BRANCHED COVERS OF CURVES WITH FIXED RAMIFICATION LOCUS

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ABSTRACT

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We examine conditions under which there exists a non-constant family of branched covers of curves over an algebraically closed field k of fixed degree and fixed ramification locus, under a notion of equivalence derived from considering linear series on a fixed source curve X. If we additionally impose that the maps are Galois, we show such a family exists precisely when the following conditions are satisfied: there is a unique ramification point, $\mathrm{char}(k) = p > 0$, and the Galois group is $(\mathbb{Z}/p\mathbb{Z})^n$ for some integer $n > 0$. In the non-Galois case, we conjecture that a given map occurs in such a family precisely when at least one ramification index is at least p. One direction of this conjecture is proven and the reverse implication is proven in several cases. We also prove a result concerning the smoothness of the moduli space of maps considered up to this notion of equivalence.

Contents

Chapter 1

Preliminaries

1.1 Overview

Let k be an algebraically closed field, and consider maps between smooth proper algebraic curves $f : X \to Y$ over k (when $k = \mathbb{C}$ this corresponds to X and Y being compact Riemann surfaces). One is often interested in maps which behave like covering spaces (i.e. are étale) except over a finite set $S \subseteq Y$, called the *branch locus*. Such a map is called a *branched cover*.

In algebraic topology, one classifies the covers of a fixed base Y. Since the base is fixed, in this situation two covers $f_i : X_i \to Y$ are considered equivalent if there is a commutative diagram:

$$\begin{tikzcd}
X_1 \arrow[rr, "\cong"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\
& Y &
\end{tikzcd}$$

Alternatively, one can consider two covers $f_i : X \to Y_i$ from a fixed space X to be equivalent if there is a commutative diagram:

$$\begin{tikzcd}
& X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
Y_1 \arrow[rr, "\cong"] & & Y_2
\end{tikzcd}$$

This latter equivalence comes naturally from the study of linear series on X, as explained in Section 1.5. Following the terminology of Brian Osserman in [Oss05], we will call the first equivalence the *covering space perspective* and the latter the *linear series perspective*. Analogs of questions originally considered in the covering space perspective can be examined, perhaps with appropriate modification, in the linear series perspective.

For instance, in the covering space perspective Kevin Coombes and David Harbater consider in [CH85] the relationship between the field of moduli and field of definition. Although for reasons of time it does not appear in this thesis, we prove that the field of moduli is always equal to the field of definition in the linear series perspective.

In the covering space perspective, one can ask when there is a non-constant family of degree d covers with fixed branch locus S. These results are summarized in Section 1.4. In the linear series perspective, the target is allowed to vary so it would not make sense to consider a fixed branch locus. Instead, we fix the ramification locus (the points on the source where the map is not étale). We will study this question in Chapter 3 and Chapter 4.

1.2 Structure of the thesis

In Section 1.3, the notation and definitions used throughout the thesis are given. The remainder of Chapter 1 is a brief exposition on the foundational material used in this thesis.

In Chapter 2 we provide the foundation upon which the results of Chapters 3 and 4 will be built. The results in Section 2.1 are from the literature. Section 2.1.1 provides an important framework for translating ramification conditions into conditions in Schubert calculus. In Section 2.2 we formally define the notion of family we are considering in Definition 2.2.1 and later prove Theorem 2.2.2 which allows us to convert theorems in the literature concerning the finiteness of a class of maps into statements about the non-existence of a non-constant family of such maps.

Chapter 3 concerns the non-Galois case. We first recall results from the literature in Section 3.1 and comment upon their proof when similar methods will be employed later in the thesis. In Section 3.3 we shift the focus from fixing the ramification indices to fixing what we refer to as the differential lengths. This shift in focus is necessary to state Conjecture 3.4.1, which postulates a necessary and sufficient condition for the existence of a non-constant family. In Section 3.5 we introduce the machinery to prove Theorem 3.5.2 concerning the smoothness of a certain moduli space of maps.

Throughout Chapter 4, in addition to the assumptions from Chapter 3 we further require that the maps between curves are Galois. The main result of this chapter is Theorem 4.4.1, which provides a necessary and sufficient condition for the existence of a non-constant Galois family. In establishing this result, we first prove Theorem 4.3.8, which provides a translation between equivalence classes of G-Galois covers with fixed source curve and G-actions on that curve. After this translation, Theorem 4.4.1 follows almost immediately in the genus greater than one case, and is proven in the genus one case in Section 4.4.1 and in the genus zero case in Section 4.4.2.

1.3 Definitions, notations, and conventions

1.3.1 Notations and conventions

The following notations will be used for frequently occurring groups: S_n is the symmetric group on n letters, A_n is the subgroup of S_n containing the even permutations, and D_{2n} is the dihedral group of order $2n$.

Let k be a field. The group $\mu_n(k) \subseteq k^\times$ consists of the solutions to $T^n - 1 = 0$. An element of order n in $\mu_n(k)$ (if it exists) will be denoted by ζ_n and referred to as a primitive nth root of unity.

Fix a positive integer n and let $GL_n(k)$ denote the group of invertible $n \times n$ matrices over k. Let $\mathrm{PGL}_n(k)$ denote the quotient of $GL_n(k)$ by the subgroup of $k^\times$-multiples of the identity matrix. The determinant homomorphism $det : GL_n(k) \to k^\times$ descends to a homomorphism $\overline{det} : \mathrm{PGL}_n(k) \to (k^\times)/(k^\times)^n$. Let $\mathrm{PSL}_n(k)$ denote the kernel of $\overline{det}$.

Let X be a scheme. For every point $x \in X$, the stalk of the structure sheaf $\mathcal{O}_X$ at x will be denoted by $\mathcal{O}_{X,x}$. The maximal ideal of $\mathcal{O}_{X,x}$ will be denoted by $\mathfrak{m}_x$. The residue field at x, which is equal to $\mathcal{O}_{X,x}/\mathfrak{m}_x$, will be denoted by $\kappa(x)$. If X is additionally assumed to be integral, the function field of X will be denoted by $\kappa(X)$.

All schemes which are the base of some family are assumed to be geometrically connected unless otherwise specified. If $f : X \to Y$ is a map between k-curves, X and Y are assumed to be smooth, proper, and geometrically connected, and further f is assumed to be non-constant.

Zero is not an element of the natural numbers, $\mathbb{N}$.

1.3.2 Definitions

Fix a field k. A morphism of k-curves $f : X \to Y$ is a *branched cover* if f is finite and generically étale. Fix a closed point $P \in X$ and let $Q = f(P)$. Let t be a generator for the maximal ideal of $\mathcal{O}_{Y,Q}$ and let $f^\# : \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$ be the homomorphism on stalks induced by f. The *ramification index of* f *at* P, e_P, is equal to $v_P(f^{\#}(t))$ where v_P is the valuation associated to the discrete valuation ring $\mathcal{O}_{X,P}$. Note that e_P is always a positive integer. If $e_P > 1$ (i.e. f is not étale at P), then P is a *ramification point* of f and f is said to be *ramified at* P; moreover Q is *branch point* of f and f is said to be *branched at* Q. The set of ramification points of f is the *ramification locus* and the set of branch points of f is the *branch locus*. Note that the ramification locus is contained in X, whereas the branch locus is contained in Y.

Let $\mathrm{char}(k) = p > 0$. Continuing with the notation from the previous paragraph, f is *tame* or *tamely ramified* at P if p does not divide e_P and is *wild* or *wildly ramified* at P if p divides e_P. For the definition of the differential length which is closely related to the ramification index, and how they compare in the tamely and wildly ramified case, see Section 3.3.

A branched cover of k-curves $X \to Y$ is *Galois* if the induced map on function fields $\kappa(X)/\kappa(Y)$ is a Galois extension of fields.

Two group homomorphisms $\varphi_1, \varphi_2 : G \to H$ are *uniformly conjugate* if there is some $h \in H$ for which $\varphi_1(g) = h\varphi_2(g)h^{-1}$ for every $g \in G$.

For the definition of a family of maps of curves used in this thesis, see Section 2.2.

1.4 Étale covers

In algebraic geometry, especially over fields which are not the complex numbers, one does not have notions of loops or covering spaces. However, by considering a notion analogous to covering spaces, one can prove theorems strikingly similar to those from algebraic topology. The analog of a covering map is an *étale morphism* $Y \to X$. One forms the *étale fundamental group* $\pi_1^{ét}(X)$ by taking the automorphisms of the inverse system of all finite étale covers of X. Recall that two covers $f_i : X_i \to Y$ are considered equivalent if there is a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[rr, "\cong"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

Consider the case where X is a smooth proper curve over a field k. Given a finite étale cover $f : V \to U$ of smooth curves, where U is an open subscheme of X, there is a unique smooth proper curve $Y \supseteq V$ and by the valuative criterion for properness an extension of f to Y, such that $f : Y \to X$ is a branched cover. Connected degree d branched covers of X which are étale over U are thus in correspondence with group homomorphisms $\pi_1^{ét}(U) \to S_d$ with transitive image, up to uniform conjugation in S_d.

When $k = \mathbb{C}$, the group $\pi_1^{ét}(U)$ is finitely generated, and is the profinite completion of the topological fundamental group of U as a Riemann surface. Therefore, there are only finitely many degree d branched covers of X with fixed branch locus.

If $\mathrm{char}(k) = p > 0$, this only holds for all d if $U = X$. The tame étale fundamental group, which classifies tame covers, is finitely generated by [Gro71, Exposé XIII Corollaire 2.12], so there are no non-constant tame families of covers which are étale over U. However, if $U \subsetneq X$, via Artin-Schreier theory one can always construct non-constant families of wild covers which are étale over U. For example, if k has characteristic $p > 0$, the maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $y^p - y = tx$ with parameter t form a family of $\mathbb{Z}/p\mathbb{Z}-$Galois covers of the x-line branched only at the point at infinity. Thus in particular there are infinitely many étale covers of degree p if $U \subsetneq X$.

1.5 Linear series

A *linear series* is a linear subspace V of the vector space of global sections of a line bundle $\mathscr{L}$ on a curve X. Two sections s_1 and s_2 of a line bundle that have no common zeroes determine a map $X \to \mathbb{P}^1_k$, taking $P \in X$ to $(s_1(P) : s_2(P))$. Conversely, a map to $\mathbb{P}^1_k$ yields two sections of a line bundle with no common zeroes, by pulling back generators for the sheaf $\mathcal{O}(1)$. Hence, in studying a linear series V, it is natural to investigate planes inside V. One can easily see that choosing a different basis for the plane corresponds to post-composing the map determined by s_1 and s_2 with a fractional linear transformation (i.e. an automorphism of the target space, $\mathbb{P}^1_k$). Therefore, planes inside a linear series correspond to maps $X \to \mathbb{P}^1$ up to post-composition with fractional linear transformations.

Motivated by the preceding paragraph, unless otherwise specified we will henceforth consider two branched covers $f_i : X \to Y_i$ from a fixed space X to be *equivalent* if there is a commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "\cong"] & & Y_2 \end{tikzcd}$$

Note that this is not the same equivalence relation from Section 1.4, where two covers are considered equivalent if there is an isomorphism of the sources. Since we are instead fixing the source X and allowing the target to vary, we shall speak of a fixed ramification locus instead of a fixed branch locus. In analogy with the covering space situation, one is led to consider the following:

Question 1.5.1. Let X be a smooth proper curve over a field k and S a finite set of closed points on X. Under what conditions does there exist a non-constant family of maps with source X of fixed degree, all with ramification locus S?

A primary focus of this thesis is finding a precise answer to this question, including understanding the behavior of such families. Based upon the results from Section 1.4, one would expect no such tame families to exist. However, as noted by Brian Osserman, the following example illustrates that this is not the case:

Example 1.5.2. ([Oss06b, Example 5.6]) Consider the family of maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ with k a field of characteristic $p > 2$ given by $y = x^{p+2} + tx^p + x$ with parameter t. For every value of t, the map is tamely ramified at the point at infinity and the $(p+1)^{st}$ roots of $-1/2$, étale elsewhere, and no distinct values of t produce equivalent maps.

However, under certain conditions it has been proven there can be no non-constant family with fixed ramification points. We will study this question further in Chapter 3. Previous results of this flavor can be found in Section 3.1.

1.6 Moduli spaces

Let S be a scheme and let Sch_S denote the category of S-schemes. For any S-scheme T, let $h_T : Sch_S \to Set$ be the contravariant functor sending an S-scheme T' to the set of S-morphisms between T' and T, $\mathrm{Hom}_{Sch_S}(T', T)$. If $F : Sch_S \to Set$ is any contravariant functor, an S-scheme M *represents* F, or is a *fine moduli space for* F if h_M is isomorphic to F. By Yoneda's lemma if M exists it is unique up to isomorphism.

A related but weaker notion is that of a coarse moduli space (see [MFK94, p. 99]). An S-scheme M is a *coarse moduli space* for a contravariant functor $F : Sch_S \to Set$ if there is a natural transformation $\eta : F \to h_M$ such that:

(1) For any algebraically closed field k, the map

$$\eta(\mathrm{Spec}(k)) : F(\mathrm{Spec}(k)) \to h_M(\mathrm{Spec}(k))$$

is a bijection.

(2) The natural transformation η is universal. That is, for any S-scheme T and natural transformation $\nu : F \to h_T$ there is a unique natural transformation $\psi : h_M \to h_T$ such that $\nu = \psi \circ \eta$.

By the universal property (2) and Yoneda's lemma, if M exists it is also unique up to isomorphism.

The concept of a moduli space is often utilized in algebraic geometry when the functor F sends an S-scheme T to the set of isomorphism classes of a certain type of family over T. In many situations, there is no fine moduli space for the functor F but nonetheless there is a coarse moduli space, as in the following example:

Example 1.6.1. Let $\mathcal{M} : Sch \to Set$ be the contravariant functor defined by $\mathcal{M}(T) = \{\text{isomorphism classes of smooth elliptic curves over } T\}$ and suppose $\mathcal{M}$ has a fine moduli space M. Consider for any algebraically closed field k of characteristic greater than three the family $\mathcal{F}$ of elliptic curves $zy^2 = x^3 - tz^3$ over $\mathbb{A}^1_k \setminus \{0\}$ with parameter t. On each fiber we may change coordinates so that the elliptic curve becomes $zy^2 = x(x - z)(x + \zeta_3 \cdot z)$, which we shall denote by E. Therefore $\mathcal{F}$ and $E \times \mathbb{A}^1_k$ induce the same morphism $\varphi : \mathbb{A}^1_k \to M$. However one can show that $\mathcal{F}$ and $E \times \mathbb{A}^1_k$ are not isomorphic elliptic curves over $\mathbb{A}^1_k$, and hence $\mathcal{M}$ cannot have a fine moduli space, since there is not a bijection between $\mathcal{M}(\mathbb{A}^1_k)$ and induced morphisms from $\mathbb{A}^1_k$.

However, $\mathcal{M}$ does have a coarse moduli space M. The existence of this coarse moduli space is of great utility, for instance because isomorphism classes of elliptic curves over an algebraically closed field k are in bijection with the k-points of M.

Chapter 2

Moduli Spaces and Families

2.1 Moduli spaces

We introduce some moduli spaces which will aid in answering questions about families. In fact, fundamental results following the definition of a family in Section 2.2 require the tools introduced in this section. Throughout this section k will be an algebraically closed field.

Let X and Y be smooth projective curves over k. Define $\mathcal{H}(X, Y)$ to be the contravariant functor from k-schemes to sets such that $\mathcal{H}(X, Y)(S) = \{S\text{-morphisms } X_S \to Y_S\}$. This functor is shown to be representable in [Gro61, p. 220–221]. More precisely, it is representable by the open subscheme of the Hilbert scheme of $X \times Y$ on which the first projection is an isomorphism. The Hilbert scheme of $X \times Y$ is a disjoint union of projective schemes parameterizing closed subschemes with a fixed Hilbert polynomial. For $\mathcal{H}(X,Y)$, the Hilbert polynomial uniquely determines the degree of the map $X \to Y$. Hence, by the taking the associated Hilbert polynomial for a given d, the functor $\mathcal{H}_d(X,Y)$ sending a k-scheme S to $\{S\text{-morphisms } X_S \to Y_S \text{ of degree } d\}$ is representable by a quasi-projective scheme $H_d(X,Y)$. By [Oss06b, Theorem A.6] there is a free action of the group scheme $\mathrm{Aut}(Y)$ on $H_d(X,Y)$ where $\varphi \in \mathrm{Aut}(Y)$ acts on the point corresponding to a map $f : X \to Y$ by sending it to the point corresponding to $\varphi \circ f : X \to Y$. In summary, we have the following:

Theorem 2.1.1. *Let k be an algebraically closed field. Fix smooth projective curves X and Y over k and $d \in \mathbb{N}$. The contravariant functor $\mathcal{H}_d(X,Y) : Sch_k \to Set$ defined by $\mathcal{H}_d(S) = \{S\text{-morphisms } X_S \to Y_S \text{ of degree } d\}$ is representable by a quasi-projective scheme $H_d(X,Y)$. Moreover, the natural action of* $\mathrm{Aut}(Y)$ *on $H_d(X,Y)$ is free.*

We are, however, ultimately interested in equivalence classes of maps. Recall that two maps $f_i : X \to Y_i$ from a fixed space X are equivalent if there is a commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "\cong"] & & Y_2 \end{tikzcd}$$

Suppose that we fix the target curve to be $\mathbb{P}^1_k$. As explained in Section 1.5, an equivalence class of degree d maps $X \to \mathbb{P}^1_k$ corresponds to the choice of an isomorphism class of a degree d line bundle $\mathscr{L}$ on X and a two dimensional linear subspace $V \subseteq H^0(X, \mathscr{L})$ such that V has no base points. A pair $(\mathscr{L}, V)$ where V is an $r+1$ dimensional subspace of the global sections of a line bundle $\mathscr{L}$ of degree d is called a *linear series of degree d and dimension r*, or a $\mathfrak{g}^r_d$. There is a fine moduli space of linear series of fixed degree and dimension on a smooth projective curve X:

Theorem 2.1.2. *([Oss11, Theorem 1.1] - Brill-Noether Theorem) Let k be an algebraically closed field. Given g, r, d, and X a smooth projective curve over k of genus g, there is a fine moduli space $G^r_d(X)$ of linear series on X of dimension r and degree d. If we set $\rho = (r+1)(d-r) - rg$, every component of $G^r_d(X)$ has dimension at least ρ. On a general smooth projective curve of genus g, $G^r_d(X)$ has dimension exactly ρ, and in particular is empty if $\rho < 0$.*

In addition to specifying the degree, we also wish to specify some ramification data. In Question 1.5.1, the ramification points must remain fixed. After fixing the ramification points and the degree of the map, there are finitely many choices of valid ramification indices at each ramification point. Therefore, if one can show in a given circumstance that the number of equivalence classes of maps is finite after additionally specifying the ramification indices, Theorem 2.2.2 below will imply that there is no non-constant family with fixed ramification points.

Let $(\mathscr{L}, V)$ be a $\mathfrak{g}^r_d$ on X, P a closed point of X, and $a_0(P) < \cdots < a_r(P)$ the integers that arise as orders of vanishing at P of sections in V. For each i, let $\alpha_i(P) = a_i(P) - i$. The $a_i(P)$ are collectively called the *vanishing sequence* of

$(\mathscr{L}, V)$ at P; the $\alpha_i(P)$ are collectively called the *ramification sequence*. The linear series is said to be *unramified* at P if $\alpha_i(P) = 0$ for every i and is otherwise said to be *ramified* at P. A linear series is *inseparable* if it is ramified at every point $P \in X$, and is otherwise said to be *separable*. We have the following bound on the ramification of a linear series:

Theorem 2.1.3. *([Oss06a, Proposition 2.4]) Let X be a smooth proper curve of genus g over an algebraically closed field k and $(\mathscr{L}, V)$ a $\mathfrak{g}^r_d$ on X. If $(\mathscr{L}, V)$ is separable, then the inequality*

$$\sum_{P \in X} \sum_{i=0}^{r} \alpha_i(P) \leq (r+1)d + \binom{r+1}{2}(2g-2)$$

holds.

Assume $r = 1$ and the linear series $(\mathscr{L}, V)$ is separable and base point free. Then $\alpha_0(P) = 0$ for every $P \in X$ and $\alpha_1(P)$ is the ramification index of the map $X \to \mathbb{P}^1_k$ induced by the linear series.

Due to the existence of inseparable linear series, a version of Theorem 2.1.2 specifying the ramification is not known in general when $\operatorname{char}(k) > 0$. For separable, base point free, one dimensional linear series, we have the following:

Theorem 2.1.4. *([Oss05, Theorem 1.2]) Let k be an algebraically closed field. Given g, d, n, and a sequence $e_1, ..., e_n$, let*

$$\rho = 2(d-1) - 2g - \sum_i (e_i - 1).$$

Then for any smooth projective curve X over k of genus g and distinct points $P_1, ..., P_n \in X$, there is a fine moduli space $G_d^{sep}(X, P_1, ..., P_n, e_1, ..., e_n)$ of separable base point free $\mathfrak{g}_d^1$'s on X corresponding to equivalence classes of separable degree d maps $X \to \mathbb{P}_k^1$ with ramification index at least e_i at each P_i. On a general curve X of genus g and n general points on X, the dimension is exactly ρ, and in particular is empty if $\rho < 0$.

Remark 2.1.5. If in addition the e_i satisfy

$$\sum_i (e_i - 1) = 2d + 2g - 2,$$

by the Riemann-Hurwitz formula we have specified the entire ramification divisor of the corresponding map $X \to \mathbb{P}_k^1$. In this case, $\rho = 0$ so that for a generic genus g marked curve $(X, P_1, ..., P_n)$ the number of maps satisfying these ramification conditions is finite. Even in the genus zero case where genericity of X is not an issue, Example 1.5.2 shows that the ramification points must be generic as well.

2.1.1 Ramification for maps $\mathbb{P}_k^1 \to \mathbb{P}_k^1$ and Schubert calculus

This section follows §0–§2 of [EH83], but over a field of arbitrary characteristic. We will examine the space $G_d^1(\mathbb{P}_k^1)$ in more detail. When $X = \mathbb{P}_k^1$, there is a unique isomorphism class of degree d line bundles on X, namely $\mathcal{O}(d)$. In this case, a $\mathfrak{g}_d^1$ will simply be a choice of a two dimensional linear subspace of $H^0(X, \mathcal{O}(d))$. The vector space $H^0(\mathbb{P}_k^1, \mathcal{O}(d))$ is isomorphic to $V = Poly_k(d)$, polynomials over k of degree at most d. Therefore, $G^1_d(\mathbb{P}^1_k) \cong Gr(2, V)$, the Grassmannian of two dimensional linear subspaces of the $d+1$ dimensional vector space V.

We will now define Schubert varieties on a Grassmannian. Fix a finite dimensional vector space W of dimension d over an algebraically closed field k. A *complete flag on* W is an increasing sequence of linear subspaces $E_\bullet : E_0 \subsetneq \cdots \subsetneq E_d = W$. Choose integers a_1 and a_2 satisfying $0 \leq a_2 \leq a_1 \leq d-1$. The *Schubert variety* $\Omega_{a_1,a_2} \subseteq Gr(2,W)$ *relative to the flag* $E_\bullet$ is $\{T \in Gr(2,W) | \dim(T \cap E_{d-1-a_1}) \geq 1, \dim(T \cap E_{d-a_2}) \geq 2\}$. The Schubert variety $\Omega_{a_1,a_2}(E_\bullet)$ is a smooth closed subvariety of $Gr(2,W)$ of codimension $a_1 + a_2$. When the flag is unambiguous, we may write Ω_{a_1,a_2}.

Fix a coordinate x on $\mathbb{P}^1_k$ and a point P unequal to ∞. For this P, define $E_i(P)$, the complete flag on $V = Poly_k(d)$, as $\mathrm{span}\{(x-P)^d, ..., (x-P)^{d-i}\}$. Define $E_i(\infty) = Poly_k(i)$. It is easy to see that a base point free linear series T is contained in $\Omega_{a_1,0}(E_\bullet(P))$ if and only if the associated map $f_T : \mathbb{P}^1_k \to \mathbb{P}^1_k$ has ramification index at least $a_1 + 1$ at P. Based on this observation, we have the following:

Proposition 2.1.6. *Fix $P_1, ..., P_n$ closed points of $\mathbb{P}^1_k$ and positive integers $e_1, .., e_n$. Every linear series corresponding to a map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ with ramification index at least e_i at P_i is contained in*

$$\bigcap_i \Omega_{(e_i-1),0}(P_i).$$

When specifying all the ramification indices so that $\sum_i (e_i - 1) = 2d-2$, we have $\sum_i \mathrm{codim}(\Omega_{(e_i-1),0}(P_i)) = \dim(Gr(2, Poly_k(d)))$. The expected number of maps is *therefore finite.*

Example 2.1.7. We shall count the number of equivalence classes of separable degree 4 maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ ramified at points $P_1, ..., P_4$ with ramification indices $e_1, e_2 = 3$ and $e_3, e_4 = 2$. By the discussion above, every such map lies in

$$\Omega_{2,0}(P_1) \cap \Omega_{2,0}(P_2) \cap \Omega_{1,0}(P_3) \cap \Omega_{1,0}(P_4).$$

Using Pieri's formula ([GH78, p. 203]), we compute the cup product of the Poincaré duals of the Schubert varieties as $\sigma_{2,0} \cup \sigma_{2,0} \cup \sigma_{1,0} \cup \sigma_{1,0} = 2\sigma_{3,3}$, so we expect there to be two equivalence classes of maps. We shall calculate the number of maps directly and compare this to the number expected from Pieri's formula.

By the Riemann-Hurwitz formula, given a separable degree 4 map with source and target the projective line, $\sum_i (e_i - 1) = 6$. Hence, if the map is tamely ramified we have specified all the ramification by specifying the ramification at $P_1, ..., P_4$. If there is any wild ramification we cannot have such a map since then $e_i - 1 < \text{length}((\Omega_{\mathbb{P}^1/\mathbb{P}^1})_{P_i})$. Therefore to have any separable degree 4 maps with these ramification indices, it must be the case that $\text{char}(k) > 3$.

Assume $\text{char}(k) > 3$ and the four ramification points are $0, 1, \infty$, and λ with ramification indices $3, 3, 2$, and 2 respectively. Since the degree of the map is only four, 0 and ∞ must map to distinct points, and by post-composing with a fractional linear transformation we may assume 0 and ∞ map to 0 and ∞ respectively. Our map must then have the form

$$\frac{x^4 + ax^3}{cx + b}$$

with $a, b, c \neq 0$. After post-composing with the fractional linear transformation $\varphi(x) = cx$, we may assume $c = 1$. To have the desired ramification, the discriminant must be a scalar multiple of $x^2(x-1)(x-\lambda)$ (for a discussion of the relationship between the ramification points and the discriminant, see Section 3.3). One checks that this condition is equivalent to a and b satisfying

$$b = \frac{\lambda}{a}, \; 2a^2 + 3(1+\lambda)a + 4\lambda = 0.$$

Therefore unless $\lambda = \pm\sqrt{32/9} + 1$ we have two equivalence classes of maps as predicted by Pieri's formula. If $\lambda = \pm\sqrt{32/9} + 1$ we have only one class of maps, and in this case the point corresponding to the map is a double point in

$$\bigcap_i \Omega_{(e_i-1),0}(P_i).$$

In characteristic 2, the plane $T = \operatorname{span}\{1, x^4\}$, which corresponds to an inseparable map, is contained in $\Omega_{3,0}(P)$ for every $P \in \mathbb{P}^1_k$ and is the unique point in $\cap\Omega_{(e_i-1),0}(P_i)$. In characteristic 3, for any choice of a, the plane $T = \operatorname{span}\{(x-a), (x-a)x^3\}$ is contained in $\Omega_{2,0}(P)$ for every P. To see this, note that T also has a basis $\{(x-a), (x-a)(x-c)^3\}$ for every affine point $P = (x-c)$. Therefore in this case $\cap_i\Omega_{(e_i-1),0}(P_i)$ again contains no separable linear series, but contains a one dimensional space of inseparable linear series with base points.

Using Schubert varieties to describe ramification conditions, we can show that the separable linear series form an open subscheme:

Proposition 2.1.8. *The points corresponding to separable linear series form an open subscheme of $G^1_d(\mathbb{P}^1_k)$. This subscheme will be denoted by $G^{sep}_d(\mathbb{P}^1_k)$ or G^{sep}_d when the $\mathbb{P}^1_k$ is understood.*

Proof. Every inseparable linear series is contained in $\Omega_{1,0}(P)$ for each closed point P. However, by Theorem 2.1.3 a separable linear series can only be ramified at finitely many points. Therefore

$$\bigcap_{P \in \mathbb{P}^1} \Omega_{1,0}(P)$$

is precisely the inseparable locus, and since it is an intersection of closed subschemes it is closed. □

2.1.2 First order deformations of a map

As is often the case, we shall use first order deformations to study moduli space properties. For the generalities in this section we follow [Ser06]. We shall work over an algebraically closed field k. The spectrum of the dual numbers over k, $\mathrm{Spec}(k[t]/(t^2))$, shall be denoted by S. Any otherwise unspecified morphism $\mathrm{Spec}(k) \to S$ will be the morphism induced from the k-homomorphism $k[t]/(t^2) \to k$ sending t to 0.

Let X be a k-scheme. A *first order deformation* of X is a flat S-scheme $\mathcal{X}$ such that the pullback of $\mathcal{X}$,

$$\begin{tikzcd}
\mathcal{X} \times_S \mathrm{Spec}(k) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & S
\end{tikzcd}$$

is isomorphic to X. The *trivial deformation* of X over S is the fiber product $X \times_k S$. A scheme X is *rigid* if it has no non-trivial first order deformations. When X is non-singular (such as in our applications), we have the following classification of first order deformations of X:

Proposition 2.1.9. *[Ser06, Proposition 1.2.9] Suppose X is non-singular. Then there is a 1-1 correspondence between isomorphism classes of first order deformations of X and $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf of X.*

There is also a notion of a first order deformation of a morphism between schemes. Let $f : X \to Y$ be a morphism of k-schemes. A *first order deformation of f* is a morphism $F : \mathcal{X} \to \mathcal{Y}$ of flat S-schemes such that the pullback of F,

$$\begin{tikzcd}
\mathcal{X} \times_S \operatorname{Spec}(k) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{Y} \times_S \operatorname{Spec}(k) \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec}(k) \arrow[r] & S
\end{tikzcd}$$

is isomorphic to f. When X and Y are non-singular, we have the following:

Proposition 2.1.10. *[Ser06, Proposition 3.4.2 (ii)] Let $f : X \to Y$ be a morphism of non-singular k-schemes. Then the first order deformations of f between $\mathcal{X} = X \times_k S$ and $\mathcal{Y} = Y \times_k S$ are in 1-1 correspondence with $H^0(\Gamma_f, \mathcal{N}_{\Gamma_f/X \times Y}) \cong H^0(X, f^*(\mathcal{T}_Y))$ where Γ_f is the graph of f and $\mathcal{N}_{\Gamma_f/X \times Y}$ is the normal bundle of $\Gamma_f \subseteq X \times_k Y$.*

Next, we will restrict to the case of a degree d map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$. This discussion will be the foundation of the results in Section 3.5. Since $\mathbb{P}^1_k$ is non-singular, by Proposition 2.1.9 its first order deformations correspond to elements of $H^1(\mathbb{P}^1_k, \mathcal{T}_{\mathbb{P}^1})$. The tangent sheaf of the projective line has no global sections, so it is rigid. Therefore, every first order deformation of f is between $X \times S$ and $Y \times S$ and hence Proposition 2.1.10 provides a correspondence between $H^0(\Gamma_f, \mathcal{N}_{\Gamma_f/\mathbb{P}^1\times\mathbb{P}^1})$ and every first order deformation of f.

Using this correspondence, we shall obtain an explicit description of the first order deformations of f:

Proposition 2.1.11. *Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a degree d map, written as a rational function $y = g(x)/h(x)$. The first order deformations of f are of the form*

$$y = \frac{g(x) + g_1(x)t}{h(x) + h_1(x)t}$$

where $\deg(g_1(x), h_1(x)) \leq d$, and every map of the above form is a first order deformation of f.

Proof. Take an element $\eta \in H^0(\Gamma_f, \mathcal{N}_{\Gamma_f/\mathbb{P}^1\times\mathbb{P}^1})$, and let $\mathcal{I}$ be the ideal sheaf of $\Gamma_f \subseteq \mathbb{P}^1_k \times \mathbb{P}^1_k$. Since $\mathcal{N}_{\Gamma_f/\mathbb{P}^1\times\mathbb{P}^1}$ is the dual of $\mathcal{I}/\mathcal{I}^2$, $\eta \in \mathrm{Hom}_{\mathcal{O}_{\Gamma_f}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\Gamma_f})$.

Write f as a rational function $y = g(x)/h(x)$, and for simplicity choose coordinates so that infinity and zero are fixed points of f. By our choice of coordinates, Γ_f is contained in two affine patches on $\mathbb{P}^1_k \times \mathbb{P}^1_k$, one away from infinity on both copies of $\mathbb{P}^1_k$ and another away from 0 on both copies of $\mathbb{P}^1_k$. Denote these affine patches by $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\widetilde{A})$ respectively, where $A = k[x, y]$ and $\widetilde{A} = k[\widetilde{x}, \widetilde{y}]$ with the relation that $x\widetilde{x} = 1$ and $y\widetilde{y} = 1$.

On $\mathrm{Spec}(A)$, Γ_f corresponds to the ideal $I = (h(x)y - g(x))$. Therefore on $\mathrm{Spec}(A)$, η restricts to an element of $\mathrm{Hom}_{A/I}(I/I^2, A/I)$. Such a homomorphism is determined by the image of the generator, $h(x)y - g(x)$. Let $\alpha_A(x, y)$ be the image. Since f is degree d, on $\mathrm{Spec}(\widetilde{A})$ a generator for the ideal of Γ_f is $\widetilde{y}\,\widetilde{x}^d(h(x)y - g(x)) \in k[\widetilde{x}, \widetilde{y}]$. Let $\alpha_{\widetilde{A}}(\widetilde{x}, \widetilde{y})$ be the image of $\widetilde{y}\,\widetilde{x}^d(h(x)y - g(x))$. By restricting to $\mathrm{Spec}(A) \cap \mathrm{Spec}(\widetilde{A})$, we see that $\alpha_{\widetilde{A}}(\widetilde{x}, \widetilde{y}) = \widetilde{y}\,\widetilde{x}^d \alpha_A(x, y)$. Therefore, since $\alpha_{\widetilde{A}}(\widetilde{x}, \widetilde{y})$ is a polynomial in $\widetilde{x}$ and $\widetilde{y}$, it must be the case that $\deg_x(\alpha_A(x, y)) \leq d$ and $\deg_y(\alpha_A(x, y)) \leq 1$. Hence, we may write $\alpha_A(x, y) = g_1(x) - h_1(x)y$ where $\deg(g_1(x), h_1(x)) \leq d$.

By following the proof of [Ser06, Proposition 3.2.1(ii)], which is the essential ingredient in Proposition 2.1.10, on $\mathrm{Spec}(A)$, η corresponds to the first order deformation of $\Gamma_f \subseteq \mathbb{P}^1_k \times \mathbb{P}^1_k$ with ideal $(h(x)y - g(x) + \alpha_A t)$. Writing this as a rational function $\mathbb{P}^1_S \to \mathbb{P}^1_S$, we have

$$y = \frac{g(x) + g_1(x)t}{h(x) + h_1(x)t}.$$

By reversing this argument, any map of the form $(g(x)+g_1(x)t)/(h(x)+h_1(x)t)$ where $\deg(g_1(x), h_1(x)) \leq d$ will produce a first order deformation on $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\widetilde{A})$ which agree on the intersection to yield a first order deformation of Γ_f. $\square$

Remark 2.1.12. Based on the form of the first order deformation in Proposition 2.1.10, since both $g_1(x)$ and $h_1(x)$ can be arbitrary polynomials of degree at most d, it may initially appear that the tangent space of the moduli space $H_d(\mathbb{P}^1, \mathbb{P}^1)$ at the point corresponding to f is $2d+2$ dimensional. However, the map which determines $g_1(x)$ and $h_1(x)$ has target A/I. Using the relation $g(x) = h(x)y$ in I, we can eliminate the $\deg(g(x))$ term from $g_1(x)$, and so the tangent space is actually $2d+1$ dimensional. This agrees with $G^1_d(\mathbb{P}^1_k)$ being $2d-2$ dimensional, since the base point free points of $G^1_d(\mathbb{P}^1_k)$ correspond to maps up to the free $\mathrm{Aut}(\mathbb{P}^1_k)$-action on the base $\mathbb{P}^1_k$, and $\mathrm{Aut}(\mathbb{P}^1_k)$ is a 3 dimensional group scheme.

2.2 Families

Let k be an algebraically closed field and X, Y_1, and Y_2 smooth projective curves over k. Recall that we consider two maps $f_i : X \to Y_i$ equivalent if there is a commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "\cong"] & & Y_2 \end{tikzcd}$$

In formulating the definition of a family, the essential point we wish to capture is that the source X is fixed but the target is allowed to vary.

Definition 2.2.1. Let k be an algebraically closed field, X a smooth projective curve over k, and S a k-scheme. A *family of degree d maps with source X over S* is a degree d S-morphism $f : X_S \to \mathcal{Y}$ where $\mathcal{Y}$ is a flat S-scheme and the geometric fibers are smooth projective curves. A family is *constant* if the fibers over every closed point of S are equivalent.

Now that we have stated the definition of a family, we can begin to properly answer questions such as Question 1.5.1. However, previous results in the literature, as well as our method of proof for Theorem 4.4.1, show that there are finitely many maps satisfying a set of conditions. Therefore we need a result to translate between statements about finitely many maps and every family being constant:

Theorem 2.2.2. *Let k be an algebraically closed field, X a smooth projective curve over k, S a connected k-scheme, $\mathcal{Y}$ an S-scheme, and $f : X_S \to \mathcal{Y}$ a family of degree d maps over S. If the maps over closed points of S lie in finitely many equivalence classes, then f is constant.*

First, we will prove Theorem 2.2.2 for $\mathcal{Y} = Y_S$ and then reduce the general case to this case. In this restricted case, we will use [Bor69, Lemma 1.8]:

Lemma 2.2.3. *(Closed orbit lemma) Let k be an algebraically closed field, G a smooth k-group of finite type over k, X a scheme of finite type over k, and $\alpha : G \times X \to X$ a group action. For $x \in X(k)$, let $\alpha_x : G \to X$ be the orbit map (i.e. $\alpha(g) = g \cdot x$). Then the set theoretic image of α_x is locally closed, and with the reduced induced scheme structure it is smooth. Moreover, the orbits of minimal dimension are closed.*

Equipped with Lemma 2.2.3, we can provide a proof of Theorem 2.2.2 in the $\mathcal{Y} = Y_S$ case:

Proposition 2.2.4. *Let k be an algebraically closed field and let X and Y be smooth projective curves over k. Let S be a connected k-scheme, and $f : X_S \to Y_S$ a family of degree d maps over S. If the maps over closed points of S lie in finitely many equivalence classes, then f is constant.*

Proof. By Theorem 2.1.1, the contravariant functor $\mathcal{H}_d(X,Y) : Sch_k \to Set$ defined by $\mathcal{H}_d(X,Y)(S) = \{S\text{-morphisms } X_S \to Y_S \text{ of degree } d\}$ is representable by a quasi-projective scheme $H_d(X,Y)$. Hence, $f : X_S \to Y_S$ corresponds to a morphism $\varphi_f : S \to H_d(X,Y)$. Since the maps over closed points of S lie in finitely many equivalence classes, $\varphi_f(S)$ is contained in finitely many $\mathrm{Aut}(Y)$-orbits. Since the action of $\mathrm{Aut}(Y)$ is free by Theorem 2.1.1, each orbit has the same dimension. By Lemma 2.2.3, each orbit is then closed. However, since S is connected, $\varphi_f(S)$ is connected as well, and must then be contained in a single $\mathrm{Aut}(Y)$-orbit. In other words, all the maps over closed points of S lie in a single equivalence class, so the family is constant. □

Using the moduli stack of genus g curves, we can reduce the general case to the situation of Proposition 2.2.4:

Proof of Theorem 2.2.2. Let $\pi : \mathcal{Y} \to S$ be the S-structure morphism. Since X is a projective curve and f is finite, $\pi^{-1}(s)$ is also a projective curve for every closed point $s \in S$. Since π is flat, the arithmetic genus of the fibers is constant and by Definition 2.2.1 the fibers are smooth. Therefore π induces a map $\varphi_\pi : S \to M_g$ to the coarse moduli space of genus g curves over k. Since S is connected and $\varphi_\pi(S)$

contains finitely many closed points by hypothesis, $\varphi_\pi(S)$ is a single point. In other words, every $\pi^{-1}(s)$ is isomorphic. Let Y denote this curve.

Next, we claim that after some étale pullback $\mathcal{Y}$ becomes trivial. That is, there exists an étale covering $\psi : S' \to S$ such that

$$\begin{tikzcd}
\mathcal{Y} \times_S S' \arrow[rr, "\cong"] \arrow[dr] & & Y_{S'} \arrow[dl] \\
& S' &
\end{tikzcd}$$

To prove this claim, let $\mathcal{M}_g$ denote the moduli stack of genus g curves over k. Since $\mathcal{Y}$ has constant fiber corresponding to a closed point of $\mathcal{M}_g$, the maps it induces from S to $\mathcal{M}_g$ and M_g factor as in the commutative diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & M_g
\end{tikzcd}$$

which induces a map $S \to \mathcal{M}_g \times_{M_g} \mathrm{Spec}(k)$. The trivial family $Y \times_{\mathrm{Spec}(k)} S$ also induces a morphism from S to $\mathcal{M}_g \times_{M_g} \mathrm{Spec}(k)$. Since $\mathcal{M}_g \times_{M_g} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a gerbe, every two objects over S are locally isomorphic in the étale topology, so there is an étale covering $\psi : S' \to S$ over which there is an isomorphism $\xi : \mathcal{Y} \times_S S' \to Y'_S$. Hence, after an étale base change we have a family

$$\begin{tikzcd}
X_{S'} \arrow[rr, "\xi \circ (f \times \psi)"] \arrow[dr] & & Y_{S'} \arrow[dl] \\
& S' &
\end{tikzcd}$$

which concludes the proof of the claim.

Denote $\xi \circ (f \times \psi)$ by g. Since $\psi : S' \to S$ is étale, if $\psi(s') = s$ then the fiber of g above s' is equivalent to the fiber of f above s. Thus, since there are finitely many equivalence classes of fibers of f, the same is true for g. However, S' may be disconnected. Therefore, to use Proposition 2.2.4, we first restrict to the connected components of S'.

Let $\mathcal{S}$ be the set of connected components of S' and $\mathcal{A}$ the subsets $\mathcal{S}' \subseteq \mathcal{S}$ such that there is a single equivalence class of fiber over closed points of

$$\coprod_{S_i \in \mathcal{S}'} S_i,$$

ordered by inclusion. Since each S_i is connected and has finitely many equivalence classes of fibers, by Proposition 2.2.4 there is only one equivalence class of fiber over S_i, so $\{S_i\} \in \mathcal{A}$ and hence $\mathcal{A}$ is non-empty. By a standard application of Zorn's lemma, $\mathcal{A}$ contains a maximal element $\mathcal{S}'$.

Suppose $\mathcal{S}' \subsetneq \mathcal{S}$ (i.e. the fibers over closed points of $g : X_{S'} \to Y_{S'}$ lie in more than one equivalence class), and let $\mathcal{T} = \mathcal{S} \setminus \mathcal{S}'$. By the maximality of $\mathcal{S}'$, it must be the case that the image under ψ of any connected component in $\mathcal{T}$ is disjoint from the image under ψ of the connected components of $\mathcal{S}'$. However, étale morphisms are open by [Gro64, Proposition 2.4.6], which implies that the images of $\mathcal{T}$ and $\mathcal{S}'$ are disjoint open subsets of S, a contradiction to the connectedness of S. It must then be the case that $\mathcal{S}' = \mathcal{S}$. Therefore the fibers over closed points of $g : X_{S'} \to Y_{S'}$ all lie in a single equivalence class, and the same is then true for the original map $f : X_S \to \mathcal{Y}$. $\square$

Remark 2.2.5. Whenever we say having a finite number of equivalence classes of fibers over a connected base implies there are no non-constant families, it will be by Theorem 2.2.2. Further when stating that there are no non-constant families it will be implicit that we are restricting to families over a connected base.

Chapter 3

Non-Galois Case

3.1 Previous results

Various results from the literature presented in this section concern the finiteness of certain classes of maps. However, using Theorem 2.2.2 we can translate these results into statements about the non-existence of a non-constant family.

3.1.1 Results over the complex numbers

When k is the complex numbers, Question 1.5.1 can be answered using a theorem of Eisenbud and Harris when X is the projective line:

Theorem 3.1.1. *([EH83, Theorem 2.3]) Let $P_1, ..., P_n \in \mathbb{P}^1_{\mathbb{C}}$ be closed points. The number of equivalence classes of degree d maps with ramification locus $P_1, ..., P_n$ is finite. Therefore, there is no non-constant family of such maps.*

This is proven within the framework discussed in Section 2.1.1. Recall that the moduli space of two dimensional degree d linear series is $2d-2$ dimensional. There are finitely many choices $e_1, ..., e_n$ of valid ramification indices for a degree d map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ with ramification locus $P_1, ..., P_n$. Thus, by Theorem 2.2.2, it suffices to show that there are finitely many equivalence classes for each possible set of ramification indices, $e_1, ..., e_n$. Each Schubert variety $\Omega_{(e_i-1),0}(P_i)$ has codimension $(e_i - 1)$, so $\sum_i \operatorname{codim}(\Omega_{(e_i-1),0}(P_i)) = 2d - 2$ by the Riemann-Hurwitz formula. Eisenbud and Harris establish that the intersection of these Schubert varieties is in fact transverse, so only finitely many points lie in the intersection.

The conclusion of Theorem 3.1.1 holds when $\mathbb{P}^1_{\mathbb{C}}$ is replaced by an arbitrary smooth proper complex curve, but the result is not available in the literature:

Theorem 3.1.2. *(Farkas) Let X be a smooth proper curve over the complex numbers and S a finite set of closed points on X. The number of equivalence classes of degree d maps with ramification locus S is finite. Therefore there are no non-constant families of such maps.*

3.1.2 Results over a field of arbitrary characteristic

Target of Genus at least two

In [dF13], de Franchis proved a strong statement about the number of maps from a fixed curve X to all curves of genus greater than one. In modern language it can be formulated as follows:

Theorem 3.1.3. *([Kan86, Theorem of de Franchis]) Let X be a smooth proper curve over a field and $S = \{f : X \to Y \mid f$ is non-constant and the genus of Y is greater than one$\}$. Consider two elements of S, $f_i : X \to Y_i$, equivalent if there is a map $g : Y_1 \to Y_2$ such that $g \circ f_1 = f_2$. Then the number of equivalence classes in S is finite.*

Since the target curve has genus at least two, $\mathrm{Aut}(Y)$ is finite. Therefore one can rephrase Theorem 3.1.3 as stating there are finitely many non-constant maps $f : X \to Y$, where the target curve Y varies over a set of representatives for the isomorphism classes of curves of genus at least two.

Source and Target $\mathbb{P}^1_k$

The following result uses the same methods as Theorem 3.1.1, but in positive characteristic. The restriction that the degree of the map is smaller than the characteristic of the field is necessary because the existence of inseparable maps in $G^1_d(\mathbb{P}^1_k)$ may cause the intersection of the Schubert varieties corresponding to imposing ramification conditions not to be transverse.

Theorem 3.1.4. *([Oss06b, Corollary 3.2]) Let k be an algebraically closed field of characteristic $p > 0$, d a positive integer less than p, and S a finite set of closed points on $\mathbb{P}^1_k$. The number of equivalence classes of degree d maps with ramification locus S is finite.*

In light of Theorem 3.1.4, it must be the case for any non-constant family of fixed degree and ramification locus that the degree is at least as large as the characteristic of the field. Indeed, this is the case in Example 1.5.2. This condition is far from sufficient, however. In fact, Theorem 3.1.5 shows that even when $d \geq p$, for a generic choice of S there are only finitely many degree d maps with ramification locus S. Hence, even when some necessary combinatorial constraints are met, a family will only exist for "special" configurations of the ramification points.

The following result shows that for a tame family to exist the ramification points must be in a restricted configuration:

Theorem 3.1.5. *([Oss06b, Theorem 3.3]) Let $e_1, ..., e_n$ be prime to p, and suppose that $\sum_i (e_1 - 1) = 2d - 2$. Then for a general choice of points $P_1, ..., P_n$, we have that the number of equivalence classes of maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree d with ramification index e_i at P_i is finite.*

This is proven using the fine moduli space $MR = MR_d(\mathbb{P}^1_k, \mathbb{P}^1_k, \{e_i\})$ for degree d maps together with a choice of points $Q_1, ..., Q_n \in \mathbb{P}^1_k$ such that the map has ramification index e_i at Q_i from [Oss06b, Theorem A.6]. There is a map *branch* : $MR \to (\mathbb{P}^1_k)^n$ sending the point corresponding $(f, Q_1, ..., Q_n)$ to $(f(Q_1), ..., f(Q_n))$. Since the tame fundamental group of the projective line minus the branch points is finitely generated by [Gro71, Exposé XIII Corollaire 2.12], the preimage of any set of points under *branch* has finitely many $\mathrm{Aut}(\mathbb{P}^1_k)$-orbits, where the action is on the source $\mathbb{P}^1_k$, corresponding to equivalence of covers. Since $\mathrm{Aut}(\mathbb{P}^1_k)$ is a three dimensional group scheme, this implies that MR is at most $n + 3$ dimensional.

Therefore by [Mum99, Theorem I.2], the generic fiber of the map $ram : MR \to (\mathbb{P}^1_k)^n$ sending the point corresponding to $(f, Q_1, ..., Q_n)$ to $(Q_1, ..., Q_n)$ is at most 3 dimensional. Since the action of $Aut(\mathbb{P}^1_k)$ on the target $\mathbb{P}^1_k$ preserves the fibers of ram and is free, this implies that the generic fiber is finite.

As an example of such a restricted configuration being necessary for a tame family, we have the following:

Example 3.1.6. Given a tame degree 4 separable map $\mathbb{P}^1_k \to \mathbb{P}^1_k$, $\sum(e_P - 1) = 6$ by the Riemann-Hurwitz formula, where the sum is taken over all closed points of source $\mathbb{P}^1_k$. Consider in particular such maps ramified at four points, $P_1, ..., P_4$, with $e_{P_1} = 4$ and $e_{P_i} = 2$ for $2 \leq i \leq 4$. If $\mathrm{char}(k) \neq 2, 3$, there is a *unique* such map. If $\mathrm{char}(k) = 2$, there is *no* such map. If $\mathrm{char}(k) = 3$, there is a *family* of such maps for an appropriate choice of P_i. Namely, after pre-composition with a fractional linear transformation, we may assume $P_1 = \infty$, $P_2 = 0$, and $P_3 = 1$. Then, every map with the required ramification is of the form $y = x^4 + tx^3 + x^2$ where t is a parameter, and no distinct values of t produce equivalent maps. To have the required ramification, it must be the case that $P_4 = -1$. Hence, after choosing three of the ramification points, for such a family to exist the fourth ramification point is completely determined.

Lastly, we have the following result which is based on results from [Moc99] concerning torally indigenous crys-stable bundles:

Theorem 3.1.7. *[Oss07, Theorem 5.3] Let k be an algebraically closed field of* *characteristic $p > 0$. Fix a positive integer d, closed points $P_1, ..., P_n$ on $\mathbb{P}^1_k$, and positive integers $e_1, ..., e_n$ such that $\sum(e_i - 1) = 2d - 2$. If all the e_i are odd and less than p, the number of equivalence classes of maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree d with ramification index e_i at P_i is finite.*

Although these are all partial results, as Example 1.5.2 shows one cannot hope for as strong of a non-existence theorem as Theorem 3.1.1 in positive characteristic, even when the ramification is tame. We shall discuss a conjectural necessary and sufficient condition for the existence of a non-constant family in Section 3.4.

3.2 Target genus one

As noted in Section 3.1, the theorem of de Franchis classifies the number of maps from a fixed curve X to curves of genus at least two. Considering linear series on X gives some finiteness results when the target is $\mathbb{P}^1_k$. In the case of maps to genus one curves, we will show:

Theorem 3.2.1. *Let X be a smooth proper curve over an algebraically closed field k. There are only finitely many equivalence classes of maps from X of fixed degree to genus one curves.*

This result essentially follows from a theorem of Kani, which first requires two definitions. Fix K a field of transcendence degree one over k. A subfield $L \subseteq K$ is called a *genus one subfield* if it corresponds to the function field of a genus one curve over k. A genus one subfield $L \subsetneq K$ is called *essential* if there is no intermediate genus one subfield $K \supsetneq L' \supsetneq L$. Let X, E, and E' be the smooth proper curves corresponding to K, L, and L' respectively. Geometrically, the existence of an intermediate genus one subfield means the morphism $X \to E$ corresponding to the field extension K/L factors through an isogeny $E' \to E$. With these definitions, we can state the crucial theorem in proving Theorem 3.2.1:

Theorem 3.2.2. *([Kan86, Theorem 4]) Let X be a smooth proper curve over a field. The number of essential genus one subfields of $\kappa(X)$ of bounded index is finite.*

First we establish a correspondence between finite index subfields of $\kappa(X)$ and equivalence classes of finite maps with source X:

Lemma 3.2.3. *Let X be a smooth proper curve over an algebraically closed field k. Finite index subfields of $\kappa(X)$ are in correspondence with equivalence classes in the linear series perspective of finite maps with source X.*

Proof. Suppose Y_1 and Y_2 are also smooth projective curves and we have finite maps $f_i : X \to Y_i$ which are equivalent. That is, there is a commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "\cong"] & & Y_2 \end{tikzcd}$$

By the equivalence of categories between smooth proper curves over k and function fields of transcendence degree one over k, this yields a commutative diagram of function fields:

$$\begin{tikzcd} & \kappa(X) & \\ \kappa(Y_1) \arrow[ur, "i_1"] & & \kappa(Y_2) \arrow[ul, "i_2"'] \arrow[ll, "\cong"'] \end{tikzcd}$$

Therefore $i_1(\kappa(Y_1)) = i_2(\kappa(Y_2))$ as subfields of $\kappa(X)$. The index of the subfield is equal to the degree of f_i.

Given a finite index subfield $L \subseteq \kappa(X)$, since the index is finite, L is also a field of transcendence degree one over k. Let Y be the smooth projective curve over k with function field L. The ambiguity of a map $i : L \to \kappa(X)$ corresponds precisely to automorphisms of L/k. Therefore the choice of a map $X \to Y$ is fixed up an automorphism of Y. That is, it gives a well-defined equivalence class of maps $X \to Y$ in the linear series perspective. These two operations are clearly inverse to each other, establishing the desired correspondence. □

Equipped with this correspondence, we can prove Theorem 3.2.1:

Proof of Theorem 3.2.1. By Theorem 3.2.2 and Lemma 3.2.3, all that is left to prove is that every essential genus one subfield of $\kappa(X)$ contains finitely many genus one subfields of bounded index. Let L be an essential subfield of $\kappa(X)$ corresponding to a genus one curve E and $n \in \mathbb{N}$. Choose a base point on E so that it is an elliptic curve. By Lemma 3.2.3, genus one subfields of L of index at most n correspond to equivalence classes of isogenies $f : E \to E'$ of degree at most n, after choosing the base point of E' to be the image of the base point of E.

The isogeny f induces a short exact sequence

$$0 \longrightarrow K \longrightarrow E \xrightarrow{f} E' \longrightarrow 0$$

where K is the kernel. This short exact sequence implies that f is K-Galois, where the points in K act on E by translation. Post-composing with $f^\vee : E' \to E$, the dual isogeny, yields a short exact sequence:

$$0 \longrightarrow E[l] \longrightarrow E \xrightarrow{f^\vee \circ f} E \longrightarrow 0$$

where $E[l]$ is the l-torsion points of E where l is the degree of f. By Theorem 4.3.8 below, finite subgroups of $\mathrm{Aut}(E)$ correspond to Galois covers with source E. Since $E[l]$ is a finite group, it has finitely many subgroups, and hence there are finitely many equivalence classes of finite maps from E to genus one curves. □

3.3 Specifying differential lengths instead of ramification indices

The results of Section 3.1 all involve fixing the ramification indices. A related notion which will behave more nicely in families is what we will call the differential length. Let $f : X \to Y$ be a finite separable morphism of smooth projective curves over an algebraically closed field k. This map induces a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow f^*(\Omega_Y) \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0\,.$$

The sheaf $\Omega_{X/Y}$ of relative differentials is torsion and its support is equal to the set of ramification points of f. For $P \in X$ a closed point, define the *differential length* l_P to be the length of $(\Omega_{X/Y})_P$ as an $\mathcal{O}_{X,P}$ module. If f is tamely ramified at P then l_P is equal to the ramification index minus one; if f is wildly ramified at P then l_P is strictly larger than that by [Hrt77, Chapter III Proposition 2.2]. Define the *discriminant divisor* to be the Weil divisor

$$Disc(f) = \sum_{P \in X} l_P \cdot P.$$

Consider the case where X and Y are projective lines, and let $f : X \to Y$ be a separable degree d map. Choose a coordinate x on the source $\mathbb{P}^1_k$ and let ∞ denote the unique pole of this coordinate function. Let U be the source projective line with ∞ removed. The divisor $Disc(f)$ is principal when restricted to U, and is generated by a unique monic polynomial in x,

$$\prod_{i=1}^{n} (x - c_i)^{l_i},$$

where the c_i are distinct and the l_i are positive integers. Let $disc_x(f)$ denote this polynomial, which depends on the coordinate function x. Two choices of coordinate functions x_1 and x_2 with the same pole yield polynomials $disc_{x_1}(f)$ and $disc_{x_2}(f)$, which are equal after applying a change of coordinates between x_1 and x_2. When the choice of coordinate is fixed, we may refer to the *discriminant of f* and omit the subscript and instead write $disc(f)$.

By the Riemann-Hurwitz formula, we have that

$$\sum_{P \in \mathbb{P}^1} l_P = 2d - 2.$$

Therefore l_∞ can be determined from the differential lengths at the affine points, which is discernible from $disc(f)$. Thus the discriminant of f is sufficient for determining the differential lengths at every point on $\mathbb{P}^1_k$. Furthermore, two maps have the same differential lengths at each point precisely when their discriminants are equal when written in the same coordinate.

After choosing coordinates on both projective lines, we have an effective method for calculating $disc(f)$, as shown in the following proposition:

Proposition 3.3.1. *Let k be an algebraically closed field and let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a separable degree d map. Choose coordinates x and y on the source and target $\mathbb{P}^1_k$ respectively. In these coordinates, write f as a rational function $y = g(x)/h(x)$ where $g(x)$ and $h(x)$ are relatively prime. The discriminant $disc(f)$ is the unique monic polynomial which is a scalar multiple of $h(x)g'(x) - g(x)h'(x)$.*

Proof. Choose a point $P = (x - c)$ unequal to infinity on the source $\mathbb{P}^1_k$. It suffices to show that the order to which $(x - c)$ divides $(h(x)g'(x) - g(x)h'(x))$ is equal to the differential length of f at P. Since dx is a generator for $\Omega_{\mathbb{P}^1,P}$, the pullback of a generator for $\Omega_{\mathbb{P}^1,f(P)}$ may be written as $\beta \cdot dx$ where $\beta \in \mathcal{O}_{\mathbb{P}^1,P}$, and the differential length of f at P is equal to the order of vanishing of β at P.

First, consider the case where $f(P) \neq \infty$. Then dy is a generator for $\Omega_{\mathbb{P}^1,f(P)}$. Since $y = g(x)/h(x)$, we have that $dy = (h(x)g'(x) - g(x)h'(x))/h(x)^2 dx$. The order of $h(x)$ at P is zero since $f(P) \neq \infty$, which implies that the differential length of f at P is equal to $\mathrm{ord}_P(h(x)g'(x) - g(x)h'(x))$. Therefore the order of the pullback of dy at $P = (x - c)$ is equal to the degree to which $(x - c)$ divides $(h(x)g'(x) - g(x)h'(x))$.

Next, consider the remaining case where $f(P) = \infty$. Then $d(1/y)$ is a generator for $\Omega_{\mathbb{P}^1, f(P)}$. Since $1/y = h(x)/g(x)$, we have that $d(1/y) = (g(x)h'(x) - h(x)g'(x))/g(x)^2 dx$. In this case we have that the order of $g(x)$ at P is zero, implying that the differential length of f at P is equal to $\mathrm{ord}_P(h'(x) - h(x)g'(x))$. Therefore the order of the pullback of $d(1/y)$ at $(x - c)$ is equal to the degree to which $(x - c)$ divides $(h(x)g'(x) - g(x)h'(x))$. $\square$

Choose a coordinate x on $\mathbb{P}^1_k$. Recall that after choosing such a coordinate, the moduli space $G_d^{sep}(\mathbb{P}^1_k)$ is naturally an open subscheme of the Grassmannian of two dimensional linear subspaces of $Poly_k(d)$, the vector space of polynomials of degree at most d in $k[x]$. Let $T \in G_d^{sep}(\mathbb{P}^1_k)$ be a plane with basis $\{g(x), h(x)\}$. The *discriminant of the basis* $\{g(x), h(x)\}$ is $h(x)g'(x) - g(x)h'(x)$. The *discriminant of* T, $disc_x(T)$, which is defined up to $k^\times$-multiplication, is the discriminant of a basis of T. The *discriminant divisor of* T, $Disc(T)$, is the unique effective divisor of degree $2d - 2$ generated away from ∞ by $disc_x(T)$.

Let $\mathbb{P}^{2d-2}_k$ be the projectivization of $Poly_k(2d - 2)$. The discriminant defines a morphism $disc : G_d^{sep}(\mathbb{P}^1_k) \to \mathbb{P}^{2d-2}_k$, since by the Riemann-Hurwitz formula the degree of the discriminant of a separable map of degree d is at most $2d - 2$. By

Proposition 3.3.1, the discriminant of a base point free linear series T is equal, up to multiplication by $k^\times$, to $disc(f)$, where $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a map corresponding to T.

As evidence for families behaving more uniformly with respect to specifying the differential lengths instead of the ramification indices, we have the following:

Example 3.3.2. We shall revisit the situation of Example 3.1.6, but specify the differential lengths instead of the ramification indices. Consider a degree 4 separable map $\mathbb{P}^1_k \to \mathbb{P}^1_k$. By the Riemann-Hurwitz formula, $\sum l_P = 6$. Consider in particular such maps ramified at four points $P_1, ..., P_4$ with $l_{P_1} = 3$ and $l_{P_i} = 1$ for $2 \leq i \leq 4$. If $\text{char}(k) \neq 2, 3$ a map with these differential lengths must be tame, and so the result from Example 3.1.6 holds here–there is a *unique* map. If $\text{char}(k) = 2$, then having a differential length of 1 is impossible, so there is *no* such map.

Finally suppose $\text{char}(k) = 3$. By pre-composing with a fractional linear transformation, we may assume $P_1 = \infty$, $P_2 = 0$, $P_3 = 1$, and the fourth point is λ. After this pre-composition, the discriminant becomes $x(x-1)(x-\lambda)$. Specifying $l_\infty = 3$ implies either the ramification index is 4 and the map is tamely ramified or the ramification index is 3 and it is wildly ramified. As before, in the tamely ramified case we have that the map up to equivalence is of the form $y = x^4 + tx^3 + x^2$ with t a parameter and further it be must be the case that $\lambda = -1$.

In the wildly ramified case, after assuming ∞ and 0 are fixed points by post-composing with a fractional linear transformation, the map must be of the form

$$y = \frac{dx^4 + ax^3 + bx^2}{x + c},$$

where $a, b, c, d \neq 0$. After post-composing with the fractional linear transformation fixing 0 and ∞ and sending 1 to d, we may assume $d = 1$. By calculating the discriminant of such a map and ensuring it has no base points, we must have

$$c \neq a,\ \lambda \neq -1,\ c = \frac{\lambda}{1+\lambda},\ b = (1+\lambda)a - \lambda.$$

Note that these conditions imply we have a one dimensional wild family when $\lambda \neq -1$. Combining this with the tame case, we have a one dimensional family with the specified differential lengths for *every* choice of the four ramification points. In contrast, when instead specifying the ramification indices, as in Example 3.1.6, only a restricted configuration of ramification points produces a map with the given ramification indices.

3.3.1 Relationship between wild and tame locus

As the following example illustrates, wild and tame maps can occur in families with the same discriminant:

Example 3.3.3. Let k be a field of characteristic 3. The two dimensional family parameterized by $\mathbb{A}^2_{s,t}$ of degree 8 maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ given by

$$y = \frac{x^4(x-1)^4}{x^4 + x - 1 + tx(x-1)} + sx^3$$

has fixed discriminant, and fibers over distinct values of (s,t) are not equivalent. When $s \neq 0$ the map is wild. There is a one dimensional tame subfamily given by $s = 0$.

In Example 3.3.3 there is a family of wildly ramified maps degenerating to a tame family when $s = 0$. This relationship between wild and tame maps holds in general, as the following theorem illustrates. In the proof we shall be using the Schubert calculus machinery developed in Section 2.1.1.

Theorem 3.3.4. *Let k be an algebraically closed field, S a finite set of closed points of $\mathbb{P}^1_k$, and d a positive integer. In $G^{sep}_d(\mathbb{P}^1_k)$, the locus of tame maps of degree d ramified precisely at S is closed in the locus of all maps ramified precisely at S.*

Proof. Recall from Section 3.3 that after choosing a coordinate x on $\mathbb{P}^1_k$, we have a map $disc : G^{sep}_d(\mathbb{P}^1_k) \to \mathbb{P}^{2d-2}_k$ which sends a linear series to its discriminant. Furthermore, a separable linear series is ramified at an affine point $P = (x - c)$ precisely when $(x - c)$ divides the discriminant, and is ramified at ∞ when the discriminant has degree less than $2d - 2$. Therefore there are finitely many choices of discriminants which correspond to a ramification locus of S.

Let $\alpha(x)$ be one such discriminant. It suffices to prove that the tame maps with discriminant $\alpha(x)$ are closed in the locus of all maps with discriminant $\alpha(x)$. Note that $disc^{-1}(\alpha(x))$ may contain linear series which do not correspond to maps (i.e. linear series with a point of S as a base point). However, every linear series with a point of S as a base point is contained in

$$\bigcup_{P \in S} \Omega_{0,1}(P).$$

Therefore we must show that the points inside of

$$disc^{-1}(\alpha(x)) \setminus \bigcup_{P \in S} \Omega_{0,1}(P)$$

corresponding to tame maps form a closed subscheme. Write

$$\alpha(x) = \prod_{i=1}^{n} (x - c_i)^{l_i}$$

where the $(x - c_i)$ are distinct and correspond to points $P_i \in S$ which are unequal to ∞. If $\alpha(x)$ is the discriminant of a tame map f of degree d, the ramification index of f at P_i must be $l_i + 1$ and the ramification index at ∞ must be $2d - 1 - \sum l_i$. Therefore the points in

$$\left(\left(\bigcap_{P_i} \Omega_{l_i,0}(P_i)\right) \bigcap \Omega_{(2d-2-\sum l_i),0}(\infty)\right) \bigcap \left(disc^{-1}(\alpha(x)) \setminus \bigcup_{P \in S} \Omega_{0,1}(P)\right)$$

correspond to the tame maps with discriminant $\alpha(x)$, and hence form a closed subscheme. $\square$

3.4 Main conjecture

With the differential length viewpoint from Section 3.3, we can state our main conjecture concerning the existence of a family in the non-Galois case:

Conjecture 3.4.1. *Let k be an algebraically closed field of characteristic $p > 0$, d a positive integer, $P_1, ..., P_n$ closed points on $\mathbb{P}^1_k$, and $l_1, ..., l_n$ positive integers such that $\sum l_i = 2d - 2$. Suppose there exists a map f in G_d^{sep} such that the differential length of f at each P_i is l_i. Then some positive dimensional subscheme of G_d^{sep} parameterizes maps with differential length l_i at each P_i if and only if for at least one i we have $l_i \geq p$.*

This conjecture implies an affirmative answer to Question 8.4 in [Oss06b]. Showing one half of Conjecture 3.4.1 is not difficult:

Proposition 3.4.2. *Let k be an algebraically closed field of characteristic $p > 0$, d a positive integer, $P_1, ..., P_n$ closed points on $\mathbb{P}^1_k$, and $l_1, ..., l_n$ positive integers such that $\sum l_i = 2d - 2$ and at least one $l_i \geq p$. Suppose there exists a map f in G_d^{sep} such that the differential length of f at each P_i is l_i. Then a positive dimensional subscheme of G_d^{sep} parameterizes maps with differential length l_i at each P_i.*

Proof. By pre-composing with an automorphism of the projective line, we may assume ∞ is one of the points P_i for which $l_{P_i} \geq p$. Let f be a representative of an equivalence class of maps with the specified differential lengths at each P_i such that ∞ is a fixed point, and represent f by a rational function $g(x)/h(x)$ with $g(x)$ and $h(x)$ relatively prime. Consider the family of maps

$$f_t(x) = \frac{g(x)}{h(x)} + t \cdot x^p = \frac{g(x) + t \cdot x^p h(x)}{h(x)}$$

with parameter $t \in k$. By our assumptions on f, each $f_t(x)$ is written in lowest terms and is a degree d map. Furthermore, one can check directly that if $f_t(x)$

and $f_{t'}(x)$ are equivalent then it must be the case that $t = t'$. The discriminant of $f_t(x)$ is $h(x)g'(x) - g(x)h'(x)$ and hence the discriminant is independent of t. Recall from Section 3.3 that two degree d separable maps have the same discriminant precisely when they have the same differential lengths at each point. Therefore the image of the morphism from $\mathbb{A}^1_t$ to G_d^{sep} induced by this family has the required properties. □

Remark 3.4.3. If $l_i \geq p^m$ with $m > 0$ for at least i, then the argument above with the family $f_t(x)$ replaced by the family

$$\frac{g(x)}{h(x)} + t_1 \cdot x^p + \cdots + t_m \cdot x^{pm}$$

can be used to produce an m dimensional subscheme of G_d^{sep} parameterizing maps with fixed differential lengths at fixed points.

Note that we do not claim that the ramification indices remain fixed. It is often the case in producing a family as above that the differential lengths are fixed but the ramification indices are not because the maps in the family vary between being wildly ramified and tamely ramified (see Theorem 3.3.4), as the following simple example illustrates:

Example 3.4.4. Let k be an algebraically closed field of characteristic $\mathrm{char}(k) = p > 0$. Consider the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $y = x^{p+1}$, which has ramification index $p+1$ at 0 and ∞. Following the procedure of Proposition 3.4.2, we construct the family of maps $y = x^{p+1} + t \cdot x^p$ with parameter $t \in k$. Since the discriminant is independent of t, the differential lengths remain fixed. However, if $t \neq 0$, the ramification index at 0 is p. One can show that in fact there is *no* non-constant family of maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree $p+1$ with fixed ramification indices $e_0, e_\infty = p+1$, and hence Conjecture 3.4.1 would be false if instead we required the ramification indices be fixed.

As evidence for the necessity of the condition in Conjecture 3.4.1, it is true when the number of ramification points is at most three:

Proposition 3.4.5. *Let k be an algebraically closed field of characteristic $p > 0$. Fix a positive integer d, points $P_1, ..., P_n$ on $\mathbb{P}^1_k$, and positive integers $l_1, ..., l_n$ such that $\sum l_i = 2d - 2$. If $n \leq 3$ and for every i we have $l_i < p$, then there are only finitely many equivalence classes of degree d maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ with differential length l_i at each P_i. Furthermore, this implies that Conjecture 3.4.1 holds when $n \leq 3$.*

Proof. Let f be such a map. Note that the condition on the differential lengths implies that all the ramification is tame. Therefore we must have $n \geq 2$.

We first examine the $n = 2$ case. By pre-composing with an automorphism of the projective line, we may assume the ramification points of f are 0 and ∞. Since f is tame of degree d, it must be the case that $l_P < d$ for every $P \in \mathbb{P}^1_k$. Therefore $l_0 = l_\infty = d-1$. By choosing a different representative in the equivalence class of f, we may assume 0 and ∞ are fixed points. The only degree d maps totally ramified at 0 and ∞ fixing those two points is of the form x^d, so in this case there is a unique equivalence class of such maps.

Finally consider the $n = 3$ case of three ramification points. The following argument is based on the proof of [Oss06b, Theorem 3.3]. By [Oss06b, Theorem A.6] there is a fine moduli space $MR = MR_d(\mathbb{P}^1_k, \mathbb{P}^1_k, \{l_i+1\})$ for degree d maps together with a choice of points $Q_1, ..., Q_n \in \mathbb{P}^1_k$ such that the map has ramification index at least $l_i + 1$ at Q_i. There are natural maps $ram, branch : MR \to (\mathbb{P}^1_k)^n$ sending the point corresponding to $(f, Q_1, ..., Q_n)$ to $(Q_1, ..., Q_n)$ and $(f(Q_1)), ..., f(Q_n))$ respectively. Furthermore, there are two natural actions of $\mathrm{Aut}(\mathbb{P}^1)_k$ on MR where $\varphi \in \mathrm{Aut}(\mathbb{P}^1_k)$ acts on the point corresponding to $(f, P_1, ..., P_n)$ by sending it to $(f \circ \varphi, \varphi^{-1}(P_1), ..., \varphi^{-1}(P_n))$ or $(\varphi \circ f, P_1, ..., P_n)$. We will refer to these actions as *acting on the source* and *acting on the target* respectively. The latter action is free. The morphism *branch* factors through the action on the source and the *ram* factors through the action on the target and is a free action. Note that here we are specifying no equivalence relation on the maps or the choice of points.

Let f be a degree d map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ with ramification index at least $l_i + 1$ at Q_i for some choice of $Q_1, ..., Q_n \in \mathbb{P}^1_k$. By the Riemann Hurwitz formula we have that

$$\sum_{Q \in \mathbb{P}^1_k} l_Q = 2d - 2$$

where l_Q is the differential length of f at Q. Since $l_{Q_i} \geq l_i$ and by hypothesis $\sum l_i = 2d - 2$, it must be the case that $l_{Q_i} = l_i$, f is tamely ramified, and f is unramified outside $Q_1, ..., Q_n$ (see Section 3.3).

Fix an n-tuple of points $(R_1, ..., R_n) \in (\mathbb{P}^1_k)^n$ and consider closed points mapping to $(R_1, ..., R_n)$ via the map $branch : MR \to (\mathbb{P}^1_k)^n$. By the previous paragraph, such a point corresponds to a tame degree d map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ with branch locus $\{f(R_1), ..., f(R_n)\}$. Let $U \subseteq \mathbb{P}^1_k$ be the open subscheme obtained by deleting this branch locus. Since the tame étale fundamental group of U is finitely generated ([Gro71, Exposé XIII Corollaire 2.12]), there are finitely many $\mathrm{Aut}(\mathbb{P}^1_k)$-orbits in MR mapping to $(R_1, ..., R_n)$ where the action is on the source. Therefore MR is at most $(n+3)$ dimensional. By [Mum99, Theorem I.2], this implies that the generic fiber of $ram : MR \to (\mathbb{P}^1_k)^n$ is at most 3 dimensional.

Suppose there is some non-empty fiber $C \subseteq MR$ of ram. Since the action of $\mathrm{Aut}(\mathbb{P}^1_k)$ on the target is free and preserves C, C must be at least 3 dimensional. However since $n = 3$, the action of $\mathrm{Aut}(\mathbb{P}^1_k)$ on the source is transitive on fibers of ram, so every fiber must be at least 3 dimensional. Since the generic fiber was shown to be at most 3 dimensional, every fiber must then be exactly 3 dimensional. Since the fibers are closed and the action of $\mathrm{Aut}(\mathbb{P}^1_k)$ on the target is free, each fiber contains finitely many $\mathrm{Aut}(\mathbb{P}^1_k)$-orbits. In other words, there are only finitely many equivalence classes of degree d maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ with differential length l_i at any fixed choice of $P_1, ..., P_n$. By Theorem 2.2.2, this implies that no positive dimensional subscheme of G_d^{sep} parameterizes maps with differential length l_i at each P_i, proving Conjecture 3.4.1 in the $n = 3$ case. □

For further evidence for Conjecture 3.4.1, in the $d < p$ case it is true by Theorem 3.1.4. When all the l_i are even and less than p it is true by Theorem 3.1.7.

3.5 Smoothness of the moduli space in the $\mathbb{P}^1$ case

We begin with some preliminary definitions. Fix $d \in \mathbb{N}$ and an effective Weil divisor D on $\mathbb{P}^1_k$ such that $\deg(D) = 2d-2$. Let $\widetilde{X_D} \subseteq G_d^{sep}$ be the set of closed points corresponding to linear series T such that $Disc(T) = D$. Fix positive integers $l_1, ..., l_n$ such that $\sum l_i = 2d-2$. Let $\widetilde{X_{(l_i)}}$ be the union of $\widetilde{X_D}$ over D of the form

$$D = \sum_{i=1}^{n} l_i \cdot P_i$$

where the $P_i \in \mathbb{P}^1_k$ are distinct. Note that the points of $\widetilde{X_{(l_i)}}$ correspond to linear series with fixed differential lengths, but where the ramification points are allowed to vary.

We first show that $\widetilde{X_D}$ and $\widetilde{X_{(l_i)}}$ are the closed points of reasonable subschemes of G_d^{sep}:

Lemma 3.5.1. *There exists a unique reduced, closed subscheme of G_d^{sep}, X_D, such that the set of closed points of X_D is $\widetilde{X_D}$. Furthermore, there exists a unique reduced, locally closed subscheme of G_d^{sep}, $X_{(l_i)}$, such that the set of closed points of $X_{(l_i)}$ is $\widetilde{X_{(l_i)}}$.*

Proof. We first prove the claim concerning X_D. Choose a coordinate function x on $\mathbb{P}^1_k$, and let U be $\mathbb{P}^1_k$ with ∞ removed. There is a unique monic polynomial $\alpha(x) \in k[x]$ generating D after restricting to U. By Section 3.3, we have a map $disc : G_d^{sep} \to \mathbb{P}^{2d-2}_k$, where $\mathbb{P}^{2d-2}_k$ is the projectivization of $Poly_k(2d-2)$, sending a linear series to its discriminant. Let X_D be the inverse image of $\alpha(x)$. It is clear that X_D is closed, reduced, and the set of its closed points is $\widetilde{X_D}$, proving the existence portion of the claim.

To establish the uniqueness of X_D, suppose X'_D is a different subscheme of G_d^{sep} satisfying the same properties as X_D. By [Hrt77, Chapter II Exc. 3.11(c)], two reduced closed subschemes of G_d^{sep} which contain the same points must be isomorphic as subschemes, so there exists a (necessarily non-closed) point P which is contained in either X_D or X'_D, but not the other. Without loss of generality assume $P \in X_D$. Since the closed points of a quasi-compact scheme are dense, P lies in the closure of the closed points of X_D. However, X_D and X'_D have the same closed points, so $P \in X'_D$ as well since X'_D is closed. This is a contradiction, so X_D must be unique.

For the remainder of this proof, use the same coordinate function x on every copy of $\mathbb{P}^1_k$. We now prove the claim concerning $X_{(l_i)}$. Let $\mathrm{Sym}^r(\mathbb{P}^1_k)$ denote the r-fold symmetric product of $\mathbb{P}^1_k$. Viewing $\mathbb{P}^{2d-2}_k$ as the projectivization of $Poly_k(2d-2) \subseteq k[x]$, we have a map (which is in fact an isomorphism) $\varphi : \mathbb{P}^{2d-2}_k \to \mathrm{Sym}^{2d-2}(\mathbb{P}^1_k)$ sending

$$f(x) = a \cdot \prod (x - c_i)^{d_i}$$

to the point corresponding to d_i copies of $(x - c_i)$ for each i and $2d - 2 - \sum d_i$ copies of ∞.

Fix positive integers $l_1, ..., l_n$ such that $\sum l_i = 2d - 2$. Let $\psi_r : \mathbb{P}^1_k \to (\mathbb{P}^1_k)^r$ be the diagonal embedding and $\psi = (\psi_{l_1}, ..., \psi_{l_n}) : (\mathbb{P}^1_k)^n \to (\mathbb{P}^1_k)^{2d-2}$. Let Z denote the image of $(\mathbb{P}^1_k)^n$ under ψ. Since $\mathbb{P}^1_k$ is separated the image under each ψ_r is closed, and thus Z is closed. Let Z' be the S_{2d-2}-orbit of Z and $\pi : (\mathbb{P}^1_k)^{2d-2} \to \mathrm{Sym}^{2d-2}(\mathbb{P}^1_k)$ the quotient map. Since Z' is closed and S_{2d-2}-invariant, $\pi(Z') = \pi(Z)$ is closed. Endow $\pi(Z)$ with the reduced induced scheme structure.

Let $Y_{(l_1,...,l_n)} = disc^{-1}(\varphi^{-1}(\pi(Z)))$. By construction, $\widetilde{X_{(l_i)}} \subseteq Y_{(l_i)}$. However, $Y_{(l_1,...,l_n)}$ may additionally contain points corresponding to linear series with discriminant divisors of the form

$$\sum_{i=1}^{n} l_i \cdot P_i$$

where the P_i are not necessarily distinct. Each such point is contained in

$$Y_{(l_1,...,l_i+l_j,...,\widehat{l_j},...,l_n)}$$

for some choice of i and j, where $\widehat{l_j}$ denotes that l_j is omitted. Let $X_{(l_i)}$ be $Y_{(l_1,...,l_n)}$ with each such

$$Y_{(l_1,...,l_i+l_j,...,\widehat{l_j},...,l_n)}$$

removed. Since each removed scheme is closed and there are finitely many choices of indices to combine, this implies that $X_{(l_i)}$ is locally closed. After endowing $X_{(l_i)}$ with the induced open subscheme structure, this establishes the existence portion of the claim.

For the uniqueness assertion, suppose $X'_{(l_i)}$ is a subscheme of G_d^{sep} also satisfying the same properties as $X_{(l_i)}$. Using the argument for the uniqueness of X_D, it must be the case that $X_{(l_i)}$ and $X'_{(l_i)}$ have the same closure, which we denote by W and endow with the reduced induced scheme structure. Both $X_{(l_i)}$ and $X'_{(l_i)}$ are locally closed, so both are open in W. Since there is a unique induced subscheme structure on a fixed open set of W, it suffices to show that $X_{(l_i)}$ and $X'_{(l_i)}$ contain the same points. To arrive at a contradiction, suppose that there exists a point P which is contained in either $X_{(l_i)}$ or $X'_{(l_i)}$, but not the other. Without loss of generality assume $P \in X_{(l_i)}$. Since $W \setminus X'_{(l_i)}$ is closed and contains P, no point in the closure of P is contained in $X'_{(l_i)}$. However the closure of P inside $X_{(l_i)}$ is quasi-compact and thus contains a closed point, a contradiction to $X_{(l_i)}$ and $X'_{(l_i)}$ containing the same closed points. Therefore $X_{(l_i)}$ is unique as claimed. $\square$

Using a generalization of the description of first order deformations in Section 2.1.2, we will prove the following smoothness result for G_d^{sep}:

Theorem 3.5.2. *Let k be an algebraically closed field of characteristic two and d a positive integer. We have the following concerning the schemes introduced in Lemma 3.5.1:*

(1) For any choice of an effective Weil divisor D on $\mathbb{P}^1_k$ such that $\deg(D) = 2d-2$, the scheme X_D is smooth.

(2) For any choice of positive integers $l_1, ..., l_n$ such that $\sum l_i = 2d-2$, the scheme $X_{(l_i)}$ is smooth.

Remark 3.5.3. The restriction on the characteristic of k, as will be explained, is an artifact of the proof. Conjecturally this restriction is unnecessary and the method of proof may be extendable to remove the characteristic restriction. However, the proof relies in an essential way on the ground field having positive characteristic, and therefore a different approach would be required in characteristic zero.

3.5.1 Lifting deformations over Artin rings

Let k be an algebraically closed field of characteristic $p > 0$ and suppose $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a separable degree d map. We will use the following result to show that the moduli space is smooth at the point corresponding to f. Note that an *infinitesimal thickening* of a scheme Y is a closed immersion $Y \subseteq Y'$ such that the ideal sheaf corresponding to Y is nilpotent.

Proposition 3.5.4. *[Hrt10, Proposition 4.6] Let X be a scheme of finite type over an algebraically closed field k. Suppose that for every morphism $f : \mathrm{Spec}(A) \to X$ with A a local Artin ring finite over k and for every infinitesimal thickening Spec$(B) \supseteq \mathrm{Spec}(A)$ there is a lifting $g : \mathrm{Spec}(B) \to X$. Then X is non-singular.*

First we show that the ring B in Proposition 3.5.4 must also be a local Artin ring:

Lemma 3.5.5. *Let A be a local Artin ring and $B \to A$ a surjective ring homomorphism with kernel I such that $I^n = 0$ for some $n \in \mathbb{N}$. Then B is also a local Artin ring.*

Proof. Since I is nilpotent, it is contained in every prime ideal of B. There is a bijection between prime ideals of B containing I and prime ideals of A given by taking the quotient. Therefore B is also a local ring. Let $\mathfrak{m}$ be the maximal ideal of A and $\mathfrak{n}$ the maximal ideal of B. Since A is an Artin ring, by [AM69, Proposition 8.6] $\mathfrak{m}^k = 0$ for some $k \in \mathbb{N}$. The inverse image of $\mathfrak{m}$ is $\mathfrak{n}$, so we have that $\mathfrak{n}^k \subseteq I$. Since $I^n = 0$, this implies that $\mathfrak{n}^{nk} = 0$; so by [AM69, Proposition 8.6], B is a local Artin ring as well. $\square$

Next, as it will be necessary for our method of proof, we will reduce Proposition 3.5.4 to lifting infinitesimal deformations of a special form:

Lemma 3.5.6. *In the situation of Proposition 3.5.4, it suffices to lift diagrams of the form*

$$\begin{tikzcd}
\mathrm{Spec}(k[t_1, ..., t_n]/I) \arrow[r] \arrow[d] & X \\
\mathrm{Spec}(k[t_1, ..., t_n]/J) \arrow[ur, dashed] &
\end{tikzcd}$$

where $I \supseteq J$*, the map is the natural quotient map,* $\sqrt{I} = \sqrt{J} = (t_1, ..., t_n)$*, and* $t_i \cdot I \subseteq J$ *for every* $1 \leq i \leq n$*.*

Proof. As in Proposition 3.5.4, suppose that A is a local Artin ring and $Spec(B) \supseteq \mathrm{Spec}(A)$ is an infinitesimal thickening of $\mathrm{Spec}(A)$. By Lemma 3.5.5, B is also a local Artin ring, finite over k. By Hilbert's Nullstellensatz, $B \cong k[t_1, ..., t_n]/J$ where $\sqrt{J} = (t_1 - a_1, ..., t_n - a_n)$ for some $a_i \in k$. Without loss of generality we may assume that $a_i = 0$ for every i and likewise that $A = k[t_1, ..., t_n]/I$ where $I \supseteq J$ and the homomorphism $B \to A$ is the natural quotient map.

Since $\sqrt{J} = (t_1, ..., t_n)$, there is a positive integer m such that if

$$\tau = \prod_{i=1}^{n} t_i^{l_i}$$

is a monomial of combined degree m, then $\tau \in J$. For $1 \leq j \leq m$, let K_j be the ideal generated by all monomials of combined degree j. Since $K_m \subseteq J$, the sequence of ideals $J_0 = I \supseteq J_1 = (J, K_1 \cdot I) \supseteq \cdots \supseteq J_m = (J, K_m \cdot I)$ terminates at an ideal equal to J. To lift to $k[t_1, ..., t_n]/J$, it then suffices to successively lift diagrams of the form

$$\begin{tikzcd} \mathrm{Spec}(k[t_1, ..., t_n]/J_{i-1}) \arrow[r] \arrow[d] & X \\ \mathrm{Spec}(k[t_1, ..., t_n]/J_i) \arrow[ur, dashed] & \end{tikzcd}$$

for $1 \leq i \leq m$. These lifts are of the desired form. $\square$

We will now characterize maps from such Artin rings with target G_d^{sep} in terms of the discriminant introduced in Section 3.3. First, we require the definition of discriminant for rational functions over Artin rings. Let A be a local Artin ring, finite over a field k. Let $f = g(x)/h(x)$ where $g(x)$ and $h(x)$ are non-zero polynomials with coefficients in A. The *discriminant of f* is $h(x)g'(x) - g(x)h'(x)$.

Proposition 3.5.7. *Let k be an algebraically closed field and $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ a degree d map. Choose coordinate functions on each copy of $\mathbb{P}^1_k$ such that f is unramified at ∞ and 0 is a fixed point. In these coordinates, represent f by a rational function $y = g(x)/h(x)$.*

Let $A = k[t_1, ..., t_n]/I$ such that $\sqrt{I} = (t_1, ..., t_n)$. Choose a basis $\{1, \tau_1, ..., \tau_k\}$ of

A over k such that each τ_i is a monomial. We have the following characterization of maps with source $\mathrm{Spec}(A)$*:*

(1) Giving a map $\mathrm{Spec}(A) \to X_{Disc(f)}$ *such that the unique closed point in* $\mathrm{Spec}(A)$ *maps to the point corresponding to f is equivalent to choosing polynomials $g_1(x), ..., g_k(x) \in k[x]$ of degree at most d and no constant term and polynomials $h_1(x), ..., h_k(x) \in k[x]$ of degree at most $d-1$ such that the discriminant of*

$$\frac{g_A(x)}{h_A(x)} = \frac{g(x) + g_1(x)\tau_1 + \cdots + g_k(x)\tau_k}{h(x) + h_1(x)\tau_1 + \cdots + h_k(x)\tau_k} \tag{3.1}$$

is equal to $disc(f)$ modulo I.

(2) Let $Disc(f) = l_1 \cdot P_1 + \cdots + l_n \cdot P_n$ with the P_i distinct. Giving a map $\mathrm{Spec}(A) \to X_{(l_i)}$ *such that the unique closed point in* $\mathrm{Spec}(A)$ *maps to the point corresponding to f is equivalent to choosing polynomials $g_1(x), ..., g_k(x) \in k[x]$ of degree at most d and no constant term and polynomials $h_1(x), ..., h_k(x) \in k[x]$ of degree at most $d-1$ such that the discriminant of*

$$\frac{g_A(x)}{h_A(x)} = \frac{g(x) + g_1(x)\tau_1 + \cdots + g_k(x)\tau_k}{h(x) + h_1(x)\tau_1 + \cdots + h_k(x)\tau_k}$$

is equal to

$$\prod_{i=1}^{n}(x - c_i + d_i(t_1, ..., t_n))^{l_i}$$

modulo I, where each d_i contains no constant term.

Proof. Recall from Section 2.1.1 that G_d^{sep} is an open subscheme of $Gr(2, Poly_k(d))$, the Grassmannian of two dimensional linear subspaces of $Poly_k(d)$. The planes which have a basis of the form $\{x^d+\alpha_{d-1}x^{d-1}+\cdots+\alpha_1 x, x^{d-1}+\beta_{d-2}x^{d-2}+\cdots+\beta_0\}$, where the α_i and β_j are arbitrary constants, form an open affine subscheme U, which is isomorphic to $\mathrm{Spec}(R)$, where $R = k[\alpha_1, ..., \alpha_{d-1}, \beta_0, ..., \beta_{d-2}]$. Since f is unramified at ∞ and 0 is a fixed point, we have that

$$\frac{g(x)}{h(x)} = \frac{x^d + a_{d-2}x^{d-2} + \cdots + a_1 x}{x^{d-1} + b_{d-2}x^{d-2} + \cdots + b_0}.$$

Therefore the plane corresponding to f is contained in U. Giving a map with source $\mathrm{Spec}(A)$ and target $X_{Disc(f)}$ or $X_{(l_i)}$ such that the unique closed point in $\mathrm{Spec}(A)$ maps to the point corresponding to f is then equivalent to giving such a map with target $X_{Disc(f)}|_U$ or $X_{(l_i)}|_U$ respectively.

We first show (i), and begin with a map $\mathrm{Spec}(A) \to X_{Disc(f)}|_U$. By Lemma 3.5.1, $X_{Disc(f)} \subseteq G_d^{sep}$ is closed. Therefore there is an ideal K such that $X_{Disc(f)}|_U$ is isomorphic to $\mathrm{Spec}(R/K)$. We shall now describe K. Let

$$\widetilde{f} = \frac{\widetilde{g(x)}}{\widetilde{h(x)}} = \frac{x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_1 x}{x^{d-1} + \beta_{d-2}x^{d-2} + \cdots + \beta_0}$$

be a map corresponding to an arbitrary closed point in U. The point corresponding to $\widetilde{f}$ is contained in $X_{Disc(f)}$ precisely when

$$\widetilde{h(x)}\widetilde{g'(x)} - \widetilde{g(x)}\widetilde{h'(x)} = a \cdot disc(f) \tag{3.2}$$

for some $a \in k^\times$. After equating powers of x in equation (3.2), we see that this yields $2d-2$ relations among the α_i and β_j which generate the ideal K.

Having reduced to a map between affine schemes, we instead consider the corresponding ring homomorphism $\varphi : k[\alpha_1, ..., \alpha_{d-1}, \beta_0, ..., \beta_{d-2}]/K \to A$. We must have that $\varphi(\alpha_i - a_i), \varphi(\beta_j - b_j) \in (t_1, ..., t_n)$ for every i and j since the unique closed point of Spec(A) maps to the point corresponding to f. This implies that the image under φ of each $(\alpha_i - a_i)$ has no constant term, and can therefore be written uniquely modulo I as $\sum c_{i,l} \cdot \tau_l$ where each $c_{i,l} \in k$. Likewise, each $\varphi(\beta_j - b_j)$ can be written uniquely modulo I as $\sum e_{j,l} \cdot \tau_l$ where each $e_{j,l} \in k$.

Write $g_i(x) = c_{d-1,i}x^{d-1} + \cdots + c_{1,i}x$ and $h_j(x) = e_{d-2,j}x^{d-2} + \cdots + e_{0,j}$. The condition that $\varphi(K) \subseteq I$ is equivalent to the discriminant of

$$\frac{g(x) + g_1(x)\tau_1 + \cdots + g_k(x)\tau_k}{h(x) + h_1(x)\tau_1 + \cdots + h_k(x)\tau_k}$$

being equal to $disc(f)$ modulo I, as desired. Reversing the argument yields the other direction, proving (i).

The proof of (ii) proceeds along similar lines, except for the caveat that $X_{(l_i)}$ is locally closed instead of closed. As noted during the construction of $X_{(l_i)}$ in the proof of Lemma 3.5.1, the closure of $X_{(l_i)}$ contains points corresponding to maps for which the ramification points coalesce. However, a map Spec$(A) \to X_{(l_i)}$ such that the unique closed point in Spec(A) maps to the point corresponding to f is equivalent to such a map with target the closure of $X_{(l_i)} \subseteq G_d^{sep}$. Therefore we may follow the same process of describing the relations in the ideal corresponding to the closure of $X_{(l_i)}$ inside U as was done for the ideal corresponding to X_D inside U. $\square$

Remark 3.5.8. We will refer to a rational function of the form in equation (3.1) as a *deformation of f over A.*

3.5.2 A differential operator in positive characteristic

Let k be an algebraically closed field of characteristic $p > 0$ and consider $V = k(x)$ a vector space over $k(x^p)$ with basis $\{1, x, ..., x^{p-1}\}$. Fix a non-zero rational function $f(x) \in k(x)$ and define the operator $T_f : k(x) \to k(x)$ by the formula $T_f(p(x)) = f'(x)p(x) - f(x)p'(x)$. This operator will be utilized in the proof of Theorem 3.5.2 since $T_f(p(x))$ is the discriminant of the rational function $f(x)/p(x)$ in the case where $f(x)$ and $p(x)$ are both polynomials.

Since differentiation is $k(x^p)$-linear, T_f is a $k(x^p)$-linear operator. We may also view T_f as a non-linear operator on $k(x)$, where $k(x)$ is viewed as a vector space over itself. The p-fold composition of T_f, $(T_f)^p$, is in fact $k(x)$-linear. By a result of Nicholas Katz in [Kat70] (which in our situation can be stated simply as [Clu03, Theorem 3.8]), the dimension of the kernel of T_f as a $k(x^p)$-linear operator is the equal to dimension of the kernel of $(T_f)^p$ as a $k(x)$-linear operator. Since $(T_f)^p$ operates on a one dimensional vector space, its kernel can be at most one dimensional. Since $T_f(f(x)) = 0$, the kernel of T_f is the $k(x^p)$-span of $f(x)$. Therefore, the image of T_f is $(p-1)$ dimensional.

Proposition 3.5.9. *For the operator T_f as defined above, if* $\mathrm{char}(k) = 2$*, then for every non-zero polynomial f the image of T_f is $k(x^2) \subseteq k(x)$. If* $\mathrm{char}(k) > 2$*, then every $(p-1)$ dimensional $k(x^p)$-vector subspace of $k(x)$ occurs as the image of T_f for some polynomial f.*

Proof. Fix a non-zero polynomial $f(x) \in k[x]$. First, consider the case where $\mathrm{char}(k) = 2$. Write

$$f(x) = \sum_{i=0}^{n} (a_i \cdot x^{2i} + b_i \cdot x^{2i+1}).$$

By direct computation, we have that

$$T_f(1) = \sum_{i=0}^{n} b_i \cdot x^{2i},\ T_f(x) = \sum_{i=0}^{n} a_i \cdot x^{2i}.$$

Since $\{1, x\}$ is a $k(x^2)$-basis for $k(x)$ and $T_f(1)$ and $T_f(x)$ both lie in $k(x^2)$, the image of T_f is indeed $k(x^2)$.

Consider the remaining case where $\mathrm{char}(k) > 2$. Viewing $\mathbb{A}^p_{k(x^p)}$ as $\mathrm{Spec}(k(x^p)[1, ..., x^{p-1}])$ and mapping $p(x)$ to the image of $T_{p(x)}$, we have a map

$$\mathbb{A}^p_{k(x^p)} \setminus \{0\} \to Gr(p-1, k(x)),$$

where $Gr(p-1, k(x))$ is the Grassmannian of $(p-1)$ planes in $k(x)$. Since this map is constant on lines in $\mathbb{A}^p_{k(x^p)} \setminus \{0\}$ and $Gr(p-1, k(x)) \cong \mathbb{P}^{p-1}_{k(x^p)}$, this descends to a map $\varphi : \mathbb{P}^{p-1}_{k(x^p)} \to \mathbb{P}^{p-1}_{k(x^p)}$. A map between projective spaces of the same dimension must be either constant or surjective, and moreover in the latter case quasi-finite.

One checks directly in characteristic greater than two that x^{p-1} lies in the image of T_x but not T_1. This implies that φ is non-constant and therefore surjective. That is, for every $(p-1)$ dimensional $k(x^p)$-vector subspace V of $k(x)$, there is some rational function $f(x)$ such that the image of T_f is V. Since multiplying $f(x)$ by a polynomial $g(x) \in k[x^p]$ does not affect the image, for an appropriate choice of $g(x)$ we have that $f(x)g(x)$ is a polynomial where the image of $T_{f(x)g(x)}$ is V, as desired. $\square$

3.5.3 Proof of the smoothness of the moduli space

To prove the smoothness result, we will make use of the following technical lemma:

Lemma 3.5.10. *Let k be a field and $f_1, ..., f_n \in k[x]$ non-zero polynomials of degrees $d_1, ..., d_n$ respectively. Assume that $d_1 \geq \cdots \geq d_n$. If $1 = \gcd(f_1, ..., f_n)$ and $h(x)$ is a polynomial of degree at most $2d_1 - 1$, then $h(x)$ can be written as a linear combination $h(x) = a_1 \cdot f_1 + \cdots + a_n \cdot f_n$ where the $a_i \in k[x]$ have degree at most $d_1 - 1$.*

Proof. It suffices to write one polynomial of every degree $0, 1, ..., (2d_1 - 1)$ in the desired form, since these will form a basis for the vector space of polynomials of degree at most $2d_1 - 1$. The set $\{x^j \cdot f_i\}$, where $0 \leq j \leq (d_1 - 1)$ and $1 \leq i \leq n$, contains polynomials of degrees $d_n, ..., (2d_1 - 1)$ written in the appropriate form. Hence, we are left to show that we can write polynomials of degrees $0, 1, ..., (d_n - 1)$ as a linear combination of the f_i's with coefficients of degree at most $d_1 - 1$. We will show this by induction. Let $S = \{f_1, ..., f_n\}$. Beginning with f_1, remove any polynomial f_i such that $1 = \gcd(S \setminus \{f_i\})$.

We will iterate on the hypothesis that we have a set of polynomials $S = \{g_1, ..., g_k\}$ of degrees $e_1 \geq \cdots \geq e_k$, such that $1 = \gcd(S)$ and $1 \neq \gcd(S \setminus \{g_i\})$ for any $1 \leq i \leq k$, and we need to write one polynomial of every degree $0, 1, ..., (e_1 - 1)$ as a linear combination of the g_i using coefficients of degree at most $e_1 - 1$. Once this iteration terminates, we will have written polynomials of degrees $0, 1, ..., (d_n - 1)$ in the requisite form.

If $k = 1$, then the degree of g_1 is zero since $\gcd(S) = 1$, and thus there is nothing left to show. If $k > 1$, the set $\{x^j \cdot g_i\}$, where $0 \leq j \leq e_1 - 1$ and $1 \leq i \leq k$, contains polynomials of degrees $e_k, ..., (2e_1 - 1)$ written in the appropriate form. We are left to write polynomials of degrees $0, 1, ..., (e_k - 1)$. By polynomial division we have

$$r = g_1 - a \cdot g_2 \tag{3.3}$$

where $\deg(a) = e_1 - e_2$ and $0 \leq \deg(r) < e_2$; $r \neq 0$ since g_2 does not divide g_1 by hypothesis. The set $\{x^j(g_1 - a \cdot g_2)\}$, where $0 \leq j \leq (e_2 - 1)$, contains polynomials of degrees $\deg(r), ..., (\deg(r) + e_2 - 1)$ written in the requisite form.

Remove g_1 from S and replace it with r. Further, remove any polynomial g_i from S such that $1 = \gcd(S \setminus \{g_i\})$. By equation (3.3), r may be written as a linear combination of the g_i using coefficients of degree at most $e_1 - e_2$. Therefore, to write polynomials of degrees $0, 1, ..., (\deg(r) - 1)$ as a linear combination of $\{g_1, ..., g_k\}$ with coefficients of degree at most $e_1 - 1$ as required, it suffices to write polynomials of degrees $0, 1, ..., e_2$ as a linear combination of the polynomials now in S using coefficients of degree at most $e_2 - 1$. This is precisely the hypothesis upon which we are iterating, so we may continue the iteration. Since at each step we are lowering the degree of one polynomial, this iteration will terminate in the case where S contains a single polynomial, which is of degree zero. $\square$

Equipped with this lemma, we can prove the main theorem of this section:

Proof of Theorem 3.5.2. Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a separable degree d map. Choose coordinate functions on each copy of $\mathbb{P}^1_k$ such that f is unramified at ∞ and 0 is a fixed point. In these coordinates, represent f by a rational function $y = g(x)/h(x)$. Let X denote the scheme X_D or $X_{(l_i)}$ from Proposition 3.5.7. To prove that X is smooth, by Lemma 3.5.6 it suffices to lift diagrams of the form

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & X \\ \mathrm{Spec}(B) \arrow[ur, dashed] & \end{tikzcd} \tag{3.4}$$

where $A = k[t_1, ..., t_n]/I$, $B = k[t_1, ..., t_n]/J$, $I \supseteq J$, the map is the natural quotient map, $\sqrt{I} = \sqrt{J} = (t_1, ..., t_n)$, and $t_i \cdot I \subseteq J$ for every $1 \le i \le n$.

Let $\{1, \tau_1, ..., \tau_r\}$ be a basis of A over k such that each τ_i is a monomial. By Proposition 3.5.7, the horizontal map in (3.4) corresponds to

$$f_A = \frac{g_A(x)}{h_A(x)} = \frac{g(x) + g_1(x)\tau_1 + \cdots + g_r(x)\tau_r}{h(x) + h_1(x)\tau_1 + \cdots + h_r(x)\tau_r},$$

a deformation of f over A, which has the appropriate discriminant modulo I. Extend $\{1, \tau_1, ..., \tau_r\}$ to a basis $\{1, \tau_1, ..., \tau_{r+l}\}$ of B over k such that each additional τ_i is a monomial. Again by Proposition 3.5.7, a lift as in (3.4) corresponds to

$$f_B = \frac{g_B(x)}{h_B(x)} = \frac{g(x) + g_1(x)\tau_1 + \cdots + g_r(x)\tau_r + g_{r+1}(x)\tau_{r+1} + \cdots + g_{r+l}(x)\tau_{r+l}}{h(x) + h_1(x)\tau_1 + \cdots + h_r(x)\tau_r + h_{r+1}(x)\tau_{r+1} + \cdots + h_{r+l}(x)\tau_{r+l}},$$

a deformation of f over B, which has the appropriate discriminant modulo J. Therefore, we must find $g_{r+1}(x), ..., g_{r+l}(x), h_{r+1}(x), ..., h_{r+l}(x)$, polynomials of degree at most d, such that the discriminant of f_B has the appropriate form modulo J.

We now make use of the fact that $\mathrm{char}(k) = 2$, and claim that

$$\gcd(T_g(1), T_g(x), T_h(1), T_h(x)) = 1.$$

To see this, let

$$q(x) = \sum_{i=0}^{s} b_i x^i$$

be an arbitrary polynomial. We have that

$$T_q(1) = \sum_{\substack{0 \le i \le s \\ 2 \nmid i}} b_i x^{i-1}, \ T_q(x) = \sum_{\substack{0 \le i \le s \\ 2 | i}} b_i x^i. \tag{3.5}$$

If $\gcd(T_g(1), T_g(x), T_h(1), T_h(x)) \neq 1$, then $(x-b)|T_g(1), T_g(x), T_h(1), T_h(x)$ for some $b \in k$. By equation (3.5), this would imply that $(x-b)|g(x), h(x)$, so that $g(x)$ and $h(x)$ would not be relatively prime. This would contradict $g(x)/h(x)$ being written in lowest terms, proving the claim.

Let $d_1 = \max\{\deg(T_g(1)), \deg(T_g(x)), \deg(T_h(1)), \deg(T_h(x))\}$. By Lemma 3.5.10, if $\beta(x)$ is a polynomial of degree at most $2d_1 - 1$, then we can write it as a linear combination

$$\beta(x) = a_1(x)T_g(1) + a_2(x)T_g(x) + a_3(x)T_h(1) + a_4(x)T_h(x), \tag{3.6}$$

where the degree of each $a_i(x)$ is at most $d_1 - 1$. By Proposition 3.5.9 the discriminant of any rational function in characteristic two is contained in $k[x^2]$, and so we restrict to the case where $\beta(x) \in k[x^2]$. Since $T_g(1), T_g(x), T_h(1)$, and $T_h(x)$ all contain terms of only even degree, we may remove all terms of odd degree from the $a_i(x)$ and equation (3.6) will still hold. Since each operator is $k(x^2)$-linear, after this adjustment we have that

$$\begin{aligned}\beta(x) &= a_1(x)T_g(1) + a_2(x)T_g(x) + a_3(x)T_h(1) + a_4(x)T_h(x) \\ &= T_g(a_1(x) + x \cdot a_2(x)) + T_h(a_3(x) + x \cdot a_4(x)). \qquad (3.7)\end{aligned}$$

If d is even, then by equation (3.5), $d_1 = d$. In this case, equation (3.7) implies that we can write every polynomial $\beta(x)$ of degree at most $2d-2$ using $a_i(x)$ of degree at most $d-1$. Conversely, if d is odd, then $d_1 = (d-1)$. By equation (3.7) we can then write every polynomial $\beta(x)$ of degree at most $2d-3$ using $a_i(x)$ of degree at most $d-2$. Since d is odd and the $a_i(x)$ contain no terms of odd degree, we moreover have that the degree of each $a_i(x)$ is at most $d-3$. By multiplying the linear combination generating a polynomial of degree $2d-4$ by x^2, we can generate a polynomial of degree $2d-2$ using $a_i(x)$ of degree at most $d-1$ as well.

Since $\{\tau_{r+1}, ..., \tau_{r+l}\}$ consists of the elements added to extend the basis of A to a basis for B, the part of the discriminant of f_A which vanishes modulo I but not modulo J may be written uniquely as

$$\sum_{m=r+1}^{r+l} \beta_m(x) \cdot \tau_m,$$

where each $\beta_m(x) \in k[x^2]$. We have shown that there are $a_i(x)$ of degree at most $d-1$ such that

$$\beta_m(x) = T_g(a_1(x) + x \cdot a_2(x)) + T_h(a_3(x) + x \cdot a_4(x)).$$

Setting $g_m(x) = a_3(x) + x \cdot a_4(x)$ and $h_m(x) = -(a_1(x) + x \cdot a_2(x))$ for each $r+1 \leq$

$m \leq r + l$, we claim that the discriminant of

$$f_B = \frac{g_B(x)}{h_B(x)} = \frac{g(x) + g_1(x)\tau_1 + \cdots + g_r(x)\tau_r + g_{r+1}(x)\tau_{r+1} + \cdots + g_{r+l}(x)\tau_{r+l}}{h(x) + h_1(x)\tau_1 + \cdots + h_r(x)\tau_r + h_{r+1}(x)\tau_{r+1} + \cdots + h_{r+l}(x)\tau_{r+l}}$$

modulo J is equal to the discriminant of $g_A(x)/h_A(x)$ modulo I. To see this, recall from Lemma 3.5.6 that $t_i \cdot I \subseteq J$ for every $1 \leq i \leq n$. This implies that $\tau_i \cdot \tau_j \in J$ for any choice of i and j. Therefore the only new terms in the discriminant of f_B which do not vanish modulo J are $\tau_m(h_m(x)g'(x) - g(x)h_m(x))$ and $\tau_m(h(x)g'_m(x) - g_m(x)h'(x))$ where $r + 1 \leq m \leq r + l$. These terms were chosen exactly to cancel with the part of the discriminant of f_A which vanishes modulo I but not modulo J. Therefore f_B is a lift to B of the desired form. □

Remark 3.5.11. The restriction on the characteristic of k is necessary because it is not in general true that $g(x)$ and $h(x)$ being relatively prime implies that $\gcd(T_g(1), ..., T_g(x^{p-1}), T_h(1), ..., T_h(x^{p-1})) = 1$. Although this condition on the greatest common divisor is not strictly necessary, it is required to invoke Lemma 3.5.10, and so a proof in higher characteristic where this does not hold would be more subtle using these techniques.

Chapter 4

Galois Case

4.1 Overview

In this chapter, we will impose the additional restriction that the branched covers considered are Galois. Recall that a branched cover of smooth proper curves $X \to Y$ is called *Galois* if the induced map on function fields $\kappa(Y) \subseteq \kappa(X)$ is a Galois field extension. Geometrically, this corresponds to the number of automorphism of X commuting with the map $X \to Y$ being the same as the degree of the cover. A branched cover of smooth proper curves $X \to Y$ is called *G-Galois* if it is Galois and $\mathrm{Aut}(X/Y)$ is isomorphic to G. The culmination of this chapter is Theorem 4.4.1, which answers the following question:

Question 4.1.1. Let X be a smooth proper curve over an algebraically closed field k, S a finite set of closed points on X, and G a finite group. Under what conditions does there exist a non-constant family of equivalence classes of G-Galois maps with source X, of fixed degree and with ramification locus S?

The first step of the proof is establishing a correspondence between equivalence classes of G-Galois branched covers with source X and G-actions on X. This is accomplished in Section 4.3. In that section it is also shown that the stabilized points of the G-action (see Definition 4.3.6) are exactly the ramification points of the induced G-Galois branched cover of curves.

With this, fixing the ramification locus S as above, we are reduced to analyzing G-actions on X with stabilized locus S. After this reduction, Question 4.1.1 can be answered almost immediately for the case when X has genus greater than one. Two essentially disjoint proofs are given for the genus zero and genus one cases.

4.2 Galois covers from the covering space perspective

Recall that in the covering space perspective two branched covers $f_i : X_i \to Y$ are considered equivalent if there is a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[rr, "\cong"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

Let X be a smooth proper curve over an algebraically closed field k and S a finite set of closed points on X. Suppose $U \subseteq X$ is an open subscheme. As explained in Section 1.4, $\pi_1^{\acute{e}t}(U)$ is an important tool in studying finite connected étale covers of U. By the valuative criterion for properness, a finite étale cover of U is equivalent to a branched cover of smooth proper curves $Y \to X$ whose branch locus is contained in $X \setminus U$.

Fix a finite group G. An equivalence class of G-Galois branched covers of X whose branch locus is contained in $X \setminus U$ is equivalent to a surjective group homomorphism $\pi_1^{\acute{e}t}(U) \to G$, up to uniform conjugation in G (see for example [Har03, Corollary 2.1.2]). If $\mathrm{char}(k) = 0$, then $\pi_1^{\acute{e}t}(U)$ is finitely generated, so there are finitely many such group homomorphism. Similarly, since the tame étale fundamental group is finitely generated by [Gro71, Exposé XIII Corollaire 2.12], there are finitely many equivalence classes of tame G-Galois covers (i.e. G-Galois covers for which only elements of G with order prime to p have fixed points) which are étale over U.

However, if $U \subsetneq X$ and $\mathrm{char}(k) = p > 0$, $\pi_1^{\acute{e}t}(U)$, is not finitely generated. In fact, there exist non-constant families of G-Galois branched covers with fixed branched points. For example, if k has characteristic $p > 0$, the maps $\mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $y^p - y = tx$ with parameter t form a family of $\mathbb{Z}/p\mathbb{Z}-$Galois covers of the x-line branched only at the point at infinity.

4.3 Correspondence between group actions and classes of covers

Recall that we will consider two branched covers of curves $X \to Y_i$ from a fixed space X to be equivalent if there is a commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "\cong"] & & Y_2 \end{tikzcd}$$

Definition 4.3.1. Let X be a smooth proper curve over an algebraically closed field k and G a finite group. A *group action* on X will be a choice of a finite subgroup of $\mathrm{Aut}(X)$. A *G-action* on X will be a choice of a subgroup $H \subseteq \mathrm{Aut}(X)$ which is isomorphic to G. The choice of such an H corresponds to an equivalence class of injective maps $i : G \to \mathrm{Aut}(X)$ where two maps are considered equivalent if their images are the same.

Remark 4.3.2. With the above definition, given $i : G \to \mathrm{Aut}(X)$, the injections in the same equivalence class as i are in correspondence with $\mathrm{Aut}(G)$, since any injection with the same image differs by an automorphism of $i(G)$ and post-composing i with an automorphism of G yields an injection with the same image. If one wished to consider injections of G into $\mathrm{Aut}(X)$ instead of equivalences classes of such injections, to preserve the correspondence with G-Galois branched covers $X \to Y$ in Theorem 4.3.8, one would need to include in the definition of a G-Galois cover an isomorphism of $\mathrm{Aut}(X/Y)$ with G. Such a definition of a G-Galois cover is used for instance in [CH85].

First, we show every G-action on a curve induces a G-Galois branched cover, and later show this construction produces a correspondence on equivalence classes of covers.

Lemma 4.3.3. *Let X be a smooth proper curve over an algebraically closed field k and G a finite group. Every G-action induces a G-Galois branched cover of smooth proper curves $X \to Y$.*

Proof. Fix a G-action on X. By the equivalence of categories between smooth proper curves over k and function fields of transcendence degree one over k, this induces a G-action on $\kappa(X)$. By taking the G-invariants of this action, we obtain a G-Galois field extension $\kappa(X)/\kappa(X)^G$. Since G is a finite group, $\kappa(X)^G$ is also a field of transcendence degree one over k. Thus, again by the equivalence of categories between function fields of transcendence degree one over k and smooth proper curves over k, this field extension induces a G-Galois branched cover $X \to Y$, where $\kappa(Y) = \kappa(X)^G$. $\square$

Remark 4.3.4. Whenever we refer to an action on X inducing a cover, it will be by the procedure in the proof of Lemma 4.3.3.

Now, we wish to relate actions of finite groups on curves to equivalence classes of covers. As the following lemma shows, distinct G-actions induce distinct classes of covers:

Lemma 4.3.5. *Suppose M and N are finite subgroups of* $\mathrm{Aut}(X)$ *whose induced covers $X \to Y_M$ and $X \to Y_N$ are isomorphic. Then $M = N$ as subgroups of* $\mathrm{Aut}(X)$.

Proof. Following the procedure in the proof of Lemma 4.3.3, the isomorphism of induced covers implies that we have a commutative diagram of fields

$$\begin{tikzcd} & \kappa(X) & \\ \kappa(X)^M \arrow[ur, "i_1"] \arrow[rr, "\varphi"] & & \kappa(X)^N \arrow[ul, "i_2"'] \end{tikzcd}$$

where φ is an isomorphism. This implies that via the embeddings i_1 and i_2, $\kappa(X)^M$ and $\kappa(X)^N$ are the same subfield of $\kappa(X)$. Let L be this subfield. Since L is the fixed field of both M and N, each is the subgroup of automorphisms of $\kappa(X)$ which fix L, so $M = N$. $\square$

Now, fix a finite group G, and consider the question of determining the equivalence classes of G-Galois covers with source X. By definition, $\mathrm{Aut}(X/Y) \cong G$, so each such cover determines a G-action on X. This observation, along with Lemma 4.3.5, shows that equivalence classes of G-Galois covers with source X are in direct correspondence with G-actions on X.

To answer Question 4.1.1, we must also reformulate ramification of G-Galois branched covers into a statement about G-actions on X. Before stating the lemma, we introduce the following definitions:

Definition 4.3.6. A closed point P on X is called a *stabilized point* of a G-action if its stabilizer under the G-action is non-trivial. The set of stabilized points of a G-action is called its *stabilized locus.* A G-action is called *stabilized point free* if its stabilized locus is empty.

Lemma 4.3.7. *Let X be a smooth proper curve over an algebraically closed field k and fix a G-action on X. The stabilized points of the G-action are exactly the ramification points of the induced G-Galois branched cover.*

Proof. Let $f : X \to Y$ be the G-Galois branched cover induced by the action. Since the cover is Galois, if $x \in X$ is a ramification point, there is a non-identity element $g \in G$ for which $g(x) = x$ and g acts trivially on the extension of residue fields $\kappa(x)/\kappa(f(x))$. However, X is defined over an algebraically closed field, so $[\kappa(x) : \kappa(f(x))] = 1$, and therefore the latter condition is trivial. $\square$

We summarize the results of this section in the following theorem:

Theorem 4.3.8. *Fix a smooth proper curve X over an algebraically closed field k and a finite group G. There is a correspondence between equivalence classes of G-Galois branched covers with source X and G-actions on X. Under this correspondence, the ramification points of the cover correspond to the stabilized points of the G-action.*

4.4 Main result in the Galois case

By Section 4.3, we reduce to studying the G-actions on X which have as stabilized points our prescribed finite set of points. We will pursue this program to prove the following:

Theorem 4.4.1. *Let X be a smooth proper curve over an algebraically closed field k, S a finite set of closed points on X, and G a finite group. Then there exist infinitely many equivalence classes of G-Galois covers with ramification locus S if and only if all the following hold:*

(1) $\mathrm{char}(k) = p > 0$

(2) $X \cong \mathbb{P}^1_k$

(3) $G \cong (\mathbb{Z}/p\mathbb{Z})^m$*, where* $m \in \mathbb{N}$

(4) $|S| = 1$.

When these conditions hold, equivalence classes of such covers ramified at S are in correspondence with rank m additive subgroups of k, as described in Theorem 4.4.18. Furthermore, when there are finitely many equivalence classes of G-Galois covers, there are no non-constant families by Theorem 2.2.2.

Remark 4.4.2. We clarify the finiteness statement of Theorem 4.4.1 to previous work. As noted in Section 3.1, even without imposing the Galois condition, [dF13] and [Kan86] imply that X has finitely many equivalence classes of maps to curves which are not $\mathbb{P}^1$. Additionally, by Theorem 3.1.2, again even without imposing the Galois condition, we have finitely many equivalence classes of maps of fixed degree to $\mathbb{P}^1_k$ with fixed ramification points if $\mathrm{char}(k) = 0$. Hence, the new content of Theorem 4.4.1 is the case of classes of Galois maps to $\mathbb{P}^1_k$ in positive characteristic.

As a corollary of Theorem 4.4.1, we have a statement about tame Galois branched covers similar to the situation from the covering space perspective, as mentioned in Section 4.2:

Corollary 4.4.3. *Let X be a smooth proper curve over an algebraically closed field k and G a finite group. Then there are only finitely many equivalence classes of G-Galois, tame covers with source X with fixed ramification points. This implies that there are no non-constant tame G-Galois families.*

This follows since the conditions of Theorem 4.4.1 require G to be acting on the projective line and have an element of order p. By Lemma 4.4.12, every automorphism of the projective line has at least one stabilized point. Hence, such a cover is wildly ramified.

As mentioned in the introduction, Theorem 4.4.1 will be proven in three cases: when the genus of X is zero, one, or greater than one. Answering Question 4.1.1 in the case where the genus of X is greater than one is quite easy, as the following remark indicates:

Remark 4.4.4. Let X be a smooth proper curve over a field k of genus greater than one. Then the automorphism group of X is finite and so there are only finitely many subgroups of $\mathrm{Aut}(X)$. Therefore, by Lemma 4.3.5, there are only finitely many equivalence classes of Galois branched covers with source X, even without specifying the ramification points or the group.

4.4.1 Genus 1 case

For the entirety of this section, fix a smooth proper genus one curve X over an algebraically closed field k, a finite set S of closed points on X, and a finite group G. Since X is a curve over an algebraically closed field, we may fix a k-rational base point 0 to consider X as an elliptic curve. Recall that $\mathrm{Aut}(X) \cong T_X \rtimes \mathrm{Aut}_0(X)$, where T_X consists of the translations of X by closed points $P \in X$ and $\mathrm{Aut}_0(X)$ is the finite group of automorphisms of X fixing the base point 0. We will denote the identity element of $\mathrm{Aut}_0(X)$ by 1.

First, we specify using the above isomorphism which automorphisms have fixed points and which are fixed point free:

Lemma 4.4.5. *Fix $\varphi \in \mathrm{Aut}(X)$. Using the isomorphism $\mathrm{Aut}(X) \cong T_X \rtimes \mathrm{Aut}_0(X)$, write $\varphi = (P, \sigma)$ where P is a closed point of X and $\sigma \in \mathrm{Aut}_0(X)$. Then φ is fixed point free precisely when $\sigma = 1$ and $P \neq 0$.*

Proof. Let $\varphi = (P, \sigma)$ be an arbitrary automorphism of X. Writing this automorphism as a function, $\varphi(Q) = \sigma(Q) + P$. The fixed points of φ are therefore the points Q such that $(1-\sigma)(Q) = P$.

If $\sigma \neq 1$, then $(1-\sigma)$ is a surjective group homomorphism with kernel K of cardinality equal to the degree of the map $(1-\sigma)$. Therefore we have a short exact sequence:

$$0 \longrightarrow K \longrightarrow X \xrightarrow{(1-\sigma)} X \longrightarrow 0.$$

There are $|K|$ preimages of P, which are the fixed points of φ, and so $\sigma \neq 1$ implies that φ is not fixed point free.

Now suppose $\sigma = 1$. Since the fixed points of φ are the points Q such that $(1-\sigma)(Q) = P$, φ has fixed points only if $P = 0$, in which case φ is the identity. □

We now show that there are finitely many stabilized point free G-actions on X:

Lemma 4.4.6. *Let G be a finite group. There are only finitely many equivalence classes of unramified G-Galois covers with source X. Stated another way, there are only finitely many G-actions on X which are stabilized point free.*

Proof. Suppose $f : X \to Y$ is an unramified G-Galois cover. Using the Riemann Hurwitz formula, Y is also a genus one curve, and after choosing the image of the base point of X as the base point of Y, it is an elliptic curve as well. Thus f is an isogeny, and we may take $f^\vee : Y \to X$, the dual isogeny. Composing these isogenies yields a short exact sequence:

$$0 \longrightarrow X[n] \longrightarrow X \xrightarrow{f^\vee \circ f} X \longrightarrow 0$$

where $X[n]$ is the n-torsion points of X where $n = |G|$. Therefore $G \subseteq \mathrm{Aut}(X)$ must be a subgroup of the group of translations by n-torsion points, and hence there are finitely many equivalence classes of unramified G-Galois covers with source X. $\square$

Next, we analyze automorphisms of X with fixed points contained in S:

Lemma 4.4.7. *There are only finitely many $\varphi \in \mathrm{Aut}(X)$ which have non-trivial fixed locus contained in S.*

Proof. Fix a closed point Q on X. Since any φ with non-trivial fixed locus contained in S must fix at least one point in S, it suffices to show that there are only finitely many automorphisms of X which fix Q. Suppose $\varphi \in \mathrm{Aut}(X)$ is one such automorphism, and suppose it is not the identity. Using the isomorphism of $\mathrm{Aut}(X)$ with $T_X \rtimes \mathrm{Aut}_0(X)$ and Lemma 4.4.5, φ is of the form (P, σ) where $\sigma \neq 1$. Since $(1-\sigma)$ is a surjective group homomorphism, we have a short exact sequence:

$$0 \longrightarrow K \longrightarrow X \xrightarrow{1-\sigma} X \longrightarrow 0$$

where K is the kernel. The fixed points of φ are precisely the points in the preimage of P under $(1-\sigma)$. Therefore P is the unique point for which Q is a fixed point of an automorphism of the form (R, σ). Since there are finitely many $\sigma \in \mathrm{Aut}_0(X)$, Q is a fixed point for only finitely many $\varphi \in \mathrm{Aut}(X)$. $\square$

Using Lemma 4.4.7, we can prove a strengthening of Theorem 4.4.1 for ramified covers in the genus 1 case:

Theorem 4.4.8. *Let X be a smooth proper genus 1 curve over an algebraically closed field k and S a finite set of closed points on X. Then there exist only finitely many group actions on X with non-empty stabilized locus contained in S.*

Proof. Since the action has non-empty stabilized locus, let φ be a non-identity element of the G-action which has fixed points. By Lemma 4.4.7, there are only finitely many automorphisms with non-trivial stabilized locus contained in S. It therefore suffices to show that only finitely many fixed point free automorphisms of X can occur in the same group action as φ.

Since φ is not the identity and has fixed points, by Lemma 4.4.5 it is of the form (P, σ) where $\sigma \neq 1$. Let ψ be a fixed point free automorphism of X. Again by Lemma 4.4.5, ψ is of the form $(Q, 1)$ where $Q \neq 0$. If φ and ψ occur in the same group action, so does $\psi \circ \varphi = (P + Q, \sigma)$. Since $\sigma \neq 1$, $(1 - \sigma)$ is a surjective group homomorphism and so it induces a short exact sequence:

$$0 \longrightarrow K \longrightarrow X \xrightarrow{(1-\sigma)} X \longrightarrow 0$$

where K is the kernel. The fixed points of $\psi \circ \varphi = (P + Q, \sigma)$ are the preimages of $P + Q$ under $(1 - \sigma)$. However, S is finite so there are only finitely many Q for which the preimage of $P + Q$ is contained in S. This implies that only finitely many ψ can occur in the same group action as φ if the stabilized locus is contained in S. □

Remark 4.4.9. Since X is an elliptic curve, the induced cover being ramified is equivalent by the Riemann Hurwitz formula to the target space being the projective line. Hence, another way to state Theorem 4.4.8 is that there are only finitely many equivalence classes of Galois branched covers from a fixed elliptic curve X to the projective line.

Theorem 4.4.1 follows as a corollary to Theorem 4.4.8 and Lemma 4.4.6 in the genus 1 case:

Corollary 4.4.10. *Let X be a smooth proper genus 1 curve over an algebraically closed field k, S a finite set of closed points on X, and G a finite group. Then there exist only finitely many equivalence classes of G-Galois branched covers with ramification locus inside S.*

4.4.2 Genus 0 case

For the entirety of this section we will assume $X = \mathbb{P}^1_k$, where k is an algebraically closed field. Furthermore, by Remark 4.4.2, we need only prove the result in positive characteristic, so we additionally assume $\mathrm{char}(k) = p > 0$.

Fix a finite group G and a finite set S of closed points on $\mathbb{P}^1_k$. To recall, we shall study the subgroups of $\mathrm{Aut}(\mathbb{P}^1)$ isomorphic to G with stabilized locus S. In doing so, we will make use of the following observation:

Remark 4.4.11. Let k be an algebraically closed field. Then $\mathrm{Aut}_k(\mathbb{P}^1) \cong \mathrm{PGL}_2(k)$. We shall fix the isomorphism between these two groups sending a fractional linear transformation $\varphi = (ax+b)/(cx+d)$ to the equivalence class of

$$M(\varphi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Under this isomorphism, φ acting on a point with projective coordinates $[x : y]$ corresponds to $M(\varphi)$ acting on the column vector $[x\ y]$.

Throughout this section, we will use the following fact:

Lemma 4.4.12. *Let $\varphi \in \mathrm{Aut}(\mathbb{P}^1_k)$ have finite order. Then φ has exactly one or two fixed points, and has exactly one fixed point when it has order p.*

Proof. Using the isomorphism with $\mathrm{PGL}_2(k)$, the fixed points of φ correspond to eigenvectors of $M(\varphi)$ up to scaling. Since k is algebraically closed, $M(\varphi)$ must have at least one eigenvector. The Jordan canonical form of a representative for φ must be one of the following:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Note that conjugating $M(\varphi)$ changes the fixed points but not the number of fixed points. The first matrix has $[1\ 0]$ and $[0\ 1]$ as eigenvectors, and hence 0 and the point at infinity as fixed points. For the second matrix, up to scaling $[0\ 1]$ is the only eigenvector, and hence the only fixed point is the point at infinity. One can easily check since $\mathrm{char}(k) = p$ that the order of the first matrix in $\mathrm{PGL}_2(k)$ is prime to p while the order of the second matrix is p. $\square$

We will divide the proof of Theorem 4.4.1 into two cases: when p does not divide $|G|$ and when it does. In [Fab12], Faber refers to the former as *p-regular groups* and the latter as *p-irregular groups*. The p-regular groups also occur as subgroups of $\mathrm{PGL}_2(\mathbb{C})$ ([Fab12, Theorem C])), but there are p-irregular subgroups of $\mathrm{PGL}_2(k)$ which do not occur as subgroups of $\mathrm{PGL}_2(\mathbb{C})$ ([Fab12, Theorem 6.1]).

To simplify the statement of theorems and proofs for the remainder of this section, we introduce the following notation:

Notation 4.4.13. For a field k, $\mu_n(k)$ will denote the nth roots of unity in $k^\times$. A primitive nth root of unity in $k^\times$ will be denoted by ζ_n.

In the p-regular case we will make use of the following classification of subgroups up to conjugacy:

Theorem 4.4.14. *([Fab12, Theorem C]) Let k be an algebraically closed field of characteristic $p > 0$. Then every finite p-regular subgroup of $\mathrm{PGL}_2(k)$ is conjugate to one of the following classes of groups:*

(1) If $p \nmid n$, $\begin{pmatrix} \mu_n(k) & 0 \\ 0 & 1 \end{pmatrix}$. This group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

(2) If $p \nmid n$ and $p \neq 2$, $\begin{pmatrix} \mu_n(k) & 0 \\ 0 & 1 \end{pmatrix} \rtimes \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$. This group is isomorphic to D_{2n}, the dihedral group of order $2n$.

(3) *If* $p \neq 2, 3$, $N \rtimes \left\langle \begin{pmatrix} 1 & \zeta_4 \\ 1 & -\zeta_4 \end{pmatrix} \right\rangle$, *where* N *is generated by* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *This group is isomorphic to* A_4.

(4) *If* $p \neq 2, 3$, *the group generated by the group in (3) and* $\begin{pmatrix} \zeta_4 & 0 \\ 0 & 1 \end{pmatrix}$. *This group is isomorphic to* S_4.

(5) *If* $p \neq 2, 3, 5$, *the group generated by* $\begin{pmatrix} \zeta_5 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & (1 - \zeta_5 - \zeta_5^{-1}) \\ 1 & -1 \end{pmatrix}$. *This group is isomorphic to* A_5.

The main utility in the statement of Theorem 4.4.14 is that it classifies groups up to conjugacy rather than isomorphism. Since we are interested in specific occurrences of G as a subgroup of $\mathrm{PGL}_2(k)$ this additional information greatly simplifies the proof of the following:

Theorem 4.4.15. *Let* k *be an algebraically closed field of characteristic* $p > 0$, S *a finite set of closed points on* $\mathbb{P}^1_k$, *and* G *a finite* p*-regular group. There are only finitely many* G*-actions on* $\mathbb{P}^1_k$ *with stabilized locus* S.

Proof. Let G be a p-regular group. Since G is p-regular, every subgroup corresponding to a G-action must be conjugate to one of the five groups listed in Theorem 4.4.14. It therefore suffices to show that there are finitely many groups conjugate to each of the listed groups with the same stabilized points. One can check directly that every group but the cyclic group in (1) has at least three stabilized points. Since an element in $\mathrm{PGL}_2(k)$ is determined by the image of three points, only finitely many elements can act as a permutation on the set of stabilized points when there are at least three. This implies except in case (1) there are finitely many conjugate groups with the same stabilized points.

Therefore, we only have the cyclic group in (1) left to consider. The stabilized points of this group are 0 and ∞. The matrices which act as a permutation on the set $\{0, \infty\}$ are of the form $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$ where $\alpha, \beta \in k^\times$. However, conjugating by a matrix of either form merely acts as an automorphism, so $\begin{pmatrix} \mu_n(k) & 0 \\ 0 & 1 \end{pmatrix}$ has no conjugate subgroups with the same stabilized points. $\square$

In the p-irregular case we will make use of the following analog of Theorem 4.4.14:

Theorem 4.4.16. *([Fab12, Theorem 6.1]) Let k be an algebraically closed field of characteristic $p > 0$. Then every finite p-irregular subgroup of $\mathrm{PGL}_2(k)$ is conjugate to one of the following classes of groups:*

(1) $\mathrm{PSL}_2(\mathbb{F}_{p^n})$ *for some $n \in \mathbb{N}$.*

(2) $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ *for some $n \in \mathbb{N}$.*

(3) The group $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix} \rtimes \begin{pmatrix} \mu_n(k) & 0 \\ 0 & 1 \end{pmatrix}$. *Here $n \in \mathbb{N}$ is prime to p and Γ is an additive subgroup of k of rank $m \in \mathbb{N}$ such that $\mu_n(k) \subseteq \Gamma$ and $\mu_n(k) \cdot \Gamma \subseteq \Gamma$.*

Two such subgroups of $\mathrm{PGL}_2(k)$ *of the same order are conjugate if and only if* $\Gamma' = \alpha \cdot \Gamma$ *for some* $\alpha \in k^\times$.

(4) If $p = 2$, *the group* $\begin{pmatrix} \mu_n(k) & 0 \\ 0 & 1 \end{pmatrix} \rtimes \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$. *Here* $n \in \mathbb{N}$ *is odd and greater than one. This group is isomorphic to the dihedral group of order* $2n$.

(5) If $p = 3$, *the group generated by* $\begin{pmatrix} \zeta_5 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & (1 - \zeta_5 - \zeta_5^{-1}) \\ 1 & -1 \end{pmatrix}$. *This group is isomorphic to* A_5.

Remark 4.4.17. To clarify a potential point of confusion, we elucidate the difference between $\mathrm{PSL}_2(k)$ and $\mathrm{PGL}_2(k)$ for a field k. The determinant map $\det : \mathrm{GL}_2(k) \to k^\times$ descends to a map $\overline{\det} : \mathrm{PGL}_2(k) \to k^\times/(k^\times)^2$. $\mathrm{PSL}_2(k)$ is the kernel of $\overline{\det}$ and need not equal $\mathrm{PGL}_2(k)$ when k is not algebraically closed.

Again the classification of subgroups up to conjugacy greatly simplifies the proof of the following theorem:

Theorem 4.4.18. *Let* k *be an algebraically closed field of characteristic* $p > 0$ *and* G *a finite* p*-irregular group. There are infinitely many* G*-actions on* $\mathbb{P}^1_k$ *with constant stabilized locus if and only if* G *is isomorphic to* $(\mathbb{Z}/p\mathbb{Z})^m$.

Each $(\mathbb{Z}/p\mathbb{Z})^m$*-action has a unique stabilized point* P. *Choosing such an action with stabilized point* P *is equivalent to choosing a rank* m *additive subgroup of* k.

Proof. Let G be a p-irregular group. Since G is p-irregular, every group corresponding to a G-action must be conjugate to one of the five groups listed in Theorem 4.4.16. As in the proof of Theorem 4.4.15, if G has at least three stabilized points there will be only finitely many G-actions with constant stabilized locus.

One can check immediately that the A_5 group from (5) and the dihedral group from (4) have at least three stabilized points.

To see that $\mathrm{PSL}_2(\mathbb{F}_{p^n})$ and $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ have at least three stabilized points, note that irrespective of the field, both groups contain the matrices $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The first fixes 0, the second ∞, and the third the square roots of -1. This yields three stabilized points in characteristic 2 and four in higher characteristics.

Therefore we have only case (3) left to consider. First, consider the case where $n > 1$. Then any subgroup of $\mathrm{PGL}_2(k)$ isomorphic to G is conjugate to a group of the form $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix} \rtimes \begin{pmatrix} \mu_n(k) & 0 \\ 0 & 1 \end{pmatrix}$ where $\mu_n(k) \subseteq \Gamma$ and $\mu_n(k) \cdot \Gamma \subseteq \Gamma$. An element of the form $\begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix}$ has 0 and ∞ as fixed points. An element of the form $\begin{pmatrix} \zeta_n & g \\ 0 & 1 \end{pmatrix}$ for g non-zero has $\frac{g}{1-\zeta_n} \neq 0$ as a fixed point. Since there are at least three stabilized points, in this case there are finitely many G-actions with constant stabilized locus as well.

Finally, suppose $n = 1$. Then G is conjugate to a group of the form $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ such that $\mathbb{F}_p \subseteq \Gamma$ and Γ is a rank m additive subgroup of k. Note that in this case $G \cong \Gamma$, which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$. The unique stabilized point of such a group is ∞. Since $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$ where $\alpha, \beta \in k$ and $\alpha \neq 0$ are precisely the matrices which fix ∞, every subgroup of $\mathrm{PGL}_2(k)$ isomorphic to G and with stabilized point ∞ is a conjugate of $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ by such a matrix. These conjugates have the form $\begin{pmatrix} 1 & \alpha \cdot \Gamma \\ 0 & 1 \end{pmatrix}$.

Conversely, let Γ be any rank m additive subgroup of k. One immediately sees that the only stabilized point of the group $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ is ∞. Choose a non-zero element $\gamma \in \Gamma$. Then $\mathbb{F}_p \subseteq \gamma^{-1} \cdot \Gamma$ and $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1 & \gamma^{-1} \cdot \Gamma \\ 0 & 1 \end{pmatrix}$ via conjugation by $\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, the subgroups of $\mathrm{PGL}_2(k)$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ and with stabilized point ∞ are of the form $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ where Γ is any rank m additive subgroup of k. $\square$

By Theorem 4.3.8, G-actions with stabilized locus S correspond to equivalence classes of G-Galois branched covers with source X and ramification locus S. Therefore, the genus 0 case of Theorem 4.4.1 follows as a corollary of Theorem 4.4.15 and Theorem 4.4.18:

Corollary 4.4.19. *Let k be an algebraically closed field, S a finite set of closed points on $\mathbb{P}^1_k$, and G a finite group. There are infinitely many equivalence classes of G-Galois branched covers with source $\mathbb{P}^1_k$ and ramification locus S if and only if all the following hold:*

(1) $\mathrm{char}(k) = p > 0$

(2) $G \cong (\mathbb{Z}/p\mathbb{Z})^m$*, where* $m \in \mathbb{N}$

(3) $|S| = 1$.

When these conditions hold, equivalence classes of such covers with ramification locus S are in correspondence with rank m additive subgroups of k.

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