BRAIDS AND OPEN BOOK DECOMPOSITIONS

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To Andrei and my parents

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ABSTRACT

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In this thesis we generalize Alexander's and Bennequin's work on braiding knots
and Markov's theorem about when two braids represent isotopic knots. We also
reprove Eliashberg's theorem on the transversal simplicity of the unknot in a tight
contact structure using braid theoretical techniques. Finally, we look at possible
changes in braid foliations induced on a surface.

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Chapter 1

Introduction

In [2], Alexander proved that any knot in $\mathbb{R}^3$ can be braided about the z-axis. In [6], a paper which marked the start of modern contact topology, leading to the Bennequin inequality and the definition of tightness, Bennequin proved the transverse case for $(\mathbb{R}^3, \xi_{std})$. Following a review of known results and background material, in Chapter 3 we generalize Bennequin's result to any closed, oriented, 3-dimensional manifold M, by looking at on open book decomposition for M together with a supported contact structure.

In [17], Markov gave an equivalent condition for two braids in $\mathbb{R}^3$ to be isotopic. This is the case if and only if the two braids differ by conjugations in the braid group and positive and negative Markov moves. In [18] Orevkov and Shevchishin proved the transversal case for $(\mathbb{R}^3, \xi_{std})$. A different proof was independently obtained by Wrinkle in [23]. In Chapter 4 we generalize Markov's theorem to any closed,

oriented, 3-dimensional manifold. We prove the transverse case and recover the topological case previously proved in [20] and [21].

In [10], Birman and Wrinkle proved that exchange reducibility implies transversal simplicity. As a consequence, Birman and Menasco's paper [8] shows that the m-component unlink is transversely simple. While exchange reducibility does not seam to work in more general settings, the unknot remains transversely simple in a tight contact structure. Eliashberg proved this fact in [11]. Later, Etnyre ([14]) proved that positive torus knots are transversely simple. In Chapter 5 we reprove Eliashberg's original result using braid theoretical techniques.

In Chapter 6 we look closer at foliations induced on an embedded surface $S \subset M^3$ by the intersection with the pages of an open book decomposition (Σ, ϕ) for M^3 and we show how adjacent saddle points can be changed with respect to the coordinate on S^1, generalizing a result of Birman and Menasco ([8]). .

Chapter 2

Background

In this chapter we review some of the definitions and results that we'll be using throughout this thesis.

2.1 Contact structures

Definition 2.1.1. Let M be a compact, oriented 3-manifold and ξ a subbundle of the tangent bundle of M such that $\xi_p = T_p M \cap \xi$ is a two dimensional subspace of $T_p M$ for all $p \in M$. Locally, ξ can be written as $\xi = \ker \alpha$ for some non-degenerate 1-form α. The plane field ξ is called a *contact structure* if $\alpha \wedge d\alpha \neq 0$.

Such a plane field is completely non-integrable, that is ξ is not tangent to any surface along an open set. If ξ is orientable then ξ can be written as $\xi = \ker \alpha$ for a global 1-form α. Depending on whether $\alpha \wedge d\alpha > 0$ or $\alpha \wedge d\alpha < 0$, ξ is called a positive or a negative contact structure. The contact structures we are working

with throughout this paper are assumed to be oriented and positive.

Definition 2.1.2. A contactomorphism between two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) is a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi_* \xi_1 = \xi_2$.

Definition 2.1.3. An embedded disk D in (M, ξ) is called an *overtwisted disk* if $T_p \partial D \subset \xi$ for all $p \in \partial D$ and D is transverse to ξ along ∂D. If there exists such a disk in (M, ξ) then ξ is called *overtwisted*. If it is not overtwisted a contact structure is called *tight*.

On $\mathbb{R}^3$, consider the two contact structures ξ_1 and ξ_2 given by the 1-forms $\alpha_1 = dz - ydx$ and $\alpha_2 = dz + r^2 d\theta$ (given in cylindrical coordinates). These two contact structures are contactomorphic and we are going to refer to them as the standard contact structure, ξ_{std}.

Unlike Riemannian geometry, contact geometry doesn't exhibit any special local behavior, as we can see from the following theorem.

Theorem 2.1.4. *(Darboux) Let (M, ξ) be a contact manifold. Every point $p \in M$ has a neighborhood that is contactomorphic to a neighborhood of the origin in $(\mathbb{R}^3, \xi_{std})$.*

A proof of this theorem can be found in [1].

Definition 2.1.5. The *Reeb vector field of α* is the unique vector field v_α on M such that

    i) $\alpha(v_\alpha) = 1$

ii) $d\alpha(v_\alpha, \cdot) = 0$

Definition 2.1.6. Let $\Sigma \subset (M, \xi)$ be an embedded surface. From the non-integrability condition of ξ it follows that $l_x = T_x\Sigma \cap \xi_x$ is a line for most points $x \in \Sigma$. The singular foliation on Σ whose leaves are tangent to the line field $l = \cup l_x$ is called the *characteristic foliation* of Σ.

The characteristic foliation of a surface Σ determines a whole neighborhood of Σ as it follows from the next theorem.

Theorem 2.1.7. *If $\Sigma_i \subset (M_i, \xi_i)$, for $i = 1, 2$ are two embedded surfaces and there exists a diffeomorphism $f : \Sigma_1 \to \Sigma_2$ which preserves the characteristic foliations then f may be isotoped to be a contactomorphism in a neighborhood of Σ_1.*

2.2 Gray's theorem

The following theorem is of vital importance in proving the generalization of Alexander's theorem on braiding links (Chapter 3).

Theorem 2.2.1. *(Gray) Let $\{\xi_t\}_{t \in [0,1]}$ be a family of contact structures on a manifold M that differ on a compact set $C \subset int(M)$. Then there exists an isotopy $\psi_t : M \to M$ such that*

i) $(\psi_t)_*\xi_0 = \xi_t$

ii) ψ_t *is the identity outside of an open neighborhood of C.*

Proof. We are going to look for ψ_t as the flow of a vector field X_t. If $\xi_t = \ker \alpha_t$, then ψ_t has to satisfy

$$\psi_t^* \alpha_t = \lambda_t \alpha_0, \tag{2.2.1}$$

for some non-vanishing function $\lambda_t : M \to \mathbb{R}^3$. By taking the derivative with respect to t on both sides the equality will still hold.

$$\frac{d}{dt}(\psi_t^* \alpha_t) = \lim_{h \to 0} \frac{\psi_{t+h}^* \alpha_{t+h} - \psi_t^* \alpha_t}{h} = \lim_{h \to 0} \frac{\psi_{t+h}^* \alpha_{t+h} - \psi_{t+h}^* \alpha_t + \psi_{t+h}^* \alpha_t - \psi_t^* \alpha_t}{h} =$$

$$= \psi_t^* (\frac{d\alpha_t}{dt}) + \psi_t^* \mathcal{L}_{X_t} \alpha_t = \psi_t^* (\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t} \alpha_t).$$

This gives

$$\psi_t^* (\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t} \alpha_t) = \frac{d\lambda_t}{dt} \alpha_0 = \frac{d\lambda_t}{dt} \frac{1}{\lambda_t} \psi^* \alpha_t \tag{2.2.2}$$

and by letting

$$h_t = \frac{d}{dt}(\log \lambda_t) \circ \psi_t^{-1} \tag{2.2.3}$$

we get

$$\psi_t^* (\frac{d\alpha_t}{dt} + d(\iota_{X_t} \alpha_t) + \iota_{X_t} d\alpha_t) = \psi_t^* (h_t \alpha_t) \tag{2.2.4}$$

If X_t is chosen in ξ_t then $\iota_{X_t} \alpha_t = 0$ and the last equality becomes

$$\frac{d\alpha_t}{dt} + \iota_{X_t} d\alpha_t = h_t \alpha_t \tag{2.2.5}$$

Applying (2.2.5) to the Reeb vector field of α_t, v_{α_t}, we find $h_t = \frac{d\alpha_t}{dt}(v_{\alpha_t})$ and have

the following equation for X_t

$$\iota_{X_t} d\alpha_t = h_t \alpha_t - \frac{d\alpha_t}{dt} \qquad (2.2.6)$$

The form $d\alpha_t$ gives an isomorphism

$$\Gamma(\xi_t) \to \Omega^1_{\alpha_t}$$

$$v \mapsto \iota_v d\alpha_t$$

Where $\Gamma(\xi_t) = \{v | v \in \xi_t\}$ and $\Omega^1_{\alpha_t} = \{\text{1-forms } \beta | \beta(v_t) = 0\}$. This implies that X_t is uniquely determined by (2.2.6) and by construction the flow of X_t works. For the subset of M where the ξ_t's agree we just choose the α_t's to agree. This implies $\frac{d\alpha_t}{dt} = 0$, $h_t = 0$ and $X_t = 0$ and all equalities hold. $\qquad \square$

2.3 Transverse and Legendrian arcs

Definition 2.3.1. In a contact manifold (M, ξ), an oriented arc $\gamma \subset M$ is called *transverse* if for all $p \in \gamma$ and ξ_p the contact plane at p, $T_p\gamma \pitchfork \xi_p$ and $T_p\gamma$ points towards the positive normal direction of the oriented plane ξ_p. If γ is a closed curve then it is called a *transverse knot*.

Definition 2.3.2. In a contact manifold (M, ξ), an arc $\gamma \subset M$ is called *Legendrian* if for all $p \in \gamma$, $T_p\gamma \subset \xi_p$, where ξ_p is the contact plane at p. If γ is a closed curve then it is called a *Legendrian knot*.

Around a transversal (or Legendrian) knot contact structures always look the same. Using Theorem 2.2.1 we can prove the following two lemmas.

Lemma 2.3.3. *A neighborhood of a transverse knot in any contact manifold is contactomorphic with a neighborhood of the z-axis in $(\mathbb{R}^3, \xi_{std})/_{z\sim z+1}$.*

Lemma 2.3.4. *A neighborhood of a Legendrian knot in any contact manifold is contactomorphic with a neighborhood of the x-axis in $(\mathbb{R}^3, \xi_{std})/_{x\sim x+1}$.*

Definition 2.3.5. Let $K \subset (M, \xi)$ be a null homologous knot and $\Sigma \subset M$ a surface such that $\partial\Sigma = K$. Let $v \in \xi|_{\partial\Sigma}$ be the restriction to K of a non-zero vector field $w \in \xi$ on Σ. Denote by K' the push-off of K in the direction of v. The *self-linking number* of K is $sl(K) = lk(K, K') = K' \cdot \Sigma$ (algebraic intersection).

2.4   Open book decompositions

Definition 2.4.1. An *open book decomposition* of M is a pair (L, π) where

  i) L is an oriented link in M called *the binding* of the open book

  ii) $\pi : M \smallsetminus L \to S^1$ is a fibration whose fiber, $\pi^{-1}(\theta)$, is the interior of a compact surface $\Sigma \subset M$ such that $\partial\Sigma = L$, $\forall\theta \in S^1$. The surface Σ is called *the page* of the open book.

Alternatively, an open book decomposition of a 3-manifold M consists of a surface Σ, with boundary, together with a diffeomorphism $\phi : \Sigma \to \Sigma$, with $\phi=$ identity near $\partial\Sigma$, such that

$$M = (\Sigma \times [0,1]/ \sim) \cup_f \coprod_i S^1 \times D^2$$

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where

$$(x, 1) \sim (\phi(x), 0).$$

Note that

$$\partial(\Sigma \times [0, 1]/\sim) = \coprod_i T_i^2,$$

each torus T_i^2 having a product structure $S^1 \times [0, 1]/\sim$. Let $\lambda_i = \{pt\} \times [0, 1]/\sim$, $\lambda_i \in T_i^2$. The gluing diffeomorphism used to construct M is defined as

$$f : \partial(\coprod_i S^1 \times D^2) \to \partial(\coprod_i T_i^2)$$

$$\{pt\} \times \partial D^2 \to \lambda_i.$$

The map ϕ is called *the monodromy* of the open book.

Theorem 2.4.2. *(Alexander, [3]) Every closed oriented 3-manifold has an open book decomposition.*

Definition 2.4.3. A contact structure ξ on M is said to be *supported* by an open book decomposition (Σ, ϕ) of M if ξ can be isotoped through contact structures so that there exists a 1-form α for ξ such that

   i) $d\alpha$ is a positive area form on each page

   ii) $\alpha(v) > 0$ for all $v \in TL$ that induce the orientation on L.

Lemma 2.4.4. *A contact structure ξ on M is supported by an open book decomposition (Σ, ϕ) if and only if (Σ, ϕ) is an open book decomposition of M and ξ can be isotoped to be arbitrarily close, as oriented plane fields, on compact subsets of the pages, to the tangent planes to the pages of the open book in such a way that after some point in the isotopy the contact planes are transverse to the binding and transverse to the pages in a neighborhood of the binding.*

Contact structures and open book decompositions are closely related. Thurston and Winkelnkemper have shown how to get contact structures from open books.

Theorem 2.4.5. *(Thurston, Winkelnkemper, [22]) Every open book decomposition (Σ, ϕ) supports a contact structure ξ_ϕ.*

Proof. Let

$$M = (\Sigma \times [0,1]/\sim) \cup_f \coprod_i S^1 \times D^2$$

given as before. We first construct a contact structure on $\Sigma \times [0,1]/\sim$ and then we extended it in a neighborhood of the binding. Let (ψ, x, θ) be coordinates near each of the binding components $((\psi, x)$ are coordinates on the page, with ψ being the coordinate along the binding, while θ is the coordinate pointing out of the page) and consider the set of forms

$$S = \{1 - \text{forms } \lambda \text{ such that } : \quad d\lambda \text{ is a volume form on } \Sigma$$

$$\lambda = (1+x)d\psi \text{ near } \partial\Sigma = L\}$$

To show that the set S is non-empty let λ_1 be a 1-form on Σ such that $\lambda_1 = (1+x)d\psi$

near $\partial\Sigma$. Let ω be a volume form on Σ such that $\omega = dx \wedge d\psi$ near $\partial\Sigma$. The form

$\omega - d\lambda_1$ is closed and since $H^2(\Sigma) = 0$, there exists a 1-form β such that $d\beta = \omega - d\lambda_1$

and $\beta = 0$ near $\partial\Sigma$. Then $\lambda = \lambda_1 + \beta$ is an element in S.

Note that for $\lambda \in S$, then $\phi^*\lambda$ is also in S.

Let λ be an element of S and consider the 1-form

$$\lambda_{(p,t)} = t\lambda_p + (1-t)(\phi^*\lambda)_p$$

on $\Sigma \times [0,1]$ where $(p,t) \in \Sigma \times [0,1]$ and take

$$\alpha_K = \lambda_{(p,t)} + K dt.$$

For sufficiently large K, α_K is a contact form and it descends to a contact form on

$\Sigma \times [0,1]/\sim$. We want to extend this form on the solid tori neighborhood of the

binding. Consider coordinates (ψ, r, θ) in a neighborhood $S^1 \times D^2$ of each binding

component. The gluing map f is given by

$$f(\psi, r, \theta) = (r - 1 + \epsilon, -\psi, \theta).$$

Pulling back the contact form α_K using this map gives the 1-form

$$\alpha_f = K d\theta - (r + \epsilon) d\psi.$$

We are looking to extend this form on the entire $S^1 \times D^2$ to a contact form of the

form $h(r)d\psi + g(r)d\theta$. This is possible if there exist functions $h, g : [0,1] \to \mathbb{R}^3$ such

that:

   i) $h(r)g'(r) - h'(r)g(r) > 0$ (given by the contact condition)

ii) $h(r) = 1$ near $r = 0$, $h(r) = -(r + \epsilon)$ near $r = 1$

iii) $g(r) = r^2$ near $r = 0$, $g(r) = K$ near $r = 1$.
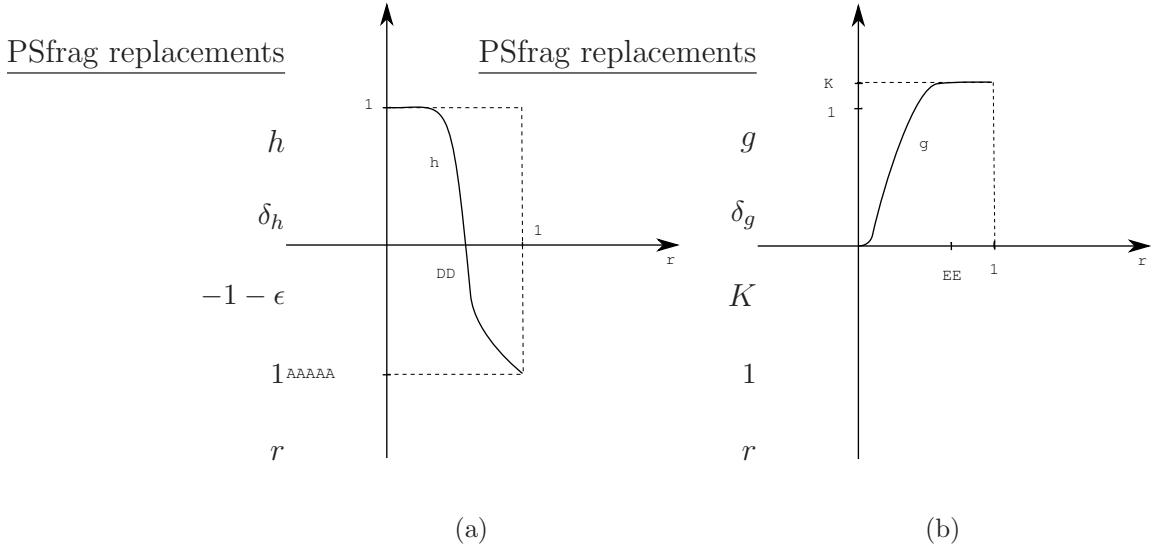
Figure 2.1: (a) h function (b) g function

The two functions h and g described in Figure 2.1 work for our purpose. The conditions i) and ii) are obviously satisfied and if δ_h and δ_g are such that $h < 0$ on $[\delta_h, 1]$ and $g = 1$ on $[\delta_g, 1]$, then iii) is satisfied as long as $\delta_h < \delta_g$. $\qquad \square$

In [15], Giroux proved there exists a correspondence between contact structures and open book decompositions.

Theorem 2.4.6. *(Giroux) Let M be a closed, oriented 3-manifold. Then there is a one to one correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilizations.*

Definition 2.4.7. A positive (negative) stabilization of an open book (Σ, ϕ) is the open book with

i) page $\Sigma' = \Sigma \cup 1 - \text{handle}$

ii) monodromy $\phi' = \phi \circ \tau_c$, where τ_c is a right-(left-)handed Dehn twist along a curve c in Σ' that runs along the added 1-handle exactly once.

Chapter 3

The generalized Alexander

theorem

3.1 Preliminaries

In [2] Alexander proved that any link in $\mathbb{R}^3$ can be isotoped to a link braided about

the z-axis. In [6] Bennequin proved the transverse case, that is that any transverse

link in $(\mathbb{R}^3, \xi_{std})$ can be transversely braided about the z-axis. The goal of this

chapter is to prove a generalization of Bennequin's result. Throughout this section

M is a closed and oriented 3-manifold.

Definition 3.1.1. Let (L, π) be an open book decomposition for M. A link $K \subset M$

is said to be *braided about* L if K is disjoint from L and there exists a parametrization

of K, $f : \coprod S^1 \to M$ such that if t is the coordinate on each S^1 then $\frac{d}{dt}(\pi \circ f) > 0$

at all time.

Our proof reduces the general case to the $(\mathbb{R}^3, \xi_{std})$ case proved by Bennequin. Below we sketch the ideas he used in his proof.

Theorem 3.1.2. *(Bennequin, [6]) Any transverse link Γ in $(\mathbb{R}^3, \xi_{std})$ is transversely isotopic to a link braided about the z-axis.*

Proof. Let (r, θ, z) be cylindrical coordinates on $\mathbb{R}^3$ and let t be the parameter on Γ. The standard contact structure is given by $\alpha = dz + r^2 d\theta$. The arcs constituting Γ are either *good* (if $\frac{d\theta}{dt} > 0$) or *bad* (if $\frac{d\theta}{dt} \leq 0$). In order to arrange Γ into a braid form, the bad arcs are moved through a transverse isotopy on the other side of the z-axis. This requires that certain wrinkling, which we describe below, is done along the bad arcs. Along a bad arc $\gamma \subset \Gamma$ we have $\frac{d\theta}{dt} < 0$ and $\frac{dz}{dt} > 0$, as the knot is transverse. The projection of γ on a large enough $r-$cylinder looks like the one in Figure 3.1. The wrinkling leaves the r coordinate unchanged and modifies the z and θ coordinates. As throughout the wrinkling $\frac{dz}{d\theta}$ increases and r remains constant the arc remains transverse to the contact planes. An arc $\gamma \subset \Gamma$ is said to be *shadowed* by another arc γ' at $(r, \theta, z) \in \gamma$ if there exists $(r', \theta, z) \in \gamma'$ with $r' < r$.

i) By introducing a wrinkle as in Figure 3.1 on the bad arc γ, it can be arranged that γ doesn't go all around the z-axis.

ii) By introducing a wrinkle as in Figure 3.1 on the bad arc γ, it can be arranged that γ is not shadowed by any other arc.

iii) A non-shadowed bad arc γ is transversely isotoped to a good arc as in Figure

17

z

θ

γ T
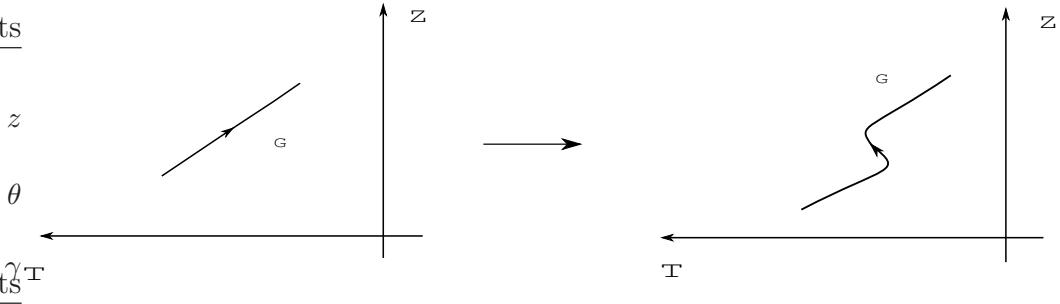
Figure 3.1: Wrinkling along a bad arc γ.

z

θ

γ
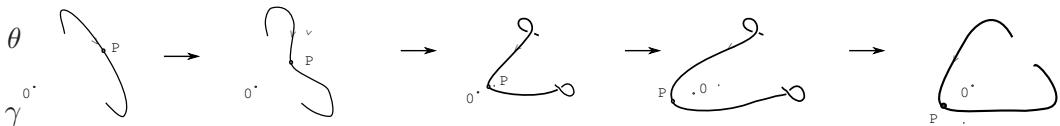
Figure 3.2: Transforming a non-shadowed bad arc into a good arc.

3.2. This isotopy keeps the θ and z coordinates of γ fixed, while it decreases the r coordinate until the arc hits the z-axis. This guarantees the trasversality as $\frac{d\theta}{dt} < 0$ makes $\frac{dz}{dt} + r^2(t)\frac{d\theta}{dt} > 0$ at all time.

iv) As the initial number of bad arcs is finite, and a whole bad arc γ can be resolved at a time ($\frac{dz}{dt} > 0$ on γ so no two parts of γ ever shadow one another throughout the process), after finitely many steps Γ is isotoped to a braid form. $\qquad\square$

3.2 The generalized Alexander theorem

Theorem 3.2.1. *Suppose (L, π) is an open book decomposition for the 3-manifold M and ξ is a supported contact structure on M. Let K be a transverse link in M. Then K can be transversely isotoped to a braid.*

Proof. The idea of the proof is to find a family of diffeomorphisms of M keeping

each page of the open book setwise fixed and taking the parts of the link where the link is not braided in a neighborhood of the binding. A neighborhood of the binding is contactomorphic to a neighborhood of the z-axis in $(\mathbb{R}^3, \xi_{std})/_{z\sim z+1}$ and there the link can be braided, according to Theorem 3.1.2.

In the neighborhood $N = S^1 \times D^2$ of each component of the binding consider coordinates (ψ, x, θ) such that $d\theta$ and $\pi^* d\theta$ agree, where $\pi^* d\theta$ is the pullback through $\pi : M \setminus L \to S^1$ of the coordinate on S^1.

As ξ is supported by the open book (L, π), ξ can be isotoped to a contact structure ξ_ϵ that is arbitrarily close to being tangent to the pages of the open book.

Consider a 1-form λ on Σ such as in Lemma 2.4.5. On $\Sigma \times [0,1]$ take $\tilde{\lambda} = (1-\theta)\lambda + \theta(\phi^* \lambda)$ and consider the family of 1-forms given by $\alpha_t = \tilde{\lambda} + K\frac{1}{t}d\theta$, where $t \in (0,1]$ and K is any large constant. This family of 1-forms descends to a family of 1-forms on $\Sigma \times [0,1]/ \sim$.

Both $\xi_1 = ker(\alpha_1)$ and ξ are contact structures supported by (L, π) and so they are isotopic. Therefore, without loosing generality, we may assume $\xi = ker(\alpha_1)$.

Note that for $t \to 0$, α_t defines a plane field almost tangent to pages.

For large enough K, the family of 1-forms $\{\alpha_t\}_t$ is a family of contact 1-forms as:

$$\alpha_t \wedge d\alpha_t = (\tilde{\lambda} + K\frac{1}{t}d\theta) \wedge (d\tilde{\lambda}) = \tilde{\lambda} \wedge d\tilde{\lambda} + K\frac{1}{t}d\theta \wedge d\tilde{\lambda} > 0$$

Note that $d\tilde{\lambda}$ is an area form on the page while $d\theta$ vanishes on the page and is positive on the positive normal to the page. This implies that the second term of

the sum is always positive and therefore α_t is a contact form for sufficiently large K. We want to extend this family to the whole M, so we need to patch in the solid tori neighborhood of the binding. Let (ψ, r, θ) be coordinates near each binding component. As in Theorem 2.4.5 the map f used to glue the solid tori is given by

$$f(\psi, r, \theta) = (r - 1 + \epsilon, -\psi, \theta).$$

Pulling back the contact forms α_t using this map gives the family of 1-forms

$$\alpha_{f,t} = K\tfrac{1}{t}d\theta - (r + \epsilon)d\psi.$$

We are looking to extend this form on the entire $S^1 \times D^2$ to a contact form of the form $h(r, t)d\psi + g(r, t)d\theta$. These two functions do exist, as we can take $h, g : [0, 1] \times (0, 1] \to \mathbb{R}^3$ with $h(r, t) = h(r)$ (as defined in the proof of Theorem 2.4.5) and $g(r, t)$ similar to $g(r)$ defined in the proof of Theorem 2.4.5 except $g(r, t)$ equals $\frac{K}{t}$ near $r = 1$. Denote the extended family of forms also by α_t and by ξ_t the family of contact structures given by $\xi_t = ker(\alpha_t)$, $t \in (0, 1]$. By Gray's theorem there exists a family of diffeomorphisms $f_t : M \to M$ such that $(f_t)_*\xi = \xi_t$. Let v_t be the Reeb vector field associated to α_t, that is the unique vector field such that $\alpha_t(v_t) = 1$ and $d\alpha_t(v_t, \cdot) = 0$.

As announced, we would like that the family $\{f_t\}_t$ fixes the pages setwise. Following the proof of Gray's theorem, f_t is given as the flow of a vector field $X_t \in \xi_t$, for which we have the following equality of 1-forms

$$\iota_{X_t} d\alpha_{t,f} = \frac{d\alpha_{t,f}}{dt}(v_t)\alpha_t - \frac{d\alpha_{t,f}}{dt} \tag{3.2.1}$$

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We already know that such a X_t exists but would need it to be tangent to the page. First notice that $\frac{d\alpha_t}{dt} = -\frac{1}{t^2}Kd\theta$ and choose some vector $v \in T\Sigma \cap \xi_t$. Applying both sides of 3.2.1 to v we get

$$d\alpha_{t,f}(X_t, v) = \frac{d\alpha_{t,f}}{dt}(v_t)\alpha_{t,f}(v) - \frac{d\alpha_{t,f}}{dt}(v) \tag{3.2.2}$$

As $v \in \xi_t = ker(\alpha_t)$ and v has no θ-component, the equality is equivalent to

$$d\alpha_t(X_t, v) = 0 \tag{3.2.3}$$

As $d\alpha_t$ is an area form on ξ_t, the above equality implies that X_t and v are linearly dependent and therefore $X_t \in T\Sigma \cap \xi_t$ ($X_t = 0$ will be 0 at singular points).

We are now looking at the singularities of X_t. On Σ_θ there are no negative elliptic singularities away from the binding as the contact planes and the planes tangent to the page almost coincide, as oriented plane fields (a negative elliptic singularity e would require ξ_e and $T_e\Sigma$ to coincide but have different orientations). Thus, for each θ, all points on Σ_θ, except for singularities of X_t and stable submanifolds of hyperbolic points, flow in finite time into an arbitrarily small neighborhood of the binding. Define S_θ as the set of points on Σ_θ that are either singularities of X_t or on stable submanifolds of hyperbolic points. Let $S = \cup S_\theta$ as θ varies from 0 to 2π.

First, note that we can arrange the monodromy map ϕ to fix the singularities on the cutting page, by thinking of ϕ as of a composition of Dehn twists away from these points. For isolated values of θ, X_t might exhibit connections between hyperbolic singularities. With these said, S has a CW structure with

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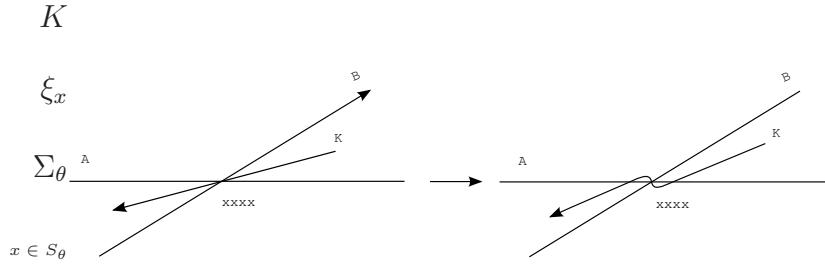
Figure 3.3: Wrinkling K in order to avoid intersections with S

1-skeleton: union of singular points and connections between hyperbolic singularities

2-skeleton: union of stable submanifolds of hyperbolic singularities

If no bad arc of K intersects S then all these arcs will be eventually pushed in a neighborhood of the binding. Before changing the contact structures through the above described diffeomorphisms we can arrange that the arcs of the link K where K is not braided avoid S by wrinkling as necessary (as in Figure 3.3). This wrinkling, which we explicitly describe below, may increase the number of arcs where the link is not braided but this is fine, as these new arcs avoid S.

By general position, we may assume $K \cap (1 - \text{skeleton of } S) = \emptyset$ and $K \pitchfork (2 - \text{skeleton of } S)$ is a finite number of points. A small neighborhood in D of a point $p \in S_\theta \cap K$ is foliated by intervals $(-\epsilon, \epsilon)$, in the same way as a small disk in the xy-plane centered at $(0,1,0)$ in $(\mathbb{R}^3, \xi_{std})$. It follows from Theorem 2.1.7 that p a has a neighborhood in M which is contactomorphic to a neighborhood of $q = (0, 1, 0)$ in $(\mathbb{R}^3, \xi_{std})$. Consider the standard (x, y, z) coordinate system in such a neighborhood. The contact plane ξ_q is given by $x + z = 0$. To make things more clear visually we make a change of coordinates (we also call the new coordinates

$(x, y, z))$ that takes this plane to the plane $z = 0$. As at p the contact plane and the plane tangent to the page almost coincide, we may assume that the tangent plane to the page at q is given by $z = \epsilon x$ and that the link K is given by $z = \frac{\epsilon}{2}$, $y = 1$ in a δ-neighborhood of q, $\delta > 0$. The wrinkling takes K to K' with the following properties:

i) K' is given by $z = \frac{3\epsilon}{2}$, $y = 1$ in a $\frac{\delta}{3}$-neighborhood of q

ii) K' is given by $z = \frac{\epsilon}{2}$, $y = 1$ outside of a $\frac{2\delta}{3}$-neighborhood of q

iii) $\frac{dz}{dx} > 0$ along K' in a δ-neighborhood of q.

While condition i) takes care of the K avoiding S along its bad zone, condition iii) takes care of the link remaining transverse throughout the wrinkling.

After making the necessary wrinklings, $f_\epsilon(K)$ has all bad regions in a neighborhood of the binding so there is a transverse isotopy K_s, $0 \leq s \leq 1$ taking $f_\epsilon(K)$ to a braid K'_ϵ (as described in Theorem 3.1.2). Then $f_\epsilon^{-1}(K_s)$, $0 \leq s \leq 1$ is the transverse isotopy we were looking for as it takes K to the braided knot $f_\epsilon^{-1}(K')$. $\qquad\square$

Chapter 4

The generalized Markov theorem

4.1 Geometric Markov theorem in an open book decomposition

Let M be a 3-dimensional, closed, oriented manifold and (L, π) an open book decomposition for M. Consider $K \subset M$ a knot braided about L, $k \subset K$ an arc that lies in a neighborhood $N(L)$ of the binding, and $D \subset N(L)$ a disk normal to the binding, with ∂D oriented according to the right hand rule.

Definition 4.1.1. With the above notations, *a positive (negative) geometric Markov move* is given by connecting ∂D and k through a half twisted band whose orientation coincides with (is opposite to) that of the page at their tangency point.

Theorem 4.1.2. *(Orevkov, Shevchishin [18]) In $(\mathbb{R}^3, \xi_{std})$ two braids represent transversely isotopic links if and only if one can pass from one braid to the other by*

braid isotopies, positive Markov moves and their inverses.

Our purpose is to prove a result similar to Theorem 4.1.2 in an open book decomposition.

Theorem 4.1.3. *(topological case) Let M be a 3-dimensional, closed, oriented manifold and (L, π) an open book decomposition for M. Let K_0 and K_1 be braid representatives of the same topological link. Then K_0 and K_1 are isotopic if and only if they differ by braid isotopies and positive and negative Markov moves and their inverses.*

This topological version has been previously proven by Skora [20] and Sundheim [21]. Our proof for this case immediately follows from the proof of the transverse case, as one does not need to worry about transversality throughout the isotopy and if transversality is not required both positive and negative Markov moves are allowed.

Theorem 4.1.4. *(transverse case) Let M be a 3-dimensional, closed, oriented manifold and (L, π) an open book decomposition for M together with a supported contact structure ξ. Let K_0 and K_1 be transverse braid representatives of the same topological link. Then K_0 and K_1 are transversely isotopic if and only if they differ by braid isotopies and positive Markov moves and their inverses.*

Proof. First we should note that an isotopy through braids is done away from the binding. As the contact planes almost coincide to the planes tangent to the

pages this isotopy is also transverse with respect to the contact structure. Let K_0 and K_1 be transverse braid representatives of the same topological knot K and $\{L_t\}_{t \in [0,1]}$ a transverse isotopy from K_0 to K_1. We parametrize the isotopy by $\mathcal{L} : \sqcup S^1 \times [0,1] \to M$, such that $\mathcal{L}_t$ defined by $s \to \mathcal{L}(s,t)$ is a parametrization of K_t, where s is the positively oriented coordinate on each S^1.

Let θ be the positive coordinate normal to the page. A *bad zone* of $\mathcal{L}$ is a connected component of the set of points in $\sqcup S^1 \times [0,1]$ for which $\frac{\partial \theta}{\partial s} \leq 0$. Denote by $\mathcal{B}$ the union of all bad zones of $\mathcal{L}$.

We would like to take all the bad zones of $\mathcal{L}$ in a neighborhood of the binding. This
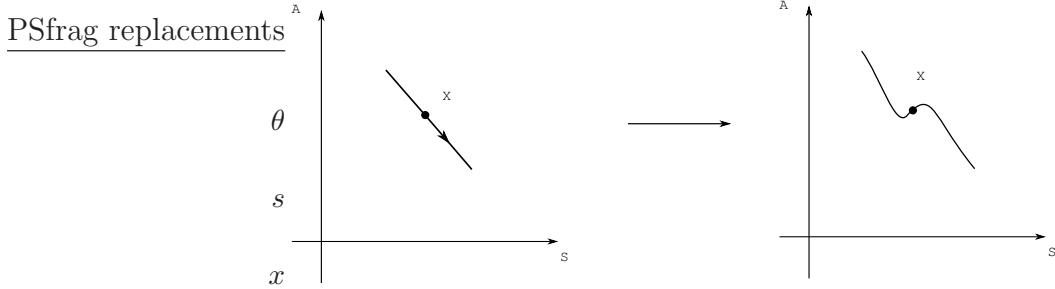
Figure 4.1: Wrinkling in order to avoid intersections between $\mathcal{B}$ and S

way the proof is reduced to the standard case proved by Orevkov and Shevchishin in [18]. For this, we need a family of diffeomorphisms of M that keep each page of the open book setwise fixed and take the bad zones of $\mathcal{L}$ in a neighborhood of the binding. We have already constructed the needed family of diffeomorphisms $\{f_t\}_t$ in the proof of Theorem 3.2.1. The f_t's are described by the flow of a family of vector fields $\{X_t\}_t$. For each θ, all points on Σ_θ, except for the set S_θ composed of singularities of X_t and stable submanifolds of hyperbolic points, flow in finite

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time into an arbitrarily small neighborhood of the binding. The isotopy $\mathcal{L}$ has to be arranged in such a way that $\mathcal{B} \cap S = \emptyset$, where S is the union of all S_θ's as θ varies from 0 to 2π. We are going to arrange that by *wrinkling* as necessary. We describe the process below.

On $\mathcal{B}$ we have $\frac{\partial \theta}{\partial s} \leq 0$. We make the arc $l \subset \mathcal{B} \cap S$ a good arc if we arrange $\frac{\partial \theta}{\partial s} > 0$ along l. To do that, for a fixed t and $x \in l$ look at the graph of θ as a function of s. Introduce a small wrinkle around x, as in Figure 4.1. For each point $x \in l$, this wrinkle is the same described in Theorem 3.2.1 and thus it can be arranged to be transverse. In [18] it was shown that this wrinkle can be done continuously for all values of t along I. $\qquad \square$

4.2 The standard braid group and Markov moves

Let S be an orientable surface and let $P = \{p_1, ..., p_n\} \subset S$ be a set of n distinct points. A *braid* on S based at P is a collection of paths $(\alpha_1, ..., \alpha_n)$, $\alpha_i : [0, 1] \to S$ such that:

i) $\alpha_i(0) = p_i$, $i = 1..n$

ii) $\alpha_i(1) \in P$, $i = 1..n$

iii) $\alpha_1(t), ..., \alpha_n(t)$ are distinct for all $t \in [0, 1]$.

The concatenation of paths defines a group structure on the set of all braids on S based at P up to homotopy. This group, which does not depend on the choice of P, is denoted by $B_n(S)$, and it is called the *braid group on n strings* in S.

This group was first introduced by Artin in [4], for $S = \mathbb{R}^2$. The braid group of the plane has the following presentation:

$$\langle \sigma_1, ..., \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| > 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i \leq n - 2 \rangle$$

where, if we project the strands on a generic vertical plane, σ_i interchanges the ith and the $i + 1$st strands and leaves all other strands unchanged.
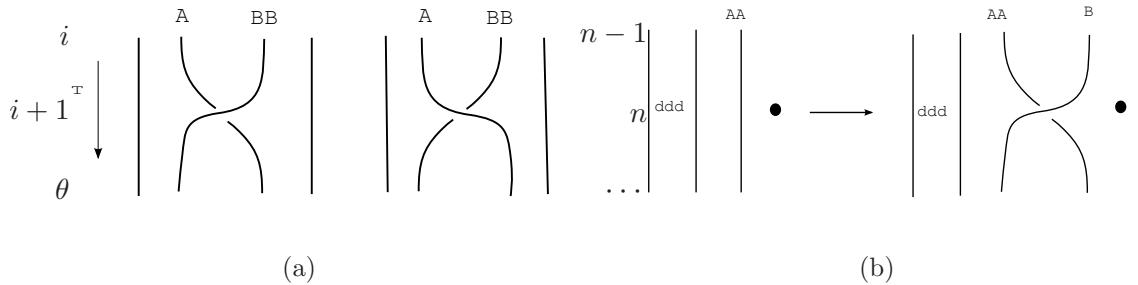
Figure 4.2: (a) Braids representing σ_i and σ_i^{-1} (b) Positive Markov move

Definition 4.2.1. With the above notations, a positive (negative) *Markov move* in $\mathbb{R}^3$ is given by the map $m : B_n \to B_{n+1}$, $b \to b \cdot \sigma_n$ ($b \to b \cdot \sigma_n^{-1}$). Geometrically, a Markov move is equivalent to a adding and extra strand linking the z-axis once.

4.3 Braids in an open book decomposition

A description of the braid group of an orientable surface is given by Bellingeri in [5]. The braid group $B_n(F)$, where F is an orientable p-punctured surface of genus $g \geq 1$ with $p \geq 1$ punctures, admits the following presentation:

$$B_n(F) = \{\sigma_1, ..., \sigma_{n-1}, a_1, ..., a_g, b_1, ..., b_g, z_1, ..., z_{p-1} | R_0, ..., R_7\},$$

where the relations $R_0, ..., R_7$ are as follows:

$$(R_0) \quad : \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \, for \, |i - j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{for} \, i \leq n - 2$$

$$(R_1) \quad : \quad a_r \sigma_i = \sigma_i a_r, 1 \leq r \leq g, i \neq 1$$

$$b_r \sigma_i = \sigma_i b_r, 1 \leq r \leq g, i \neq 1$$

$$(R_2) \quad : \quad \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1^{-1}, 1 \leq r \leq g$$

$$\sigma_1^{-1} b_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} b_r \sigma_1^{-1}, 1 \leq r \leq g$$

$$(R_3) \quad : \quad \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1, s < r$$

$$\sigma_1^{-1} b_s \sigma_1 b_r = b_r \sigma_1^{-1} b_s \sigma_1, s < r$$

$$\sigma_1^{-1} a_s \sigma_1 b_r = b_r \sigma_1^{-1} a_s \sigma_1, s < r$$

$$\sigma_1^{-1} b_s \sigma_1 a_r = a_r \sigma_1^{-1} b_s \sigma_1, s < r$$

$$(R_4) \quad : \quad \sigma_1^{-1} a_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} a_r \sigma_1, 1 \leq r \leq g$$

$$(R_5) \quad : \quad z_j \sigma_i = \sigma_i z_j, i \neq 1, j = 1, ..., p - 1$$

$$(R_6) \quad : \quad \sigma_1^{-1} z_i \sigma_1 a_r = a_r \sigma_1^{-1} z_i \sigma_1, 1 \leq r \leq g, i = 1, ..., p - 1, n > 1$$

$$\sigma_1^{-1} z_i \sigma_1 b_r = b_r \sigma_1^{-1} z_i \sigma_1, 1 \leq r \leq g, i = 1, ..., p - 1, n > 1$$

$$(R_7) \quad : \quad \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1, j = 1, ..., p - 1, j < l$$

$$(R_8) \quad : \quad \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1}, j = 1, ..., p - 1$$

To get a feel of how these generators look and interact, think of the braiding as being done in a small enough neighborhood of one boundary component (here the

monodromy map is the identity) and of the generators z_i, a_i and b_i as being elements in π_1(page) (see Figure 4.3). These generators should not be though of as lying on a specific page but intersecting the pages transversely between different values of θ. A generator given by the topology of the page can be parametrized by $c :$ $[0,1] \to \Sigma \times [\theta_1, \theta_2]$, $c(t) = (\gamma(t), \delta(t))$, where $\gamma : [0,1] \to \Sigma$ and $\delta : [0,1] \to [\theta_1, \theta_2]$ is strictly increasing. While the stabilizations given by the Markov moves will always be assumed to be performed on the nth strand of a n-strand braid, the loops representing the generators given by the topology of the page will always be assumed to come out of the first strand of a braid.
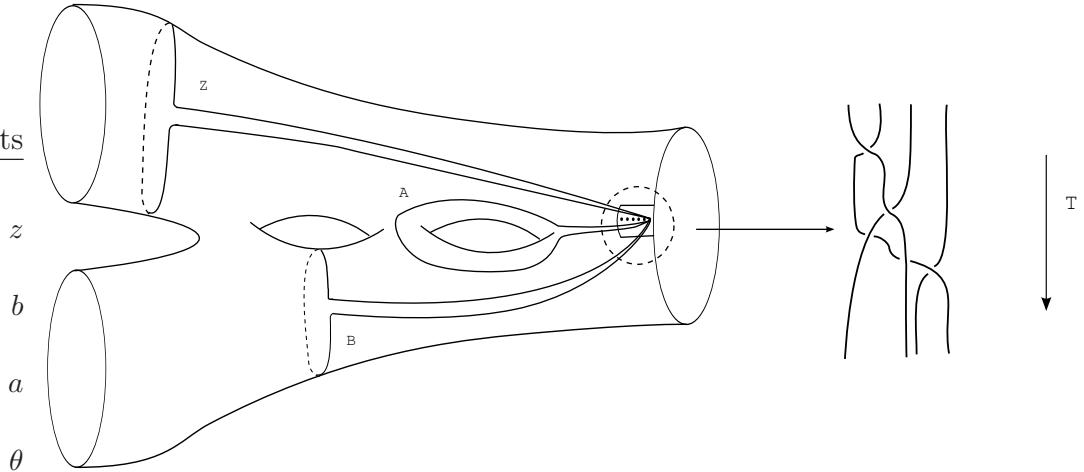

Figure 4.3: Generators of the braid group of a surface.

In an open book decomposition the Markov moves are defined in the neighborhood of each binding component in the same way as in the standard model. The action corresponding to the conjugation in the standard model should take into

account the monodromy map. We are going to fix the monodromy page, call it Σ_0, and we are going to read all the braid words starting at this page and moving in the increasing θ direction.

First, there is an action given by $b \to \sigma \cdot b \cdot \sigma^{-1}$, where σ is a word in the braid generators σ_i's. This is not influenced by the monodromy map, as ϕ is identity near the $\partial \Sigma$. We are going to refer to this as to a *b-conjugation*.

Second, there is an action given by $b \to c \cdot b \cdot \phi^{-1}(c^{-1})$, where c is any of the a_i, b_i, z_i or their inverses. We are going to refer to this as to a *t-conjugation*. Note that by applying the monodromy map to the loop representing $\phi^{-1}(c^{-1})$ when passing through Σ_0 we get a loop representing c^{-1}.

4.4 Algebraic interpretation of the Markov theorem in an open book decomposition

In this section we make the connection between the braid isotopies mentioned in section 4.1 and the elements of the braid group introduced in the previous section.

Theorem 4.4.1. *Let M be a 3-dimensional, closed, oriented manifold and (L, π) an open book decomposition for M. Let B_0 and B_1 be closures of two elements of the braid group. Then B_0 and B_1 are isotopic as braids in M if and only if they differ by b- and t-conjugations in the braid group. They represent the same topological*

knot type if and only if they differ by b- and t-conjugations and stabilizations in the braid group.

Proof. Away from the binding the contact planes and the planes tangent to the surface almost coincide, thus a braid isotopy is equivalent to a transverse isotopy. We are going to have a look at the two different types of actions.

For a *b-conjugation*: If we see the braid generators in a small enough neighborhood of the binding, such a conjugation will preserve the braid isotopy class. The arcs involved in the conjugation may be assumed close enough to the circle $r = \epsilon$, $z = 0$, which is in braid form for $\epsilon > 0$. When projected on the cylinder of radius R, where R is large enough that all points p on the strands involved in the conjugation are such that $r(p) < R$, the braid conjugation represents a sequence of type II Reidemeister moves.

For a *t-conjugation*: In this case, the conjugation is a sequence of conjugations with the a, b and z generators or their inverses. Let $(p, \theta_1) \in \Sigma_{\theta_1}$ be the starting point of a loop l_c representing the element c and $(p, \theta_2) \in \Sigma_2$ be its ending point. Let $(p, \theta_2) \in \Sigma_{\theta_2}$ be the starting point of a loop $l_{\phi^{-1}(c^{-1})}$ representing $\phi^{-1}(c^{-1})$ and $(p, \theta_3) \in \Sigma_{\theta_3}$ be its ending point. We want to isotop $l_c \cdot l_{\phi^{-1}(c^{-1})}$ to the curve $\{p\} \times [\theta_1, \theta_3]$. The cutting page Σ_0 interposes itself between l_c and $l_{\phi^{-1}(c^{-1})}$. As we move $l_{\phi^{-1}(c^{-1})}$ through Σ_0 and apply the monodromy map ϕ to it we get a new curve $l'_{c^{-1}}$ representing c^{-1}. We want to isotop $l_c \cdot l'_{c^{-1}}$ to the curve $\{p\} \times [\theta_1, \theta_3]$. This is certainly possible, as the the arc $l_c \cdot l'_{c^{-1}}$ is disjoint form Σ_0 and can be isotoped in

$\Sigma \times [\theta_1, \theta_3]$ around the topology of the page to $\{p\} \times [\theta_1, \theta_3]$. This isotopy can be realized so that the θ coordinate is left unchanged and thus it is an isotopy through braids.

We now look at the converse. We assume that two braids are isotopic in the complement of the binding and want to see that they are related by b- and t-conjugations. Fix again the cutting page Σ_0 and consider a braid β which may be assumed to intersect Σ_0 in a small neighborhood of L_1. We want to arrange that the isotopy fixes the endpoints of β on the cutting page. If this is the case, the isotopy can be seen in $\Sigma \times [0, 1]$ and the and the problem is reduced to the product case discussed in Section 4.3 and thus are represented by the same word in the braid group.

If the endpoints of β on the cutting page are not fixed, we can modify the isotopy in such a way that they remain fixed. Given an isotopy β_t from $\beta_0 = \beta$ to β_1 we construct a new isotopy β_t' where $\beta_0' = \beta_0$ and β_1' is β_1 after some b- and t-conjugations and such that the intersection of β_t' with Σ_0 is fixed. Say that the endpoint of one of the strands of β, describes a curve $\gamma \subset \Sigma_0$ troughout the isotopy. First, modify the isotopy such that right at the endpoint of γ, the strand strand looks like $\{p\} \times [0, \epsilon]$. Then, slide $\{p\} \times \{\frac{\epsilon}{2}\}$ along $\Sigma \times \{\frac{\epsilon}{2}\}$ back in a small neighborhood of L_1 such that $\{p\} \times [0, \epsilon]$ is replaced by $\gamma_1 \cdot \gamma_1^{-1} \subset \Sigma \times [0, \epsilon]$. Now, by shifting the whole isotopy by ϵ in the negative θ direction we find that the endpoint of the given strand on Σ_0 is left unchanged by the new isotopy. The isotopy β_t is

left unchanged on the other strands. We repeat the process for all other strands. Notice that we can do this without having the corresponding γ strands interfere one with the other, by nesting them with respect to the θ direction.

In the geometric version of the theorem we talked about stabilizations, without regard for the involved binding components. In building the algebra behind it we assumed that the braid elements σ_i lie in a neighborhood of some chosen component of the binding, L_1. The Markov moves are therefore obvious when performed with respect to this binding component, but not so if they are done with respect to a different binding component. We would like to see how is this second stabilization written in terms of the generators of the braid group.


(a)

(b)

(c)

(d)

(e)

(f)
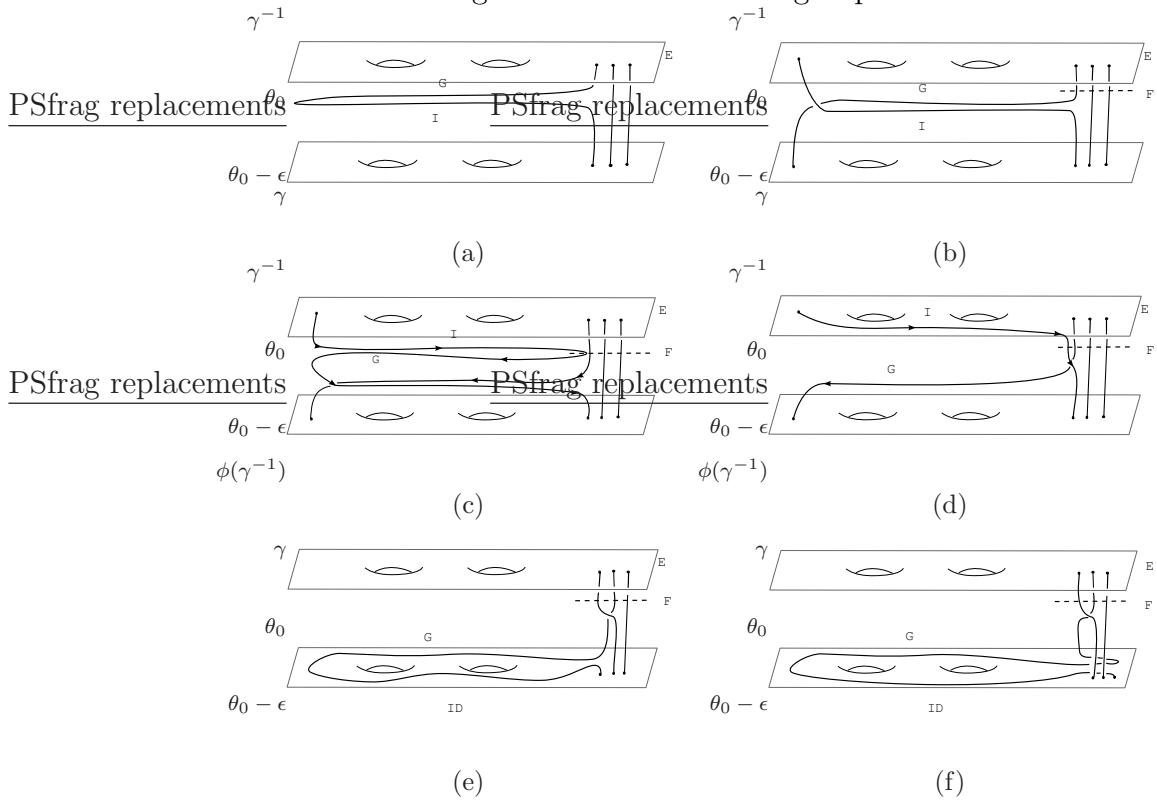
Figure 4.4: Stabilization about a second binding component

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Let l_n be the nth strand of the braid, the outermost one when viewed with respect to the L_1 binding component. Consider the cutting page Σ_0 given by $\pi^{-1}(\theta_0)$ and a curve $\alpha : [0,1] \to \Sigma \times [\theta_0 - \epsilon, \theta_0]$, $\alpha(t) = (\gamma(t), \delta(t))$, where $\gamma : [0,1] \to \Sigma$ and $\delta : [0,1] \to [\theta_0 - \epsilon, \theta_0]$ is strictly increasing, such that $\gamma(1)$ belongs in a small neighborhood of a different binding component, L_2. We may assume that $l_n \cap \Sigma \times [\theta_0 - \epsilon, \theta_0 + \epsilon]$ is of the form $\{p\} \times [\theta_0 - \epsilon, \theta_0 + \epsilon]$. We isotop l_n such that $\{p\} \times [\theta_0 - \epsilon, \theta_0 + \epsilon]$ follows $\gamma \cdot \gamma^{-1}$ in $\Sigma \times [\theta_0 - \epsilon, \theta_0 + \epsilon]$ and perform a Markov move with respect to L_2 that can be either positive or negative (see Figure 4.4(a,b)). Figure 4.4(b) pictures the braid just after a negative Markov move. We look at this case, as the positive one is similar. We want that the monodromy map ϕ fixes the points of intersection of the braid with the cutting page. For this, following the earlier procedure, we isotop the intersection of the resulting arc with Σ_0 back along γ to a neighborhood of L_1 (as in Figure 4.4(c)). After an additional isotopy outside of a neighborhood of the binding and after passing γ through the cutting page Σ_0 (as in Figure 4.4(d,e)), we find that, in the braid group, a negative stabilization with respect to L_2 is given by $b \to b \cdot \sigma_n^{-1} \cdot w$ where w is a word representing the loop $\gamma \cdot \phi(\gamma^{-1})$. Similarly, a positive stabilization with respect to L_2 is given by $b \to b \cdot \sigma_n \cdot w$ where w is a word representing the loop $\gamma \cdot \phi(\gamma^{-1})$. $\square$

Chapter 5

On the transversal simplicity of

the unknot

In [10], Birman and Wrinkle proved that exchange reducibility implies transversal simplicity. As a consequence, Birman and Menasco's paper [8] shows that the m-component unlink is transversely simple. While exchange reducibility does not seem to work in more general settings, the unknot remains transversely simple in a tight contact structure. Eliashberg proved this fact in [11]. Later, Etnyre proved in [14] that positive torus knots are transversely simple. In this chapter we reprove Eliashberg's original theorem using braid theoretical techniques.

5.1 Braid foliations

Definition 5.1.1. Let (M, ξ) be a 3-dimensional contact manifold. A topological class of knots $\mathcal{T}$ is called *transversely simple* if any two transverse representatives of $\mathcal{T}$ having the same self-linking number are transversely isotopic.

Further, we are looking at how an embedded surface in M may sit with respect to an open book decomposition (Σ, ϕ) for M. This ideas were first introduced and studied by Birman and Menasco. Let $U \subset M$ be an embedded unknot together with an embedded disk D such that $\partial D = U$. By Theorem 3.2.1, U may be assumed to be braided about the binding of (Σ, ϕ). The intersection of D with the pages of the open book induce a foliation on D, the *braid foliation*.
We may assume that D is in general position in the following sense:

i) D intersects the binding L transversely, finitely many times in such a way that up to orientation the binding and the normal direction to the disk coincide

ii) In a neighborhood of such an intersection point D is radially foliated (as in Figure 5.5(a))

iii) All but finitely many pages Σ_θ meet D transversely and those which do not are tangent to Σ_θ at points which are either saddle points or local extremes with respect to the parameter θ.

iv) The tangency points described in iii) are all on different pages

v) U has a neighborhood $N(U)$ in M such that $N(U) \cap D$ is foliated by arcs transverse to U. Also, the oriented foliation lines go from inside D transversely towards U in this neighborhood.

We would like to isotop D such that there are no singularities given by local extremes with respect to the parameter θ. This requires few more definitions and results.

Definition 5.1.2. Let $\mathcal{F}$ be an oriented singular foliation on Σ. Let $\Gamma \subset \Sigma$ be a properly embedded 1-manifold. We say that Γ divides $\mathcal{F}$ if

1) $\Gamma \pitchfork \mathcal{F}$

2) $\Sigma \setminus \Gamma = \Sigma_+ \sqcup \Sigma_-$, $\Sigma_\pm \neq \emptyset$

3) $\exists$ a vector field $\overrightarrow{w}$ on Σ and a volume form ω on Σ such that

    i) $\overrightarrow{w}$ directs $\mathcal{F}$

    ii) $\overrightarrow{w}$ is expanding ω on Σ_+ and contracting ω on Σ_-

    iii) $\overrightarrow{w}$ points out of Σ_+(and into Σ_-).

Definition 5.1.3. A vector field $\overrightarrow{v}$ on a contact manifold (M, ξ) is called a *contact vector field* $\overrightarrow{v}$ if its flow preserves ξ.

Definition 5.1.4. A surface $\Sigma \subset (M, \xi)$ is said to be *convex* if there exists a contact vector field $\overrightarrow{v}$ transverse to Σ.

Theorem 5.1.5. *For a convex surface Σ as above there exists a dividing set Γ given by the points on Σ where $\overrightarrow{v} \in \xi$.*

Lemma 5.1.6. *(Legendrian Realization Principle) Let γ be an embedded 1-manifold on a convex surface Σ such that each component of $\Sigma \setminus \gamma$ intersects a dividing set Γ. Then Σ can be isotoped through convex surfaces such that γ is Legendrian.*

Lemma 5.1.7. *Let (Σ, ϕ) be an open book decomposition for M and ξ a supported tight contact structure. Let $S \subset M$ be an embedded surface. Then S can be isotoped in such a way that its braid foliation has no tangency points given by local extremes with respect to the parameter θ.*

Proof. Assume there exists a tangency point p of S with a page, given by a local extreme with respect to θ. We are going to show that S can be isotoped in such a way that the singularity at p is eliminated. Near p the surface S intersects the pages in concentric circles c_θ with $c_\theta = \partial D_\theta$, $D_\theta \subset \Sigma_\theta$ a family of disks. Moving through the fibration away from the tangency point we must arrive at a circle $\gamma_0 \subset \Sigma_0$ containing a singular point p_0. There are three cases we need to consider.

Case 1: The curve γ_0 bounds a disk in Σ_0 (as in Figure 5.1(a)). In this case, using the 3-ball

$$B = \coprod_\theta D_\theta$$

composed by disks in between p and γ_0, we can isotop the part of the surface on the side of γ_0 containing p so that both p and p_0 are eliminated.

Case 2: The curve γ_0 is not trivial in homology (as in Figure 5.1(b)). In this case γ_0 is non separating so it satisfies the hypothesis of Lemma 5.1.6 and if the page would have been a closed surface γ_0 could have been Legendrian realized. If so, the

γ_0

δ

δ_1

γ_0

γ_0

δ_2

(a)   (b)   (c)
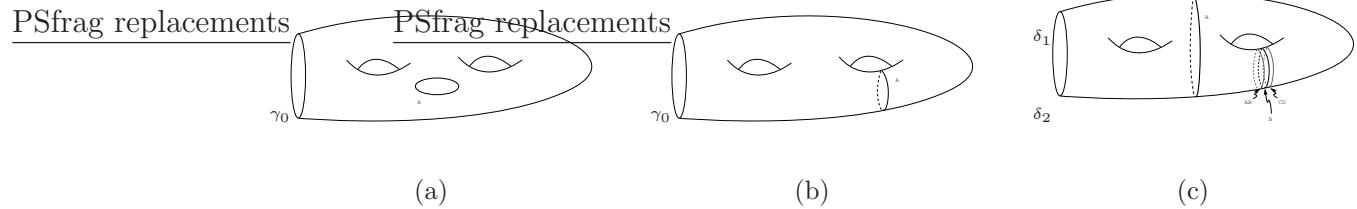
Figure 5.1: (a) Homotopically trivial γ (b) Homologically trivial γ not bounding a disk (c) Homologically essential γ

disk bounded by the Legendrain curve γ_0 on S would be an overtwisted disk, contradicting the hypothesis. We can still apply Lemma 5.1.6, by taking $\Sigma' = \Sigma_0 \cup_{id_L} \Sigma_{\frac{1}{2}}$ and looking at $\gamma_0 \subset \Sigma'$. The surface Σ' is closed, with dividing curve $\Gamma = L$.

Case 3: The curve γ_0 is trivial in homology but does not bound a disk in Σ_0 (as in Figure 5.1(c)). Consider Σ' as defined above. The curve $\gamma_0 \subset \Sigma'$ may not satisfy the hypothesis of Lemma 5.1.6, as there might be a component of $\Sigma' \setminus \gamma_0$ which does not intersect the dividing set, Γ. If so, then consider a non-separating curve δ in this component. The curve δ can be Legendrian realized. This can be done by looking at an annulus neighborhood of δ. This annulus has a neighborhood contactomorphic to a neighborhood N of the annulus in the xy-plane in $(\mathbb{R}^3, \xi_{std})/_{x \sim x+1}$. The neighborhood N can be replaced with a new neighborhood which introduces two new singularity lines and two new components to the dividing set. The new surface has a dividing set $\Gamma' = \Gamma \cup \{\delta_1, \delta_2\}$, where δ_1 and δ_2 are curves parallel to δ, in a small neighborhood of δ (as in Figure 5.1(c)). With respect to Γ', γ_0 satisfies the hypothesis of Lemma 5.1.6 and therefore can be Legendrian realized. This trick was used by Honda in [16] and is called super Legendrian realization. As in Case 2

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this gives rise to an overtwisted disk. $\qquad\qquad$ $\square$

In Chapter 2 we have introduced the notion of a characteristic foliation for an embedded surface $S \subset M$. This foliation and the braid foliation described above have the same distribution of singularities along S. In a neighborhood of the binding L the contact structure looks like the standard one, $dz + r^2 d\theta$ and by perturbing S we can make the normal to S and the binding align at the intersection points, having a small disk around the intersection point radially foliated, as in Figure 5.5(a). Away from the binding the contact planes almost coincide with the planes tangent to the pages and thus, modulo a small perturbation the two foliations also coincide here. The two foliations are topologically conjugate by a homeomorphism closed to the identity. This means that from the perspective of the pictures they are the same.

Consider D a spanning disk for an unknot in M. It follows from the general position requirements and Lemma 5.1.7 that D can be isotoped in such a way that the singularities of its braid foliation are either elliptic, given by the intersections with the binding (as in Figure 5.2(a,b)), or hyperbolic, given by saddle points of D (as in Figure 5.2(c,d)). Further, condition iv) satisfied by a disk in general position, implies that there are no foliation lines connecting two hyperbolic singularities in the braid foliation of D. A generic braid foliation for D looks like the one in Figure 5.3. In this picture, the elliptic singularities are denoted by filled dots while the hyperbolic singularities are denoted by empty dots.
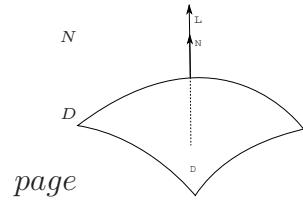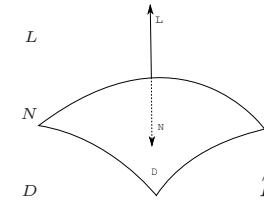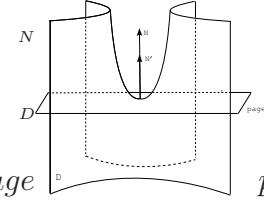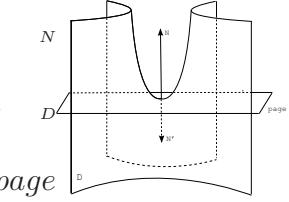
(a)   (b)   (c)   (d)

Figure 5.2: (a,b)Positive and negative elliptic singularities (c,d) Positive and negative hyperbolic singularities
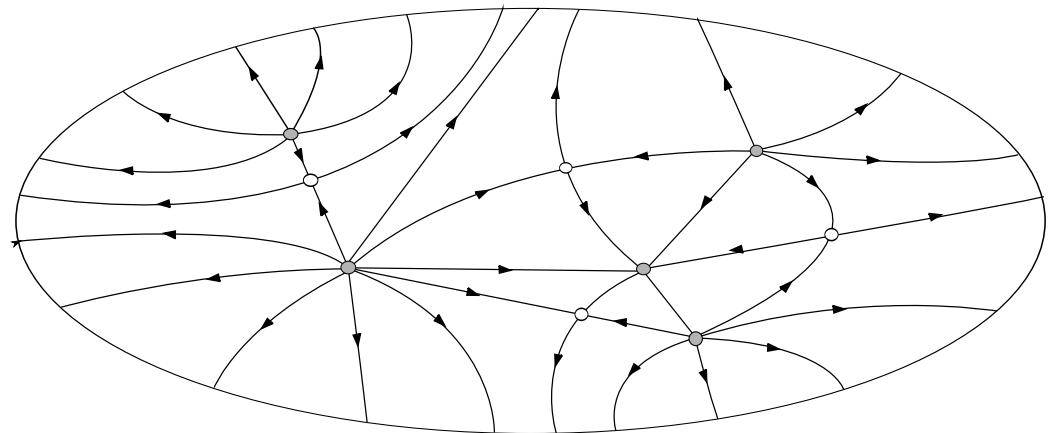

Figure 5.3: Induced braid foliation on D.

A sign can be assigned to each elliptic or hyperbolic singularity in the following way. An elliptic singularity is positive (negative) if the binding intersects the disk D in the direction consistent with (opposite to) N, the normal direction to D (as in Figure 5.2(a,b)). Define e_+ (e_-) to be the number of positive (negative) elliptic singularities.

A hyperbolic singularity is positive (negative) if the normal N to D at the intersection point, is consistent with (opposite to) the normal direction to the page at that point, N' (as in Figure 5.2(c,d)). Define h_+ (h_-) to be the number of positive (negative) hyperbolic singularities.

Definition 5.1.8. Let e be an elliptic singularity in the braid foliation of D, as above. The *valence of e*, $v(e)$, is defined as the number of hyperbolic singularities adjacent to e.

In [6], Bennequin found a way of writing the self-linking number in terms of the singularities in a foliation.

Lemma 5.1.9. *Let U be an unknot spanning the embedded disk D and consider the braid foliation of D as described above. Then $sl(U) = (e_- - h_-) - (e_+ - h_+)$.*

Definition 5.1.10. Associate to a foliation $\mathcal{F}$ of D the number

$$c(\mathcal{F}) = e_+ + e_- + h_+ + h_-,$$

that is the number of its singularities, and call $c(\mathcal{F})$ the *complexity of $\mathcal{F}$*.

5.2 Transverse stabilizations and braid destabilizations

Lemma 5.2.1. *Let e be a negative elliptic singularity in the braid foliation of the disk D, $\partial D = U$, and let h be an adjacent negative hyperbolic singularity that is also adjacent to U. Then U can be transversely isotoped (as in Figure 5.4(a)) so that both e and h are eliminated.*

Proof. The foliation we see on the disk V is the braid foliation. By slightly perturbing it we get the characteristic foliation on V. This foliation determines how the contact structure looks like in a neighborhood of V in M. Using Theorem 2.1.7, we look for a disk V_0 embedded in $(\mathbb{R}^3, \xi_{std})$, with the same characteristic foliation. Consider the disc $V_0 \subset (\mathbb{R}^3, \xi_{std})$ traced by the isotopy

$$(s,t) \rightarrow (t - 3s^2, st - s^3, 18s), \ t \in [-1, 1], \ s \in [-1.1, 1.1]$$

The disk V_0 has the same characteristic foliation as V and for each fixed value of t, the curve $s \rightarrow (t - 3s^2, st - s^3, 18s)$, $s \in [-1.1, 1.1]$ remains transverse to the contact planes, as the form $dz - ydx$ evaluates to $18 + 6s^2(t - s^2) > 0$ on the tangent vector $\langle -6s, t - 3s^2, 18 \rangle$. $\qquad \square$

Lemma 5.2.2. *Let e be an elliptic singularity in the braid foliation of the disk D, $\partial D = U$, such that $v(e) = 1$ and let h be its only adjacent hyperbolic singularity. If h and e are of the same sign then U can be transversely isotoped so that both e and*

h are eliminated. We are going to refer to this isotopy as to a braid destabilization.

Proof. As the foliation is oriented in such a way that the flowlines go towards U from inside D, e can only be a positive singularity. Up to choice of orientation on the disk D the situation is identical to that of the previous lemma, except the t-parameter should be chosen with the opposite orientation.

□

PSfrag replacements

U

(a)

PSfrag replacements

U

(b)
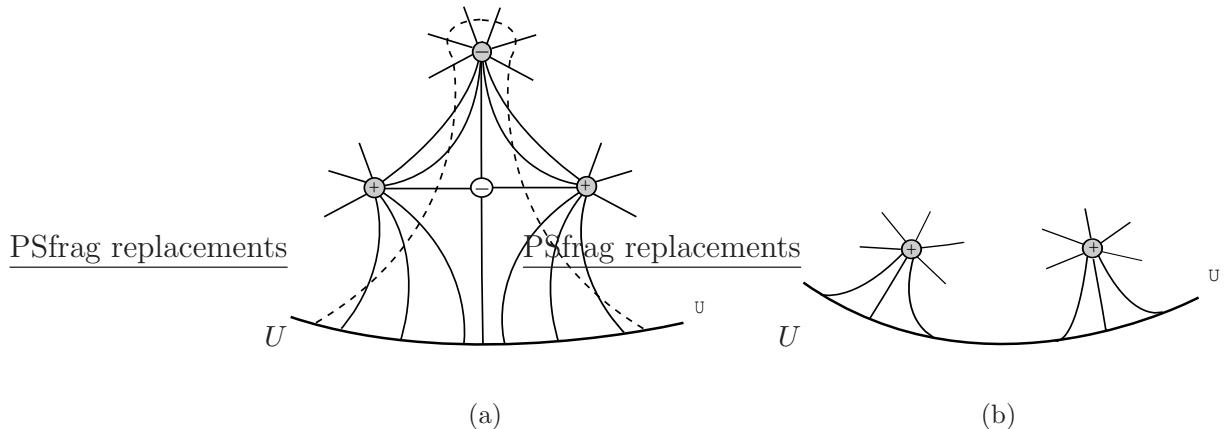
Figure 5.4: Part of D before and after the elimination of a negative hyperbolic singularity adjacent to U.

Definition 5.2.3. A knot K is said to be obtained through a *transverse stabilization* of a transverse knot K' if $K = \delta \cup \beta$ and $K' = \delta \cup \beta'$ and $\beta \cup \beta'$ bounds a disk with one positive elliptic and one negative hyperbolic singularity (as in Figure 5.5(b)). Here the term *transverse* refers to the position with respect to the characteristic foliation.

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Note that through a transverse stabilization the self-linking number is decreased by 2. The following is a well known lemma, but it's only sketched in the present literature. We give a more detailed proof.

Lemma 5.2.4. *If K_1 and K_2 are obtained through a transverse stabilization from two transversely isotopic knots K_1' and K_2', then K_1 and K_2 are themselves transversely isotopic.*

Proof. Since K_1' and K_2' are transversely isotopic we may assume $K_1' = K_2' = K'$. Let D_1 and D_2 be the disks given by the two stabilizations (as the shaded disk in Figure 5.5(c)). Let e_1, h_1 and e_2, h_2 be the pairs of singularities on D_1 and D_2 and α_1, α_2 the two Legendrian arcs formed by the stable manifolds of h_1 and h_2. We may transversely isotop K' so that $K_1 \setminus K'$ lies arbitrarily close to α_1 and similarly for K_2 and α_2. We can see this in Figure 5.5(c). The pictured part of the knot can be brought towards the α-arc, as indicated by arrows, remaining transverse to the characterstic foliation. The claim in the lemma follows from the following three additional lemmas. $\square$

Lemma 5.2.5. *There exists a contact isotopy preserving K' taking the endpoint of α_1 in D_1 to the endpoint of α_2 in D_2.*

Lemma 5.2.6. *There exists a contact isotopy preserving K' taking α_1 to α_2.*

Proof. Consider p_1 the common starting point of α_1 and α_2 together with a neighborhood $N(p_1)$ contactomorphic to a neighborhood of the origin in $(\mathbb{R}^3, \xi_{std})$, $N(0)$.

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|		
(a)	(b)	(c)
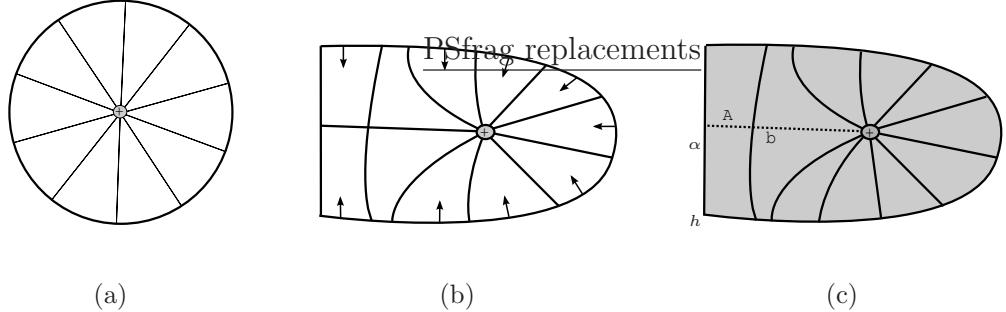
Figure 5.5: (a) Disk foliation of complexity 1 (b) Valence 1 elements in E_U (c) Stabilization disk

Consider ξ_{std} given by the 1-form $dz + r^2 d\theta$. We identify $\alpha_1 \cap N(p_1)$ with the arc $N(0) \cap (\theta = \theta_1) \cap (z = 0)$ and $\alpha_2 \cap N(p_1)$ with the arc $N(0) \cap (\theta = \theta_2) \cap (z = 0)$, for $\theta_1 \neq \theta_2$. There is an isotopy given by $\psi_t(r, \theta, z) = (r, (1-t)\theta_1 + t\theta_2, z), t \in [0, 1]$, that takes $N(0) \cap (\theta = \theta_1) \cap (z = 0)$ to $N(0) \cap (\theta = \theta_2) \cap (z = 0)$. Pulling this back through the contactomorphism gives an isotopy between $\alpha_1 \cap N(p_1)$ and $\alpha_2 \cap N(p_1)$. A compactness argument completes the proof.

$\square$

Lemma 5.2.7. *Any two stabilizations of K' along a fixed Legendrian arc are transversally isotopic.*

Proof. Consider D_1 and D_2 the two stabilization disks that coincide along the Legendrian arc α. By a small deformation we can assume they coincide in a small neighborhood of the α arc. As described above, the knot of can be transversely isotoped to such a neighborhood. $\square$

5.3 In a tight contact structure the unknot is transversely simple

Theorem 5.3.1. *Let (M, ξ) be a 3-dimensional contact manifold supported by the open book decomposition (L, π). If (M, ξ) is tight then the unknot is transversely simple.*

Proof. Considering an arbitrary representative of the unknot, $U \subset M$, and $D \subset M$ an embedded disk with $\partial D = U$. Using Lemma 5.2.1, the braid foliation on D may be assumed to have no couples consisting of a negative hyperbolic singularity and a negative elliptic singularity adjacent to U.

First, we show that any transverse representative of the unknot of maximal self-linking number can be transversally isotoped to the trivial braid, that is a braid bounding a disk foliated as in Figure 5.5(a). This proves the theorem in the maximal self-linking number case. Second, we prove the theorem for arbitrary self-linking number.

Assume now that U is a transverse representative of the unknot of maximal self-linking number. The goal is to show that the foliation can be changed into one of minimal complexity, i.e. such that $c(\mathcal{F}) = 1$. We are going to see that due to the maximality of $sl(U)$, the foliation on D has no negative hyperbolic singularity with both unstable manifolds going towards U. Having already eliminated the negative hyperbolic singularities adjacent to U (as in Lemma 5.2.1), all singular

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points adjacent to U have positive sign.

Denote by E_U the set of elliptic singularities adjacent to U. The elements of E_U are connected in between them through stable manifolds of positive elliptic singularities. For $e \in E_U$ define $v_+(e)$ to be the number of such connections. If for all $e \in E_U$, $v_+(e) \geq 2$, then the graph with vertices the elements of E_U and edges the above connections exhibits a cycle. This cycle is the boundary of an overtwisted disk. As ξ is tight, there must exist an element $e' \in E_U$ with $v_+(e') \in \{0, 1\}$.

If E_U contains e' with $v_+(e') = 0$, then the disk D is foliated as in Figure 5.5(a) (as it has only one connected component) and therefore $c(\mathcal{F}) = 1$.

If E_U contains e' with $v_+(e') = 1$, then a neighborhood of e' is foliated like in Figure 5.5(b). In this case the complexity of the foliation can be reduced by 2 through a braid destabilization as in Lemma 5.2.2.


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L_1

L_2
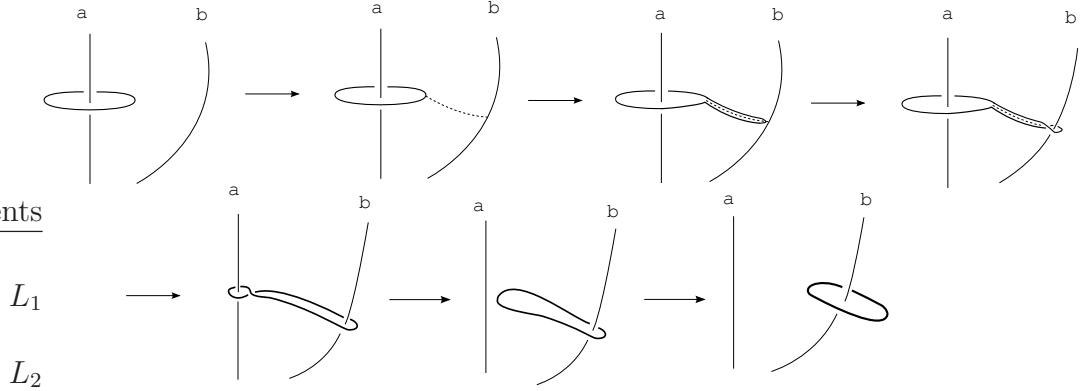
Figure 5.6: Isotopy between trivial unknots linking different binding components.

If the foliation on D has negative hyperbolic singularities with both unstable

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manifolds going towards U, we look at an inner most such singularity. Following Lemma 5.2.1, its unstable manifolds together with part of U will bound a disk foliated in a such a way that all singularities adjacent to the boundary are positive. The same argument as above implies that U must have been obtained from some unknot U' through a transverse destabilization as in Definition 5.2.3, contradicting the maximality of the self-linking number.

If two transversal representatives of the unknot both have maximal self-linking number, they are transversely isotopic, as by the above process can be both braid destabilized by transverse isotopies to a trivial representative, with induced braid foliation as the on in Figure 5.5(a). If the two trivial unknots are linking the same binding component then one can isotop one to the other by first shrinking both in a small enough neighborhood of the respective component, that looks like the a neighborhood of the z-axis in $(\mathbb{R}^3, \xi_{std})/_{z\sim z+1}$. But what if through the destabilization process the two unknots end up linking two different binding components, L_1 and L_2? In this case one of the unknots can be first dragged towards the opposite binding component, linked through a braid stabilization (the reversed process described by Lemma 5.2.2) with this and then freed from linking the initial binding component through a braid destabilization. This isotopy is described in Figure 5.6. To guarantee that the unknot remains transverse throughout the process the part of the unknot used to link L_1 is dragged towards L_2 so that it remains in a small neighborhood of an arc between $U \cap \Sigma_0$ and L_2 in a chosen page Σ_0. Using the

x

y

z

$-\epsilon$

ϵ

$t = 1/4$

$t = 1/2$

$t = 3/4$

$t = 1$

$t = 0$
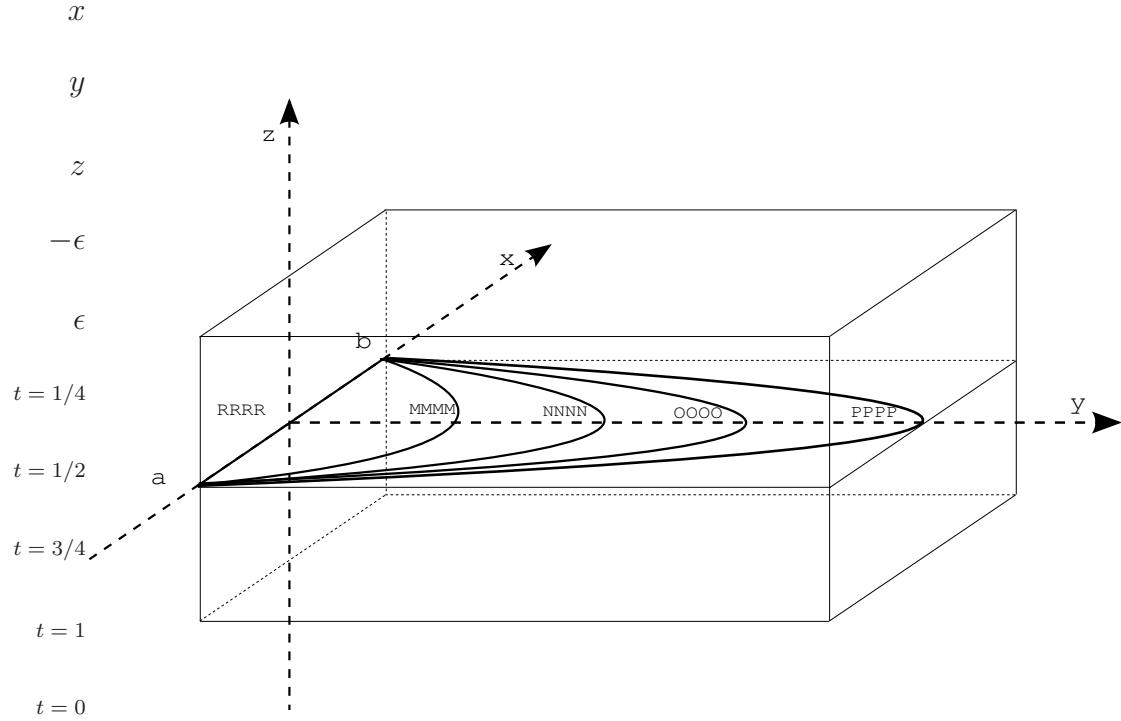
Figure 5.7: Arc isotopy from one binding component to another.

(x, y, z) coordinates for this neighborhood (where (x, y) are coordinates on the page and z is the coordinate normal to the page) the actual isotopy can be described by:

$$f_t(x, y, z) = (x, t(1 - (\tfrac{x}{\epsilon})^2), z), \text{ for } t \in [0, 1] \text{ (see Figure 5.7)}$$

Assume now that U is a transverse unknot which does not have maximal self linking. In the braid foliation of D there must exist negative hyperbolic singularities having both unstable manifolds going towards U. Otherwise, as above, after a transverse isotopy, D can be assumed to be foliated as in Figure 5.5(a), thus contradicting the non-maximality of the self-linking number. Using the same argument as in the maximal self-linking case, the part of the disk bounded by the unstable manifolds of an innermost one such singularity together with part of U (α_1 and δ

in Figure 5.5(c)) may be assumed to contain a single singularity, a positive elliptic singularity.

Therefore, in this case, U can be obtained from U_0, a transversal representative of the unknot with $sl(U_0) = sl(U) + 2$, through a transverse stabilization, as described in Definition 5.2.3. An inductive argument completes the proof. $\square$

Chapter 6

Surface changes

In the beginning of Chapter 5 we described a generic braid foliation induced on a disk D bounding the unknot U. We also described ways of changing this foliation through certain stabilizations and destabilizations. In this chapter we are concerned with a different type of changes in a foliation, involving adjacent hyperbolic singularities of the same sign. Originally studied by Birman and Menasco in [8] for $(\mathbb{R}^3, \xi_{std})$, this change in foliation is a key ingredient in proving that the unknot is exchange reducible in $(\mathbb{R}^3, \xi_{std})$.

6.1 Few preliminaries

Consider an open book decomposition (Σ, ϕ) for the 3-manifold M, a null-homologous knot $K \subset M$, and a surface S such that $\partial S = K$. We are going to look at the braid foliation on S, induced by the intersection with the pages. Following a general

PSfrag replacements

K K K

(a)   (b)   (c)   (d)   (e)
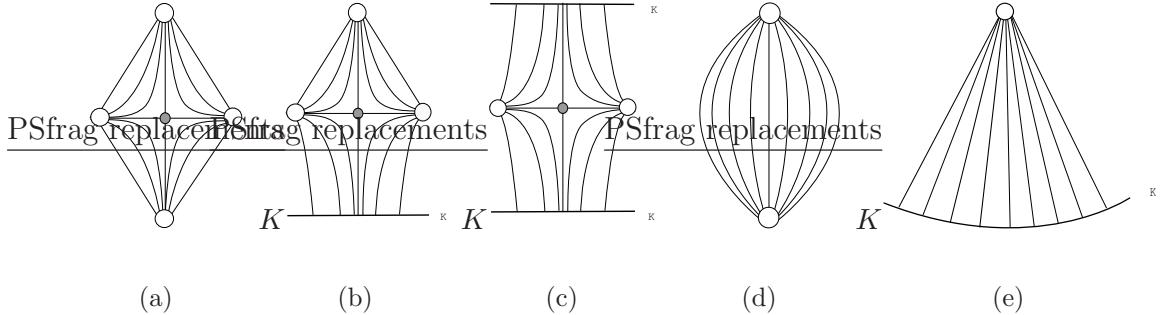
Figure 6.1: Possible tiles in the decompositions of S.

position argument as the one on Chapter 5 and Lemma 5.1.7, there is a natural way of decomposing S into tiles, according to the singularities of this foliation. This decomposition was studied by Birman and Menasco in the standard case. The tiles we are considering are either neighborhoods of a hyperbolic singularity (as in Figure 6.1$(a-c)$) or foliated disks as in Figure 6.1(d,e).

To each tile of type (a), (b) or (c) we will associate a sign, the same as the one of its singularity.

6.2 Changing order of adjacent saddle points

Theorem 6.2.1. *With the above notations, assume a decomposition for S contains tiles T_1 and T_2 of the same sign which are adjacent at an arc going between two elliptic points. Let h_1 and h_2 be the two hyperbolic singularities contained in T_1 and T_2 and θ_1, $\theta_2 \in S^1$ such that $h_1 \in S \cap \Sigma_{\theta_1}$ and $h_2 \in S \cap \Sigma_{\theta_2}$ Then there is an isotopy F taking S to a surface S' such that:*

i) $F(T_1)$ and $F(T_2)$ are adjacent regions of the same sign in the decomposition

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of S'

ii) *The decomposition of S outside of a small neighborhood of $T_1 \cup T_2$ coincides with that of S' outside of a small neighborhood of $F(T_1) \cup F(T_2)$*

iii) $F(h_1) \in S \cap \Sigma_{\theta_2}$ *and* $F(h_2) \in S \cap \Sigma_{\theta_1}$

Proof. Roughly, this theorem says that given two adjacent hyperbolic singularities, of the same sign (as in Figure 6.2(a)) their order with respect to the θ coordinate can be changed (as in Figure 6.2(b)). In Figure 6.2 the pages are viewed as horizontal planes. We are going to show that the two singularities can be moved past one another by showing the existence of a family of disks that can be used to isotop the

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surface. The proof of the existence of such disks relies on observing that in a small neighborhood of the two tiles, T_1 and T_2, the picture can always be standardized, independent of the open book decomposition.

Consider the two hyperbolic points h_1 and h_2 as in the hypothesis and without
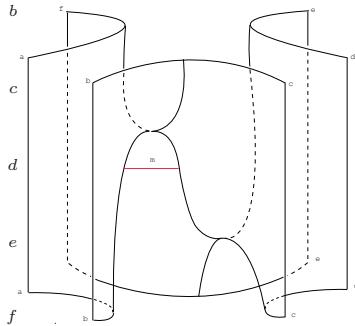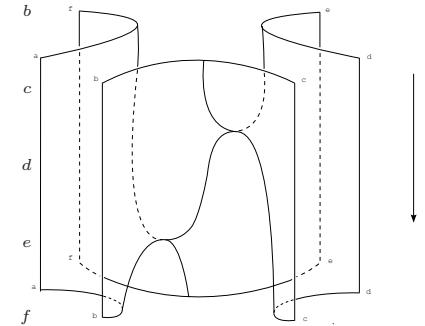


(a)   (b)

Figure 6.2: Neighborhood of adjacent saddle points of the same sign before and after the isotopy.

loss of generality assume they are both positive. As we proceed in the fibration past h_1 and h_2 the traces described by S on consecutive pages in a small enough neighborhood of the two singularities are according to the sequence in Figure 6.3. In this pictures, the signs are labeling the surface S according to its positive or negative side. Thus, independent of the open book decomposition for M, a small disk $D \subset T_1 \cup T_2$ with $h_1, h_2 \in D$ has an embedding as in Figure 6.2(a). The sequence in Figure 6.3 represents local pictures in each Σ_θ around the two singularities. The arcs depicted for some value of θ are subsets of $S \cap \Sigma_\theta$. The set of curves $S \cap \Sigma_\theta$ may have a complicated behavior outside of this neighborhood, depending on the open book, but the local pictures are the same.

Following the proof for the $(\mathbb{R}^3, \xi_{std})$ case in [7] we are looking for a family of curves $\{\alpha_\theta\}_{\theta \in [\theta_1, \theta_2 + \epsilon]}$ as shown in Figure 6.2(a) with interior disjoint from S and K. The union of the α_θ's is a disk V which we are going to use to isotop S.

Call β and β_1 the arcs in the foliation obtained as a consequence of passing through h_1, with β being the arc between the two elliptic singularities common to T_1 and T_2, as in the hypothesis. Consider $\alpha_\theta \subset \Sigma_\theta$ an arc between β and β_1. Pick α_θ so that it's endpoints lie in the small neighborhood D. As β doesn't surger between θ_1 and θ_2 we can choose α_θ for all $\theta \in [\theta_1, \theta_2]$. As both h_1 and h_2 are positive, the traces of D on consecutive pages have to be as in Figure 6.3 (S has to 'meet itself' on the negative side for a positive hyperbolic singularity). Note that for $\theta_2 > \theta > \theta_2 + \epsilon$ one endpoint of α_θ will no longer be in D, but only on Σ_θ. The important aspect is
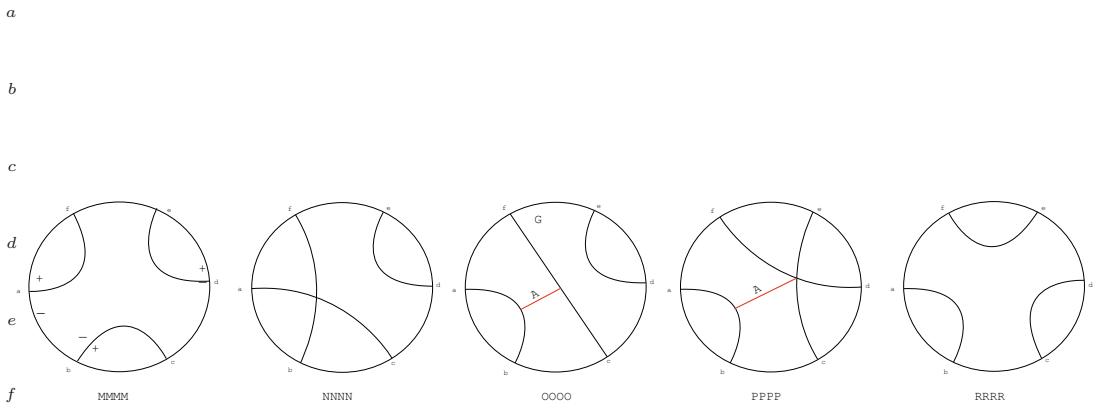
Figure 6.3: Traces of a neighborhood of adjacent saddle points of the same sign on different Σ_θ's.

that that the α_θ's can be considered beyond $\theta = \theta_3$ without its interior intersecting D. This means that the α_θ arc may be considered for $\theta \in [\theta_2, \theta_2 + \epsilon]$.

For the family of arcs $\{\alpha_\theta\}_{\theta \in [\theta_1, \theta_2 + \epsilon]}$, consider the family of disk neighborhoods V_θ parametrized by $\mathcal{L}_\theta : [0, 1] \times (-\delta, \delta)$ such that $\alpha_\theta = \mathcal{L}_\theta([0, 1] \times \{0\})$. Consider the 3-ball given by the union

$$V = \bigcup V_\theta, \text{ where } \theta \in [\theta_1, \theta_2 + \epsilon].$$

Then $\text{int}(V) \bigcap D = \emptyset$. If V has interior disjoint from S and K, we can use V to isotop D and change the order of h_1 and h_2. If there are parts of K or S intersecting V, we can first isotop these forward in the fibration, outside of $V' = \bigcup V_\theta$, where $\theta \in [\theta_1, \theta_2 + \frac{\epsilon}{2}]$ and then isotop D and change the order of h_1 and h_2. $\qquad \square$

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