A TWISTED NONABELIAN HODGE CORRESPONDENCE

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ABSTRACT

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We prove an extension of the nonabelian Hodge theorem in which the underlying objects are *twisted* torsors over a smooth complex projective variety. On one side of this correspondence the twisted torsors come equipped with an action of a sheaf of twisted differential operators in the sense of Beĭlinson and Bernstein. On the other, we endow them with appropriately defined twisted Higgs data.

The proof we present here is completely formal, in the sense that we do not delve into the analysis involved in the classical nonabelian Hodge correspondence. Instead, we use homotopy-theoretic methods, especially the theory of principal ∞-bundles, to reduce our statement to previously known results of Simpson.

Contents

Chapter 1

Introduction: the case of twisted vector bundles

1.1 Twisted vector bundles

Let X be a smooth projective variety over $\mathbb{C}$, considered either as a scheme with the étale topology or as a complex analytic space endowed with the classical topology. Given $\alpha \in H^2(X, \mathcal{O}_X^\times)$, we can always choose an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X such that there exists a Čech 2-cocycle $\underline{\alpha} = \{\alpha_{ijk}\} \in \check{Z}^2(\mathfrak{U}, \mathcal{O}_X^\times)$ representing the class α[1]. The following definition goes back to Giraud's work on nonabelian cohomology [Gir71].

Definition 1.1.1. *An* α-twisted sheaf *on* X *is a collection*

$$\left(\underline{\mathcal{E}} = \{\mathcal{E}_i\}_{i\in I},\ \underline{g} = \{g_{ij}\}_{i,j\in I}\right)$$

of sheaves $\mathcal{E}_i$ *of* $\mathcal{O}_X$*-modules on* U_i*, together with isomorphisms* $g_{ij} : \mathcal{E}_j|_{U_{ij}} \to \mathcal{E}_i|_{U_{ij}}$

[1] In the analytic case, take a good open cover; for the étale topology, see [Mil80, Theorem III.2.17].

satisfying $g_{ii} = \mathrm{id}_{\mathcal{E}_i}$, $g_{ij} = g_{ji}^{-1}$, *and the* $\underline{\alpha}$*-twisted cocycle condition*

$$g_{ij} g_{jk} g_{ki} = \alpha_{ijk} \, \mathrm{id}_{\mathcal{E}_i}$$

along U_{ijk} *for any* $i, j, k \in I$. *Given two* α*-twisted sheaves* $\left(\underline{\mathcal{E}}, \underline{g}\right)$ *and* $\left(\underline{\mathcal{F}}, \underline{h}\right)$, *a morphism between them is given by a collection*

$$\underline{\varphi} = \{\varphi_i\}_{i \in I}$$

of morphisms $\varphi_i : \mathcal{E}_i \to \mathcal{F}_i$ *intertwining the transition functions, i.e., such that*

$$\varphi_i g_{ij} = h_{ij} \varphi_j .$$

Although this definition uses a particular representative for the class α, it can be proven [Căl00] that the category of such objects is independent —up to equivalence— of the choice of cover $\mathfrak{U}$ and representing cocycle $\underline{\alpha}$, justifying our choice of labelling them with the class α in sheaf cohomology instead of with the cocycle $\underline{\alpha}$ itself.

In this article we will be concerned not with the whole category of α-twisted sheaves but with the subgroupoid ${}_{\alpha}\mathfrak{Vec}_n(X)$ of α-twisted vector bundles of fixed rank n and isomorphisms between them; to wit, those α-twisted sheaves $\left(\underline{\mathcal{E}}, \underline{g}\right)$ for which each $\mathcal{E}_i$ is a vector bundle of rank n, with morphisms $\underline{\varphi} : \left(\underline{\mathcal{E}}, \underline{g}\right) \to \left(\underline{\mathcal{F}}, \underline{h}\right)$ the invertible ones.

One important thing to notice is that, for a fixed class $\alpha \in H^2(X, \mathcal{O}_X^{\times})$, there might be no α-twisted vector bundles of rank n. In general this is a difficult question having to do with the relationship between the Azumaya Brauer group of X and its cohomological Brauer group [Gro68b]. A short survey of these issues can be found in [DP08, Section 2.1.3]. Here we will content ourselves with pointing out that ${}_{\alpha}\mathfrak{Vec}_n(X)$ is empty unless α is n-torsion —a fact that will come up again below. Indeed, suppose that there is an object $(\underline{\mathcal{E}}, \underline{g}) \in {}_{\alpha}\mathfrak{Vec}_n(X)$. Then, $\det \underline{g} = \{\det g_{ij}\}_{i,j\in I}$ provides an element of $\check{C}^1(\mathfrak{U}, \mathcal{O}_X^\times)$ whose Čech differential equals $\underline{\alpha}^n$, so that α^n is the trivial class in $H^2(X, \mathcal{O}_X^\times)$.

Observe too that we can produce an honest, untwisted $\mathbb{P}^{n-1}$-bundle on X by projectivizing all the locally defined vector bundles: the twisting $\underline{\alpha}$ goes away because it is contained in the kernel of the map $GL_n \to \mathbb{P}GL_n$ —which coincides with the center of GL_n. This pattern will appear once more when we introduce connections and Higgs fields in the remainder of this introduction.

1.2 Twisted connections and twisted Higgs fields

1.2.1 The classical nonabelian Hodge correspondence.

Let $\mathcal{E}$ be a vector bundle on X. Recall that a *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$ satisfying the Leibniz rule $\nabla(fv) = v \otimes df + f\nabla v$ for all sections $f \in \mathcal{O}_X, v \in \mathcal{E}$. There is a natural extension of a connection ∇ to a $\mathbb{C}$-linear map $\nabla^{(1)} : \mathcal{E} \otimes \Omega_X^1 \to \mathcal{E} \otimes \Omega_X^2$ defined through the graded Leibniz identity

$$\nabla^{(1)}(v \otimes \alpha) = v \otimes d\alpha - \nabla v \wedge \alpha, \qquad \text{for } v \in \mathcal{E}, \alpha \in \Omega_X^1.$$

The composition $\nabla^{(1)} \circ \nabla$ is then $\mathcal{O}_X$-linear, and we define the *curvature* $C(\nabla)$ of ∇ to be its image under the standard duality isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega_X^2) \cong \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{O}_X, \Omega_X^2 \otimes \mathcal{E}nd(\mathcal{E})\big) \cong \Gamma\big(X, \Omega_X^2 \otimes \mathcal{E}nd(\mathcal{E})\big). \quad (1.2.1)$$

A connection is said to be *flat* if its curvature vanishes, and then it is customary to call the pair $\big(\mathcal{E},\nabla\big)$ a *flat vector bundle*. The difference of any two connections ∇_1 and ∇_2 on the same vector bundle $\mathcal{E}$ is an $\mathcal{O}_X$-linear map and can also be considered as a 1-form with values in the endomorphism bundle of $\mathcal{E}$ through

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}\otimes\Omega^1_X)\cong\mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{O}_X,\Omega^1_X\otimes\mathcal{E}nd(\mathcal{E})\big)\cong\Gamma\big(X,\Omega^1_X\otimes\mathcal{E}nd(\mathcal{E})\big).$$

If $\nabla_2-\nabla_1=\omega$, their curvatures are related by $C(\nabla_2)-C(\nabla_1)=d\omega-\omega\wedge\omega$, where $\omega\wedge\omega$ corresponds under (1.2.1) to the composition

$$\mathcal{E}\xrightarrow{\omega}\mathcal{E}\otimes\Omega^1_X\xrightarrow{\omega\otimes\mathrm{id}}\mathcal{E}\otimes\Omega^1_X\otimes\Omega^1_X\xrightarrow{\mathrm{id}\otimes(-\wedge-)}\mathcal{E}\otimes\Omega^2_X.$$

On the other hand, a *Higgs bundle* is a pair $\big(\mathcal{F},\phi\big)$ of a vector bundle $\mathcal{F}$ together with an $\mathcal{O}_X$-linear map $\phi:\mathcal{F}\to\mathcal{F}\otimes\Omega^1_X$ —usually referred to as a *Higgs field*— such that

$$0=\phi\wedge\phi\in\Gamma\big(X,\Omega^2_X\otimes\mathcal{E}nd(\mathcal{E})\big),$$

where $-\phi\wedge\phi$ is called the *curvature* $C(\phi)$ of ϕ in analogy with the case of a connection. Two Higgs fields ϕ_1 and ϕ_2 on the same vector bundle $\mathcal{F}$ also differ by a 1-form with values in $\mathcal{E}nd(\mathcal{F})$, say $\phi_2-\phi_1=\omega$, with their curvatures coupled by the equation $C(\phi_2)-C(\phi_1)=-\omega\wedge\omega$

The nonabelian Hodge theorem [Sim92] establishes an equivalence between the category of flat vector bundles on X and a certain full subcategory of the category of Higgs bundles on the same variety. The latter is specified by two conditions on objects:

- The first one is purely topological: the components of the first and second Chern characters of $\mathcal{F}$ along the hyperplane class $[H]\in H^2(X,\mathbb{C})$ of X should vanish:

$$\mathrm{ch}_1(\mathcal{F}) \cdot [H]^{\dim X - 1} = 0 = \mathrm{ch}_2(\mathcal{F}) \cdot [H]^{\dim X - 2}.$$

Equivalently, the first and second Chern *classes* vanish along $[H]$.

- The second condition involves the holomorphic structure of both the underlying vector bundle and the Higgs field: a Higgs bundle $(\mathcal{F}, \phi)$ is said to be *semistable* if for every subbundle $\mathcal{F}' \subseteq \mathcal{F}$ preserved by the Higgs field —i.e, such that $\phi(\mathcal{F}') \subseteq \mathcal{F}' \otimes \Omega^1_X$— we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. Here the *slope* μ of a vector bundle is defined as the quotient of its degree by its rank.

The first of these conditions implies the vanishing of the slope of any Higgs bundle in this subcategory, since $\mathrm{ch}_1(\mathcal{F}) \cdot [H]^{\dim X - 1}$ is its degree; the second condition then reduces to saying that any ϕ-invariant subbundle of $\mathcal{F}$ has non-positive degree.

1.2.2 Twisted connections.

If we want a generalization of this result to twisted vector bundles, the first question should be how to endow an α-twisted vector bundle $(\underline{\mathcal{E}}, \underline{g})$ with something like a flat connection. The obvious answer would be to equip each $\mathcal{E}_i$ with a flat connection $\nabla_i : \mathcal{E}_i \to \mathcal{E}_i \otimes \Omega^1_{U_i}$ in such a way that the following diagram is commutative for every pair of indices $i, j \in I$:

$$\begin{tikzcd} \mathcal{E}_j\big|_{U_{ij}} \arrow[r, "\nabla_j"] \arrow[d, "g_{ij}"] & \mathcal{E}_j \otimes \Omega^1_{U_j}\big|_{U_{ij}} \arrow[d, "g_{ij}"] \\ \mathcal{E}_i\big|_{U_{ij}} \arrow[r, "\nabla_i"] & \mathcal{E}_i \otimes \Omega^1_{U_i}\big|_{U_{ij}} \end{tikzcd}$$

We can restate this last condition in a more compact way as $\nabla_i - g_{ij} \nabla_j \, g_{ij}^{-1} = 0$.

There is, however, a natural weakening of these requirements that yields a more general and interesting class of objects: namely, to allow for the locally defined connections to

- differ on double intersections by 1-forms with values in the center of the appropriate endomorphism bundle, and

- have nonzero central curvature.

This amounts to choosing cochains

$$\underline{\omega} = \{\omega_{ij}\} \in \check{C}^1(\mathfrak{U}, \Omega^1_X) \quad \text{and} \quad \underline{F} = \{F_i\} \in \check{C}^0(\mathfrak{U}, \Omega^2_X)$$

diagonally embedded in $\check{C}^1\big(\mathfrak{U}, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})\big)$ and $\check{C}^0\big(\mathfrak{U}, \Omega^2_X \otimes \mathcal{E}nd(\mathcal{E})\big)$, respectively, and demanding the ∇_i to satisfy the equations

$$\nabla_i - g_{ij} \nabla_j \, g_{ij}^{-1} = \omega_{ij} \quad \text{and} \quad C(\nabla_i) = F_i.$$

Of course we cannot pick $\underline{\omega}$ and $\underline{F}$ arbitrarily. Rather there is a set of compatibility conditions coming from the fact that the ∇_i are connections:

$$\begin{aligned} \omega_{ik} = \omega_{ij} + \omega_{jk} - d\log \alpha_{ijk} & \quad \text{on } U_{ijk} \\ F_i - F_j = d\omega_{ij} & \quad \text{on } U_{ij} \\ dF_i = 0 & \quad \text{on } U_i \end{aligned} \tag{1.2.2}$$

It turns out that these are precisely the relations needed to make the triple $\big(\underline{\alpha}, \underline{\omega}, \underline{F}\big)$ into a Čech 2-cocycle in hypercohomology of the multiplicative de Rham complex of X:

$$\mathrm{dR}_X^{\mathbb{G}_m} := \Big[\mathcal{O}_X^\times \xrightarrow{d\log} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots\Big].$$

To fix notation, recall that, given a complex $(E^\bullet, d)$ of sheaves on X, its hypercohomology with respect to a cover $\mathfrak{U}$ is defined as the cohomology of the total complex of the double complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\check{C}^1(\mathfrak{U}, E^0) \arrow[r, "d"] \arrow[u, "\delta"'] & \check{C}^1(\mathfrak{U}, E^1) \arrow[r, "d"] \arrow[u, "-\delta"'] & \check{C}^1(\mathfrak{U}, E^2) \arrow[r, "d"] \arrow[u, "\delta"'] & \cdots \\
\check{C}^0(\mathfrak{U}, E^0) \arrow[r, "d"] \arrow[u, "\delta"'] & \check{C}^0(\mathfrak{U}, E^1) \arrow[r, "d"] \arrow[u, "-\delta"'] & \check{C}^0(\mathfrak{U}, E^2) \arrow[r, "d"] \arrow[u, "\delta"'] & \cdots
\end{tikzcd}$$

Inspired by the geometrization of degree 2 cohomology classes that we will discuss in subsequent sections, we make the following definition.

Definition 1.2.1. *A* flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe over X *is the data of a 2-cocycle* $\big(\underline{\alpha}, \underline{\omega}, \underline{F}\big) \in \check{\mathbb{Z}}^2\big(\mathfrak{U}, \mathrm{dR}_X^{\mathbb{G}_m}\big)$.

Flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbes form a strict 2-category —a 2-groupoid, in fact—, with the 1-morphisms given by Čech 1-cochains, and the 2-morphisms by Čech 0-cochains. We should warn the reader that this category is not independent of the choice of cover $\mathfrak{U}$. Nevertheless, taking a refinement $\mathfrak{V}$ of $\mathfrak{U}$ produces a functor from the category of flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbes to that of flat $\mathfrak{V}$-$\mathbb{G}_m$-gerbes that is injective at the level of equivalence classes. We will later give a more abstract definition —that of a *flat* $\mathbb{G}_m$*-gerbe* (Definition 4.2.2)— that does not depend on the choice of a cover.

The notions of flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbes and flat $\mathbb{G}_m$-gerbes have appeared in similar form in the literature before, under the names of Dixmier-Douady sheaves of groupoids with connective structure and curving [Bry08], bundles gerbes with connection and curvature [Mur96], and gerbs with 0- and 1-connection [Cha98]. Our approach in this introduction is similar to that of Chaterjee, while the point of view in the remainder is closer in spirit to the first two references.

Definition 1.2.2. *A* basic[2] vector bundle on a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\big(\underline{\alpha}, \underline{\omega}, \underline{F}\big)$ *over* X *is*

[2]Classically these are known as *weight 1 twisted vector bundles.* In the more general setting in

a collection $\left(\underline{\mathcal{E}}, \underline{\nabla}, \underline{g}\right)$ *consisting of an* α*-twisted vector bundle* $\left(\underline{\mathcal{E}}, \underline{g}\right)$ *on* X *together with connections* ∇_i *on* $\mathcal{E}_i$ *satisfying the equations*

$$\nabla_i - g_{ij}\nabla_j\, g_{ij}^{-1} = \omega_{ij} \quad \textit{and} \quad C(\nabla_i) = F_i.$$

Having fixed a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\left(\underline{\alpha}, \underline{\omega}, \underline{F}\right)$, we can consider the groupoid of basic vector bundles on it. Its morphisms are given by those isomorphisms of the underlying α-twisted vector bundles that commute with the connections. A straightforward computation shows that an element of $\check{\mathbb{C}}^1\left(\mathfrak{U}, \mathrm{dR}_X^{\mathbb{G}_m}\right)$ giving an equivalence between two flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbes also furnishes an equivalence of their respective categories of basic vector bundles. Arguing as in [Căl00], it is possible to show that refining the cover $\mathfrak{U}$ does not affect this category up to equivalence.

If the cocycle $\underline{\alpha}$ is composed of locally constant functions —so that $d\log\underline{\alpha} = 0$—, the equations (1.2.2) satisfied by the 1- and 2-form parts of a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe do not involve $\underline{\alpha}$. In this case, we can separate the data of a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe into what we might call a $\mathfrak{U}$-$\mathbb{G}_m$*-gerbe* without any qualifiers —i.e., $\underline{\alpha}$ itself[3]— and the pair $\left(\underline{\omega}, \underline{F}\right)$. The latter, which defines a class in $\check{\mathbb{H}}^1(\mathfrak{U}, \Omega_X^1 \to \Omega_X^{2,\mathrm{cl}})$, gives rise to a sheaf of twisted differential operators (TDOs) on X in the sense of Beĭlinson and Bernstein [BB93], and basic vector bundles on $\left(\underline{\alpha}, \underline{\omega}, \underline{F}\right)$ can then be described as α-twisted vector bundles equipped with an action of this sheaf of TDOs.

The fact that α is n-torsion implies that we can always choose a representing cocycle $\underline{\alpha}$ that is locally constant. Not only that, but, given a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe, we can always find an equivalent one for which the part in $\check{Z}^2(\mathfrak{U}, \mathcal{O}_X^\times)$ is indeed locally constant. This shows that we can realize basic vector bundles on flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbes

which we wish to prove our result, twisted torsors might not be classified simply by a weight; hence the need for a different terminology. See section 2.2, and Definition 2.2.1 in particular.

[3] We could then refer to α-twisted vector bundles as *basic vector bundles on the* $\mathfrak{U}$-$\mathbb{G}_m$*-gerbe* $\underline{\alpha}$ (see Prop. 2.2.2).

as twisted vector bundles with an action of a sheaf of TDOs. More is true, in fact: the class of $(\underline{\omega}, \underline{F})$ in hypercohomology —equivalently, the TDO it yields— must also be n-torsion. The pervasiveness of this torsion phenomena will be explicated thoroughly in section 5.1

There is yet one more thing that carries over from the case of bare twisted vector bundles of section 1.1: projectivizing kills all central data, so that a basic vector bundle of rank n on a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe yields an honest, untwisted $\mathbb{P}^{n-1}$-bundle with flat connection on X.

1.2.3 Twisted Higgs fields.

After the discussion above, it is clear how we should proceed on the Higgs bundle side. Given an α'-twisted vector bundle $(\underline{\mathcal{E}}', \underline{g}')$, pick cochains

$$\underline{\omega}' = \{\omega'_{ij}\} \in \check{C}^1(\mathfrak{U}, \Omega^1_X) \quad \text{and} \quad \underline{F}' = \{F'_i\} \in \check{C}^0(\mathfrak{U}, \Omega^2_X),$$

equip each $\mathcal{E}'_i$ with a Higgs field ϕ_i and require that they fulfill the equations

$$\phi_i - g'_{ij}\phi_j\,(g'_{ij})^{-1} = \omega'_{ij} \quad \text{and} \quad C(\phi_i) = F'_i.$$

The compatibility conditions in this case are

$$\begin{aligned} \omega'_{ik} &= \omega'_{ij} + \omega'_{jk} && \text{on } U_{ijk} \\ F'_i &= F'_j && \text{on } U_{ij} \end{aligned}$$

which say that the triple $\left(\underline{\alpha}', \underline{\omega}', \underline{F}'\right)$ assembles into a Čech 2-cocycle in hypercohomology of the multiplicative Dolbeault complex

$$\mathrm{Dol}_X^{\mathbb{G}_m} := \left[\mathcal{O}_X^{\times} \xrightarrow{0} \Omega_X^1 \xrightarrow{0} \Omega_X^2 \xrightarrow{0} \cdots\right].$$

The following parallel Definitions 1.2.1 and 1.2.2, and the comments below those about the corresponding categories apply verbatim, as well as our recurring remark about projectivization.

Definition 1.2.3. *A* Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe *over X is a 2-cocycle*

$$\left(\underline{\alpha}', \underline{\omega}', \underline{F}'\right) \in \check{\mathbb{Z}}^2\left(\mathfrak{U}, \mathrm{Dol}_X^{\mathbb{G}_m}\right)$$

.

Definition 1.2.4. *A* basic vector bundle on a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\left(\underline{\alpha}', \underline{\omega}', \underline{F}'\right)$ *over X is a collection $\left(\underline{\mathcal{E}}', \underline{\phi}, \underline{g}'\right)$ consisting of an α'-twisted vector bundle $\left(\underline{\mathcal{E}}', \underline{g}'\right)$ on X together with Higgs fields ϕ_i on $\mathcal{E}_i'$ satisfying the equations*

$$\phi_i - g_{ij}'\phi_j\left(g_{ij}'\right)^{-1} = \omega_{ij}' \quad \textit{and} \quad C(\phi_i) = F_i'.$$

1.2.4 The nonabelian Hodge correspondence for twisted vector bundles.

We are finally in a position to state a form of the main theorem of this paper.

Corollary 1.2.5. *Let $\left(\underline{\alpha}, \underline{\omega}, \underline{F}\right)$ be a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe over X. Then there is a Higgs*

$\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\left(\underline{\alpha}', \underline{\omega}', \underline{F}'\right)$ over X for which there is a fully faithful functor

$$\left\{\begin{matrix} \textit{Basic vector bundles of rank } n \\ \textit{on } \left(\underline{\alpha}, \underline{\omega}, \underline{F}\right) \end{matrix}\right\} \hookrightarrow \left\{\begin{matrix} \textit{Basic vector bundles of rank } n \\ \textit{on } \left(\underline{\alpha}', \underline{\omega}', \underline{F}'\right) \end{matrix}\right\}$$

Conversely, given a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\left(\underline{\alpha}', \underline{\omega}', \underline{F}'\right)$ there exists a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe $\left(\underline{\alpha}, \underline{\omega}, \underline{F}\right)$ such that the same conclusion holds.

From our claims above that the categories of basic vector bundles on a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe and on a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe are independent of the choice of cover and cocycles, we can deduce the same about this statement. However, its explicit dependence on them becomes truly problematic in trying to provide a proof based on the classical nonabelian Hodge correspondence, for the latter necessitates of the compactness assumption —that is, we cannot just break up this twisted correspondence into local pieces. We will provide below an explicitly cover- and cocycle-independent version of the above statement —Theorem 5.1.1— from which it can be proved.

1.3 Outlook

In this introduction we have chosen to present the case of twisted vector bundles for two reasons: because it is the simplest one —in fact, the one from which we drew our intuition— and because it is the case some of whose objects —flat ($\mathfrak{U}$-)$\mathbb{G}_m$-gerbes, concretely— have appeared before in the literature.

The full generality of our result is expressed in terms of a (connected) linear algebraic group H over $\mathbb{C}$ together with an abelian subgroup A contained in its center —nothing is lost by assuming that A is in fact the whole center. The objects of interest are then twisted H-torsors over X —the definition of which (Definition 2.2.1) is the evident adaptation of Definition 1.1.1—, with the twisting given by a class in $H^2(X, A)$. The category of twisted vector bundles can then be recovered from that of GL_n-torsors by taking the associated (twisted) fiber bundles with fiber $\mathbb{C}^n$. The reader is well-advised to keep in mind this case ($H = GL_n, A = \mathbb{G}_m$), where most of the statements should feel familiar —and be well-known for $\alpha = 0$.

The remainder of this article is organized as follows.

Section 2 is devoted to developing the language of A-gerbes. These (1-)stacks provide a geometric realization of degree 2 cohomology classes, and twisted torsors on their base can be recast in terms of honest torsors on them. This first step removes the need for a specific choice of cover and cocycles, although the definitions are still in terms of them.

The higher homotopical tools that we review in section 3 have a two-fold objective. On one hand, they allow us to give a completely cover- and cocycle-free formulation of our problem. On the other, viewing the gerbes of section 2 as principal ∞-bundles [NSS12a] over their base opens up a host of very powerful techniques that enable us to reduce statements about twisted torsors to simpler ones about untwisted torsors and gerbes; chief among them is the existence of a classifying (2-)stack for our gerbes.

In section 4 we recall the classical statements of abelian and nonabelian Hodge theory from the perspective of ∞-stacks. In particular, the introduction of the de Rham and Dolbeault stacks of X —denoted by X_{dR} and X_{Dol}, respectively— brings connections and Higgs fields into our framework of principal ∞-bundles. The naturality of our choice of allowing these connections and Higgs fields to differ by something central and to have central curvature is then made manifest. Section 4.3 examines the passage between the algebraic and the analytic worlds. Thanks to a series of comparison theorems between the étale and analytic topologies, we can rest assured that our arguments in section 5 work in both cases.

The core of our proof is contained in this last section. We start (section 5.1) by going back to the original case of twisted vector bundles, now seen in the light of all the tools developed in previous sections. The failure of the obvious strategy of proof —as well as how it needs to be modified to hopefully work— is already visible in this simple example. We return to the general case in section 5.2, where we analyze the implementation of this modification. Section 5.3 finally brings everything together, leading us to the correct stability conditions on the Higgs side that make our theorem true.

Chapter 2

A more geometric perspective

2.1 Gerbes

Definition 2.1.1 ([Gir71]). *Let $\mathfrak{Y}$ be a (1-)stack over X. We say that $\mathfrak{Y}$ is a* gerbe over X *if it is locally nonempty and locally connected.*

The first of these conditions means that there exists an open cover $\{U_i \to X\}_{i \in I}$ of X such that the canonical maps $\mathfrak{Y} \times_X U_i \to U_i$ all have global sections, while the second one ensures that we can choose it so that the groupoid of global sections of each $\mathfrak{Y} \times_X U_i \to U_i$ has a single isomorphism class.

Definition 2.1.2 ([Gir71, Bre90]). *Let G be a linear algebraic group over $\mathbb{C}$*[1]*, and let $\mathfrak{Y}$ be a (1-)stack over X. We say that $\mathfrak{Y}$ is a* G-gerbe over X *if it is locally isomorphic to $BG \times X$.*

Equivalently, we have $\mathfrak{Y} \times_X U_i \simeq BG \times U_i$ over U_i for each $i \in I$ in an suitable cover. As a sanity check, note that a G-gerbe is a gerbe.

[1] Going forward we will often use the same letter for denoting a linear algebraic group over $\mathbb{C}$ and the sheaf on X obtained by pullback. Thus we write $\mathbb{G}_m$-gerbes for what some authors (e.g., [DP08]) call $\mathcal{O}^\times$-gerbes

The attentive reader might have noticed that the concept of a G-gerbe bears a striking similarity to that of a fiber bundle, only now the fiber BG and the structure group $\underline{\mathrm{Aut}}(BG)$ are bona fide stacks —rather than 0-truncated objects. Such line of thought can actually be formalized into the notion of an ∞-bundle [NSS12a], which we will explore in section 3.

However we can manage with the theory of crossed modules [Whi46] to prove that G-gerbes are classified by $H^1(X, \underline{\mathrm{Aut}}(BG))$ [Bre90]. Here the automorphism stack of BG —which goes also by the name of automorphism 2-group of G [BL04]— is represented by the crossed module $G \to \mathrm{Aut}(G)$, and H^1 refers to crossed module cohomology.

The exact sequence of crossed modules

$$1 \longrightarrow \Big[G \to \mathrm{Inn}(G)\Big] \longrightarrow \Big[G \to \mathrm{Aut}(G)\Big] \longrightarrow \Big[1 \to \mathrm{Out}(G)\Big] \longrightarrow 1 \qquad (2.1.1)$$

induces an exact sequence of pointed sets

$$H^1(X, G \to \mathrm{Inn}(G)) \longrightarrow H^1(X, G \to \mathrm{Aut}(G)) \xrightarrow{\beta} H^1(X, \mathrm{Out}(G)). \qquad (2.1.2)$$

Given a G-gerbe $\mathfrak{Y}$ over X, the $\mathrm{Out}(G)$-torsor classified by $\beta([\mathfrak{Y}])$ is called the *G-band of $\mathfrak{Y}$*.

Definition 2.1.3. *A G-gerbe over X is called a* G-banded gerbe *if its G-band is the* trivial $\mathrm{Out}(G)$*-torsor on X.*

In case $G = A$ is *abelian*, the diagram (2.1.1) simplifies to

$$1 \longrightarrow \Big[A \to 1\Big] \longrightarrow \Big[A \to \mathrm{Aut}(A)\Big] \longrightarrow \Big[1 \to \mathrm{Aut}(A)\Big] \longrightarrow 1$$

and the sequence (2.1.2) yields

$$0 \longrightarrow H^2(X, A) \longrightarrow H^1(X, A \to \mathrm{Aut}(A)) \xrightarrow{\beta} H^1(X, \mathrm{Aut}(A)).$$

Proposition 2.1.4. *A-banded gerbes over X are classified by $H^2(X, A)$.*

In the remainder we will only deal with A-banded gerbes, so we will abuse terminology and call them simply A-gerbes, or even just gerbes if the group A is clear from the context.

2.2 Torsors on gerbes

2.2.1 Presentations of gerbes

We briefly recall here a presentation of A-gerbes by gluings of the local pieces $BA \times U_i$. Much of the material and the notation in this section is borrowed from [DP08], which the reader is encouraged to consult for an extended exposition.

Given a class $\alpha \in H^2(X, A)$, denote by ${}_{\alpha}X$ the A-gerbe that it classifies. With the choice of a cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X and a representing cocycle $\underline{\alpha} = \{\alpha_{ijk}\}_{i,j,k \in I} \in \check{Z}^2(\mathfrak{U}, A)$, we have compatible groupoid presentations of X,

$$\left(\mathfrak{R} := \bigsqcup\nolimits_{i,j \in I} U_{ij} \overset{\mathfrak{s}}{\underset{\mathfrak{t}}{\rightrightarrows}} U := \bigsqcup\nolimits_{i \in I} U_i \ , \mathfrak{m}, \mathfrak{i}, \mathfrak{e} \right), \tag{2.2.1}$$

and ${}_{\alpha}X$,

$$\left(R := \bigsqcup\nolimits_{i,j \in I} U_{ij} \times A \overset{s}{\underset{t}{\rightrightarrows}} U = \bigsqcup\nolimits_{i \in I} U_i \ , \boldsymbol{m}, \boldsymbol{i}, \boldsymbol{e} \right). \tag{2.2.2}$$

The maps in (2.2.1) are the obvious ones. Those in the presentation (2.2.2) of ${}_{\alpha}X$ are as follows.

- The source and target morphisms come from those of (2.2.1) and the canonical projection $\pi : R \to \mathfrak{R}$ as $\boldsymbol{s} = \boldsymbol{\mathfrak{s}} \circ \pi$ and $\boldsymbol{t} = \boldsymbol{\mathfrak{t}} \circ \pi$;

- The identity and the inverse are given by the formulas

$$
\begin{array}{ccc}
U \xrightarrow{\quad \boldsymbol{e} \quad} R & \qquad\qquad & R \xrightarrow{\quad \boldsymbol{i} \quad} R \\
(x,i) \longmapsto ((x,i,i),1_A) & & ((x,i,j),a) \longmapsto ((x,j,i),a^{-1})
\end{array}
$$

- Finally, the multiplication $\boldsymbol{m}$ is the one encoding the class of the gerbe:

$$
\begin{array}{c}
R \times_{\boldsymbol{t},U,\boldsymbol{s}} R \xrightarrow{\qquad \boldsymbol{m} \qquad} R \\
((x,i,j),a) \times ((x,j,k),b) \longmapsto ((x,i,k),\alpha_{ijk}(x)ab)
\end{array}
$$

We denote by $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ the canonical projections $R \times_{\boldsymbol{t},U,\boldsymbol{s}} R \rightrightarrows R$ onto the first and second factors, respectively.

2.2.2

Let H be a linear algebraic group over $\mathbb{C}$ containing A as a closed subgroup of its center. Given the groupoid presentation (2.2.2) of ${}_{\alpha}X$, descent theory tells us that an H-torsor on ${}_{\alpha}X$ is given by an H-torsor $P \to U$ together with an isomorphism $\boldsymbol{j} : \boldsymbol{t}^*P \longrightarrow \boldsymbol{s}^*P$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & \boldsymbol{p}_1^*\boldsymbol{s}^*P & \boldsymbol{p}_1^*\boldsymbol{t}^*P \arrow[l, "\boldsymbol{p}_1^*\boldsymbol{j}"'] \arrow[dr, equal] & \\
\boldsymbol{m}^*\boldsymbol{s}^*P \arrow[ur, equal] & & & \boldsymbol{p}_2^*\boldsymbol{s}^*P \\
 & \boldsymbol{m}^*\boldsymbol{t}^*P \arrow[ul, "\boldsymbol{m}^*\boldsymbol{j}"] \arrow[r, equal] & \boldsymbol{p}_2^*\boldsymbol{t}^*P \arrow[ur, "\boldsymbol{p}_2^*\boldsymbol{j}"'] &
\end{tikzcd}
\tag{2.2.3}
$$

The map $\boldsymbol{j}$ breaks up into a collection of isomorphisms of H-torsors

$$\left\{h_{ij} : \boldsymbol{t}^* P\big|_{U_{ij}\times A} \longrightarrow \boldsymbol{s}^* P\big|_{U_{ij}\times A}\right\}_{i,j\in I}$$

These induce isomorphisms $h_{ij}(x, a)$ of the fiber H over each point (x, a) of the base $U_{ij} \times A$ commuting with the tautological right action of H on itself; that is to say, each $h_{ij}(x, a)$ acts as left multiplication by an element of H. In terms of these the cocycle condition (2.2.3) results in the equation

$$h_{ij}(x, a)h_{jk}(x, b) = h_{ik}(x, \alpha_{ijk}(x)ab) \tag{2.2.4}$$

for $x \in U_{ijk}$.

Though the H-torsors $\boldsymbol{s}^*P = \pi^*\boldsymbol{\mathfrak{s}}^*P$ and $\boldsymbol{t}^*P = \pi^*\mathbf{t}^*P$ obviously descend to $\mathfrak{R}$, there is in general no hope for $\boldsymbol{j}$ to go along too unless we impose some additional constraint on it. Taking into account that we are performing descent along the A-torsor $\pi : R \to \mathfrak{R}$, it is clear that what we need to do is equip $\boldsymbol{s}^*P$ and $\boldsymbol{t}^*P$ with A-equivariant structures —i.e., lifts of the A-action on R to the total space of these torsors— and ask for $\boldsymbol{j}$ to be equivariant with respect to them. But there are two natural choices coming from the trivial and tautological maps from A to H. Indeed, since $\boldsymbol{s}^*P$ and $\boldsymbol{t}^*P$ are H-torsors, they come equipped with a right action of H preserving the fibers of their respective projections to R. Composing this action with the two maps above yields fiberwise actions of A on $\boldsymbol{s}^*P$ and $\boldsymbol{t}^*P$, respectively. Collating these with the A-action on the base R yields the required A-equivariant structures on these torsors. With this choice, $\boldsymbol{j}$ is equivariant if

$$h_{ij}(x, a)b = h_{ij}(x, ab).$$

Defining

$$\left\{h_{ij} : \mathbf{t}^*P\big|_{U_{ij}} \longrightarrow \mathfrak{s}^*P\big|_{U_{ij}}\right\}_{i,j\in I}$$

by $h_{ij}(x) := h_{ij}(x,1)$ —this abuse of notation should not cause any confusion—, equation (2.2.4) is equivalent to $h_{ij}h_{jk} = \alpha_{ijk}h_{ik}$, which, in turn, implies —and is implied by— the conditions

$$h_{ii} = 1, \quad h_{ij} = h_{ji}^{-1}, \quad \text{and} \quad h_{ij}h_{jk}h_{ki} = \alpha_{ijk}. \tag{2.2.5}$$

Definition 2.2.1. *We say that an H-torsor $(P, \boldsymbol{j})$ on ${}_\alpha X$ is* basic *if $\boldsymbol{j}$ is A-equivariant for the trivial and tautological A-equivariant structures on $\boldsymbol{t}^*P$ and $\boldsymbol{s}^*P$, respectively.*

Proposition 2.2.2. *The category of α-twisted H-torsors on X is equivalent to the category of basic H-torsors on the A-gerbe ${}_\alpha X$.*

2.2.3 Obstruction gerbes.

Let K denote the quotient H/A (which is again a linear algebraic group: see Section 5.2). Given a K-torsor $Q \to X$, we can ask whether there exists an H-torsor giving rise to it through the obvious map in the long exact sequence of cohomology

$$H^1(X,A) \longrightarrow H^1(X,H) \longrightarrow H^1(X,K) \xrightarrow{ob_A} H^2(X,A).$$

It is clear that a necessary and sufficient condition is for the A-gerbe classified by $ob_A([Q \to X])$ to be trivial —i.e., globally equivalent to $BA \times X$. We call the latter the *obstruction A-gerbe* of the K-torsor $Q \to X$. If it is not trivial, we can only lift $Q \to X$ to an $ob([Q \to X])$-twisted H-torsor on X —the reason why some authors use the term *lifting gerbe*. To show this, choose a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X that trivializes $Q \to X$, so that the K-torsor is specified solely by the transition functions

$\underline{k} = \{k_{ij}\}_{i,j\in I} \in \check{Z}^1(\mathfrak{U}, K)$. Over each open U_i, our K-torsor is trivial and hence liftable to the trivial H-torsor on U_i. Over each double intersection U_{ij}, we can lift the transition function k_{ij} to an element $h_{ij} \in \Gamma(U_{ij}, H)$. But in general we cannot make $\underline{h} = \{h_{ij}\} \in \check{C}^1(\mathfrak{U}, H)$ into a 1-cocycle; rather,

$$\underline{a} = \Big\{a_{ijk} := h_{ij}h_{jk}h_{ki}\Big\}_{i,j,k\in I} \in \check{Z}^2(\mathfrak{U}, A)$$

provides a representative for the class of the obstruction gerbe of $Q \to X$. Different choices of lifts h_{ij} produce different representatives for the class $[\underline{a}] \in \check{H}^2(\mathfrak{U}, A)$. On the other hand, notice that an α-twisted H-torsor $S \to X$ induces an untwisted K-torsor on X by change of fiber, namely $S \times_H K \to X$.

Chapter 3

Higher homotopical machinery

As discussed in section 2.1, it is possible to see gerbes as fiber bundles with stacky fibers. A first indication of it is the fact that, if $\mathfrak{Y}$ is an A-gerbe over X, we have $\mathfrak{Y} \times_X \mathfrak{Y} \simeq \mathfrak{Y} \times BA$, an expression very similar to that in the classical definition of a torsor over an algebraic group. In that case, representability of the moduli functor requires that we keep track of automorphisms of the objects. This is, in fact, what led to the introduction of stacks in [DM69]. But once we allow the fiber itself to become a stack, representability forces us to up the homotopical ante and consider also automorphisms between automorphisms; that is, we need a 2-stack.

The theory of higher stacks was originally envisioned by Grothendieck [Gro83] and developed over the last few decades, among others, by Jardine [Jar87], Simpson [Sim96a], Toën-Vezzosi [TV05, TV08] and Lurie [Lur09]. We provide here a cursory look at this theory, following the particularly succint review of it in [NSS12a]. We encourage the reader to consult the above references for further details.

3.1 ∞-topoi

In the same way that ordinary Grothendieck (1-)topoi should be thought of as categories of sheaves of sets on a site, ∞-topoi are intimately related to ∞-categories of sheaves of higher homotopy types in which the gluings occur up to homotopy. Among all ∞-categories, ∞-topoi are characterized by the ∞-*Giraud axioms* [NSS12a, Definition 2.1].

Below we will need the following result.

Lemma 3.1.1 ([Lur09, Lemma 5.5.5.12]). *Let $\mathcal{C}$ be an ∞-category and let $f : Y \to X$, $g : Z \to X$ be morphisms in $\mathcal{C}$. Then there is a natural identification of $\mathbf{Map}_{\mathcal{C}_{/X}}(f, g)$ with the homotopy fiber of the map*

$$\mathbf{Map}_{\mathcal{C}}(Y, Z) \to \mathbf{Map}_{\mathcal{C}}(Y, X)$$

induced by composition with g, where the fiber is taken over the point corresponding to f.

3.1.1 1-localic ∞-topoi and their hypercompletions.

Let $(\mathcal{C}, \tau)$ be a Grothendieck site admitting finite limits. There are two closely related variants of the ∞-topos of ∞-stacks over it:

- We denote by $\mathsf{St}(\mathcal{C}, \tau)^\wedge$ the hypercomplete ∞-topos presented by both the local injective [Jar87] and local projective [Bla01, Dug01] model structures on the category $[\mathcal{C}^{\mathrm{op}}, \mathsf{sSet}]$ of simplicial presheaves on $\mathcal{C}$. These can also be realized as left Bousfield localizations at the collection of all hypercovers of the global injective and global projective model structures, respectively, on $[\mathcal{C}^{\mathrm{op}}, \mathsf{sSet}]$ [DHI04].

- Localizing any of the global model structures at the smaller class of Čech hypercovers yields the so-called *1-localic* ∞-topos of ∞-stacks on $(\mathcal{C}, \tau)$ [Lur09, Definition 6.4.5.8], which we simply denote by $\mathsf{St}(\mathcal{C}, \tau)$.

From their descriptions as left Bousfield localizations it should come as no surprise that there is a geometric morphism of ∞-topoi,

$$\mathsf{St}(\mathcal{C}, \tau)^\wedge \leftrightarrows \mathsf{St}(\mathcal{C}, \tau) ,$$

in which the right adjoint is fully faithful. This exhibits $\mathsf{St}(\mathcal{C}, \tau)^\wedge$ as a reflective sub-∞-category of $\mathsf{St}(\mathcal{C}, \tau)$, and is referred to as its *hypercompletion* (see the discussion above [Lur09, Lemma 6.5.2.9]). Its objects —which are then said to be *hypercomplete*— can be described as those ∞-stacks that satisfy descent with respect to all hypercovers, as opposed to only with respect to Čech hypercovers. [Lur09, Theorem 6.5.3.12]. This difference between descent and hyperdescent disappears if we only work with truncated objects —that is, ∞-stacks whose homotopy sheaves vanish above some finite level—; indeed, these are always hypercomplete [Lur09, Lemma 6.5.2.9]. Notice too that this implies that the 1-topoi of 0-truncated objects of these two ∞-topoi coincide: they can be realized as (the nerve of) the classical 1-topos of sheaves of sets on $(\mathcal{C}, \tau)$:

$$\tau_{\leq 0}\mathsf{St}(\mathcal{C}, \tau)^\wedge \simeq \tau_{\leq 0}\mathsf{St}(\mathcal{C}, \tau) \simeq \mathsf{Sh}(\mathcal{C}, \tau).$$

The key technical advantage of the 1-localic version over its hypercompletion lies in the fact that geometric morphisms are determined by their 0-truncations. The concrete statement, letting $\mathrm{Fun}_*(\mathcal{X}, \mathcal{Y})$ denote the ∞-category of geometric morphisms between two ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$ (with the right adjoint mapping $\mathcal{X}$ to $\mathcal{Y}$), is the following.

Proposition 3.1.2 ([Lur09, Lemma 6.4.5.6]). *For any ∞-topos $\mathcal{X}$, restriction induces an equivalence*

$$\mathrm{Fun}_*(\mathcal{X}, \mathsf{St}(\mathcal{C}, \tau)) \longrightarrow \mathrm{Fun}_*(\tau_{\leq 0}\mathcal{X}, \tau_{\leq 0}\mathsf{St}(\mathcal{C}, \tau))\,.$$

In the succeeding sections, we will encounter several sites together with functors between them that are classically known to induce geometric morphisms between their associated 1-topoi of sheaves of sets —they are either continuous or cocontinuous in the terminology of [$\mathrm{SGA4_I}$]. We will then immediately be able to lift this geometric morphisms to the 1-localic ∞-topoi of ∞-stacks over them.

The reader that feels more comfortable working in the hypercomplete ∞-topos $\mathsf{St}(\mathcal{C}, \tau)^\wedge$ can breathe a sigh of relief knowing that these two ∞-topoi coincide when the 1-topos of their 0-truncated objects has enough points —which is certainly the case for the sites that we examine in section 4.1.

3.2 Principal ∞-bundles

Theorem 3.2.1 ([NSS12a, Theorem 3.19]). *For all $\mathfrak{X}, \mathbf{B}G \in \mathbf{H}$ there is a natural equivalence of ∞-groupoids*

$$G\mathrm{Bund}(X) \simeq \mathbf{H}(\mathfrak{X}, \mathbf{B}G)$$

A bundle $P \to \mathfrak{X}$ is mapped to a morphism $X \to \mathbf{B}G$ such that $P \to \mathfrak{X} \to \mathbf{B}G$ is a fiber sequence.

Corollary 3.2.2. *For all $\mathbf{B}G \in \mathbf{H}$ there is a natural equivalence of ∞-categories*

$$G\mathrm{Bund} \simeq \mathbf{H}_{/\mathbf{B}G}$$

A morphism of bundles

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

is mapped to a homotopy coherent triangle

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & \mathbf{B}G & \end{tikzcd}$$

such that the columns in

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ \mathfrak{X} \arrow[r] \arrow[d] & \mathfrak{Y} \arrow[d] \\ \mathbf{B}G \arrow[r, equal] & \mathbf{B}G \end{tikzcd}$$

are fiber sequences.

3.3 Gerbes as principal ∞-bundles

3.3.1

Let H be a linear algebraic group over $\mathbb{C}$, and A an abelian subgroup contained in its center. Denote by K the quotient H/A in the category of sheaves on $\mathsf{Aff}_{\mathbb{C},\text{ét}}$. The following definition is the culmination of the process of geometrization of degree 2 cohomology classes that we set out to achieve in section 2.

Definition 3.3.1. *An* A-gerbe over a stack $\mathfrak{X}$ *is a principal* BA*-bundle over* $\mathfrak{X}$*.*

By Theorem 3.2.1, the category of such objects is equivalent to the mapping space $\mathbf{Map}\big(\mathfrak{X}, B^2A\big)$. It is a 2-category, with equivalence classes given by

$$\pi_0\,\mathbf{Map}\big(\mathfrak{X}, B^2A\big) \cong H^2(\mathfrak{X}, A)$$

Following the notation of section 2.2.1, we denote by ${}_\alpha\mathfrak{X}$ the A-gerbe over $\mathfrak{X}$ classified by $\alpha \in \mathbf{Map}\big(\mathfrak{X}, B^2A\big)_0$.

In case $\mathfrak{X}$ is our original smooth complex projective variety, X, this last definition coincides with Definition 2.1.3 above. Even then, it has the advantage of being independent of the choice of an open cover of the base.

3.3.2

Consider now the exact sequence of sheaves $1 \to A \to H \to K \to 1$. It gives rise to a long fiber sequence of stacks —the *Puppe sequence*—

$$1 \to A \to H \to K \to BA \to BH \to BK \to B^2A,$$

that exhibits BH —the classifying stack of H-torsors— as an A-gerbe over BK. It is now easy to see how the equivariance condition in Definition 2.2.1 translates into our higher homotopical framework.

Definition 3.3.2. *A basic* H*-torsor on an* A*-gerbe* ${}_\alpha\mathfrak{X}$ *is an* H*-torsor over* ${}_\alpha\mathfrak{X}$ *whose classifying morphism is* BA*-equivariant.*

The description of mapping spaces in overcategories of Lemma 3.1.1 yields a presentation of the (1-)category of basic H-torsors over an A-gerbe ${}_\alpha\mathfrak{X}$ as a limit:

$$\mathbf{Map}_{BA}\big({}_\alpha\mathfrak{X}, BH\big) \simeq \lim \left\{ \begin{tikzcd} & * \arrow[d, "\alpha"] \\ \mathbf{Map}\big(\mathfrak{X}, BK\big) \arrow[r, "ob_A"] & \mathbf{Map}\big(\mathfrak{X}, B^2A\big) \end{tikzcd} \right\} \tag{3.3.1}$$

Hence a basic H-torsor on ${}_\alpha\mathfrak{X}$ is given by a K-torsor on $\mathfrak{X}$ together with an equivalence between the obstruction A-gerbe of the latter (cf. section 2.2.3) and ${}_\alpha\mathfrak{X}$. In particular, any two basic H-torsors on ${}_\alpha\mathfrak{X}$ having the same underlying K-torsor on $\mathfrak{X}$ differ by an A-torsor on $\mathfrak{X}$.

Chapter 4

Classical Hodge theory

4.1 Towards cohesive structures

In this section we recall the theory of the de Rham construction for ∞-stacks of [ST97]. Almost all of the material is contained in *loc.cit.* and we claim no originality on it. The main novelty resides in the pervasive use of the language of ∞-categories. On one hand, this streamlines certains aspects of the theory. On the other, it allows us to state all of the results at the ∞-categorical level and not only at the level of their homotopy categories.

4.1.1

Let $(\mathcal{C}, \tau)$ be any one of the following Grothendieck sites:

- $(\mathsf{Aff}_{\mathbb{C}}, \text{ét})$: the category of affine complex schemes equipped with the étale topology,
- $(\mathsf{Aff}_{\mathbb{C},\text{ft}}, \text{ét})$: the full subcategory of the above consisting of affine schemes of finite type over $\operatorname{Spec}\mathbb{C}$ with the induced topology, or

- $(\mathsf{An}, \text{ét})$: the site of complex analytic spaces endowed with the topology in which covers are jointly surjective collections of local isomorphisms —also known as the analytic étale topology.

Of course these three sites are intimately related: the first two in the obvious fashion, and the last two through the analytification functor —of which we will say more in Section 4.3. In this section we generically refer to an object in any of these categories as a *representable space*.

For any of the choices above, let $\mathcal{C}_{\text{red}}$ be the full subcategory of $\mathcal{C}$ consisting of geometrically reduced representable spaces —made into a site by giving it the induced topology. The inclusion functor j possesses a right adjoint, red, that exhibits the first as a coreflective subcategory of the second:

$$\mathcal{C}_{\text{red}} \underset{j}{\overset{\text{red}}{\leftrightarrows}} \mathcal{C} \tag{4.1.1}$$

The usual yoga of functoriality of categories of presheaves [SGA4$_{\text{I}}$, Exposé I, §5] yields an adjoint quadruple,

$$j_! \quad \dashv \quad j^* \cong \text{red}_! \quad \dashv \quad j_* \cong \text{red}^* \quad \dashv \quad j^! := \text{red}_*, \tag{4.1.2}$$

between the categories $[\mathcal{C}^{\text{op}}, \mathsf{Set}]$ and $[\mathcal{C}_{\text{red}}^{\text{op}}, \mathsf{Set}]$ of presheaves of sets on $\mathcal{C}$ and $\mathcal{C}_{\text{red}}$, respectively. Since both j and red are continuous and cocontinuous [SGA4$_{\text{I}}$, Exposé III] , these four functors descend to give another adjoint quadruple of functors, this time between their respective categories of sheaves of sets. Furthermore, $\mathcal{C}$ and $\mathcal{C}_{\text{red}}$ both possess finite limits, and hence we can use Proposition 3.1.2 to lift the last three functors[1] of (4.1.2) to their 1-localic ∞-topoi of ∞-stacks:

$$\mathsf{St}(\mathcal{C}_{\mathrm{red}}, \tau) \begin{smallmatrix} \xleftarrow{\quad j^! \quad} \\ \xhookrightarrow{\quad j_* \quad} \\ \xleftarrow{\quad j^* \quad} \end{smallmatrix} \mathsf{St}(\mathcal{C}, \tau) \tag{4.1.3}$$

Denote by $\mathrm{Red} = j \circ \mathrm{red}$ the idempotent comonad associated to the pair (4.1.1), which sends a representable space to its induced reduced representable subspace. The adjoint quadruple (4.1.2) induces an adjoint triple

$$\mathrm{Red}_! \cong j_! \circ j^* \quad \dashv \quad \mathrm{Red}^* \cong j^* \circ j_* \quad \dashv \quad \mathrm{Red}_* \cong j_* \circ j^!$$

of endofunctors on $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$; again, the last two can be lifted all the way to the ∞-level:

$$\mathsf{St}(\mathcal{C}, \tau) \begin{smallmatrix} \xrightarrow{\quad \delta \quad} \\ \xleftarrow{\quad (-)_{\mathrm{dR}} \quad} \end{smallmatrix} \mathsf{St}(\mathcal{C}, \tau) \tag{4.1.4}$$

where, following the notation of [ST97, Proposition 3.3], we denote the functors Red^* and Red_* (at the level of the ∞-topoi) by $(-)_{\mathrm{dR}}$ and δ, respectively. The first of these receives the name of *de Rham functor*, and the image of an ∞-stack $\mathfrak{X} \in \mathsf{St}(\mathcal{C}, \tau)$ under it is its *de Rham stack*, $\mathfrak{X}_{\mathrm{dR}}$.

The counit $\mathrm{Red} \to \mathrm{id}$ of the comonad Red induces a natural transformation $\mathrm{id} \to (-)_{\mathrm{dR}}$ of ∞-functors. We say that an ∞-stack $\mathfrak{X}$ is *formally smooth* if the natural morphism $\mathfrak{X} \to \mathfrak{X}_{\mathrm{dR}}$ is an effective epimorphism —that is, if its de Rham stack is (equivalent to) the ∞-colimit of the Čech nerve of $\mathfrak{X} \to \mathfrak{X}_{\mathrm{dR}}$. If $\mathfrak{X}$ is 0-truncated this definition coincides with the classical one, which is nothing but the infinitesimal lifting property.

[1]Since j does not preserve finite limits, $j_!$ does not either and hence the pair $j_! \dashv j^*$ is *not* a geometric morphism.

4.1.2

Let $*$ denote the terminal category. We can extend (4.1.1) to a diagram

$$
\begin{tikzcd}
\mathcal{C}_{\mathrm{red}} \arrow[rr, hook, shift right, "j"'] \arrow[dr, shift right, "\pi_{\mathrm{red}}"'] & & \mathcal{C} \arrow[ll, shift right, "\mathrm{red}"'] \arrow[dl, shift left, "\pi"] \\
& * \arrow[ul, shift right, "i_{\mathrm{red}}"'] \arrow[ur, shift left, "i"] &
\end{tikzcd}
$$

Here π (resp., π_{red}) is the unique functor to $*$, and its right adjoint i (resp., i_{red}) takes the unique object of $*$ to the terminal object of $\mathcal{C}$ (resp., $\mathcal{C}_{\mathrm{red}}$). The following relations are easy to check:

$$\pi_{\mathrm{red}} \circ \mathrm{red} = \pi, \quad \pi \circ j = \pi_{\mathrm{red}}, \quad j \circ i_{\mathrm{red}} = i, \quad \mathrm{red} \circ i = i_{\mathrm{red}}. \tag{4.1.5}$$

As before, we obtain an adjoint quadruple of functors,

$$\pi_! \quad \dashv \quad \pi^* \cong i_! \quad \dashv \quad \pi_* \cong i^* \quad \dashv \quad \pi^! := i_* \tag{4.1.6}$$

$$(\text{resp.,} \qquad \pi_{\mathrm{red},!} \quad \dashv \quad \pi^*_{\mathrm{red}} \cong i_{\mathrm{red},!} \quad \dashv \quad \pi_{\mathrm{red},*} \cong i^*_{\mathrm{red}} \quad \dashv \quad \pi^!_{\mathrm{red}} := i_{\mathrm{red},*} \quad), \tag{4.1.7}$$

between the appropriate categories of presheaves of sets. Now, i (resp., i_{red}) is both continuous and cocontinuous and hence the last three functors in (4.1.6) (resp., (4.1.7)) descend to the categories of sheaves of sets[2], and then lift via Proposition 3.1.2 to the 1-localic ∞-topoi of ∞-stacks. From its avatar as i^* (resp., i^*_{red}) it is obvious that π_* (resp., $\pi_{\mathrm{red},*}$) is nothing but the canonical functor of global sections, which we denote by Γ (resp., Γ_{red}) following the standard terminology; its left adjoint,

[2]It is easy to get fooled into thinking that π (resp., π_{red}) preserves covers. The fact that this is not true stems from the fact that there are empty representable spaces in $\mathcal{C}$ (resp. $\mathcal{C}_{\mathrm{red}}$).

π^* (resp., π^*_{red}), is then the extension to ∞-stacks of the locally constant sheaf functor, which we denote by const (resp. $\mathrm{const}_{\mathrm{red}}$). All in all, we have the following diagram of ∞-functors:

$$\begin{tikzcd}
\mathsf{St}(\mathcal{C}_{\mathrm{red}},\tau) \arrow[rr, hook, "j_*"] \arrow[dr, "\Gamma_{\mathrm{red}}"] & & \mathsf{St}(\mathcal{C},\tau) \arrow[ll, bend right, "j^!"'] \arrow[ll, bend left, "j^*"] \arrow[dl, "\Gamma"'] \\
& \infty\mathsf{Gpd} \arrow[ul, hook, bend left, "\mathrm{const}_{\mathrm{red}}"] \arrow[ul, bend right, "\pi^!_{\mathrm{red}}"'] \arrow[ur, hook', bend right, "\mathrm{const}"'] \arrow[ur, bend left, "\pi^!"] &
\end{tikzcd} \tag{4.1.8}$$

The equalities (4.1.5) yield a natural equivalence of geometric morphisms of ∞-topoi,

$$j_* \circ \mathrm{const}_{\mathrm{red}} \simeq \mathrm{const} \quad \dashv \quad \Gamma \simeq \Gamma_{\mathrm{red}} \circ j^!, \tag{4.1.9}$$

which, together with the counit of the adjunction $\mathrm{const}_{\mathrm{red}} \dashv \Gamma_{\mathrm{red}}$, implies the existence of a natural transformation

$$\mathrm{dis} := \mathrm{const} \circ \Gamma \simeq j_* \circ \mathrm{const}_{\mathrm{red}} \circ \Gamma_{\mathrm{red}} \circ j^! \longrightarrow j_* \circ j^! \simeq \delta.$$

from the functor that associates to a complex-analytic ∞-stack the constant ∞-stack on its global sections to the right adjoint to de Rham functor. In the case $(\mathcal{C},\tau) = (\mathsf{An}, \text{ét})$, the relationship between these two is extremely simple for stacks that are deloopings of 0-truncated group objects.

Proposition 4.1.1 ([ST97, Proposition 3.3])**.**

(a) For any complex Lie group G, $\delta(BG) \simeq B(\mathrm{dis(G)})$.

(b) For any abelian complex Lie group A, $\delta(B^nA) \simeq B^n(\mathrm{dis(A)})$.

4.1.3

Although the functor const (resp., $\mathrm{const}_{\mathrm{red}}$) might not have a further left adjoint — and even if it does, the latter is never induced by $\pi_!$ (resp., $\pi_{\mathrm{red},!}$)— it does preserve finite limits and hence possesses a pro-left adjoint [SGA4$_{\mathrm{I}}$, Exposé I, §8.11] known as the *fundamental pro-∞-groupoid* functor [Hoy13] (see also [Lur12, Appendix A] and [Sim96b]):

$$\Pi_\infty : \mathsf{St}(\mathcal{C}, \tau) \longrightarrow \mathrm{Pro}(\infty\mathsf{Gpd}) \qquad (\text{resp.,} \quad \Pi_{\infty\mathrm{red}} : \mathsf{St}(\mathcal{C}_{\mathrm{red}}, \tau) \longrightarrow \mathrm{Pro}(\infty\mathsf{Gpd}) \quad) .$$

We will not go further in the study of this functor. Here we simply note that the natural equivalence (4.1.9) of geometric morphisms implies a natural equivalence of ∞-functors,

$$\Pi_{\infty\mathrm{red}} \circ j^* \simeq \Pi_\infty, \tag{4.1.10}$$

that we will use below in a cohomology computation.

4.1.4

We close this section with a comment about its title. Inspired by work of Lawvere [Law05, Law07] in the context of 1-topoi, Schreiber defines a *cohesive ∞-topos* [Sch11, Section 2.2] to be an ∞-topos H whose global sections geometric morphism extends to a quadruple

$$\begin{tikzcd} \mathsf{H} \arrow[r, shift left=3] \arrow[r, "\Gamma", shift left=1] \arrow[r, "\Pi_\infty"', shift right=3] & \infty\mathsf{Gpd} \arrow[l, hook', shift right=5] \arrow[l, hook', "\mathrm{const}"', shift left=1] \end{tikzcd} \tag{4.1.11}$$

of adjoint ∞-functors in which both adjoints to Γ are fully faithful ∞-functors, and Π_∞ preserves products. In similar form, this structure already appears in [ST97] and [KR00].

We have come close to exhibiting our ∞-topoi $\mathsf{St}(\mathcal{C},\tau)$ as cohesive ∞-topoi. Indeed, we are only missing that the pro-left adjoint to const should actually land in $\infty\mathsf{Gpd}$ rather than in $\mathrm{Pro}(\infty\mathsf{Gpd})$, and that it preserves finite products. It seems altogether possible that this is true in the analytic category, for [ST97, Section 2.16] constructs this functor explicitly at the level of the homotopy category. It is doubtful, though, that the same holds in the algebraic context, since there Π_∞ should encode, among other things, the étale fundamental group, which is often a true pro-algebraic group.

If $j_!$ in (4.1.2) also lifts to the ∞-level, (4.1.3) realizes what Schreiber calls an *infinitesimal cohesive neighborhood* [Sch11, Section 2.4]. Although we do believe that this should be the case for any and all of our choices of $(\mathcal{C},\tau)$, we do not have a proof of this fact.

4.2 The de Rham and Dolbeault stacks of a smooth projective variety

In this section we restrict ourselves to the analytic topology. The algebraic counterpart is the concern of the next section.

4.2.1 The classical nonabelian Hodge correspondence.

For X a smooth projective variety, the $(n+1)$-simplices of the Čech nerve of $X \to X_{\mathrm{dR}}$ are given by the formal completion of the main diagonal in $X^{\times n}$. Furthermore, the ∞-colimit of this simplicial object agrees with the ordinary colimit of its 1-truncation, and so X_{dR} can be simply realized as the quotient of X by the formal completion of the diagonal in $X \times X$:

$$X_{\mathrm{dR}} = [(X \times X)^{\wedge}_{\Delta} \rightrightarrows X]$$

Coherent sheaves over X_{dR} are then easily seen to be the same thing as crystals of coherents sheaves on X in the sense of Grothendieck [Gro68a]: that is, vector bundles with flat connection.

On the other hand, Higgs bundles can be codified as vector bundles on the so-called *Dolbeault stack* of X. The latter is defined as the quotient of X by the formal completion of the zero section in the total space of its tangent bundle [Sim97a]:

$$X_{\mathrm{Dol}} := [TX^{\wedge}_{0} \rightrightarrows X]$$

We can thus say that the nonabelian Hodge theorem [Sim92] establishes an equivalence between the categories of vector bundles on X_{dR} and that of vector bundles on X_{Dol} satisfying the appropriate conditions of semistability and vanishing of rational Chern classes.

One important remark is that this correspondence intertwines the monoidal structures on both sides. Hence we can use the Tannakian formalism to extend it to the setting of G-torsors for G a linear algebraic group over $\mathbb{C}$. Indeed, the category of G-torsors on X_{dR} (resp., X_{Dol}) can be seen to be equivalent to the category of tensor functors from $\mathrm{Rep}(G)$ —the category of representations of G— to that of vector bundles on X_{dR} (resp., X_{Dol}) [Sim92, Section 6]. In this language, a G-torsor on X_{Dol} is said to be semistable and to have zero rational Chern classes if so does the image of any representation of G.

Theorem 4.2.1 ([Sim92])**.** $\mathbf{Map}\big(X_{\mathrm{dR}}, BG\big) \simeq \mathbf{Map}\big(X_{\mathrm{Dol}}, BG\big)^{\mathrm{ss},0}$

4.2.2

Since the de Rham and Dolbeault stacks codify flat connections and Higgs fields for torsors, we can expect them to do the same for gerbes. Hence, given A an abelian linear algebraic group over $\mathbb{C}$, we make the following definitions.

Definition 4.2.2. *A flat A-gerbe over X is an A-gerbe over X_{dR}.*

Definition 4.2.3. *A Higgs A-gerbe over X is an A-gerbe over X_{Dol}.*

The cohomology of X_{dR} (resp., X_{Dol}) with abelian coefficients can be expressed in terms of the de Rham (resp., Dolbeault) complex of X with coefficients in A:

$$\mathrm{dR}_X^A := \left[A \xrightarrow{a \mapsto a^{-1}da} \Omega_X^1 \otimes \mathfrak{a} \xrightarrow{d} \Omega_X^2 \otimes \mathfrak{a} \xrightarrow{d} \cdots \right] \tag{4.2.1}$$

$$\Big(\text{resp.,} \quad \mathrm{Dol}_X^A := \left[A \xrightarrow{0} \Omega_X^1 \otimes \mathfrak{a} \xrightarrow{0} \Omega_X^2 \otimes \mathfrak{a} \xrightarrow{0} \cdots \right] \quad \Big). \tag{4.2.2}$$

Lemma 4.2.4. $\pi_i \, \mathbf{Map}\big(X_{\mathrm{dR}}, B^k A\big) \cong \mathbb{H}^{k-i}\big(X, \mathrm{dR}_X^A\big)$.

Proof. The holomorphic Poincaré lemma asserts that dR_X^A is a resolution of the sheaf $\mathrm{dis}(A)$. Using (4.1.4) and Proposition 4.1.1, we have

$$\begin{aligned}\pi_i \, \mathbf{Map}\big(X_{\mathrm{dR}}, B^k A\big) \cong \pi_i \, \mathbf{Map}\big(X, \delta(B^k A)\big) \cong \pi_i \, \mathbf{Map}\big(X, B^k(\mathrm{dis}(A))\big) \\ = H^{k-i}\big(X, \mathrm{dis}(A)\big) \cong \mathbb{H}^{k-i}\big(X, \mathrm{dR}_X^A\big).\end{aligned}$$

□

Lemma 4.2.5. $\pi_i \, \mathbf{Map}\big(X_{\mathrm{Dol}}, B^k A\big) \cong \mathbb{H}^{k-i}\big(X, \mathrm{Dol}_X^A\big)$.

Proof. An abelian linear algebraic group factors as a direct sum of copies of $\mathbb{G}_m$, $\mathbb{G}_a$ and a finite abelian group, which is itself a direct sum of copies of μ_n for various values of n (see section 5.2.1 for the full justification of this claim in the analytic topology). Since delooping commutes with finite products, it is enough to check each of these cases separately. For $A = \mathbb{G}_a$ the calculation is classical (see, e.g., [Sim97b, Proposition 3.1]) and we omit the proof.

Let F be a discrete group —a class to which finite groups belong—, so that $F \simeq \mathrm{dis}(F)$. Since $\mathrm{dis} = \mathrm{const} \circ \Gamma$ and const has Π_∞ as a pro-left adjoint, we have

$$\mathbf{Map}\big(\mathfrak{X}, B^k F\big) \simeq \mathbf{Map}\big(\mathfrak{X}, \mathrm{const}\, B^k(\Gamma F)\big) \simeq \mathbf{Map}\big(\Pi_\infty \mathfrak{X}, B^k(\Gamma F)\big);$$

that is, the cohomology of a stack $\mathfrak{X}$ with coefficients in F depends only on its fundamental pro-∞-groupoid. We now claim that

$$\Pi_\infty X_{\mathrm{Dol}} \simeq \Pi_\infty X, \tag{4.2.3}$$

which implies the lemma for $A = F$.

In order to prove (4.2.3), we proceed by a series of reductions[3]. For the first one

[3]Pun intended

we observe that X_{Dol} is defined as the quotient of X by $TX_0^\wedge$, and that Π_∞ commutes with ∞-colimits. Hence it is enough to check that

$$\Pi_\infty\big(TX_0^\wedge\big)^{\times_X m} \simeq \Pi_\infty X.$$

for every $m \geq 1$. But $TX_0^\wedge$ is given as the colimit of the infinitesimal neighborhoods $TX_0^{[s]}$ of the zero section in TX, and so we only need to show that

$$\Pi_\infty\big(TX_0^{[s]}\big)^{\times_X m} \simeq \Pi_\infty X.$$

for every $m \geq 1$ and $s \geq 1$. Now, since both X and $\big(TX_0^{[s]}\big)^{\times_X m}$ are representable spaces (in the terminology of the last section), and $\mathrm{Red}\big(TX_0^{[s]}\big)^{\times_X m} \cong X$, we have

$$\begin{aligned} j^*\left(\big(TX_0^{[s]}\big)^{\times_X m}\right) &= \mathrm{red}\left(\big(TX_0^{[s]}\big)^{\times_X m}\right) \\ &\cong \mathrm{red}\circ j\circ \mathrm{red}\left(\big(TX_0^{[s]}\big)^{\times_X m}\right) \cong \mathrm{red}(X) = j^*(X) \end{aligned}$$

and (4.1.10) finishes the proof:

$$\Pi_\infty\big(TX_0^{[s]}\big)^{\times_X m} \simeq \Pi_{\infty\mathrm{red}}\circ j^*\left(\big(TX_0^{[s]}\big)^{\times_X m}\right) \simeq \Pi_{\infty\mathrm{red}}\circ j^*(X) \simeq \Pi_\infty X.$$

Finally, the exponential sequence and the cases of $\mathbb{G}_a$ and $\mathbb{Z}$ —a discrete group for sure— imply the statement for $A = \mathbb{G}_m$. $\square$

Notice that the case $A = \mathbb{G}_m$, $k = 2$ of these calculations show that Definitions 4.2.2 and 4.2.3 provide our coveted cover- and cocycle-independent versions of Definitions 1.2.1 and 1.2.3 —which can be recovered through the usual comparison of Čech and sheaf cohomology.

4.2.3 The Hodge correspondence for gerbes.

Since the coefficient groups of gerbes are abelian, we might expect that abelian Hodge theory should relate their flat and Higgs versions. There is however one important restriction, which is, in fact, the only thing that makes the twisted correspondence we aim to prove non-trivial; namely, that A cannot contain any algebraic torus. Indeed, there is no hope of establishing an equivalence between $\mathbf{Map}\big(X_{\mathrm{dR}}, B^2\mathbb{G}_m\big)$ and $\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2\mathbb{G}_m\big)$ or a full subcategory thereof, for the automorphism 1-categories of objects in these (2-)categories are given by

$$\Omega\,\mathbf{Map}\big(X_{\mathrm{dR}}, B^2\mathbb{G}_m\big) \simeq \mathbf{Map}\big(X_{\mathrm{dR}}, B\mathbb{G}_m\big)$$

and

$$\Omega\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2\mathbb{G}_m\big) \simeq \mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{G}_m\big),$$

respectively. But Theorem 4.2.1 establishes an equivalence between the first of these and the full subcategory of the second on the degree zero Higgs $\mathbb{G}_m$-torsors (the semistability condition is trivially in this case, as is the vanishing of the second rational Chern class: see below). This shows that any relation between $\mathbf{Map}\big(X_{\mathrm{dR}}, B^2\mathbb{G}_m\big)$ and $\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2\mathbb{G}_m\big)$ would have to involve restrictions not only on objects, but also on 1-morphisms.

Proposition 4.2.6. *Suppose $A \cong \mathbb{G}_a^{\oplus m} \oplus F$, where F is a finite group. Then,*

$$\mathbf{Map}\big(X_{\mathrm{dR}}, B^2 A\big) \simeq \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2 A\big).$$

Proof. As in the proof of Lemma 4.2.5, it is enough to show the statement independently for the cases $A = \mathbb{G}_a$ and $A = \mu_n$. The first follows from abelian Hodge theory, while the second holds true because μ_n has trivial Lie algebra, and hence

$$\mathbf{Map}(X_{\mathrm{dR}}, B^2\mu_n) \simeq \mathbf{Map}(X, B^2\mu_n) \simeq \mathbf{Map}(X_{\mathrm{Dol}}, B^2\mu_n). \qquad \square$$

4.2.4

Although Proposition 4.2.6 concerns gerbes, it also contains within itself a statement about the category of A-torsors on X_{Dol}, since the latter appears as the automorphism 1-category of the distinguished object of $\mathbf{Map}(X_{\mathrm{Dol}}, B^2A)$. Namely, that the conditions on semistability and vanishing of Chern classes of Theorem 4.2.1 are always satisfied. As we remarked above, this is not quite so for $\mathbb{G}_m$-torsors on X_{Dol}: we do need to impose that they are of degree zero —semistability follows from it.

Indeed, a $\mathbb{G}_m$-torsor on X_{Dol}, $\mathcal{L}$, sends an irreducible representation of $\mathbb{G}_m$ — that is, a character— to a Higgs line bundle, for which the semistability condition is obviously vacuous, as is the vanishing of the second Chern class. If we require that the first rational Chern class of these line bundles coming from irreps also vanishes, then $\mathcal{L}$ sends an arbitrary representation of $\mathbb{G}_m$ to a direct sum of degree zero Higgs line bundles, which is certainly semistable.

We can also prove the claim for $A = \mu_n$ and $A = \mathbb{G}_a$ without invoking the Hodge correspondence for gerbes. In the first case we can use the same argument of the last paragraph, complemented with the observation that the Higgs line bundles we obtain now are torsion, and so their first Chern class vanishes automatically.

For $A = \mathbb{G}_a$ the resulting vector bundles are unipotent, and the Higgs fields on them, nilpotent. These Higgs bundles can hence be written as successive extensions of the trivial line bundle equipped with the zero Higgs field. We can finish by observing that the category of semistable Higgs bundles with vanishing first and second rational Chern classes is closed under extensions.

4.3 Analytification

The main purpose of this section is to show that the Hodge correspondences — Theorem 4.2.1 and Proposition 4.2.6— also hold in the algebraic setting. Our results here will allow us to be work in section 5 without explicit reference to the étale or analytic topologies.

4.3.1 Finiteness conditions on the algebraic side.

Let $a : \mathsf{Aff}_{\mathbb{C},\mathrm{ft}} \hookrightarrow \mathsf{Aff}_{\mathbb{C}}$ denote the inclusion functor of the category of complex affine schemes of finite type into that of all complex affine schemes. With respect to the étale topologies on both sides, a is both continuous and cocontinuous (and preserves finite limits), and hence the triple of adjoint functors $a_! \dashv a^* \dashv a_*$ between categories of presheaves of sets descends to their respective categories of sheaves of sets, and ultimately lifts to their 1-localic ∞-topoi of ∞-stacks via Proposition 3.1.2:

$$\mathsf{St}(\mathsf{Aff}_{\mathbb{C},\mathrm{ft}}, \text{ét}) \underset{\hookrightarrow\; a_* \;\longrightarrow}{\overset{\hookrightarrow\; a_! \;\longrightarrow}{\longleftarrow\; a^* \;\longrightarrow}} \mathsf{St}(\mathsf{Aff}_{\mathbb{C}}, \text{ét})$$

An object $\mathfrak{X} \in \mathsf{St}(\mathsf{Aff}_{\mathbb{C}}, \text{ét})$ is said to be *almost locally of finite type* [Gai11] if it belongs to the essential image of $a_!$. Because $a_!$ preserves ∞-colimits and finite ∞-limits, it is easy to see from their constructions that the de Rham and Dolbeault stacks of a smooth projective variety (as outlined in section 4.2) do belong in this subcategory, as is the case for linear algebraic groups over $\mathbb{C}$.

4.3.2 The analytification functor.

Given a complex affine scheme *of finite type*, we can produce a complex analytic space à la Serre [Ser56] in a functorial manner. This analytification functor, an :

$\mathsf{Aff}_{\mathbb{C},\mathrm{ft}} \to \mathsf{An}_{\mathbb{C}}$, is obviously continuous with respect to the étale topologies on both sides and preserves finite limits; the same procedure we have used several times now produces an adjoint pair of ∞-functors

$$\mathsf{St}(\mathsf{Aff}_{\mathbb{C},\mathrm{ft}}, \text{ét}) \underset{\leftarrow \text{an}^* -}{\overset{- \text{an} \to}{\rightleftarrows}} \mathsf{St}(\mathsf{An}_{\mathbb{C}}, \text{ét})$$

The left adjoint, which we have denoted an abusing terminology, is induced by $\text{an}_!$ and extends the original analytification functor to all those ∞-stacks on the étale site that are almost locally of finite type.

Given a smooth complex projective variety X, denote by X_{dR} (resp. X_{Dol}) its de Rham (resp. Dolbeault) stack as constructed in the étale topology following the recipe of section 4.2.1. Considering X as a complex-analytic space through the analytification functor above, we can also look at its de Rham (resp,. Dolbeault) stack, this time constructed in the analytic sense; we denote the latter by $X^{\mathrm{an}}_{\mathrm{dR}}$ (resp., $X^{\mathrm{an}}_{\mathrm{Dol}}$). Because analytification commutes with ∞-colimits and finite ∞-limits, we have

$$\text{an}(X_{\mathrm{dR}}) \simeq X^{\mathrm{an}}_{\mathrm{dR}} \qquad \Big(\text{resp., an}(X_{\mathrm{Dol}}) \simeq X^{\mathrm{an}}_{\mathrm{Dol}}\Big)$$

Moreover, if G is a linear algebraic group over $\mathbb{C}$ (resp. A is an abelian linear algebraic group over $\mathbb{C}$), considered as an étale scheme, denote by G^{an} (resp. A^{an}) its analytification.

The following Proposition —which we will implicitly use throughout section 5— expresses the fact that the categories we are interested in are the same in the algebraic and analytic cases, so that Theorem 4.2.1 and Proposition 4.2.6 also hold in the étale topology.

Proposition 4.3.1. *For G a linear algebraic group over $\mathbb{C}$, and $A \cong \mathbb{G}_a^{\oplus m} \oplus F$, where F is a finite group, we have*

1. $\mathbf{Map}\big(X_{\mathrm{dR}}, BG\big) \simeq \mathbf{Map}\big(X_{\mathrm{dR}}^{an}, BG^{an}\big)$

2. $\mathbf{Map}\big(X_{\mathrm{Dol}}, BG\big) \simeq \mathbf{Map}\big(X_{\mathrm{Dol}}^{an}, BG^{an}\big)$

3. $\mathbf{Map}\big(X_{\mathrm{dR}}, B^kA\big) \simeq \mathbf{Map}\big(X_{\mathrm{dR}}^{an}, B^kA^{an}\big)$

4. $\mathbf{Map}\big(X_{\mathrm{Dol}}, B^kA\big) \simeq \mathbf{Map}\big(X_{\mathrm{Dol}}^{an}, B^kA^{an}\big)$

Proof. For a proof of (i), we refer the reader to [Sim96a, Theorem 9.2]. Statement (ii) follows from Serre's GAGA [Ser56], which implies that analytic Higgs vector bundles are in fact algebraic.

For the case $A = \mathbb{G}_a$, (iii) follows from the usual comparison theorem of de Rham cohomology in the algebraic and analytics settings [Gro68a], while (iv) is once again a direct consequence of GAGA. The comparison theorem between étale and analytic cohomology [Mil80, Theorem 3.12] proves both (iii) and (iv) in case A is a finite abelian group. □

Chapter 5

The twisted correspondence

5.1 Torsion phenomena in the vector bundle case

With all the technical baggage of the last three sections under our belt, we return here to the case of vector bundles —or, rather, GL_n-torsors. Our discussion in this section will serve to illustrate the chief difficulty in the obvious approach to proving a twisted nonabelian Hodge correspondence and point out the main idea of the workaround.

5.1.1

Recall that in our introduction we defined twisted vector bundles, twisted connections and twisted Higgs fields in terms of a cover $\mathfrak{U}$ of our smooth complex projective variety X. We condensed all of these concepts into a few definitions, that we detail in the following table.

de Rham side	Dolbeault side
Flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe (Definition 1.2.1)	Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe (Definition 1.2.3)
Basic vector bundle on a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe (Definition 1.2.2)	Basic vector bundle on a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe (Definition 1.2.4)

Our work in the last three sections was partly aimed at giving cover- and cocycle-independent versions of all of these definitions. We remarked at the end of section 4.2.2 that we had succeeded in doing so for the objects in the first row of the table above. But upon seeing Definitions 4.2.2 and 4.2.3 the reader should have immediately been able to guess the appropriate counterpart for the definitions in the second row —by simply remembering Definition 3.3.2.

de Rham side	Dolbeault side
Flat $\mathbb{G}_m$-gerbe (Definition 4.2.2)	Higgs $\mathbb{G}_m$-gerbe (Definition 4.2.3)
Basic GL_n-torsor on a flat $\mathbb{G}_m$-gerbe (Definition 3.3.2)	Basic GL_n-torsor on a Higgs $\mathbb{G}_m$-gerbe (Definition 3.3.2)

Indeed, suppose $\left(\underline{\alpha}, \underline{\omega}, \underline{F}\right) \in \check{\mathbb{Z}}^2\left(\mathfrak{U}, \mathrm{dR}_X^{\mathbb{G}_m}\right)$ is a flat $\mathfrak{U}$-$\mathbb{G}_m$-gerbe, and let $\theta \in \mathbf{Map}\left(X_{\mathrm{dR}}, B^2\mathbb{G}_m\right)_0$ be a flat $\mathbb{G}_m$-gerbe[1]. We claim that if the obvious compatibility condition between these two pieces of data —namely, that the image of

[1]We will no longer distinguish terminologically a gerbe from its classifying morphism.

$\big[\big(\underline{\alpha},\underline{\omega},\underline{F}\big)\big]\in\check{\mathbb{H}}^2\big(\mathfrak{U},\mathrm{dR}_X^{\mathbb{G}_m}\big)$ in sheaf cohomology coincides with $[\theta]\in\mathbb{H}^2\big(X,\mathrm{dR}_X^{\mathbb{G}_m}\big)$—, then the category of basic vector bundles on $\big(\underline{\alpha},\underline{\omega},\underline{F}\big)$ is equivalent to that of basic GL_n-torsors on ${}_\theta(X_{\mathrm{dR}})$. The proof of this fact is a rather messy and unilluminating calculation and so we omit it in the hopes that the reader will find it intuitively obvious. Of course, the parallel statement about basic vector bundles on Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbes and basic GL_n-torsors on Higgs $\mathbb{G}_m$-gerbes also holds.

The cover- and cocycle-dependent version of our main result as we gave it in the introduction (Corollary 1.2.5) follows then from what will be the final form of our theorem —at least for the case of vector bundles— by a similarly tedious computation.

Theorem 5.1.1. *Let $\theta\in\mathbf{Map}\big(X_{\mathrm{dR}},B^2\mathbb{G}_m\big)_0$ be a flat $\mathbb{G}_m$-gerbe over X. Then there is a Higgs $\mathbb{G}_m$-gerbe over X, $\widetilde{\theta}\in\mathbf{Map}\big(X_{\mathrm{Dol}},B^2\mathbb{G}_m\big)_0$, for which there is an equivalence*

$$\mathbf{Map}_{B\mathbb{G}_m}\big({}_\theta(X_{\mathrm{dR}}),BGL_n\big)\simeq\mathbf{Map}_{B\mathbb{G}_m}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}),BGL_n\big)^{\mathrm{ss}}$$

Conversely, given $\widetilde{\theta}\in\mathbf{Map}\left(X_{\mathrm{Dol}},B^2\mathbb{G}_m\right)_0$ we can find $\theta\in\mathbf{Map}\left(X_{\mathrm{dR}},B^2\mathbb{G}_m\right)_0$ such that the same conclusion holds.

The semistability conditions that define the right hand side of this correspondence as a full subcategory of the category of basic GL_n-torsors on ${}_{\widetilde{\theta}}(X_{\mathrm{Dol}})$ are rather difficult to state at this point. We will build the requisite language to do so in section 5.2, and in section 5.3 they will finally be revealed.

5.1.2

Let $\theta\in\mathbf{Map}\big(X_{\mathrm{dR}},B^2\mathbb{G}_m\big)_0$ be a flat $\mathbb{G}_m$-gerbe over X, and $\widetilde{\theta}\in\mathbf{Map}\big(X_{\mathrm{Dol}},B^2\mathbb{G}_m\big)_0$ a Higgs $\mathbb{G}_m$-gerbe over the same variety. According to (3.3.1), we can express the categories of basic GL_n-torsors on one and the other as limits:

$$\mathbf{Map}_{B\mathbb{G}_m}\big({}_{\theta}(X_{\mathrm{dR}}), BGL_n\big) \simeq \lim \left\{ \begin{tikzcd} & * \arrow[d, "\theta"] \\ \mathbf{Map}\big(X_{\mathrm{dR}}, B\mathbb{P}GL_n\big) \arrow[r, "ob_{\mathbb{G}_m}"] & \mathbf{Map}\big(X_{\mathrm{dR}}, B^2\mathbb{G}_m\big) \end{tikzcd} \right\}$$

$$\mathbf{Map}_{B\mathbb{G}_m}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}), BGL_n\big) \simeq \lim \left\{ \begin{tikzcd} & * \arrow[d, "\widetilde{\theta}"] \\ \mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{P}GL_n\big) \arrow[r, "ob_{\mathbb{G}_m}"] & \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2\mathbb{G}_m\big) \end{tikzcd} \right\}$$

Notice how, as we mentioned several times in the introduction, a basic GL_n-torsor on ${}_{\theta}(X_{\mathrm{dR}})$ (resp., ${}_{\widetilde{\theta}}(X_{\mathrm{Dol}})$) determines an honest, untwisted $\mathbb{P}GL_n$-torsor on X_{dR} (resp., X_{Dol}).

The natural attempt at proving Theorem 5.1.1 would be to try to relate the terms in one of these limits to the matching ones in the other in a functorial manner. One of the comparisons is easy: the classical nonabelian Hodge correspondence (Theorem 4.2.1) provides an equivalence

$$\mathbf{Map}\big(X_{\mathrm{dR}}, B\mathbb{P}GL_n\big) \simeq \mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{P}GL_n\big)^{\mathrm{ss},0}$$

between the category of flat $\mathbb{P}GL_n$-torsors on X and the full subcategory of the category of Higgs $\mathbb{P}GL_n$-torsors on X on the semistable objects with vanishing rational Chern classes. This is, in fact, one of the two stability conditions we will need to impose on the Dolbeault side of our correspondence. However, as we saw in section 4.2.3, the Hodge correspondence fails for $\mathbb{G}_m$-gerbes.

Observe, though, that the above presentation of $\mathbf{Map}_{B\mathbb{G}_m}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}), BGL_n\big)$ as a limit implies that automorphisms of $\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2\mathbb{G}_m\big)$ enter into it at the level of objects: restricting these automorphisms would then be a restriction on objects of the category of basic GL_n-torsors on ${}_{\widetilde{\theta}}(X_{\mathrm{Dol}})$, and so the hope for the existence of a twisted nonabelian Hodge correspondence is not all lost.

5.1.3

It is the torsion phenomena that we referred to in the introduction that allows us to bypass this difficulty; they all stem from the fact that the determinant map $\det : GL_n \to \mathbb{G}_m$ is a surjective group homomorphism that remains surjective when restricted to its center. These surjectivity properties allow us to write the following commutative diagram of linear algebraic groups and homomorphisms with exact rows and columns:

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & & \\ 1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & SL_n \arrow[r] \arrow[d] & \mathbb{P}GL_n \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "(-)^n"] & GL_n \arrow[r] \arrow[d, "\det"] & \mathbb{P}GL_n \arrow[r] & 1 \\ & \mathbb{G}_m \arrow[r, equal] \arrow[d] & \mathbb{G}_m \arrow[d] & & \\ & 1 & 1 & & \end{tikzcd} \tag{5.1.1}$$

From the Puppe sequences associated to the exact sequences above, we deduce a diagram of ∞-stacks, in which the rows and the column are fiber sequences:

$$\begin{tikzcd} BSL_n \arrow[r] \arrow[d] & B\mathbb{P}GL_n \arrow[r, "ob_{\mu_n}"] \arrow[d, equal] & B^2\mu_n \arrow[d, "ob^{\mu_n}_{\mathbb{G}_m}"] \\ BGL_n \arrow[r] & B\mathbb{P}GL_n \arrow[r, "ob_{\mathbb{G}_m}"] & B^2\mathbb{G}_m \arrow[d, "(-)^n"] \\ & & B^2\mathbb{G}_m \end{tikzcd} \tag{5.1.2}$$

The second named horizontal map is the universal obstruction $\mathbb{G}_m$-gerbe for $\mathbb{P}GL_n$-torsors, in the sense that, for every ∞-stack $\mathfrak{X}$, it induces the map that associates to a $\mathbb{P}GL_n$-torsor on $\mathfrak{X}$ its obstruction $\mathbb{G}_m$-gerbe (see section 2.2.3). The fact that this map factors through $B^2\mu_n$ shows that the nth-power of the obstruction $\mathbb{G}_m$-gerbe of a $\mathbb{P}GL_n$-torsor is always trivializable.

Lemma 5.1.2. *Let $\alpha \in \mathbf{Map}\big(\mathfrak{X}, B^2\mathbb{G}_m\big)_0$ be a $\mathbb{G}_m$-gerbe over an ∞-stack $\mathfrak{X}$. Then, the category of basic GL_n-torsors on ${}_\alpha\mathfrak{X}$ is empty unless α^n is a trivializable $\mathbb{G}_m$-gerbe.*

Proof. If α^n is not trivializable, the image of the two morphisms into $\mathbf{Map}\big(\mathfrak{X}, B^2\mathbb{G}_m\big)$ in

$$\mathbf{Map}_{B\mathbb{G}_m}\big({}_\alpha\mathfrak{X}, BGL_n\big) \simeq \lim \left\{ \begin{tikzcd} & * \arrow[d, "\alpha"] \\ \mathbf{Map}\big(\mathfrak{X}, B\mathbb{P}GL_n\big) \arrow[r, "ob_{\mathbb{G}_m}"] & \mathbf{Map}\big(\mathfrak{X}, B^2\mathbb{G}_m\big) \end{tikzcd} \right\}$$

land in different connected components. □

This provides a uniform explanation for all the occurrences of torsion in the introduction, from that of the bare, underlying $\mathbb{G}_m$-gerbe to that of the twisted connections and twisted Higgs fields. It also suggests how we might get around the fact that the Hodge correspondence does not hold for $\mathbb{G}_m$-gerbes: by using the universal obstruction μ_n-gerbe (the first named horizontal map in (5.1.2)).

5.2 Torusless gerbes and rectifiability

As promised in the introduction, the most general form of our main theorem involves basic H-torsors on A-gerbes over the de Rham and Dolbeault stacks of X, where H is a (connected) linear algebraic group over $\mathbb{C}$, and A a closed subgroup of its center. Once again, we defer the statement of the stability conditions on the Dolbeault side until section 5.3.

Theorem 5.2.1 ((cf. Theorem 5.1.1)). *Let $\theta \in \mathbf{Map}\big(X_{\mathrm{dR}}, B^2A\big)_0$ be a flat A-gerbe over X. Then there is a Higgs A-gerbe over X, $\widetilde{\theta} \in \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A\big)_0$, for which there is an equivalence*

$$\mathbf{Map}_{BA}\big({}_{\theta}(X_{\mathrm{dR}}), BH\big) \simeq \mathbf{Map}_{BA}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}), BH\big)^{\mathrm{ss}}$$

Conversely, given $\widetilde{\theta} \in \mathbf{Map}\left(X_{\mathrm{Dol}}, B^2A\right)_0$ we can find $\theta \in \mathbf{Map}\left(X_{\mathrm{dR}}, B^2A\right)_0$ such that the same conclusion holds.

5.2.1

In this section we deal with linear (equivalently, affine) algebraic groups over $\mathbb{C}$. Their theory, as worked out in, e.g., [DG70], regards them as sheaves on the big fppf site of $\operatorname{Spec}\mathbb{C}$. Our first result says that we can lift the whole theory to the étale topology.

Lemma 5.2.2. *Let G be a linear algebraic group over $\mathbb{C}$, and N a closed normal subgroup. Then,*

1. *N is a linear algebraic group,*
2. *the fppf quotient G/N is representable by a linear algebraic group, and*
3. *the fppf quotient G/N is also a quotient in the étale topology.*

Proof. The first statement is obvious, since a closed subscheme of an affine scheme is affine. The second is [DG70, III, §3, 5.6]. To prove the third one, notice that the quotient map $G \to G/N$ is an fppf morphism [DG70, III, §3, 2.5 a)] with smooth fibers —all of them are isomorphic to N, which is smooth by a theorem of Cartier's [DG70, II, §6, 1.1 a)]—, hence smooth; and smooth morphisms have sections étale-locally. $\square$

Should we want to stay on the algebraic —by which of course we mean étale— side, the preceding lemma is all we need. If, however, we want to work in the analytic topology, we have to take one more step and use the analytification functor of section 4.3. But because the latter is exact and all of our constructions rely only on finite limits and colimits, the statements from their structure theory —for which we refer to [Bor91] (see also [Mil12a, Mil12b])— that we will use come through without any problem.

5.2.2

Let H be a *connected* linear algebraic group over $\mathbb{C}$, A a closed subgroup contained in its center, and $K = H/A$. Because A is abelian it decomposes as a product $A \cong F \times \mathbb{G}_m^{\oplus r} \times \mathbb{G}_a^{\oplus s}$ with F finite abelian.

Proposition 5.2.3. *There is a surjective homomorphism of linear algebraic groups $\kappa : H \to \mathbb{G}_m^{\oplus r}$ such that the composition $A \hookrightarrow H \xrightarrow{\kappa} \mathbb{G}_m^{\oplus r}$ is also surjective.*

Proof. We can assume that A coincides with the center of H; otherwise compose the homomorphism constructed below with a projection onto a torus of the appropriate rank in such a way that the induced map from the Lie algebra of A to that of the torus becomes an isomorphism.

Write H as an extension of its maximal reductive quotient H_{red} by its unipotent radical R_uH:

$$1 \to R_uH \to H \to H_{\text{red}} \to 1$$

The structure theorem of reductive groups gives a decomposition of H_{red} as the almost-direct product of its radical RH_{red} and its derived subgroup $\mathcal{D}H_{\text{red}}$:

$$1 \to RH_{\text{red}} \cap \mathcal{D}H_{\text{red}} \to RH_{\text{red}} \times \mathcal{D}H_{\text{red}} \to H_{\text{red}} \to 1$$

Here RH_{red} is the maximal subtorus in $Z(H_{\text{red}})$, and the intersection $RH_{\text{red}} \cap \mathcal{D}H_{\text{red}}$ is finite.

The canonical projection $RH_{\text{red}} \times \mathcal{D}H_{\text{red}} \longrightarrow RH_{\text{red}}$ descends to a surjective homomorphism

$$H_{\text{red}} \longrightarrow \frac{RH_{\text{red}}}{RH_{\text{red}} \cap \mathcal{D}H_{\text{red}}}$$

that is obviously surjective when restricted to its center. Notice that the target of this map is again a torus, of the same rank as RH_{red}. Composing with the projection $H \to H_{\text{red}}$ provides the required homomorphism. $\square$

Remark 5.2.4. We insist on the connectedness assumption on H only because we do not know whether we can extend this last proposition to the non-connected case. The requirement is superfluous whenever we can lift the homomorphism above from the connected component of the identity to the whole group. Since these fit in the exact sequence

$$1 \to H^0 \to H \to \pi_0 H \to 1,$$

all of our results hold, for example, for groups for which the latter is a split sequence.

It is an instructive exercise to follow the steps of this last proof in the case $H = GL_n$, for it exactly reconstructs the determinant map. Remember that it was the surjectivity of this map —and that its restriction to the center of GL_n— that allowed us to construct (5.1.1). Since κ enjoys the same properties, we have another commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & A' \arrow[r] \arrow[d] & H' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] \arrow[d, "\kappa"] & H \arrow[r] \arrow[d, "\kappa"] & K \arrow[r] & 1 \\
 & \mathbb{G}_m^{\oplus r} \arrow[r, equal] \arrow[d] & \mathbb{G}_m^{\oplus r} \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd} \tag{5.2.1}$$

Here we slightly abuse notation by denoting the restriction of κ to A by the same letter. There are two important remarks that we should make about the kernel A' of the latter:

- By construction, κ is trivial when restricted to the unipotent part of A, which is hence fully contained in A'.
- On the other hand, the restriction of κ to the non-unipotent part of A induces an isomorphism at the level of Lie algebras, which in turn leads us to the crucial statement about A': it contains no torus part. In other words, A' satisfies the hypothesis of Proposition 4.2.6.

In analogy with (5.1.2) we have the following diagrams of ∞-stacks, where again the rows and the column are fiber sequences:

$$\begin{tikzcd}
BH' \arrow[r] \arrow[d] & BK \arrow[r, "ob_{A'}"] \arrow[d, equal] & B^2A' \arrow[d, "ob_a^{A'}"] \\
BH \arrow[r] & BK \arrow[r, "ob_A"] & B^2A \arrow[d, "\kappa"] \\
 & & B^2\mathbb{G}_m^{\oplus r}
\end{tikzcd} \tag{5.2.2}$$

It is now clear what takes on the role that torsion had in the vector bundle case.

Definition 5.2.5. *We say that an A-gerbe over $\mathfrak{X}$ is* κ-torsion *if its image under the map*

$$\mathbf{Map}\big(\mathfrak{X}, B^2A\big) \xrightarrow{\kappa} \mathbf{Map}\big(\mathfrak{X}, B^2\mathbb{G}_m^{\oplus r}\big),$$

is a trivializable $\mathbb{G}_m^{\oplus r}$-gerbe.

Lemma 5.2.6 ((cf. Lemma 5.1.2))**.** *Let $\alpha \in \mathbf{Map}\big(\mathfrak{X}, B^2A\big)_0$ be an A-gerbe over an ∞-stack $\mathfrak{X}$. Then, the category of basic H-torsors on ${}_\alpha\mathfrak{X}$ is empty unless ${}_\alpha\mathfrak{X}$ is κ-torsion.*

5.2.3

We now seek to give a different presentation of the category of basic H-torsors on a κ-torsion A-gerbe —one that drops the explicit dependence on the category of A-gerbes in favor of that of A'-gerbes, for which the Hodge correspondence holds.

Consider the antidiagonal actions of A' on $A' \times A$ and $H' \times A$. They give rise to an exact sequence of augmented simplicial objects in the category of linear algebraic groups over $\mathbb{C}$:

$$\begin{tikzcd}
1 \arrow[r] & A' \times A /\!\!/ A' \arrow[r] \arrow[d, "m"] & H' \times A /\!\!/ A' \arrow[r] \arrow[d, "m"] & K /\!\!/ * \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & A \arrow[r] & H \arrow[r] & K \arrow[r] & 1
\end{tikzcd} \tag{5.2.3}$$

The augmentations of each of these are in fact the quotient maps of the corresponding actions. Notice that, although the action of A' described by the leftmost simplicial object seems to be twisted, it is actually isomorphic to the trivial A'-action on $A' \times A$.

The Puppe sequences of the rows in (5.2.3) yield the folllowing diagram of augmented simplicial objects in the appropriate ∞-topos of ∞-stacks:

$$
\begin{tikzcd}
BK /\!\!/ * \arrow[r, "(ob_{A'}, *)/\!\!/ *"] \arrow[d] & B^2A' \times B^2A /\!\!/ B^2A' \arrow[d, "m"] \\
BK \arrow[r, "ob_A"] & B^2A
\end{tikzcd}
\tag{5.2.4}
$$

Here the augmentation of the simplicial object on the left is trivially an effective epimorphism, while that of the simplicial object on the right is so because it is induced by the quotient of the trivial A'-torsor on A. The horizontal maps are composed of the universal obstruction A'- and A-gerbe for K-torsors, and the trivial maps that pick the natural points out in B^2A' and B^2A.

Now, for any ∞-stack $\mathfrak{X}$, apply the ∞-functor $\mathbf{Map}(\mathfrak{X}, -)$ to (5.2.4). After choosing a κ-torsion A-gerbe on $\mathfrak{X}$, $\alpha \in \mathbf{Map}(\mathfrak{X}, B^2A)_0$, and a lift, $\alpha' \in \mathbf{Map}(\mathfrak{X}, B^2A')_0$, to an A'-gerbe on $\mathfrak{X}$, we obtain can extend the resulting diagram of augmented simplicial ∞-groupoids to the following:

$$
\begin{tikzcd}
\mathbf{Map}(\mathfrak{X}, BK) /\!\!/ * \arrow[r, "(ob_{A'}, *)/\!\!/ *"] \arrow[d] & \begin{matrix} \mathbf{Map}(\mathfrak{X}, B^2A') \\ \times \mathbf{Map}(\mathfrak{X}, B^2A) \end{matrix} /\!\!/ \mathbf{Map}(\mathfrak{X}, B^2A') \arrow[d, "m"] & * /\!\!/ * \arrow[l, "(\alpha', *)/\!\!/ *"'] \arrow[d] \\
\mathbf{Map}(\mathfrak{X}, BK) \arrow[r, "ob_A"] & \mathbf{Map}(\mathfrak{X}, B^2A) & * \arrow[l, "\alpha"']
\end{tikzcd}
\tag{5.2.5}
$$

Once again, each vertical map in this diagram realizes the quotient of the groupoid object above it. Taking limits along the rows finally yields:

$$\begin{array}{c} \dfrac{\mathbf{Map}_{BA'}\big({}_{\alpha'}\mathfrak{X}, BH'\big) \\ \times \mathbf{Map}\big(\mathfrak{X}, BA\big)}{} \Big/\!\!\Big/ \mathbf{Map}\big(\mathfrak{X}, BA'\big) \\ \big\downarrow m \\ \mathbf{Map}_{BA}\big({}_{\alpha}\mathfrak{X}, BH\big) \end{array} \tag{5.2.6}$$

This last map is unfortunately *not* an effective epimorphism[2]. It turns into one, however, if we restrict to certain connected components of the target.

5.2.4

Before exploring this last claim, let us describe what the objects and morphisms in (5.2.6) look like in concrete terms. From the realization (3.3.1) of the category of basic H-torsors on ${}_{\alpha}\mathfrak{X}$ as a limit, we see that its objects are given by pairs

$$\Big(Q \to \mathfrak{X},\ ob_A(Q \to \mathfrak{X}) \xrightarrow{\gamma} {}_{\alpha}\mathfrak{X}\Big) \tag{5.2.7}$$

of a K-torsor on $\mathfrak{X}$ —which we dub the *underlying K-torsor* of the pair— together with an equivalence between its obstruction A-gerbe and ${}_{\alpha}\mathfrak{X}$. A similar description could be made of basic H'-torsors on ${}_{\alpha'}\mathfrak{X}$, but there is a slightly different characterization of these that will prove useful later.

Indeed, recall from (5.2.2) that the following is a fiber sequence of ∞-groupoids:

$$\mathbf{Map}\big(\mathfrak{X}, B^2A'\big) \xrightarrow{ob_A^{A'}} \mathbf{Map}\big(\mathfrak{X}, B^2A\big) \xrightarrow{\kappa} \mathbf{Map}\big(\mathfrak{X}, B^2\mathbb{G}_m^{\oplus r}\big).$$

We can read this as saying that A'-gerbes can be seen as (κ-torsion) A-gerbes together with the choice of a trivialization of their image under κ. In particular, think of the obstruction A'-gerbe of a K-torsor $Q \to \mathfrak{X}$ as its obstruction A-gerbe together with

[2]Effective epimorphisms constitute the *left* part of an orthogonal factorization system in any ∞-topos, which is not necessarily preserved by ∞-limits [Lur09, Section 5.2.8]

a trivialization

$$\kappa\big(ob_A(Q \to \mathfrak{X})\big) \xrightarrow{ob_{A'}} \mathfrak{X} \times B\mathbb{G}_m^{\oplus r}.$$

Similarly, a lift α' of α can be expressed as the choice of a trivialization

$$\kappa\big({}_\alpha\mathfrak{X}\big) \xrightarrow{\alpha'} \mathfrak{X} \times B\mathbb{G}_m^{\oplus r}.$$

In this language, to give a basic H'-torsor on ${}_{\alpha'}\mathfrak{X}$ is the same as giving a triple

$$\left(Q \to \mathfrak{X},\ ob_A(Q \to \mathfrak{X}) \xrightarrow{\gamma} {}_\alpha\mathfrak{X},\ \begin{tikzcd} \kappa\big(ob_A(Q \to \mathfrak{X})\big) \arrow[rr, "\kappa(\gamma)"] \arrow[dr, "ob_{A'}"'] & \arrow[d, phantom, "\Downarrow F"] & \kappa\big({}_\alpha\mathfrak{X}\big) \arrow[dl, "\alpha'"] \\ & \mathfrak{X} \times B\mathbb{G}_m^{\oplus r} & \end{tikzcd} \right) \tag{5.2.8}$$

The existence of a homotopy F filling the last diagram is what says that γ is part of an equivalence of A'-gerbes —given by the choice of such a homotopy.

Notice that the first two pieces of data in (5.2.8) define a basic H-torsor on ${}_\alpha\mathfrak{X}$ (5.2.7). Forgetting the homotopy F is precisely the fortgetful functor

$$\mathbf{Map}_{BA'}\big({}_{\alpha'}\mathfrak{X}, BH'\big) \xrightarrow{\pi} \mathbf{Map}_{BA}\big({}_\alpha\mathfrak{X}, BH\big) \tag{5.2.9}$$

obtained by taking limits along the rows of the following diagram:

$$\begin{tikzcd} \mathbf{Map}\big(\mathfrak{X}, BK\big) \arrow[r, "ob_{A'}"] \arrow[d, equal] & \mathbf{Map}\big(\mathfrak{X}, B^2A'\big) \arrow[d, "ob_A^{A'}"] & * \arrow[l, "\alpha'"'] \arrow[d, equal] \\ \mathbf{Map}\big(\mathfrak{X}, BK\big) \arrow[r, "ob_A"] & \mathbf{Map}\big(\mathfrak{X}, B^2A\big) & * \arrow[l, "\alpha"'] \end{tikzcd}$$

In the other direction, completing a pair (Q, γ) to a triple (Q, γ, F) is not a trivial task, but rather a strong condition on (Q, γ) —and one that depends on the choice of α'. Indeed, for a fixed α', we have two trivializations of $\kappa\big(ob(Q \to \mathfrak{X})\big)$: namely,

$\alpha' \circ \kappa(\gamma)$ and $ob_{A'}$. But the category of trivializations of a trivializable $\mathbb{G}_m^{\oplus r}$-gerbe is (noncanonically) equivalent to $\mathbf{Map}(\mathfrak{X}, B\mathbb{G}_m^{\oplus r})$, and we have no reason to expect the two to belong to the same connected component.

On the other hand, the category of equivalences between two (equivalent) A-gerbes is (again, noncanonically) equivalent to $\mathbf{Map}(\mathfrak{X}, BA)$, and so we have an action map

$$\begin{array}{ccc} \mathbf{Map}_{BA}({}_{\alpha}\mathfrak{X}, BH) \times \mathbf{Map}(\mathfrak{X}, BA) & \xrightarrow{\sigma} & \mathbf{Map}_{BA}({}_{\alpha}\mathfrak{X}, BH) \\ \big((Q,\gamma), \mathcal{L}\big) & \longmapsto & (Q, \mathcal{L}\gamma) \end{array} \tag{5.2.10}$$

We can finally write the multiplication map m in (5.2.6) as the appropriate composition of the forgetful functor (5.2.9) and this last map:

$$m : \begin{array}{ccccc} \begin{array}{c}\mathbf{Map}_{BA'}({}_{\alpha'}\mathfrak{X}, BH') \\ \times \mathbf{Map}(\mathfrak{X}, BA)\end{array} & \xrightarrow{\pi \times \mathrm{id}} & \begin{array}{c}\mathbf{Map}_{BA}({}_{\alpha}\mathfrak{X}, BH) \\ \times \mathbf{Map}(\mathfrak{X}, BA)\end{array} & \xrightarrow{\sigma} & \mathbf{Map}_{BA}({}_{\alpha}\mathfrak{X}, BH) \\ \big((Q,\gamma,F), \mathcal{L}\big) & \longmapsto & \big((Q,\gamma), \mathcal{L}\big) & \longmapsto & (Q, \mathcal{L}\gamma) \end{array}$$

For sake of completeness, let us also give an expression for the action of $\mathbf{Map}(\mathfrak{X}, BA')$ in (5.2.6). Seeing A'-torsors as pairs $(\mathcal{L}', \mu)$ consisting of an A-torsor and a trivialization of its image under κ —just as we did for A'-gerbes—, it is given by

$$\begin{array}{ccc} \begin{array}{c}\mathbf{Map}_{BA'}({}_{\alpha'}\mathfrak{X}, BH') \\ \times \mathbf{Map}(\mathfrak{X}, BA)\end{array} \times \mathbf{Map}(\mathfrak{X}, BA') & \xrightarrow{\mu} & \begin{array}{c}\mathbf{Map}_{BA'}({}_{\alpha'}\mathfrak{X}, BH') \\ \times \mathbf{Map}(\mathfrak{X}, BA)\end{array} \\ \big((Q,\gamma,F), \mathcal{L}, (\mathcal{L}', \mu)\big) & \longmapsto & \big((Q, \mathcal{L}'\gamma, F \circ \mu), (\mathcal{L}')^{-1}\mathcal{L}\big) \end{array} \tag{5.2.11}$$

5.2.5

With these descriptions in hand, we return to the claim in the last paragraph of section 5.2.3.

Definition 5.2.7. *A basic H-torsor on ${}_{\alpha}\mathfrak{X}$ is said to be α'-*rectifiable *if it belongs to the image of* $\mathbf{Map}_{BA'}\big({}_{\alpha'}\mathfrak{X}, BH'\big) \times \mathbf{Map}\big(\mathfrak{X}, BA\big)$ *under the multiplication map m in* (5.2.6).

In other words, the multiplication map is an effective epimorphism onto the category of α'-rectifiable basic H-torsors on ${}_{\alpha}\mathfrak{X}$, which, in turn, realizes the latter as the quotient

$$\mathbf{Map}_{BA}\big({}_{\alpha}\mathfrak{X}, BH\big)^{\alpha'\text{-rect}} \simeq \frac{\mathbf{Map}_{BA'}\big({}_{\alpha'}\mathfrak{X}, BH'\big) \times \mathbf{Map}\big(\mathfrak{X}, BA\big)}{\mathbf{Map}\big(\mathfrak{X}, BA'\big)}$$

Proposition 5.2.8. *A basic H-torsor on ${}_{\alpha}\mathfrak{X}$ is α'-rectifiable if and only if the obstruction A'-gerbe of its underlying K-torsor is equivalent to ${}_{\alpha'}\mathfrak{X}$.*

Proof. Necessity is clear, so we just need to prove sufficiency. Let (Q, γ) be a basic H-torsor on ${}_{\alpha}\mathfrak{X}$, and suppose $ob'_A(Q \to \mathfrak{X})$ is equivalent to ${}_{\alpha'}\mathfrak{X}$. This amounts to the existence of a pair

$$\left(ob_A(Q \to \mathfrak{X}) \xrightarrow{\phi} {}_{\alpha}\mathfrak{X}, \quad \begin{tikzcd} \kappa\big(ob_A(Q \to \mathfrak{X})\big) \arrow[rr, "\kappa(\phi)"] \arrow[dr, "ob_{A'}"'] & \arrow[d, phantom, "\Downarrow H"] & \kappa\big({}_{\alpha}\mathfrak{X}\big) \arrow[dl, "\alpha'"] \\ & \mathfrak{X} \times B\mathbb{G}_m^{\oplus r} & \end{tikzcd} \right)$$

Since the category of equivalences between two (equivalent) A-gerbes is equivalent to $\mathbf{Map}\big(\mathfrak{X}, BA\big)$, we can find an A-torsor $\mathcal{L}$ such that $\phi \cong \mathcal{L}\gamma$. Hence $m\big((Q, \phi, H), \mathcal{L}\big) = (Q, \mathcal{L}\phi) \cong (Q, \gamma)$. $\square$

Definition 5.2.9. *Let $[\alpha] \in \pi_0 \mathbf{Map}\big(\mathfrak{X}, B^2A\big)$. We define $L(\mathfrak{X})([\alpha])$ as the set of equivalence classes of liftings of the A-gerbe ${}_{\alpha}\mathfrak{X}$ to an A'-gerbe:*

$$L(\mathfrak{X})([\alpha]) := \Big\{ [\alpha'] \in \pi_0 \mathbf{Map}\big(\mathfrak{X}, B^2A'\big) \,\Big|\, ob_A^{A'}([\alpha']) = [\alpha] \Big\}$$

Corollary 5.2.10. $\mathbf{Map}_{BA}\big({}_{\alpha}\mathfrak{X}, BH\big) \simeq \coprod_{[\alpha'] \in L(\mathfrak{X})([\alpha])} \mathbf{Map}_{BA}\big({}_{\alpha}\mathfrak{X}, BH\big)^{\alpha'\text{-}rect}$

Proof. We only need to observe that Proposition 5.2.8 implies that every basic H-torsor (Q, γ) is $ob_{A'}(Q \to \mathfrak{X})$-rectifiable. $\square$

Let us make two quick remarks about this sets of equivalence classes of liftings.

- The concept is only meaningful if ${}_{\alpha}\mathfrak{X}$ is κ-torsion, for otherwise $L(\mathfrak{X})([\alpha])$ is empty. Nevertheless, Corollary 5.2.10 remains true in this case (cf. Lemma 5.2.6).

- Even if if ${}_{\alpha}\mathfrak{X}$ is κ-torsion, it is not necessarily true that every element in $L(\mathfrak{X})([\alpha])$ is hit by an element of $\pi_0 \mathbf{Map}\big(\mathfrak{X}, BK\big)$; that is,

$$\mathbf{Map}_{BA}\big({}_{\alpha}\mathfrak{X}, BH\big)^{\alpha'\text{-rect}}$$

  might be empty for some $[\alpha'] \in L(\mathfrak{X})([\alpha])$. We could therefore restrict the index set of the union in Corollary 5.2.10 to the intersection of $L(\mathfrak{X})([\alpha])$ and the image of

$$\pi_0 \mathbf{Map}\big(\mathfrak{X}, BK\big) \xrightarrow{ob_{A'}} \pi_0 \mathbf{Map}\big(\mathfrak{X}, B^2 A'\big),$$

  without altering its validity.

5.3 Stability conditions

5.3.1

Choose a pair

$$\theta' \in \mathbf{Map}\big(X_{\mathrm{dR}}, B^2 A'\big)_0, \qquad \widetilde{\theta'} \in \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2 A'\big)_0$$

consisting of a flat A'-gerbe on X and a Higgs A'-gerbe on X that are related to each other under the Hodge correspondence for gerbes (Proposition 4.2.6), and denote by

$$\theta := ob_A^{A'}(\theta') \in \mathbf{Map}\big(X_{\mathrm{dR}}, B^2A\big)_0, \qquad \widetilde{\theta} := ob_A^{A'}(\widetilde{\theta'}) \in \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A\big)_0$$

the induced (κ-torsion) A-gerbes.

Definition 5.3.1. *A basic H'-torsor on ${}_{\widetilde{\theta'}}(X_{\mathrm{Dol}})$ is called* semistable *if its underlying K-torsor on X_{Dol} is semistable and has zero first and second rational Chern classes. We denote by $\mathbf{Map}_{BA'}\big({}_{\widetilde{\theta'}}(X_{\mathrm{Dol}}), BH'\big)^{\mathrm{ss}}$ the full subcategory of the category of basic H'-torsors on ${}_{\widetilde{\theta'}}(X_{\mathrm{Dol}})$ on the semistable objects.*

Proposition 5.3.2. $\mathbf{Map}_{BA'}\big({}_{\theta'}(X_{\mathrm{dR}}), BH'\big) \simeq \mathbf{Map}_{BA'}\big({}_{\widetilde{\theta'}}(X_{\mathrm{Dol}}), BH'\big)^{\mathrm{ss}}$

Proof. This is just the obvious attempt that we described in section 5.1.2 —only now the Hodge correspondence does hold for A'-gerbes. □

Definition 5.3.3. *A $\widetilde{\theta'}$-rectifiable H-torsor on ${}_{\widetilde{\theta}}(X_{\mathrm{Dol}})$ is called* semistable *if it is the image under the multiplication map m in (5.2.6) of a semistable basic H'-torsor on ${}_{\widetilde{\theta'}}(X_{\mathrm{Dol}})$ and an A-torsor on X_{Dol} with zero first Chern class. We denote the category of these objects by*

$$\mathbf{Map}_{BA}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}), BH\big)^{\widetilde{\theta'}\text{-}rect,\mathrm{ss}}$$

Observe that, owing to the discussion in section 4.2.4, we do not need to explicitly require semistability for the A-torsor on X_{Dol} —vanishing of the first Chern class is enough.

Since the multiplication map m in (5.2.6) is a quotient map, there is the question of whether this definition of semistability depends on the choice of inverse image. Any two objects of $\mathbf{Map}_{BA'}\big({}_{\widetilde{\theta'}}(X_{\mathrm{Dol}}), BH'\big) \times \mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big)$ giving rise to the same basic H-torsor on ${}_{\widetilde{\theta}}(X_{\mathrm{Dol}})$ differ by the action of an A'-torsor on X_{Dol} as in (5.2.11).

However, the latter are always of degree zero because A' contains no algebraic torus, and they remain of degree zero after taking their image under $ob_A^{A'}$. Hence, if one of the objects consists of a semistable basic H'-torsor on ${}_{\widetilde{\theta'}}(X_{\mathrm{Dol}})$ and an A-torsor on X_{Dol} with zero first Chern class, so does the other. In other words, the action of $\mathbf{Map}\big(X_{\mathrm{Dol}}, BA'\big)$ preserves the subcategory

$$\begin{aligned}\mathbf{Map}_{BA'}\big({}_{\widetilde{\theta'}}(X_{\mathrm{Dol}}), BH'\big)^{\mathrm{ss}} \times \mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big)^0 \\ \subseteq \mathbf{Map}_{BA'}\big({}_{\widetilde{\theta'}}(X_{\mathrm{Dol}}), BH'\big) \times \mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big).\end{aligned}$$

Corollary 5.3.4. $\mathbf{Map}_{BA}\big({}_{\theta}(X_{\mathrm{dR}}), BH\big)^{\theta'\text{-}rect} \simeq \mathbf{Map}_{BA}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}), BH\big)^{\widetilde{\theta'}\text{-}rect,\mathrm{ss}}$

Proof. Once again, the obvious approach works: the terms in the definitions of both of the categories on both sides of this equivalence,

$$\mathbf{Map}_{BA}\big({}_{\theta}(X_{\mathrm{dR}}), BH\big)^{\theta'\text{-rect}} \simeq \frac{\mathbf{Map}_{BA'}\big({}_{\theta'}(X_{\mathrm{dR}}), BH'\big) \times \mathbf{Map}\big(X_{\mathrm{dR}}, BA\big)}{\mathbf{Map}\big(X_{\mathrm{dR}}, BA'\big)}$$

and

$$\mathbf{Map}_{BA}\big({}_{\widetilde{\theta}}(X_{\mathrm{Dol}}), BH\big)^{\widetilde{\theta'}\text{-rect,ss}} \simeq \frac{\mathbf{Map}_{BA'}\big({}_{\widetilde{\theta'}}(X_{\mathrm{Dol}}), BH'\big)^{\mathrm{ss}} \times \mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big)^0}{\mathbf{Map}\big(X_{\mathrm{Dol}}, BA'\big)},$$

exactly correspond to each other under Theorem 4.2.1. □

5.3.2

In the last section we proved something that puts us very close to the statement of Theorem 5.2.1. Indeed, for any choice of a κ-torsion flat A-gerbe on X, $\theta \in$

$\mathbf{Map}\big(X_{\mathrm{dR}}, B^2A\big)_0$, Corollary 5.2.10 gives us the decomposition

$$\mathbf{Map}_{BA}\big({}_{\theta}(X_{\mathrm{dR}}), BH\big) \simeq \coprod_{[\theta'] \in L(X_{\mathrm{dR}})([\theta])} \mathbf{Map}_{BA}\big({}_{\theta}(X_{\mathrm{dR}}), BH\big)^{\theta'\text{-rect}}$$

By Corollary 5.3.4, each one of the pieces on the right hand side of the last equation is equivalent to the category of semistable $\widetilde{\theta'}$-rectifiable basic H-torsors on ${}_{ob_A^{A'}(\widetilde{\theta'})}(X_{\mathrm{Dol}})$, where $\widetilde{\theta'}$ is related to θ' through the Hodge correspondence for gerbes (cf. 5.3.1). Our first remark is that all of the $[ob_A^{A'}(\widetilde{\theta'})] \in \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A\big)$ coincide.

Lemma 5.3.5. *Let $[\theta] \in \pi_0\,\mathbf{Map}\big(X_{\mathrm{dR}}, B^2A\big)$ be an equivalence class of κ-torsion flat A-gerbes on X. There is a unique equivalence class of κ-torsion Higgs A-gerbes on X, $[\widetilde{\theta}] \in \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A\big)$, for which there is a injective function*

$$L(X_{\mathrm{dR}})([\theta]) \hookrightarrow L(X_{\mathrm{Dol}})([\widetilde{\theta}])$$

Proof. Consider the following diagram, constructed out of the Puppe sequence of the exact sequence $0 \to A' \to A \to \mathbb{G}_m^{\oplus r} \to 0$ of abelian linear algebraic groups over $\mathbb{C}$ coming from (5.2.1).

$$\begin{tikzcd}
\pi_0\,\mathbf{Map}\big(X_{\mathrm{dR}}, BA\big) \arrow[r, "\cong"] \arrow[d] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big)^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big) \arrow[d] \\
\pi_0\,\mathbf{Map}\big(X_{\mathrm{dR}}, B\mathbb{G}_m^{\oplus r}\big) \arrow[r, "\cong"] \arrow[d] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{G}_m^{\oplus r}\big)^0 \arrow[r, hook] \arrow[d] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{G}_m^{\oplus r}\big) \arrow[d] \\
\pi_0\,\mathbf{Map}\big(X_{\mathrm{dR}}, B^2A'\big) \arrow[r, "\cong"] \arrow[d, "ob_A^{A'}"] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A'\big) \arrow[r, equal] \arrow[d, "ob_A^{A'}"] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A'\big) \arrow[d, "ob_A^{A'}"] \\
\pi_0\,\mathbf{Map}\big(X_{\mathrm{dR}}, B^2A\big) & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A\big) \arrow[r, equal] & \pi_0\,\mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A\big)
\end{tikzcd} \tag{5.3.1}$$

Notice that the first and last columns are, by construction, exact sequences of abelian groups; although the middle one is not exact, the composition of any two maps in it is zero. The maps from the first column to the second are provided by the appropriate Hodge correspondences.

For $[\theta] \in \pi_0 \, \mathbf{Map}\big(X_{\mathrm{dR}}, B^2 A\big)$, look at all its possible liftings through $ob_A^{A'}$ —that is, at $L(X_{\mathrm{dR}})([\theta])$. Since any two elements in the latter set differ by an element in $\pi_0 \, \mathbf{Map}\big(X_{\mathrm{dR}}, B\mathbb{G}_m^{\oplus r}\big)$, their images under the Hodge correspondence differ by an element of $\pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{G}_m^{\oplus r}\big)^0$. These images then map to a well-defined element $[\widetilde{\theta}] \in \pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2 A\big)$, thus defining a function $L(X_{\mathrm{dR}})([\theta]) \to L(X_{\mathrm{Dol}})([\widetilde{\theta}])$. The injectivity of the latter is clear. □

This last map is in general not surjective. Indeed, the upper right square in (5.3.1) being cartesian, we have

$$L(X_{\mathrm{dR}})([\theta]) \cong \frac{\pi_0 \, \mathbf{Map}\big(X_{\mathrm{dR}}, B\mathbb{G}_m^{\oplus r}\big)}{\pi_0 \, \mathbf{Map}\big(X_{\mathrm{dR}}, BA\big)} \cong \frac{\pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{G}_m^{\oplus r}\big)^0}{\pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big)^0} \hookrightarrow \frac{\pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, B\mathbb{G}_m^{\oplus r}\big)}{\pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, BA\big)} \cong L(X_{\mathrm{Dol}})([\widetilde{\theta}])$$

However, it becomes an isomorphism when restricted to the images under $ob_{A'}$ of the category of K-torsors on X_{dR}, on one side, and that of the category of semistable K-torsors on X_{Dol} with zero first and second rational Chern classes, on the other:

$$\begin{aligned} &\operatorname{im}\Big(\pi_0 \, \mathbf{Map}\big(X_{\mathrm{dR}}, BK\big) \xrightarrow{ob_{A'}} \pi_0 \, \mathbf{Map}\big(X_{\mathrm{dR}}, B^2 A'\big)\Big) \cap L(X_{\mathrm{dR}})([\theta]) \\ &\qquad \cong \operatorname{im}\Big(\pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, BK\big)^{\mathrm{ss},0} \xrightarrow{ob_{A'}} \pi_0 \, \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2 A'\big)\Big) \cap L(X_{\mathrm{Dol}})([\widetilde{\theta}]) \end{aligned}$$

This fact follows easily from the commutative diagram

$$\begin{tikzcd}
\pi_0 \mathbf{Map}\big(X_{\mathrm{dR}}, BK\big) \arrow[r, "\cong"] \arrow[d, "ob_{A'}"] & \pi_0 \mathbf{Map}\big(X_{\mathrm{Dol}}, BK\big)^{\mathrm{ss},0} \arrow[d, "ob_{A'}"] \\
\pi_0 \mathbf{Map}\big(X_{\mathrm{dR}}, B^2A'\big) \arrow[r, "\cong"] & \pi_0 \mathbf{Map}\big(X_{\mathrm{Dol}}, B^2A'\big)
\end{tikzcd}$$

mediated by the nonabelian Hodge correspondence, and the Hodge correspondence for gerbes. Because of the last remark of section 5.2.5, this observation finishes the proof of Theorem 5.2.1. In particular, it allows us to find, given an equivalence class of κ-torsion Higgs A-gerbes on X, an equivalence class of κ-torsion flat A-gerbes for which the conclusion of Lemma 5.3.5 holds.

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