A STUDY OF THE GEOMETRY OF THE DERIVED CATEGORY

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ABSTRACT

A STUDY OF THE GEOMETRY OF THE DERIVED CATEGORY

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This thesis is concerned with the study of the derived category of coherent sheaves on an algebraic variety. It pursues the rigidity of the derived category through the group of autoequivalences and the dimension of the derived category by way of tilting objects.

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Chapter 1

Introduction

The collection of all sheaves of $\mathcal{O}$-modules is a fundamental geometric invariant of an algebraic variety. Traditionally one studies the abelian category of $\mathcal{O}$-modules which is a powerful and robust structure containing complete information about the variety. However, a standard problem with this method of study is that the natural geometric and algebraic constructions (e.g., pushforward, pullback, and tensor product) are not always exact and so do not preserve this abelian category. Derived functors and eventually the derived category were introduced as a means of reconciling this lack of exactness. Beyond creating a conceptual framework for understanding the behavior of non-exact functors, derived categories provide a new important flexibility: they can serve as a replacement of the classical notion of a space and provide a setting for the exploration of noncommutative geometry. This thesis revolves around a fundamental issue that arises during the course of this exploration: how and what kind of geometry is encoded in the derived category of sheaves on an algebraic variety?

In chapter one, we explore the derived category of coherent sheaves on a variety through its group of autoequivalences. More precisely, we show that a scheme of finite type over a field is determined by its bounded derived category of coherent sheaves together with a collection of autoequivalences corresponding to an ample family of line bundles. In particular for a quasi-projective variety we need only a single autoequivalence. This imposes strong conditions on the Fourier-Mukai partners of a projective variety. Namely, if X is any smooth projective variety over $\mathbb{C}$, we have a representation ρ of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X))$ on $\mathrm{H}^*(X)$. Now if ker $\rho = 2\,\mathbb{Z} \times \mathrm{Pic}^0(X) \ltimes \mathrm{Aut}^0(X)$ then we are able to conclude that X has finitely many Fourier-Mukai partners. In particular we are able to show that abelian varieties have finitely many Fourier-Mukai partners. We also show that from the derived category of coherent sheaves on a scheme of finite type over a field one can recover the full subcategory of objects with proper support. As applications we show that an abelian variety can be recovered from its derived category of coherent D-modules and that a smooth variety with the property that the canonical bundle restricted to any proper subvariety is either ample or anti-ample can be recovered from its derived category of coherent sheaves (generalizing a well-known theorem of Bondal and Orlov).

Chapter two is part of an ongoing joint-work with Matthew Ballard studying the dimension of the derived category of coherent sheaves on a variety. Here we study the global dimension of the endomorphism algebra of a tilting object. We prove that if the tilting object is a sheaf then the global dimension is bounded below by the dimension of the variety and above by twice the dimension of the variety. In particular, if T is a tilting object on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $\mathrm{Ext}^i(T \otimes \omega_X, T) = 0$ for $i > 0$ then we show that the dimension of the derived category of coherent sheaves on X is equal to the dimension of X. This ends up being related to the relationship between the dimension of the derived category of coherent sheaves on X and the dimension of the derived category of coherent sheaves on the total space of a vector bundle over X. When the vector bundle is the canonical bundle and X is Fano, this is related to noncommutative crepant resolutions of the anti-canonical ring of X. As is turns out the dimension of any noncommutative crepant resolution of a affine gorenstein variety is always equal to the dimension of the variety. We also show that if $\hat{X}$ is a resolution of rational singularities then $\dim D^b_{coh}(\hat{(X)}) \geq \dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. And that if C is a smooth orbicurve then $\dim D^b_{coh}(C) = 1$.

Chapter 2

Reconstructions and some finiteness results for Fourier-Mukai partners

2.1 Introduction

In [5], Bondal and Orlov prove that a smooth projective variety with ample or anti-ample canonical bundle can be reconstructed from its derived category of coherent sheaves. This reconstruction uses the fact that the Serre functor corresponds to an ample or anti-ample line bundle (together with a shift). There is nothing special about the the Serre functor in this reconstruction except that it is intrinsic to the category. The starting point for this chapter is to instead take the data of both the derived category and a functor which comes from an ample line bundle. Using this data, one can reconstruct the scheme. More generally we shall prove the following,

Theorem 1. *Let X be a divisorial scheme of finite type over a field. Then X can be reconstructed* *from its derived category of coherent sheaves together with a collection of autoequivalences corresponding to an ample family of line bundles. Let Y be a divisorial scheme of finite type over a field, $F : D^b_{coh}(X) \to D^b_{coh}(Y)$ an equivalence, $\{\mathcal{A}_i\}$ an ample family of line bundles on X, and $\{\mathcal{M}_i\}$ any collection of line bundles on Y. If $F^{-1} \circ (\bullet \otimes \mathcal{M}_i) \circ F = (\bullet \otimes \mathcal{A}_i)$ then $X \cong Y$.*

In order to prove this theorem one uses the fact that objects fixed under the collection of autoequivalences are supported on finite sets of points and from there recovers objects isomorphic to structure sheafs of closed points. These points form the set X and one proceeds by recovering the Zariski topology on this set followed by the structure sheaf. The crucial step being the recovery of objects supported on finite sets of points. In fact, there are other situations in which one is able to recover these objects. For example suppose X is a variety such that the only proper closed subvarieties are finite sets of points. Then to recover structure sheaves of points, one first could try to recover all objects supported on proper subvarieties. In fact we are able to do just that. Let us denote by $\mathfrak{Prop}(X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ consisting of objects supported on proper subvarieties of X. We will prove the following,

Theorem 2. *Let X be a k-variety. Then $\mathfrak{Prop}(X)$ is equivalent to the full subcategory of $D^b_{coh}(X)$ consisting of objects $A \in D^b_{coh}(X)$ with the property that $\mathrm{Hom}(A, B)$ is finite dimensional over k for all $B \in D^b_{coh}(X)$. Hence as either k-linear graded or k-linear triangulated categories, $\mathfrak{Prop}(X)$ can be recovered from $D^b_{coh}(X)$.*

On a smooth variety, $\mathfrak{Prop}(X)$ also comes equipped with a Serre functor. Mimicking the ideas of Bondal and Orlov and combining them with the above theorem, one can once again recover the structure sheaves of points on any smooth variety with ample or anti-ample canonical bundle. Thus such spaces can be reconstructed from their derived category (this eliminates the projectivity assumption made by Bondal and Orlov). More generally we prove the following theorem,

Theorem 3. *Let X and Y be divisorial varieties. Let $\omega_{X_{sm}}$ be the canonical bundle of the smooth locus of X. Suppose that any proper closed positive dimensional subvariety Z is contained in X_{sm}*

and that for any such Z, $\omega_{X_{sm}}$ restricted to Z is either ample or anti-ample. Then X can be reconstructed from its derived category (as a k-linear graded category). Furthermore if $D^b_{coh}(X)$ is equivalent to $D^b_{coh}(Y)$ then $X \cong Y$.

The moduli space of integrable local systems on an abelian variety is such a space (in fact all proper subvarieties are zero-dimensional in this case). D. Arinkin observed that due to the equivalence of categories between coherent sheaves on this moduli space and D-modules on the dual abelian variety [22, 31, 36], one can in fact recover an abelian variety from its derived category of D-modules. Hence as a corollary to the above theorem we have the following theorem which we attribute to Arinkin.

Theorem 4 (Arinkin)**.** *An abelian variety A can be reconstructed from its derived category of coherent D-modules. If two abelian varieties A and B have equivalent derived categories of coherent D-modules then $A \cong B$*

It has been conjectured by D. Orlov that this statement is true for any variety.

Theorem 1 tells us that the schemes which are derived equivalent to X are encoded in the autoequivalences of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. This idea can be applied to abelian varieties because the autoequivalences of their derived categories are well understood. It is believed conjecturally that if you consider the derived category of coherent sheaves on a fixed variety there are only finitely many isomorphism classes of varieties with an equivalent derived category. We prove the following special case of this conjecture,

Theorem 5. *Let X be a smooth projective variety over $\mathbb{C}$ and ρ be the representation of $\mathrm{Aut}(D^b_{coh}(X))$ on $\mathrm{H}^*(X,\mathbb{Q})$. Let,*

$$\ker\rho = 2\,\mathbb{Z} \times \mathrm{Pic}^0(X) \ltimes \mathrm{Aut}^0(X),$$

then the number of projective Fourier-Mukai partners of X is bounded by the number of conjugacy classes of maximal unipotent subgroups of $\rho(\mathrm{Aut}(D^b_{coh}(X)))$. In particular since $\mathrm{im}\,\rho$ is an arithmetic group it is finite.

It is well known that this condition holds for abelian varieties [13, 25, 12]. For abelian varieties with Neron-Severi group equal to $\mathbb{Z}$ we calculate an explicit bound which turns out to be the number of inequivalent cusps for the action of $\Gamma_0(N)$ on the upper half plane (N is a certain invariant of the abelian variety). Our proof of this conjecture extends the well-known fact that any abelian variety is derived equivalent to at most finitely many abelian varieties. Shortly after the posting of the original version of this paper, Huybrechts and Nieper-Wisskirchen proved that any Fourier-Mukai partner of an abelian variety is in fact an abelian variety. This result is also proved in an unpublished thesis by Fabrice Rosay. The results of Huybrechts, Nieper-Wisskirchen, and Rosay, are in fact stronger than the bound we calculate in some cases.

2.2 Reconstruction

We begin by proving that the derived category together with an ample family of line bundles determines the scheme. Equivalently one could use the dual family of line bundles by switching to the inverse functors but we omit such statements. We also make all the statements for the bounded derived category of coherent sheaves. For affine schemes or quasi-projective schemes over a field it is known that the bounded derived category of coherent sheaves can be recovered from the derived category of quasi-coherent sheaves. It is precisely the full subcategory of locally cohomologically finitely presented objects. In fact the statement is true for a larger class of schemes satisfying a certain technical condition (see [34] for details). If one instead took the bounded derived category of quasi-coherent sheaves on a separated noetherian scheme, then once again one could recover the bounded derived category of coherent sheaves as the full subcategory of compact objects [34]. Thus we could equally well make the statements below for the derived category of quasi-coherent sheaves over an affine scheme or a quasi-projective variety or for the bounded derived category of quasi-coherent sheaves over a separated noetherian scheme but once again we omit such statements.

As a matter of convention, a variety always means an integral scheme of finite type over a field k, all half exact functors such as tensor products and pullbacks acting on objects in the derived category are taken to be derived functors unless otherwise stated, and any functor between derived categories is taken to be graded. The reconstructions below (only) necessitate the graded structure and sometimes the k-linear structure, so when we say that X can be reconstructed from $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, we mean that X is determined by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ as a (k-linear) graded category and when we posit equivalences $F : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ we require that F is a (k-linear) graded functor.

We begin by recalling the notion of an ample family of line bundles (see [37] 6.II 2.2.3).

Definition 2.2.1. *Let X be a quasi-compact, quasi-separated scheme and $\{\mathcal{A}_i\}$ be a family of invertible sheaves on X. $\{\mathcal{A}_i\}$ is called an* ample family of line bundles *if it satisfies the following equivalent conditions:*

a) *The open sets X_f for all $f \in \Gamma(X, \mathcal{A}_i^{\otimes n})$ with $i \in I$, $n > 0$ form a basis for the Zariski topology on X.*

b) *There is a family of sections $f \in \Gamma(X, \mathcal{A}_i^{\otimes n})$ such that the X_f form an affine basis for the Zariski topology on X.*

c) *There is a family of sections $f \in \Gamma(X, \mathcal{A}_i^{\otimes n})$ such that the X_f form an affine cover of X.*

d) *For any quasi-coherent sheaf $\mathcal{F}$ and $i \in I$, $n > 0$ let $\mathcal{F}_{i,n}$ denote the subsheaf of $\mathcal{F} \otimes \mathcal{A}_i^{\otimes n}$ generated by global sections. Then $\mathcal{F}$ is the sum of the submodules $\mathcal{F}_{i,n} \otimes \mathcal{A}_i^{\otimes -n}$.*

e) *For any quasi-coherent sheaf of ideals $\mathcal{F}$ and $i \in I$, $n > 0$, $\mathcal{F}$ is the sum of the submodules $\mathcal{F}_{i,n} \otimes \mathcal{A}_i^{\otimes -n}$.*

f) *For any quasi-coherent sheaf $\mathcal{F}$ of finite type there exist integers $n_i, k_i > 0$ such that $\mathcal{F}$ is a quotient of $\bigoplus_{i \in I} \mathcal{A}_i^{\otimes -n_i} \otimes \mathcal{O}_X^{k_i}$.*

g) *For any quasi-coherent sheaf of ideals $\mathcal{F}$ of finite type there exist integers $n_i, k_i > 0$ such that $\mathcal{F}$ is a quotient of $\bigoplus_{i \in I} \mathcal{A}_i^{\otimes -n_i} \otimes \mathcal{O}_X^{k_i}$.*

A scheme which admits an ample family of line bundles is called divisorial. All smooth varieties are divisorial (see [37] 6.II). More generally any normal noetherian locally $\mathbb{Q}$-factorial scheme with affine diagonal is divisorial [6].

We fix the following notation: X is a noetherian scheme, $\{\mathcal{A}_i\}$ is a finite ample family of line bundles on X, and A_i is the autoequivalence of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ which corresponds to tensoring with the sheaf $\mathcal{A}_i$. We use multi-index notation so that $\mathcal{A}^d := \mathcal{A}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{A}_r^{\otimes d_r}$ for $d \in \mathbb{N}^r$.

Definition 2.2.2. *An object $P \in D^b_{coh}(X)$ is called a* point object *with respect to a collection of autoequivalences $\{A_i\}$ if the following hold:*

i) $A_i(P) \cong P$ *for all i,*

ii) $\mathrm{Hom}^{<0}(P, P) = 0$,

iii) $\mathrm{Hom}^0(P, P) = k(P)$ *with $k(P)$ a field.*

Proposition 2.2.3. *Let $\{\mathcal{A}_i\}$ be an ample family of line bundles on a noetherian scheme X. Then P is a point object with respect to $\{A_i\}$ if and only if $P \cong \mathcal{O}_x[r]$ for some $r \in \mathbb{Z}$ and some closed point $x \in X$.*

Proof. Any structure sheaf of a closed point is clearly a point object. On the other hand suppose P is a point object for $\{A_i\}$. Let $\mathcal{H}_j$ be the j^{th} cohomology sheaf of P. Consider the map $\mu_f : P \to P \otimes \mathcal{A}^d$ given by multiplication by $f \in \Gamma(X, \mathcal{A}^d)$. Since P is a point object μ_f is either 0 or an isomorphism for any f. In particular, the induced map, $\mu_{f,j}$ on $\mathcal{H}_j$ is either 0 or an isomorphism. If $\mu_{f,j} = 0$ we have that $\mathrm{Supp}(\mathcal{H}_j) \subseteq Z(f)$ and if $\mu_{f,j}$ is an isomorphism then $\mathrm{Supp}(\mathcal{H}_j) \cap Z(f) = \emptyset$.

Now suppose that x, y are two distinct points in the support of $\mathcal{H}_j$. Since $\{\mathcal{A}_i\}$ is an ample family, the opens X_f with $f \in \Gamma(X, \mathcal{A}_i^d)$ form a basis for the Zariski topology of X. Thus there exists a function $f \in \Gamma(X, \mathcal{A}_i^d)$ for some i such that f vanishes on x but not on y, yielding a contradiction. It follows that $\mathcal{H}_j$ is supported at a point. The result then follows from [13] Lemma 4.5 (this lemma uses the noetherian assumption). $\square$

We have shown that one can recover the structure sheaves of points up to shift, now we wish to recover the line bundles up to shift. This motivates the following definition:

Definition 2.2.4. *An object $L \in D^b_{coh}(X)$ is called* invertible *for a set S if for all $P \in S$ there exists an $n_P \in \mathbb{Z}$ such that :*

$$\mathrm{Hom}(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_P \\ \\ 0 & \text{otherwise} \end{cases}$$

An invertible object with respect to a collection of autoequivalences $\{A_i\}$ is an invertible object for the set of point objects with respect to $\{A_i\}$. Let $S := \{\mathcal{O}_x[n] | x \in X$ is a closed point and $n \in \mathbb{Z}\}$. It follows from the proof of Proposition 2.4 in [5] that if X is a noetherian scheme, any invertible object $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for S is isomorphic to $\mathcal{L}[t]$ for some line bundle $\mathcal{L}$ and some $t \in \mathbb{Z}$ (the set of shifted line bundles).

Lemma 2.2.5. *A divisorial scheme of finite type over a field can be recovered from $D^b_{coh}(X)$ together with the full subcategory of objects with zero dimensional support. Furthermore if Y is a divisorial scheme of finite type over a field and $F : D^b_{coh}(X) \to D^b_{coh}(Y)$ is an equivalence which maps objects with zero dimensional support to objects with zero-dimensional support then $X \cong Y$.*

We provide two proofs of the lemma, the later proof is meant to follow more closely that found in [5] and in particular the variation of that proof found in [13].

Proof. Let S denote the full subcategory of objects with zero dimensional support. Consider objects $P \in S$ satisfying:

i) $\mathrm{Hom}^{<0}(P, P) = 0$,

ii) $\mathrm{Hom}^0(P, P) = k(P)$ with $k(P)$ a field.

Call this class of objects T. By Lemma 4.5 of [13], all such objects are isomorphic to $\mathcal{O}_x[r]$ for some $r \in \mathbb{Z}, x \in X$. As noted above, invertible objects with respect to T are precisely the objects isomorphic to shifted line bundles. Let $p^N_T := \{P \in T | \mathrm{Hom}(N, P) = k(P)\}$ where N is a fixed invertible object. We may assume $p_T^N = \{\mathcal{O}_x | x \in X\}$. We call this set X_0 and proceed by recovering the Zariski topology on this set. The line bundles are now

$$l_T^N := \{L | L \text{ is invertible and } \mathrm{Hom}(L, P) = k(P)\ \forall P \in p_T^N\}.$$

Now given any two objects $L_1, L_2 \in l_T^N$, and $\alpha \in \mathrm{Hom}(L_1, L_2)$ we get an induced map,

$$\alpha_P^* : \mathrm{Hom}(L_2, P) \to \mathrm{Hom}(L_1, P).$$

Then denote by X_α the subset of those objects $P \in p_{\{A_i\}}^N$ for which $\alpha_P^* \neq 0$. Then X_α is the complement of the zero-locus of α. By assumption letting α run over all morphisms in $\mathrm{Hom}(L_1, L_2)$ and L_1, L_2 run over all line bundles, we get a basis for the Zariski topology on X_0. From this set it is easy to see that one can add prime ideals for each irreducible closed subset to recover X together with its Zariski topology.

From here the two proofs diverge. For each open set $U \subseteq X$ we consider the full subcategory

$$D_U := \{A \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \,| \mathrm{Hom}(A, P[i]) = 0 \forall P \in U_0, i\}.$$

This is the subcategory of objects supported on $X \backslash U$. Localizing we reconstruct $D^b_{\mathrm{coh}}(U)$ i.e. $D^b_{\mathrm{coh}}(U)$ is the Verdier quotient of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ by D_U. Hence we can reconstruct the structure sheaf on X as $\mathcal{O}_X(U) := \mathrm{Hom}_{D^b_{\mathrm{coh}}(U)}(N, N)$.

The second proof requires X to be a quasi-projective variety over an algebraically closed field and uses the triangulated structure of the category. It proceeds as follows: for every object $P \in p_T^N$ we can consider morphisms $\psi \in \mathrm{Hom}(P, P[1])$. Each such morphism produces an exact triangle $\mathcal{E}_\psi \to P \to P[1]$.

Now consider finite dimensional vector subspaces $V \subseteq \mathrm{Hom}(N, M)$ for some $M \in l_T^N$ such that,

i) For all $P, Q \in p_T^N$ there exists $f \in V$ such that $f_P^* : \mathrm{Hom}(M, P) \to \mathrm{Hom}(N, P) \neq 0$ and $f_Q^* : \mathrm{Hom}(M, Q) \to \mathrm{Hom}(N, Q) = 0$,

ii) For all $P \in p_T^N$ and all $\psi \in \mathrm{Hom}(P, P[1])$ there exists a $\phi \in \mathrm{Hom}(M, \mathcal{E}_\psi)$ such that $\mathrm{Hom}(N, \bullet)(\phi)(W_P) \neq 0$ where $W_P := \{v \in V | v_P^* = 0\}$.

The first condition says that V separates points the second says that it separates tangent vectors hence V gives an embedding into projective space and we recover the scheme structure on X.

Now when $F : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ is an equivalence and $F^{-1} \circ (\bullet \otimes \mathcal{M}_i) \circ F = (\bullet \otimes \mathcal{A}_i)$. Then F takes point objects with respect to $\{(\bullet \otimes \mathcal{A}_i)\}$ to point objects with respect to $\{(\bullet \otimes \mathcal{M}_i)\}$. Since for any $y \in Y, n \in \mathbb{Z}$ the objects $\mathcal{O}_y[n]$ are point objects with respect to $\{(\bullet \otimes \mathcal{M}_i)\}$ we have that $F(\mathcal{O}_y[n]) \cong \mathcal{O}_x[r]$. Hence we get a set theoretic map $X \to Y$. This map is clearly injective as F is an equivalence. Furthermore it is surjective since the collection of objects $\{\mathcal{O}_y[n] | y \in Y, n \in \mathbb{Z}\}$ has the property that for any object $B \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ there exists an object $\mathcal{O}_y[n]$ such that $\mathrm{Hom}(B, \mathcal{O}_y[n]) \neq 0$. After getting a bijection, all of the above reconstructions are identical. In particular the sections of $\{\mathrm{Hom}(N, N \otimes \mathcal{M}_i^d) | d \in \mathbb{N}^r)$ form a basis for the topology on Y hence $\{(\bullet \otimes \mathcal{M}_i)\}$ is also an ample family. It follows that $X \cong Y$. □

Remark 2.2.6. *Notice that the proof applies to a larger class of schemes than just those of finite type over a field. Instead suppose X is a noetherian scheme and let X_0 be the set of closed points of X. The proof requires that the prime ideals are in bijection with irreducible closed subsets of the set of closed points. Many of the proofs below apply to this situation as well.*

We now arrive at an analogous theorem to that in [5],

Theorem 2.2.7. *Let X be a divisorial scheme of finite type over a field. Then X can be reconstructed from its derived category of coherent sheaves together with a collection of autoequivalences corresponding to an ample family of line bundles. Let Y be a divisorial scheme of finite type over a field, $F : D^b_{coh}(X) \to D^b_{coh}(Y)$ an equivalence, $\{\mathcal{A}_i\}$ an ample family of line bundles on X, and $\{\mathcal{M}_i\}$ any collection of line bundles on Y. If $F^{-1} \circ (\bullet \otimes \mathcal{M}_i) \circ F = (\bullet \otimes \mathcal{A}_i)$ then $X \cong Y$.*

Proof. This follows immediately from Proposition 3.3 and Lemma 3.5.

Once again we provide a second proof which resembles more closely the one found in [5]. For this proof we use the ample family of line bundles to reconstruct X as an open subset of its multigraded coordinate ring as follows. First we recover the Zariski topology on X as above, recall that this requires the choice of an invertible object N. From this object we recover the multigraded coordinate ring as $S := \bigoplus_{d \in \mathbb{N}^r} \mathrm{Hom}(N, A^d(N))$. Note that $\forall \alpha \in S$ we have the open set X_α. Now we use the open embedding of X into $\mathrm{Proj}(S)$ to recover the scheme structure. Recall from [6] that $\mathrm{Proj}(S)$ is defined as the union of the affine sets S_α where $\alpha \in S$ is relevant. As we have recovered S, we can consider just the relevant $\alpha \in S$ and hence recover $\mathrm{Proj}(S)$. Now for a divisorial variety X, there is an open embedding $X \to \mathrm{Proj}(S)$ ([6] Theorem 4.4). Now it is a fact that for relevant α, X_α is an open subset of S_α with equality if and only if X_α is affine ([6] Corollary 4.5 and Corollary 4.6). Now for each $x \in X$, as $\{x\}$ is closed, we can write x as a complement $x = (\bigcup_{\beta \in I_x} X_\beta)^c$ for some indexing set I_x. So given a relevant $\alpha \in S$ we have that X_α is affine if and only if every closed point of S_α is expressed as $(\bigcup_{\beta \in I_x} (S_\alpha)_\beta)^c \subseteq \mathrm{Proj}(S)$ for the indexing set corresponding to some $x \in X_\alpha$. This recovers the affine X_α. The scheme structure on X is recovered as the open subscheme of $\mathrm{Proj}(S)$ given by the union of the affine $X_\alpha = S_\alpha$. □

Remark 2.2.8. *If X and Y are divisorial schemes of finite type over a a field and $F : D^b(X) \to D^b(Y)$ is an equivalence, then any ample family of line bundles $\{\mathcal{A}_i\}$ on X induces a collection of autoequivalences $\{F \circ (\bullet \otimes \mathcal{A}_i) \circ F^{-1}\} \in Aut(D^b_{coh}(Y))$. The category $D^b_{coh}(Y)$ together with this family of autoequivalences will reconstruct the space X via the above procedure. Thus spaces with equivalent derived categories are somehow encoded in the autoequivalences of the category. However, it is not clear to the author whether or not it is possible to give nice categorical conditions on a collection of autoequivalences that insures it comes from an ample family of line bundles.*

Corollary 2.2.9. *Suppose X is a scheme of finite type over a field with ample trivial bundle i.e. X is quasi-affine. Then X can be reconstructed from $D^b_{coh}(X)$. If Y is any divisorial scheme of finite type over a field such that $D^b_{coh}(X)$ is equivalent to $D^b_{coh}(Y)$ then $X \cong Y$.*

Proof. The functor $(\bullet \otimes \mathcal{O}_X)$ is the identity functor, hence for any equivalence $F : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ we have $F^{-1} \circ (\bullet \otimes \mathcal{O}_Y) \circ F \cong (\bullet \otimes \mathcal{O}_X)$. □

Our aim now is to reconstruct an abelian variety from its derived category of D-modules, let

$\mathfrak{Prop}(X)$ denote the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ consisting of objects supported on proper subvarieties of X.

Theorem 2.2.10. *Let X be a k-variety. Then $\mathfrak{Prop}(X)$ is equivalent to the full subcategory of $D^b_{coh}(X)$ consisting of objects $A \in D^b_{coh}(X)$ with the property that* $\operatorname{Hom}(A, B)$ *is finite dimensional over k for all $B \in D^b_{coh}(X)$. Hence as either k-linear graded or k-linear triangulated categories, $\mathfrak{Prop}(X)$ can be recovered from $D^b_{coh}(X)$.*

Proof. Suppose $A^\bullet$ is supported on a proper subscheme. Consider the spectral sequence,

$$E_2^{(p,q)} = \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(\mathrm{H}^{-q}(A^\bullet), B^\bullet[p]) \Rightarrow \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A^\bullet, B^\bullet[p+q]).$$

Each term in the spectral sequence is a finite dimensional vector space hence $\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A^\bullet, B^\bullet)$ is finite dimensional.

On the other hand, suppose $A^\bullet$ is supported on a non-proper subscheme. Let m be the greatest integer such that $\mathrm{H}^m(A^\bullet)$ is supported on a non-proper subscheme. Since the support is not proper, there exists an affine curve $C \subseteq \operatorname{Supp}(\mathrm{H}^m(A^\bullet))$. Let i denote the inclusion map, $i : C \to \operatorname{Supp}(\mathrm{H}^m(A^\bullet))$. We compute (the pullback and pushforward are not derived, we just take the sheaf concentrated in degree zero),

$$\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A^\bullet, i_* i^* \mathrm{H}^m(A^\bullet)).$$

Using the same spectral sequence as before we have,

$$E_2^{(p,q)} = \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(\mathrm{H}^{-q}(A^\bullet), i_* i^* \mathrm{H}^m(A^\bullet)[p]) \Rightarrow \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A^\bullet, i_* i^* \mathrm{H}^m(A^\bullet)[p+q]).$$

Notice that since C is affine $E_2^{(0,-m)} \cong \operatorname{Hom}(i^*\mathrm{H}^m(A^\bullet), i^*\mathrm{H}^m(A^\bullet))$ is the endomorphism ring of a module supported on all of C. Hence this is an infinite dimensional vector space and furthermore all the terms below it are finite dimensional. Hence $E_\infty^{(0,-m)}$ is infinite dimensional. Thus there is a filtration of $\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A^\bullet, i_* i^* \mathrm{H}^m(A^\bullet)[-m])$ which contains an infinite dimensional vector space. Hence

$$\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A^\bullet, i_* i^* \mathrm{H}^m(A^\bullet)[-m])$$

is infinite dimensional. □

Remark 2.2.11. *This statement is formally similar to a result of Orlov's in [27]. This result says that the category* $\mathfrak{Perf}(X)$ *formed by perfect complexes can be recovered from the (unbounded) derived category of coherent sheaves. That is, the triangulated subcategory formed by perfect complexes is precisely the full subcategory of homologically finite objects.*

Theorem 2.2.12. *Let* X *and* Y *be divisorial varieties. Let* $\omega_{X_{sm}}$ *be the canonical bundle of the smooth locus of* X*. Suppose that any proper closed positive dimensional subvariety* Z *is contained in* X_{sm} *and* $\omega_{X_{sm}}$ *restricted to* Z *is either ample or anti-ample, then* X *can be reconstructed from its derived category (as a* k*-linear graded category). Furthermore if* $D^b_{coh}(X)$ *is equivalent to* $D^b_{coh}(Y)$ *then* $X \cong Y$*.*

Proof. By Theorem 3.8 we can recover $\mathfrak{Prop}(X)$ from $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Now consider all properly supported objects which are point objects with respect to the identity. Call this class $\mathcal{C}$. Notice that the structure sheaf of any proper subvariety is in $\mathcal{C}$. Suppose $A \in \mathcal{C}$ is orthogonal to all other objects in $\mathcal{C}$ then the support of A is not contained in any positive dimensional proper subvariety and by Lemma 4.5 of [13] it must be the structure sheaf of a point. Hence we have recovered the structure sheaves of points (up to shift) which are not contained in a positive dimensional proper subvariety. Now take the (left) orthogonal to this collection of structure sheaves in $\mathfrak{Prop}(X)$. This is the category of objects whose support is contained in a positive dimensional proper subvariety. This category comes equipped with a Serre functor given by $(\cdot \otimes \omega_{X_{sm}})[\dim X_{sm}]$. By assumption (after shifting) this functor acts as tensoring with an ample or anti-ample line bundle. It follows from the proof of Proposition 3.3 that using this functor we can recover the objects isomorphic to structure sheaves of closed points (up to shift) which are contained in a positive dimensional proper subvariety. Hence we can recover the structure sheaves of closed points on X (up to shift). The statement follows from the proof of Lemma 3.5. □

Theorem 2.2.13 (Arinkin)**.** *An abelian variety* A *can be reconstructed from its derived category* *of coherent D-modules. If two abelian varieties A and B have equivalent derived categories of coherent D-modules then $A \cong B$*

Proof. Let $\mathfrak{g} = \mathrm{H}^1(A, \mathcal{O}_A)$. Then there is a tautological extension

$$0 \to \mathfrak{g}^* \otimes \mathcal{O}_A \to \mathcal{E} \to \mathcal{O}_A \to 0$$

which corresponds to the identity of $\mathrm{End}(\mathfrak{g}^*) = \mathrm{Ext}^1(\mathcal{O}_A, \mathfrak{g}^* \otimes \mathcal{O}_A)$. Let $A^\natural$ be the $\mathfrak{g}^*$-principal bundle associated to the extension $\mathcal{E}$. Then the derived category of D-modules on the dual abelian variety $\hat{A}$ is equivalent to the category of coherent sheaves on $A^\natural$ [22, 36, 31].

We now show that the space $A^\natural$ has only finite sets of points as proper subvarieties. Hence from either of the previous corollaries we can recover $A^\natural$ and use Hodge theory to recover A then dualize to recover $\hat{A}$.

Suppose $P \subseteq A^\natural$ is a proper subvariety. Since $A^\natural$ is an affine bundle over A, the projection $\pi : P \to A$ is finite and the pullback of $A^\natural$ to P will have a section i.e. it will be a trivial affine bundle. Now $A^\natural$ is represented by the ample class $\mathrm{Id} \in \mathrm{End}(\mathfrak{g}^*) = \mathrm{Ext}^1(\mathcal{O}_A, \mathfrak{g}^* \otimes \mathcal{O}_A) = \mathrm{H}^1(A, \Omega^1_A)$. Now since $\pi : P \to A$ is finite, the projection of $\pi^*(\mathrm{Id}) \in \mathrm{H}^1(P, \pi^*\Omega^1_A)$ onto $\mathrm{H}^1(P, \Omega^1_P)$ is also ample. The only way an ample class on P can be zero is if it is a finite set of points. □

Remark 2.2.14. *Actually over $\mathbb{C}$, $A^\natural$ is a Stein space and hence we see immediately that finite sets of points are the only proper closed subvarieties.*

2.3 Autoequivalences and Fourier-Mukai partners

In this section we consider the case of smooth projective varieties. For projective varieties, a single ample line bundle gives an ample family. Furthermore due to a famous result of Orlov [26] generalized by Canonaco and Stellari [9], any equivalence between derived categories of smooth projective varieties is a Fourier-Mukai transform. Hence in what follows when both varieties in question are smooth and projective we say only that they are equivalent and often use the fact that the equivalence is a Fourier-Mukai transform. Now suppose that we have two varieties X and Y. Any equivalence

$$F : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$$

induces an isomorphism of the groups of autoequivalences,

$$\begin{aligned} F_* : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) &\to \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)) \\ \Phi \qquad &\mapsto \quad F \circ \Phi \circ F^{-1}. \end{aligned}$$

Suppose $\mathcal{A}$ is an ample line bundle on X. We saw above that if $F_*(A) = (\bullet \otimes \mathcal{M})$ then $X \cong Y$. In fact when F is a Fourier-Mukai transform we can say more. Namely, if a Fourier-Mukai transform Φ_P takes skyscraper sheaves of points to shifted skyscraper sheaves of points we have that $\Phi_P \cong \gamma_* \circ (\bullet \otimes \mathcal{N})[s] =: (\gamma, \mathcal{N})[s]$, for some isomorphism $\gamma : X \xrightarrow{\sim} Y$, $\mathcal{N} \in \mathrm{Pic}(X)$ and $s \in \mathbb{Z}$ [13, 16]. Hence if $F_*(A) = (\bullet \otimes \mathcal{M})$ then $F \cong (\gamma, \mathcal{N})[s]$. For the sake of applications we also want to consider the more general situation in which $F_*(A) = (\tau, \mathcal{L})[r]$. First we show that if $F_*(A) = (\tau, \mathcal{L})[r]$ then $r = 0$ and τ^n is the identity for some $n \in \mathbb{Z}$. Replacing A by $\underbrace{(\bullet \otimes \mathcal{A}) \circ \cdots \circ (\bullet \otimes \mathcal{A})}_{n \text{ times}}$ we get the following,

Lemma 2.3.1. *Let X and Y be a smooth projective varieties. Let $\mathcal{A}$ be an ample line bundle on X, $\tau \in \mathrm{Aut}(Y)$, and $\mathcal{L} \in \mathrm{Pic}\ Y$ and suppose we have an equivalence $F : D^b_{coh}(X) \cong D^b_{coh}(Y)$ and $F_*((\bullet \otimes \mathcal{A})) = (\tau, \mathcal{L})[r]$ for some $r \in \mathbb{Z}$. Then $F \cong (\gamma, \mathcal{N})[s]$ for some line bundle $\mathcal{N} \in \mathrm{Pic}(Y)$, an isomorphism $\gamma : X \xrightarrow{\sim} Y$, and $s \in \mathbb{Z}$.*

Proof. Take an arbitrary $A \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ then if $r \neq 0$ by considering the homology sheaves of A we notice that A is not isomorphic to $\tau_*(A \otimes \mathcal{L})[r] = (\tau, \mathcal{L})[r](A)$. However for any $y \in Y, \mathcal{O}_y \cong (\mathcal{O}_y \otimes \mathcal{A})$. Hence $r = 0$.

Let $F \cong \Phi_P$ and $F^{-1} \cong \Phi_Q$ for some $P, Q \in D^b(X \times Y)$. Then $P \boxtimes Q \in D^b(X \times X \times Y \times Y)$ defines an equivalence $\Phi_{P \boxtimes Q} : D^b(X \times X) \to D^b(Y \times Y)$ [13]. It follows easily from the formula for composition of Fourier-Mukai transforms that $\Phi_{P \boxtimes Q}(S) \cong T$ where $F_*(\Phi_S) = \Phi_T$ with Φ_S

and Φ_T autoequivalences. Let Δ be the diagonal map and $\delta_n^\tau := \tau_* \mathcal{L} \otimes \tau_*^2 \mathcal{L} \otimes ... \otimes \tau_*^n \mathcal{L}$. Then we have,

$$\underbrace{(\bullet \otimes \mathcal{A}) \circ \cdots \circ (\bullet \otimes \mathcal{A})}_{n \text{ times}} \cong \Phi_{\Delta_*(\mathcal{A}^{\otimes n})} \text{ and } \underbrace{(\tau, \mathcal{L}) \circ \cdots \circ (\tau, \mathcal{L})}_{n \text{ times}} \cong \Phi_{(\mathrm{id} \times \tau^n)_* \delta_n^\tau}$$

Therefore $\Phi_{P \boxtimes Q}(\Delta_*(\mathcal{A}^{\otimes n})) \cong (\mathrm{id} \times \tau^n)_* \delta_n^\tau$ by uniqueness of the Fourier-Mukai kernel. Let Z_n denote the fixed locus of τ^n. Then we have,

$$\mathrm{Ext}^i_X(\mathcal{O}_X, \mathcal{A}^{\otimes n}) \cong \mathrm{Ext}^i_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{A}^{\otimes n}) \cong \mathrm{Ext}^i_{X \times X}(\Phi_{P \boxtimes Q}(\Delta_* \mathcal{O}_X), \Phi_{P \boxtimes Q}(\Delta_* \mathcal{A}^{\otimes n})) \cong$$

$$\mathrm{Ext}^i_{Y \times Y}(\Delta_* \mathcal{O}_Y, (\mathrm{id} \times \tau^n)_* \delta_n^\tau) \cong H^i(Z_n, \Delta^*(\mathrm{id} \times \tau^n)_* \delta_n^\tau).$$

In particular there exists an n such that $Z_n \neq \emptyset$. Then for $z \in Z_n$, $\mathcal{O}_z$ is a point object with respect to $\tau^n \circ (\bullet \otimes \delta_n^\tau)$. Hence $\Phi_Q(\mathcal{O}_z) \cong \mathcal{O}_x[r]$ for some $x \in X, r \in \mathbb{Z}$. Using [13] Corollary 6.12, we gain the existence of an open set U and a morphism $f : U \to X$ such that for any $y \in U$ we have that $\Phi_Q(\mathcal{O}_y) \cong \mathcal{O}_{f(y)}[r]$. Moreover, f must be injective since F is an equivalence. Let $v \in f(U)$, then as $\mathcal{O}_v \otimes \mathcal{A}^{\otimes n} \cong \mathcal{O}_v$ we have $\tau_*^n(\mathcal{O}_{f^{-1}(v)} \otimes \delta_n^\tau) \cong \mathcal{O}_{\tau^n(f^{-1}(v))} \cong \mathcal{O}_{f^{-1}(v)}$. Therefore $f^{-1}(v)$ is a fixed point. Therefore the fixed locus contains U, but as it is closed and X is irreducible, the fixed locus is the whole space, i.e. $\tau^n = \mathrm{id}$. □

Remark 2.3.2. *If $\mathcal{A}$ is not ample then the above is not necessarily true: if $\mathcal{P} \in D^b_{coh}(A \times \hat{A})$ is the Poincar line bundle on an abelian variety A and t_a is translation by $a \in A$, then $\Phi^*_{\mathcal{P}}(t_a) = (\bullet \otimes \mathcal{L})$ where $\mathcal{L}$ is a degree zero line bundle.*

To illustrate how such a statement can be utilized to bound the number of Fourier-Mukai partners of a given variety we provide some easy corollaries here,

Corollary 2.3.3. *The number of projective Fourier-Mukai partners of a smooth projective variety is bounded by the number of conjugacy classes of maximal abelian subgroups of* $\mathrm{Aut}(D^b_{coh}(X))$.

Proof. The Picard group is always abelian and hence the Picard group of any Fourier-Mukai partner is contained in one of these conjugacy classes (under some equivalence). If the Picard groups of two Fourier-Mukai partners, Y and Z, are contained in the same conjugacy class then by modifying an equivalence we may assume the two Picard groups lie in the same maximal abelian subgroup. In particular under a suitable equivalence we have $\mathrm{Pic}(Y) \subseteq \mathrm{Aut}(D^b_{\mathrm{coh}}(Z))$ commutes with an ample line bundle on Z. Hence any element of $\mathrm{Pic}(Y)$ is of the form $(\gamma, \mathcal{N})[s]$ as an element of $\mathrm{Aut}(D^b_{\mathrm{coh}}(Z))$. In particular an ample line bundle in $\mathrm{Pic}(Y)$ is mapped to an element of this form. □

Using the same reasoning, one could also say,

Corollary 2.3.4. *Suppose X is a smooth projective variety such that for every $v \in \mathrm{Aut}(D^b_{coh}(X))$ there exists a power of v under composition which is conjugate to $(\gamma, \mathcal{N})[s]$ for some $\gamma \in \mathrm{Aut}(X)$, $\mathcal{N} \in \mathrm{Pic}(X)$, and $s \in \mathbb{Z}$. Then X has no non-trivial Fourier-Mukai partners.*

Proof. Suppose Y is a smooth projective variety and $F : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Let $\mathcal{A}$ be an ample line bundle on Y. Then by hypothesis there exists an n such that $F \circ ((\bullet \otimes \mathcal{A}^n)) \circ F^{-1} = t^{-1} \circ (\gamma, \mathcal{N})[s] \circ t$, for some $t \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X))$. So we have $(F \circ t)^*((\bullet \otimes \mathcal{A}^n)) = (\gamma, \mathcal{N})[\mathrm{s}]$. □

For example, for all projective varieties with ample or anti-ample canonical bundle we have that

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) = \mathbb{Z} \times \mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)$$

where $\mathbb{Z}$ acts by the shift functor, the proof of this statement can be found in [5] or can be seen directly the fact that the Serre functor commutes with all autoequivalences. Likewise we see that we can reconstruct such varieties as our earlier reconstruction theorem is just a generalization of the result in [5]. In any case, we see that such varieties have no non-trivial Fourier-Mukai partners. The ideas of the above corollaries lead us to our main result but first we need a lemma,

Lemma 2.3.5. *Let X be a smooth projective variety over $\mathbb{C}$ and ρ be the representation of $\mathrm{Aut}(D^b_{coh}(X))$ on $\mathrm{H}^*(X, \mathbb{Q})$. The image of $\rho : \mathrm{Aut}(D^b_{coh}(X)) \to \mathrm{Gl}(\mathrm{H}^*(X, \mathbb{Q}))$ is an arithmetic group.*

Proof. The Fourier-Mukai autoequivalences act on the topological K-theory of the space [16]. Topological K-theory is a finitely generated abelian group and its image under the Mukai vector map is a full sublattice of $\mathrm{H}^*(X, \mathbb{Q})$. Hence the image of ρ preserves this full sublattice. □

We are now ready to prove our main result,

Theorem 2.3.6. *Let X be a smooth projective variety over $\mathbb{C}$ and ρ be the representation of $\mathrm{Aut}(D^b_{coh}(X))$ on $\mathrm{H}^*(X, \mathbb{Q})$. If $\ker \rho = 2\,\mathbb{Z} \times \mathrm{Pic}^0(X) \ltimes \mathrm{Aut}^0(X)$ then the number of projective Fourier-Mukai partners of X is bounded by the number of conjugacy classes of maximal unipotent subgroups of $\rho(\mathrm{Aut}(D^b_{coh}(X)))$. In particular since $\mathrm{im}\ \rho$ is an arithmetic group it is finite.*

Proof. First observe that given any equivalence of categories between $F : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ the conjugation F_* induces an isomorphism of exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \rho_X & \longrightarrow & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) & \xrightarrow{\rho_X} & \mathrm{im}\ \rho_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker \rho_Y & \longrightarrow & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)) & \xrightarrow{\rho_Y} & \mathrm{im}\ \rho_Y & \longrightarrow & 0 \end{array}$$

In particular we have an isomorphism $F_* : \ker \rho_X \xrightarrow{\sim} \ker \rho_Y$. A theorem of Rouquier (see [13, 33]) states that F_* induces an isomorphism of algebraic groups $F_* : \mathrm{Pic}^0(X) \ltimes \mathrm{Aut}^0(X) \xrightarrow{\sim} \mathrm{Pic}^0(Y) \ltimes \mathrm{Aut}^0(Y)$. The theorem is also proved by Rosay in his thesis. Hence the condition that $\ker \rho = 2\,\mathbb{Z} \times \mathrm{Pic}^0(X) \ltimes \mathrm{Aut}^0(X)$ is true for all Fourier-Mukai partners of X.

Let Y and Z be two projective Fourier-Mukai partners of X with ample line bundles $\mathcal{A}_Y$ and $\mathcal{A}_Z$ and fix equivalences $F : Y \to X, G : Z \to X$. Notice that the action of an ample line bundle on the cohomology of a space is given by multiplication by the Chern character and thus is unipotent. Hence $\rho(F_*(\bullet \otimes \mathcal{A}_Y)) := y$ and $\rho(G^*(\bullet \otimes \mathcal{A}_Z)) := z$ are both unipotent. Now suppose they lie in the same conjugacy class of maximal unipotent subgroups. Then by altering one of the equivalences by an autoequivalence, we may assume they lie in the same maximal unipotent subgroup. Now the lower central series of any unipotent group terminates, in particular the commutator $[y^{-1}, [y^{-1}, [y^{-1}, ...[y^{-1}, z]] = 1$. Denote this commutator by b_0, removing the last commutator denote the next expression by b_1 i.e. $b_1 := [y^{-1}, [y^{-1}, ...[y^{-1}, z]]$.

Define b_i similarly so that we have $y^{-1}b_iyb_i^{-1} = b^{i-1}$ or $b_iyb_i^{-1} = yb^{i-1}$. For each i choose an element B_i such that $\rho_X(F_*(B_i)) = b_i$ and similarly define ψ so that $\rho_X(F_*(\psi)) = z$. By induction assume that $B_i \cong (\mathcal{N}_i, \gamma_i)[s_i]$ for some line bundle $\mathcal{N}_i \in \mathrm{Pic}(Y)$ $\gamma_i \in \mathrm{Aut}(Y)$ and $s_i \in \mathbb{Z}$. Since $b_0 = 1$ this the induction hypothesis is satisfied for $i = 0$. Now pulling back the equation $b_iyb_i^{-1} = yb^{i-1}$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ we have $B_{i*}((\bullet \otimes \mathcal{A}_Y)) = (\bullet \otimes \mathcal{A}_Y) \circ (\mathcal{N}_{i-1}, \gamma_{i-1})[2r_{i-1}] = (\gamma^*(\mathcal{A}_Y) \otimes \mathcal{N}_{i-1}, \gamma_{i-1})[2r_{i-1}]$. Hence by Lemma 4.1 $B_i \cong (\mathcal{N}_i, \gamma)[s_i]$ for some $\mathcal{N}_i \in \mathrm{Pic}(Y)$, $\gamma_i \in \mathrm{Aut}(Y)$, and $s_i \in \mathbb{Z}$. We conclude that $\psi \cong (\mathcal{N}, \gamma)[s]$ for some $\mathcal{N} \in \mathrm{Pic}(Y)$, $\gamma \in \mathrm{Aut}(Y)$, and $s \in \mathbb{Z}$. Finally, $F_*^{-1} \circ G_*((\bullet \otimes \mathcal{A}_Z)) = \psi \circ \phi$ with $\phi \in \ker \rho_Y$. Hence $(F^{-1} \circ G)_*((\bullet \otimes \mathcal{A}_Z)) = (\mathcal{L}, \tau)[t]$ for some $\mathcal{L} \in \mathrm{Pic}(Y)$, $\tau \in \mathrm{Aut}(Y)$, and $t \in \mathbb{Z}$. Applying Lemma 4.1 one more time we get $Y \cong Z$ hence the result. The fact that the number of conjugacy classes of maximal unipotent subgroups of an arithmetic group is finite is well-known[1]. □

Remark 2.3.7. *A weakness of this result is that many important examples do not satisfy the hypotheses. For example on an even dimensional variety the square of a spherical twist acts trivially on cohomology [38] and similarly any $\mathbb{P}^n$-twist acts trivially on cohomology [15]. However one may be able to overcome this problem. For example, if one could show that for any projective variety X there exists a splitting, s, of $Aut(D^b_{coh}(X))/Pic^0(X) \ltimes Aut^0(X) \twoheadrightarrow im\ \rho$ such that there exists an ample line bundle $\mathcal{A}$ with $[(\bullet \otimes \mathcal{A})] \in s(im\ \rho)$ then the result would hold for all projective varieties. Or perhaps if one could show such a result for certain types of kernels e.g. those generated by squares of spherical twists and $2\,\mathbb{Z} \times Pic^0(X) \ltimes Aut^0(X)$, then the result would hold for those varieties with those types of kernels.*

We now apply our theorem to the case of abelian varieties. The autoequivalences of the derived category of an abelian variety have been satisfactorily described in [12] and [25]. It is this understanding of autoequivalences that allows us to declare the number of Fourier-Mukai partners

[1]Stated this way the result can be found as [23] Corollary 9.38. It is equivalent to Theorem 9.37 of [23] which says that for any parabolic subgroup P, and arithmetic group Γ of an algebraic group G the double-coset space $\Gamma \backslash G_{\mathbb{Q}}/P_{\mathbb{Q}}$ is finite. The latter statement can be found for example in [3] Theorem 15.6 or [32] Theorem 13.26.

of an abelian variety to be finite and give an explicit bound in the case where the Neron-Severi group of the abelian variety is $\mathbb{Z}$.

Theorem 2.3.8. *Let A be an abelian variety over $\mathbb{C}$. Then the number of Fourier-Mukai partners of A is finite. Furthermore suppose that the Neron-Severi group of A is $\mathbb{Z}$. Let $\mathcal{L}$ be a generator of the Neron-Severi group of A and $\mathcal{M}$ be a generator of the Neron-Severi group of $\hat{A}$. As ample bundles, $\mathcal{L}$ and $\mathcal{M}$ induce isogenies $\Phi_{\mathcal{L}}$ and $\Phi_{\mathcal{M}}$ moreover $\Phi_{\mathcal{L}} \circ \Phi_{\mathcal{M}} := N \cdot \mathrm{Id}$. Then the number of smooth projective Fourier-Mukai partners of A is bounded by $\sum_{d|N} \phi(\gcd(d, \frac{N}{d}))$. If N is square free then all projective Fourier-Mukai partners are abelian varieties and the bound is attained, i.e. the number of such partners is just $\sum\limits_{d|N} \phi(\gcd(d, \frac{N}{d})) = \sum\limits_{d|N} 1 = 2^s$ where s is the number of prime factors of N.*

Proof. The conditions of Theorem 4.5 are satisfied for abelian varieties [13, 25, 12]. We show how to calculate the bound when $\mathrm{NS}(A) = \mathbb{Z}$. Let $U(A) := \{M \in \mathrm{Aut}(A \times \hat{A}) |\ M^{-1} = \det(M)\overline{M^{-1}}\}$ denote the Polishchuk group. For simplicity we start with the case $\mathrm{End}(A) = \mathbb{Z}$, so that $U(A) = \Gamma_0(N)$. For $N = 1$ we let $\Gamma_0(1) := \mathrm{Sl}_2 \mathbb{Z}$, so that this case is also included. For abelian varieties, im ρ is commonly notated as $\mathrm{Spin}(A)$ we use this convention.

Now we reduce the study of maximal unipotent subgroups of $\mathrm{Spin}(A)$ to maximal unipotent subgroups of $\Gamma_0(N)$ as follows. For an abelian variety we have the following diagram [12, 13],

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathbb{Z}/2\mathbb{Z} & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & 2\mathbb{Z} \times A \times \hat{A} & \longrightarrow & \mathrm{Aut}(D^b(A)) & \xrightarrow{\rho} & \mathrm{Spin}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} \times A \times \hat{A} & \longrightarrow & \mathrm{Aut}(D^b(A)) & \longrightarrow & \Gamma_0(N) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

We have an isomorphism of algebras $\mathrm{End}(\mathrm{H}^*(A)) \cong \mathrm{Cl}(\Lambda, Q)$ where $\Lambda := H_1(A) \oplus H_1(\hat{A})$ and Q is the canonical quadratic form [12]. Hence $F_*((\bullet \otimes \mathcal{A}))$ for some ample line bundle $\mathcal{A}$ corresponds to $1 + N \in \mathrm{Cl}(\Lambda, Q)$. This induces an action on Λ given by $v \mapsto (1+N)v(1 - N + N^2 - N^3 + ...)$. Hence we also have a unipotent element of $\mathrm{Aut}(A \times \hat{A}) = \Gamma_0(N)$. This element is non-trivial since the kernel of the map $\mathrm{Spin}(A) \to \Gamma_0(N)$ is just $\{\pm 1\}$. Taking powers of this element yields a collection of non-trivial conjugacy classes of unipotent matrices in $\Gamma_0(N)$. Now suppose that two unipotent elements of $\mathrm{Spin}(A)$ are conjugate in $\Gamma_0(N)$ then they are conjugate up to sign in $\mathrm{Spin}(A)$ but this means they are conjugate because the negative of a unipotent is not unipotent. Furthermore the maximal unipotent subgroups of $\Gamma_0(N)$ are infinite cyclic. Therefore if two unipotent elements of $\mathrm{Spin}(A)$ lie in the same conjugacy class of maximal unipotent subgroup in $\Gamma_0(N)$ some power of them is conjugate and hence this is also true in $\mathrm{Spin}(A)$. Thus the number of maximal unipotent subgroups of $\mathrm{Spin}(A)$ is bounded by the number of maximal unipotent subgroups of $\Gamma_0(N)$.

Now a matrix in $\Gamma_0(N)$ is unipotent if and only if the trace is 2 if and only if the action of the matrix on the upper half plane is parabolic. Such a matrix fixes a unique cusp on the boundary of the half plane. We identify two such cusps $z_1 \sim z_2 \Leftrightarrow \exists \gamma \in \Gamma_0(N)$ such that $\gamma(z_1) = z_2$. Now suppose $z_1 \sim z_2$ then for a unipotent matrix, $B \in \Gamma_0(N)$, B fixes z_1 if and only if $\gamma^{-1}B\gamma$ fixes z_2. Notice also that all powers of B fix z_1. Hence the number of conjugacy classes of unipotent matrices together will all their powers is in bijection with the number of classes of cusps. This is a well studied phenomenon. The number of such classes of cusps is precisely the bound given, $\sum_{d|N} \phi(\gcd(d, \frac{N}{d}))$ ([10], pg. 103).

If we allow complex multiplication we note that the condition to be in $U(A)$ is that $A^{-1} = \det(A)\overline{A^{-1}}$. For a unipotent matrix, this just means the matrix is real. Since $NS(A) = \mathbb{Z}$ being real means that we have integer entries, therefore unipotent matrices can be taken to lie in $\Gamma_0(N)$. □

Chapter 3

Dimensions of Triangulated Categories

In [34], R. Rouquier introduced the notion of dimension of a triangulated category. Roughly, this is the infimum over all generators of the minimal number of triangles it takes to get from the generator to any other object.

Under some mild hypotheses on X, R. Rouquier has shown that the dimension of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is always finite, it is bounded below by the dimension of the variety and on a smooth variety it is bounded above by twice the dimension of the variety [34].

The following conjecture is due to D. Orlov,

Conjecture 3.0.9. *Let X be a variety. Then* $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \dim(X)$.

In [34], Rouquier has shown that Conjecture 3.0.9 is true for smooth affine varieties, projective space, and smooth quadrics. Recently D. Orlov has proven that this conjecture is true for algebraic curves [28].

Below we will see that if a variety X possesses a tilting object the dimension of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is bounded above by the global dimension of the endomorphism algebra of the tilting object. The following theorem describes the global dimension of the endomorphism algebra of a tilting object on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$,

Theorem 3.0.10. *Suppose X is a smooth variety over k and T is a tilting object on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Let i be the largest i for which $\mathrm{Ext}^i(T, T \otimes \omega_X^{-1}) \neq 0$. Then the Hochschild dimension of $\mathrm{End}(X)$ is equal to $\dim(X) + i$. In particular, if $i = 0$ then $\mathrm{gd}(\mathrm{End}(X)) = \dim(X) = \mathrm{hd}(\mathrm{End}(X))$ and if X is proper over a perfect field then $\mathrm{gd}(\mathrm{End}(X)) = \dim(X) + i = \mathrm{hd}(\mathrm{End}(X))$.*

As a corollary we prove that Conjecture 3.0.9 is true for Fano toric Deligne-Mumford stacks of dimension two and/or Picard number less than or equal to two.

We also prove in this chapter that Conjecture 3.0.9 holds for Hirzebruch surfaces and affine varieties with rational Gorenstein singularities.

3.1 Preliminaries

In this section we introduce some of the necessary background and gather the theorems which will be of importance to us later on.

3.1.1 Generalities on Triangulated Categories

Definition 3.1.1. *Let X be an algebraic variety. An object $T \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is called a* ***tilting object*** *if the following three conditions hold:*

1. $\mathrm{Hom}(T, T[i]) = 0$ for all $i > 0$;
2. T *generates* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
3. $\mathrm{End}(T)$ *is finitely generated and coherent.*

Proposition 3.1.2. *Let T be a tilting object and $A := \mathrm{End}(T)$. Then the functors $\mathrm{RHom}(T, \bullet)$ and $\bullet \otimes_A T$ define an equivalence $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \cong \mathrm{D}^{\mathrm{b}}(\text{mod-}A)$.*

Proof. Consider the composition $\Psi := \mathrm{RHom}(T, \bullet \otimes_A T)$. We have,

$$\Psi(A) = \mathrm{RHom}(T, A \otimes_A T) \cong A$$

and clearly this isomorphism induces an isomorphism,

$$\mathrm{Hom}(A, A) \cong \mathrm{Hom}(\Psi(A), \Psi(A)).$$

Therefore Ψ is the identity on all complexes of finitely generated free A-modules. But every object in $D^b(\text{mod-}A)$ is isomorphic to such a complex. One argues similarly for $\mathrm{RHom}(T, \bullet) \otimes_A T$. □

Remark 3.1.3. *Below we will define the dimension of a triangulated category. If* $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ *is finite then* $\mathrm{End}(T)$ *is automatically finitely generated. This is true when* X *is over a perfect field and separated[34].*

Definition 3.1.4. *A* k*-linear functor* S *from a* k*-linear category* $\mathcal{C}$ *to itself is called a* ***Serre functor*** *if, for any pair of objects,* X *and* Y*, from* $\mathcal{C}$*, there exists an isomorphism of* k*-vector spaces,*

$$\mathrm{Hom}_{\mathcal{C}}(Y, X)^* \cong \mathrm{Hom}_{\mathcal{C}}(X, S(Y)),$$

which is natural in X *and* Y*.*

A Serre functor, if it exists, is determined uniquely up to natural isomorphism [13].

3.1.2 Dimension of a Triangulated Category

Let $\mathcal{T}$ be a triangulated category. For a subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the full subcategory of $\mathcal{T}$ whose objects are summands of direct sums of shifts of objects in $\mathcal{I}$. For two subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$ we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of objects $X \in \mathcal{T}$ such that there is a distinguished triangle $X_1 \to X \to X_2 \to X_1[1]$ with $X_i \in \mathcal{I}_i$. Further set $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$. By setting $\langle \mathcal{I} \rangle_1 := \langle \mathcal{I} \rangle$ we are able to inductively define $\langle \mathcal{I} \rangle_n := \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle$.

Remark 3.1.5. *Do we want to start the indexing from* 0 *or* 1*?*

Definition 3.1.6. *G is a **strong generator** of a triangulated category $\mathcal{T}$ if there exists an n such that $\mathcal{T} = \langle G \rangle_n$.*

Definition 3.1.7. *Let X be an object in $\mathcal{T}$. The **generation time** of X, denoted $\circleddash(X)$, is*

$$\circleddash(X) := \min \{n \in \mathbb{N} \,|\, \mathcal{T} = \langle X \rangle_{n+1}\}.$$

It is set to ∞ if X is not a strong generator.

Definition 3.1.8. *The **dimension** of a triangulated category $\mathcal{T}$ is the minimal generation time.*

The following lemma follows immediately from the definition.

Lemma 3.1.9. *Let $F : \mathcal{T} \to \mathcal{R}$ be a functor between triangulated categories. Furthermore suppose every object from R is isomorphic to a direct summand of an object from the essential image of F. Then $\dim(\mathcal{R}) \leq \dim(\mathcal{T})$.*

In the case of the lemma, we say that F has dense image.

The following theorem relates the generation time of an algebra to its global dimension (see [8, 21]),

Theorem 3.1.10 (Christensen-Krause-Kussin)**.** *Suppose A is a right-coherent algebra and let $\mathcal{T} = \mathrm{D^b}(\text{mod-}A)$. Then $\circleddash(A) = \mathrm{gd}(A)$.*

Definition 3.1.11. *Let A be a k-algebra. The **Hochschild dimension** of A, denoted $\mathrm{hd}(A)$ is the projective dimension of A as an $A \otimes_k A^{op}$-module.*

Lemma 3.1.12. $\circleddash(A) = \mathrm{gd}(A) \leq \mathrm{hd}(A)$.

Proof. The equality is above. A resolution of the diagonal gives a functorial resolution of any object yielding the inequality. □

The following result is classical - a proof can be found in [34].

Lemma 3.1.13. *Suppose A is a finite-dimensional algebra over a perfect field k. Then $\mathrm{hd}(A) = \mathrm{gd}(A)$.*

We will use the following lemma to compute the Hochschild dimension.

Lemma 3.1.14. *Let A be an algebra over k with finite Hochschild dimension. Then* $\mathrm{hd}(A)$ *is the maximal i with* $\mathrm{Ext}^i_{A\otimes_k A^{op}}(A, A\otimes_k A)$ *nonzero.*

Proof. Take a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$$

of A as $A \otimes_k A^{op}$-module and let i_0 be the maximal i so that $\mathrm{Ext}^i_{A\otimes_k A^{op}}(A, A\otimes_k A)$ is nonzero. It is clear that i_0 must be less than or equal to n. If i_0 is strictly less than n, then $\mathrm{Ext}^n_{A\otimes_k A^{op}}(A, P)$ is zero for any projective module P. Thus, the map $P_n \to P_{n-1}$ must split allowing us to shorten the projective resolution. □

For a scheme we have an analogous definition:

Definition 3.1.15. *Let X be a k-variety. Consider objects of the form $G \boxtimes H \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \times X)$ such that the diagonal, $\Delta_*(\mathcal{O}_X)$, is in $\langle G \boxtimes H\rangle_{n+1}$. The **diagonal dimension** of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, denoted $\dim_\Delta(X)$ is the minimal such n.*

The diagonal dimension has the following nice properties,

Lemma 3.1.16. *Let X be a k-variety. One has:*

1. $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \leq \dim_\Delta(X)$
2. $\dim_\Delta(X \times Y) \leq \dim_\Delta(X) + \dim_\Delta(Y)$,
3. *If X is smooth then* $\dim_\Delta(X) \leq 2 \cdot \dim(X)$

Proof. The first two are clear, the third is proven in [34]. □

3.2 Tilting Objects

Theorem 3.2.1. *Suppose X is a Gorenstein variety over k and T is a tilting object on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Let i be the largest i for which $Ext^i(T, T\otimes\omega_X^{-1}) \neq 0$. Then the Hochschild dimension of* $\mathrm{End}(X)$

is equal to $\dim(X)+i$. *In particular, if* $i=0$ *then* $\mathrm{gd}(\mathrm{End}(X)) = \dim(X) = \mathrm{hd}(\mathrm{End}(X))$ *and if* X *is proper over a perfect field then* $\mathrm{gd}(\mathrm{End}(T)) = \dim(X)+i = \mathrm{hd}(\mathrm{End}(X))$.

Proof. Let $A := \mathrm{End}(T)$ and let $T^\vee := R\mathcal{H}om(T, \mathcal{O}_X)$. Then $A \otimes_k A^{op} \cong \mathrm{End}(T \boxtimes T^\vee)$. By Proposition 3.1.2, this yields an equivalence of categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \times X) \cong \mathrm{D}^{\mathrm{b}}(\text{mod-}A \otimes_k A^{op})$ under which $\mathcal{O}_\Delta$ corresponds to A with its natural bimodule structure and $T \boxtimes T^\vee$ corresponds to the free module $A \otimes_k A^{op}$. Hence,

$$\mathrm{Ext}^j(A, A \otimes_k A^{op}) \cong \mathrm{Hom}(\mathcal{O}_\Delta, T \boxtimes T^\vee[j])$$

$$\cong \mathrm{Hom}(\mathcal{O}_X, \Delta^!(T \boxtimes T^\vee)[j] \cong \mathrm{Hom}(T, T \otimes \omega_X^v[j - \dim(X)])$$

By hypothesis $\mathrm{Ext}^j(A, A \otimes_k A^{op}) = 0$ for $j > \dim(X)+i$. A standard argument shows that the Hochschild dimension of A is $\dim(X)+i$. The rest follows from Theorem 3.1.10 and Lemma 3.1.13.

□

Lemma 3.2.2. *Let* X *be a variety possessing a tilting object* T. *Then the diagonal dimension of* X *is less than the Hochschild dimension of* $\mathrm{End}(T)$.

Corollary 3.2.3. *Let* X *be a variety possessing a tilting bundle* T *with* $\mathrm{Ext}^i_X(T, T \otimes \omega_X^{-1})$ *zero for* i *greater than zero. The diagonal dimension of* X *is equal to the dimension of* X *is equal to the dimension of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Remark 3.2.4. *It would be surprising to the authors if there were a smooth and proper variety over a perfect field whose diagonal dimension exceeded the dimension of the variety.*

3.2.1 Blow-ups

Let $\mathcal{B}_t$ be any blow-up of $\mathbb{P}^2$ at any finite set of reduced points and $\pi : \mathcal{B}_t \to \mathbb{P}^2$ be the projection. Let E_i be the exceptional divisors. We set $T_1 := \pi^*\mathcal{O} \oplus \pi^*\mathcal{O}(1) \oplus \pi^*\mathcal{O}(2) \oplus \pi^*\mathcal{O}(E_1) \oplus \cdots \oplus \pi^*\mathcal{O}(E_t)$ and $T_2 := \pi^*\mathcal{O} \oplus \pi^*\mathcal{O}(1) \oplus \pi^*\mathcal{O}(2) \oplus \pi^*\mathcal{O}_{E_1} \oplus \cdots \oplus \pi^*\mathcal{O}_{E_t}$

Proposition 3.2.5. *If $t \leq 3$ and the points are not colinear then* $\mathrm{gd}(\mathrm{End}(T_1)) = 2$*, whereas if* $t > 3$ *or* $t = 3$ *and the points are colinear then* $\mathrm{gd}(\mathrm{End}(T_1)) = 3$*. Furthermore* $\mathrm{gd}(\mathrm{End}(T_2)) = 3$ *for all* $\mathcal{B}_t$*.*

Proof. This is a simple calculation using Theorem 3.2.1. □

Remark 3.2.6. *We see from the above example that the global dimension of a tilting object is not invariant under mutation.*

3.2.2 Pullback Tilting Objects

Proposition 3.2.7. *Suppose T is tilting object on a (non-proper) gorenstein Calabi-Yau X. Then* $\dim(X) = \dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \mathrm{gd}(T)$*.*

Proof. This follows immediately from Theorem 3.2.1. □

Let ω_X denote the canonical bundle on X.

Definition 3.2.8. *Let X and Y be varieties and $\pi : Y \to X$ an affine morphism. We say that a tilting object is* pullback with respect to π *if the pullback of the tilting object remains tilting. We simply say that an exceptional collection is* pullback *if it is pullback with respect to the canonical projection* $\pi : \mathrm{Tot}(\omega_X) \to X$*.*

Clearly a tilting object T is pullback if and only if,

$$\mathrm{Ext}^k_X(T, T \otimes \omega_X^{\otimes p}) = 0 \text{ for } k \neq 0 \text{ and } p \leq 0$$

Notice that in particular if T is pullback then it satisfies the conditions of Theorem 3.2.1 with $i = 0$. Thus $\mathrm{gd}(T) = \dim(X)$. Notice that ω_X is Calabi-Yau hence by Proposition 3.2.7. $\mathrm{gd}(\pi^* T) = \dim(X) + 1$.

In general one has the following.

Proposition 3.2.9. *Suppose $\mathcal{L}$ is a line bundle on X. And $T \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a tilting object which is pull-back with respect to the projection from the total space of $\mathcal{L}$ to X. Let π denote the projection from the total space of $\mathcal{L}$ to X. Then* $\mathrm{gd}(\mathrm{End}(\pi^*T, \pi^*T)) \geq \mathrm{gd}(\mathrm{End}(T)) + 1$. *Moreover if X is proper and $\mathcal{L}$ is sufficiently anti-ample then T is automatically pullback and* $\mathrm{gd}(\mathrm{End}(\pi^*T, \pi^*T)) = 2\mathrm{dim}(X) + 1$.

Proof. When $\mathcal{L}$ is a sufficiently anti-ample line bundle we have,

$$\mathrm{Ext}^i(\pi^*T, \pi^*T \otimes \omega_{\mathcal{L}}^{-1}) \cong \mathrm{Ext}^i(T, T \otimes \omega_X^{-1} \otimes \bigoplus_{n \leq 1} \mathcal{L}^{\otimes n}) = \mathrm{Ext}^i(T, T \otimes \omega_X^{-1}).$$

Using Theorem 3.2.1, the statement that $\mathrm{gd}(\mathrm{End}(\pi^*T, \pi^*T)) \geq \mathrm{gd}(\mathrm{End}(T)) + 1$ follows from the fact that $\mathrm{Ext}^i(T, T \otimes \omega_X^{-1}$ is a summand of $\mathrm{Ext}^i(\pi^*T, \pi^*T \otimes \omega_{\mathcal{L}}^{-1})$. The final statement follows from considering $\mathrm{Ext}^i(\pi^*T, \pi^*T \otimes \mathcal{L})$ and applying Serre duality on X while choosing $\mathcal{L}$ sufficiently ample.. □

Remark 3.2.10. *More generally one can show that for any generator $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, π^*G takes at least one more step.*

When X is Fano and $\mathcal{L}^n = \omega_X$ for some $n \geq 0$ and T is a tilting bundle, then Proposition 7.2 of [40] states that $\mathrm{End}(\pi^*T)$ is a noncommutative crepant resolution of the homogeneous coordinate ring of $\mathcal{L}$. As it turns out, any noncommutative crepant resolution A of a affine gorenstein variety S will have global dimension equal to the dimension of S (see [39] Theorem 2.2). In this situation once again Theorem 3.2.1 is trivially verified.

3.2.3 Toric Fano Deligne-Mumford Stacks

In [7], Borisov and Hua construct explicit full strong exceptional collections of line bundles for all toric Fano Deligne-Mumford stacks of Picard number at most two or dimension at most two. We now prove that the corresponding tilting bundles are pullback. Hence we have the following,

Theorem 3.2.11. *Suppose that X is a toric Fano Deligne-Mumford stack of Picard number at most two or dimension at most two. Then there exists a pullback tilting bundle (which is a sum of line bundles). In particular if X is a variety,* $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \dim(X) = \mathrm{gd}(\mathrm{End}(T)) = \mathrm{hd}(\mathrm{End}(T))$ *and the anti-canonical ring of X has a noncommutative crepant resolution which is derived equivalent to the total space of ω_X.*

Proof. The setup in [7] is as follows: S is a finite set of line bundles and $T := \bigoplus_{\mathcal{L}\in S} \mathcal{L}$, the terminology below can be found in [7] as well. We have $\omega_X^{-1} = \mathcal{O}_X(E_1 + ... + E_n)$.

Case 1: T is the generator appearing in [7] Proposition 5.1. For any two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in S$, we have $\deg \mathcal{L}_2 \otimes \mathcal{L}_1^{-n} > \deg(K)$. Hence $\deg \mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \omega_X^{-n} = \deg \mathcal{L}_2 \otimes \mathcal{L}_1^{-1} + n \cdot \deg(-K) > \deg(K)$. Hence it is acyclic by [7] Proposition 4.5.

Case 2: T is the generator appearing in [7] Theorem 5.11. By [7] Proposition 5.7, there are three forbidden cones corresponding to the subsets $\emptyset, I_+$ and I_- of $\{1, ..., n\}$. For any two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in S$ let $\mathcal{L} = \deg \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$. Since $\mathcal{L}$ is not in the forbidden cone corresponding to the empty set neither is $\mathcal{L} \otimes \omega_X^{-n}$. Furthermore $|\alpha(\mathcal{L} \otimes \omega_X^{-n})| = |\alpha(\mathcal{L})| \leq \frac{1}{2} \sum_{i\in I_+} \alpha_i$. Hence as in the proof of Proposition 5.8, $\mathcal{L}$ does not lie in the forbidden cones I_+ and I_-.

Case 3: T is the generator appearing in [7] Theorem 7.3. For any two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in S$ let $\mathcal{L} = \deg \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$. Suppose $\mathcal{L} \cong \mathcal{O}_X(\sum_{i=1}^n x_i E_i)$. As in the proof of [7] Proposition 7.2 $\sum r_i x_i > -1$. Hence $\mathcal{L} \otimes \omega_X^{-n} \cong \mathcal{O}_X(\sum_{i=1}^n (x_i + 1)E_i$ and $\sum r_i(x_i + n) = \sum r_i x_i + n \sum r_i = \sum r_i x_i + n > -1$. Therefore $\mathcal{L} \otimes \omega_X^{-n}$ is not in the forbidden cone corresponding to the empty set. Now $\mathcal{L}$ and $\mathcal{L} \otimes \omega_X^{-n}$ have the same image in $\widehat{\mathrm{Pic}}_{\mathbb{R}} \mathbb{P}_\Sigma)$. Therefore they do not intersect the other forbidden cones because they do not intersect this cones under their projections to $\widehat{\mathrm{Pic}}_{\mathbb{R}} \mathbb{P}_\Sigma$.

□

Remark 3.2.12. *Notice that the above theorem is false for the non-Fano toric surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ for $m \geq 4$ by Theorem 3.2.17 and also false for the non-toric Del Pezzos $\mathcal{B}_t$ for $4 \leq t \leq 8$ by Proposition 3.2.5.*

3.2.4 Weighted Projective Spaces and Projective Bundles

Let $X_{m,n} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(m))$ for $m \geq 0$. Let $\pi : X_{m,n} \to \mathbb{P}^n$ be the projection and F be the class of the fiber of $X_{m,n}$ i.e. $\pi^*\mathcal{O}(1) = \mathcal{O}(F)$. Similarly let S denote the class of the zero section of the total space of $\mathcal{O}_{\mathbb{P}^n}(m)$ under the natural inclusion $\mathcal{O}_{\mathbb{P}^n}(m) \to X_{m,n}$ i.e the relative bundle $\mathcal{O}_{X_{m,n}}(1) = \mathcal{O}(S)$. Consider the object,

$$T := \mathcal{O} \oplus \mathcal{O}(F) \oplus \cdots \oplus \mathcal{O}(nF) \oplus \mathcal{O}(S) \oplus \mathcal{O}(S+F) \oplus \cdots \mathcal{O}(S+nF).$$

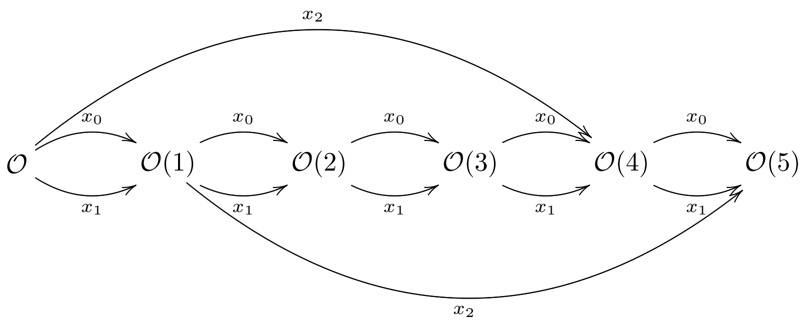
Proposition 3.2.13. *T is a tilting generator. If $m \leq n+1$ then $\mathrm{gd}(\mathrm{End}(T)) = n+1$ and if $m \geq n+2$ then $\mathrm{gd}(\mathrm{End}(T)) = 2n+1$. Furthermore when $m \geq 2n+2$ any tilting bundle S (or more generally any tilting object which has a summand that is a vector bundle) has global dimension equal to either $2n+1$ or $2n+2$.*

Proof. One easily verifies T is a tilting generator. The global dimension of $\mathrm{End}(T)$ is a simple calculation using Theorem 3.2.1. Finally S has a summand that is a vector bundle then $S \otimes \mathrm{RHom}(S, \mathcal{O}_X)$ has $\mathcal{O}$ as a summand. Since $\omega_{X_{m,n}}^{-1}$ has higher cohomology when $m \geq 2n+2$, the last statement also follows from Theorem 3.2.1. □

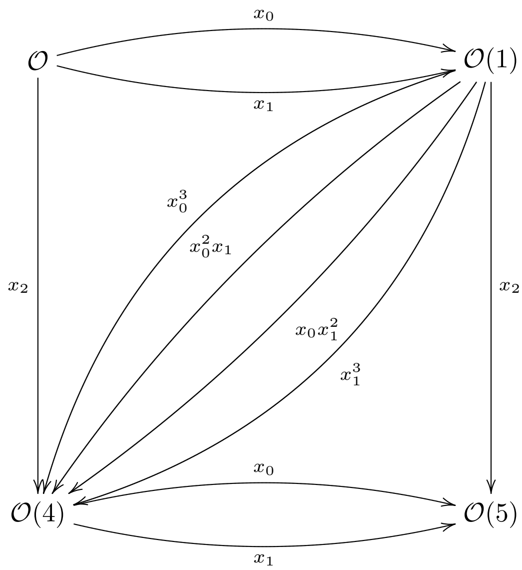
Despite the above proposition we are able to show that $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{m,n}) = \dim(X_{m,n}) = n+1$. The dimension is achieved by a generator which is not tilting. Let us denote stacky weighted projective space by $\mathbb{P}(a_0 : \cdots : a_n)$. The category of coherent sheaves on this space is described in [1]. The following lemma is inspired by [1],

Lemma 3.2.14. *$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{m,n})$ is an admissible subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\mathbb{P}(\underbrace{1 : \cdots : 1}_{(n+1)\text{-times}} : m)$*

Proof. $\mathbb{P}(1 : \cdots : 1 : m)$ has as a strong full exceptional collection consisting of the line bundles $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(m+n)$. The following quiver (with relations implicit from the labeling) describes the endomorphism algebra of the collection in the case of $\mathbb{P}(1 : 1 : 4)$.



The degrees of x_0 and x_1 are one and the degree of x_2 is four. Let $m > n$. Consider the strong exceptional collection formed by the line bundles $\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n), \mathcal{O}(m), \mathcal{O}(m+1), \cdots, \mathcal{O}(m+n)$. The quiver associated to this exceptional collection is exactly the quiver for the exceptional collection for $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(m))$ given above. In the case of $\mathbb{P}(1:1:4)$, we take $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(4), \mathcal{O}(5)$ and get the following quiver



which is the quiver (with relations) for $\mathbb{F}_4$.

Let $E = \sum_{i=0}^{n} \mathcal{O}(i) \oplus \mathcal{O}(m+i)$. Then $\mathrm{RHom}(E, -) : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}(1:\cdots:1:m)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{m,n})$ is an exact and essentially surjective functor. The left adjoint to $\mathrm{RHom}(E, -)$ is $-\otimes E$. $-\otimes E$. $-\otimes E$ is full and faithful. Thus, smallest triangulated category, closed under direct summands, containing E is isomorphic to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{m,n})$. Since both categories possess Serre functors, $-\otimes E$ also possesses a left adjoint, and, consequently, $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{m,n})$ is an admissible subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}(1:\cdots:1:m)$.

□

Lemma 3.2.15. $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\mathbb{P}(a_0 : \cdots : a_n) = n$

Proof. This is a special case of Theorem 3.2.11. □

Remark 3.2.16. *The above Lemma can also be realized as follows. Let μ_r denote the group of r^th roots of unity and consider the diagonal action of $G := \mu_{a_0} \times \cdots \times \mu_{a_n}$ on $\mathbb{P}^n$. Then $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\mathbb{P}(a_0 : \cdots : a_n)$ is equivalent to the bounded derived category of G-equivariant sheaves on $\mathbb{P}^n$. One easily verifies that the terms of the Beilinson resolution have a natural ΔG-equivariant structure such that the morphisms are ΔG invariant with respect to this structure (see [18]). Hence the category of G-equivariant has an n-step generator.*

Theorem 3.2.17. *Conjecture 3.0.9 holds for $X_{m,n}$.*

Proof. As noted in the proof of Lemma 3.2.14, $\mathrm{RHom}(E, -)$ is essentially surjective. Hence by Lemma 3.1.9, $\dim\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{m,n}) \leq \dim\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}(a_0 : \cdots : a_n)) = n+1$ by Lemma 3.2.15. □

Remark 3.2.18. *If one considers noncommutative deformations of weighted projective space $\mathbb{P}_\theta(a_0 : \cdots : a_n)$ as in [1], one can obtain the same upper bound $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}_\theta(a_0 : \cdots : a_n) \leq n$ using their Proposition 2.7. Similarly for the corresponding noncommutative deformations of $X_{m,n}$ we have $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{\theta,m,n}) \leq n+1$. However, as these spaces are noncommutative, a good lower bound is unknown.*

3.3 Rational Singularities

In this section we will need to make use of the ghost lemma and its converse. This lemma appears in many places for example [33, 29].

Lemma 3.3.1 (Ghost Lemma)**.** *Let $\mathcal{T}$ be a triangulated category and let*

$$H_1 \xrightarrow{f_1} H_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} H_n \xrightarrow{f_n} H_{n+1}$$

be a sequence of morphisms between cohomological functors $\mathcal{T}^{op} \longrightarrow Ab$. For every i, let $\mathcal{I}_i$ be a subcategory of $\mathcal{T}$ closed under shifts and on which f_i vanishes. Then $f_n \cdots f_1$ vanishes on $\mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_{n-1}$.

Corollary 3.3.2 (Rouquier)**.** *Let $\mathcal{T}$ be a triangulated category, $M \in \mathfrak{Ob}\mathcal{T}$. Let $n \in \mathbb{N}$ such that there is a sequence of morphisms in $\mathcal{T}$,*

$$N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} N_n \xrightarrow{f_n} N_{n+1}$$

such that

1. *the composition $f_n \cdots f_1 \neq 0$*
2. *$Hom_{\mathcal{T}}(M, -)(f_i) = 0$ for any i,*

Then $N_1 \notin \langle M \rangle_n$. In particular if such a sequence exists for any M, then $\dim \mathcal{T} \geq n$.

Proof. Set $\mathcal{I}_i = \langle M \rangle$ and $H_i = \mathrm{Hom}(-, N_i)$ in the ghost lemma. □

We have the following converse to the ghost lemma due to Opperman [30].

Proposition 3.3.3 (Opperman)**.** *Let $\mathcal{T} \subseteq \mathcal{S}$ be triangulated categories satisfying the following propeties:*

1. *$\mathcal{S}$ is cocomplete*
2. *for any $T, T_i \in \mathcal{T}$, any morphism $T \to \coprod_{i \in I} T_i$ factors through a subcoproduct $\oplus_{i \in I_0} T_I$. Where I_0 is finite.*

Let $\mathcal{I}_i (1 \leq i \leq n)$ be skeletally small subcategories of $\mathcal{T}$ closed under shifts. Let $X \in \mathfrak{Ob}\mathcal{T} \backslash \mathfrak{Ob}\mathcal{I}_i \diamond \cdots \diamond \mathcal{I}_n$.

Then there is a sequence of morphisms

$$X = S_0 \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} S_n$$

in $\mathcal{S}$ such that

1. *For any* $L \in \mathcal{I}_i, Hom(L,-)(F_i) = 0$

2. $f_n \cdots f_1 \neq 0$

In the context of derived categories of coherent sheaves on an algebraic variety we can rephrase the above as follows. Given objects $\mathcal{P} \in D^b_{coh}(X \times Y)$ and $\mathcal{Q} \in D^b_{coh}(Y \times Z)$ we define the convolution,

$$\mathcal{Q} \circ \mathcal{P} := \pi_{XZ_*}(\pi^*_{XY} \otimes \pi^*_{YZ}\mathcal{Q}),$$

where π_{XZ}, π_{XY}, and π_{YZ} are the projections $X \times Y \times Z$ to $X \times Z$, $X \times Y$, and $Y \times Z$ respectively. Let G be a generator of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and G^* be the derived dual. Consider the object $G \boxtimes G^* \in D^b_{coh}(X \times X)$, there is a natural trace map from $G \boxtimes G^*$ to $\mathcal{O}_\Delta$. Denote by $\mathcal{P}_G$ the cone of this map. The map to the cone induces a map $f_n : \mathcal{O}_\Delta \to \underbrace{\mathcal{P}_G \circ \mathcal{P}_G \circ \cdots \circ \mathcal{P}_G}_{n}$ Denote by Φ_{f_n} the natural transformation between the Fourier-Mukai functors corresponding to these kernels. The ghost lemma together with the proof of the converse gives,

Proposition 3.3.4 (Geometric "Ghost Lemma")**.**

$$A \in \langle G \rangle_{n+1} \textit{ if and only if } \Phi_{f_n}(A) = 0$$

.

Lemma 3.3.5. *Suppose X is a scheme such that coh(X) has enough locally frees. Let $A, B \in D^b_{coh}(X)$. Suppose $A \xrightarrow{f} B$ is a non-zero morphism. Then there exists and object $\hat{A} \in \mathfrak{Perf}(X)$ and a morphism $\hat{A} \xrightarrow{\psi} A$ such that the composition $f \circ \psi$ is non-zero.*

Proof. Using some quasi-isomorphism, we may assume that A is a complex of locally frees bounded on the right. Consider the spectral sequence,

$$E_2^{p,q} = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(H^{-q}(A), B[p]) \Rightarrow \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}(A, B[p+q])$$

. Notice that this spectral sequence is bounded above and below and to the left. Hence this spectral sequence terminates in finitely many steps n. Now f is represented by some non-zero cocycle in $E_n p, q$. If we choose a sufficiently large truncation then this part of the spectral sequence with be completely unaffected and $f \circ \psi$ will be represented by the same non-zero cocycle under the appropriate identifications. □

We follow Opperman [30] in defining the notion of dimension of a subcategory of a triangulated category.

Definition 3.3.6. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{C} \subseteq \mathcal{T}$. Let $M \in \mathfrak{Ob}\mathcal{T}$. We define the $M - level$ of $\mathcal{C}$ to be*

$$M - level_{\mathcal{T}}\mathcal{C} = min\{n \in \mathbb{N}\,|\mathcal{C} \subseteq \langle M\rangle_{n+1}\}$$

and the dimension of $\mathcal{C}$ to be

$$dim_{\mathcal{T}}\mathcal{C} = min_{M \in \mathfrak{Ob}\mathcal{T}} M - level_{\mathcal{T}}\mathcal{C}.$$

Proposition 3.3.7. *Suppose X is a separated noetherian scheme of finite type and coh(X) has enough locally frees. Then $dim_{D^b_{coh}(X)}\mathfrak{Perf}(X) = dim\, D^b_{coh}(X)$.*

Proof. Clearly $\dim_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}\mathfrak{Perf}(X) \geq \dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Now suppose $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = n$ and let M be a minimal generator of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ i.e. $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle M\rangle_{n+1}$. Take $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \setminus \langle M\rangle_n$. In Opperman's theorem set $\mathcal{I}_i = \langle M\rangle$ $\mathcal{T} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $\mathcal{S} = D(q - coh)$. For $1 \leq i \leq n-1$ we get that there exists a sequence of morphisms,

$$X = S_0 \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} S_n$$

in $\mathcal{S}$ such that

1. For any $\mathrm{RHom}(M, -)(f_i) = 0$
2. $f_n \cdots f_1 \neq 0$.

But by the lemma we may replace X by $\hat{X} \in \mathfrak{Perf}(X)$ so that $f_n \cdots f_1 \circ \psi \neq 0$ where $\hat{X} \xrightarrow{\psi} X$. Now replace f_1 by $f_1 \circ \psi$ and notice that $\mathrm{RHom}(M, -)(f_1 \circ \psi) = \mathrm{RHom}(M, -)(f_1) \circ \mathrm{RHom}(M, -)(\psi) = 0$.

Hence we have

$$\hat{X} = W_0 \xrightarrow{g_1} W_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} W_n$$

in $\mathcal{S}$ such that

1. For any $\mathrm{RHom}(M, -)(g_i) = 0$

2. $g_n \cdots g_1 \neq 0$.

Applying the ghost lemma gives the theorem.

□

Corollary 3.3.8. *Suppose X is a variety with rational singularities and Y is a minimal resolution of singularities. Then dim $D^b_{coh}(X) \leq$ dim $D^b_{coh}(Y)$.*

Proof. Let $\pi : Y \to X$ be the map given by the resolution. Let $A \in \mathfrak{Perf}X$. Let G be a minimal generator for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$. Now since X has rational singularities we have $\mathrm{R}\pi_* \mathrm{L}\pi^*(A) \cong A$. It follows that $\mathrm{R}\pi_* G - \mathrm{level}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)} \mathfrak{Perf}(X) \leq G - \mathrm{level}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) = \dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$. Now one applies the previous propostion. □

One can also show,

Theorem 3.3.9. *Let X be a smooth variety and G be a finite group acting on X X/G be the quotient variety. Then $\dim D^b_{coh}(X) \geq \dim D^b_{coh} X/G$.*

Proof. The proof is as above only now $\mathcal{O}_X$ is a summand of $\mathrm{R}\pi_* \mathcal{O}_X = \pi_* \mathcal{O}_X$. □

Using the Ghost Lemma one can also proove,

Lemma 3.3.10. *Suppose X is a smooth variety and V is a vector bundle on X. Then,*

$$dim(D^b_{coh}(X)) \leq dim(D^b_{coh}(Tot(V))).$$

Moreover if G is an s-step generator of $D^b_{coh}(X)$ then the (derived) pullback of G to $D^b_{coh}(Tot(V))$ is at least an $s+1$-step generator.

Proof. Let $\pi : \text{Tot}(V) \to X$ be the projection and $i : X \to \text{Tot}(V)$ be the inclusion of the zero-section. Notice that both maps are affine and the (derived) composition $\pi_* \circ i_* \cong Id$.

Claim: for any $A \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), i^* i_* A \cong \bigoplus \bigwedge^k \mathcal{V}^*[k] \otimes A$.

$\pi^* V$ has a canonical section that vanishes along the zero section which is regular. Hence the we have Koszul resolution of $i_* \mathcal{O}_X$,

$$0 \to \bigwedge^r \pi^* V^* \to \cdots \to \pi^* V^* \to \mathcal{O}_{\mathrm{Tot(V)}} \to i_* \mathcal{O}_X$$

Now let R_j denote the shifted truncation $0 \to \bigwedge^r \pi^* V^* \to \cdots \bigwedge^j \pi^* V^*$. Then we have exact triangles for each i,

$$R_{j-1} \to \bigwedge^j \pi^* V^* \to R_j[-1].$$

Notice that these objects are adapted to tensor product and now for any object $A \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ consider the exact triangle,

$$R_{j-1} \otimes i_* A \to \bigwedge^j \pi^* V^* \otimes i_* A \to R_i[-1] \otimes i_* A.$$

Since the maps in the Kozsul resolution vanish along X the first map is zero. Therefore one has isomorphisms,

$$R_{j-1} \otimes i_* A[1] \oplus \bigwedge^j \pi^* V^* \otimes i_* A \cong R_i[-1] \otimes i_* A.$$

Together these give the isomorphism $i_* \mathcal{O}_X \otimes i_* A \cong \bigoplus \bigwedge^k \pi^* V^*[k] \otimes i_* A)$ and so we have,

$$i_* i^* i_* A \cong i_* \mathcal{O}_X \otimes i_* A \cong \bigoplus \bigwedge^k V^*[k] \otimes i_* A) \cong i_*(\bigoplus i^* \pi^* \bigwedge^k V^* \otimes A) \cong i_*(\bigoplus V^* \otimes A).$$

As π is affine, we may apply π_* to both sides of this isomorphism to get $i^* i_* A \cong \bigoplus \bigwedge V^* \otimes A$.

Hence we see that $i^* : D^b_{coh}(\text{Tot}(V) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ has dense image and by Lemma 3.1.9 $\dim(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) \leq \dim(D^b_{coh}(\text{Tot}(V)))$.

Now suppose G is an s-step generator of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. By Proposition 3.3.3 there exists a sequence of morphisms $A_1 \to \cdots \to A_s$ such that $f_{s-1} \circ \cdots \circ f_1 \neq 0$ $\text{Hom}(G, \bullet)(f_i) = 0$. For any object

$A \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ consider the triangle, $R_1 \to \mathcal{O}_{\mathrm{Tot}(V)} \to i_*\mathcal{O}_X$. Tensor this triangle with π^*A to acheive, $R_1 \otimes \pi^*A \to \pi^*A \to i_*A$. Denote by g_A the map from $i_*A \to R_1 \otimes \pi^*A[1]$. Now for any $h \in \mathrm{Hom}(\pi^*G, i_*A) \cong \mathrm{Hom}(\pi^*G, i_*A)$ notice that h factors as $\pi^*G \to \pi^*A \to i_*A$. Hence $h \circ g_A = 0$. This means that the map g_A is a ghost map for the generator π^*G i.e. $\mathrm{Hom}(G, \bullet)(g_A) = 0$ for all A. Now consider the sequence,

$$i_*A_1 \to \cdots \to i_*A_n \to R_1 \otimes \pi^*A_n[1]$$

. Since all maps are ghost we need only show the total composition $g_A \circ i_*f_{s-1} \circ \cdots \circ i_*f_1 \neq 0$. Consider $i^*(g_A \circ i_*f_{s-1} \circ \cdots \circ i_*f_1) = i^*g_A \circ i^*i_*f_{s-1} \circ \cdots \circ i^*i_*f_1$. One easily verifies that for the map $i^*i_*f_i : i^*i_*A_i \cong \bigoplus \bigwedge^k V^*[k] \otimes A_i \to i^*i_*A_{i+1} \cong \bigoplus \bigwedge^k V^*[k] \otimes A_{i+1}$ one has $i^*i_*f_i \cong \bigoplus \mathrm{Id}_{\wedge^k V^*[k]} \otimes f_i$. Furthermore the map $i^*g_A : \bigoplus \bigwedge^k V^*[k] \otimes A_n \to i^*(R_1 \otimes \pi^*A_n)[1] \cong i^*R_1 \otimes A_n[1] \cong \bigoplus_{k \geq 1} \bigwedge^k V^*[k] \otimes A_n$ is easily seen to be to projection. Hence the composition is clearly nonzero. $\square$

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