous argument gives a finite étale cover V of U such that we have a commutative diagram

$$\begin{array}{ccccc} D_U = C \cap U \times V & \longrightarrow & V & & \\ \downarrow & & \downarrow & & \\ C \cap U & \longrightarrow & U & \longrightarrow & X \end{array}$$

Taking Y to be the normalisation of X in the function field $K(V)$ of V over $K(U) = K(X)$, we have that $V = Y \times U$. Note that $D \longrightarrow C$ is normal since C is locally Noetherian, as well as assumed to be normal, and the cover is finite étale. Thus D is equal to the normalisation of C in $k(D_U)$. Now compare this with $D' = C \times Y$, a normal cover of C since the étaleness of $Y \longrightarrow X$ is preserved under base change to C. D' is birational to D_U and to D', which implies $D = D'$; then, the claim follows.

Now to the case where $C \subset X$ may be non-regular, so that $\widetilde{C}$ is not necessarily a subscheme of X. We have $i_C : \widetilde{C} \longrightarrow C \longrightarrow X$, and are given a cover $\widetilde{D}$ of $\widetilde{C}$. Since $\widetilde{C}$ and C are birational, they have the same function field. So let D be the normalisation of C in $k(D)$, and apply the first part to $D \longrightarrow C$. Then $D = C \times Y$ for some finite étale cover Y, and since $\widetilde{D} = D \times \widetilde{C}$, the claim follows by associativity of the fibre product.

In conclusion, we have that $N(C) = \widetilde{i}_C^{-1}(\widetilde{i}_C(N_C)) = N_C$. Now if z is a regular point of C, then there is a unique point $\pi_1(\widetilde{z})$ of $\widetilde{C}$ lying above $z \in C$, and we have a natural isomorphism $\pi_1(\widetilde{z}) \cong \pi_1(z)$ fitting into the following diagram:

$$\begin{array}{ccc} \pi_1(\widetilde{z}) & \longrightarrow & \pi_1(\widetilde{C}) \\ \cong \downarrow & \nearrow & \\ \pi_1(z) & & \end{array}$$

A diagram chase comparing $N(z)$ to N_z now easily proves the claim. □

Returning to the proof of Theorem 2.3.15, we have $N(x_i) = N_{x_i}$ for all i. Let $N(C') = \widetilde{i}_{C'}^{-1}(N)$ denote the preimage under the natural map $\widetilde{i}_{C'} : \pi_1(\widetilde{C'}) \to \pi_1(X)$, M' its image in G, and M the image of N in G. If $\widetilde{x}_i$ is the unique point of $\widetilde{C}$ lying above x_i, we have a diagram:

$$\begin{array}{ccccc} \pi_1(\widetilde{C}) & \longrightarrow & G & \longleftarrow & \pi_1(\widetilde{C'}) \\ & \nwarrow & \uparrow & \nearrow & \\ & & \pi_1(\widetilde{x}_i) & & \end{array} \qquad \begin{array}{ccccc} N_C & \longrightarrow & M, M' & \longleftarrow & N_{C'} \\ & \nwarrow & \uparrow & \nearrow & \\ & & N_{x_i} & & \end{array}$$

Noting that the image of N_{x_i} in G must be the same for both triangles, we have, in particular, that $Frob_{x_i} \in M$ iff $Frob_{x_i} \in M'$ for all i. Since the x_i were picked such that the $\{Frob_{x_i}\}$ are all the conjugacy classes of G, this implies that $M = M'$, and the diagram commutes. Since C' was irreducible in $Y \longrightarrow X$, we have $N_x = N'(x)$ by Lemma 2.3.18. Now since $N(x)$ and $N'(x)$ both map to $M = M'$ when composing the canonical morphism $\widetilde{i}_x : \pi_1 x \longrightarrow \pi_1(X)$ with the projection

$\pi_1(X) \longrightarrow G$, they must be equal. Thus we have $N_x = N(x)$ as claimed. This finishes the proof of the theorem. $\square$

Chapter 3

The Wiesend Idèle Class Group

In this chapter, we let X be an arithmetical scheme, and define the Wiesend idèle class group for both the flat and variety case (cf. Definition 2.1.2). We define the reciprocity homomorphism and establish some functorial properties.

3.1 Definitions and some Functiorial Properties

Let X be an arithmetical scheme. As before, we let $|X|$ denote the set of closed points of X and let $C \subset X$ be a curve. If X is in the flat case, then we have two possibilities: If the image of the structural morphism $C \longrightarrow Spec\,\mathbb{Z}$ is a point $p \in Spec\,\mathbb{Z}$ then C is called *vertical*. Otherwise, $C \longrightarrow Spec\,\mathbb{Z}$ has dense image. Then the regular compactification $P(C)$ of C is isomorphic to some order of $Spec\,R$, where $R \subset O_K$ is an order inside the ring of integers of a number field K at some element f. If C is regular, then the regular compactification $P(C)$ of C is isomorphic to $Spec\, O_K$.

Definition 3.1.1. Let $C \subset X$ be a curve inside an arithmetical scheme X. If the structural morphism of C factors through $Spec\ \mathbb{F}_p$ for some prime p, C is called a *vertical* curve. Otherwise, C is called *horizontal*.

Note that if X is a variety, then all curves are vertical. For a vertical curve, we denote by C_∞ the finite set of (normalized) discrete valuations of $K(C)$ without a center on $\widetilde{C}$. If C is horizontal, we let C_∞ be the finite set of discrete valuation of $K(C)$ corresponding to the points without a center on $\widetilde{C}$ together with the finite set of archimedean places of $K(C)$.

Now recall (e.g. from [9, Remark 8.3.19]) the following:

Fact 3.1.2. Let C be any integral curve. There is a one-to-one correspondence between the set of discrete valuations on its function $K(C)$ having center on C and the set of closed points in the normalisation $\widetilde{C}$ of C.

It is given by associating to each discrete valuation its unique center; its inverse is given by associating to a closed point x the valuation ν_x it defines.

For a scheme X, let the idèle group (after Wiesend) be the abelian topological group group defined by

$$\mathcal{J}_X = \bigoplus_{x \in |X|} \mathbb{Z}.x \oplus \bigoplus_{C \subset X} \bigoplus_{\nu \in C_\infty} K(C)^\times_\nu$$

with the direct sum topology. Note that this is a countable direct sum of locally compact abelian groups (cf. [21, Section 7, Remarks after 1st Definition]). A typical element is of the form $((n_x.x)_x, (t_{C,\nu})_{(C,\nu)})$, where the indices x, C, ν run over the set of closed points, of curves contained in X and valuations in C_∞, respectively, and where at most finitely many components are non-zero.

Remark 3.1.3. If X is an arithmetical scheme with idèle group $\mathcal{J}_X$, then $\mathcal{J}_X$ is Hausdorff but not necessarily locally compact. If $C \subset X$ is a horizontal curve on X, let C_∞^{arch} denote the set of archimedean places. If $C \subset X$ is vertical, set $C_\infty^{arch} = \emptyset$. Then the subgroup

$$\mathcal{J}_X^1 = \bigoplus_{C \subset X} \bigoplus_{\nu \in C_\infty^{arch}} K(C)_\nu^\times$$

is the connected componenet of the identity.

Definition 3.1.4. Let $f : X \longrightarrow Y$ be a morphism of arithmetical schemes. We define the *induced morphism* $f_{\mathcal{J}} : \mathcal{J}_X \longrightarrow \mathcal{J}_Y$ *on ideal class groups* as follows:

1) The image $y = f(x)$ of a closed point $x \in X$ is a closed point, and the extension of residue fields $k(x)/k(y)$ is finite. So let $f_{\mathcal{J}}(1.x) = [k(x) : k(y)] \,.\, y$.

2) For $C \subset X$ a curve, we either have $f(C) = y$ a closed point, or $f(C)$ is dense in a curve $D \subset Y$.

2a) In the first case, since closed points of integral varieties of finite type over $\mathbb{Z}$ have finite residue field, C is a variety over some $\mathbb{F}_{p^n} = k(y)$. For ν a valuation on $K(C)$, let $k(\nu) = O_\nu/m_\nu$ denote the residue field, a finite extension of $k(y)$. Now f gives rise to an embedding $k(y) \hookrightarrow k(\nu)$, so for $t \in K(C)_\nu^\times$, we can define the image under $f_{\mathcal{J}}$ as $f_{\mathcal{J}}(t) = \nu(t)[k(\nu) : k(y)].y$.

2b) In the second case: Let D be the closure of $f(C)$ in Y, then f gives rise to the finite extension $K(C)/K(D)$, and ν restricts to a valuation ω on $K(D)$. If ω does not have a center in $\widetilde{D}$, then $\mathcal{J}_Y$ has a factor $K(D)_\omega^\times$, so and we can define $f_\mathcal{J} : K(C)_\nu \longrightarrow K(D)_\omega$ by the norm $N_{K(C)_\nu/K(D)_\omega}$.

2c) Lastly, if ω has a center z on $\widetilde{D}$, then $\mathcal{J}_Y$ does not contain the summand $K(D)_\omega^\times$, but instead the discrete summand $\mathbb{Z}.z$, so we define $f_\mathcal{J}(t) = \nu(t)\,[k(\nu) : k(z)]\,.\,z$ for all $t \in K(C)_\nu^\times$.

Finally, define $f_\mathcal{J} : \mathcal{J}_X \longrightarrow \mathcal{J}_Y$ by setting the image of $f_\mathcal{J}$ equal to the sum of the images of the components, as they were defined above. Then $\mathcal{J}$ is a continuous homomorphism by definition.

Remark 3.1.5. The identity morphism $X \longrightarrow X$ induces the identity $\mathcal{J}_X \longrightarrow \mathcal{J}_X$. Moreover, the composition of two induced morphisms is the induced morphism of the composition. Thus we get a functor $\mathcal{J}$ from the category of arithmetical schemes over $\mathbb{Z}$ to the category of Wiesend idèle groups with induced morphism $f_\mathcal{J}$.

We shall use the functorial properties of the induced morphisms repeatedly. Among other things, they imply that if the diagram

$$\begin{array}{ccc} C & \xrightarrow{f^2} & D \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle f^3} \\ X & \xrightarrow[f^4]{} & Y \end{array}$$

of schemes over S is commutative, then so is the diagram

$$\begin{array}{ccc} \mathcal{J}_C & \xrightarrow{f^2_{\mathcal{J}}} & \mathcal{J}_D \\ {\scriptstyle f^1_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle f^3_{\mathcal{J}}} \\ \mathcal{J}_X & \xrightarrow[f^4_{\mathcal{J}}]{} & \mathcal{J}_Y \end{array}$$

of induced morphisms on the idèle groups.

For a curve $C \subset X$ with normalisaton $\widetilde{C}$, the composition i_C of the canonical maps

$$\widetilde{C} \longrightarrow C \longrightarrow X$$

thus induces a morphism

$$(i_C)_{\mathcal{J}} : \mathcal{J}_{\widetilde{C}} \longrightarrow \mathcal{J}_X \tag{3.1.1}$$

There also exists a natural inclusion map $j_C : K(C)^{\times} \longrightarrow J_{\widetilde{C}}$ given by componentwise inclusion:

$$t \mapsto ((\nu_x(t).\, x), (i_\nu(t))_\nu)\,,$$

where ν_x is the discrete valuation associated to the closed point $x \in C$, and $i_\nu : K(C) \longrightarrow K(C)_\nu$ is the inclusion of the function field in its completion at ν. Composing j_C with i_{C*}, and taking the direct sum over all curves $C \subset X$, we get a map

$$j = \Sigma_C\,(i_{C*} \circ j_C) : \sum_{C \subset X} K(C)^{\times} \hookrightarrow \mathcal{J}_X. \tag{3.1.2}$$

Definition 3.1.6. Let X be an arithmetical scheme of Kronecker dimension $dim(X) \geq 2$, or a regular curve. Then the *Wiesend idèle class group* of X is defined to be the quotient of $\mathcal{J}_X$ by the image of j:

$$\mathcal{C}_X = \mathcal{J}_X / \, im(j) = \left(\bigoplus_{x \in |X|} \mathbb{Z}.x \oplus \bigoplus_{C \subset X} \bigoplus_{\nu \in C_\infty} K(C)_\nu^\times \right) / \sum_{C \subset X} im(i_{C*} \circ j_C)$$

together with the quotient topology. For an arithmetical scheme of dimension zero, i.e. a point x, we define analogously $\mathcal{C}_x = \mathcal{J}_x = \mathbb{Z}.x$.

Remark 3.1.7. For an arithmetical scheme X with class group $\mathcal{C}_X$, let $\mathcal{D}_X$ denote the connected component of the identity. Then $\mathcal{D}_X$ is equal to the closure of the image of the subgroup $\mathcal{J}_X^1$ in $\mathcal{C}_X$ defined in 3.1.3 (also see [21, Section 7]).

Proposition 3.1.8. *For a morphism $f : X \longrightarrow Y$, the induced map $f_\mathcal{J}$ on the ideal class groups descends to a continuous homomorphism $f_* : \mathcal{C}_X \longrightarrow \mathcal{C}_Y$ on the class groups.*

Proof. To show this, we have to prove that $im(j)$ is contained in the kernel of the canonical map $\overline{f}_\mathcal{J} : \mathcal{J}_X \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{C}_Y$. Since $im(j) = \Sigma_{C \subset X} \, im(i_{C*} \circ j_C)$, this amounts to showing that each $(i_{C*} \circ j_C)(K(C)^\times)$ is is contained in the kernel of $\overline{f}_\mathcal{J}$.

There are several cases to consider:

1) $f(C) = y$ is a closed point of Y. Then for $t \in K(C)^\times$, we have

$$\begin{aligned} (f_\mathcal{J}|_C \circ j_C)(t) &= f_\mathcal{J}|_C(((i_\nu(t))_{\nu \in C_\infty}, (\nu_x(t))_{x \in \widetilde{C}}) && (3.1.3) \\ &= \sum_{\nu \in C_\infty} \nu(t)[k(\nu) : k(y)].y + \sum_{x \in \widetilde{C}} \nu_x(t)[k(x) : k(y)].y \\ &= (\sum_{\nu \in V_{K(C)}} \nu(t)[k(\nu) : k(y)]).y \\ &= \; (deg(div(t)).y = 0 && (3.1.4) \end{aligned}$$

where $V_{K(C)}$ denotes the set of places on $K(C)$. We obtain a canonical diagram

$$\begin{array}{ccccc} K(C)^\times & \xrightarrow{j_C} & \mathcal{J}_{\widetilde{C}} & \xrightarrow{i_{C*}} & \mathcal{J}_X \\ \downarrow & & \downarrow{\scriptstyle (f|_C)_\mathcal{J}} & & \downarrow{\scriptstyle f_\mathcal{J}} \\ 0 & \longrightarrow & \mathbb{Z}.y & \xhookrightarrow{can.} & \mathcal{J}_Y \ , \end{array}$$

where the first square commutes by the above computation, and the second square commutes as the middle vertical map is just $f_\mathcal{J} \circ i_{C*}$ restricted to $\mathcal{J}_{\widetilde{C}}$. Thus, the outer rectangle also commutes, and the image of $K(C)^\times$ is in the kernel of $\overline{f}_\mathcal{J}$, as required.

2) $f(C)$ is dense inside a curve D of Y. There exists a unique $\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{D}$ induced by f such that the following diagrams commute:

$$\begin{array}{ccccc} \widetilde{C} & \longrightarrow & C & \longrightarrow & X \\ {\scriptstyle \exists! h}\,\vdots & & \downarrow{\scriptstyle f|_C} & & \downarrow{\scriptstyle f} \\ \widetilde{D} & \longrightarrow & D & \longrightarrow & Y. \end{array}$$

Since the maps $j_C : K(C)^\times \longrightarrow \mathcal{J}_C$ and $j_D : K(D)^\times \longrightarrow \mathcal{J}_D$ factor through $\mathcal{J}_{\widetilde{C}} \longrightarrow \mathcal{J}_C$ and $\mathcal{J}_{\widetilde{D}} \longrightarrow \mathcal{J}_D$, respectively, we get diagrams

$$\begin{array}{ccccccc} K(C)^\times & \xrightarrow{j_C} & \mathcal{J}_{\widetilde{C}} & \longrightarrow & \mathcal{J}_C & \longrightarrow & \mathcal{J}_X \\ {\scriptstyle N_{K(C)/K(D)}}\downarrow & & \downarrow{\scriptstyle h_\mathcal{J}} & & \downarrow{\scriptstyle (f|_C)_\mathcal{J}} & & \downarrow{\scriptstyle f_\mathcal{J}} \\ K(D)^\times & \xrightarrow{j_D} & \mathcal{J}_{\widetilde{D}} & \longrightarrow & \mathcal{J}_D & \longrightarrow & \mathcal{J}_Y. \end{array}$$

We show that the left square is commutative:

Indeed, we show that $h_\mathcal{J} \circ j_C = j_D \circ N_{K(C)/K(D)}$ agree on every direct summand of $\mathcal{J}_{\widetilde{D}}$. We have a direct summand for every discrete valuation ω on $K(D)$. If ω does not have a center on $\widetilde{D}$, then the ω-summand is $K(D)^\times_\omega$, and if $\omega = \nu_{\widetilde{y}}$ for some center $\widetilde{y} \in \widetilde{D}$, then the ω-summand is $\mathbb{Z}.\widetilde{y}$. Let p_ω and $p_{\widetilde{y}}$ denote the corresponding projections.

Recall that for a valuation $\nu \in V_{K(C)}$ $i_\nu : K(C)^\times \longrightarrow K(C)_\nu^\times$ denotes the canonical inclusion $t \in K(C)^\times$. For $t \in K(C)^\times$, we let $s = N_{K(C)/K(D)}(t) \in K(D)^\times$, then

$$j_C(t) = ((\nu_x(t))_{x \in |\widetilde{C}|}), (i_\nu(t))_{\nu \in C_\infty}),$$

$$j_D(s) = ((\nu_y(s))_{y \in |\widetilde{D}|}), (i_\omega(s))_{\omega \in D_\infty}).$$

1) First consider the case of an ω-summand with $\omega \in D_\infty$. First note that a ν-summand gets mapped into the ω-summand iff ν is a valuation in C_∞ and $\nu|_{K(D)} = \omega$. As ω does not have a center on $\widetilde{D}$, any ν on $K(C)$ restricting to ω on $K(D)$ does not have a center on $\widetilde{C}$. In particular, any valuation restricting to ω is automatically already in C_∞.

Furthermore, $h_{\mathcal{J}}$ restricted to $K(C)_\nu^\times$ is the local norm $N_{K(C)_\nu/K(D)_\omega} : K(C)_\nu^\times \longrightarrow K(D)_\omega^\times$. Thus in $K(D)_\omega^\times$, we have

$$\begin{aligned}
(p_\omega \circ j_D \circ N_{K(C)/K(D)})(t) &= (p_\omega \circ j_D)(N_{K(C)/K(D)}(t)) \\
&= i_\omega(N_{K(C)/K(D)}(t)) \\
&= N_{K(C)/K(D)}(t)
\end{aligned}$$

for all $t \in K(C)_\nu^\times$, and

$$(p_\omega \circ h_{\mathcal{J}} \circ j_C)(t) = \sum_{\nu:\nu|_{K(D)}=\omega} h_{\mathcal{J}}((i_\nu(t)) = \sum_{\nu:\nu|_{K(D)}=\omega} N_{K(C)_\nu/K(D)_\omega}(t)$$

for all $t \in K(C)_\nu^\times$. Therefore, showing that $h_{\mathcal{J}} \circ j_C$ and $j_D \circ N_{K(C)/K(D)}$ agree on $K(D)_\omega$ amounts to the well-known fact that global norms are the product of local norms (for a proof, see e.g. [12, Ch. III]):

$$N_{K(C)/K(D)}(t) = \sum_{\nu:\nu|_{K(D)}=\omega} N_{K(C)_\nu/K(D)_\omega}(t) \text{ for all } t \in K(C)_\nu^\times.$$

2) Now consider ω-summands for $\omega = \nu_{\widetilde{y}}$ with center $\widetilde{y} \in \widetilde{D}$. Then $K(C)_\nu^\times$ maps into the ω-summand $\mathbb{Z}.\widetilde{y}$ iff we have $\nu|_{K(D)} = \omega$. Furthermore, we get a contribution from the discrete $\widetilde{x}$-summand $\mathbb{Z}.\widetilde{x}$ if and only if $\widetilde{f}(\widetilde{x}) = \widetilde{y}$, which is equivalent to the condition $\nu_{\widetilde{x}}|_{K(D)} = \nu_{\widetilde{y}}$. We have

$$(p_\omega \circ j_D \circ N_{K(C)/K(D)})(t) = (p_{\widetilde{y}} \circ j_D)(N_{K(C)/K(D)}(t)) = v_{\widetilde{y}}(N_{K(C)/K(D)}(t)) \,.\, \widetilde{y}$$

and

$$\begin{aligned}
(p_{\widetilde{y}} \circ \widetilde{F} \circ j_D)(t) &= \sum_{\nu\in C_\infty : \nu|_{K(D)}=\nu_{\widetilde{y}}} \nu(t)\ [k(\nu) : k(\widetilde{y})] \,.\, \widetilde{y} + \sum_{\widetilde{x}:f(\widetilde{x})=\widetilde{y}} \nu_{\widetilde{y}}(t)\ [k(\widetilde{x}) : k(\widetilde{y})] \,.\, \widetilde{y} \\
&= \left(\sum_{\nu\in C_\infty : \nu|_{K(D)}=\nu_{\widetilde{y}}} \nu(t)\ [k(\nu) : k(\widetilde{y})] + \sum_{\nu_{\widetilde{x}} : \widetilde{x}\in\widetilde{C}, \nu_{\widetilde{x}}|_{K(D)}=\nu_{\widetilde{y}}} \nu_{\widetilde{y}}(t)\ [k(\widetilde{x}) : k(\widetilde{y})] \right) .\, \widetilde{y} \\
&= \left(\sum_{\nu : \nu|_{K(D)}=\nu_{\widetilde{y}}} \nu(t)\ [k(\nu) : k(\widetilde{y})] \right) .\, \widetilde{y}
\end{aligned}$$

Thus, showing that $\widetilde{F} \circ j_C$ and $j_D \circ N_{K(C)/K(D)}$ agree on the $\widetilde{y}$-summand amounts to showing that

$$v_{\widetilde{y}}(N_{K(C)/K(D)}(t)) = \left(\sum_{\nu : \nu|_{K(D)}=\nu_{\widetilde{y}}} \nu(t)[k(\nu) : k(\widetilde{y})] \right)$$

which is again a fact of basic number theory of global fields. $\square$

Definition 3.1.9. Let $f : X \longrightarrow Y$ be a morphism of regular arithmetical schemes, and let $f_* : \pi_1(X) \longrightarrow \pi_1(Y)$ be the induced morphism on fundamental groups. Then we denote the image subgroup $f_*(\mathcal{C}_X)$ inside $\mathcal{C}_Y$ by $\mathcal{N}\mathcal{C}_X$, the and call it the *f-norm subgroup.*

We shall also make use of the following proposition:

Proposition 3.1.10. *Let X be a regular arithmetical scheme, and let $\mathcal{J}_X$, $\mathcal{C}_X$ denote the Wiesend idèle group, respectively the Wiesend idèle class group of X. Then $\mathcal{J}_X$ is generated by the images of $\mathcal{J}_{\widetilde{C}}$ under the canonical maps i_{C*}, and $\mathcal{C}_X$ is generated by the images of $\mathcal{C}_{\widetilde{C}}$ under i_{C*}.*

Proof. The proof easily reduces to the following

Claim 3.1.11. If X is a regular arithmetical scheme, then for every point $x \in X$, there exists a curve $C \subset X$ such that x is a regular point on C.

. Indeed, let x be a point of X. Since being regular is an open condition, by shrinking X if necessary, we may reduce to a regular open affine $U = Spec\,A$ contained in X, where x corresponds to a prime ideal $\mathfrak{p} \subset A$. Let $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ be the local ring at x. As x is a regular point of X, we can choose $(t_1, \ldots, t_d)$, all elements of $A_{\mathfrak{p}}$, such that they form a system of local parameters at x. (Replacing t_i by a multiple by an element of $A - \mathfrak{p}$ if necessary, we may assume that $t_i \in A$ for all i.) By [9, Theorem 2.5.15], $B := A_{\mathfrak{p}}/(t_1, \ldots, t_{d-1})$ is regular, local and of dimension 1, and therefore integral; the closure of its generic point in $Spec\, A$ is thus an integral curve C.

Now x is a regular point of $Spec\, B$ iff $t_r \notin (t_1, \ldots, t_{r-1})^2$ in $A_{\mathfrak{p}}$ (cf. [9, 4.2.15]). Letting $\overline{\mathfrak{m}_{\mathfrak{p}}} = \overline{(t_d)}$ denote the image of $\mathfrak{m}_{\mathfrak{p}}$ under the natural surjection $A \longrightarrow B$, then if t_d were contained in $(t_1, \ldots, t_d)^2$, $\{t_1, \ldots, t_d\}$ would not be an independent set of generators of $\mathfrak{m}_{\mathfrak{p}}$ modulo ${\mathfrak{m}_{\mathfrak{p}}}^2$, i.e. not a coordinate system of $A_{\mathfrak{p}}$. Thus x is a regular point of $Spec\, B$. □

3.2 The Reciprocity Homomorphism

In this section, we show the existence of a homomorphism $\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$, which will be called the *reciprocity homomorphism*. This morphism is a generalisation of the Artin map from classical class field theory: For X a regular curve, ρ_X will coincide with the Artin map as defined in [12, Section VI.7]. The details of this are shown in the next section.

We shall construct the homomorphism ρ_X by first defining a map ψ at the level of the idèle group: $\psi : \mathcal{J}_X \longrightarrow \pi_1^{ab}(X)$, and then show that ψ factors through $\mathcal{C}_X$. We define as follows:

1) For a closed point $x \in X$, set $\psi(1.x) = Frob_x$, the Frobenius element of $\pi_1^{ab}(X)$ associated to x.

2) For a curve $C \subset X$, recall that $\widetilde{C}$ denotes the normalisation of C, and i_C the canonical homomorphism $i_C : \widetilde{C} \longrightarrow X$ (cf. 2.2.13). Also recall from 3.1.1 that

$(i_C)_{\mathcal{J}} : \mathcal{J}_{\widetilde{J}} \longrightarrow \mathcal{J}_X$ denotes the induced morphism on idèle groups .

If ν a valuation of $K(C)$ without a center on $\widetilde{C}$, we let $\psi = (i_C)_{\mathcal{J}} \circ \rho_C$, where $\rho_C : \mathcal{J}_{\widetilde{C}} \longrightarrow \pi_1^{ab}(\widetilde{C})$ is the reciprocity homomorphism associated to the normal curve $\widetilde{C}$ (cf. [12, Ch.]).

Then ψ is trivial on the image of $j : \sum_C K(C)^\times \longrightarrow \mathcal{J}_X$, so it descends to a homomorphism $\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$.

Definition 3.2.1. Let X be a regular arithmetical scheme. Then the continuous homomorphism $\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$ defined above is called the *reciprocity homomorphism* of X.

Let $f : X \longrightarrow Y$, and recall from Section 2.2 that $\widetilde{f}$ denotes the induced morphism on fundamental groups. Recall also that the abelianised fundamental groups $\pi_1^{ab}(X)$, $\pi_1^{ab}(Y)$ are well-defined without reference to geometric basepoints.

Notation 3.2.2. Given f, $\widetilde{f}$ as above, we let $\bar{f} : \pi_1^{ab}(X) \longrightarrow \pi_1^{ab}(Y)$ denote the induced morphism on the abelianised fundamental groups.

Proposition 3.2.3. *The recprocity morphism is compatible with induced homomorphisms on the class and fundamental groups. I.e. if we let $f : X \longrightarrow Y$ is a morphism of regular arithmetical schemes, then the diagram*

$$\begin{array}{ccc} \mathcal{C}_X & \xrightarrow{f_*} & \mathcal{C}_Y \\ {\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ \pi_1^{ab}(X) & \xrightarrow{\bar{f}} & \pi_1^{ab}(Y) \end{array}$$

commutes.

Proof. It is most convenient to show this at the level of the idèle groups, i.e. to show that the diagram

$$\begin{array}{ccc} \mathcal{J}_X & \xrightarrow{f_{\mathcal{J}}} & \mathcal{J}_Y \\ {\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ \pi_1^{ab}(X) & \xrightarrow{\bar{f}} & \pi_1^{ab}(Y) \end{array}$$

commutes. We then check commutativity componentwise on every summand $\mathcal{C}_X$ as follows:

1) If x is a closed point, then so is $y = f(x)$. Then $k(y) \simeq \mathbb{F}_q$ and $k(x) \simeq \mathbb{F}_{q^n}$ are finite fields, $\pi_1^{ab}(x) = \pi_1(x))$, $\mathcal{C}_x = \mathbb{Z}.\, x$ and similarly for y. The reciprocity morphisms are

$$\rho_x : \mathbb{Z}.\, x \longrightarrow \pi_1(x)$$

$$\rho_y : \mathbb{Z}.\, y \longrightarrow \pi_1(y)\,.$$

Here, $\rho_x : \mathbb{Z}.\, x \longrightarrow \pi_1^{ab}(x)$ maps $1.\, x \mapsto Frob_x$, where $Frob_x = Frob_{\mathbb{F}_{q^n}}$ denotes the Frobenius associated to the finite field $k(x)$. Similarly, $Frob_y = Frob_{\mathbb{F}_q}$, so we have $Frob_x^n = Frob_y$ in $\pi_1(y)$; thus the upper square in the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}.x & \longrightarrow & \mathbb{Z}.y \\ \downarrow & & \downarrow \\ \pi_1(x) & \longrightarrow & \pi_1(y) \\ \downarrow & & \downarrow \\ \pi_1^{ab}(X) & \longrightarrow & \pi_1^{ab}(Y) \end{array} \qquad \begin{array}{ccc} 1.x & \longrightarrow & n.y \\ \downarrow & & \downarrow \\ Frob_x & \longrightarrow & Frob_x = Frob_y^n \\ \downarrow & & \downarrow \\ Frob_x & \longrightarrow & Frob_x = Frob_y^n \end{array}$$

By a slight abuse of notation, we have let $Frob_y$ also denote the image of $Frob_y$ under the canonical homomorphism $\pi_1(x) \longrightarrow \pi_1^{ab}(X)$. We shall keep this notation also in the following paragraphs. Commutativity of the lower square follows from functoriality of π_1 and π_1^{ab} (cf. Section 2.2 or [19, Ch. 5]).

2) Now let $C \subset X$ be a curve, and recall the canonical morphism $i_C : \widetilde{C} \longrightarrow C \longrightarrow X$ (cf. 2.2.13). Recall also the notation for induced morphisms on the idèle and idèle class groups (cf. Definition 3.1.4, Proposition 3.1.8 and (3.1.1)).

2.1) $f(C)$ is a closed point y. Then for any valuation $\nu \in C_\infty$, the induced map $f_{\mathcal{J}}$ on the idèle groups maps $\sum K(C)_\nu \subset \mathcal{J}_X$ into the y-summand $\mathbb{Z}.y \subset \mathcal{J}_Y$. By the functorial properties of induced maps on the abelianised fundamental group, the square

$$\begin{array}{ccc} \pi_1^{ab}(\widetilde{C}) & \xrightarrow{(f|_C)_*} & \pi_1(y) \\ {\scriptstyle i_{C*}}\downarrow & & \downarrow{\scriptstyle i_y} \\ \pi_1^{ab}(X) & \xrightarrow{f_*} & \pi_1^{ab}(Y) \end{array}$$

commutes. Moreover, by the definition of the Artin map on idèle class group with restricted ramification (cf. [12, Section]), the square

$$\begin{array}{ccc} K(C)_\nu^\times & \longrightarrow & \mathbb{Z}.y \\ {\scriptstyle \rho_C}\downarrow & & \downarrow{\scriptstyle \rho_y = Frob_y^{(.)}} \\ \pi_1^{ab}(\widetilde{C}) & \xrightarrow{(f|_C)_*} & \pi_1(y) \end{array}$$

commutes for any valuation ν.

2.2) $f(C)$ is dense inside a curve $D \subset Y$. Then the associated function field $K(C)$ is an extension of $K(D)$, and $\omega := \nu|_{K(D)}$ is a valuation of $K(D)$ for all valuations ν on $K(C)$. We distinguish two cases:

2.2.1) If ω does not have a center on $\widetilde{D}$, then $\omega \in D_\infty$, and $f_{\mathcal{J}}$ restricted to $K(C)_\nu^\times$ is given by the norm $N = N_{K(C)_\nu/K(D)_\omega} : K(C)^\times \longrightarrow K(D)_\omega^\times$. We show that the following diagram commutes:

$$\begin{array}{ccc} K(C)_\nu^\times & \xrightarrow{N} & K(D)_\omega^\times \\ \downarrow{\scriptstyle \rho_C} & & \downarrow{\scriptstyle \rho_D} \\ \pi_1^{ab}(C) & \xrightarrow{(f|_C)_*} & \pi_1^{ab}(D) \end{array}$$

Indeed, let $t = \pi^n u$ be an element of $K(C)_\nu^\times$ and let $f = [k(y) : k(x)]$. Then $N(t) = u'\pi^{fn}$, where u' is a unit of $O_{K(D)}$, so $\rho_D(N(t)) = Frob_{k(\omega)}^{fn}$, where $Frob_{k(\omega)}$ is the generator of $Gal(\overline{\mathbb{F}}_p/k(\omega))$ inside $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

On the other hand, $\rho_C(t) = Frob_{k(\omega)}^n$. Viewing $\pi_1^{ab}(C)$ and $\pi_1^{ab}(D)$ as the Galois groups of the maximal unramified outside C_∞ (resp. D_∞) extension of $K(C)$ (resp. $K(D)$), $(f|_C)_*$ is just the restriction map to $K(C)$ (resp. $K(D)$), under which $Frob_{k(\nu)}$ gets mapped to $Frob_{k(\omega)}^f$. So $(f|_C)_*(Frob_{k(\nu)}) = Frob_{k(\omega)}^f$ and the above diagram commutes, as claimed.

2.2.2) Now assume that ω has a center on the normalisation $\widetilde{D}$, say y. Then ω is the valuation ν_y associated to y, and $k(\nu)$ is a finite extension of $k(\omega) = k(y)$. We need to show commutativity of

$$\begin{array}{ccc} K(C)_\nu^\times & \xrightarrow{f_{\mathcal{J}}} & \mathbb{Z}.y \\ \downarrow{\scriptstyle \rho_C} & & \downarrow{\scriptstyle \rho_D} \\ \pi_1^{ab}(C) & \xrightarrow{(f|_C)_*} & \pi_1(y)\,. \end{array}$$

Indeed, let π be a uniformiser of $K(C)_\nu$. For $t \in K(C)^\times$, write $t = u\pi^n$, where u is a unit of $K(C)_\nu$. We have $f_{\mathcal{J}}(t) = \nu(t)[k(\nu) : k(y)].\, y$, which gets mapped to ${Frob_y}^{n[k(\nu):k(y)]}$ in $\pi_1(y)$.

On the other hand, we have $\rho_C(t) = {Frob_{k(\nu)}}^n$, so since $Frob_{k(\nu)} = Frob_y^{[k(\nu):k(y)]}$, the diagram commutes as claimed.

□

3.3 The Base Cases of the Induction Argument: Reciprocity for Regular Schemes of Dimension One and Zero

In this section, we summarise the reciprocity laws for arithmetical schemes of dimensons one and zero. These will form the base case for induction on dimension in the proof of the Key Lemma in Chapter 5.

In dimension one, this amounts to a reformulation of results of classical class field theory with restricted ramification in the language of schemes, which we give below. We begin with the dimension zero case, where reciprocity is essentially clear from the definitions.

An arithmetical scheme of dimension zero is equal to $Spec\, k$ for some finite field k. If x is a closed point in some higher-dimensional arithemtic scheme X, then the residue field $k(x)$ is a finite field, and $x \simeq Spec\, k(x)$. The fundamental group of this scheme is abelian, and isomorphic to $\widehat{\mathbb{Z}}$, the procyclic group generated by the Frobenius automorphism $Frob_x$ of the finite field $k(x)$. The class group of x is equal to $\mathcal{C}_x = \mathcal{J}_x = \mathbb{Z}.\, x$, and the reciprocity morphism $\rho_x : \mathcal{C}_x \longrightarrow \pi_1(x)$ maps $1.\, x \mapsto Frob_x$. We get the following

Proposition 3.3.1. *Let x be a zero-dimensional arithmetical scheme. Then ρ_x induces an exact sequence*

$$0 \longrightarrow \mathcal{C}_x \xrightarrow{\rho_x} \pi_1(x) \longrightarrow \hat{\mathbb{Z}}/\mathbb{Z} \longrightarrow 0\,.$$

The following corollary summarises the key statements for a zero-dimensional class field theory:

Corollary 3.3.2. *Let x be an arithmetical scheme of dimension zero and $\mathcal{C}_x \simeq \mathbb{Z}\, x$ the idèle class group of X.*

1. *There exists a one-to-one correspondence between open subgroups of $\mathcal{C}_x$ and open subgroups N of the fundamental group $\pi_1(x)$; it is given by $\rho_x^{-1}(N) \leftarrow N$. The inverse correspondence is given by $H \mapsto \overline{\rho_x(H)}$, where the bar denotes the topological closure. If $H < \mathcal{C}_x$ corresponds to $N < \pi_1(x)$, then*

$$\mathcal{C}_C/H \simeq \pi_1(x)/N\ .$$

2. *The norm subgroups of $\mathcal{C}_x$ are precisely the open subgroups of finite index.*

3. If $y \longrightarrow x$ is an étale connected cover, then the reciprocity map gives rise to an isomorphism

$$\mathcal{C}_x/\mathcal{N}\mathcal{C}_y \xrightarrow{\simeq} Gal(y/x).$$

Proof. The claims follow directly from Proposition 3.3.1; we show the details of 3.3.2.1: Open subgroups of $\mathcal{C}_x$ are of the form $n\mathbb{Z}.\,x$ for non-negative integers n, and open subgroups of $\pi_1(x)$ are of the form $\overline{< Frob_x^n >}$, where the bar denotes the topological closure. Then clearly, the association

$$N = \overline{< Frob_x^n >} \mapsto \rho_x^{-1}(N) = n\mathbb{Z}.\,x$$

gives a one-to-one correspondence between open subgroups of $\mathcal{C}_x$ and $\pi_1(x)$.

The inverse correspondence is given by associating to a subgroup $H = n\mathbb{Z}.\,x \leq \mathcal{C}_x$ the topological closure of the image subgroup: $N = \overline{\rho_x(H)} = \overline{< Frob_x^n >}$. □

Now let C be a regular arithmetical curve. Then the required results are given by the classical class field theory of (number or function) field extensions with restricted ramification, as derived in [13, Chapter VIII].

There are two distinct cases: Either C is a curve over $\mathbb{F}_p$ for some prime p, or C is open in $Spec\,\mathcal{O}_K$ for some number field K.

We first consider the case where C is a regular curve over $\mathbb{F}_p$, which is the main focus of this thesis. Then $K(C)$ is a function field, i.e. a finite extension of $\mathbb{F}_p(t)$. C is either proper or affine. Let $P(C)$ be the regular compactification of C, and let $S = P(C)\backslash C$ be the complement of C. Then S is non-empty if and only if C is affine. The elements of S are closed points, i.e. of codimension one in C. We can identify S with the set of valuations without a center on C:

Proposition 3.3.3. *There exists a contravariant equivalence of categories of connected étale covers of C to finite, unramified-outside-S extensions of $K(C)$.*

Proof. We give the contravariant functor which defines the equivalence of categories: It is given by associating to a connected étale cover $Y \longrightarrow C$ its function field $K(Y)$, which is unramified over C by definition.

The inverse functor is given by associating to a finite extension L of $K(C)$ the normalisation C_L of C in L. Indeed, the normalisation morphism $C_L \longrightarrow C$ is finite, and if L is unramified at the primes contained in C, then $C_L \longrightarrow C$ will be étale. □

Remark 3.3.4. Now recall that the the maximal unramified-outside-of-S extension of $K(C)$ is defined as the union of all finite field extensions of $K(C)$ which are unramified at primes not in S. Let it be denoted by K_S, then Proposition 3.3.3 implies that $\pi_1(C) \simeq Gal(K_S/K(C))$

Remark 3.3.5. As before, the choice of a geometric basepoint for $\pi_1(C)$ amounts to fixing a separable closure of $K(C)$.

We have $\mathcal{J}_C = \oplus \mathbb{Z}.x \oplus_{\nu \in S} K(C)_\nu^\times$, and thus $\mathcal{C}_C \simeq C_S(K)$ is the S-idèle class group of $K(C)$ as defined in [13, VIII.3]. Then [13, Thm. 8.3.14] yields the following:

Theorem 3.3.6. *If C is a regular curve over a finite field, then there exists a* *canonical exact sequence*

$$0 \longrightarrow \mathcal{C}_C \xrightarrow{\rho_C} \pi_1^{ab}(C) \longrightarrow \hat{\mathbb{Z}}/\mathbb{Z} \longrightarrow 0 \ .$$

If $\mathcal{C}_C^0$ and $\pi_1^{ab,0}(C)$ denote the degree-0 parts of the class and fundemental group, respectively, then we obtain surjectivity of the reciprocity morphism restricted to $\mathcal{C}_C^0$, and thus an isomorphism

$$\mathcal{C}_C^0 \simeq \pi_1^{ab,0}(C) \ .$$

In the flat case, Theorem [13, 8.3.12] implies the following:

Theorem 3.3.7. *If C is a flat arithmetical curve, let $\mathcal{D}_C$ denote the connected component of the identity in the Wiesend idèle class group $\mathcal{C}_C$. Then there exists a canonical exact sequence*

$$0 \longrightarrow \mathcal{D}_C \longrightarrow \mathcal{C}_C \xrightarrow{\rho_C} \pi_1^{ab}(C) \longrightarrow 0 \ .$$

Corollary 3.3.8. *Let C be a regular arithmetical curve.*

1. *There exists a one-to-one correspondence between open subgroups of $\mathcal{C}_C$ and open subgroups N of $\pi_1^{ab}(C)$; it is given by $\rho_C^{-1}(N) \leftarrow N$. The inverse correspondence is given by $H \mapsto \overline{\rho_C(H)}$, where the bar denotes the topological closure.*

   If $H < \mathcal{C}_C$ corresponds to $N < \pi_1^{ab}(C)$, then

$$\mathcal{C}_C/H \simeq \pi_1^{ab}(C)/N \ .$$

2. *The norm subgroups of* $\mathcal{C}_C$ *are precisely the open subgroups of finite index.*

3. *If* $D \longrightarrow C$ *is an étale connected cover,* $D' \longrightarrow C$ *the maximal abelian subcover, then* $\mathcal{N}\mathcal{C}_D = \mathcal{N}\mathcal{C}_{D'}$ *and the reciprocity map gives rise to an isomorphism*

$$\mathcal{C}_C/\mathcal{N}\mathcal{C}_D \xrightarrow{\simeq} Gal(D'/C).$$

Proof. In the flat case, the first statement is immediate from the surjection of the reciprocity homomorphism; here, taking the topological closure is a trivial operation. In the function field case, the first statement follows directly from the exact sequence and the topology of $\hat{\mathbb{Z}}/\mathbb{Z}$.

For the second statement, we note that for an étale cover $f : D \longrightarrow C$ of regular curves, the induced morphism $f_* : \mathcal{C}_D \longrightarrow \mathcal{C}_C$ is identical to the natural morphism $C_S(K(D)) \longrightarrow C_S(K(C))$ of S-class groups which is induced by the norm of the classical idèle class group, $N_{K(D)/K(C)} : I_{K(D)} \longrightarrow I_{K(C)}$. Then, Corollaries [13, 8.3.13] and [13, 8.3.15] show that the norm subgroups are precisely the open subgroups of finite index.

Lastly, we observe that the norms subgroup $\mathcal{N}\mathcal{C}_D$ correspond to a unique abelian étale cover of C by 3.3.8.2, which proves the third statement. □

Chapter 4

From the Main Theorem to the Key Lemma

In this chapter, we introduce the main result of this thesis, the Main Theorem 4.3.4, which will yield the desired generalisation of known reciprocity laws to all open subvarieties of $\mathbb{P}^n_k$.

We begin begin by introducing *induced covering data*, the link between covering data considerations and subgroups of the Wiesend idèle class group. We show that any open subgroup $H < \mathcal{C}_X$ of finite index gives rise to an induced covering datum $\mathcal{D}_H$, and call H *finitely realisable* if $\mathcal{D}_H$ is effective with a finite realisaion. We obtain a one-to-one correspondence between finitely realisable subgroups and open subgroups of the fundamental group.

The central result is Theorem 4.3.4, which concerns open subvarieties X of $\mathbb{P}^n_k$.

It identifies the realisable open subgroups of finite index $\mathcal{C}_X$ as those which are geometrically bounded (cf. Definition 4.2.4). As a corollary, we obtain a one-to-one correspondence between open, geometrically bounded subgroups of finite index and open subgroups of the fundamental group, as well as a description of the kernel and image of the reciprocity morphism $\rho_X : \mathcal{C}_X \longrightarrow \pi_1^{ab}(X)$. We also obtain a description of the norm subgroups of $\mathcal{C}_X$.

In the last section, we show that the proof of the Main Theorem for an open subvariety $X \subset \mathbb{P}_k^n$ can be reduced to proving the Key Lemma. The Key Lemma for open subvarieties $X \subset \mathbb{P}_k^n$ will be shown in Chapters 5 and 6.

4.1 Introducing induced covering data

In this section, we provide the link between covering data considerations and subgroups of the Wiesend idèle class group: Any open subgroup of finite index $H < \mathcal{C}_X$ gives rise to an *induced covering datum* $\mathcal{D}_H$. We give a proof that if the induced covering datum $\mathcal{D}_H$ is effective with realisation X_N, the image of N in the abelianised fundamental group is the image of the subgroup H.

Construction 4.1.1. Let X be a regular arithmetical scheme, and let $H < \mathcal{C}_X$ be an open subgroup of the Wiesend idèle class group. Let $H_C := i_{C*}^{-1}(H)$ and $H_x := i_{x*}^{-1}(H)$ denote the preimages in $\mathcal{C}_{\widetilde{C}}$ and $\mathcal{C}_x \simeq \mathbb{Z}.x$, respectively. By the reciprocity law of the base cases (cf. Section 3.3), the images $\rho_{\widetilde{C}}(H_C) = \overline{N}_C$ and $\rho_x(H_x) = N_x$ are open subgroups of $\pi_1^{ab}(\widetilde{C})$, respectively $\pi_1(x)$. Let N_C denote the subgroup of $\pi_1(\widetilde{C})$ corresponding to $\overline{N}_C \leq \pi_1^{ab}(\widetilde{C})$.

Definition 4.1.2. Let X be a regular arithmetical scheme, and let $H < \mathcal{C}_X$ be an open subgroup of the Wiesend idèle class group. Then the *induced covering datum* $\mathcal{D}_H$ is the covering datum $\mathcal{D}_H := (N_C, N_x)_{C,x}$ defined by the subgroups N_C, N_x defined in the construction above.

Let x be a point on the curve $C \subset X$. To show that the induced datum is a covering datum, we must prove that $\mathcal{D}_H$ satisfies the compatibility condition for a covering datum at (x, C).

If $\widetilde{x}$ is a point of $x \times \widetilde{C}$, then the commutative square

$$\begin{array}{ccc} \widetilde{x} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow \\ x & \longrightarrow & C \end{array}$$

induces commutative squares at the level of class groups and fundamental groups:

$$\begin{array}{ccc} \mathcal{C}_{\widetilde{x}} & \longrightarrow & \mathcal{C}_{\widetilde{C}} \\ \downarrow & & \downarrow \\ \mathcal{C}_x & \longrightarrow & \mathcal{C}_C \end{array} \qquad \begin{array}{ccc} \pi_1(\widetilde{x}) & \longrightarrow & \pi_1(\widetilde{C}) \\ \downarrow & & \downarrow \\ \pi_1(x) & \longrightarrow & \pi_1(C) \end{array} .$$

Let $H_{\widetilde{x}}$ denote the preimage of H in $\mathcal{C}_{\widetilde{x}}$. Then $H_{\widetilde{x}}$ is the preimage of H_C and of H_x under the natural morphisms above. By the functoriality of the reciprocity morphism, see the diagram below. Thus, $\rho_{\widetilde{x}}(H_{\widetilde{x}}) = N_{\widetilde{x}}$ is actually equal to the preimage of N_x and N_C in $\pi_1(\widetilde{x})$. In particular, these two concide whenever x is a point on C, and the compatibility condition for a covering datum is satisfied.

$$\begin{array}{ccccc} \mathcal{C}_x & \longleftarrow & \mathcal{C}_{\widetilde{x}} & \longrightarrow & \mathcal{C}_{\widetilde{C}} \\ \downarrow{\scriptstyle \rho_x} & & \downarrow{\scriptstyle \rho_{\widetilde{x}}} & & \downarrow{\scriptstyle \rho_{\widetilde{C}}} \\ \pi_1(x) & \longleftarrow & \pi_1(\widetilde{x}) & \longrightarrow & \pi_1(\widetilde{C}) \end{array} .$$

Thus, the datum $\mathcal{D}_H$ defines a covering datum on X.

Definition 4.1.3. If X is an arithmetical scheme, $H < \mathcal{C}_X$ an open subgroup, the covering datum $\mathcal{D}_H$ defined above is called the *covering datum induced by H.*

Remark 4.1.4. Let X be an arithmetical scheme, and recall the definititions made in 2.3.3.

1. If $H < \mathcal{C}_X$ of finite index, then induced covering datum $\mathcal{D}_H$ is of index bounded by m.

2. If $H < \mathcal{C}_X$ is such that $\mathcal{C}_X/H$ is cyclic, then $\mathcal{D}_H$ is a of cyclic index.

In the following, if $N < \pi_1(X)$, we again denote by $\overline{N}$ its image in the abelianisation $\pi_1^{ab}(X)$.

Proposition 4.1.5. *Let X be a regular arithmetical scheme, and let $H < \mathcal{C}_X$ be an open subgroup of the ideal class group. If the induced covering datum $\mathcal{D}_H$ is effective with realisation Y_N, then $\rho_X^{-1}(\overline{N}) = H$, and reciprocity induces an isomorphism*

$$\mathcal{C}_X/H \simeq \pi_1^{ab}(X)/\overline{N}\,.$$

Proof. Let H'_C be the preimage of N in $\mathcal{C}_X$, and H'_C, H'_x the preimages of H in $\pi_1(\widetilde{C})$, $\pi_1(x)$ in the commutative diagram

$$\begin{array}{ccc} \pi_1^{ab}(\widetilde{C}) & \longrightarrow & \pi_1^{ab}(X) \\ \uparrow{\scriptstyle \rho_{\widetilde{C}}} & & \uparrow{\scriptstyle \rho_X} \\ \mathcal{C}_{\widetilde{C}} & \xrightarrow{i_{C*}} & \mathcal{C}_X \end{array}$$

where the arrows in the top rows are induced by functoriality of the fundamental group descended to the abelianisation. By the class field theory for arithmetic curves, we then have $H'_C = H_C$ for all curves $C \subset X$. Since the images of $\mathcal{C}_{\widetilde{C}}$ generate $\mathcal{C}_X$, this implies that $H = H'$.

To prove the second part of the Proposition, let $C \subset X$ be any curve and recall again the canonical induced morphism $\widetilde{i}_C : \pi_1(\widetilde{C}) \longrightarrow \pi_1(X)$. If $N \triangleleft \pi_1(X)$ is an open subgroup, let $N_C := \widetilde{i}_C^{-1}(N)$ denote its preimage in $\pi_1(\widetilde{C})$. By Lemma 2.1.14, there exists a curve $C \subset X$ such that the injection $\pi_1(\widetilde{C})/N_C \hookrightarrow \pi_1(X)/N$ is an isomorphism. Let $\overline{N}$ $\overline{N}_C$ denote the images of N, respectively N_C in the abelianised fundamental groups, and recall from above that $\rho_X^{-1}(N) = H$. Then we obtain a diagram of injections

$$\begin{array}{ccc} \mathcal{C}_X/H & \xhookrightarrow{\rho_X} & \pi_1^{ab}(\overline{N}) \\ \uparrow & & \uparrow{\scriptstyle \simeq} \\ \mathcal{C}_{\widetilde{C}}/H_C & \xhookrightarrow{\rho_{\widetilde{C}}} & \pi_1^{ab}(\widetilde{C})/\overline{N}_C \end{array}$$

From classical class field theory for curves (cf. Corollary 3.3.8), we have that the Artin map $\rho_{\widetilde{C}}$ induces an isomorphism

$$\overline{\rho}_{\widetilde{C}} : \mathcal{C}_{\widetilde{C}}/H_C \simeq \pi_1^{ab}(\widetilde{C})\,.$$

Thus, all the arrows in the diagram must be isomorphisms, as claimed. □

4.2 Geometrically bounded covering data

In this section, we define the notion of a bounded covering datum on an arithmetical variety X over the finite field k. The condition for boundedness shall make use of the degree of a curve; for this end, we shall first set the discussion in projective space:

Notations/Remark 4.2.1. If not otherwise explicitly stated, we will from now work in the following context:

1. We consider $U \overset{i}{\hookrightarrow} X$, an open, dense quasi-projective and regular subset (which exists by Fact 2.1.10), say $U \subset \mathbb{P}^n_k$. Let $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ be the canonical embedding. Then replacing U by $U \cap \mathbb{A}^n_k$ if necessary, we may assume that $U \hookrightarrow \mathbb{A}^n_k \subset \mathbb{P}^n_k$. Let $\overline{U} \subset \mathbb{P}^n_k$ be the closure of U in $\mathbb{P}^n_k$.

2. We consider only those curves $C \subset U$ which are regular curves on $\overline{U}$, i.e. curves such that the regular compactification $P(C)$ is contained in $\overline{U} \subset \mathbb{P}^n_k$. (Note that since k is perfect, regularity and smoothness are equivalent conditions for curves over k.) Then $P(C) = \overline{C}$, the topological closure of C in $\mathbb{P}^n_k$.

3. Note that for curves $C \subset U$ as above, the arithmetical genus $g_a(P(C))$ is the the same as the geometric genus g_C of $P(C)$.

4. Given a positive integer d, we also let $g(n, d)$ denote the integer from Fact 2.1.9.

5. In the above context, let $\mathcal{D}'$ be a covering datum on X, and consider the pullback $\mathcal{D} := i^*(\mathcal{D}')$, a covering datum on U. For every $N_C \in \mathcal{D}$, consider the étale cover $f_C : Y_C^{\mathcal{D}} \longrightarrow C$, thus $\deg f_C = [\pi_1(C) : N_C]$. Notice that $Y_C^{\mathcal{D}} \longrightarrow C$ is smooth, and let $g_{Y_C^{\mathcal{D}}}$ denote the geometric genus of (the completion of) C. Let k_C be the field of constants of $Y_C^{\mathcal{D}}$. Finally, let $\deg_{k_C} R_C$ denote the degree of the ramification divisor of the ramification locus of f_C over k_C. Then Hurwitz's formula gives (cf. [2, Cor. IV.2.4] and [9, Section. 7.4]):

$$\begin{aligned} g_{Y_C} &= (\deg f_C)(g_C - 1) + 1 + \frac{1}{2}(\deg_{k_C} R_C) \qquad (4.2.1) \\ &= [\pi_1(C) : N_C](g_C - 1) + 1 + \frac{1}{2}(\deg_{k_C} R_C) \end{aligned}$$

where R_C is the branch locus of f_C inside the compactification $\overline{Y}_C^{\mathcal{D}}$ of $Y_C^{\mathcal{D}}$.

6. Recall also that a covering datum $\mathcal{D}$ on U is of bounded index if there exists a positive integer m such that $[\pi_1(\widetilde{C}) : N_C] \leq m$ for all curves $C \subset X$ (see Definition 2.3.3). If this is the case, the maximal genus of a cover $Y_C^{\mathcal{D}} \longrightarrow C$ in the covering datum is solely determined by $\deg_{k_C} R_C$, the degree of the ramification divisors R_C over the field of definition k_C of f_C:

$$g_{Y_C} \leq m(g_C - 1) + 1 + \frac{1}{2}(\deg_{k_C} R_C)$$

Definition 4.2.2. Let $U \overset{i}{\hookrightarrow} \mathbb{P}^n_k$ be a quasi-projective arithmetical variety, $\mathcal{D}$ a covering datum on U. Then $\mathcal{D}$ is called *geometrically bounded* if for every positive integer d, there exists a positive integer $\delta = \delta(d)$ such that the following holds:

For all curves C as in Notation 4.2.1 that satisfy $\deg_{k_C} C \le d$, the cover $Y_C^{\mathcal{D}} \longrightarrow C$ defined by $\mathcal{D}$ satisfies $g_{Y_C^{\mathcal{D}}} \le \delta(d)$

Remark 4.2.3. Contrary to the definition of covering data of bounded index, this is a condition only involving the regular curves C in U with $\overline{C} \subset \overline{U} \subset \mathbb{P}^n_k$ regular; no assumption is made about the behaviour of the covering datum at curves with non-regular compactifications in U.

Definition 4.2.4. Let X be an arithmetical variety, and let $H < \mathcal{C}_X$ be an open subgroup of the Wiesend idèle class group. We say that *the subgroup H is geometrically bounded* if the induced covering datum $\mathcal{D}_H$ on X is geometrically bounded.

Recall from 4.2.1 that we let $\overline{C}$ denote the (regular) closure of C in $\overline{U}$, and that C is open in $\overline{C}$. Since f_C is étale over C, the complement $\overline{C} \backslash C = \{x_1, \ldots, x_n\}$ contains the branch locus R_C of f_C.

The degree of R_C can then be given in terms of the higher ramification groups above the points x_j: If G^i_y is the i-th ramification group at y, a point lying over some x_j, we have

$$\deg R_C = \sum_{j=1}^{n} \sum_{y \in f_C^{-1}(x_j)} \left(\sum_{i=0}^{\infty} |G^i_y| - 1 \right) [k(y) : k_C] \ .$$

Definition 4.2.5. In notations as in (4.2.1), let $Y_C^{\mathcal{D}} \longrightarrow C$ be a cover of regular curves, and let $y \in \overline{Y}_C^{\mathcal{D}}$ be a point. The *ramification number* m_y of y is defined to be the biggest integer m_y such that the m_y-th ramification group G^{m_y} is non-zero.

If $Y_C^{\mathcal{D}} \longrightarrow C$ is assumed to be Galois, we have $m_y = m_{x_j}$ for all $y \in f_C^{-1}(x_j)$. Then since G_y^i is a subgroup of G_C for all j and i, the sum $\sum_{i=0}^{\infty} | \left(G_y^i| - 1\right)$ is bounded above by $m_j \cdot (|G_C| - 1)$. Note also that the set S_C given by $S_C = \{y : y \in f_C^{-1}(x_j) \text{ for some } j\}$ is finite and contains the ramification locus.

So let $M = \max_{y \in S_C} [k(y) : k]$, and $m = \max_j m_{x_j}$, then

$$\deg_{k_C} R_C \leq n \cdot m \cdot M \cdot (|G_C| - 1) \ . \tag{4.2.2}$$

Here, $n = n(C)$ is the number of points in $\overline{C} \backslash C$, a constant depending on the curve C once the embedding $i : U \hookrightarrow \mathbb{P}_k^n$ is fixed. Similarly, we have $M = M(C)$ is a constant depending only on C once i is fixed.

The proposition below shows that for all regular curves, we may replace $n(C)$, $M(C)$ by upper bounds which only depends on the degree of the curve.

Proposition 4.2.6. *Let $U \subset \mathbb{P}_k^n$ be quasi-projective arithmetical variety over the finite field k and let $\mathcal{D}$ be a covering datum on U. For every positive integer d, there exist integers $N = N(d)$, $M = M(d)$ such that*

1. *$n(C) \leq N(d)$ whenever $\overline{C} \subset \overline{U}$ is a regular curve $\deg_k C \leq d$.*
2. *$M(C) \leq M(d)$ whenever $\overline{C} \subset \overline{U}$ is a regular curve $\deg_k C \leq d$.*

Proof. 1. We have $\overline{C}\backslash C = \left(\overline{U}\backslash U\right) \cap \overline{C} = \{x_1, \ldots, x_{n(C)}\}$. Then since by intersection theory,

$$\deg_k \left(\overline{U}\backslash U\right) \cdot \deg_k \overline{C} = \sum_{j=1}^{n(C)} mult_j(x_j)[k(x_j) : k] \geq n(C) \tag{4.2.3}$$

In particular, for any regular curve $C \subset U$ of degree $\deg_k C \leq d$, we have

$$n(C) \leq \deg_k \overline{C} \cdot \deg \overline{U}\backslash U \leq d \cdot \deg_k \overline{U}\backslash U \ .$$

2. For all $y \in S_C$, we have

$$[k(y) : k_C] = [k(y) : k(x_j)] \cdot [k(x_j) : k] \leq \deg f_C \cdot [k(x_j) : k]$$

for some $x_j \in \overline{C}\backslash C$. Noting that

$$[k(x_j) : k] \leq \deg_k \overline{C}\backslash C \leq \deg_k \overline{U}\backslash U \cdot \deg_k \overline{C} \leq d \cdot \deg_k \overline{U}\backslash U \,,$$

for all $x_j \in \overline{C}\backslash C$, we get

$$M(C) \leq d \cdot \deg f_C \cdot \deg_k \overline{U}\backslash U \,, \tag{4.2.4}$$

as claimed.

□

Definition 4.2.7. Let $U \subset \mathbb{P}^n_k$ be a quasi-projective arithmetical variety over the finite field k, and let $\mathcal{D}$ be a covering datum on U. Let $C \subset U$ be a regular curve whose regular compactification $P(C)$ is contained in $\overline{U}$ (cf. Notations 4.2.1). Recall that k_C denotes the field of constants of $Y_C^{\mathcal{D}} \longrightarrow C$, thus a finite extension of k.

Suppose that $\overline{C}$ has degree $\deg_{k_C} \overline{C} \leq d$. The set of *ramification numbers of* $\mathcal{D}$ is defined to be the set of the ramification numbers $m_{(y,C)}$ of points $y \in \overline{Y}_C^{\mathcal{D}}$ above points of $\overline{C} \backslash C$.

The ramification numbers of $\mathcal{D}$ are *uniformly bounded* if there exists a positive integer M such that $m_{(y,C)} \leq M$ for all pairs (C, y) of curves $C \subset U$ and closed points $y \in \overline{Y}_C^{\mathcal{D}} \backslash Y_C^{\mathcal{D}}$.

Then we get the following:

Proposition 4.2.8. *Let $\mathcal{D}$ be covering datum of bounded index h on a quasi-projective arithmetical scheme U. Then $\mathcal{D}$ is geometrically bounded if and only if the ramification numbers of $\mathcal{D}$ are uniformly bounded.*

Proof. Let $C \subset U$ be a regular curve of degree $\deg_{k_C} \leq d$, and let $Y_C^{\mathcal{D}} \longrightarrow C$ be the cover induced by $\mathcal{D}$. Putting together Equation (4.2.2) and Proposition 4.2.6, we have

$$\deg_{k_C} R_C \leq N(d) \cdot m_C \cdot M(d) \cdot (|G_C| - 1) \leq N(d) \cdot m_C \cdot M(d) \cdot (h - 1) \quad (4.2.5)$$

where $m_C = \max_y \{m_{C,y}\}$ is the maximum of all the ramification numbers of points above $\overline{C} \backslash C$. Clearly, if the ramification numbers of $\mathcal{D}$ are not uniformly bounded, there exist curves C_n with at least one point $y_n \in Y_{C_n}^{\mathcal{D}}$ such that $m_{(C_n, y_n)} \overset{n \to \infty}{\longrightarrow} \infty$. Then

$$g_{Y_{C_n}^{\mathcal{D}}} = \deg f_C (g_C - 1) + 1 + \frac{1}{2} \deg_{k_C} R_C \overset{n \to \infty}{\longrightarrow} \infty \,.$$

Conversely, if there exists m such that $m_{(C,y)} \leq m$ for all pairs of curves C and closed points y lying above $\overline{C}\backslash C$ in $Y_C^{\mathcal{D}} \longrightarrow C$, then

$$\deg_{k_C} R_C \leq N(d) \cdot m \cdot M(d) \cdot (|G_C| - 1) \ .$$

and thus

$$\begin{aligned} g_{Y^{\mathcal{D}}_{C_n}} &= \deg f_C(g_C - 1) + 1 + \frac{1}{2}\deg_{k_C} R_C \\ &\leq h(g_C - 1) + 1 + \frac{N(d)\, m\, M(d)\,(h-1)}{2} \end{aligned}$$

whenever C is a curve with $\deg_{k_C} C \leq d$. □

Now consider a Galois étale cover $Y \longrightarrow U$. As above, let $\overline{U} \subset \mathbb{P}^n_k$ denote the closure of U in projective space, and $\overline{C}$ the closure of a regular curve $C \subset U$. If $Y = U_N$ corresponds to the open normal subgroup $N \lhd \pi_1(U)$, let $\mathcal{D}^N$ be the corresponding covering datum.

Proposition 4.2.9. *Let $U \subset \mathbb{P}^n_k$ be a regular quasi-projective arithmetical variety, and let $f : Y \longrightarrow U$ be the étale cover corresponding to the open normal subgroup $N \lhd \pi_1(U)$. Then the induced covering datum $\mathcal{D}^N$ is geometrically bounded.*

Proof. Let $\overline{Y}$ be the normalisation of $\overline{U}$ in $K(Y)$, then f extends to a finite cover $\overline{f} : \overline{Y} \longrightarrow \overline{U}$. As the cover is étale over U, the ramification locus R is contained in $\overline{U}\backslash U$ ([9, Section 8.3, Ex. 2.15]). Now let $C \subset U$ be a regular curve of degreee $\leq d$, $\overline{C}$ its compactification in $\overline{U}$ and R_C the ramification locus of $f_C : \overline{Y} \times \overline{C} \longrightarrow \overline{C}$. As f_C is étale over C, we have $R_C \subseteq \overline{Y}\backslash Y$. Noting that $\deg_k R_C = [k_C : k] \deg_{k_C} R_C$, we get

$$\deg_k R_C = (\deg_k \overline{C}) \cdot (\deg_k R) \leq d\,[k_C : k]\,(\deg_k R)\ .$$

for any curve of degree $\deg_{k_C} C \leq d$. Then

$$\begin{aligned} g_{Y_C} &= (\deg f_C)(g_C - 1) + 1 + \frac{1}{2}\deg_{k_C} R_C \\ &\leq (\deg f)(g(n,d) - 1) + 1 + \frac{\deg_k R_C}{2[k_C : k]} \\ &\leq (\deg f)(g(n,d) - 1) + 1 + \frac{d}{2}(\deg_k R) \end{aligned}$$

is bounded by constants depending only on invariants of U, invariants of the given étale cover $Y \longrightarrow U$ and the maximal degree d of the curve over the field of definition k_C of the pullback $Y \times C \longrightarrow C$, as claimed. □

Corollary 4.2.10. *Let U be as in Proposition 4.2.9, $\mathcal{D}$ a covering datum on U. If $\mathcal{D}$ is effective with a finite realisation, then $\mathcal{D}$ is geometrically bounded.*

Proof. Let $X_N \longrightarrow X$ be the finite realisation of $\mathcal{D}$, then $\mathcal{D}^N = \mathcal{D}$. The covering datum $\mathcal{D}^N$ is geometrically bounded by the proposition. □

Now we define the notion of a geometrically bounded covering data for any regular arithmetical scheme, and obtain the analogous statement to Corollary 4.2.10.

Definition 4.2.11. Let X be a regular scheme over a finite field, and $\mathcal{D}$ a covering datum on X. Then $\mathcal{D}$ is *geometrically bounded* on X if there exists a quasi-projective regular open subscheme $U \overset{i}{\hookrightarrow} X$ such that the pullback $i^*(\mathcal{D})$ is geometrically bounded.

Remark 4.2.12. If the Main Theorem 4.3.4 holds for the scheme X, i. e. if any geometrically bounded covering datum on X is finitely realisable, then the above condition is equivalent to requiring that the pullback $i^*(\mathcal{D})$ to *any* open quasi-projective subscheme $U \subseteq X$ is geometrically bounded. Indeed, if $\mathcal{D}$ is an induced covering datum that is effective with finite realisation, let $U' \overset{i'}{\hookrightarrow} X$ be another quasi-projective regular open subscheme. Apply Lemma 2.3.13.3 to U' and $U \cap U'$, then the pullback $i'^*(\mathcal{D})$ is also finitely realisable, and thus geometrically bounded by Corollary 4.2.10.

Corollary 4.2.13. *Let $X \in Sch(\mathbb{F}_p)$ be any regular arithmetical variety, $\mathcal{D}$ a covering datum on X. If $\mathcal{D}$ is effective with a finite realisation, then $\mathcal{D}$ is geometrically bounded.*

Proof. Let $U \overset{i}{\hookrightarrow} X$ be a quasi-projective regular open subscheme, and let $Y \longrightarrow X$ be the étale cover realising $\mathcal{D}$. Then $i^*(\mathcal{D})$ is effective with finite realisation $Y \times U$, so $i^*(\mathcal{D})$ is geometrically bounded by Corollary 4.2.10.

□

4.3 Realisable subgroups of the class groups

In this section, we define the (finitely) realisable open subgroups of the Wiesend class group $\mathcal{C}_X$ of an arithmetical scheme X as those open subgroups whose induced covering datum is (finitely) realisable. We shall show that there exists a one-to-one correspondence of these subgroups with open subgroups of finite index in $\pi_1^{ab}(X)$, and provide an explicit description of these groups as the bounded open subgroups of finite index of $\mathcal{C}_X$, which will be proven in the following chapters. This explicit description and its proof is the essential feature of the Main Theorem, and is the main result of this thesis.

Definition 4.3.1. An open subgroup $H < \mathcal{C}_X$ of finite index is called *finitely realisable* with realisation $\overline{N}$ if the induced covering datum is effective with finite realisation Y_N.

The following lemma shows that finite realisations of an induced covering datum are unique. In particular, a covering datum induced by an open subgroup of the class group has at most one finite realisation.

Lemma 4.3.2. *Let X be a regular arithmetical scheme, and let $\overline{N}_1$, $\overline{N}_2$ be two open subgroups of $\pi_1^{ab}(X)$. Then the following are equivalent:*

1) $\overline{N}_1 \subset \overline{N}_2$

2) $\rho_X^{-1}(\overline{N}_1) \subset \rho_X^{-1}(\overline{N}_2)$

3) $(\rho_X \circ i_{C*})^{-1}(\overline{N}_1) \subset (\rho_X \circ i_{C*})^{-1}(\overline{N}_2)$ *for all curves* $C \subset X$

4) $(\rho_X \circ i_{x*})^{-1}(\overline{N}_1) \subset (\rho_X \circ i_{x*})^{-1}(\overline{N}_2)$ *for all closed points* $x \in |X|$ *. In particular, an induced covering datum has at most one finite realisation.*

Proof. (Taken from [5].) The implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) are clear since every closed point is a regular point on some curve. If Condition 4) holds, then in particular, $N_{1,x} \subset N_{2,x}$ for all closed points x, so the cover associated to $N_1/N_1 \cap N_2$ is completely split. Hence by Lemma 2.1.12, it is trivial, which implies 1). $\square$

Proposition 4.3.3. *Let X be a regular arithmetical scheme. Then the map $\overline{N} \mapsto \rho_X^{-1}(\overline{N})$ defines a one-to-one correspondece between finitely realisable open subgroups of $\mathcal{C}_X$ and open subgroups $\overline{N}$ of the abelianised fundamental group $\pi_1^{ab}(X)$.*

Proof. A correspondence is one-to-one if and only if it has a well-defined, two-sided inverse correspondence.

We define an inverse correspondence as follows: If $H < \mathcal{C}_X$ is an open subgroup of the class group which is finitely realisable, say by $f_N : X_N \longrightarrow X$, where $N \triangleleft \pi_1(X)$, map $H \mapsto \overline{N}$, where $\overline{N}$ is the image of N in $\pi_1^{ab}(X)$. Then, if $\mathcal{D}_H$ is finitely realisable with finite realisation N, the realisation is unique by Lemma 4.3.2. Thus, the inverse correspondence is well-defined. By Proposition 4.1.5, we then also have $\rho_X^{-1}(\overline{N}) = H$.

Conversely, if $\overline{N}$ is an open normal subgroup of $\pi_1^{ab}(X)$, $\rho_X^{-1}(\overline{N})$ is an open subgroup of $\mathcal{C}_X$. Let N be the preimage of $\overline{N}$ in $\pi_1(X)$. Let H_C, H_x be the preimages of $\rho_X^{-1}(\overline{N})$ in $\mathcal{C}_{\widetilde{C}}$ and $\mathcal{C}_x$, respectively, then by reciprocity for curves and points, we have $\rho_{\widetilde{C}}^{-1}(N(C)) = H_C$ for all curves $C \subset X$, $\rho_x^{-1}(N(x)) = H_x$ for all closed points $x \in |X|$.

$$\begin{array}{ccc} \mathcal{C}_{\widetilde{C}} & \xrightarrow{\rho_{\widetilde{C}}} & \pi_1^{ab}(\widetilde{C}) \\ {\scriptstyle i_{C*}}\downarrow & & \downarrow{\scriptstyle \widetilde{i}_C} \\ \mathcal{C}_X & \xrightarrow{\rho_X} & \pi_1^{ab}(X) \end{array} \qquad \begin{array}{ccc} \mathcal{C}_x & \xrightarrow{\rho_x} & \pi_1^{ab}(x) \\ {\scriptstyle i_{x*}}\downarrow & & \downarrow{\scriptstyle \widetilde{i}_x} \\ \mathcal{C}_X & \xrightarrow{\rho_X} & \pi_1^{ab}(x) \end{array}$$

In particular, H_C is realised by $N(C)$, H_x is realised by $N(x)$ for all curves and closed points x and we have $\mathcal{D}_{\rho_X^{-1}(\overline{N})} = \mathcal{D}^N$. Since N realises $\mathcal{D}^N$ by definition, this shows that N realises $\mathcal{D}_{\rho_X^{-1}(\overline{N})}$.

□

We can now state the Main Theorem of this thesis. Let X be a regular arithmetical variety, and recall from that an open subgroup $H \leq \mathcal{C}_X$ induces a covering datum $\mathcal{D}_H$.

Main Theorem 4.3.4. *Let k be a finite field, and let $X \subset \mathbb{P}^n_k$ be an open k-subariety. Then an open subgroup $H \leq \mathcal{C}_X$ is finitely realisable if and only if the induced covering datum $\mathcal{D}_H$ geometrically bounded.*

Remark 4.3.5. Lemma 4.2.13 showed that finitely realisable subgroups are geometrically bounded, and the converse will be shown for open affine subsets of $\mathbb{P}^n_k$ in the next chapter.

Remark 4.3.6. In the flat case of an arithmetical scheme X, the analogue of this theorem is the content of Wiesend's paper [21], supplemented by the work of Kerz-Schmidt ([5], [6]).

Let X be an arithmetical scheme with idèle class group $\mathcal{C}_X$. Recall from 3.1.9 that a subgroup is a norm subgroup iff $H = f_*(\mathcal{N}\mathcal{C}_Y)$ for some étale cover $Y \longrightarrow X$.

Corollary 4.3.7. *Let $X \subset \mathbb{P}^n_k$ be an open subvariety, then the following hold:*

1) There exists a one-to-one correspondence between open and geometrically bounded subgroups $H < \mathcal{C}_X$ of finite index in the Wiesend idèle class group and open subgroups $\overline{N}$ of $\pi_1^{ab}(X)$; it is given by $\rho_X^{-1}(\overline{N}) \mapsto \overline{N}$. Then ρ_X is a continuous injection with dense image in $\pi_1^{ab}(X)$.

2) A subgroup of finite index $\mathcal{C}_X$ is a norm subgroup if and only if it is realisable with a finite realisation, which is the case if and only if it is open and geometrically bounded.

3) If $f : X'' \longrightarrow X$ is an étale connected cover, $X' \longrightarrow X$ the maximal abelian subcover, then $\mathcal{N}\mathcal{C}_{X''} = \mathcal{N}\mathcal{C}_{X'}$ and the reciprocity map gives rise to an isomorphism

$$\mathcal{C}_X / \mathcal{N}\mathcal{C}_{X''} \xrightarrow{\simeq} Gal(X'/X).$$

Proof. We begin by proving Part 2 and note that the first equivalence is trivial. Let $H < \mathcal{C}_X$ be a norm subgroup of finite index, then $H = \mathcal{N}(f_*)$ for some étale cover $Y \longrightarrow X$. Note hat norm subgroups are open since the local norm is an open map, and the induced map $f_* : \mathcal{C}_Y \longrightarrow \mathcal{C}_X$ was defined as the sum of local norms.

Since a norm subgroup $\mathcal{N}(f_*)$ is realisable the cover f it is induced from, by Lemma 4.2.9, the norm subgroup $\mathcal{N}(f_*) < \mathcal{C}_X$ is geometrically bounded. In particular, a norm group of finite index is open, of finite index and geometrically bounded, as claimed. The Main Theorem 4.3.4 gives the converse statement.

Part 3 follows from 2 : Let $\mathcal{N}(f_*) = f_*(\mathcal{C}_{X''})$ denote the f-norm subgroup, and let $\mathcal{D}_{\mathcal{N}(f_*)}$ be the induced covering datum. Then by Part 2 of this corollary, $\mathcal{D}_{\mathcal{N}(f_*)}$ is realisable, and the realisation $f_N : X_N \longrightarrow X$ is finite since it must be a subcover of f. By Remark 4.1.5, we have

$$\mathcal{C}_X/\mathcal{N}(f_*) \simeq \pi_1^{ab}(X)/N \simeq Gal(X'/X)$$

In particular, f_N is abelian.

Lastly, we show Part 1: The one-to-one correspondence follows from Proposition 4.3.3 and Part 2 of the corollary. The kernel of ρ_X is the connected component of the identity (cf. [21, Section 8]), which is equal to $\{1\}$ for arithmetic varieties by Remark 3.1.7. As $\pi_1^{ab}(X)$ is a profinite group, the one-to-one correspondence implies that ρ_X has dense image. □

4.4 The Key Lemma

Let X be a regular arithmetical variety, and $\mathcal{D}$ a covering datum on X. We recall that a cover of curves is tamely ramified if it is tamely ramified at all closed points x (cf. [9, Definition 4.15]).

Definition 4.4.1. Following Wiesend, we say that a covering datum $\mathcal{D}$ is *curve-tame* or *curve-tamely ramified* if the covers $Y_C^{\mathcal{D}} \longrightarrow \widetilde{C}$ are tamely ramified for all curves $C \subset X$. If $\mathcal{D}$ is not tamely ramified, it is called *wildly ramified*.

Remark 4.4.2. If $\mathcal{D}$ is tame, the covers $Y_C \longrightarrow \widetilde{C}$ must have degrees prime to p, or be étale covers.

Definition 4.4.3. A cover $Y_C \longrightarrow \widetilde{C}$ which has no non-trivial tamely ramified subcovers is called *purely wildly ramified.*

In this section, we shall reduce the proof of the Main Theorem 4.3.4 to a Key Lemma, which is split into two statements - one about at most tamely ramified covering data, the other dealing with index-p^m covering data which may be purely wildly ramified:

Key Lemma 4.4.4. *Let k be a finite field of characteristic p, and let $X \subset \mathbb{P}^n_k$ be an open subvariety. Then the following hold:*

1. *If $\mathcal{D}$ is a geometrically bounded covering datum of cyclic prime-power index p^m, then $\mathcal{D}$ is effective with a finite realisation.*
2. *If $\mathcal{D}$ is a covering datum with cyclic prime-power index l^m, where $l \neq p$, then $\mathcal{D}$ is effective with finite realisation.*

Part 1) of the Key Lemma shall be shown in the next chapter using Artin-Schreier-Witt Theory, and Chapter 6 is devoted to proving 4.4.4.2 using Kummer Theory.

In the remaining part of this chapter, we show that Key Lemma 4.4.4 implies Main Theorem 4.3.4. Actually, once can prove a stronger assertion as follows:

Proposition 4.4.5. *Assume that the Key Lemma 4.4.4 holds for a regular arithmetical k-variety X. Then the Main Theorem 4.3.4 holds for X.*

Proof. By Fact 2.1.10 and Lemma 2.3.13, we may assume without loss of generality that X is a regular quasi-projective arithmetical variety. We start by proving the following lemma:

Lemma 4.4.6. *Let $\mathcal{D}$ be a tame covering datum on any regular quasi-projective arithmetical variety X. Then $\mathcal{D}$ is geometrically bounded.*

Proof. Let $C \subset X$ be a regular curve as in Notation 4.2.1, and let $f_C : Y_C^{\mathcal{D}} \longrightarrow C$ be the cover of C defined by $\mathcal{D}$. Recall that we let $m_{(C,y)}$ denote the ramification number of a regular point y above a point $x \in \overline{C}$. Since f_C is tame, we have $m(C,y) \leq 1$, the ramification numbers of $\mathcal{D}$ are uniformly bounded (cf. Definition 4.2.7).

The lemma then follows directly from Proposition 4.2.8. $\square$

Lemma 4.4.7. *Let X be an arithmetical scheme, and let $H < \mathcal{C}_X$ be an open subgroup of finite index of the Wiesend idèle class group. Let H be an open subgroup of finite index such that all H' containing H with cyclic factor group $\mathcal{C}_X/H'$ are finitely realisable. Then H is finitely realisable.*

Proof. If $H < \mathcal{C}_X$ is open and of finite index, then $\mathcal{C}_X/H$ is a finite abelian group. By the structure theorem for finite abelian groups, we have $\mathcal{C}_X/H \simeq \Pi_i C_i$ where $C_1, \dots, C_r$ are cyclic groups of prime power order. Let $H_i < \mathcal{C}_X$ be the kernel of

$\mathcal{C}_X \longrightarrow C_i$. Then H_i is of cyclic prime-power index, and we show that H is realisable if and only if H_i is realisable for all i.

If H is finitely realisable, then the realisation $f : X' \longrightarrow X$ is a finite cover which, in particular, trivialises $\mathcal{D}_H$. Recall that $i_C : \widetilde{C} \longrightarrow X$ denotes the composition of the normalisation with the inclusion morphism, and that i_{C*} : denotes the induced morphism on the Wisend idèle class groups. Let $H_C := i_{C*}^{-1}(H)$, $H_{iC} := i_{C*}^{-1}(H_i)$ denote the preimages in $\mathcal{C}_{\widetilde{C}}$, and set $N(C) := \rho_{\widetilde{C}}(H_C)$, $N_i(C) := \rho_{\widetilde{C}}(H_{iC})$.

For $x \in |X|$ a closed point, define similarly H_x, H_{ix}, $N_x := \rho_x(H_x)$, $N_{ix} := \rho_x(H_{ix})$. Note that in this notation, the covering datum induced by H consists of the groups $N(C)$, $N(x)$, and $\mathcal{D}_{H_i} = (N_i(C), N_i(x))_{(C,x)}$.

Since $H_C \subset H_{iC}$ for all curves $C \subset X$, we also have $N(C) \subset N_i(C)$ for all i. In particular, by Corollary 2.3.9, if f trivialises $\mathcal{D}_H$, f also trivialises $\mathcal{D}_{H_i}$. By Theorem 2.3.15, this implies that H_i is realisable.

Conversely, assume that H_i is realisable for all i. As $\mathcal{C}_X/H_i \simeq \mathbb{Z}/m_i\mathbb{Z}$ is cyclic, we may assume that it is realised by an open normal subgroup $N_i < \pi_1(X)$: We have $\rho_X^{-1}(N_i) = H_i$. Then $N := \bigcap_{i=1}^n N_i$ is open and normal as a finite intersection of open normal subgroups of $\pi_1(X)$. Let Y_N be the corresponding étale cover of X. Then we have

$$\rho_X^{-1}(N) = \rho_X^{-1}(\cap_i N_i) = \cap_i \rho_X^{-1}(N_i) = \cap_i H_i = H \ ,$$

so N realises H. $\square$

Coming back to the proof of Proposition 4.4.5, let X be an arithmetical scheme such that the Key Lemma holds, and let $\mathcal{D}$ be a geometrically bounded covering datum on X. Now let $H_i < \mathcal{C}_X$ be the subgroups defined in the proof of Lemma 4.4.7, i.e. $H_i = ker(\mathcal{C}_X \longrightarrow C_i)$, where $\mathcal{C}_X/H \simeq \Pi_i C_i$ is written as the product of finite cyclic groups.

Then $\mathcal{D}_{H_i}$ is a subdatum of $\mathcal{D}_H$ (cf. Definition 2.3.4, i.e. the cover $Y_{\widetilde{C}}^{\mathcal{D}_{H_i}} \longrightarrow \widetilde{C}$ is always a subcover of $Y_{\widetilde{C}}^{\mathcal{D}_H} \longrightarrow \widetilde{C}$.

Recall from Definition 4.2.5 that the ramification numbers of a covering datum $\mathcal{D}$ on X were defined as the last jumps of the lower ramification filtrations associated to the cover of curves defined by $\mathcal{D}$. Also recall that the lower ramification filtration is well-behaved for subcovers (cf. [17, Prop IV.2]); thus the ramification numbers of $\mathcal{D}_{H_i}$ are bounded by those of $\mathcal{D}_H$. Since $\mathcal{D}_H$ is geometrically bounded by assumption, Lemma 4.2.8 implies that $\mathcal{D}_{H_i}$ is also geometrically bounded. Thus we can apply the Key Lemma 4.4.4, and assume that H_i is realisable.

Then Lemma 4.4.7 gives the desired result. □

Chapter 5

Wildly ramified Covering Data

In this chapter, we begin with a review of the Artin-Schreier-Witt Theory of finite k-algebras and its connection to cyclic $\mathbb{Z}/p^m\mathbb{Z}$-étale covers of the corresponding affine schemes. We then prove the Key Lemma in the case of affine space $X = \mathbb{A}^n_k$, and proceed to the case where $X \subset \mathbb{A}^n_k$ is any open subset.

Specifically, in both cases we construct several pro-étale covers which weakly trivialise the given covering datum $\mathcal{D}$ of cyclic index p^m on X. Then, we show that the associated "intersection cover", as described in the proposition below, is in fact, finite over X. This way, we obtain an étale cover of $X \subset \mathbb{A}^n_k$ which trivialises the covering datum, and by Proposition 2.3.13 this means that $\mathcal{D}$ is also effective with a finite realisation. Thus, $\mathcal{D}$ is realised by an étale cover, which finishes the proof of Part 1) of the Key Lemma.

We then construct several pro-étale covers which weakly trivialise the given covering datum $\mathcal{D}$ of cyclic index p^m on X. Then, we show that the associated "intersection cover", as described in the proposition below, is in fact, finite over X. This way, we obtain an étale cover of $X \subset \mathbb{A}_k^n$ which trivialises the covering datum, and by Proposition 2.3.13 this means that $\mathcal{D}$ is also effective with a finite realisation. Thus, $\mathcal{D}$ is realised by an étale cover, which finishes the proof of Part 1) of the Key Lemma.

A key observation for all of these proofs is the following:

Proposition 5.0.8. *Let $\{N_i\}_{i\in I}$ be a family of closed subgroups of $\pi_1(X)$ such that each of the associated pro-étale covers $X_i = X_{N_i} \longrightarrow X$ weakly trivialises the covering datum $\mathcal{D}$. Then the* intersection cover $X_N \longrightarrow X$ *corresponding to the closed subgroup*

$$N = \overline{< N_i >_{i\in I}}$$

generated by the N_i also weakly trivialises $\mathcal{D}$.

Proof. Using characterisation 2) of Corollary 2.3.9, the proof is straightforward: $X_M \longrightarrow X$ weakly trivialises $\mathcal{D}$ if and only if we have an inclusion $N_x \supseteq \widetilde{i}_x^{-1}(M)$ in $\pi_1(x)$ for all closed points $x \in X$. By assumption, we have $N_x \supseteq \widetilde{i}_x^{-1}(N_i)$ for all i. Since N_i is closed, it follows that

$$N_x \supseteq \overline{< \widetilde{i}_x^{-1}(N_i) >_i}\,,$$

proving the Proposition. □

5.1 A Review of Artin-Schreier-Witt Theory

5.1.1 Artin-Schreier Theory

Let k be a field of characteristic $p > 0$, and let A be a finitely generated k-algebra. Let $\widetilde{A}$ be a universal cover of A, i.e. the integral closure of A inside the maximal separable algebraic extension $\widetilde{K}/K$ of the function field in which A is unramified.

Let $\wp : \widetilde{A} \longrightarrow \widetilde{A}$ be the map sending $x \mapsto x^p - x$ then $\ker \wp = \mathbb{Z}/p$, then we obtain a short exact sequence of $\pi_1(Spec\, A)$-modules

$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \widetilde{A} \stackrel{\wp}{\longrightarrow} \widetilde{A} \longrightarrow 1\,.$$

The long exact sequence of cohomology (cf. [11, Prop. 4.12]) then gives rise to a short exact sequence

$$0 \longrightarrow A/\wp(A) \longrightarrow \pi_1(Spec\, A') \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

Thus we get a canonical isomorphism

$$\Psi : A/\wp(A)^m \stackrel{\simeq}{\longrightarrow} Hom(\pi_1(Spec\, A), \mathbb{Z}/p\mathbb{Z})\,.$$

Note that a group homomorphism $f : \pi_1(Spec\, A) \longrightarrow \mathbb{Z}/p$ has kernel of index p if and only if its image is non-trivial. Conversely, any normal index-p subgroup of $\pi_1(Spec\, A)$ gives rise to such a homormisphm (i.e. the induced canonical projection). Thus, we see that $Hom(\pi_1(Spec\, A), \mathbb{Z}/p)$ and thus $A/\wp(A)$ classifies all $\mathbb{Z}/p$-covers of $Spec\, A$.

The natural $\mathbb{F}_p$-vector space structure on $A/\wp(A)$ is given by $a.[f] := [af]$ for $a \in \mathbb{F}_p$, $[f] \in A/\wp(A)$. Indeed, if $f' \in [f]$ is another representative, then $f' = f + h^p - h$ for some $h \in A$. Since $a^p = a$ for all $a \in \mathbb{F}_p$, we then have that $af' = af + ah^p - ah = af + (ah)^p - (ah) \sim af$, as required. Recalling the natural $\mathbb{F}_p$-vector space structure of $Hom(\pi_1(Spec\, A), \mathbb{F}_p)$, it follows immediately that Ψ is an $\mathbb{F}_p$-linear map.

Remark 5.1.1. Let $a \in \mathbb{F}_p$, $f \in A$, $H_f(z) = z^p - z - f$. Since $a^p = a$ for all $a \in \mathbb{F}_p$, we have $H_{af}(z) = aH_f(a^{-1}z)$, so $(H_f) = (H_{af})$ as ideals in $A[z]$. Thus f and af define the same cover of $Spec\, A$, which we denote by $Y_{[f]}$.

Conversely, if $Y_{[f]} = Y_{[f']}$ is a non-trivial $\mathbb{Z}/p$-cover of $Spec\, A$, then $N_f = \ker(\pi_f)$ and $N_{f'} = \ker(\pi_{f'})$ are equal. Thus the surjection $\pi_f : \pi_1(Spec\, A) \twoheadrightarrow \mathbb{F}_p$ gives rise to an isomorphism $\overline{\pi}_f : \pi_1(Spec\, A)/N_{f'} \simeq \mathbb{F}_p$.

$$\begin{array}{ccc} \pi_1(Spec\, A)/N_{f'} & \xrightarrow{\overline{\pi}_f} & \mathbb{F}_p \\ & \searrow^{\overline{\pi}_{f'}} & \downarrow \cdot a \\ & & \mathbb{F}_p \end{array}$$

Then, since $\overline{\pi}_{f'}$ also gives such an isomorphism, $\overline{\pi}_{f'} \circ \overline{\pi}_f^{-1}$ is an automorphism of the additive group $\mathbb{F}_p$, and as such representable by an element $a \in \mathbb{F}_p^*$. Therefore, $\pi_{f'} = a{\cdot}\pi_f$. As Ψ is an $\mathbb{F}_p$-linear map, $a\pi_f$ corresponds to $[af]$, so we have $[f'] = [af]$.

We thus obtain a one-to-one correspondence between $\mathbb{F}_p$-vector subspaces of the form $\mathbb{F}_p.[f]$ and $\mathbb{Z}/p$ covers of $Spec\, A$:

$$\mathbb{F}_p.[f] \mapsto Y_{[f]} \leftrightarrow N_f = \ker(\chi_f)$$

More generally, we have:

Proposition 5.1.2. *Let A be a finitely generated algebra over a finite field k. Then there is a one-to-one inclusion-reversing correspondence of $\mathbb{F}_p$-vector subspaces $M = \bigoplus_{i\in I} \mathbb{F}_p.[f_i]$ of $A/\wp(A)$ and exponent-p étale covers of Spec A:*

$$M \mapsto Y_M \leftrightarrow N_M := \cap_{f\in M} N_f \, .$$

M is finitely generated of rank n if and only if Y_M is a $(\mathbb{Z}/p)^n$-extension of $\mathbb{A}^n_k$.

Proof. For finitely generated modules M, the assertion is clear from the remarks preceding the proposition and the fact that the isomorphism ψ is compatible with addition. To get the general case, note that an infinitely generated submodule of is the direct limit of its finitely generated submodules, and that a p-exponent cover of $\mathbb{A}^n$ is the inverse limit of its finite subcovers. □

Some relevant properties of the correspondence are as follows:

1. If $\{f_i\}_{i\in I}$ is any generating set for M, then $N_M = \bigcap_{i\in I} N_{f_i}$: Indeed, if $f = \sum_{i=1}^m a_i.f_i$ with $a_i \in \mathbb{F}_p$, then by the additive property of Ψ, $N_f \subset \cap_i N_{f_i}$, so $N_M \subset \bigcap_{i\in I} N_{f_i}$. The other inclusion is trivial.

   In particular, the submodule $M_{f,g}$ generated by two elements f, g corresponds to the cover $Y_{f,g}$ defined by $N_f \cap N_g$. Furthermore, since $f + g \in M_{f,g}$, Y_{f+g} is a subcover of $Y_{f,g}$.

2. The correspondence is inclusion-reversing: If $M \subset M'$, let I be a basis of M over $\mathbb{F}_p$, and I' a basis of M' containing I (such a basis can be found

by completing I to a basis of M' - see [8, Theorem III.5.1]). Then clearly, $N_M = \bigcap_{f \in I} N_f \supset \bigcap_{f \in I'} N_f = N_{M'}$ as claimed.

3. For any two submodules M, M', we have

   a) $N_{M+M'} = N_M \cap N_{M'}$, and

   b) $N_{M \cap M'} =< N_M, N'_M >$

We finish by summarising some properties of $\mathbb{Z}/p\mathbb{Z}$-covers of regular curves:

Let C be a regular curve over a finite field k, and let $A = \mathcal{O}_C(C)$. Assume we are given a $\mathbb{Z}/p\mathbb{Z}$- étale cover $\phi : Y_{[f]} \longrightarrow C$ with Artin-Schreier representative $[f] \in A/\wp(A)$. Let $\overline{\phi} : \overline{Y}_{[f]} \longrightarrow \overline{C}$ denote the induced cover on their regular compactifications, and recall that $Y_{[f]}$, $\overline{Y}_{[f]}$ are the normalisations of C, $\overline{C}$ in the function field $K(Y_{[f]}) := K(C)[z]/(H_f(z))$, where $H_f(z) = z^p - z - f$.

Lemma 5.1.3. *In the situation above, let $x \in \overline{C}$ be a closed point, and let ν_x denote the associated valuation. Recall that $Y_{[f]}$ can be specified by any of the elements in the equivalence class $[f] \in A/\wp(A)$.*

1. *$\overline{\phi}$ is étale at x if and only if there exists $f' \in [f]$ such that $\nu_x(f') \geq 0$. If this is the case, then $\nu_x(f') \geq 0$ for all $f' \in [f]$.*

2. *$\overline{\phi}$ is totally ramified above x if and only if there exists $f' \in [f]$ is such that $\nu_x(f') < 0$.*

3. *If* $\overline{\phi}$ *is totally ramified above* x*, then let* $f'' \in [f]$ *be such that* $\nu_x(f'') = \min_{f' \in [f]}\{\nu_x(f')\}$*. Then* $\gcd(\nu_x(f'), p) = 1$*, and the ramification number of the unique point* y *above* x *is given by* $m_{(C,y)} = -\nu_x(f')$*.*

Proof. If C is a curve over the finite field k, then $K(C)$ is an algebraic function field over k. Then 1 is shown in [18, 3.7.8.a)], and 2 follows from [18, 3.7.7] and [18, 3.7.8.b,c)]. □

Corollary 5.1.4. *Let* k *be any finite field, and let* $\phi : \overline{Y}_{[f/g^a]} \longrightarrow \mathbb{P}^1_k$ *be a* $\mathbb{Z}/p\mathbb{Z}$*-cover of curves that is étale over the open subscheme* $D_+(\overline{g}) \subset \mathbb{P}^1_k$*. If* $\mathbb{P}^1_k \backslash D_+(\overline{g}) = \{x_1, \ldots, x_l\}$ *denotes the set of closed points at which* ϕ *is ramified. Let* $K(Y_{[f/g^a]})$ *denote the function field of* $Y_{[f/g^a]}$*. Let* $\nu_j \in V(K(Y_{[f/g^a]}))$ *denote the valuation associated to* x_j*, and let* m_j *denote the ramification number associated to a point above* x_j*. Then*

1. ϕ *is étale at* x_j *if and only if* $\nu_j(f/g^a) \geq 0$*, and totally ramified iff* $\nu_j(f/g^a) < 0$*.*

2. *If* $\nu_j(f/g^a) < 0$ *is relatively prime to* p*, then we have* $m_j = -\nu_j(f/g^a)$*.*

3. *One can always find a representative of* $f'/g^{a'}$ *of* $[f/g^a]$ *such that for all* j*,* $\nu_j(f'/g^{a'})$ *is either zero or relatively prime to* p*.*

5.1.2 Artin-Schreier-Witt Theory

We begin with a quick primer on Witt vectors, further details of which can be found in [12, II.4] and [13, VI.I]. As in the previous section, k is a finite field of characteristic p, and A a finitely generated k-algebra with separable closure $\widetilde{A}$.

In the following sections, we shall either set $A = k[x_1, \ldots, x_n]$, or $A = A_G := k[x_1, \ldots, x_n]_G$, the localisation of $k[x_1, \ldots, x_n]$ at the set $\{G^a : a \in \mathbb{Z}\}$

Recall that the *m-th Witt polynomial* $W_m \in A[X_0, \ldots, X_m]$ is defined as

$$W_m(X_0, \ldots, X_m) = X_0^{p^m} + pX_1^{p^{m-1}}, \ldots, p^m X_0 \,.$$

By induction on m, one can show that there exist universal polynomials $S_m \in A[X_0, \ldots, X_m, Y_0, \ldots, Y_m, S_1, \ldots S_{m-1}]$ such that

$$W_m(X_0, \ldots, X_{m-1}) + W_m(Y_0, \ldots, Y_{m-1}) = W_m(S_0, \ldots, S_{m-1}) \,.$$

Indeed, we may define

$$S_0 = X_0 + Y_0 \,,$$

$$S_1 = \frac{1}{p}(X_0^p + Y_0^p - S_0^p) + X_1 + Y_1 = \frac{1}{p}(X_0^p + Y_0^p - (X_0 + Y_0)^p) + X_1 + Y_1 \,,$$

and in general

$$S_m = \frac{1}{p^m}\left(W_{m-1}(X_0, \ldots, X_{m-1}) + W_{m-1}(Y_0, \ldots, Y_{m-1}) - W_m(S_0, \ldots, S_{m-1})\right) + X_m + Y_m \,.$$

By induction, it is now clear that $S_m \in A[X_0, \ldots, X_m, Y_0, \ldots, Y_m]$. Similarly, one can show that there exist polynomials $P_m \in A[X_0, \ldots, X_m, Y_0, \ldots, Y_m]$ such that

$$W_m(X_0, \ldots, X_m) \cdot W_m(Y_0, \ldots, Y_m) = W_m(P_0, \ldots, P_m) \,.$$

Definition 5.1.5. Let A be a finitely generated k-algebra, where k is a finite field. The *ring* $W_m(A)$ *of Witt vectors of length* m *over* A is defined to have underlying set consisting of systems of length m over A, with ring operations as follows:

Letting $\mathbf{X} = (X_0, \ldots, X_{m-1})$ and $\mathbf{Y} = (Y_0, \ldots, Y_{m-1})$ be two elements of A^m, we set

$$\begin{aligned} \mathbf{X} \oplus \mathbf{Y} &:= (S_0(\mathbf{X}, \mathbf{Y}), \ldots, S_{m-1}(\mathbf{X}, \mathbf{Y}))\,, \text{ and} \\ \mathbf{X} \cdot \mathbf{Y} &:= (P_0(\mathbf{X}, \mathbf{Y}), \ldots, P_{m-1}(\mathbf{X}, \mathbf{Y}))\,. \end{aligned}$$

An element of $W_m(A)$ is denoted by $\mathbf{f} = [f_0, \ldots, f_{m-1}]$.

Remark 5.1.6. Since $char(A) = p$, we have $[f_0, \ldots, f_{m-1}]^p = [f_0^p, \ldots, f_{m-1}^p]$ for all Witt vectors $[f_0, \ldots, f_{m-1}] \in W_m(A)$.

Since the Witt polynomials S_m and P_m are universal, we clearly have an inclusion $W_k(A) \subset W_m(A)$ given by $[f_0, \ldots, f_{k-1}] \mapsto [f_0, \ldots, f_{k-1}, 0, \ldots, 0]$. Conversely, we also have a projection $W_m(A) \longrightarrow W_k(A)$:

Notation 5.1.7. For $\mathbf{f} = [f_0, \ldots, f_{m-1}] \in W_m(A)$, $k \leq m$, let $\mathbf{f}^{(k)}$ denote the *truncated Witt vector* $[f_0, \ldots, f_{k-1}] \in W_k(A)$.

Now let $\sigma_m(X_0, \ldots, X_{m-1}) = S_m - X_m - Y_m$, then $\sigma_m \in A[X_0, Y_0, \ldots, Y_{m-1}]$. The following two Lemmas are immediate (using induction):

Lemma 5.1.8. *The polynomial* $\sigma_j(X_0, \ldots, X_{j-1}, Y_0, \ldots, Y_{j-1})$ *is homogeneous of degree* p^j.

Lemma 5.1.9. $\deg_{X_j} \sigma_k = \deg_{Y_j} \sigma_k \leq p^{k-j}$.

Define a homomorphism $\wp : W_m(A) \longrightarrow W_m(A)$ by setting $\wp([a_0, \ldots, a_m]) = [a_0, \ldots, a_m]^p \ominus [a_0, \ldots, a_m]$, where $\ominus$ denotes the subtraction operation of Witt vectors. By Remark 5.1.6, we have

$$\begin{aligned} \ker \wp &= \{[a_0, \ldots, a_m] : a_i^p = a_i \text{ for all } i\} \\ &= \{[a_0, \ldots, a_m] : a_i \in \mathbb{F}_p \text{ for all } i\} \\ &= W_m(\mathbb{F}_p) \\ &\simeq \mathbb{Z}/p^m\mathbb{Z}\,. \end{aligned}$$

Then we have an exact sequence of $\pi_1(Spec\,\widetilde{A})$-modules

$$0 \longrightarrow \mathbb{Z}/p^m \longrightarrow W_m(\widetilde{A}) \xrightarrow{\wp} W_m(\widetilde{A}) \longrightarrow 0\,.$$

Applying the long exact sequence of étale cohomology, since $W_m(A)$ is affine, the connecting homomorphism induces an isomorphism

$$\Psi : W_m(A)/\wp(W_m(A)) \xrightarrow{\simeq} Hom(\pi_1(Spec\,A), \mathbb{Z}/p^m\mathbb{Z})\,. \tag{5.1.1}$$

The isomorphism Ψ makes it possible to associate to each Witt vector $\mathbf{f}$ an exponent-p^m cyclic cover of $Spec\,A$:

Let $\mathbf{f} = [f_0, \ldots, f_m] \in W_m(A)$. The homomorphism

$$\Psi(\mathbf{f}) : \pi_1(Spec\,A) \longrightarrow \mathbb{Z}/p^m\mathbb{Z}$$

has image $p^{m-s}\mathbb{Z}/p^m\mathbb{Z} \simeq \mathbb{Z}/p^s\mathbb{Z}$, where $s \le m$, and corresponds to the normal subgroup $\ker \psi_{\mathbf{f}}$ of index p^s, which in turn defines a Galois $\mathbb{Z}/p^s\mathbb{Z}$-cover of $Spec\,A$, and denoted by $Y_{[\mathbf{f}]}$.

Definition 5.1.10. Let $\mathbf{f} = [f_0, \ldots, f_m] \in W_m(A)$. The cover $Y_{[\mathbf{f}]} \longrightarrow Spec\, A$ defined above is called the *cover associated to* $[f]$.

Remark 5.1.11. For $\mathbf{f} = [f_0, \ldots, f_{m-1}] \in W_m(A)/\wp(W_m(A))$, the associated cover $Y_{[f]} \longrightarrow Spec\, A$ is a proper $\mathbb{Z}/p^m\mathbb{Z}$-cover if and only if $f_0 \notin \wp(A)$.

Noting that $Hom(\pi_1(Spec\, A), \mathbb{Z}/p^m\mathbb{Z})$ has a natural structure as a $\mathbb{Z}/p^m\mathbb{Z}$-module, we can define a natural $\mathbb{Z}/p^m\mathbb{Z}$-module structure on $W_m(A)/\wp(W_m(A))$ so that Ψ becomes a $\mathbb{Z}/p^m\mathbb{Z}$-module homomorphism:

For $\mathbf{a} \in W_m(\mathbb{F}_p)$, $\mathbf{f} \in W_m(A)$, we have $\mathbf{a}^p = \mathbf{a}$, and define $\mathbf{a} \cdot [\mathbf{f}] := [\mathbf{af}]$.

Now if $\mathbf{f}'$ is another representative of $[\mathbf{f}]$, then $\mathbf{f}' = \mathbf{f} \oplus \mathbf{h}^p \ominus \mathbf{h}$ for some $\mathbf{h} \in W_m(A)$. Then

$$\mathbf{af}' = \mathbf{a} \cdot (\mathbf{f} \oplus \mathbf{h}^p \ominus \mathbf{h}) = \mathbf{af} \oplus \mathbf{ah}^p \ominus \mathbf{h} = \mathbf{af} \oplus (\mathbf{ah})^p \ominus (\mathbf{ah}) \sim \mathbf{af}\,,$$

i.e. the action is well-defined.

As in the previous section, we can then generalise the correspondence induced by Ψ to the following proposition:

Proposition 5.1.12. *Let A be any finitely generated algebra over a finite field k. Then there is a one-to-one inclusion-reversing correspondence of $\mathbb{Z}/p^m\mathbb{Z}$-submodules $M = \bigoplus_{i \in I} W_m(\mathbb{F}_p).[\mathbf{f}_i]$ of $W_m(A)/\wp(W_m(A))$ and cyclic p^s étale covers of $Spec\, A$, $s \leq m$:*

$$M \mapsto Y_M \leftrightarrow N_M := \cap_{f \in M} N_f\,.$$

M is finitely generated of rank n if and only if Y_M is a finite cover of $Spec\,A$.

Proof. For $\mathbf{f} \in W_m(A)/\wp(W_m(A)$, let $Y_{[\mathbf{f}]} \longrightarrow Spec\,A$ denote the cover associated to $\mathbf{f}$. We begin by analysing the structure of $Y_{[f]}$.

As $Y_{[\mathbf{f}]}$ is integral, it is given as the normalisation of X inside the field extension defined by adjoining to $K(A)$ elements $\{y_0, \ldots, y_{m-1}\}$ such that $\wp(y_0, \ldots, y_{m-1}) = \mathbf{f}$, i.e. such that

$$\begin{aligned} y_0^p - y_0 &= f_0, \\ y_1^p - y_1 &= -\sigma_1(y_0^p, -y_0) + f_1, \\ &\ldots, \\ y_{m-1}^p - y_{m-1} &= -\sigma_{m-1}(y_0^p, \ldots, y_{m-2}^p, -y_0, \ldots, -y_{m-2}) + f_{m-1}\,. \end{aligned}$$

The first equation determines a $\mathbb{Z}/p\mathbb{Z}$-cover $Y_{[f_0]}$ of $Spec\,A$, the second equation a $\mathbb{Z}/p\mathbb{Z}$ cover of $Y_{[f_0]}$, etc. Thus we get $Y_{[\mathbf{f}]}$ as a tower of successive $\mathbb{Z}/p\mathbb{Z}$ extensions over the original scheme $Spec\,A$ (see the graphic below).

Notation 5.1.13. Let $\mathbf{f} \in W_m(A)$ be a Witt vector of length m, and recall that $\mathbf{f}^{(k)}$ denotes the truncated Witt vector $[(f_0, \ldots, f_{k-1}]$ of length k associated to $\mathbf{f}$. Then the intermediate scheme defined by $\mathbf{f}^{(k)}$ is a $\mathbb{Z}/p^k$-cover of $Spec\,A$, and denoted by $Y_{[\mathbf{f}^{(k)}]}$.

Then the tower of intermediate covers can be written as follows:

$$
\begin{array}{c}
Y_{[\mathbf{f}]} = Y_{[(f_0,\ldots,f_{m-1})]} \\
\big\downarrow \mathbb{Z}/p\mathbb{Z} \\
Y_{[\mathbf{f^{(m-2)}}]} \\
\big\downarrow \mathbb{Z}/p\mathbb{Z} \\
\vdots \\
\big\downarrow \mathbb{Z}/p\mathbb{Z} \\
Y_{[f_0]} \\
\big\downarrow \mathbb{Z}/p\mathbb{Z} \\
Spec\, A
\end{array}
\qquad
\begin{array}{l}
Y_{[\mathbf{f^{(m-2)}}]} \xrightarrow{\mathbb{Z}/p^{m-1}\mathbb{Z}} Spec\, A \\
Y_{[\mathbf{f}]} \xrightarrow{\mathbb{Z}/p^{m}\mathbb{Z}} Spec\, A
\end{array}
$$

Here, at each intermediate step, $Y_{[\mathbf{f}^{(k+1)}]} \longrightarrow Y_{[\mathbf{f}^{(k)}]}$ is equal to the normalisation of $Y_{[\mathbf{f}^{(k)}]}$ in $K(Y_{[\mathbf{f}^{(k)}]})[y_{k+1}]/(H_{[\mathbf{f}^{(k+1)}]})$, where

$$H_{\mathbf{f}^{(k+1)}}(y_0,\ldots,y_{k-1}) = y_k^p - y_k + \sigma_k(y_0^p,\ldots,y_{k-1}^p,-y_0,\ldots,y_{k-1}) - f_{k-1} \text{ and}$$

$$K(Y_{[\mathbf{f}^{(k)}]}) = A[y_0,\ldots,y_{k-1}]/\left(H_{[f_0]}(y_0),\ldots,H_{[\mathbf{f}^{(k-1)}]}(y_{k-1})\right) .$$

Now recall from Remark 5.1.1 that in the prime case $m=1$, for $a_0 \in \mathbb{F}_p^\times$ we have $H_{f_0}(a_0^{-1}y_0) = a_0^{-1}H_{a_0f_0}(y_0)$. Similarly, using Lemma 5.1.8, if $a_{k-1} \neq 0$, we have

$$\begin{aligned}
& H_{\mathbf{f}^{(k)}}(a_{k-1}^{-1}y_0, a_{k-1}^{-1}y_1,\ldots,a_{k-1}^{-1}f_{k-1}) \\
= \;& (a_{k-1}^{-1})^p y_{k-1}^p - a_{k-1}^{-1}y_{k-1} + \sigma_k((a_{k-1}^{-1})^p y_0,\ldots,(a_{k-1}^{-1})^p y_0, -a_{k-1}^{-1}y_0,\ldots,-a_{k-1}^{-1}y_{k-1}) - f_{k-1} \\
= \;& a_{k-1}^{-1}\left(y_{k-1}^p - y_{k-1} + \sigma_k(y_0^p,\ldots,y_{k-1}^p,-y_0,\ldots,-y_{k-1}) - a_{k-1}f_{k-1}\right) \\
= \;& a_{k-1}^{-1}H_{(\mathbf{af})^{(k)}}(y_0,\ldots,y_{k-1})
\end{aligned}$$

Therefore, whenever $\mathbf{a} \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, we have $(H_{(\mathbf{af})^{(k)}}) = (H_{\mathbf{f}^{(k)}})$ as ideals for all $k = 1,\ldots,m$, and thus $Y_{[\mathbf{f^{(k)}}]} = Y_{[(\mathbf{af})^{(\mathbf{k})}]}$ for all k. In particular, we have $Y_{[\mathbf{f}]} = Y_{[\mathbf{af}]}$ for all $\mathbf{a} \in W_m(\mathbb{F}_p)$. Conversely, if the $\mathbb{Z}/p^m\mathbb{Z}$-covers $Y_{[\mathbf{f}]}$, $Y_{[\mathbf{f}']}$ are equal for two elements $\mathbf{f}, \mathbf{f}' \in W_m(A)/\wp(W_m(A))$, we must have

$$\Psi(\mathbf{f}) = \Psi(\mathbf{f}') : \pi_1(Spec\, A) \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}\,.$$

Then $N_{\mathbf{f}} = \ker \Psi(\mathbf{f})$ and $N_{\mathbf{f}'} = \ker \Psi(\mathbf{f}')$ are equal, and $\Psi(\mathbf{f})$ gives rise to an isomorphism $\overline{\Psi}_{\mathbf{f}} : \pi_1(Spec\, A)/N_{\mathbf{f}'} \simeq \mathbb{Z}/p^m\mathbb{Z}$.

$$\begin{array}{ccc} \pi_1(Spec\, A)/N_{\mathbf{f}'} & \xrightarrow{\overline{\Psi}_{\mathbf{f}}} & \mathbb{Z}/p^m\mathbb{Z} \\ & \overset{\overline{\Psi}_{\mathbf{f}'}}{\searrow} & \downarrow \cdot\mathbf{a} \\ & & \mathbb{Z}/p^m\mathbb{Z} \end{array}$$

Then, since $\overline{\Psi}(\mathbf{f}')$ also gives such an isomorphism, $\overline{\Psi}(\mathbf{f}') \circ \overline{\Psi}(\mathbf{f})^{-1}$ is an automorphism of the additive group $\mathbb{Z}/p^m\mathbb{Z}$, and as such representable by an element $\mathbf{a} \in (\mathbb{Z}/p^m\mathbb{Z})^{\times}$. Therefore, $\overline{\Psi}(\mathbf{f}') = \mathbf{a} \cdot \overline{\Psi}(\mathbf{f})$. As we showed Ψ to be an $\mathbb{Z}/p^m\mathbb{Z}$-linear map, $\mathbf{a}\overline{\Psi}(\mathbf{f})$ corresponds to $[\mathbf{af}]$, so we have $[\mathbf{f}'] = [\mathbf{af}]$.

Thus we get a one-to-one correspondence between $\mathbb{Z}/p^m\mathbb{Z}$-submodules $W_m(\mathbb{F}_p).[\mathbf{f}]$ of Artin-Schreier-Witt space $W_m(A)/\wp(W_m(A))$, and $\mathbb{Z}/p^s\mathbb{Z}$-covers of $Spec\, A$, where $s = |\{i : f_i \notin \wp(A)\}| \leq m$.

$$W_m(\mathbb{F}_p).[\mathbf{f}] \mapsto Y_{[\mathbf{f}]} \leftrightarrow N_{\mathbf{f}} = \ker(\Psi([\mathbf{f}]))$$

The statement about finitely generated submodules and their ranks follows immediately, as do the statements about general submodules when taking direct limits (see the proof of Proposition 5.1.2 for details). □

Remark 5.1.14. $Y_{[\mathbf{f}]}$ has degree p^m over $Spec\, A$ if and only if $f_0 \notin \wp(A)$. If this is the case, we call $[\mathbf{f}]$ a *primitive* element of $W_m(A)/\wp(W_m(A))$, and $W_m(\mathbb{F}_p).[\mathbf{f}]$ a primitive simple submodule of $W_m(A)/\wp(W_m(A))$.

Definition 5.1.15. A submodule $M \subset W_m(A)/\wp(W_m(A))$ is called *primitive* if every simple submodule is contained in a primitive simple submodule.

We note that under the correspondence, primitive submodules correspond to covers with Galois group a direct sum of copies of $\mathbb{Z}/p^m\mathbb{Z}$.

We finish by defining some submodules of Artin-Schreier-Witt space:

Definition 5.1.16. Let Δ, m be positive integers, and let A_G denote the localisation of $k[x_1, \ldots, x_n]$ at the set $\{G^a : a \in \mathbb{N}\}$. Define subsets of Artin-Schreier-Witt space $W_m(A_G)/\wp(W_m(A_G))$ by

$$M_{\Delta,m}(A_G) = \left\langle \left[\left(\frac{F_0}{G^{a_0}}, \ldots, \frac{F_{m-1}}{G^{a_{m-1}}}\right)\right] \;:\; p^{m-j-1}a_j \leq \Delta \text{ for all } j, \right.$$
$$\left. p^{m-j-1}\left(\deg_{x_i} F_j - a_j \deg_{x_i} G_j\right) \leq \Delta \text{ for all } i \right\rangle .$$

$$M^i_{\Delta,m}(A_G) = \left\langle \left[\left(\frac{F_0}{G^{a_0}}, \ldots, \frac{F_{m-1}}{G^{a_{m-1}}}\right)\right] \;:\; p^{m-j-1}a_j \leq \Delta \text{ for all } j, \right.$$
$$\left. p^{m-j-1}\left(\deg_{x_i} F_j - a_j \deg_{x_i} G_j\right) \leq \Delta \text{ for all } j \neq i \right\rangle .$$

Proposition 5.1.17. *Let $M_{\Delta,m}(A_G)$, $M^i_{\Delta,m}(A)G)$ be the subsets of Artin-Schreier-Witt space defined above. Then $M_{\Delta,m}(A_G)$ and $M^i_{\Delta,m}(A_G)$ are $\mathbb{Z}/p^m$-submodules of $W_n(A_{\mathbf{G}})/\wp(W_n(A_{\mathbf{G}}))$.*

Proof. The addition in $W_m(A_{\mathbf{G}})/\wp(W_m(A_{\mathbf{G}}))$ is inherited from the addition of Witt Vectors, so

$$\begin{aligned}
&\left[\left(\frac{F_0}{G^{a_0}}, \ldots, \frac{F_{m-1}}{G^{a_{m-1}}}\right)\right] + \left[\left(\frac{F'_0}{G^{a'_0}}, \ldots, \frac{F'_{m-1}}{G^{a'_{m-1}}}\right)\right] \\
= &\left[\left(\frac{F_0}{G^{a_0}} + \frac{F'_0}{G^{a'_0}}, \frac{F_1}{G^{a_1}} + \frac{F'_1}{G^{a'_1}} + \sigma_1(\frac{F_0}{G^{a_0}}, \frac{F'_0}{G^{a'_0}}), \ldots \;\; \frac{F_{m-1}}{G^{a_{m-1}}} + \frac{F'_{m-1}}{G^{a'_{m-1}}} + \sigma_{m-1}(\ldots, \frac{F_{m-2}}{G^{a_{m-2}}}, \ldots, \frac{F'_{m-2}}{G^{a'_{m-2}}})\right)\right]
\end{aligned}$$

Now A_j be the power of G in the denominator of $\sigma_j(\ldots, \frac{F_{j-1}}{G^{a_{j-1}}}, \ldots, \frac{F'_{j-1}}{G^{a'_{j-1}}})$. By Lemma 5.1.9, $A_j \leq \max_{l<j}\{p^{j-l}a_l, p^{j-l}a'_l\}$. Since we assumed $p^{m-l-1}a_l$, $p^{m-l-1}a'_l < \Delta$ for all l, this implies that $p^{m-j-1}A_j \leq \Delta$.

Now let $\widetilde{a}_j$ be the power of G in the denominator of the j-th coordinate in the sum, then $\widetilde{a}_j$ is bounded by $\widetilde{a}_j \leq \max\{a_j, a'_j, A_j\} \leq \frac{\Delta}{p^{m-j-1}}$, as required.

Similarly, since by assumption $\deg_{x_j}(F_l/G^{a_l})$,

$$\deg_{x_j}(F'_l/G^{a'_l}) \leq \Delta/p^{m-l-1}$$

for all j and l, Lemma 5.1.9 implies that

$$\deg_{x_j}(\sigma_j(\ldots, \frac{F_{j-1}}{G^{a_{j-1}}}, \ldots, \frac{F'_{j-1}}{G^{a'_{j-1}}})) \leq \max_{l<j} p^l \Delta/p^{m-l-1} = \Delta/p^{m-j-1}$$

for all j (or for all $j \neq i$).

Thus, $M_{\Delta,m}(A_G)$ and $M^i_{\Delta,m}(A_G)$ are additive subgroups. It's clear that the action of $\mathbb{Z}/p^m$ on $W_m(A)/\wp(W_m(A))$ leaves the degree of numerators and denominators of any element $\frac{F}{G^a}$ unchanged, so $M_{\Delta,m}(A_G)$, $M^i_{\Delta,m}(A_G)$ are submodules.

□

Definition 5.1.18. For $m = 1$, we obtain the vector subspaces

$$M_\Delta(A_G) := M_{\Delta,1}(A_G) = \left\langle [F/G^a] : a \leq \Delta, \deg_{x_i} F \leq a \deg_{x_i} G + \Delta \text{ for all } i \right\rangle \subset A_G/\wp(A_G)$$

$$M^i_\Delta(A_G) = M^i_{\Delta,1} = \left\langle [F/G^a] : a \leq \Delta, \deg_{x_j} F \leq a \deg_{x_j} G + \Delta \text{ for all } j \neq i \right\rangle \subset A_G/\wp(A_G)\,.$$

Similarly, if we set $G = 1$, then $A_G = A := k[x_1, \dots, x_n]$ and we obtain analogous submodules for A:

$$M_{\Delta,m}(A) := \left\langle [F_0, \dots, F_{m-1}] : p^{m-s-1} \deg_{x_i} F_s \leq \Delta \text{ for all } i, s \right\rangle$$

$$M^i_{\Delta,m}(A) := \left\langle [F_0, \dots, F_{m-1}] : p^{m-s-1} \deg_{x_j} F_s \leq \Delta \text{ for all } s, \text{ for all } j \neq i \right\rangle ,$$

both of which are $\mathbb{Z}/p^m$-submodules of $W_m(A)/\wp(W_m(A))$.

5.2 Covering data of Prime Index

In this section, we consider the case $m = 1$, where $\mathcal{D}$ is a bounded covering datum of prime index p. We show that for $X = \mathbb{A}^n_k$ and, more generally, for any $X \subset \mathbb{P}^n_k$ an open subset, a geometrically bounded covering datum is realisable.

5.2.1 Affine n-space $\mathbb{A}^n_k$

We begin by considering the affine n-space $X = \mathbb{A}^n_k$. Let $A = k[x_1, \dots, x_n]$ and let $\mathcal{D}$ be a geometrically bounded covering datum of cyclic index p on $X = Spec\, A$. This special case formed the starting point of the investigation, and already contains most of the strategies used for the more general cases. The goal will be to show the following Proposition:

Proposition 5.2.1. *Let k be a finite field of characteristic p and let $\mathcal{D}$ be a bounded covering datum of index p on $\mathbb{A}^n_k$. Then there exists an étale cover of $\mathbb{A}^n_k$*

realising $\mathcal{D}$.

Throughout the section, we shall make much use of the following, particularly simple fibrations of affine space:

Definition 5.2.2. Let $A_i = k[x_1, \ldots, \hat{x}_i, \ldots, x_n]$ and let $\phi_i : A_i \longrightarrow A$, $\phi_i(x_j) = x_j$ be the natural inclusions, then the induced morphism

$$\Phi_i : \mathbb{A}^n_k \longrightarrow \mathbb{A}^{n-1}_k$$

are called the *projection fibrations onto the coordinate hyperplanes.*

Fix a projection fibration Φ_i. A closed point $\omega \in \mathbb{A}^{n-1}_k$ corresponds to a maximal ideal $m_\omega = (g_1, \ldots, \hat{g}_i, \ldots, g_n)$, where $g_j = g_j(x_1, \ldots, \hat{x}_i, \ldots, x_j)$ is a polynomial in j variables if $i > j$, and in $j-1$ variables if $i \leq j$. Letting $k(\omega) \simeq A/m_\omega$ denote the residue field of ω, the fiber $C_{\omega,i}$ of Φ_i above ω is isomorphic to $Spec\, k(\omega)[x_i] \simeq \mathbb{A}^1_{k(\omega)}$.

As before, we let $Y^{\mathcal{D}}_{C_{\omega,i}} \longrightarrow C_{\omega,i}$ denote the cover induced by the subgroup $N_{C_{\omega,i}} \triangleleft \pi_1(C_{\omega,i})$ of the covering datum $\mathcal{D}$. Recall the notations established in Section 5.1, then by Proposition 5.1.2, there exists a polynomial $f_{\omega,i} \in k(\omega)[x_i]$ such that $Y_{C_{\omega,i}} = Y_{[f_{\omega,i}]}$ is the Artin-Schreier cover associated to $[f_{\omega,i}] \in A/\wp(A)$. $Y_{[f_{\omega,i}]}$ will be referred to as an *Artin-Schreier representative* of the cover $Y_{C_{\omega,i}}$.

Now consider the natural surjection

$$p_i : A \twoheadrightarrow k(\omega)[x_i]/\wp(k(\omega)[x_i])$$

which gives a map whose kernel clearly contains $\wp(A)$, thus inducing a surjection

$$\pi_i : A/\wp(A) \twoheadrightarrow k(\omega)[x_i]/\wp(k(\omega)[x_i])\ .$$

For $F \in A$ mapping to $f_{\omega,i} \in k(\omega)[x_i]$, we thus have $\pi_i([F]) = [f_{\omega,i}]$. For any such F, the base change of the cover Y_F over $C_\omega \hookrightarrow \mathbb{A}^n$ is just $Y_{[f_{\omega,i}]}$ by construction. In particular, if $Y_{[f_{\omega,i}]}$ is the cover induced by an index-p normal subgroup $N_{C_{\omega,i}}$ of the covering datum $\mathcal{D}$, then $Y_{[F]}$ trivialises the covering datum $\mathcal{D}$ over C_ω.

Let $F \in A$, then we denote by $\deg_{x_i}(F)$ the degree of F considered as a polynomial of $k(x_1, \ldots, \hat{x}_i, \ldots, x_n)[x_i]$.

Lemma 5.2.3. *In the notation established above, let $f \in [f_{\omega,i}]$ be a representative of degree d. Then there exists a class $[F] \in \pi^{-1}(f)$ containing an element F such that $deg_{x_i}(F) \leq d$.*

Proof. Write $k(\omega) = k(\theta_1, \ldots, \hat{\theta}_i, \theta_n)$, where θ_{j+1} is such that

$$k(x_1, \ldots, \hat{x}_i, \ldots, x_{j+1})/(g_1, \ldots, \hat{g}_i, \ldots, g_{j+1}) = k[x_1, \ldots, \hat{x}_i, \ldots, x_j]/(g_1, \ldots, \hat{g}_i, \ldots, g_j)(\theta_{j+1}) \,.$$

Then any $a \in k(\omega)$ can be written as $a = \sum a_I \theta^I$, where $I = (\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{n-1})$ is a multi-index such that $\sum \alpha_j \leq [k(\omega) : k]$. If we now define an element of A by $F_a(x_1, \ldots, \hat{x}_i, \ldots, x_n) := \sum a_I \, x^I$, then $p(F_a) = a$ in $k(\omega)$. Moreover, given $f(x_i) = a_d x_i^d + \ldots + a_0 \in k(\omega)[x_i]$, we can define $F(x_1, \ldots, x_n) = F_{a_d} x_i^d + \ldots F_{a_0}$ such that $p_i(F(x_1, \ldots, x_n)) = f(x_i)$. Since $deg(f) = d$, we have $deg_{x_i}(F) = d$, and $\pi_i([F]) = f$ by the remark preceding the claim. $\square$

Recall that we set $A = k[x_1, \ldots, x_n]$, and recall from 5.1.18 the definitions of $M_\Delta(A)$ and $M_\Delta^i(A)$.

Lemma 5.2.4. *Let $A = k[x_1, \ldots, x_n]$ and fix a positive integer Δ. Let $M_i \subset M_\Delta^i(A)$ be vector subspaces in Artin-Schreier space $A/\wp(A)$. If we set $M = \cap M_i$, then*

$M \subset M_\Delta(A)$, i.e. every equivalence class in M contains an element G such that $\deg_{x_i} G \le \Delta$.

Proof. As $M \subset M_i \subset M^i_\Delta(A)$ for all i, every equivalence class of M contains an element F_i such that $\deg_{x_i}(F_i) \le D$. We have $F_i = F_j + h^p - h$ for some $h \in A$. Write

$$F_i = \sum_\alpha a_\alpha x_i^\alpha,$$
$$F_j = \sum_{\alpha'} b_{\alpha'} x_i^{\alpha'},$$

where the coefficents a_α and $b_{\alpha'}$ are polynomials in $k[x_1, \dots, \hat{x}_i, \dots, x_n]$, and consider the terms of F_i that are of highest x_i-degree D. Note that for any non-constant $h \in k[x_1, \dots, x_n]$, the monomials of highest x_i-degree in $h^p - h$ contain all x_k to a power divisible by p. Thus, if D is not divisible by p, or if a_D contains a monomial whose exponents are not all multiples of p, and $F_j + h^p - h$ is equal to F_i, then $h^p - h$ must be of x_i-degree strictly smaller than D. In particular, $a_D x_i^D$ is left unchanged by subtracting $h^p - h$, and must thus be equal to the corresponding terms in F_j. In particular, we must have equality of degrees in x_i: $\deg_{x_i}(F_i) = \deg_{x_i}(F_j)$.

Now consider the case where p divides D, and all exponents in the terms of a_D are divisible by p: Let c be the highest power of p dividing all exponents of terms in a_D, and such that p^c divides D. Then we can write

$$a_D = a'_D(x_1^{p^c}, \dots, x_n^{p^c}) = \sum_I a_I x_1^{p^c\alpha_1} \dots x_n^{p^c\alpha_n},$$

where the sum goes over all multi-indices $I = (p^c\alpha_1, \dots, p^c\alpha_n)$ of size $n-1$. Then let a'_I be such that ${a'^{p^c}}_I = a'_I$, and define $h = \sum_I a'_I x^{D'}$, where $D = p^c D'$. Clearly,

$$h^{p^c} - h = (h^{p^c} - h^{p^{c-1}}) + (h^{p^c-1} - h^{p^{c-2}}) + \ldots + (h^p - h) \in \wp(k[x_1, \ldots, x_n])$$

and $F'_i = F_i + (h^{p^c} - h) \sim F_i$ has $\deg_{x_i}(F'_i) < d$. Repeating the process if necessary, we thus obtain F'_i of x_i-degree D' such that either p does not divide D', or such that $a_{D'}$ contains a term in which a variable occurs with an exponent not divisible by p. We are thus reduced to the first case, and get

$$\deg_{x_i}(F_j) = \deg_{x_i}(F'_i) \leq \deg_{x_i}(F_i) \leq D\,.$$

Now use this procedure to compare F_1 to all F_i, then F_1 must satisfy $\deg_{x_i}(F_1) \leq \deg_{x_i}(F_i) \leq D$ for all i, as required. □

Proposition 5.2.5. *Let k be a finite field, and let $\Phi_i : \mathbb{A}^n_k \longrightarrow \mathbb{A}^{n-1}_k$ be the ith coordinate fibration. Let $\mathcal{D}$ be a geometrically bounded covering datum of index p on $\mathbb{A}^n_k$, and let $Y^{\mathcal{D}}_{C_{\omega,i}} \longrightarrow C_{\omega,i}$ be the cover of $C_{\omega,i}$ defined by $\mathcal{D}$. Then for all ω and i, there exists an Artin-Schreier representative $f_{\omega,i} \in k(\omega)[x_i]$ such that $Y_{[f_{\omega,i}]} = Y^{\mathcal{D}}_{C_{\omega,i}}$ and a positive integer Δ such that $\deg_{x_i}(f_\omega) \leq \Delta$.*

Proof. Since $\mathcal{D}$ is a covering datum of index p, for all ω and i, we have $Y^{\mathcal{D}}_{C_{\omega,i}} = Y_{[f_{\omega,i}]}$ for some $f_{\omega,i} \in k(\omega)[x_i]$. Let $s = \deg_{x_i}(f_{\omega,i})$. Whenever $s > 0$ and $p|s$, the highest term is of f is of the form $a x_i^{ps'}$. We may then make a change of variables $z \mapsto z - a' x_i^{s'}$, where $a'^p = a$, and thus replace the term of highest degree of f by its pth root. Repeating if necessary, we may without loss of generality assume that $(s, p) = 1$.

Recall that $Y_{[f_{\omega,i}]}$ denotes the normalisation of $\mathbb{A}^1_{k(\omega)}$ in

$$K(f_{\omega,i}) := k(\omega)(x_i)[z]/(z^p - z - f_{\omega,i}(x_i)) .$$

The regular compactification of $\mathbb{A}^1_{k(\omega)}$ is $\mathbb{P}^1_{k(\omega)}$; let $\overline{Y}_{[f_{\omega,i}]}$ denote the normalisation of $\mathbb{P}^1_{k(\omega)}$ in $K(f_{\omega,i})$. We have a commutative diagram

$$\begin{array}{ccc} Y_{[f_{\omega,i}]} & \longrightarrow & \overline{Y}_{[f_{\omega,i}]} \\ {\scriptstyle f_{C_{\omega,i}}}\downarrow & & \downarrow{\scriptstyle \overline{f}_{C_{\omega,i}}} \\ \mathbb{A}^1_{k(\omega)} & \longrightarrow & \mathbb{P}^1_{k(\omega)} \end{array} ,$$

where all covers are defined over $k(\omega)$. The ramification locus $R_{C_{\omega,i}}$ either consists of the point x_∞ at infinity corresponding to the fractional ideal $(1/x_i) \subset k(\omega)(x_i)$, or is empty.

Since $\mathcal{D}$ is assumed to be geometrically bounded, by Proposition 4.2.8, there exists a positive integer such that the ramification numbers $m_{y_\infty}(\omega, i) \leq \Delta$. By Corollary 5.1.4, we get that $\deg_{x_i} f_{\omega,i} \leq \Delta$ for all ω, i. □

Remark 5.2.6. Using Lemma 5.1.4 and the fact that the genera of degree-p covers of affine lines can easily be computed, we can also provide a direct proof of Proposition 5.2.5 which does not make use of Proposition 4.2.8:

Indeed, the degree of $C_{\omega,i}$ as a subvariety of $\mathbb{P}^n_k$ is given by $\deg_k C_{\omega,i} = [k(\omega) : k]$, and $f_{C_{\omega,i}} : Y_{C_{\omega,i}} \longrightarrow C_{\omega,i}$ is defined over $k_{C_{\omega,i}} = k(\omega)$, so we have $\deg_{k_{C_{\omega,i}}} C_{\omega,i} = 1$ for all fibers $C_{\omega,i}$ of one of the coordinate fibrations Φ_i.

As above, we let $y_\infty \in Y_{[f_{\omega,i}]}$ denote the unique point above the point x_∞ of

$\mathbb{P}^1_{k(\omega)} \backslash \mathbb{A}^1_{k(\omega)}$, then $[k(y_\infty) : k(x_\infty)] = 1$. Note also that $k(x_\infty) = k(\omega)$, and that we have $g_{C_{\omega,i}} = g_{\mathbb{A}^1_{k(\omega)}} = 0$ for all fibers $C_{\omega,i}$, $\overline{C}_{\omega,i} \backslash C_{\omega,i} = \mathbb{P}^1_{k(\omega)} \backslash \mathbb{A}^1_{k(\omega)}$. Then Hurwitz's formula (cf. 4.2.1) gives

$$\begin{aligned} g_{Y_{[f_{\omega,i}]}} &= p(g_{C_{\omega,i}} - 1) + 1 + \frac{1}{2} \deg_{k(\omega)} R_C \\ &= p(g_{C_{\omega,i}} - 1) + 1 + \frac{1}{2} \sum_{i=0}^{\infty} (|G_{y_\infty}| - 1) \, [k(y_\infty) : k(\omega)] \\ &= 1 - p + \frac{1}{2} \sum_{i=0}^{m_{y_\infty}} (|G_{y_\infty}| - 1) \\ &= \frac{(m_{y_\infty} - 1)(p-1)}{2} \end{aligned}$$

By Lemma 5.1.4, we have

$$g_{Y_{[f_{\omega,i}]}} = \frac{(s-1)(p-1)}{2} \tag{5.2.1}$$

Since $\mathcal{D}$ is geometrically bounded, Definition 4.2.2 gives a constant $\delta = \delta(1)$ such that $g_{Y_{C_{\omega,i}}} \leq \delta$ for all $\omega \in \mathbb{A}^{n-1}_k$, for all coordinate fibrations i. Thus, s is bounded by

$$s \leq \frac{2\delta}{p-1} + 1 \,,$$

which proves the Proposition as claimed.

Proof of Proposition 5.2.1. Following the notation established above, let $Y^{\mathcal{D}}_{C_{\omega,i}} \longrightarrow C_{\omega,i}$ denote the induced by the covering datum $\mathcal{D}$, and for each ω and i, let $f_{\omega,i} \in A^{\omega,i}$ be an Artin-Schreier representative of $Y^{\mathcal{D}}_{C_{\omega,i}}$.

Recall that $C_{\omega,i} \simeq \mathbb{A}^1_{k(\omega)}$ denotes the fiber of the ith projection fibration onto a coordinate hyperplane, and that $Y_{[f_{\omega,i}]} \longrightarrow C_{\omega,i}$ is thus defined over $k(\omega)$. Thus, all the fibers have degree $\deg_{k_C} C_{\omega,i} = \deg_{k(\omega)} \mathbb{A}^1_{k(\omega)} = 1$ over their field of definition $k_C = k(\omega)$. Thus, by Proposition 5.2.5, there exists a positive integer Δ such that $\deg(f_{\omega,i}) \leq \Delta$ for all ω, i.

From Lemma 5.2.3, we may now find lifts $F_{\omega,i} \in \pi^{-1}([f_{\omega,i}])$ such that $\deg_{x_i} F_{\omega,i} = \deg_{x_i} f_{\omega,i} \leq \Delta$ for all ω, for all $i = 1, \ldots, n$. Let $M_i = \oplus_\omega \mathbb{F}_p.[F_{\omega,i}]$ be the subspace of $A/\wp(A)$ generated by the lifts $\{F_{\omega,i}\}$ and recall the definition of $M_\Delta(A)$ and $M^i_\Delta(A)$ from Definition 5.1.18. Then $M_i \subset M^i_\Delta(A)$ for all i and $Y_{M_i} \longrightarrow \mathbb{A}^n$ trivialises all $N_{C_{\omega,i}}$ by construction.

The following proposition shows that Y_{M_i} is a weak trivialisation of $\mathcal{D}$, i.e. Y_{M_i} trivialises N_x for all closed points x contained in a fiber $C_{\omega,i}$ of Φ_i:

Proposition 5.2.7. *Let $f : X \longrightarrow X'$ be a fibration with regular fibers, $\mathcal{D}$ a covering datum on X. If $Y \longrightarrow X$ is a pro-étale cover trivialising $\mathcal{D}$ on each fiber of f, then Y weakly trivialises $\mathcal{D}$.*

Proof. Every closed point $x \in |X|$ is contained in some fiber, say C_ω for some $\omega \in X'$. As all the fibers are regular, x is always a regular point of C_ω. Then the proposition follows directly from Lemma 2.3.18. □

Now let $M = \overline{\langle M_i \rangle_i}$, then the associated p-elementary cover Y_M also weakly trivialises $\mathcal{D}$ by Theorem 5.0.8.

By Lemma 5.2.4, M is contained in $M_\Delta(A) = \langle [F] : \deg_{x_i} F \leq \Delta \text{ for all } i \rangle$. Any class in $M_\Delta(A)$ has a representative of total degree $\leq n\Delta$, so M_Δ is finitely generated. Thus M is of finite rank, so $Y_M \longrightarrow \mathbb{A}_k^n$ trivialises $\mathcal{D}$ by Theorem 2.3.10. Then $\mathcal{D}$ is realisable with a finite realisation by Theorem 2.3.15.

□

5.2.2 The Case of Open Subsets of $\mathbb{P}_k^n$

Let k be a finite field. In this section, we consider open subsets $X \subset \mathbb{P}_k^n$, and show that bounded covering data of prime index p on such X are finitely realisable (Theorem 5.2.11).

Since $\mathbb{P}_k^n$ is Noetherian, we can write X as a finite union of principal open subsets: $X = \cup_{j=1}^m D_+(\overline{G}_j)$, where for any j, $\overline{G}_j \in k[X_0, \ldots, X_N]$ is a homogeneous polynomial and defines a hypersurface of $\mathbb{P}_k^n$. We let $A = k[x_1, \ldots, x_n]$, and consider the open affine subset $\mathbb{A}_k^n = D_+(X_0) \simeq \mathit{Spec}\, A$ of $\mathbb{P}_k^n$. The intersection of a simple open $D_+(\overline{G}_1)$ with $\mathbb{A}^n$ is given by $D(G)$, where $G \in A$ is the dehomogenisation of $\overline{G}_1$.

Let A_G denote the localisation of A at the set $\{G^a : a \in \mathbb{N}\}$. Then $\mathit{Spec}\, A_G \simeq D(G) \subset \mathbb{A}_k^n$.

Definition 5.2.8. Let $\phi_i : k[x_i] \hookrightarrow A$ be the inclusion, then we shall call the induced morphism $\Phi_i : \mathbb{A}_k^n \longrightarrow \mathbb{A}_k^1$ the *projection fibration onto the i-th coordinate axis.*

If $\omega \in \mathbb{A}_k^1$ is a closed point with residue field $k(\omega) = k[x_i]/(h_i(x_i))$, let $A^{\omega,i}$ denote the ring $k(\omega)[x_1, \ldots, \hat{x}_i, \ldots, x_n]$. Then the fiber above the closed point $\omega \in \mathbb{A}_k^1$ is given by $C_{\omega,i} \simeq Spec\, A^{\omega,i} \subset \mathbb{A}^n_k$.

Recall from Section 5.1.1 that the surjection $p_{\omega,i} : A \twoheadrightarrow A^{\omega,i}$ induces a surjection of Artin-Schreier spaces

$$\pi_{\omega,i} : A/\wp(A) \twoheadrightarrow A^{\omega,i}/\wp(A^{\omega,i})$$

such that $\pi_{\omega,i}([F]) = f$ for any $F \in p^{-1}_{\omega,i}(f)$.

Now let $g^{\omega,i} = g$ be the image of G under $p_{\omega,i}$, and let $f = p_{\omega,i}(F)$ be the image of a polynomial $F \in A$. Then $p_{\omega,i}$ induces a natural surjection

$$p'_{\omega,i} : A_{\mathbf{G}} \twoheadrightarrow A^{\omega,i}_{\mathbf{g}}$$

given by sending $F/G^a \mapsto f/g^a$. Composing with the natural projection, we get a surjection $\overline{p}'_{\omega,i} : A_G \twoheadrightarrow A^{\omega,i}_g/\wp(A^{\omega,i}_g)$ whose kernel contains $\wp(A_G)$. Thus we get a surjection of Artin-Schreier spaces

$$\pi'_{\omega,i} : A_G/\wp(A_G) \twoheadrightarrow A^{\omega,i}/\wp(A^{\omega,i}) \tag{5.2.2}$$

such that $\pi'_{\omega,i}([F/G^a]) = f/g^a$ for any $F \in p'^{-1}_{\omega,i}(f)$.

Lemma 5.2.9. *Given* $[f/g^a] \in A^{\omega,i}_g/\wp(A^{\omega,i}_g)$*, there exists a preimage* $[F/G^a] \in \pi_i^{-1}([f/g^a])$ *which has a representative such that* $\deg_{x_j} F = \deg_{x_j} f$ *for all* $j \neq i$*.*

Proof. We prove the claim by constructing $F \in p_i^{-1}(f)$ such that $\deg_{x_j} F = \deg_{x_j} f$ for all $j \neq i$. Then $\pi'_{\omega,i}([F/G^a]) = f/g^a$ so F/G^a is the required representative.

Let θ be a primitive element of $k(\omega)$ over k, i.e. such that $k(\omega) = k(\theta)$, then we can write any $a \in k(\omega)$ as $a = \sum_{k=0}^{m-1} \alpha_k \theta^k$, where $m = [k(\omega) : k]$ and $\alpha_k \in k$. Now let $F_a(x_i) = \sum_{i=0}^{m-1} \alpha_k x_i^k$, then F_a maps to a under the surjection $k[x_i] \twoheadrightarrow k(\omega)$.

Writing f as $f(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \sum_I a_I x^I x_i^k$, where the sum goes over multiindices I of size $n-1$, and $x^I = x_1^{\alpha_1} \ldots \hat{x}_i \ldots x_n^{\alpha_{n-1}}$ for the multiindex $I = (\alpha_1, \ldots, \alpha_{n-1})$, we may define F by $F(x_1, \ldots, x_n) = \sum_I F_{a_I} x^I x_i^k$. Then $p_i(F) = f$ and therefore $\pi_i([F]) = [f]$ as claimed.

Noting that we have constructed F such that $\deg_{x_j} F = \deg_{x_j} f$ for all $j \neq i$, and such that $\deg_{x_i} F \leq [k(\omega) : k]$ concludes the proof of the claim. $\square$

Recall from Section 5.1.1 that Artin-Schreier space $A_G/\wp(A_G)$ classifies p-exponent covers of $Spec\, A_G$. Also recall the special vector subspaces $M_\Delta(A_G)$, $M_\Delta^i(A_G)$ of $A_G/\wp(A_G)$ (Definition 5.1.18).

Lemma 5.2.10. *Let A_G be the localisation of $A = k[x_1, \ldots, x_n]$ at the set $\{G^a : a \in \mathbb{Z}\}$, and fix a positive integer Δ. Let $M_i \subset M_\Delta^i(A_G)$ be vector subspaces of the associated Artin-Schreier space $A_G/\wp(A_G)$. If we set $M = \cap M_i$, then $M \subset M_\Delta(A_G)$, i.e. every equivalence class in M contains an element F/G^a such that $a \leq \Delta$, and $\deg_{x_j} F \leq a \deg_{x_j}(G)$ for all j.*

Proof. As $M \subset M_i$ for all i, any equivalence class contains an element F_i/G^{a_i} such that $\deg_{x_j} F_i \leq a_i \deg_{x_j} G$ for all $j \neq i$. For all pairs (i, j) such that $j \neq i$, there thus exists some element F'/G^b such that

$$\frac{F_i}{G^{a_i}} + \frac{F'^p}{G^{pb}} - \frac{F'}{G^b} = \frac{F_j}{G^{a_j}}$$

Let $\widetilde{a}_i = \max\{a_i, bp\}$, then $a_j \leq \widetilde{a}_i$ and the above is equivalent to

$$F_i G^{\widetilde{a}_i - a_i} + F'^p G^{\widetilde{a}_i - bp} - F' G^{\widetilde{a}_i - p} = F_j G^{\widetilde{a}_i - a_j} \tag{5.2.3}$$

Bringing the terms containing F' to the other side, we have equality of x_i-degrees as follows:

$$\deg_{x_i} F_i = \max\{p \deg_{x_i} F' + (a_i - bp) \deg_{x_i} G, \deg_{x_i} F' + (a_i - p) \deg_{x_i} G, \deg_{x_i} F_j + (a_i - a_j) \deg_{x_j} G\} .$$

So if $\deg_{x_i} F' \leq b \deg_{x_i} G$, then this implies that

$$\deg_{x_i} F_i \leq \max\{a_i \deg_{x_i} G, \deg_{x_i} F_j + (a_i - a_j) \deg_{x_i} G\} .$$

and therefore

$$\deg_{x_i} F_i \leq a_i \deg_{x_i} G .$$

If this is the case, we can take $F/G^a = F_i/G^{a_i}$ and are done. So now assume that $\deg_{x_i} F' > b \deg_{x_i} G$. Then the middle term of the left-hand side in (5.2.3) has bigger x_i-degree than the term to its right.

If $\deg_{x_i} F_i \leq \deg_{x_i} F_j + (a_i - a_j) \deg_{x_i} G$, then $\deg_{x_i} F_i \leq a_i \deg_{x_i} G$, so there is nothing to be shown. Thus assume that $\deg_{x_i} F_i > \deg_{x_i} F_j + (a_i - a_j) \deg_{x_i} G$. Then the middle term has to cancel the terms of highest x_i-degree of the term to the left; in particular, the x_i-degree of the first and middle term must be equal. This implies that

$$\deg_{x_i} F_i = (a_i - bp) \deg_{x_i} G + p \deg_{x_i} F' .$$

In particular, p must divide $\deg_{x_i} F_i - a_i \deg_{x_i} G$. Now, we define the i-order, an order on the terms of a polynomial $\in k[x_1, \ldots, x_n]$, as follows. The lexicographical order on polynomials in $k[x_1, \ldots, x_n]$ has considers the x_1-degree first, then the x_2-degree etc. In the i-order of terms in a polynomial, we firstly then consider the degree in x_i first, and the remaining degrees in lexicographical order. Then we can write

$$\begin{aligned} F_i &= \alpha_0 x_1^{\alpha_1} \ldots x_n^{\alpha_n} + (\text{terms of lower } i\text{-order, same } x_i\text{-degree}) + (\text{terms of lower } x_i\text{-degree})\,, \\ G &= \beta_0 x_1^{\beta_1} \ldots x_n^{\beta_n}\, (\text{terms of lower } i\text{-order, same } x_i\text{-degree}) + (\text{terms of lower } x_i\text{-degree})\,, \\ F' &= \alpha_0 x_1^{\alpha'_1} \ldots x_n^{\alpha'_n} + (\text{terms of lower } i\text{-order, same } x_i\text{-degree}) + (\text{terms of lower } x_i\text{-degree})\ , \end{aligned}$$

and we must have that $\alpha_l - a_i\beta_l = p\,(\alpha'_l - b\beta_l)$ for all l. In particular, p must divide $\alpha_l - a_i\beta_l$ for all l.

Thus we can write $\alpha_l - a_i\beta_l = pN_l$, and let $N'_l = N_l - \beta_l$. Then setting $b' = \lfloor \frac{a_i}{p} \rfloor$, we have $b'p \leq a_i$. Let $F' = \gamma'_0 x_1^{N'_1} \ldots x_n^{N'_n}$, where γ'_0 is such that $\alpha_0 = \gamma_0'^p \beta_0^{(bp-a_i)}$. Then we can replace $\frac{F_i}{G^{a_i}}$ by

$$\frac{F_i + F'^p G^{a_i - bp} - F' G^{a_i - b}}{G_i^{a_i}}\,,$$

thereby replacing the term of highest i-order in the numerator F'_i by a lower-order term. For all $j \neq i$, one now checks that the x_j-degree of F'_i is still bounded by the x_j-degree of the denominator: $\deg_{x_j} F'_i \leq a'_i \deg_{x_j} G$. Also, we have chosen b' so that $b'p \leq a_{i,k}$, so that the power of G in the denominator remains unchanged: $a'_i = a_i \leq D$. Thus we may repeat this process until we have replaced all terms of highest x_i-degree by terms of lower x_i-degree, and we repeat further until we get $\frac{F'_i}{G^{a'_i}}$ such that $p \not| \alpha_l - a_i\beta_l$ for some l, or such that $\deg_{x_i} F'_i$ is less than that of the denominator. In the latter case, we get that $\deg_{x_i} F_i' \leq a_i \deg_{x_i} G$, and can take $F_i'/G^{a_i'}$ as the required representative. In the former case, we again compare $F_i'/G^{a_i'}$ to the representative F_j/G^{a_j}:

$$F_i' G^{\widetilde{a}_i - a_i'} + F'^p G^{\widetilde{a}_i - bp} - F' G^{\widetilde{a}_i - b} = F_j G^{\widetilde{a}_i - a_j}.$$

The condition $p \not| \alpha_l - a_i' \beta_l$ ensures that no term of $F'^p G^{a' i - b}$ can match the term of highest x_i-degree of $F_i G^{\widetilde{a}_i - a_i}$; thus $\deg_{x_i} F_i' = \deg_{x_i} F_j \leq a_j \deg_{x_i} G$ and we may take $F_i'/G^{a_i'}$ as the required representative.

□

We can now state and prove the following Theorem:

Theorem 5.2.11. *Let k be a finite field of characteristic $char(k) = p$, and let X be an open subset of $\mathbb{P}^n_k$, and $\mathcal{D}$ a geometrically bounded covering datum of prime index p on X. Then there exists an étale cover $Y \longrightarrow X$ realising $\mathcal{D}$.*

Proof. We have $\mathbb{P}^n_k \simeq Proj\, k[X_0, \ldots, X_n]$. As above, we write $X = \cup_{j=1}^m D_+(\overline{G}_j)$, where for any j, $\overline{G}_j$ is a homogeneous polynomial in $X_0, \ldots, X_N$ and defines a hypersurface of $\mathbb{P}^n_k$. By Propositions 2.3.13 and 2.3.15, it will suffice to find a trivialisation of $\mathcal{D}$ over $D_+(\overline{G}_1)$, an open subscheme.

Now let $\Phi_i : \mathbb{A}^n_k \longrightarrow \mathbb{A}^1_k$ be the projection onto a coordinate axis, as defined in Definition 5.2.8. For a closed point $\omega \in \mathbb{A}^1_k$, let $H_{\omega,i} \simeq \mathbb{A}^{n-1}_{k(\omega)}$ denote the fiber above ω, and set $H'_{\omega,i} = H_{\omega,i} \cap X$.

Then $H'_{\omega,i}$ is the fiber of $\Phi_i|_X : X \longrightarrow \mathbb{A}^1_k$, the restriction of Φ_i to X.

Construction 5.2.12 (Induction Construction). The fiber $H'_{\omega,i}$ of $\Phi_i|_X$ above a closed point $\omega \in \mathbb{A}^1_i$ is of dimension $n-1$ and isomorphic to $D(g) \subset \mathbb{A}^{n-1}_{k(\omega)}$, where $g = g^{\omega,i}$ is the image of G under the canonical surjection $p_{\omega,i} : A \twoheadrightarrow A^{\omega,i}$, which is given by "modding out by $(h_i(x_i))$". Let $\mathcal{D}_{\omega,i}$ denote the restriction of $\mathcal{D}$ to $H'_{\omega,i}$. Then $\mathcal{D}_{\omega,i}$ is a geometrically bounded covering datum of index bounded by p on $H'_{\omega,i}$.

Thus we may use induction on the dimension n of X to prove Theorem 5.2.11. The induction hypothesis is that $\mathcal{D}_{\omega,i}$ has a finite realisation $Y_{\omega,i} \longrightarrow H'_{\omega,i}$, and the base case is given by $n = 2$, where the fibers of Φ_i are curves on which $\mathcal{D}_{\omega,i}$ will be trivially realisable. Indeed, we have $H_{\omega,i} \simeq \mathbb{A}^1_{k(\omega)}$, $H'_{\omega,i} \simeq D(g) \subset \mathbb{A}^1_{k(\omega)}$ and $\mathcal{D}_{\omega,i}$ is just the p-exponent cover $Y^{\mathcal{D}}_{H'_{\omega,i}} \longrightarrow H'_{\omega,i}$ cover of regular curves induced by $\mathcal{D}$.

We note that since $\mathcal{D}_{\omega,i}$ is of index bounded by p, the realisation $Y_{\omega,i}$ of $\mathcal{D}_{\omega,i}$ is either trivial or a $\mathbb{Z}/p$-cover of $D(g)$. By the Artin-Schreier Theory (cf. Section 5.1.1), we have $Y_{\omega,i} = Y_{[f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}]}$ for some element $f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}} \in A_g^{\omega,i}$.

Key Properties of this construction are summarised in the following lemma:

Lemma 5.2.13. *Let k be a finite field, let $\mathcal{D}$ be a geometrically bounded covering datum of index $p = char(k)$ on $X = D_+(\overline{G}) \subset \mathbb{P}^n_k$, and identify $\mathbb{A}^n_k$ with $V_+(X_0)$. Let Φ_i be the projection fibration onto the ith coordinate axis, let $H_{\omega,i}$ be the fiber above a closed point $\omega \in \mathbb{A}^1_k$. Set $H'_{\omega,i} = H_{\omega,i} \cap X \subset \mathbb{A}^n_k$, and let $\mathcal{D}_{\omega,i}$ denote the restriction of $\mathcal{D}$ to $H'_{\omega,i}$. If $\mathcal{D}_{\omega,i}$ is effective with a finite realisation $Y_{\omega,i} \longrightarrow C'_{\omega,i}$, then there exists a positive integer Δ such that*

1. *$Y_{\omega,i} = Y_{[f_{\omega,i}/g^{a_{\omega,i}}]}$ with $a_{\omega,i} \leq \Delta$ for all ω,i.*

2. $\deg_{x_j} f_{\omega,i} \leq a_{\omega,i} \deg_{x_j} g_{\omega,i} + \Delta$ *for all* $j \neq i$.

Proof of 5.2.13.1. If $\mathcal{D}$ is geometrically bounded, let Δ be the constant from Proposition 4.2.2, i.e. such that the ramification number $m_{(C,y)}$ associated to any closed point y above $\overline{C}\backslash C$ on the compactification of a curve $C \subset X$ is bounded by Δ: $m_{(C,y)} \leq \Delta$ for all pairs (C, y).

Note that if $\mathcal{D}$ is of index p, then $\mathcal{D}_{\omega,i}$ is of index bounded by p. If $\mathcal{D}_{\omega,i}$ is trivial, the lemma is trivially true: Let $g = 1$, $a = 0$, $f_{\omega,i} \in \wp(A)$ then $Y_{[f_{\omega,i}]} \longrightarrow C$ realises $\mathcal{D}_{\omega,i}$, and any positive integer Δ satisfies the condition of the lemma.

So if there exist ω and i such that $a_{\omega,i} > \Delta$, then $\mathcal{D}_{\omega,i}$ is of index p, and its realisation is a proper $\mathbb{Z}/p\mathbb{Z}$-cover $Y_{[f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}]} \longrightarrow H'_{\omega,i}$. We want to find a curve $C \subset H'_{\omega,i}$ on which we have $m_{(C,x)} > \Delta$ for some closed point $x \in \overline{C}\backslash C$.

We let $f = f_{\omega,i}$, $g = g_{\omega,i}$ and $a = a_{\omega,i}$, and let $j \in \{1, \ldots, n\}$, $j \neq i$. Consider the projection fibration onto the jth coordinate hyperplane $\Phi_j : \mathbb{A}^{n-1}_{k(\omega)} \longrightarrow \mathbb{A}^{n-2}_{k(\omega)}$. Let $C_{\eta,j} \subset \mathbb{A}^{n-1}_{k(\omega)}$ be the fiber above a closed point $\eta \in \mathbb{A}^{n-2}_{k(\omega)}$.

Define $A^{\eta,j} := k(\eta)[x_1, \ldots, \hat{x}_i, \hat{x}_j, \ldots, x_n]$, then we have a surjection

$$A^{\omega,i} \twoheadrightarrow A^{\eta,j} .$$

Denote the images of f, g by f_η, g_η. Now let $A^{\omega,i}_g$, $A^{\eta,j}_{g_\eta}$ denote the localisations at the sets $\{g^b : b \in \mathbb{Z}\}$ and $\{g_\eta^b : b \in \mathbb{Z}\}$, respectively, then we can extend the above surjection to a surjection

$$p_{\eta,j} : A^{\omega,i}_g \twoheadrightarrow A^{\eta,j}_{g_\eta} \text{ , which maps}$$

$$\frac{f}{g^a} \mapsto \frac{f_\eta}{{g_\eta}^a}.$$

Then since $Y_{[f/g^a]} \longrightarrow H'_{\omega,i}$ realises $\mathcal{D}_{\omega,i}$ on $H'_{\omega,i}$, the pullback $Y_{[\overline{f}/\overline{g}^a]} \longrightarrow C_{\eta,j}$ realises the covering datum over $C_{\eta,j}$.

$$\begin{array}{ccc} Y_{[\overline{f}/\overline{g}^a]} = Y \times_{C\eta,j} Y_{[f/g^a]} & \longrightarrow & Y_{[f/g^a]} \\ \downarrow & & \downarrow \\ C_{\eta,j} & \longrightarrow & H'_{\omega,i}. \end{array}$$

Now write $f/g^a = f'/g_1^{a_1} \cdots g_l^{a_l}$, where f' and g_j are pairwise relatively prime for all j. Then there is a one-to-one correspondence between the g_j and the points of codimension one x_j in $\mathbb{P}^{n-1}_{k(\omega)} \backslash D_+(\overline{g})$ which correspond to finite primes of $\mathbb{A}^{n-1}_{k(\omega)}$. Let ν_{x_j} be the valuation associated to g_j and x_j, then $\nu_{x_j}(f/g^a) = -a_j \leq -\Delta$.

Claim 5.2.14. Let $\overline{f}' := p_{\eta,j}(f')$, $\overline{g}_r = p_{\eta,j}(g_r)$ for $r = 1, \ldots, l$ denote the images of f' and g_r in $A^{\eta,j}$. Then $\overline{f}$ is relatively prime to $\overline{g}_r$ for all r.

Proof. Since f' and g_r are relatively prime in $A^{\omega,i}$, there exist polynomials h_r,$h'_r \in A^{\omega,i}$ such that $h_r\, f' + h'_r\, g_r = 1$. Letting $\overline{h}_r$, $\overline{h}'_r$ denote the images of h_r, h'_r in $A^{\eta,j}$, we have $\overline{h}_r\, \overline{f}'_j + \overline{h}'_r\, \overline{g}_r = 1$, as claimed. ☐

Now we have $p_{\eta,j}(\frac{f}{g^a}) = \frac{\overline{f'}}{\overline{g}_1^{a_1} \cdots \overline{g}_l^{a_l}} = \frac{\overline{f}}{\overline{g}^a}$. Then if y is a closed point of $\overline{C}_{\eta,j} \backslash C_{\eta,j}$ lying in the closure $V(g_r)$ of x_r, we thus have $\nu_y(\overline{f}/\overline{g}^a) = \nu_y(\overline{f}'/\overline{g}_1^{a_1} \cdots \overline{g}_l^{a_l}) \leq -a_r$

By Lemma 5.1.4, if y' denotes a point of lying above y we then have $m_{C_{\eta,j},y'} = -\nu_y(f/g^a) = a_j > \Delta$, which contradicts the uniform bound on ramification numbers of $\mathcal{D}_{\omega,i}$.

□

Proof of 5.2.13.2. Let $\overline{H}_{\omega,i} \simeq \mathbb{P}^{n-1}_{k(\omega)}$ denote the closure of $H'_{\omega,i}$ in $\mathbb{P}^n_k$, and let $\overline{H}'_{\omega,i}$ denote the closure of $H'_{\omega,i}$ in $X = D_+(\overline{G})$.

Recall that $\mathcal{D}$ is a covering datum defined on all of $X = D_+(\overline{G})$, and not just on the affine part $X \cap V_+(X_0) \simeq D(G)$. Similarly, $\mathcal{D}_{\omega,i}$ is defined on all of $\overline{H}_{\omega,i}$. So if $Y_{\omega,i} \longrightarrow H'_{\omega,i}$ realises $\mathcal{D}_{\omega,i}$ then the normalisation $\overline{Y}_{\omega,i}$ of $\overline{H}'_{\omega,i}$ is étale at $\overline{H}'_{\omega,i} \backslash H'_{\omega,i} \simeq V_+(X_0) \cap \overline{H}'_{\omega,i}$.

Since $\mathcal{D}$ is geometrically bounded by assumption, we let Δ be the positive integer defined in Proposition 4.2.2.

We argue by contradiction, and assume that there exist ω, i, $j \neq i$ be such that $\deg_{x_j} f_{\omega,i} > a_{\omega,i} \deg_{x_j} g_{\omega,i} + \Delta$ in $Y_{[f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}]}$.

Now let $\Phi'_j : \mathbb{A}^{n-1}_{k(\omega)} \longrightarrow \mathbb{A}^{n-2}_{k(\omega)}$ be the projection fibration onto the jth hyperplane. For $\eta \in \mathbb{A}^{n-2}_{k(\omega)}$ a closed point, let $C_{\eta,j} = Spec\, k(\eta)[x_i] \simeq \mathbb{A}^1_{k(\eta)}$ be the fiber above η, and note that $\overline{C}_{\eta,j} \simeq \mathbb{P}^1_{k(\eta)}$. Denote $C_{\eta,j} \cap X$ by $C'_{\eta,j}$. Let $A^{\eta,j} = k(\eta)[x_j]$, then we have the canonical surjection

$$p_{\eta,j} : A^{\omega,i} \twoheadrightarrow A^{\eta,j} .$$

Since $\cup_\eta C_{\eta,j} = \mathbb{A}^{n-1}_{k(\omega)}$,we have $\cup_\eta \overline{C}_{\eta,j} = \overline{H}_{\omega,i}$ and $\cup_\eta \left(\overline{C}_{\eta,j} \backslash C_{\eta,j}\right) = \overline{H}_{\omega,i} \backslash H_{\omega,i}$. Therefore

$$\cup_\eta \left(\left(\overline{C}_{\eta,j} \backslash C_{\eta,j}\right) \cap X\right) = \left(\overline{H}_{\omega,i} \backslash H_{\omega,i}\right) \cap X .$$

We have two cases: If the left-hand side is non-empty, there exists an $\eta \in \mathbb{A}^{n-2}_{k(\omega)}$ such that $\left(\overline{C}_{\eta,j} \backslash C_{\eta,j}\right) \cap X = \mathbb{P}^1_{k(\eta)} \backslash \mathbb{A}^1_{k(\omega)} \cap X$ is non-empty; let x_∞ denote its unique point. Note that since $x_\infty \in X$, $\overline{Y}_{[f/g^a]} \longrightarrow \overline{H}_{\omega,j}$ must be étale at x_∞. In particular, if $\overline{f} := p_{\eta,j}(f)$, $\overline{g} := p_{\eta,j}(g)$, then the pullback to $\overline{C}_{\eta,j}$, given by $\overline{Y}_{[\overline{f}/\overline{g}^a]} \longrightarrow \overline{C}_{\eta,j}$, must be étale at $X_\infty \in \overline{C}_{\eta,j}$.

We have $C_{\eta,j} \simeq Spec\, k(\eta)[x_j]$, so we let $\nu_{x_\infty} = \nu_{1/x_j}$ denote the valuation associated to point x_∞ at infinity, then

$$\begin{aligned} \nu_{1/x_j}(\overline{f}/\overline{g}^a) &= -\deg_{x_j}(\overline{f}/\overline{g}^a) = -\deg_{x_j}(f/g^a) \\ &= a\deg_{x_j} g - \deg_{x_j} f < -\Delta < 0 \end{aligned}$$

Then by Lemma 5.1.3, $\overline{Y}_{[\overline{f}/\overline{g}^a]} \longrightarrow \overline{C}_{\eta,j}$ is not étale at x_∞, a contradiction.

Now consider the case where $\overline{Y}_{[f/g^a]} \longrightarrow \overline{H}_{\omega,j}$ is not necessarily étale at the point at infinity of any fiber $C_{\eta,j}$. By definition of Δ in Proposition 4.2.2, we must still have that $m_{(C,x)} \leq \Delta$ for all pairs of closed points x on the closure of a curve $C \subset X$.

But by Lemma 5.1.4, $m_{C,x_\infty} = -\deg_{x_j}(\overline{f}/\overline{g}^a) > \Delta$, a contradiction.

□

Returning to the Proof of Theorem 5.2.11, by Lemmas 5.2.13 and 5.2.13, there exists a positive integer Δ such that each cover $Y_{\omega,i} \longrightarrow H'_{\omega,i}$ has an Artin-Schreier representative $f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}$ such that for all ω and i, we have $a_{\omega,i} \leq \Delta$ and $\deg_{x_j} f_{\omega,i} \leq a_{\omega,i}\deg_{x_j} g_{\omega,i} + \Delta$ for all $j \neq i$.

By Lemma 5.2.9, we may always choose a lift $[F_{\omega,i}/G^{a_{\omega,i}}]$ of $[f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}]$ such that $\deg_{x_j} F_{\omega,i} = \deg_{x_j} f_{\omega,i}$ for all $j \neq i$. So fix i and let $M_i =< [F_{\omega,i}/G^{a_{\omega,i}}] : \omega \in \mathbb{A}^1_k >$ be the vector subspace of $A_G/\wp(A_G)$ generated by these lifts. Then the associated cover $Y_{M_i} \longrightarrow X$ trivialises $\mathcal{D}$ on each of the fibers $H'_{\omega,i}$ by construction. By Lemma 5.2.7, it follows that Y_{M_i} weakly trivialises the covering datum $\mathcal{D}$ for all i. Theorem 5.0.8 shows that the cover Y_M associated to $M = \overline{< M_i >_i}$ also weakly trivialises $\mathcal{D}$.

Recall Definition 5.1.18 of the vector subspaces $M_\Delta(A_G)$, $M^i_\Delta(A_G)$ of Artin Schreier space$A_G/\wp(A_G)$ and note that since k is a finite field, $M_\Delta(A_G)$ is finite. By Lemma 5.2.10, $M \subset M_\Delta(A_G)$, so M is also finite.

Therefore, $f_M : Y_M \longrightarrow X$ is a finite étale cover of X weakly trivialising $\mathcal{D}$. By Lemma 2.3.10, f_M trivialises $\mathcal{D}$, so the covering datum $\mathcal{D}$ is effective with a finite realisation by Theorem 2.3.15. The realisation is étale above X by Lemma 2.1.15, so all induction assumptions are satisfied.

□

5.3 Covering data of Prime Power Index

In this section, we let $X \subset \mathbb{P}^n_k$ be an open subset of, and consider geometrically bounded covering data $\mathcal{D}$ of cyclic index p^m on X. The goal is to show the following:

Theorem 5.3.1. *Let k be a finite field with characteristic $char(k) = p$, and let X be an open subset of $\mathbb{P}^n_k$. Let $\mathcal{D}$ a geometrically bounded covering datum of cyclic index p^m on X. Then there exists an étale cover of $Y \longrightarrow X$ realising $\mathcal{D}$.*

Remark 5.3.2. Setting $X = \mathbb{A}^n_k \subset \mathbb{P}^n_k$, the theorem equals includes the case where

$\mathcal{D}$ is a covering datum of cyclic index p^m on $\mathbb{A}^n_k$.

As before, we let $\Phi_i : \mathbb{A}^n_k \longrightarrow \mathbb{A}^1_k$ denote the projection fibration onto the ith coordinate axis. Denote by $H_{\omega,i} \simeq \mathbb{A}^{n-1}_{k(\omega)}$ the fiber above a closed point $\omega \in \mathbb{A}^1_k$, and by $H'_{\omega,i} = H_{\omega,i} \cap D(G) \simeq D(g)$ the intersection of the fiber with X. If G denotes the dehomogenisation of $\overline{G}$ at X_0, then $g = g^{\omega,i}$ is the image of G under $p_{\omega,i} : A \twoheadrightarrow A^{\omega,i}$.

Then as in the Induction Construction 5.2.12, we assume by induction on n that the restriction $\mathcal{D}_{\omega,i}$ of $\mathcal{D}$ to $H'_{\omega,i}$ is realisable. As such, it is either trivial or a $\mathbb{Z}/p^s$-cover of $C'_{\omega,i}$ for some $s \leq m$; in either case it is represented by some $[\mathbf{f}] = [\frac{f_0}{g^{a_0}}, \ldots, \frac{f_{m-1}}{g^{a_{m-1}}}] \in W_m(A_g)$. As in 5.2.12, the base case is given by the covering datum itself.

Lemma 5.3.3. *Let $X = D_+(G)$ be a simple open subset of $\mathbb{P}^n_k$, let $\mathcal{D}$ be a geometrically bounded covering datum of cyclic prime power index p^m on X, and let $\mathcal{D}_{\omega,i}$ be the restriction of $\mathcal{D}$ to $H'_{\omega,i}$. Assume that $\mathcal{D}_{\omega,i}$ is realised by an Artin-Schreier-Witt cover $Y_{\omega,i} = Y_{[\mathbf{f}]}$, where $[\mathbf{f}] = [\frac{f_0}{g^{a_0}}, \ldots, \frac{f_{m-1}}{g^{a_{m-1}}}] \in W_m(A_G)/\wp(A_G)$.*

Then there exists a postive integer Δ such that

1. *$p^{m-s-1}a_s \leq \Delta$ for all s and*
2. *$p^{m-s-1}\left(\deg_{x_j} f_s - a_s \deg_{x_j} g\right) \leq \Delta$ for all s, for all $j \neq i$.*

Proof. We proceed by induction on the exponent m: For $m = 1$, it was shown in Lemmas 5.2.13 and 5.2.3 that we can take Δ to be the bound on ramification numbers given by Lemma 4.2.2.

So now assume that the lemma holds for $m-1$: If $\mathcal{D}$ is a geometrically bounded covering datum on X, then there exists a positive integer Δ' such that whenever $\mathcal{D}_{\omega,i}$ is realisable by a $\mathbb{Z}/p^m\mathbb{Z}$-cover $Y_{[\mathbf{f'}]} \longrightarrow H'_{\omega,i}$ for some element $[\mathbf{f'}] = [\frac{f'_0}{g^{a'_0}}, \ldots, \frac{f'_{m-2}}{g^{a'_{m-2}}}]$ of $W_{m-1}(A_G)/\wp(W_{m-1}(A_G))$, we have that

$$p^{m-s-2}a'_s \leq \Delta' \text{ for all } s, \text{ and}$$
$$p^{m-s-2}\left(\deg_{x_j} f'_s - a'_s \deg_{x_j} g\right) \leq \Delta' \text{ for all } s, \text{ for all } j \neq i\,. \tag{5.3.1}$$

Recall that for $\mathbf{f} \in W_m(A_G)$, $k = 1, \ldots m-1$, we let $\mathbf{f^{(k)}}$ denote the truncated Witt vector of length k, and that we have a tower of succcessive $\mathbb{Z}/p\mathbb{Z}$-covers

$$Y_{[\mathbf{f}]} \longrightarrow Y_{[\mathbf{f^{(m-1)}}]} \longrightarrow \ldots \longrightarrow Y_{[\mathbf{f_0}]} \longrightarrow H'_{\omega,i}\,. \tag{5.3.2}$$

Then applying (5.3.1) to the truncated Witt vector $[\mathbf{f^{(m-1)}}]$, we get

$$p^{m-s-2}a_s \leq \Delta' \text{ for all } s, \text{ and}$$
$$p^{m-s-2}\left(\deg_{x_j} f_s - a_s \deg_{x_j} g\right) \leq \Delta' \text{ for all } s, \text{ for all } j \neq i\,. \tag{5.3.3}$$

Now let $\Phi'_j : \mathbb{A}^{n-1}_{k(\omega)} \longrightarrow \mathbb{A}^{n-2}_{k(\omega)}$ be the projection fibration onto jth coordinate hyperplane of $H_{\omega,i} \simeq \mathbb{A}^{n-1}_{k(\omega)}$. Let $\eta \in \mathbb{A}^{n-2}_{k(\omega)}$ be a closed point, recall that we set $A^{\omega,i} = k(\omega)[x_1, \ldots, \hat{x}_i, \ldots, x_n]$, and define $A^{\eta,j} = k(\eta)[x_j]$. Let $g = g^{\eta,j}$ is the image of G under the combined surjections:

$$p_{\eta,j} \circ p_{\omega,i} : A \twoheadrightarrow A^{\omega,i} \twoheadrightarrow A^{\eta,j}\,,$$

let $C_{\eta,j} \simeq Spec\, k(\eta)[x_j] = \mathbb{A}^1_{k(\eta)}$ denote the fiber above η and set $C'_{\eta,j} := C_{\eta,j} \cap X \simeq D(g)$.

For an element $f \in A^{\omega,i}$, we set $\overline{f} := p_{\eta,j}(f)$, and note that $\deg_{x_j} f = \deg_{x_j} \overline{f}$.

For $\mathbf{f} \in W_m(A_G)$, we set $\overline{\mathbf{f}} := p_{\eta,j}(\mathbf{f}) = [\frac{\overline{f}_0}{\overline{g}^{a_0}}, \ldots, \frac{\overline{f}_{m-1}}{\overline{g}^{a_{m-1}}}]$. Then the pullback of the tower (5.3.2) to $C'_{\eta,j}$ is given by

$$Y_{[\overline{\mathbf{f}}]} \longrightarrow Y_{[\overline{\mathbf{f}}^{(\mathbf{m-1})}]} \longrightarrow \ldots \longrightarrow Y_{[\overline{\mathbf{f}}_{\mathbf{0}}]} \longrightarrow C'_{\eta,j}\,, \tag{5.3.4}$$

where $\overline{\mathbf{f}}^{(\mathbf{s})}$ denotes the truncated Witt vector of length s associated to $\overline{\mathbf{f}}$.

Recall the definition 4.2.5 of the ramification number $m_{(C'_{\eta,j},z)}$ of $Y_{[\overline{\mathbf{f}}]} \longrightarrow C'_{\eta,j}$ at a closed point $z \in Y_{[\overline{\mathbf{f}}]}$, then $m_{(C'_{\eta,j},z)}$ is equal to the last lower jump of the ramification filtration $\{G^i_z\}$ associated to z (cf. [14, 2.4]). Then by [14, Lemma 3.1], $m_{(Y_{[\overline{\mathbf{f}}]},z)}$ is equal to the unique lower jump of the ramification filtration $\{G^{i'}_z\}$ of $Y_{[\overline{\mathbf{f}}_{\omega,i}]} \longrightarrow Y_{[\overline{\mathbf{f}}^{(\mathbf{m-1})_{\omega,\mathbf{i}}}]}$, i.e. the ramification number $m'_{(C'_{\eta,j},z)}$ of z in the uppermost $\mathbb{Z}/p\mathbb{Z}$-subcover of the tower (5.3.4).

Recall from the proof of 5.1.12 that $Y_{[\overline{\mathbf{f}}_{\omega,i}]} \longrightarrow Y_{[\overline{\mathbf{f}}^{(\mathbf{m-1})_{\omega,\mathbf{i}}}]}$ is defined by

$$y^p_{m-1} - y_{m-1} = -\sigma_{m-1}(y^p_0, \ldots, y^p_{m-2}, y_0, \ldots, y_{m-2}) + \frac{f_{m-1}}{g^{a_{m-1}}}\,,$$

where the y_s are the elements generating $K(Y_{[\overline{\mathbf{f}}^{(\mathbf{s})}_{\omega,i}]})$ over $K(Y_{[\overline{\mathbf{f}}^{(\mathbf{s-1})}_{\omega,i}]})$.

Now let $z_{l-1} \in Y_{[\mathbf{f}^{(\mathbf{l})}_{\omega,\mathbf{i}}]}$ be the image of z under $Y_{[\overline{\mathbf{f}}_{\omega,i}]} \longrightarrow Y_{[\overline{\mathbf{f}}^{(\mathbf{l})}_{\omega,i}]}$ for $l = m, \ldots, 1$, and let $x \in C'_{\eta,j}$ be the image of y under $Y_{[\overline{\mathbf{f}}_{\omega,i}]} \longrightarrow C'_{\eta,j}$. By Lemma 5.1.3, we have

$$m'_{(C'_{\eta,j},z)} = -\nu_{z_{m-2}}(\sigma_{m-1}(y^p_0, \ldots, y^p_{m-2}, y_0, \ldots, y_{m-2}) + \frac{f_{m-1}}{g^{a_{m-1}}})\,. \tag{5.3.5}$$

Recall that $\mathcal{D}$ is geometrically bounded, and that we defined Δ to be the constant from Lemma 4.2.2, then Δ is such that $m'_{(C'_{\eta,j},z_{m-2})} = m_{(C'_{\eta,j},z)} \leq \Delta$. Since σ_j is homogeneous of degree p^j, and $\nu_{z_{m-2}}(\sigma_{m-1}(y^p_0, \ldots, y^p_{m-2}, y_0, \ldots, y_{m-2})) < 0$, we have

$$
\begin{aligned}
\nu_{z_{m-2}}(\sigma_{m-1}(y_0^p,\ldots,y_{m-2}^p,y_0,\ldots,y_{m-2})) &= \min_{l<m-1}\{p^{m-1}\nu_{z_{m-2}}(y_l^p),p^{m-1}\nu_{z_{m-2}}(y_l)\} \\
&= p^m \min_{l<m-1}\{\nu_{z_{m-2}}(y_l)\} \\
&= p^m \min_{l<m-1}\{\frac{1}{e(z_{m-2}/z_l)}\ \nu_{z_l}(y_l)\} \\
&\geq p^m \min_{l<m-1}\{\nu_{z_l}(y_l)\} \\
&\geq p^m \min_{l<m-1}\{p\nu_{z_l}(\frac{f_l}{g^{a_l}})\} \\
&\geq p^{m+1} \min_{l<m-1}\{\nu_{x_l}(\frac{\overline{f}_l}{\overline{g}^{a_l}})\} \\
&= p^{m+1} \min_{l<m-1}\{\frac{1}{e(z_l/x)}\ \nu_x(\frac{\overline{f}_l}{\overline{g}^{a_l}})\} \\
&\geq p^{m+1} \min_{l<m-1}\{\nu_x(\frac{\overline{f}_l}{\overline{g}^{a_l}})\}
\end{aligned}
$$

By the induction assumption, we have $\nu_x(\frac{f_l}{g^{a_l}}) \geq -\frac{\Delta'}{p^{m-l-2}}$. This implies that

$$
\begin{aligned}
\nu_{z_{m-2}}(\sigma_{m-1}(y_0^p,\ldots,y_{m-2}^p,y_0,\ldots,y_{m-2})) &= p^{m+1}\min_{l<m-1}\{\nu_x(\frac{f_l}{g^{a_l}})\} \\
&\geq -p^{m+1}\max_{l<m-1}\{\frac{\Delta'}{p^{m-l-2}}\} \\
&\geq -p^{m+1}\Delta'
\end{aligned}
$$

Thus by (5.3.5),

$$
\begin{aligned}
-\nu_{z_{m-2}}(\frac{f_{m-1}}{g^{a_{m-1}}}) &= \max\{-\nu_{z_{m-2}}(\sigma_{m-1}(y_0^p,\ldots,y_{m-2}^p,y_0,\ldots,y_{m-2})),m'_{(C'_{\eta,j},z)}\} \\
&= \max\{p^{m+1}\Delta',m_{(Y_{[\overline{\mathbf{f}}]},z)}\} \\
&\leq \max\{p^{m+1}\Delta',\Delta\}\,.
\end{aligned}
$$

and therefore,

$$\begin{aligned}-\nu_x(\frac{f_{m-1}}{g^{a_{m-1}}}) &= -e(z_{m-2}/x)\nu_{z_{m-2}}(\frac{f_{m-1}}{g^{a_{m-1}}}) \\ &\leq p^{m-1}\nu_{z_{m-2}}(\frac{f_{m-1}}{g^{a_{m-1}}}) \\ &\leq \max\{p^{2m}\Delta', \Delta\}\,.\end{aligned}$$

Let us define $\Delta'' := \max\{p^{2m}\Delta', \Delta\}$. For x the point at infinity of $C_{\eta,j}$, we then obtain $\deg_{x_j}(f_{m-1}/g^{a_{m-1}}) = \deg_{x_j}(\overline{f}_{m-1}/\overline{g}^{a_{m-1}}) \geq \Delta''$. Since $\Delta'' > p\Delta'$, we have from 5.3.1 that $p^{m-s-1}\left(\deg_{x_j} f_s - a_s \deg_{x_j} g\right) \leq \Delta''$ as well, which proves 5.3.3.2.

To show 5.3.3.1, assume that $\frac{f_{m-1}}{g^{a_{m-1}}}$ is written in lowest terms: $g_{m-1} \not| f_{m-1}$. Then write $\frac{f_{m-1}}{g^{a_{m-1}}} = \frac{f'}{g_1^{\alpha_1}\cdots g_l^{\alpha_l}}$ such that g_r is irreducible and relatively prime to f' for all r. Then we have $\frac{\overline{f}_{m-1}}{\overline{g}^{a_{m-1}}} = \frac{\overline{f}'}{\overline{g}_1^{\alpha_1}\cdots\overline{g}_l^{\alpha_l}} \in A^{\eta,j}$. Now let $x \in \overline{C}_{\eta,j}\backslash C'_{\eta,j}$ be a point in $V(g_r)$, then

$$a_{m-1} \leq \alpha_r = \nu_{g_r}\left(\frac{f_{m-1}}{g^{a_{m-1}}}\right) \leq \nu_x\left(\frac{\overline{f}_{m-1}}{\overline{g}^{a_{m-1}}}\right) \leq \Delta''$$

and we have $p^{m-s-1}a_s \leq p\Delta' < \Delta''$ for all $s < m-1$ from the induction assumption 5.3.1.

□

By Lemma 5.3.3, we may thus assume that $\mathcal{D}_{\omega,i}$ is realised by a cover with Artin-Schreier-Witt representative $[\mathbf{f}] = [\frac{f_0}{g^{a_0}}, \ldots, \frac{f_{m-1}}{g^{a_{m-1}}}] \in W_m(A_G)/\wp(A_G)$ such that for all ω, for all i, we have

$$p^{m-s-1}a'_s \leq \Delta' \text{ for all } s, \text{ and}$$

$$p^{m-s-1}\left(\deg_{x_j} f'_s - a'_s \deg_{x_j} g\right) \leq \Delta' \text{ for all } s, \text{ for all } j \neq i \tag{5.3.6}$$

for some fixed positive integer Δ'.

Now let $\mathbf{F}_{\omega,i,s} = \frac{F_{\omega,i,s}}{G^{a_{\omega,i,s}}}$ be the representative of a preimage of $[\frac{f_s}{g^{a_s}}]$ defined by Lemma 5.2.3, and set $\mathbf{F}_{\omega,i} = (\mathbf{F}_{\omega,\mathbf{i},\mathbf{0}}, \ldots, \mathbf{F}_{\omega,\mathbf{i},\mathbf{m-1}}) \in W_m(A_G))$. Then since $\deg_{x_j} \frac{F_{\omega,i,s}}{G^{a_{\omega,i,s}}} = \deg_{x_j} \frac{f_{\omega,i,s}}{g^{a_{\omega,i,s}}}$, and the exponents a_s of G in the denominators remain unchanged, $[\mathbf{F}_{\omega,i}] \in M_{\Delta,m}(A_G)$ for all $\omega \in \mathbb{A}^1_k$.

Now let $M_i := \langle W_m(\mathbb{F}_p).[\mathbf{F}_{\omega,i}]\rangle_\omega$ be the $W_m(\mathbb{F}_p)$-module generated by the $\mathbf{F}_{\omega,i}$, then $M_i \subset M^i_{\Delta',m}(A_G)$. By construction, the associated p^m- exponent cover $Y_{M_i} \longrightarrow X$ trivialises the covering datum $\mathcal{D}$ on the fibers $H'_{\omega,i}$ of $\Phi|_X$. By Lemma 5.2.7, Y_{M_i} thus weakly trivialises $\mathcal{D}$ over $X \cap \mathbb{A}^n_k$. The cover Y_M associated to $M = \cap_i M_i$ then weakly trivialises $\mathcal{D}$ over $X \cap \mathbb{A}^n_k$ by Theorem 5.0.8.

By Lemma 5.3.4 (see below), $M \subset M_{D,m}(A_G)$ is finitely generated. Then Lemma 2.3.10 implies that Y_M trivialises $\mathcal{D}$ over $X \cap \mathbb{A}^n_k$, and the covering datum $\mathcal{D}$ is effective with a finite realisation by Corollary 2.3.16 of Theorem 2.3.15. The realisation of a covering datum is étale above X by Lemma 2.1.15, so all induction assumptions are satisfied.

Lemma 5.3.4. *Let A_G be the localisation of $A = k[x_1, \ldots, x_n]$ at the set $\{G^a : a \in \mathbb{Z}\}$, and fix a positive integer Δ. For $i = 1, \ldots, n$, assume we are* *given vector subspaces $M_i \subset M^i_{\Delta,m}(A_G)$ of the associated Artin-Schreier-Witt space $W_m(A_G)/\wp(W_m(A_G))$. If we set $M = \cap M_i$, then $M \subset M_{p\Delta,m}(A_G)$.*

Proof. We must show that every equivalence class in M contains a representative $(\mathbf{F}_0, \ldots, \mathbf{F}_{m-1})$ such that if $\mathbf{F}_s = \frac{F_s}{G^{a_s}}$,

$$p^{m-s-1}\left(\deg_{x_j} F_s - a_s \deg_{x_j} G\right) \leq p\Delta \text{ for all } j \text{ and}$$

$$p^{m-s-1} a_s \leq p\Delta \text{ for all } j \text{ and } s\ ,.$$

Since $M \subset M_i \subset M^i_{\Delta,m}(A_G)$ for all $i = 1, \ldots, n$, $[\mathbf{F}]$ has representatives $\mathbf{F}_i = (\mathbf{F}_{i,0}, \ldots, \mathbf{F}_{i,m-1})$ such that if $\mathbf{F}_{i,s} = \frac{F_{i,s}}{G^{a_{i,s}}}$, then

$$p^{m-s-1} a_{i,s} \leq \Delta \text{ and}$$

$$p^{m-s-1}\left(\deg_{x_j} F_{i,s} - a_{i,s} \deg_{x_j} G\right) \leq \Delta$$

for all s, for all $j \neq i$.

We proceed by induction on m: For $m = 1$, this was shown in Sections 5.2.1 and 5.2.2: We found $\mathbf{F} = \frac{F}{G^a}$ such that $[\mathbf{F}] \in M_{D,1}$ realises $\mathcal{D}$.

So assume that any $\mathbb{Z}/p^{m-1}\mathbb{Z}$-cover can be realised by an element of $M_{\Delta,m-1}(A_G)$. In particular, for $[\mathbf{F}] \in M$, the $\mathbb{Z}/p^{m-1}\mathbb{Z}$-cover $Y_{[\mathbf{F}_i^{(m-1)}]}$ has a representative $(\mathbf{H}_0, \ldots, \mathbf{H}_{m-2}) \in M_{\Delta,m-1}(A_G)$.

Then we have $(\mathbf{H}_0, \ldots, \mathbf{H}_{m-2}) \ominus \mathbf{F}_i^{(m-1)} \in \wp(W_m(A))$, so we can replace $\mathbf{F}_i$ by $\mathbf{F}'_i = \mathbf{F}_i \oplus (\mathbf{H}_0, \ldots, \mathbf{H}_{m-1}) \ominus \mathbf{F}_i^{(m-1)} = (\mathbf{H}_0, \ldots, \mathbf{H}_{m-2}, \mathbf{F}'_{m-1})$. Here,

$$\mathbf{F}'_{m-1} = \mathbf{F}_{i,m-1} + \sigma_{m-1}(\mathbf{H}_0, \ldots, \mathbf{H}_{m-2}, \mathbf{F}_{i,0}, \ldots, \mathbf{F}_{i,m-2}) = \frac{F'_{m-1}}{G^{a'_{m-1}}}\ .$$

By construction of $\mathbf{F}_i$, for all $j \neq i$, we have $\mathbf{F}_{i,m-1} = \frac{F_{i,m-1}}{G^{a_{i,m-1}}}$ such that $a_{i,m-1} \leq \Delta$ and $p\left(\deg_{x_j} F_{i,m=1} - a_{i,m-1} \deg_{x_j} G\right) \leq \Delta$.

Writing $\mathbf{H}_s = \frac{H_s}{G^{b_s}}$, we can apply the induction assumption to get

$$\left(\deg_{x_j} H_s - b_s \deg_{x_j} G\right) \leq \frac{\Delta}{p^{m-s-2}} \text{ for all } j, \text{ for all } s = 0, \ldots, m-2\,,$$

$$b_s \leq \frac{\Delta}{p^{m-s-2}} \text{ for all } s = 0, \ldots, m-2\,.$$

By Lemma 5.1.9, this implies that the exponent of G in $\sigma_{m-1}(\mathbf{H}_0, \ldots, \mathbf{H}_{m-2}, \mathbf{F}_{i,0}, \ldots, \mathbf{F}_{i,m-2})$ is at most $p\Delta$, and that

$$\deg_{x_j}(\sigma_{m-1}(\mathbf{H}_0, \ldots, \mathbf{H}_{m-2}, \mathbf{F}_{i,0}, \ldots, \mathbf{F}_{i,m-2})) \leq p\Delta\ .$$

It follows that $\mathbf{F}'_i = [F'_0/G^{a'_0}, \ldots, F'_{m-1}/G^{a'_{m-1}}]$ satisfies

$$p^{m-s-1} a'_s \leq p\Delta \text{ for all } s \text{ and}$$

$$p^{m-s-1}\left(\deg_{x_j} F'_s - a'_s \deg_{x_j}\right) \leq p\Delta G \text{ for all } j \neq i \text{ for all } s\,,$$

so that $[\mathbf{F}'_i] \in M_{p\Delta,1} \subset A/\wp(A)$.

Now we compare $\mathbf{F}'_i$ to $\mathbf{F}'_j$: We must have $\mathbf{F}'_i \ominus \mathbf{F}'_j \in \wp(W_m(A))$. But $\mathbf{F}'_i \ominus \mathbf{F}'_j = (0, \ldots, 0, \mathbf{F}'_{i,m-1} - \mathbf{F}'_{j,m-1})$, so this implies that $[\mathbf{F}'_{i,m-1}] = [\mathbf{F}'_{j,m-1}]$ in $A/\wp(A)$. By Lemma 5.2.10, this implies that $\mathbf{F}'_{i,m-1} \in M_{p\Delta,1}(A_G)$.

Thus there exists an element $\mathbf{H}_{m-1} \in [\mathbf{F}_{i,m-1}]$ such $\mathbf{H}_{m-1} = \frac{H}{G^A}$, where $\deg_{x_j} H - A \deg_{x_j} G \leq p\Delta$ and $A \leq p\Delta$. Then $\mathbf{F}'_i \sim (\mathbf{F}_0, \ldots, \mathbf{F}_{m-2}, \mathbf{H})$, so $(\mathbf{F}_0, \ldots, \mathbf{F}_{m-2}, \mathbf{H})$ is the required representative of $[\mathbf{F}]$. □

Chapter 6

Tamely ramified covering data revisited

In this chapter, we prove Part 2 of the Key Lemma 4.4.4:

Theorem 6.0.5. *Let k be a finite field of characteristic p, and let $X \subset \mathbb{A}^n_k$ be an open regular subscheme. If $\mathcal{D}$ is any covering datum with cyclic prime-power index l^m, where $l \neq p$, then $\mathcal{D}$ is effective with finite realisation.*

As noted previously, these results are already known from the works of Wiesend, Kerz and Schmidt ([21], [5]). We give a new proof using Kummer Theory and families of étale covers (cf. Theorem 5.0.8). The strategy of proof is modelled on that of Chapter 5, with Kummer Theory replacing Artin-Schreier-Witt Theory. By Lemma 4.4.6, no considerations of geometric boundedness conditions are necessary.

6.1 A review of Kummer Theory

Let A be an integrally closed domain which is a finitely generated k-algebra. Given an integer m relatively prime to $char(k) = p$, enlarge k to a finite field k' containing the m-th roots of unity, and consider $A' = A \otimes k'$. Let $\widetilde{A}'$ be a universal cover of A', i.e. the integral closure of A in the maximal seperable algebraic extension of the function field $K = K(A)$ in which A' does not ramify. Consider the short exact sequence of $\pi_1(Spec\, A')$-modules

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{A}'^{\times} \xrightarrow{n} \widetilde{A}'^{\times} \longrightarrow 1\,.$$

The long exact sequence of cohomology then gives rise to a short exact sequence

$$0 \longrightarrow A'^{\times}/\left(A'^{\times}\right)^m \longrightarrow Hom(\pi_1(Spec\, A'), \mu_m) \longrightarrow Pic(A') \longrightarrow 0\,,$$

where $Pic(A')$ denotes the Picard group of the ring A' (see [11, Prop. 4.11] for details). In particular, if $Pic(A') = 0$, we get a canonical isomorphism

$$\Psi : A'^{\times}/\left(A'^{\times}\right)^m \xrightarrow{\simeq} Hom(\pi_1(Spec\, A'), \mu_m)\,.$$

Remark 6.1.1. In our situation, the rings considered will always be affine rings of arithmetical varieties. For such rings, the Picard group is isomorphic to the class group of Cartier divisors (cf. [9]), which is finitely generated. Therefore, after possibly inverting an element of the ring, the ring A' can be assumed to have trivial Picard group $Pic(A') = 0$. From now on, we only consider ring A' satisfying this hypothesis.

Definition 6.1.2. We call $\mathcal{K}_m(A') := A'^{\times}/\,(A'^{\times})^m$ the *Kummer space of level m of* A', and note the natural μ_m-module structure on $\mathcal{K}(A')$ given by $\zeta_n^r \cdot [a] = [a]^r = [a^r]$.

Now let $\Delta_a = \{a^r : r \in \mathbb{Z}\}$ be the subgroup of $\mathcal{K}(A')$ generated by an element a of $A'^{\times}$. Note that we have $\Delta_a = \Delta_{a'}$ for two cyclic subgroups of order n if and only if $g^m a = a'^r g'^m$ for some elements $g, g' \in A'^{\times}$, r such that $(r, m) = 1$. (Without loss of generality, $r < m$.)

In general, Δ_a is a subgroup of order k dividing m. $\Psi(a)$ then has kernel N_a, a normal subgroup of index k, and image isomorphic to $\mathbb{Z}/k\mathbb{Z}$. Let Y_a denote the étale cover corresponding to N_a, then Y_a is equal to the normalisation of $Spec\,A'$ in the field $K(\sqrt[m]{\Delta_a})$.

If $Y_{[a]} = Y_{[a']}$ is a $\mathbb{Z}/n\mathbb{Z}$-cover of $Spec\,A'$, then $N_a = \ker(\Psi(a))$ and $N_{a'} = \ker(\Psi(a'))$ are equal, and of full index n. In particular, $\Psi(a)$ also factors through N_a giving rise to an isomorphism $\overline{\Psi(a)} : \pi_1(Spec\,A')/N_{a'} \simeq \mu_m$. Thus we get

$$\begin{array}{ccc} \pi_1(Spec\,A)/N_{a'} & \xrightarrow{\overline{\Psi(a')}} & \mu_m \\ & \searrow^{\overline{\Psi(a)}} & \downarrow r \\ & & \mu_m \end{array}$$

Then $\overline{\Psi(a)} \circ \overline{\Psi(a)^{-1}}$ is an isomorphism of μ_m, i.e. represented by exponentiating with an element r relatively prime to m. Thus we can write $(\Psi(a))^r = \Psi(a')$. Since Ψ is compatible with the μ_m-module structure, this implies that $[a^r] = [a']$, which is equivalent to $\Delta_a = \Delta_{a'}$ for the associated cyclic subgroups of order n in $\mathcal{K}(A')$.

Conversely, if $\Delta_a = \Delta_{a'} \subset A'^{\times}/\,(A'^{\times})^n$ are cyclic subgroups of order m, then clearly $K(\sqrt[m]{\Delta_a}) = K(\sqrt[m]{\Delta_{[a']}})$, so $Y_{[a]} = Y_{[a']}$.

Thus Ψ induces a one-to-one correspondence of simple μ_n-submodules of order k dividing m with $\mathbb{Z}/k\mathbb{Z}$-covers of $Spec\, A'$:

$$\Delta_a \mapsto Y_{[a]} \leftrightarrow N_a < \pi_1(Spec\, A')$$

Here, $Y_{[a]}$ corresponds to the normal subgroup N_a of index k in $\pi_1(Spec\, A')$.

More generally, if $M \subset \mathcal{K}_m(A')$ is a μ_m-module, then we can associate to M the normalisation Y_M of $Spec\, A'$ in $K(\sqrt[n]{M})$, the field obtained by adjoining to K all m-th roots of elements in M. Then Y_M is the étale cover associated to the normal subgroup $N_M = \cap_{a \in M} N_a$.

If M is finite, then N_M is a finite intersection of normal subgroups with finite index, and thus also of finite index. If this is the case, then $Y_M \longrightarrow Spec\, A'$ is a (finite) étale cover.

Conversely, if $Y \longrightarrow Spec\, A$ is an étale Galois étale cover of exponent m, then we use the Galois correspondence to write Y as the composition of r $\mathbb{Z}/k_i\mathbb{Z}$-covers $Y_{[a_i]}$, where k_i divides m for $i = 1, \ldots, r$. (Here, r is the rank of the Galois group as a $\mathbb{Z}/m\mathbb{Z}$-module.) Then Y corresponds to the subgroup $N := \cap_i N_{a_i}$ and is equal to the normalisation of $Spec\, A$ in the compositum of the function fields $K(\sqrt[m]{\Delta_{a_i}})$. Writing $M = \langle [a_i] : i = 0, \ldots, r \rangle \subset \mathcal{K}_m(A')$, this means that the function field of Y can be obtained by adjoining all m-th roots of elements in M to K, i.e. $K(Y) = K(\sqrt[m]{M})$.

We thus obtain the following:

Proposition 6.1.3. *Let A' be any finitely generated algebra over a finite field k' containing the mth roots of unity. Then there is a one-to-one inclusion-reversing correspondence of μ_m-submodules $M = \Sigma_{i\in I}\Delta_{a_i}$ of $\mathcal{K}_m(A)$ and Galois pro-étale covers of exponent m of $\operatorname{Spec} A'$:*

$$M \mapsto Y_M \leftrightarrow N_M := \cap_{a\in M} N_a \,.$$

M is finitely generated of rank r if and only if Y_M is an exponent m-étale cover of $\mathbb{A}^n_{k'}$ whose Galois group has r generators.

Proof. The infinite case follows from the finitely generated case by taking inverse limits (as in Section 5.1). □

The properties of the correspondence are summarised as follows:

1. For $M \subset \mathcal{K}_m(A')$, the associated pro-étale cover Y_M is the normalisation of K in $K(\sqrt[n]{M})$, the field obtained by adjoining all m-roots of elements in M.

2. If $\{a_i\}_{i\in I}$ is any generating set for $M \subset \mathcal{K}_m(A')$, then $N_M = \bigcap_{i\in I} N_{a_i}$: Indeed, if $f = \prod_{i=1}^m f_i^{r_i}$ with r_i relatively prime to m, then since Ψ is multiplicative, $N_a \subset \cap_i N_{a_i}$, so $N_M \subset \bigcap_{i\in I} N_{a_i}$. The other inclusion is trivial.

3. For any two submodules M, M', we have

   a) $MM' \leftrightarrow N_M \cap N_{M'}$, and

   b) $M \cap M' \leftrightarrow < N_M, N'_M >$

We now define several important submodules of Kummer space $\mathcal{K}_m(A')$ for the free k'-aglebra $A' = k'[x_1, \ldots, x_n]$ and its localisation $A_G = k'[x_1, \ldots, x_n]_G$ at some element $G \in A$.

For $A = k[x_1, \ldots, x_n]$, we let as above k' be a finite extension of k containing the mth roots of unity, and $A' = k'[x_1, \ldots, x_n]$. We define a submodule of Kummer m-space by setting $M_D(A') =< [f] : \deg_{x_j} f \leq D$ for all $j >$. Then M_D is finitely generated and finite, as every element has a representative contained in the finite set $\{f^r \in A : \deg_{x_j} \leq D$ for all j$, r < m$ relatively prime to $m\}$.

We also define $M_D^i(A') =< [f] : \deg_{x_j} f \leq D$ for all $j \neq i >$, an (infinitely generated) μ_m-submodule of $\mathcal{K}(A')$, and show:

Proposition 6.1.4. *In the above notations, the intersection of the spaces $M_D^i(A')$ over all i equal to $M_{D'}(A')$ for some positive integer D', and thus finite.*

Proof. By assumption, each equivalence class $[F] \in M_{D'}(A')$ contains an element F_i of $M_D^i(A)$, i.e. such that $\deg_{x_j} F_i \leq D$ for all $j \neq i$. If $G^m | F_i$ for some polynomial G, without loss of generality replace F_i by $F_i' = F_i/G^m$. Then $\deg_{x_j} F_i' \leq \deg_{x_j} F_i$ for all j, and thus $\deg_{x_j} F_i \leq D$ for all $j \neq i$. Then we have $F_i' = F_j^r H^m$ for some polynomial $H \in A'$ and some $r < m$ that is relatively prime to m. But F_i' is not divisible by any n-th power of a polynomial, so H must be constant, and we have $\deg_{x_i} F_i' \leq r \deg_{x_i} F_j \leq mD$. Thus we have $F_i' \in M_{mD}(A)$.

The other inclusion is trivial by definition. ∎

Let A_G' denote the localisation of A' at some element of A', and consider the

Kummer space $\mathcal{K}(A_G)$. Define μ_m-submodules

$$M_D(A_G) =< [F/G^a] : a \leq D, \deg_{x_j} F \leq a \deg_{x_j} G \text{ for all } j > \quad \text{and}$$

$$M_D^i(A_G) =< [F/G^a] : a \leq D, \deg_{x_j} F \leq a \deg_{x_j} G \text{ for all } j \neq i > ,$$

then $M_D(A_G)$ is again finite as it is finitely generated as a $\mathbb{Z}$-module. Similar to above, we may now show that:

Proposition 6.1.5. *In notations as above, the intersection of the spaces $M_D^i(A'_G)$ over all i contained in $M_{D'}(A'_G)$ for some positive integer D', and thus finite.*

Proof. By assumption, each equivalence class $[F/G^a] \in M_{D'}(A'_G)$ contains elements $F_i/G^{a_i} \in M_D^i(A_G)$, i.e. such that F_i and G are relatively prime, and such that $a_i \leq D$ and $\deg_{x_j} F_i \leq a_i \deg_{x_j} G$ for all $j \neq i$. If $H^m | F_i$ for some polynomial H, without loss of generality replace F_i/G^{a_i} by F'_i/G^{a_i}, where $F'_i = F_i/H^m$. Now we may assume that F'_i is not divisble by the m-th power of any polynomial, and since $\deg_{x_j} F'_i \leq \deg_{x_j} F_i$ for all j, it is still true that $\deg_{x_j} F_i \leq a_i \deg_{x_j} G$ for all $j \neq i$. Now we have

$$\frac{F'_i}{G^{a_i}} = \left(\frac{F_j}{G^{a_j}}\right)^r \left(\frac{H}{G^b}\right)^m$$

for some polynomial $H \in A'$ and some $r < m$ that is relatively prime to m.

Write $\frac{H}{G^b}$ in lowest terms, then we have two cases: If $b > 0$, then H and G are relatively prime and $a_i = ra_j + mb$. Multiplying through by G^{a_i}, we have $F'_i = F_j^r H^m$. But F'_i is not divisible by any n-th power of a polynomial, so then H must be constant, and we have $\frac{F'_i}{G^{a_i}} = \frac{F_j^r}{G^{ra_j}}$. As the F_l and G are relatively prime by assumption, this implies that $a_i = ra_j \leq mD$. Taking degrees on both sides, we also get that $\deg_{x_i} F'_i = r \deg_{x_i} F_j \leq ra_j \deg_{x_i} G = a_i \deg_{x_i} G$. Thus we have $F'_i \in M_{mD}(A_G)$, as required.

If $b = 0$, then write $H = G^c H'$ with H' relatively prime to G, then $F'_i = F_j^r (G^c H')^m$ As before, this implies that $G^c H'$ is contant, so $c = 0$ and H' is constant and we get that $F'_i \in M_{mD}(A_G)$ as above. □

6.2 Tamely ramified covering data of cyclic factor group

In this section, we shall show that for $X = \mathbb{A}^n_k$ or $X \subset \mathbb{P}^n_k$ an open subvariety, any finite open subgroup $H < \mathcal{C}_X$ whose index is prime to $char(k) = p$ is realisable, and thus a preimage. Together with the results of the previous chapter, this proves the Key Lemma 4.4.4.

We note that our construction will not use any theorems on geometric finiteness, contrary to the proof given for the tame variety case in [5].

Theorem 6.2.1. *Let k be a finite field of characteristic p, and let $X \subset \mathbb{P}^n_k$ be an open subvariety. If $\mathcal{D}$ is a covering datum of cyclic index m on X such that $(m, p) = 1$, then $\mathcal{D}$ is realisable with a finite realisation.*

Proof. We already showed in the previous chapter that it suffices to trivialise $\mathcal{D}$ over an affine open subset of the form $D(G) \subset \mathbb{A}^n_k \simeq D_+(X_0)$. We again make use of the fibrations $\Phi_i : \mathbb{A}^n_k \longrightarrow \mathbb{A}^1_k$, let ω denote a closed point of $\mathbb{A}^1_k$ with residue field $k(\omega) \simeq k[x_i]/(h_i(x_i))$, and $C_{\omega,i}$ the fiber of Φ_i over ω.

As before, we denote $X \cap C_{\omega,i}$ by $C'_{\omega,i}$, and assume by induction that $\mathcal{D}_{\omega,i}$ is realisable by a Galois cover $Y_{[f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}]}$ whose equivalence classe $[f_{\omega,i}/g_{\omega,i}^{a_{\omega,i}}] \in \mathcal{K}_m(A_G)$ has a representative such that $a_{\omega,i} \leq D$ and $\deg_{x_j} f_{\omega,i} \leq a \deg_{x_j} g_{\omega,i}$ for all $j \neq i$. As before, the base case is given by our assumption that $\mathcal{D}$ is a covering datum that is defined on the projective closure $D_+(\overline{G})$, and thus étale at the point at infinity of a curve $D(g) \subset \mathbb{A}^1_k$. This ensures that $\deg_x f \leq a \deg_x g$ in this case. Furthermore, as $\mathcal{D}$ is tame and of index bounded by m, it is geometrically bounded by Lemma 4.4.6, so we may assume that $a \leq D$ universally for some D. Recalling the canonical surjections $A \twoheadrightarrow A^{\omega,i}$, where $A^{\omega,i} = k(\omega)[x_1, \ldots, \hat{x}_i, \ldots, x_n]$, and $A_G \twoheadrightarrow A^{\omega,i}_{g_{\omega,i}}$. They clearly induce surjections of Kummer spaces analogously to those of Artin-Schreier space:

$$p_{\omega,i} : \mathcal{K}_m(A) \twoheadrightarrow \mathcal{K}_m(A^{\omega,i}) \text{ and}$$

$$\pi_{\omega,i} : \mathcal{K}_m(A_G) \twoheadrightarrow \mathcal{K}_m(A^{\omega,i}_g)\,.$$

Then the lemma below is proven entirely analously to Lemma 5.2.9 of the Artin-Schreier case:

Lemma 6.2.2. *Given* $[f/g^a] \in \mathcal{K}_m(A^{\omega,i}_g)$*, there exists a preimage* $[F/G^a] \in \pi^{-1}_{\omega,i}([f/g^a])$ *which has a representative such that* $\deg_{x_j} F = \deg_{x_j} f$ *for all* $j \neq i$*.*

So for each ω, i, we let $F_{\omega,i} \in A$ denote this representative, and set $M_i = \langle [F_{\omega,i}/G^{a_{\omega,i}}] : \omega \in \mathbb{A}^1_k \rangle$. Then as before, M_i corresponds to a pro-étale cover of X trivialising all the fibers of Φ_i. As the fibers are regular, every closed point of X is a regular point of a fiber. So $\mathcal{D}$ is trivialised at all closed points, i.e. Y_{M_i} weakly trivialises $\mathcal{D}$ on X. Combining Theorem 5.0.8 and Property 4) of the Kummer correspondence 6.1.3, this implies that the cover $Y_M \longrightarrow X$ corresponding to $M = \cap_i M_i$ also weakly trivialises $\mathcal{D}$. $M \subset M_D(A_G)$ is finite, so Y_M is an étale cover. By Proposition 2.3.10, it gives a full trivialisation of $\mathcal{D}$, and is thus realised by an element $[F/G^a] \in M \subset M_D(A_G)$. In particular, we have $a \leq D$ and $\deg_{x_j} F \leq a \deg_{x_j} G$ for all j, so all the induction assumptions are satisfied.

□

Remark 6.2.3. As remarked in Section 5.5.2, this includes the case of affine n-space $X = \mathbb{A}^n_k$ since we then have $X = D_+(\overline{G})$, $\overline{G}(X_0, \ldots, X_n) = X_0$.

Bibliography

[1] L. Gruendken, L. Hall-Selig et al.: *Semi-direct Galois covers of the affine line*, Women in Numbers Research Directions in Number Theory, Fields Institute Communications **60** (2011), 201-212

[2] R. Hartshorne: Algebraic Geometry, Springer, 1977

[3] G. Halphen: Mémoire sur la classication des courbes gauches algébriques, J. Ec. Polyt. **52** (1882), 1-200

[4] N. Katz, S Lang: *Finiteness Theorems in Geometric Class Field Theory*, L'Enseignement Mathematique (1982), 285-319

[5] M. Kerz, A. Schmidt: *Covering data and higher dimensional class field theory*, J. of Number Theory **129** (2009), 2569-2599

[6] M. Kerz, A. Schmidt: *On different notions of tameness in arithmetical geometry*, arXiv:0807.0979v2 (2009)

[7] S. Lang: Fundamentals of Diophantine Geometry, Springer 1983

[8] S. Lang: Algebra, Revised Third Edition, Springer 2002

[9] Q. Liu: Algebraic Geometry and arithmetical Curves, Oxford University Press, 2002

[10] Matsumura, Commutative Algebra, 2nd edn, Benjamin, New York 1980

[11] J. S. Milne: Étale Cohomology, Princeton Mathematical Series No. 33, Princeton University Press, 1980

[12] J. Neukirch: Algebraic Number Theory, Springer, 1999

[13] J. Neukirch: Cohomology of Number Fields, Springer, 2000

[14] A. Obus, R. Pries: Wild cyclic-by-tame extensions, Arxiv preprint, 2009

[15] J.-P. Serre: Algebraic Groups and Class Fields, Springer 1988

[16] J.-P. Serrer: Abelian l-adic representations and elliptic curves, reprinted in Wellesley, MA: A K Peters, Ltd., 1998

[17] J.-P. Serre: Corps locaux, Hermann, 1968

[18] Henning Stichtenoth.:Algebraic function elds and codes, Second Edition, Graduate Texts in Mathematics, Vol. 254, Springer 2009

[19] T. Szamuely: Galois Groups and Fundamental Groups, Cambridge University Press, 2009

[20] S. Mac Lane: Categories for the Working Mathematician, 1998

[21] G. Wiesend: *Class field theory for arithmetic schemes*, Mathematische Zeitschrift, **256** No. 4 (2007), 717-729