ORTHOGONAL EPSILON CONSTANTS FOR TAME ACTIONS OF FINITE GROUPS ON SURFACES

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ABSTRACT

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Darren Glass

Ted Chinburg

In this thesis we suppose G is a finite group acting tamely on a regular projective curve $\mathcal{X}$ over $\mathbb{Z}$. Let V be an orthogonal representation of G of dimension 0 and trivial determinant. Our main result determines the sign of the ϵ-constant $\epsilon(\mathcal{X}/G, V)$ in terms of data associated to the archimedean place and to the crossing points of irreducible components of finite fibers of $\mathcal{X}$, subject to certain standard hypotheses about these fibers. In the course of the proof we associate to V and the action of G on $\mathcal{X}$ an element $\mu(\mathcal{X}, G, V)$ of order two in the Brauer group of $\mathbb{Q}$. Such invariants have been defined by Saito for orthogonal motives of even weight. By contrast, the relevant motive in this paper is $(H^1(\mathcal{X}) \otimes V)^G$ which is symplectic of weight 1.

Contents

Chapter 1

Introduction

This chapter will state the main questions and results of the thesis, specify notation, and give some background. Let $\mathcal{X}$ be an arithmetic scheme of dimension $d+1$ which is flat, regular, and projective over $\mathbb{Z}$. We suppose that $f : \mathcal{X} \to Spec(\mathbb{Z})$ is the structure morphism and that its fibres are all of dimension d. Let G be a finite group which acts tamely on $\mathcal{X}$ in the sense that for each closed point $x \in X$, the order of the inertia group of x is relatively prime to the residue characteristic of x. We will discuss tameness in detail in Chapter 5, as well as give examples. Define $\mathcal{Y}$ to be the quotient scheme $\mathcal{X}/G$. We assume that $\mathcal{Y}$ is regular, and that for all finite places v the fiber $\mathcal{Y}_v = (\mathcal{X}_v)/G = \mathcal{Y} \otimes_{\mathbb{Z}} (\mathbb{Z}/p(v))$ has normal crossings and smooth irreducible components with multiplicities relatively prime to the residue characteristic of v. Finally, let V be a representation of G over $\overline{\mathbb{Q}}$.

Associated to this data, we can define a ζ-function and an L-function, both functions of a complex variable s. The zeta function $\zeta(s, \mathcal{Y}, V)$ has the form $\sum_{n=1}^{\infty} a_n(V) n^{-s}$, with the $a_n(V)$ being algebraic numbers associated to the action of G on the closed points of $\mathcal{X}$. In particular:

$$\zeta(s, \mathcal{Y}, V) = \prod_{y \in (\mathcal{Y})^0} det(1 - F_x s | V^{I_x})^{-1}$$

where the product runs over all closed points of $\mathcal{Y}$, and in each term, x is an arbitrary point of $\mathcal{X}$ lying over y, I_x is the inertia group of x in G, and F_x is the Frobenius automorphism of the point x. That is, F_x is the unique element of G_x/I_x which induces the automorphism $\alpha \to \alpha^q$ of the residue field $k(x)$. The function $\zeta(s, \mathcal{Y}, V)$ does not depend on the choice of x over y.

The L-function $L(s, \mathcal{Y}, V)$ is a product $\Gamma_V(s) \cdot \zeta(s, \mathcal{Y}, V)$ in which the function $\Gamma_V(s)$ is a product of finitely many functions of the form $\Gamma(as+b)$ in which a and b are constants and $\Gamma(s)$ is the classical Γ-function. The L-function is conjectured to have a functional equation of the form:

$$L(s, \mathcal{Y}, V) = \epsilon(\mathcal{Y}, V) A(\mathcal{Y}, V)^{-s} L(d+1-s, \mathcal{Y}, V^*)$$

in which $A(\mathcal{Y}, V)$ is a positive integer called the conductor, V^* is the dual representation of V and the ϵ-constant $\epsilon(\mathcal{Y}, V)$ is a nonzero algebraic number. In recent years, many authors have studied the problem of determining these ϵ constants, which may be defined unconditionally after we choose an auxiliary prime ℓ. In the next chapter, we will discuss some known results, as well as how $\epsilon(\mathcal{Y}, V)$ can be computed.

This thesis concerns the case where V is an orthogonal representation, meaning that there is a non-degenerate symmetric G-invariant bilinear form $V \times V \to \overline{\mathbb{Q}} \subseteq \mathbb{C}$ (where we fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$). In order to get the strongest results, we will furthermore make the technical hypotheses that V is a virtual representation of trivial determinant and dimension zero. In other words, V will be a linear combination of orthogonal representations such that the weighted sum of their dimensions is zero and the product of their determinants is trivial.

I can now state in general terms the main result of this thesis.

Theorem 1.1. *If $d = 1$ and V is an orthogonal virtual representation of degree zero and trivial determinant then the sign of the constant $\epsilon(\mathcal{Y}, V) \in \mathbb{R}^*$ can be determined from the ϵ-constant $\epsilon_\infty(\mathcal{Y}, V)$ and from the restriction of the G-cover $\mathcal{X} \to \mathcal{Y}$ over the finite set of closed points z of $\mathcal{Y}$ where two distinct irreducible components of a fiber of $\mathcal{Y}$ over $Spec(\mathbb{Z})$ intersect.*

The constant $\epsilon_\infty(\mathcal{Y}, V)$ which comes up in this formulae is the archimedean ϵ-constant defined by Deligne in §8 of [D1] using the action of the group G and of complex conjugation on the Hodge cohomology groups $H^{p,q}(\mathcal{X}, \mathbb{C})$. Chapter Two of this thesis recalls this and other definitions of ϵ-constants, as well as work done by Deligne, Frohilch, Queyrut, Chinburg, Erez, Pappas, and Taylor in computing ϵ-constants associated to situations similar to those in Theorem 1.1. In Chapter 3, we make a more precise statement of the main theorem and prove it. The proof uses formulae of Saito, Classfield theory, and several of the results discussed in Chapter 2.

Chapter Four discusses work of Cassou-Nogues, Erez, and Taylor which extends work of Serre related to tame coverings. In particular, we define a class $\mu(\mathcal{X}, G, V) \in H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ which is associated to the situation we are working in. We then discuss a connection between the class $\mu(\mathcal{X}, G, V)$, and a class that they define in an analogous situation. Finally, in Chapter Five we discuss the tameness hypotheses required for my results, and give examples of computations.

Chapter 2

Background and Motivation

In this chapter, we look at some of the work that others have done in order to compute ϵ-constants in various situations. We will also describe an application of this research to the conjecture of Birch and Swinnerton-Dyer.

Fröhlich and Queyrut look at computing ϵ-constants in the case where $\mathcal{X}$ and $\mathcal{Y}$ are of relative dimension 0 over $\mathbb{Z}$ and V is an orthogonal representation. In [FQ] they are able to prove the following result:

Theorem 2.1. *If $d = 0$ and V is an orthogonal representation of G, then $\epsilon(\mathcal{Y}, V)$ is positive.*

Due to the assumption that $d = 0$, this problem can be rephrased in terms of extensions of rings of integers of number fields, and this is the context in which Fröhlich and Queyrut proved their result, which was originally conjectured by Serre. Their result does not require the tameness hypothesis which was made in our formulation of the problem.

One can define ϵ-constants associated not only to the situation described in §1, but also in the more general situation of motives. We refer the reader to [D3] for the definition of and basic results on motives. Saito is able to prove the following positivity result on ϵ-constants associated to motives in [tS2]:

Theorem 2.2. *Let M be an orthogonal motive of even weight. Then the global epsilon factor $\epsilon(M)$ is positive.*

It is conjectured that the global ϵ-factors associated to all orthogonal motives are positive, though it is not known in general for the case of motives of weight one. We now recall some elements of Deligne's theory of local constants, which is essential for our work.

Definition 2.3. *Let $\mathcal{X}$ and $\mathcal{Y} = \mathcal{X}/G$ be as above. Let V be any virtual complex representation of G.*

a. Let $\epsilon_{v,0}(\mathcal{Y}, V)$ be the Deligne local constant defined in [CEPT1]. (see also [D1]). In particular, the definition of $\epsilon_{v,0}(\mathcal{Y}, V)$ requires that one chooses an auxiliary prime $\ell \neq v$, a nontrivial continuous complex character of $\mathbb{Q}_v$ which we denote by ψ_v and *a Haar measure dx_v on $\mathbb{Q}_v$. In the case where V has trivial determinant and is of dimension 0, then $\epsilon_{v,0}(\mathcal{Y}, V)$ is independent of these choices (see Proposition 2.4.1 of [CEPT1]). This term is well-defined for $v = \infty$ as well as for finite places v.*

b. *Let X be a variety of dimension d which is defined over a finite field of characteristic p. Let ℓ be a prime different from p and let $j_\ell : \overline{\mathbb{Q}_\ell} \to \mathbb{C}$ be an embedding. Finally, define V_ℓ to be a virtual representation of G over $\overline{\mathbb{Q}_\ell}$ such that $j_\ell(\chi_{V_\ell}) = \chi_V$. Define $\epsilon(X, V) = j_\ell(det(-F|(H^*_{et}(\overline{\mathbb{F}_p} \times_{\mathbb{F}_p} X, \mathbb{Q}_\ell) \otimes V_\ell^*)^G))$, where F is the geometric Frobenius automorphism. This number is independent of all choices.*

c. *For finite places v of $\mathbb{Q}$, we let $\epsilon_v(\mathcal{Y}, V) = \epsilon_{v,0}(\mathcal{Y}, V)\epsilon(\mathcal{Y}_v, V)$, where $\epsilon(\mathcal{Y}_v, V)$ is defined as an ϵ-constant over a finite field. Furthermore,in the case where $v = \infty$, we let $\epsilon(\mathcal{Y}_v, V) = 1$ so that in particular $\epsilon_\infty(\mathcal{Y}, V) = \epsilon_{\infty,0}(\mathcal{Y}, V)$.*

d. *The global ϵ-constant associated to V is defined by $\epsilon(\mathcal{Y}, V) = \prod_v \epsilon_v(\mathcal{Y}, V)$ where the product is over all places v of $\mathbb{Q}$.*

The ϵ-constants associated to varieties defined over finite fields are studied by Chinburg, Erez, Pappas, and Taylor in [CEPT3], where in particular the following theorem is shown.

Theorem 2.4. *Let X be a variety of dimension d defined over a finite field. If d is even (respectively, if d is odd) and if V is a symplectic (respectively an orthogonal) representation of G, then $\epsilon(X, V)$ is positive.*

A symplectic representation is a representation V which is equipped with a non-degenerate alternating G-invariant bilinear form. Papers of Chinburg, Erez, Pappas, and Taylor such as [CEPT1] and [CEPT2] prove results on computing ϵ-constants associated to arithmetic schemes in the case where V is a symplectic representation. Their results include the following theorem:

Theorem 2.5. *Suppose that $d = 1$, and that $\mathcal{X}/H$ is regular for every non-cyclic subgroup H of order four in G. (In particular, this condition holds whenever G is either a cyclic group or a generalized quaternion group). Finally, let V be a symplectic representation of G. Then $\epsilon(\mathcal{Y}, V)$ is positive. Furthermore, for each place v of $\mathbb{Q}$, $\epsilon_v(\mathcal{Y}, V) > 0$.*

In order to prove Theorem 2.5, they show that under their hypotheses both $\epsilon_{v,0}(\mathcal{Y}, V)$ and $\epsilon(\mathcal{Y}_v, V)$ are determined by an equivariant Euler characteristic $\chi \in K_0(\mathbb{F}_p G)$, so that they have the same sign and therefore that $\epsilon_v(\mathcal{Y}, V) = \epsilon_{v,0}(\mathcal{Y}, V)\epsilon(\mathcal{Y}_v, V)$ is positive. They ask if such a theorem holds whenever d is odd, although there are counterexamples when $d = 0$.

Calculating ϵ-constants would have several important implications. One of the most striking relates to the Birch-Swinnerton-Dyer conjecture. In particular, an equivariant version of this conjecture says that if $\mathcal{Y}$ is an arithmetic surface and if V is an irreducible representation, then $ord_{s=1}L(s, \mathcal{Y}, V)$ is equal to the multiplicity of V in the $\mathbb{C}G$-module

$\mathbb{C}\otimes(\mathrm{Pic}^0(\mathcal{X})(\mathbb{Q}))$, where $\mathrm{Pic}^0(\mathcal{X})$ is the group of divisor classes on $\mathcal{X}$ which have degree zero on the general fiber of $\mathcal{X}$.

More precisely, this implies that if V is a self-dual representation (i.e. $V = V^*$), then $\epsilon(\mathcal{Y}, V) > 0$ if and only if V has even multiplicity in the $\mathbb{C}G$-module $\mathbb{C} \otimes \mathrm{Pic}^0(\mathcal{X})$. Thus, if $\epsilon(\mathcal{Y}, V) < 0$, then $\mathbb{C}\otimes(\mathrm{Pic}^0(\mathcal{X}))$ must have a non-trivial V-isotypic part. This property can be rephrased in terms of the existence of integer solutions to the systems of equations defining the Jacobian of $\mathcal{X}$. In other words, if the Birch-Swinnerton-Dyer conjecture is true, then the sign of an ϵ-constant will predict whether or not certain equations have integer solutions. Theorem 1.1 makes it significantly easier to compute various ϵ-constants, and thus to test further cases of the Birch-Swinnerton-Dyer conjecture.

For this application as well as in other situations it is not the actual ϵ-constant we are interested in computing but merely the sign of this constant. We set $W(\mathcal{Y}, V) = sign(\epsilon(\mathcal{Y}, V))$ and call this the root number of V.

Chapter 3

Main Results

3.1 Reduction To Fibral Computations

Let $\mathcal{X}$, G, $\mathcal{Y} = \mathcal{X}/G$ be as in §1. Let S be the set of all finite places v of $\mathbb{Q}$ where either the fiber $\mathcal{Y}_v = \mathcal{Y} \otimes_{\mathbb{Z}} (\mathbb{Z}/p(v))$ is not smooth or the map $\pi : \mathcal{X} \to \mathcal{Y}$ is ramified. Let D' be a horizontal divisor on $\mathcal{Y}$ such that $D' + \mathcal{Y}_T = K_{\mathcal{Y}} + \mathcal{Y}_S^{red}$, where $K_{\mathcal{Y}}$ is a canonical divisor on $\mathcal{Y}$, $\mathcal{Y}_S^{red}$ is the sum of the reductions of the fibers of $\mathcal{Y}$ at the places in S, T is a finite set of finite places of $\mathbb{Q}$ which is disjoint from S, and $\mathcal{Y}_T$ is the sum of the (necessarily reduced) fibers of $\mathcal{Y}$ over the places in T. Thus $O_{\mathcal{Y}}(D' + \mathcal{Y}_T)$ is isomorphic to the twist $\omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_S^{red})$ of the relative dualizing sheaf $\omega_{\mathcal{Y}/\mathbb{Z}}$ by $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}_S^{red})$. We further wish to choose D' so that it intersects the non-smooth fibers $\mathcal{Y}_v$ of $\mathcal{Y}$ transversally at smooth points on the reduction of $\mathcal{Y}_v$. We can choose such a D' after a suitable base change due to the moving lemma proven as Proposition $9.1.3$ in [CEPT1]. The choice of this canonical divisor is not unique, but our calculation will show that the results are independent of the choice of D'.

Remark 3.1. *As stated above, we can only choose a horizontal divisor D' with the desired properties after a suitable base change. Thus, we need to consider how base changes will affect the ϵ-constants. To be precise about how we make the base change, we will choose an odd prime ℓ which is not in the set of bad primes S, and we denote by N_∞ the cyclotomic $\mathbb{Z}_\ell$ extension of $\mathbb{Q}$. Because we have chosen $\ell \notin S$, this base extension is étale over S, and the pullback of a canonical divisor remains canonical up to a multiple of the fiber of $\mathcal{Y}$ over ℓ. Proposition $9.1.3$ of [CEPT1] shows that a horizontal divisor D' with the required properties exists after a base extension to a the ring of integers of a finite extension of $\mathbb{Q}$ inside N_∞. This base extension, which we now fix, is of degree a power of ℓ. Since ℓ is not in the set S, the Hasse-Davenport Theorem together with Lemma $9.4.1$ of [CEPT1] shows that the epsilon constants we will consider for the base change are the ℓ^a-th power of the corresponding constants before the base change. So to consider sign information, we are free to make a base change of the above kind. If we were interested in preserving more than just the sign of the ϵ-constant we can achieve this by placing a more strict congruence condition on the prime ℓ.*

Lemma 3.2. *For the infinite place, $\epsilon_{\infty,0}(D', V) = 1$*

This lemma is an immediate corollary to Proposition 5.4.2 of [CEPT1]. In particular, this proposition says that if d is odd then the archimedean epsilon constant associated to the canonical divisor and to any representation V of trivial determinant and dimension zero is equal to one. D' differs from the canonical divisor only by vertical fibers, and thus the result applies.

Lemma 3.3. *With $\mathcal{Y}$, D', and V chosen as above, $\epsilon_{v,0}(\mathcal{Y}, V) = \epsilon_{v,0}(D', V)$ for all finite places v of $\mathbb{Q}$.*

Proof: For all places $v \in S$, this follows directly from [CEPT1]. Their proof involves comparing Gauss sums with different arguments. In particular, let $\mathcal{C}_v$ be the set of irreducible components of $\mathcal{Y}_v^{red}$. For each $C_i \in \mathcal{C}_v$ they define κ_i, a Gauss sum associated to the restriction of the representation V to the inertia group of the generic point of C_i. Furthermore, they define c_i to be the ℓ-adic Euler characteristic with compact support of the open subscheme of C_i consisting of points which are nonsingular in $\mathcal{Y}_v^{red}$. They then note that the formulae developed by Saito in [tS] imply that $\epsilon_{v,0}(\mathcal{X}, V) = \prod_{i \in \mathcal{C}_v} \kappa_i(V)^{c_i}$.

Next they show that for each C_i we can compute that $deg_{C_i}(\mathcal{O}_{\mathcal{Y}}(K_{\mathcal{Y}} + \mathcal{Y}_S^{red})) = -c_i f_i$, where f_i is the index of the constant field extension $[F_i : \mathbb{F}_p]$. Changing views, we let δ' be a point where $\mathcal{Y}_v^{red}$ intersects the horizontal divisor D'. We can define Gauss sums $\kappa_{\delta'}$ in a similar way to the above defined κ_i, such that, in particular, $\kappa_{\delta'} = \kappa_i^{[k(\delta):F_i]}$. Furthermore, the local epsilon constant $\epsilon_{v,0}(D', V)$ is given by $\prod_{\delta' \in D' \cap \mathcal{Y}_v^{red}} \kappa_{\delta'}$ (see [tS] p. 416). The proof of the lemma in this case now reduces to counting intersection numbers and verifying that κ_i occurs as a factor the same number of times in both $\epsilon_{v,0}(\mathcal{Y}, V)$ and $\epsilon_{v,0}(D', V)$.

For the finite places v which are not in S the argument is similar. It is only the intersection multiplicities of D' with certain vertical divisors that matters, and these numbers do not change in the event that we add new vertical fibers into the divisors. For this reason, the appearance of $\mathcal{Y}_T$ in the equality $D' + \mathcal{Y}_T = K_{\mathcal{Y}} + \mathcal{Y}_S^{red}$ makes no difference in the argument. ■

With these lemmas in hand, we can make the following series of calculations:

$$\begin{aligned}
\epsilon(\mathcal{Y}, V) &= \prod_v \epsilon_v(\mathcal{Y}, V) \\
&= \epsilon_\infty(\mathcal{Y}, V) \prod_{v finite} \epsilon_{v,0}(\mathcal{Y}, V)\epsilon(\mathcal{Y}_v, V) \\
&= \epsilon_\infty(\mathcal{Y}, V) \prod_{v finite} \epsilon_{v,0}(D', V)\epsilon(\mathcal{Y}_v, V) \\
&= \epsilon_\infty(\mathcal{Y}, V)\epsilon_{\infty,0}(D', V) \prod_{v finite} \epsilon_{v,0}(D', V)\epsilon(D'_v, V)\epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V) \\
&= \epsilon(D', V)\epsilon_{\infty,0}(\mathcal{Y}, V) \prod_{v finite} \epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V) \qquad (3.1)
\end{aligned}$$

In these calculations, $D'_v = D' \otimes_{\mathbb{Z}} \mathbb{Z}/p(v)$ is the finite collection of closed points of D' lying above the finite place v of $\mathbb{Q}$.

Lemma 3.4. *$\epsilon(D', V)$ is positive.*

Proof: D' is a one-dimensional object, and the restriction of V to D' will still be an orthogonal representation. By applying the theorem of Fröhlich-Queyrut to the normalization of D' (which we denote by $(D')^{\#}$), we get that $\epsilon((D')^{\#}, V)$ is positive. Now, because the definition of local constants involves only the Galois action on general fibers, $\epsilon_{v,0}((D')^{\#}, V) = \epsilon_{v,0}(D', V)$. Thus, we are only concerned with the difference between the terms $\epsilon((D')^{\#}_v, V)$ and $\epsilon(D'_v, V)$, all of which come about from the singular points z of D'. The action of G is étale at these points, and thus we can compute the local constants at these points as $\epsilon(y, V) = det(-F|(H^0(y, \mathbb{Q}_\ell) \otimes V)^{G_x} = det(V)(\pi_{\mathcal{Y}_v^{red},y})$, which is equal to one due to our hypotheses that V has trivial determinant. ■

Thus, we have reduced the calculation of the sign of $\epsilon(\mathcal{Y}, V)$, which is an inherently two-dimensional calculation, to a collection of fibral computations $\epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V)$ for each finite place v, and a calculation for the archimedean component $\epsilon_{\infty,0}(\mathcal{Y}, V)$.

3.2 The One-Component Case

Theorem 3.5. *Let $\mathcal{X}, \mathcal{Y}, D'$ be as above and let V be an orthogonal virtual representation of dimension 0 and trivial determinant. Furthermore, assume v is a finite place of $\mathbb{Q}$ such that $\mathcal{Y}_v^{red}$ is irreducible. Then $\epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V) = 1$.*

Proof: Assume that $\mathcal{Y}_v^{red}$ consists of a single component. Then $\mathcal{Y}_v^{red}$ is smooth by hypothesis. Let c be an irreducible component of $\mathcal{X}_v$ with generic point μ_c. Let G_{μ_c} be the Galois group acting on the generic point of c, and I_{μ_c} be the inertia group at the generic point of c. Then we have that $I_{\mu_c} \subseteq G_{\mu_c} \subseteq G$. We denote I_{μ_c} by I. We know from our tameness hypotheses that the order of I is relatively prime to v, and we further know that I is a cyclic group. The specific structure of I is discussed in detail in the Appendix to [CEPT1].

We begin by computing $\epsilon(\mathcal{Y}_v, V) = \prod_i det(-F|(H^i(\overline{\mathbb{Z}/v\mathbb{Z}} \otimes_{\mathbb{Z}/v\mathbb{Z}} \mathcal{Y}_v, \mathbb{Q}_\ell) \otimes V)^G)^{(-1)^{i+1}}$, where F is the Frobenius element as described above. We know by our hypotheses that the cover $\mathcal{X}_v^{red} \to \mathcal{Y}_v^{red}$ is a tame G_{μ_c}/I-cover of smooth curves over $\mathbb{Z}/p\mathbb{Z}$. Furthermore, the action of G/I on $\mathcal{X}_v^{red}$ is étale because $I = I_{\mu_c} = I_{\mathcal{X},x}$, the inertia group of the point x, for all points $x \in \mathcal{X}_v^{red}$. This implies that $\epsilon(\mathcal{Y}_v, V) = \epsilon(\mathcal{Y}_v^{red}, V^I)$. I acts trivially on the cohomology group $H^*(\mathcal{X}_v^{red}, \mathbb{Q}_\ell)$, so Saito's formulae in [tS] imply that $\epsilon(\mathcal{Y}_v, V)$ can be calculated as $det(V^I)(K_{\mathcal{Y}_v^{red}})$, where $K_{\mathcal{Y}_v^{red}}$ is the canonical divisor on $\mathcal{Y}_v^{red}$. The terms $K_{\mathcal{Y}_v^{red}}$ are well defined, as we have assumed that for all finite v the irreducible components of $\mathcal{Y}_v^{red}$ are themselves smooth.

Next we look at the term $\epsilon(D'_v, V)$. Let D be the preimage of D' in $\mathcal{X}$, and let ℓ be a prime different from v. Let $I_{D,x}$ be the cyclic inertia group of a point x lying above the points in $\mathcal{Y}_v \cap D'$ (note that this is independent of which point x we choose). Because D'_v is zero dimensional, we know that $\epsilon(D'_v, V) = \prod_{y \in D'_v} \epsilon(y, V)$ where

$$\epsilon(y, V) = det(-F|(H^0(\pi^{-1}(y)^{red}, \mathbb{Q}_\ell) \otimes V)^G)$$

if we view π as the cover $D_v \to D'_v$. We know that $\pi^{-1}(y)^{red} = (y \times_{D'_v} D_v)^{red} = x \times_{G_x} G$. In particular, this implies that

$$H^0(\pi^{-1}(y)^{red}, \mathbb{Q}_\ell) \otimes V = (Ind_{G_x}^G H^0(x, \mathbb{Q}_\ell)) \otimes V$$

Because $I_{\mu_c} = I_{\mathcal{X},x}$ for all points x, we can see that

$$Ind_{G_x}^G H^0(x, \mathbb{Q}_\ell) = Infl_{G/I_{\mathcal{X},x}}^G Ind_{G_x/I_{\mathcal{X},x}}^{G/I_{\mathcal{X},x}} H^0(x, \mathbb{Q}_\ell)$$

Recall that $I = I_{\mathcal{X},x}$ acts trivially on $H^0(x, \mathbb{Q}_\ell)$. This allows us to compute that

$$\begin{aligned}
\epsilon(y, V) &= det(-F|(H^0(\pi^{-1}(y)^{red}, \mathbb{Q}_\ell) \otimes V)^G) \\
&= det(-F|(Infl_{G/I}^G Ind_{G_x/I}^{G/I} H^0(x, \mathbb{Q}_\ell) \otimes V)^G) \\
&= det(-F|(Ind_{G_x/I}^{G/I} H^0(x, \mathbb{Q}_\ell) \otimes V^I)^{G/I}) \\
&= \epsilon(y, V^I)
\end{aligned}$$

where $\epsilon(y, V^I)$ is the local constant associated to the G/I cover $\mathcal{X}_v^{red} \to \mathcal{Y}_v^{red}$.

This last term is in turn equal to $det(V^I)(\pi_{\mathcal{Y}_v^{red},y})$, where $\pi_{\mathcal{Y}_v^{red},y}$ is the local uniformizer from classfield theory since $\mathcal{X}_v^{red} \to \mathcal{Y}_v^{red}$ is an unramified G/I cover. Finally, we can put these terms together to get that $\epsilon(D'_v, V) = det(V^I)(D' \cap \mathcal{Y}_v^{red})$, where $D' \cap \mathcal{Y}_v^{red}$ is viewed as a divisor on $\mathcal{Y}_v^{red}$.

Lemma 3.6. *Under the above hypotheses, $D' \cap \mathcal{Y}_v^{red}$ is a canonical divisor on $\mathcal{Y}_v^{red}$.*

If we are able to prove this lemma, we will have shown that $\epsilon(D'_v, V) = det(V^I)(K) = \epsilon(\mathcal{Y}_v, V)$, so in particular $\epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V) = 1$, and Theorem 3.5 will be proven.

In order to prove Lemma 3.6, recall that we chose D' so that $\mathcal{O}_\mathcal{Y}(D' + \mathcal{Y}_T) = \omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_S^{red})$. We note that if we look at the two exact sequences:

$$0 \to \mathcal{O}_\mathcal{Y}(-\mathcal{Y}_v^{red}) \to \mathcal{O}_\mathcal{Y} \to \mathcal{O}_{\mathcal{Y}_v^{red}} \to 0$$

$$0 \to \mathcal{O}_\mathcal{Y}(D' - \mathcal{Y}_v^{red}) \to \mathcal{O}_\mathcal{Y}(D') \to \mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_v^{red}} \to 0$$

we get that for all primes v, $\mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_v^{red}}$ is the same as $\mathcal{O}_{\mathcal{Y}_v^{red}}(D' \cap \mathcal{Y}_v^{red})$. Furthermore, for those primes v which are in S (and in particular are not in T), we further get that $\mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_v^{red}} = \mathcal{O}_\mathcal{Y}(D' + \mathcal{Y}_T)|_{\mathcal{Y}_v^{red}}$. We now are able to make the following computation for all $v \in S$ such that $\mathcal{Y}_v^{red}$ is irreducible:

$$\begin{aligned}
\mathcal{O}_{\mathcal{Y}_v^{red}}(D' \cap \mathcal{Y}_v^{red}) &= \mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_v^{red}} \\
&= \mathcal{O}_\mathcal{Y}(D' + \mathcal{Y}_T)|_{\mathcal{Y}_v^{red}} \\
&= \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{red})|_{\mathcal{Y}_v^{red}} \\
&= \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_v^{red})|_{\mathcal{Y}_v^{red}} \\
&= \omega_{\mathcal{Y}_v^{red}}
\end{aligned}$$

In other words, for such v, $D' \cap \mathcal{Y}_v^{red}$ is a canonical divisor on $\mathcal{Y}_v^{red}$ under these assumptions.

It remains to show that Lemma 3.6 holds for primes w outside of the set S. We know that for such w, the fibres $\mathcal{Y}_w$ are reduced and smooth and that the local equations have a nice form. This implies in particular that $\mathcal{Y}_w^{red} = \mathcal{Y}_w$ is a principal divisor and thus that $\mathcal{O}_\mathcal{Y}(\mathcal{Y}_w^{red})$ is isomorphic to $\mathcal{O}_\mathcal{Y}$.

Recall that by definition we have that $D' + \mathcal{Y}_T = K_\mathcal{Y} + \mathcal{Y}_S^{red}$. This tells us that

$$D' + \mathcal{Y}_T - \mathcal{Y}_S^{red} + \mathcal{Y}_w^{red} = K_{\mathcal{Y}/\mathbb{Z}} + \mathcal{Y}_w^{red}$$

and therefore that

$$\mathcal{O}_\mathcal{Y}(D' + \mathcal{Y}_T - \mathcal{Y}_S^{red} + \mathcal{Y}_w^{red})|_{\mathcal{Y}_w^{red}} = \omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_w^{red})|_{\mathcal{Y}_w^{red}}$$

The right hand side is equal to $\omega_{\mathcal{Y}_w^{red}}$ by the adjunction formula. To calculate the left hand side, we observe that w is not in S by hypotheses, although it may be in T. Thus there is an integer m which depends on the multiplicity of w in T,such that the following calculations hold:

$$\begin{aligned}
\mathcal{O}_\mathcal{Y}(D' + \mathcal{Y}_T - \mathcal{Y}_S^{red} + \mathcal{Y}_w^{red})|_{\mathcal{Y}_w^{red}} &= \mathcal{O}_\mathcal{Y}(D' + m\mathcal{Y}_w^{red})|_{\mathcal{Y}_w^{red}} \\
&= \mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_w^{red}} \otimes \mathcal{O}_\mathcal{Y}((\mathcal{Y}_w^{red})^{\otimes m})|_{\mathcal{Y}_w^{red}} \\
&\cong \mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_w^{red}} \otimes \mathcal{O}_\mathcal{Y} \\
&\cong \mathcal{O}_\mathcal{Y}(D')|_{\mathcal{Y}_w^{red}} \\
&= \mathcal{O}_\mathcal{Y}(D' \cap \mathcal{Y}_w^{red})
\end{aligned}$$

which proves Lemma 3.6 and therefore Theorem 3.5. ■

Remark 3.7. *Note that we can identify $det(V^I)$ with a character of order 1 or 2 of $\mathrm{Pic}_{Weil}(\mathcal{Y}_v^{red})$. Thus, $\epsilon(\mathcal{Y}_v^{red}, det(V^I)) = 1$, as it's the ratio of epsilon constants associated to zeta functions. In particular we can show that both $\epsilon(\mathcal{Y}_v, V)$ and $\epsilon(D', V)$ are trivial, which would give us another way of proving Theorem 3.5. However, for what follows it is more illuminating to instead consider what their ratio is, and in particular how close each is to being of the form $det(V^I)(K)$.*

Remark 3.8. *One should note, however, that our hypotheses that V is of trivial determinant does not imply that V^I is of trivial determinant, as one might speculate. A simple example where one can see this is if we allow $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Representation theory tells us that there are three nontrivial irreducible representations on this group χ_1, χ_2, and $\chi_1\chi_2$ in addition to the trivial representation χ_0. All four of these representations are of dimension one. Let V be the virtual representation $\chi_1 + \chi_2 + \chi_1\chi_2 - 3\chi_0$, which is of trivial determinant and dimension zero. Now, let $I \subseteq G$ be the second copy of $\mathbb{Z}/2\mathbb{Z}$, in which case $V^I = \chi_1 - 3\chi_0$, which does not have trivial determinant (or degree zero, for that matter).*

3.3 Partial Trivializations and the Canonical Cycles

In this section we will describe in detail the relative canonical cycle associated to line bundles with partial trivializations, as defined by Takeshi Saito in [tS], as well as other machinery which we will need in order to compute the terms $\epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V)$ in the case where $\mathcal{Y}_v^{red}$ consists of more than one component.

Definition 3.9. *Let $\mathcal{D}$ be a divisor on a scheme X and let $\{\mathcal{D}_i\}_{i\in I}$ be the set of irreducible components of $\mathcal{D}$. A locally free sheaf $\mathcal{E}$ on X is said to be partially trivialized on $\mathcal{D}$ if there exists a family $\rho = (\rho_i)$ of $\mathcal{O}_{\mathcal{D}_i}$-morphisms $\rho_i : \mathcal{E}|_{\mathcal{D}_i} \to \mathcal{O}_{\mathcal{D}_i}$ such that for all subsets $J \subset I$, the map $\rho_J = \bigoplus_{i\in J} \rho_i : \mathcal{E}|_{\mathcal{D}_J} \to \mathcal{O}^J_{\mathcal{D}_J}$ is surjective.*

Given a partial trivialization of the sheaf $\mathcal{E}$ of rank n on X, Saito defines the relative top chern class $c_n(\mathcal{E}, \rho) \in H^{2n}(X \; mod \; \mathcal{D}, \mathbb{Z}_q(n))$ based on an idea of Anderson in [A]. In particular, Saito notes that there is a canonical isomorphism

$$\Phi : H^{2n}(X mod \mathcal{D}, \mathbb{Z}_q(n)) \to H^{2n}(V mod \Delta, \mathbb{Z}_q(n))$$

where V is the covariant vector bundle associated to the dual of $\mathcal{E}$. We also have a natural map from $H^0(X, \mathbb{Z}_q) \to H^{2n}(V \; mod \; \Delta, \mathbb{Z}_q(n))$. Let $[0]$ be the image of the class $1 \in H^0(X, \mathbb{Z}_q)$ under this map, and then Saito defines the relative top chern class to be the inverse image of $[0]$ under the canonical isomorphism Φ above. Relative top chern classes satisfy nice functorial properties, and the relative top chern class is mapped to the normal top chern class under the canonical map $H^{2n}(X \; mod \; \mathcal{D}, \mathbb{Z}_q(n)) \to H^{2n}(X, \mathbb{Z}_q(n))$. Furthermore, the following corollary of Proposition 1 in [tS] gives us a way to compare the relative top chern classes associated to two different partial trivializations.

Corollary 3.10. *Let X be a $\mathbb{F}_p$-scheme, and let $(\mathcal{E}, \rho)$ be a partially trivialized locally free sheaf on X. Let $\sigma_i = f_i^{-1} \cdot \rho_i : \mathcal{E}|_{\mathcal{D}_i} \to \mathcal{O}_{\mathcal{D}_i}$ where f_i comes from $\mathbb{F}_p^*$, so that $\sigma = (\sigma_i)$ is another partial trivialization of $\mathcal{E}$. Finally, let $\mathcal{E}_i = Ker(\rho_i)$ so that $\rho|_{\mathcal{D}_i}$ is a partial trivialization of $\mathcal{E}_i$. Then we can compute the difference between the relative top chern classes as*

$$c_n(\mathcal{E}, \rho) - c_n(\mathcal{E}, \sigma) = \sum \{f_i\} \cup c_{n-1}(\mathcal{E}_i, \rho|_{\mathcal{D}_i})$$

In section 2 of [tS], Saito uses the construction of relative top chern classes to define the relative canonical cycle.

Definition 3.11. *Let $\mathcal{D}$ be a divisor with simple normal crossings on a variety X of dimension n defined over a perfect field F of characteristic p, and let $U = X - \mathcal{D}$. Let $\Omega^1_{X/F}(log \; \mathcal{D})$ be the locally free $\mathcal{O}_X$-module of rank n of differential 1-forms on X with logarithmic poles along $\mathcal{D}$. Then the cycle*

$$c_{X,U} = (-1)^n c_n(\Omega^1_{X/F}(log\mathcal{D}), res)$$

is called the relative canonical cycle. It lies inside the cohomology with compact support $H^{2n}_c(Xmod \; \mathcal{D}, \hat{\mathbb{Z}}'(n))$, where $\hat{\mathbb{Z}}' = \prod_{q\neq p} \mathbb{Z}_q$. The relative canonical cycle has degree equal to $\chi(U_{\overline{F}}) = \Sigma(-1)^q dim H^q_c(U_{\overline{F}}, \overline{\mathbb{Q}}_l)$. Note that this definition differs from that of S. Saito in [sS], but only up to a change in sign.

Saito observes that one can also define a relative top chern class (and hence a relative canonical cycle) sitting inside of $H^n(X\, mod\, \mathcal{D}, \mathbb{G}_m)$, the divisor class group with modulus D, and in particular we can define $c_{X,U}$ as an element of $H^n(X\ mod\ \mathcal{D}, \mathbb{G}_m)$ in the case when $n = 1$. For our work we will want to consider the case where X is one of the components of $\mathcal{Y}_v^{red}$, and therefore is of dimension one. We will look at the relative top chern class $c_{X,U}$ lying inside the generalized class group

$$H^1(X mod \mathcal{D}, \mathbb{G}_m) = [(\oplus_{x \notin \mathcal{D}} \mathbb{Z}) \oplus (\oplus_{x \in \mathcal{D}} K^*/U_x^1)]/K^*$$

where K is the fraction field of X and $U_x^1 = 1 + m_x$. In particular, the class $c_{X,U}$ can be computed in the following way (see the example in §1 of [tS]). Let ω be a nontrivial rational section of $\Omega_X^1(log\ \mathcal{D})$ such that for all points $x \in \mathcal{D}$, $ord_x(\omega) = -1$ and $res_x(\omega) = 1$ then the relative canonical cycle represented by the class of the zero cycle which is supported off of $\mathcal{D}$ given by

$$c_{X,U} = -\sum_{x \in U} ord_x(\omega) \cdot [x]$$

Proposition 3.12. *Let $\mathcal{Y}_v^{red}$ consist of two components F' and G'. Let D' be a horizontal divisor chosen as in the previous sections.*

(1) There is a canonical isomorphism $\phi : \mathcal{O}_{F'}(D' \cap F') \to \omega_{F'}(F' \cap G')$ up to multiplication by a global unit.

(2) The global section $1 \in \Gamma(\mathcal{O}_{F'}(D' \cap F'))$ maps under ϕ to an element $\gamma \in \Gamma(\omega_{F'}(F' \cap G'))$ such that $ord_x(\gamma) = 1$ if $x \in F' \cap D'$, $ord_x(\gamma) = -1$ if $x \in F' \cap G'$, and $ord_x(\gamma) = 0$ otherwise.

(3) Set $a_x = res_x(\gamma)$ for all $x \in F' \cap G'$. Then $c_{F',U_{F'}}$ is such that $-c_{F',U_{F'}}$ is the class in $[(\oplus_{x \in (F'-G')} \mathbb{Z}) \oplus (\oplus_{x \in F' \cap G'} K^*/U_x^1)]/K^* = H^1(F'\ mod\ (F' \cap G'), K)$ of the element

$$c = (\oplus_{x \in D' \cap F'} 1 \in \mathbb{Z}) \oplus (\oplus_{x \in F' - D' - G'} 0 \in \mathbb{Z}) \oplus (\oplus_{x \in F' \cap G'} a_x)$$

The proof of part (1) follows from carrying through a series of calculations analagous to those in the proof of Lemma 3.6. In particular,

$$\begin{aligned}
\mathcal{O}_{F'}(D' \cap F') &= \mathcal{O}_{\mathcal{Y}}(D')|_{F'} \\
&= \mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)|_{F'} \\
&\cong \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red})|_{F'} \\
&= \omega_{\mathcal{Y}}(\mathcal{Y}_v^{red})|_{F'} \\
&= [\omega_{\mathcal{Y}}(F') \otimes \mathcal{O}_{\mathcal{Y}}(G')]|_{F'} \\
&= \omega_{\mathcal{Y}}(F')|_{F'} \otimes \mathcal{O}_{\mathcal{Y}}(G')|_{F'} \\
&= \omega_{F'} \otimes \mathcal{O}_{F'}(F' \cap G') \\
&= \omega_{F'}(F' \cap G')
\end{aligned}$$

To prove parts (2) and (3) of the proposition, we set $X = F'$ and $\mathcal{D} = F' \cap G'$. Proving these statements is then just a matter of calculating the various orders and residues of γ given that we know them for the element $1 \in \Gamma(\mathcal{O}_{F'}(D' \cap F'))$. Explicitly, they can be computed by following the residue map on elements of the sheaves through the equalities and congruences in the calculations above. Note that all of the isomorphisms are unique with the exception of $\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T) \cong \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red})$. This map, while not unique, is well-defined up to multiplication by a global unit, and therefore when we look at classes mod K^* the discrepancy will not matter.

This proposition gives us an explicit way to construct the relative canonical class in our situation. In particular, the a_x terms come about because of the difference in natural partial trivializations on the sheaves $\mathcal{O}_{F'}(D' \cap F')$ and $\omega_{F'}(F' \cap G')$ associated to the restriction map $\mathcal{O}_{F'}(D' \cap F') \to \mathcal{O}_{F'}(D' \cap F')|_{F' \cap G'} = \mathcal{O}_{F' \cap G'}$ and the residue maps res_x.

For the computations in the next section, we will need the following definitions.

Definition 3.13. *Define the following classes which lie in the generalized class group* $H^1(F' \; mod \; (F' \cap G'), K)$.

a. *Define an element* $\lambda \in (\oplus_{x \in (F'-G')} \mathbb{Z}) \oplus (\oplus_{x \in F' \cap G'} K^*/U_x^1)$ *which has components equal to* $1 \in \mathbb{Z}$ *at all points* $x \in D' \cap F'$, *equal to* $0 \in \mathbb{Z}$ *at all points* x *in* $F' - D' - G'$ *and equal to the identity in* K^*/U_x^1 *for all points* $x \in F' \cap G'$. *We then look at the class* $[\lambda]_{F'} \in H^1(F' \; mod \; (F' \cap G'), K)$, *which is the first relative chern class of the line bundle* $\mathcal{O}_{F'}(D' \cap F')$ *with partial trivializations. One can define* $[\lambda]_{G'}$ *in a similar way.*

b. *Let* $\delta_{F'}$ *be the class in* $H^1(F' \; mod \; (F' \cap G'), K)$ *which corresponds to the element* $\delta = (\oplus 0) \oplus (\oplus a_x) \in (\oplus_{x \in F'-G'} \mathbb{Z}) \oplus (\oplus_{x F' \cap G'} K^*/U_x^1)$. *In other words, this element is trivial at all places corresponding to* $x \notin F' \cap G'$ *and for those places which correspond to points* $x \in F' \cap G'$ *consists of the terms* a_x *coming about as the difference between the partial trivializations of* $\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)$ *and* $\omega_{\mathcal{Y}}(\mathcal{Y}_S^{red})$, *as found in the above characterization of* $c_{F', U_{F'}}$. *Note that* $\delta_{F'}$ *can be thought of as the quotient of* $c_{F', U_{F'}}$ *and* $[\lambda]_{F'}$. *One can define* $\delta_{G'}$ *in a similar way.*

3.4 The General Case

We have shown that in the situation where $\mathcal{Y}_v^{red}$ consists of a single component F' then the fibral contribution to the root number is positive. Now we will consider the next case, where $\mathcal{Y}_v^{red}$ consists of two irreducible components, say F' and G'. Note that in particular this implies that $v \in S$. Recall that from Equation 3.1 above we are interested in comparing $\epsilon(D'_v, V)$ and $\epsilon(\mathcal{Y}_v, V)$. By our initial assumptions, D' intersects $\mathcal{Y}_v^{red}$ in smooth points of $\mathcal{Y}_v^{red}$, so in particular we get that the set $D' \cap F' \cap G' = \emptyset$

Define $I_1 = I_{\mu(F)}$ and $I_2 = I_{\mu(G)}$, where F and G are components of $\mathcal{X}_v^{red}$ lying above F' and G' respectively. Then $det(V^{I_1})$ is a character of the Galois group of the cover $F \to F'$, which will be tame with respect to the divisor $F' \cap G'$. Classfield theory says that we can therefore view $det(V^{I_1})$ as a character of the ray class group of F' with conductor $F' \cap G'$. We wish to define the term $det(V^{I_1})(\pi_{D',y})$ for points $y \in F'$. In order to do so, we view $det(V^{I_1})$ as a character of the ideles $J_{F'}$ of F'. In other words, it is an idele class character modulo the conductor, which will be supported on $F' \cap G'$. We then define $det(V^{I_1})(\pi_{D',y})$ to be the value of $det(V^{I_1})$ on the idele $(1, \dots, 1, \pi_{D',y}, 1, \dots)$ which is trivial away from y. This is well defined as the conductor of $det(V^{I_1})$ does not involve y and the difference between two local uniformizers is a unit.

If we define $det(V^{I_1})(D' \cap F')$ to be equal to the product $\prod_{y \in D' \cap F'} det(V^{I_1})(\pi_{D',y})$, then this term will be independent of the choices of uniformizers as all components are unramified, and we are able to make the following calculation:

$$\begin{aligned} \epsilon(D'_v, V) &= \prod_{y \in D' \cap \mathcal{Y}_v^{red}} \epsilon(y, V) \\ &= \prod_{y \in D' \cap F'} \epsilon(y, V) \prod_{y \in D' \cap G'} \epsilon(y, V) \\ &= \prod_{y \in D' \cap F'} det(V^{I_1})(\pi_{D',y}) \prod_{y \in D' \cap G'} det(V^{I_2})(\pi_{D',y}) \\ &= det(V^{I_1})(D' \cap F') \cdot det(V^{I_2})(D' \cap G') \end{aligned} \tag{3.2}$$

Recall that in the case where $\mathcal{Y}_v^{red}$ consisted of a single component F', we were able to show that $\epsilon(D'_v, V) = det(V^I)(D' \cap F')$. In that case Lemma 3.6 showed that our hypothesis on D' implied that $D' \cap F'$ was a canonical divisor on F'. The preceding section showed that in this more complicated case, while $D' \cap F'$ is not a canonical divisor on F', it is close to being one. To make this precise requires the results of the previous section. In particular, when viewed as an idele class character, $det(V^{I_1})$ breaks into components $det(V^{I_1})_x$ which are unramified for all $x \notin F' \cap G'$, and therefore we get

$$det(V^{I_1})(D' \cap F') = \prod_{y \in D' \cap F'} det(V^{I_1})_y(\pi_{D',y}) = det(V^{I_1})([\lambda]_{F'})$$

Therefore Equation 3.2 says that in the case where V is an orthogonal virtual representation of dimension 0 and trivial determinant, $\mathcal{Y}_v^{red}$ consists of two components F' and G' and D' is chosen as above, then we have that

$$\epsilon(D'_v, V) = det(V^{I_1})([\lambda]_{F'}) det(V^{I_2})([\lambda]_{G'}) \tag{3.3}$$

Switching gears, we now want to take a look at the term $\epsilon(\mathcal{Y}_v, V)$. For the moment, we will assume that $F' \cap G'$ consists of a single point z. We begin by looking at the two exact sequences:

$$0 \to U = \mathcal{Y}_v^{red} - z \to \mathcal{Y}_v^{red} \to z \to 0$$

$$0 \to U \to \widehat{\mathcal{Y}_v^{red}} = F' \amalg G' \to \{z_{F'}, z_{G'}\} \to 0$$

where $z_{F'}$ (respectively $z_{G'}$) is the point z thought of as sitting just on F' (respectively G'). Epsilon factors are multiplicative within exact sequences as well as in disjoint unions, so these sequences imply that

$$\begin{aligned}\epsilon(\mathcal{Y}_v^{red}, V) &= \epsilon(U, V)\epsilon(z, V) \\ &= \frac{\epsilon(F', V)\epsilon(G', V)}{\epsilon(z_{F'}, V)\epsilon(z_{G'}, V)}\epsilon(z, V) \end{aligned} \tag{3.4}$$

To continue, we must consider the $\epsilon(F', V)$ term. In order to compute this term we use the following result proven by Saito in [tS]

Lemma 3.14. *Let X, U be as in Definition 3.11, and let the action of G be étale on U. Then $\prod_{y \in U} \epsilon_y(X, V) = det(V)(c_{X,U})$*

Applying this lemma to our situation, we are able to make the following computation:

$$\begin{aligned}\epsilon(F', V) &= \epsilon(F', V^{I_1}) \\ &= \prod_{y \in (F')^0} \epsilon_y(F', V^{I_1}) \\ &= \epsilon_z(F', V^{I_1}) \prod_{y \neq z} (\epsilon_y(F', V^{I_1})) \\ &= \epsilon_z(F', V^{I_1}) det(V^{I_1})(c_{F', U_{F'}}) \\ &= \epsilon_{0,z}(F', V^{I_1})\epsilon(z_{F'}, V^{I_1}) det(V^{I_1})(c_{F', U_{F'}}) \end{aligned} \tag{3.5}$$

Plugging Equation 3.5 (and the analogous formula for $\epsilon(G', V)$) into Equation 3.4 gives that

$$\epsilon(\mathcal{Y}_v^{red}, V) = \epsilon_{0,z}(F', V^{I_1})\epsilon_{0,z}(G', V^{I_2}) det(V^{I_1})(c_{F', U_{F'}}) \det(V^{I_2})(c_{G', U_{G'}})\epsilon(z, V)$$

which we we can combine with Equation 3.3 to get that

$$\frac{\epsilon(\mathcal{Y}_v, V)}{\epsilon(D'_v, V)} = \frac{det(V^{I_1})(c_{F', U_{F'}}) \det(V^{I_2})(c_{G', U_{G'}})}{det(V^{I_1})([\lambda]_{F'}) det(V^{I_2})([\lambda]_{G'})}\epsilon_{0,z}(F', V^{I_1})\epsilon_{0,z}(G', V^{I_2})\epsilon(z, V)$$

Note that

$$\frac{det(V^{I_1})(c_{F', U_{F'}})}{det(V^{I_1})([\lambda]_{F'})} = det(V^{I_1})(\delta_{F'})$$

where δ is the class defined in Definition 3.13.

Considering a slightly more general case, in which we still only have two components, but where $F' \cap G'$ consists of more than one point, it is clear that all of the calculations will follow through and we will get that

$$\frac{\epsilon(Y_v, V)}{\epsilon(D'_v, V)} = det(V^{I_1})(\delta_{F'}) det(V^{I_2})(\delta_{G'}) \prod_{z \in F' \cap G'} \epsilon_{0,z}(F', V^{I_1})\epsilon_{0,z}(G', V^{I_1})\epsilon(z, V)$$

If we have more than two components in $\mathcal{Y}_v^{red}$ then the bookkeeping becomes more complicated but the mathematics does not. We first set up the necessary notation. Let C_i be the components of $\mathcal{Y}_v^{red}$. Furthermore, let $C_{i,j} = C_i \cap C_j$, $Z = \cup_{i \neq j} C_{i,j}$ be the collection of all intersection points and let U_{C_i} be the open set consisting of $C_i - Z$. Finally, let I_i be the inertia group associated to C_i as above. We are still interested in computing $\epsilon(\mathcal{Y}_v, V)$ and $\epsilon(D'_v, V)$. Let λ_v and δ_{v,C_i} be the classes λ and δ defined above for a particular class v and a particluar component C_i. In particular, recall that δ_{v,C_i} can be calculated purely from looking at points $z \in Z$

For the latter, the computation works just as it did before, as we know that if $i < j$ the $C_{i,j}$ are disjoint from each other as well as from D'. We obtain that

$$\begin{aligned}\epsilon(D'_v, V) &= \prod_i det(V^{I_i})(C_i \cap D') \\ &= \prod_i det(V^{I_i})([\lambda_v]_{C_i})\end{aligned}$$

To compute $\epsilon(\mathcal{Y}_v, V)$ we need to use the following exact sequences:

$$0 \to U = \mathcal{Y}_v^{red} - Z \to \mathcal{Y}_v^{red} \to Z \to 0$$

$$0 \to U \to \widehat{\mathcal{Y}_v^{red}} = \amalg_i C_i \to \amalg_{i \neq j} C_{i,j} \to 0$$

where $\amalg_{i \neq j} C_{i,j}$ can be thought of as the set consisting of two copies of Z, with each point considered as sitting once on each of the two C_i which it comes from originally. We can now use these sequences as well as the above calculations of $\epsilon(C_i, V)$ to get that

$$\begin{aligned}\epsilon(\mathcal{Y}_v^{red}, V) &= \epsilon(U, V)\epsilon(Z, V) \\ &= \prod_i \epsilon(C_i, V) \prod_{z \in Z} \frac{\epsilon(z, V)}{\epsilon(z_{C_{i_1}}, V)\epsilon(z_{C_{i_2}}, V)} \\ &= \prod_i det(V^{I_i})(c_{C_i, U_{C_i}}) \prod_{z \in Z} \epsilon_{0,z}(C_{i_1}, V^{I_{i_1}})\epsilon_{0,z}(C_{i_2}, V^{I_{i_2}})\end{aligned}$$

where we think of $z \in Z$ as lying on $C_{i_1} \cap C_{i_2}$. If we put all of these calculations together we get the following result.

Theorem 3.15. *Under all of the above hypotheses and notation, we get that for all v,*

$$\frac{\epsilon(\mathcal{Y}_v, V)}{\epsilon(D'_v, V)} = \prod_i det(V^{I_i})(\delta_{v,C_i}) \prod_{z \in Z} \epsilon_{0,z}(C_{i_1}, V^{I_{i_1}})\epsilon_{0,z}(C_{i_2}, V^{I_{i_2}})\epsilon(z, V)$$

where both of these products are equal to one if the set Z is empty.

Combining Equation (3.1), Lemma 3.4 and Theorem 3.15 gives a precise form of Theorem 1.1. Note that other than the term $\epsilon_\infty(\mathcal{Y}, V)$, the other terms depend only on the crossing points of the components of fibers $\mathcal{Y}_v^{red}$.

Chapter 4

Connections to Other Work

In this chapter we define an element of the Brauer group $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ related to the results of the previous section and prove a connection between it and the Galois theoretic invariant $w_2(\pi)$ defined by Cassou-Nogues, Erez, and Taylor.

4.1 Definition of $\mu(\mathcal{X}, G, V)$

In order to define the invariant $\mu(\mathcal{X}, G, V) \in H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ we must first impose an additional condition on our horizontal divisor D'. In particular, we must assume that D' is chosen so that when we calculate the a_x terms by the residue maps as in Definition 3.13, they are all equal to 1. We will call such a choice of D' a nice divisor.

Given any horizontal divisor D' as in Chapter 3, it is possible to find a nice divisor which is close to it due to the following moving lemma, which is proven in section 4.3. In particular, this shows that nice divisors D' always exist and therefore that our class $\mu(\mathcal{X}, G, V)$ will be well-defined.

Lemma 4.1. *There exists a meromorphic function h on $\mathcal{Y}_v^{red}$ such that the divisor of h intersects the special fibers $\mathcal{Y}_v^{red}$ transversally at smooth points away from D'_v and such that h takes on prescribed values at the singular points of $\mathcal{Y}_v^{red}$. In particular, given a horizontal divisor D' as in the previous section, the divisor $D' + div(h)$ will have residue maps equal to one at the crossing points of components of $\mathcal{Y}_v^{red}$.*

It is clear that for all nice divisors D', the classes δ_{v,C_i} defined in Definition 3.13 are the same and in particular that the terms $det(V^{I_i})(\delta_{v,C_i})$ are independent of the choice of a nice divisor. This allows us to prove the following.

Proposition 4.2. *In the case where V is an orthogonal representation of dimension equal to zero and trivial determinant and where D' is a nice divisor, the local constant $\epsilon_v(D', V)$ is independent of the choice of D'. In particular, the element $\mu(\mathcal{X}, G, V)$ in the global Brauer group $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ whose local invariant at the place v is given by the sign of $\epsilon_v(D', V)$ is well-defined.*

Proof: If V is of dimension 0 and trivial determinant, the proposition follows from the statement of Theorem 3.15. For any fixed place v of $\mathbb{Q}$, it is clear that the right hand side of the equation in Theorem 3.15 is independent of our choice of a nice canonical divisor D', as the $det(V^{I_i})(\delta_{v,C_i})$ terms are. Furthermore, it is clear that $\epsilon(\mathcal{Y}_v, V)$ is independent of our choice of D'. Thus, it follows from the theorem that $\epsilon_v(D', V)$ is independent of the choice of D'. Next, we note that lemma 3.3 tells us that $\epsilon_{v,0}(D', V)$ must also be independent of our choice of D'. Therefore it must be the case that $\epsilon_v(D', V) = \epsilon_{v,0}(D', V)\epsilon(D'_v, V)$ is independent of the choice of D'. The product of all of the $\epsilon_v(D', V)$ is equal to $\epsilon(D', V)$, which must be equal to one from the Fröhlich-Queyrut theorem. This tells us that we can define an element $\mu(\mathcal{X}, G, V)$ in $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ by setting the local component at the prime v to be equal to the sign of $\epsilon_v(D', V)$. ■

4.2 The connection to $w_2(\pi)$

In this section, we will consider the relationship between the class $\mu(\mathcal{X}, G, V)$ lying in $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ which we defined in proposition 4.2 and the Stiefel-Whitney class $w_2(\pi) \in H^2(Y_{et}, \mathbb{Z}/2\mathbb{Z})$ associated to the cover $\pi : \mathcal{X} \to \mathcal{Y}$ which is considered by Cassou-Nogues, Erez, and Taylor in [CNET]. To begin this comparison, we describe their construction.

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a tamely ramified cover of degree n, where $\mathcal{X}$ and $\mathcal{Y}$ are regular schemes and $\mathcal{Y}$ is connected. Furthermore, we must make the technical assumption that the ramification indices are all odd. Cassou-Nogues, Erez, and Taylor use Grothendeick's equivariant cohomology theory to define an invariant $w_2(\mathcal{X}/\mathcal{Y}) = w_2(\pi) \in H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z})$ associated to this situation. Their definition generalizes to define classes $w_i(\pi)$ which lie in $H^i(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z})$ for all positive integers i, but in this thesis we will only be interested in w_2. These terms are generalized Stiefel-Whitney classes, and are obtained by pulling back the universal Hasse-Witt classes defined by Jardine using classifying maps related to a quadratic form E. The precise definition of E uses the existence of a locally free sheaf $\mathcal{D}_{\mathcal{X}/\mathcal{Y}}^{-1/2}$ whose square is the inverse different of the covering $\mathcal{X}/\mathcal{Y}$. In the case of unramified coverings $\mathcal{D}_{\mathcal{X}/\mathcal{Y}}^{-1/2} = \mathcal{O}_{\mathcal{X}}$.

In [CNET] they prove the following equality in $H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z})$, which is an analog of a theorem of Serre:

$$w_2(\pi_*(\mathcal{D}_{\mathcal{X}/\mathcal{Y}}^{-1/2}, Tr_{\mathcal{X}/\mathcal{Y}})) = w_2(\pi) + (2) \cup (d_{\mathcal{X}/\mathcal{Y}}) + \rho(\mathcal{X}/\mathcal{Y})$$

where $\rho(\mathcal{X}/\mathcal{Y})$ is defined by the ramification of $\mathcal{X}/\mathcal{Y}$, $d_{\mathcal{X}/\mathcal{Y}}$ is the function field discriminant, and the left hand side of the equation is the Hasse-Witt invariant associated to the square-root of the inverse different bundle. Note that if we look at the one-dimensional version of this formula the middle term on the right hand side becomes trivial. Therefore, in the case of étale covers of curves the formula reduces to

$$w_2(\pi) = w_2(\pi_*(\mathcal{D}_{\mathcal{X}/\mathcal{Y}}^{-1/2}, Tr_{\mathcal{X}/\mathcal{Y}})) = w_2(E)$$

where $w_2(E)$ is the second Hasse-Witt invariant associated to the square root of the inverse different, as described in detail in [CNET].

Let D' be a choice of a canonical divisor on $\mathcal{Y}$ in the sense of the previous chapters, and let $i : D' \hookrightarrow \mathcal{Y}$ be the natural inclusion. An étale covering of $\mathcal{Y}$ naturally restricts to give an étale covering of D'. We now have the following natural maps

$$i^* : H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z}) \to H^2(D'_{et}, \mathbb{Z}/2\mathbb{Z})$$

$$res : H^2(D'_{et}, \mathbb{Z}/2\mathbb{Z}) \to H^2_{et}(\mathbb{Q}(D'), \mathbb{Z}/2\mathbb{Z}) = H^2_{gal}(\overline{\mathbb{Q}}/\mathbb{Q}(D'), \mathbb{Z}/2\mathbb{Z})$$

$$cor : H^2_{gal}(\overline{\mathbb{Q}}/\mathbb{Q}(D'), \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$$

where the latter two maps are restriction and corestriction in the sense of Serre (for details see Chapter VII of [Se]). Composing these maps gives a natural map

$$H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$$

We denote the image of the class $w_2(\pi) \in H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z})$ under this map by $\tilde{w}_2(\pi) \in H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$. At first glance it appears as though this element may depend on our choice of canonical divisor D'. However, we want to show that it does not depend on this choice and furthermore that the element $\tilde{w}_2(\pi)$ is connected in a natural way to the element $\mu(\mathcal{X}, G, V)$. Recall that $\mu(\mathcal{X}, G, V)$ is defined by letting the local invariant at the place v be given by the sign of $\epsilon(D'_v, V)$ but also turns out to be independent of our choice of a canonical divisor D'.

Of course, the class $\mu(\mathcal{X}, G, V)$ depends on the choice of a representation V of G. The natural representation to consider is R, the regular representation of the group G. In particular, the nicest possible theorem would say that if V were the regular representation of G, $\mu(\mathcal{X}, G, V)$ would equal to $\tilde{w}_2(\pi)$. However,we have only shown that $\mu(\mathcal{X}, G, V)$ is a well-defined class in the case where V is of dimension zero and of trivial determinant, neither of which holds for R. So instead of setting $V = R$, we consider the representation $V = R - det(R) - T^{n-1}$, where $det(R)$, the determinant of the regular representation, is a character whose order is either one or two, T is the trivial representation and n is the degree of the cover $\mathcal{X}/\mathcal{Y}$. This choice of V is an orthogonal representation, and it has trivial determinant and dimension 0. In this case, we can prove the following theorem:

Theorem 4.3. *Assume that we are in the above situation, and in particular that* $V = R - det(R) - T^{n-1}$. *Let* $\mathcal{Y}_1/\mathcal{Y}$ *be either the trivial cover or the subcover of* $\mathcal{X}/\mathcal{Y}$ *of degree 2, depending on whether* $det(R)$ *is of order* 1 *or* 2 *respectively. Then as classes in* $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$*, we have the equality*

$$\mu(\mathcal{X}, G, V) = \tilde{w}_2(\mathcal{X}/\mathcal{Y}) - \tilde{w}_2(\mathcal{Y}_1/\mathcal{Y}) - (n-1)\tilde{w}_2(\mathcal{Y}/\mathcal{Y})$$

The proof of this theorem relies on the interpretation of each side of the equation as a Stiefel-Whitney class. In particular, Cassou-Nogues, Erez, and Taylor show that the element $w_2(\pi)$ is the Hasse-Witt invariant associated to the full covering of surfaces. Thus, when we restrict the class to the one-dimensional divisor D' we see that the element $i^*(w_2(\pi)) \in H^2(D'_{et}, \mathbb{Z}/2\mathbb{Z})$ is equal to the Stiefel-Whitney class associated to the form $E' = (\mathcal{D}_{D/D'}^{-1/2}, Tr_{D/D'})$ on the canonical divisor D' of $\mathcal{Y}$. This follows as a generalization to étale cohomology of results of Fröhlich in [F], which allow us to associate the class $i^*(w_2(\pi))$ to G-extensions of the ring of integers of the residue field of the generic point of D'.

Next we make use of the results of Deligne which allow us to interpret local Stiefel-Whitney classes in terms of local root numbers. In particular, the following lemma is shown in [D2]:

Lemma 4.4. *Let $d = 1$, so that the fibers $\mathcal{X}_v$ and $\mathcal{Y}_v$ are all one dimensional schemes. Furthermore, let V be an orthogonal virtual representation of dimension zero and trivial determinant. Under these hypotheses, the local root number $W(V_v) = sign(\epsilon_v(\mathcal{Y}, V))$ is equal to $exp(2\pi i cl(sw_v))$, where sw_v is the local Stiefel-Whitney class, and $cl(sw_v) \in \{0, 1/2\} \subset \mathbb{Q}/\mathbb{Z}$.*

In other words, in characteristic not equal to two, the sign of the ϵ-constants $\epsilon_v(D', V)$ of the representation on the one dimensional horizontal divisor D' are determined by whether or not the classes $w_2(\pi)$ are trivial in the Brauer group, and $\epsilon_v(D', V)$ is automatically positive when $v = 2$. One can see that these are exactly the terms which come up in the computation of the class of Cassou-Nogues, Erez, and Taylor.

In particular, $\epsilon_v(D', V) = \epsilon_v(D', R)\epsilon_v(D', det(R))$ is the same as the local Hasse-Witt invariants. However, we are working with étale covers of curves and so from the results of [CNET] discussed above, these Hasse-Witt invariants are simply the images of the appropriate classes $w_2(\pi)$. This proves Theorem 4.3. ■

4.3 Proof of Lemma 4.1

The proof of Lemma 4.1 involves a generalized version of Bertini's Theorem. For now, let us assume that X is a smooth curve defined over an infinite field k and let us choose a finite set of points $p_1, \dots, p_m$ on X. We define the divisor $p = \sum_i p_i$. Furthermore, let us choose constants c_i which lie in the residue field $k(p_i)$ of the points p_i. Finally, let us choose Λ to be an effective very ample divisor on X of large degree which is supported off of p. We look at the group of global sections $H^0(X, \mathcal{O}_X(\Lambda))$ and let $f_0, \dots, f_t$ be a basis of this group. This basis defines a projective embedding from X into $\mathbb{P}^t_k$ whose projective coordinates we will write as $x_0, \dots, x_t$.

We wish to prove that there exist linear forms l_0 and l_1 in the variables x_i such that the following properties hold:

(1) For $j = 0, 1$, let H_j be the hyperplane defined by $l_j = 0$ in $\mathbb{P}^t_k$. Then $H_j \cap X$ is a finite set of closed points which is regular and disjoint from $\{p_1, \dots, p_m\}$. Furthermore, we wish to choose the l_j so that $H_1 \cap H_2 \cap X$ to be empty.

(2) It follows from (1) that the function $l_1/l_0|_X$ is in $\mathcal{O}_{X,p_i}$ for each i. We wish to impose the additional conditions that the image of l_1/l_0 in each $k(p_i)$ are the prescribed constants c_i.

The classical version of Bertini's Theorem (Theorem II.8.18 of [H]) tells us that there exist linear forms l_0 so that H_0 satisfies condition (1). We now fix one choice of such an l_0, and we will attempt to construct an l_1 so that the pair satisfies properties (1) and (2). We begin by looking at the set V consisting of all linear forms such that $\{l_0, l_1\}$ satisfy condition (2). In other words,

$$V = \{l = a_0x_0 + \ldots + a_tx_t \mid \forall j, \frac{l}{l_0}|_X(p_j) = c_j \in k(p_j)\}$$

This V will be an affine space over k. Furthermore, because we chose the divisor Λ to have high degree it follows from a Riemann-Roch argument that V is of codimension m inside of $H^0(X, \mathcal{O}_X(\Lambda))$.

For each point $x \in X$, we now define a set $V_x \subseteq V$ which consists of all linear forms $l \in V$ so that the hyperplane defined by $l = 0$ has contact order strictly bigger than 1 at x. In other words, V_x will consist of those linear forms who do not intersect X nicely at the point x. We can again use the Riemann-Roch theorem to show that for almost all choices of x, we get that the dimension of V_x is equal to $dim\ V - 2$.

Let $U = X - \{p_1, \ldots, p_m\}$ so that U is an affine curve, and define $T \subseteq U \times V$ to be the set of all pairs (x, l) such that $x \in U$ and $l \in V_x$. We have seen that the projection map $\pi : T \to U$ is surjective and for almost all $x \in U$ (in particular for those points such that $k(x) = k$), we see that the fiber $\pi^{-1}(x)$ is an affine space whose dimension is equal to $dim\ V - 2$. In particular, this shows that T is irreducible and that the dimension of T is equal to $dim\ V - 1$. But this shows us that the natural projection map $\gamma : T \to V$ must not be surjective.

In particular, we can choose some element $l_1 \in V$ which is not in the image of γ. This statement says that the hyperplane H defined by $\{l_1 = 0\}$ is such that $H \cap U$ is regular and, since $l_1 \in V$, we know that $l_1/l_0(p_i) = c_i \in k(p_i)$, and thus that l_1 and l_o satisfy conditions (1) and (2) above.

The above argument has used the hypothesis that X is a smooth curve. However, as long as X is a reduced curve with smooth irreducible components which have normal crossings, then the same argument will hold as long as we include these crossing points in the set of $\{p_i\}$. Instead of using the normal Riemann-Roch theorem we must use the version for singular curves described on p.298 of [H], and the rest of the argument will hold.

In order to prove lemma 4.1 we will use this generalized version of Bertini's theorem applied to $X = \mathcal{Y}_v^{red}$. Specifically, we choose the set of points $\{p_1, \ldots, p_m\}$ to include the crossing points of components of $\mathcal{Y}_v^{red}$ as well as the points in $D' \cap \mathcal{Y}_v^{red}$. The above argument then allows us to find a meromorphic function h where we can specify the values of the function $h = l_1/l_0$ at the crossing points of $\mathcal{Y}_v^{red}$ so that the residues that come up when we consider $D'' = D' + div(h)$ are all equal to one and D'' intersects $\mathcal{Y}_v^{red}$ in the prescribed manner.

Chapter 5

Tameness and Examples

Throughout this thesis, we have assumed that we have a finite group acting tamely on an arithmetic surface. In this chapter we will discuss in detail the idea of tameness and construct some concrete examples.

The concept of tameness first shows up in number theory, where we define an extension of local fields to be tamely ramified if the ramification index is relatively prime to the characteristic of the field. This definition generalizes to the case of covering maps of varieties. In particular, let us consider a finite morphism of varieties $V \to U$ and a point u of codimension one in U which is normal, so that the local ring $\mathcal{O}_{U,u}$ is a discrete valuation ring. We then say that $V \to U$ is tamely ramified over u if for each point $v \in V$ lying above u, the extension $\mathcal{O}_{V,v}/\mathcal{O}_{U,u}$ of DVR's is tame, in the usual sense of having a separable residue field extension and ramification of order prime to the residue characteristic of u.

Throughout this thesis, we have been working in the setting first developed by Grothendeick and Murre in [GM]. For this definition of tameness, we define X to be a regular and connected noetherian scheme and $D \subset X$ to be a divisor with normal crossings, and let $U = X - D$. Define $D_1, \dots, D_n$ to be the irreducible components of D and $\eta_1, \dots, \eta_n$ to be the generic points of the D_i respectively. Let $U' \to U$ be a finite étale morphism, and let X' be the normalization of U'. We say that the covering $U' \to U$ is tame if the D_i are all regular and if the extensions of discrete valuation rings associated to the local rings on X of the generic points η_i are tamely ramified in the sense of number theory.

This definition of tameness has many nice properties, but in practice it can be difficult to test. There is a related concept known as numerical tameness, which was defined by Chinburg and Erez in [CE].

Definition 5.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a G-cover of normal schemes. The cover f is said to be numerically tame if for each point $y \in \mathcal{Y}$ there exists a scheme $\mathcal{Y}'$ and a $\mathcal{Y}'$-scheme Z such that:*

a. There exists a flat morphism $\mathcal{Y}' \to \mathcal{Y}$ whose image contains y.

b. The structure morphism $Z \to \mathcal{Y}'$ is an H-covering, where H is a finite group with order relatively prime to the residue characteristic of $\mathcal{Y}'$

c. There exists a G-equivariant isomorphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \simeq (Z \times_{\mathcal{Y}'} G_{\mathcal{Y}'})/H$ induced by a homomorphism $H \to G$.

In [CE], the authors show that a cover that is tamely ramified in codimension one with respect to a divisor with normal crossings in the sense of Grothedeick and Murre is numerically tame. Numerical tameness has the advantage of being a local condition which can be detected by étale base change. Furthermore, if $\mathcal{X}$ is a regular model of a curve X, then a group G acts in a numerically tame way on $\mathcal{X}$ if and only if it does on the minimal model $\mathcal{X}^{min}$. In an unpublished paper of Seon-In Kwon [K], she proves the following theorem:

Theorem 5.2. *Let X be an elliptic curve over K and let $\mathcal{X}$ be the minimal model of X over $\mathcal{O}_K$. Consider the action of a group $G \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \subset X(K)$ of torsion points on $\mathcal{X}$. Then the action of G on $\mathcal{X}$ is numerically tame if and only if for each place v of $\mathcal{O}_K$ whose residue characteristic p divides the order of G, the following conditions are satisfied:*
(i) The minimal model $\mathcal{X}$ has good or multiplicative reduction at v.
(ii) The Zariski closure in $\mathcal{X}$ of the p-Sylow subgroup G_p of G is smooth over $Spec\mathcal{O}_K$.
In particular, these conditions imply that $gcd(n, m) = 1$.

This theorem provides us with a set of concrete criteria for checking when a finite group acts tamely on the integral model of an elliptic curve. In particular, the second condition asks us to compute the p-torsion points of the minimal model over p, and check that they do not coalesce when we reduce mod p.

We will now show an example of a computation of the orthogonal ϵ-constants associated to the tame action of a finite group on a surface. In order to do so, we will need to calculate terms $\epsilon(z, V)$, where $z \in \mathcal{Y}$ is a closed point defined over a finite field. Section 2.5 of [CEPT1] gives us the following way of making this computation.

Lemma 5.3. *Let x be a point of $\mathcal{X}$ over a point $y \in \mathcal{Y}$ which has finite residue field. Furthermore, let F_x be the arithmetic Frobenius element lying in G. Then $\epsilon(y, V) = det(V^{I_x})(-F_x)$, where $I_x \subseteq G$ is the inertia group of the point x.*

Example 5.4. *Let X be the elliptic curve given by the equation $y^2 + xy + y = x^3$. This equation is minimal over every prime $p \in Spec(\mathbb{Z})$, and thus it defines $\mathcal{X}$, the minimal model over $\mathbb{Z}$. The torsion subgroup of X is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and the torsion points of order three are $(0, 0)$ and $(0, -1)$. We wish to check to see whether or not the action of $G \cong \mathbb{Z}/3\mathbb{Z}$ is numerically tame by the criteria in Theorem 5.2. In order to do this, we first note that the discriminant of $\mathcal{X}$ is $-26 = -1 \cdot 2 \cdot 13$, and thus $\mathcal{X}$ has good reduction at 3. Furthermore, using the algorithm of Tate as explained in [T] we see that $\mathcal{X}$ has multiplicative reduction at 2 and at 13 (and in particular the fibers have Kodaira type $I1$).*

Next we check condition (ii). $G_3 = G = \{(0,0),(0,-1),\mathbf{0}\}$, and these points clearly do not coalesce mod q for any prime q (and in particular for $q = 3$). Thus, this action of G on $\mathcal{X}$ does satisfy the appropriate conditions.

Therefore the action of G is in fact numerically tame on $\mathcal{X}$. Furthermore, it follows from formulae of Velu in [V] that $\mathcal{Y} = \mathcal{X}/G$ is an integral model of the elliptic curve defined by the equation $y^2 + xy + y = x^3 - 5x - 8$. The fibers of $\mathcal{Y}$ are also nonsingular with the exceptions of the fibers at $v = 2, 12$ which are both of Kodaira type $I3$. However, $\mathcal{Y}$ is not regular, and thus the results of Chapter 3 do not apply and in fact the ϵ-constants are not well-defined. However, due to Theorem 3.8 of Kwon's dissertation [K2], we know that after a finite number of blow-ups on the singular fibers we can blow up $\mathcal{X}$ in a way such that the action of the group G extends to a tame action of G on the blow-up $\mathcal{X}_1$, and the quotient $\mathcal{Y}_1 = \mathcal{X}_1/G$ is in fact regular. This theorem applies because all of the local decomposition groups must be subgroups of $\mathbb{Z}/3\mathbb{Z}$ and in particular must be cyclic of degree $n \leq 3$.

Next we need to define a representation V of $\mathbb{Z}/3\mathbb{Z}$ satisfying certain properties. We know from representation theory that there are two distinct nontrivial one-dimensional characters of $\mathbb{Z}/3\mathbb{Z}$ of order three. Let us define V_1 to be the sum of these characters and V_2 to be $2\chi_0$, where χ_0 is the trivial character. We then define V to be $V_1 - V_2$. The sum of two characters which are complex conjugate is an orthogonal representations, so V will be orthogonal. It also is not hard to see that V has dimension zero and trivial determinant.

In general, computing $\epsilon(\mathcal{Y}_1, V)$ might be difficult, but in light of Theorem 1.1, the computation simplifies greatly. In particular, we only need to compute $\epsilon_{\infty,0}(\mathcal{Y}_1, V)$, $det(V^{I_j})(\delta_{v,C_i})$ for $v = 2, 13$, and the terms

$$\epsilon_{0,z}(C_{i_1}, V^{I_{i_1}})\epsilon_{0,z}(C_{i_2}, V^{I_{i_2}})\epsilon(z, V)$$

at the singular points above the primes $p = 2$ and $p = 13$. For the above choice of the representation V, we can see that $det(V^I)$ is trivial for all possible inertia groups I. More precisely, $det(V_j^I)$ will be trivial for $j = 1, 2$. If I acts trivially on V_j this is obvious, as the $det(V_j)$ are both in fact trivial. On the other hand, if I acts nontrivially on V_j, then V_j^I will be trivial as the kernels of both characters which make up V_j are the same, and thus $det(V_j^I)$ will be trivial as well.

Let us first look at the part of the calculation of $\epsilon(\mathcal{Y}_1, V)$ coming from the fiber of $\mathcal{Y}_1$ above the prime 2. Denote the three components of $\mathcal{Y}_2$ by $F_1, F_2, and F_3$. Let I_i be the inertia group associated to F_i. In particular, $det(V^{I_i})$ is trivial in each of these cases for the reasons described above. Thus, the $det(V^{I_j})(\delta_{2,C_i})$ terms are equal to one. Many of the $\epsilon_{0,z}(C_i, V^{I_i})$ terms will also immediately be equal to one as many of the V^{I_i} terms are themselves trivial. To compute the others, we use the formulae of Saito in [tS]. Because we are looking at cases where $det(V_i^I)$ is trivial, these formulae reduce the computation of $\epsilon_{0,z}(C_i, V^{I_i})$ to the computation of a Gauss sum $\tau_{C_i}(V^{I_i})$. We now use the fact that our representation V is the sum of a representation and its complex conjugate. A formula of Lang ([L] p.92) tells us that the Gauss sum associated to the representation $W + \overline{W}$ has the same sign as the representation W evaluated at -1. Applying this to our situation we find that $\tau_{C_i}(V^{I_i})$ is positive.

The above paragraph holds for the points above $p = 13$ as well, so we can ignore those terms. We can now use Lemma 5.3 in order to compute the $\epsilon(z, V)$ terms. In particular, the fact that all of the $det(V^I)$ terms are trivial tells us that these terms are also positive. To summarize, we have that $\epsilon(\mathcal{Y}_1, V) = \epsilon_{\infty,0}(\mathcal{Y}_1, V)$.

In Theorem 4.0.1 of [CPT] they show that the Euler characteristics and the character functions ζ_S associated to the action of a finite group on a minimal model over $\mathbb{Z}$ of an elliptic curve over $\mathbb{Q}$ satisfying certain properties (which our example does satisfy) are trivial. This is because the group must act trivially on the various $H^{p,q}$ pieces of the Hodge structure. But this in turn shows that $\epsilon_{\infty,0}(\mathcal{Y}_1, V)$ is trivial, and thus $\epsilon(\mathcal{Y}_1, V)$ is positive.

We note that many of the computations in the last example will hold whenever we are in the case of one of Kwon's examples. In particular, it will often be the case that we can show that $det(V^{I_i})$ is trivial. Combining this with the results in [CPT] shows that $\epsilon(\mathcal{Y}, V)$ is trivial for a large number of examples where $\mathcal{Y}$ is a blowup of the minimal model of an elliptic curve with G acting tamely as in [K].

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