Automorphisms and Calabi-Yau Threefolds

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"Iucundi acti labores"

Cicero, De finibus

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ABSTRACT

Automorphisms and Calabi-Yau Threefolds

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We generate and classify two families of Calabi-Yau threefolds by taking quotients of threefolds by symplectic groups.

First we generalize the Vafa-Witten construction by allowing all abelian extensions of the multiplicative automorphisms of fixed elliptic curves. We classify and compute the fundamental groups and Hodge numbers of the corresponding Calabi-Yau threefolds.

Secondly, we study non-symplectic automorphisms of order 3 of $K3$ surfaces. We analyze the fixed locus and we focus on automorphisms fixing a curve of genus 1 or 2. Using this information we generalize the construction of Borcea and Voisin and compute the invariants of the threefolds which we obtain.

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Chapter 1

Introduction

1.1 A brief history

Up to the early eighties people knew few examples of Calabi-Yau threefolds. Mori's program was still young and higher dimensional varieties with trivial canonical class hadn't sparked yet a lot of interests. The stimulus came from outside mathematics. Over the last decades physicists were trying to develop a theory unifying all fundamental forces. Although they had been successful in uniting electromagnetism with the weak and the strong force, gravity remained out of reach. In the early seventies, building on ideas of Kaluza, some physicists made the assumption that particles could be considered as eigenmodes of vibration of tiny strings in a space of larger dimension. Some symmetries exhibited by Veneziano confirmed the model. String theory was born. It was nevertheless not a smooth start. Many issues remained (anomaly cancellation etc.) and only in the early eighties, with the second string revolution, were the big hindrances lifted. Among the different string theories (united through M-theory) type II theory and heterotic theory are based on ten dimensional spaces fibered on four dimensional Minkowski space-time. (There is sometimes some additional data such as vector bundles on the ten dimensional space.) The six dimensional gap (that is the fiber) is made of a variety which satisfies certain conditions. People came to realize that one of the possible varieties were three dimensional Calabi-Yau threefolds. [1] This revived the interest in Calabi-Yau threefolds in the mathematical community.

1.2 Vafa-Witten Construction

One of the first papers where the physics was related to the geometry of the fiber is the paper by Dixon, Harvey, Vafa and Witten ([9]). This paper was soon followed by another paper by Vafa and Witten ([19]) where they study a toy model obtained by taking the quotient of a three torus by a group of automorphisms where each element acts by inverting exactly two of the tori. Each element acts symplectically and therefore the quotient has a volume form. It is easy to see from there that it is a Calabi-Yau threefold. Although the paper had paved the way, its authors had not found a satisfactory model. Faraggi and his collaborators, inspired by

[1]For the story it is funny to know that since mathematicians knew few Calabi-Yau threefolds, physicists were optimistic that there weren't too many of them

the work of Vafa and Witten, created a "realistic fermionic model" by taking a further quotient by translations. Inspired by the results, Donagi and Faraggi ([11]) classified all models coming from direct extensions of the group of Vafa and Witten by translations. Donagi and Wendland have pushed the classification further by allowing extensions which are not direct product but surject onto the Vafa-Witten group ([13]).

In the first chapter of the thesis we extend the construction by using elliptic curves with complex multiplication. The automorphisms of order higher than 2 and fixing the origin must be of order 3,4 or 6 and act each, on a unique curve. If we take the three torus obtained by multiplying three copies of any of those curves we get a three torus with a $(\mathbb{Z}/n\mathbb{Z})^3$ action. We take the subgroup fixing the volume form, and extend it with translations on the torus which commute with the multiplicative action. We call the resulting group together with its action on the three torus an *abelian Vafa-Witten group* of order n. Analogously, the resulting orbifold is called abelian Vafa-Witten orbifold. Although there are a priori thousands of such groups, we show that in fact they fall in a small number of isomorphism classes:

Theorem 1.2.1. *There are* 11 *isomorphism classes of abelian Vafa-Witten groups for* n *equal to* 3 *or* 4*. For* n *equal to* 6*, there is a unique group.*

All these groups give rise to orbifolds with different Hodge diamond (with one exception).

As it is physically interesting to have a non-trivial fundamental group, we have also investigated the topology of the above orbifolds and we saw that:

Proposition 1.2.2. *Among all above orbifolds, only one has non-trivial fundamental group.*

1.3 Borcea-Voisin Construction

The second chapter of my thesis extends the construction of Borcea and Voisin ([2],[20]) by studying higher order automorphism of K3 surfaces. Their construction, although simple, gives many nice examples which come in mirror pairs. Hopefully an extension could eventually lead to other mirror pairs.

Borcea and Voisin construct Calabi- Yau threefolds using an elliptic curve and a K3 surface with an involution. They take the product of the surface and the curve and take the quotient by the product involution. K3 surfaces with involutions had been studied earlier by Nikulin and there is a clear understanding of the relation between their intrinsic geometry and their involutions ([10]).

In order to generate Calabi-Yau manifolds which would hopefully have non-trivial fundamental group, a plus for the heterotic construction, I allow the $K3$ surfaces to have non-symplectic automorphisms of order 3. Automorphisms of order 4 and 6 will be analyzed in a later paper. We work with non-symplectic automorphism which, although they do not preserve the $(2,0)$ form of the $K3$ surface, when combined with an appropriate action on an elliptic curve will preserve the volume form on the product variety. To identify the Calabi-Yau threefolds that they will yield, we have to understand the automorphisms of the $K3$ surfaces and, in particular their fixed loci. The fixed locus consists in a disjoint union of smooth curves and points. There is a natural scission of the problem depending on the highest genus of the fixed curves. In this chapter we study the cases where the genus is 1 or 2 using geometric arguments. When there is a curve of genus 1, we are lead to study automorphisms of elliptically fibered $K3$'s.

An elliptic automorphism of a $K3$ surface, i.e. an automorphism which preserves the fibration, preserves two fibers, one of which is the fixed elliptic curve. The other curve could be a smooth or a singular fiber consisting of a configuration of transversal rational curves. When the fiber is singular, a local study shows that the non-symplectic $\mathbb{Z}/3\mathbb{Z}$ action induces an essentially unique action on those $\mathbb{P}^1$'s. In the late eighties, Miranda and Persson classified configurations of singular fibers on elliptic rational surfaces ([16],[17]). We construct the automorphisms, in a bottom to top philosophy, by using triple base change of rational elliptic surfaces, ramified over a smooth fiber and a singular fiber. In this manner, we get all possible $K3$ surfaces with non-symplectic $\mathbb{Z}/3\mathbb{Z}$ fixing an elliptic curve. The result is a collection of surfaces which yield Calabi-Yau threefolds whose Hodge numbers are restricted to three types:

Proposition 1.3.1. *The Calabi-Yau threefolds obtained using a non-symplectic au-*

tomorphism of order three fixing an elliptic curve have their Hodge number of type $(46, 10)$,$(57, 9)$,$(68, 8)$*. All these orbifolds have trivial fundamental group.*

If a non-symplectic automorphism fixes a curve a genus 2, we inherit a map to $\mathbb{P}^2$ which is birational to a double cover ramified along a sextic. The automorphism of the $K3$ surface descends to an automorphism of the projective plane. This automorphism fixes a line (which is the image of the genus 2 curve) and a point. We can therefore conclude that the fixed locus of the automorphism on the $K3$ consists of components of the fibers above the fixed point and fixed line of $\mathbb{P}^2$. The geometry of the fixed locus depends essentially on the location of the fixed point with respect to the branch locus. By studying the possible branch curves we deduce all possible $K3$ surfaces which admit an order 3 non-symplectic automorphsim:

Proposition 1.3.2. *A non-symplectic automorphism of order 3 on a* $K3$ *containing a curve of genus 2 in its fixed locus also fixes two other isolated points or one fixed line or* 2 *fixed lines and* 4 *isolated points*

These $K3$ give us Calabi-Yau threefolds with Hodge numbers : $(19, 19)$ or $(41, 17)$.

◇

Chapter 2

Vafa-Witten Orbifolds

2.1 Preamble

In this chapter we take quotients of three tori by symplectic actions coming from multiplicative symmetries of order 3,4,6 on each torus. The aim is to create orbifolds with non trivial fundamental group as this could lead to interesting physics. We we look at abelian automorphism of the three-tori which surject onto the multiplicative symmetry group and classify them. We show that there are 11 families for the symmetries of order 3 and 4 (the case of order 6 doesn't admit any abelian extension) (Proposition 2.3.1). We compute the Hodge structure of the resulting three-folds and show that only one family out of the 23 has a non-trivial fundamental group.

In the first section we explain the construction. In the second one, we classify all possible actions up to isomorphism. In the last two parts of this chapter, we

Figure 2.1: Elliptic curves with automorphism of order 4 (resp. 3 and 6) and the 2 (resp. 3 and 1) fixed points.

compute the Hodge numbers and fundamental groups of the varieties.

2.2 Construction of the Vafa-Witten Orbifolds

Let E_n be an elliptic curve endowed with an $\mathbb{Z}/n\mathbb{Z}$ symmetry ($n \in \{3, 4, 6\}$) corresponding to multiplication by the n^{th} root of unity. We can identify E_n with the complex plane $\mathbb{C}$ modulo a lattice subgroup $\Lambda_n = \mathbb{Z} \oplus \omega_n \mathbb{Z}$ where $\omega_3 = \omega_6 = e^{\frac{i\pi}{3}}$ and $\omega_4 = \sqrt{-1}$. If we write $[x]$ for the class in $\mathbb{C}/\Lambda_i$ of $x \in \mathbb{C}$, the action is given explicitly by

$$[x] \mapsto [\zeta_n x]$$

where ζ_n is a primitive n^{th} root of unity. This multiplication is well defined as multiplying by ζ_n preserves the lattice Λ_n.

Since we have an action of $\mathbb{Z}/n\mathbb{Z}$ for $n \in \{3, 4, 6\}$ on E_n, there is an action of $(\mathbb{Z}/n)^3$ on $X_n = E_n \times E_n \times E_n$. The latter action restricts to an action of $(\mathbb{Z}/n)^2$, kernel of the addition map $(\mathbb{Z}/n\mathbb{Z})^3 \to \mathbb{Z}/n\mathbb{Z}$, on X_n :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})^2 & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})^3 & \xrightarrow{+} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & & \searrow & \big\downarrow & & & & \\ & & & & Aut(X_n) & & & & \end{array}$$

We call the group $(\mathbb{Z}/n\mathbb{Z})^2$, together with the above action on X_n, the n^{th} Vafa-Witten group.

The elliptic curve E_n possesses a natural additive group structure. We denote by T_n the subgroup of E_n consisting of elements which are fixed under multiplication by an n^{th} root of unity. The cardinality of T_n is 3 for $n = 3$; 2 for $n = 4$ and 1 for $n = 6$; the fixed points are represented in figure 2.1. Note that all T_n are cyclic.

We can synthesize the situation in the following table:

n	ω_n	ζ_n	T_n
3	$e^{\frac{i\pi}{3}}$	$e^{\frac{2i\pi}{3}}$	$\mathbb{Z}/3\mathbb{Z}$
4	i	i	$\mathbb{Z}/2\mathbb{Z}$
6	$e^{\frac{i\pi}{3}}$	$e^{\frac{i\pi}{3}}$	$\{0\}$

Consider $V_n = (\mathbb{Z}/n\mathbb{Z})^2 \times T_n^3$, the direct product of the Vafa-Witten group with its group of fixed points. Let $\mathcal{H}_n$ be the set of subgroups of V_n which restrict onto the Vafa-Witten group. In the subsequent sections we will classify and study the Hodge structure of orbifolds obtained by taking the quotient of X_n by elements of $\mathcal{H}_n$.

2.2.1 Notation

We will write out the elements of V_n as sextuples $(m_1, m_2, m_3; a_1, a_2, a_3)$ with the $m_k \in \mathbb{Z}/n\mathbb{Z}$, adding up to zero, and the $a_k \in \mathbb{Z}/k\mathbb{Z}$ where k is the order of T_n. The action of $g = (m_1, m_2, m_3; a_1, a_2, a_3)$ on $z = ([z_1], [z_2], [z_3]) \in X_n$ is

$$g.z = ([\zeta_i^{m_1} z_1 + a_1 t_n], [\zeta_i^{m_2} z_2 + a_2 t_n], [\zeta_i^{m_3} z_3 + a_3 t_n])$$

where t_n is a generator of T_n. We will call the m_i's the *twist* part, and the a_i's the *shift* part.

2.3 Isomorphism classes of $\mathcal{H}_n$

In this section we will classify the elements of $G \in \mathcal{H}_n$ up to isomorphism. The classification will be made according to the *rank*, that is the number of generators of G minus 2. We get the following classification :

Proposition 2.3.1. *The groups $\mathcal{H}_n$ contain only finitely isomorphism classes of groups. The number of classes is* 11 *for $\mathcal{H}_3$ and $\mathcal{H}_4$ and* 1 *for $\mathcal{H}_6$. (the results are sorted by rank in table 2.1)*

2.3.1 General Lemmas

To identify groups, we will make recurrent use of the following lemma:

Table 2.1: Number of isomorphism classes in $\mathcal{H}_n$

n	rank	# of classes
3,4	0	4
	1	5
	2	1
	3	1
6	0	1

Lemma 2.3.2. *Let G be an element of $\mathcal{H}_4$ (resp. $\mathcal{H}_3$) then G has at least two generators. The first two generators can be taken of the form $g_1 = (1,2,0;*,*,*)$ and $g_2 = (2,0,1;*,*,*)$ (resp. $g_1 = (1,3,0;*,*,*)$ and $g_2 = (3,0,1;*,*,*)$). If there are more than two generators, then they can be taken of the form $g_{i>2} = (0,0,0;*,*,*)$.*

Proof. Since G must surject onto the corresponding Vafa-Witten group, it has at least two generators. These two generators g_1, g_2 can be choosen to be the lifts of the generators of the Vafa-Witten group. For $n = 4$ (resp. $n = 3$) the Vafa-Witten group admits as generators $(1,2,0)$ and $(2,0,1)$ (resp. $(1,3,0)$ and $(3,0,1)$) so the nature of the first two generators is settled.

Let g_i be another generator of G. Since the twist part of g_1 and g_2 generates the Vafa-Witten group, there exists a word w in the first two generators so that

$w.g_i = (0, 0, 0; *, *, *)$. We can now substitute g_i by $w.g_i$. $\square$

The most powerful tool to identify groups will be conjugation by torsion elements of ${E_n}^3$. For simplicity we will work on a single torus. Consider the transformation $z \mapsto \zeta_i^a z + \tau t_i$. We will conjugate it with the translation by λ, any given point of E_n:

$$(\zeta_i^a(z + \lambda) + \tau t_i) - \lambda = \zeta_i^a z + \tau t_i + (\zeta_i^a \lambda - \lambda)$$

If $a \neq 0$, we can choose a λ such that $\tau t_i + (\zeta_i^a \lambda - \lambda) = 0$, which in turn implies that our transformation is conjugate to $\zeta_i^a z$.

2.3.2 Order 3

rank 0

There are a priori $(3^3)^2 = 9801$ possible choices of $g_1 = (1, 2, 0; a_1, a_2, a_3)$ and $g_2 = (2, 0, 1; b_1, b_2, b_3)$. Using conjugation and symmetry of the elliptic curve, it is easy to see that in fact we can restrict ourselves to at most 8 cases:

Lemma 2.3.3. *Without loss of generality we can assume g_1 and g_2 to be of the form $(1, 2, 0; 0, 0, a_3)$ and $(2, 0, 1; b_1, b_2, 0)$ with $a_3, b_1, b2 \in \{0, 1\}$.*

Proof. Let G be a group with generators $g_1 = (1, 2, 0; a_1, a_2, a_3)$ and
$g_2 = (2, 0, 1; b_1, b_2, b_3)$. We will find an isomorphism mapping G to G', where $G' \in \mathcal{H}_3$ and whose generators are as in the statement of the lemma.

Let α_1 (resp. α_2, β_3) be a 3 torsion element of E_n such that $\zeta^1 \alpha_1 - \alpha_1 + a_1 = 0$

(resp. $\zeta^2\alpha_2 - \alpha_2 + a_2 = 0$, $\zeta^1\beta_3 - \alpha_3 + b_3 = 0$). We will conjugate the group by the element $(0,0,0;\alpha_1,\alpha_2,\alpha_3)$. The element g_1 will map to the element $(1,2,0;0,0,a_3)$ and g_2 will map to $(2,0,1;b'_1,b_2,0)$.

Since t_3 and $2t_3$ are symmetric, we can assume that a_3, b_1, b_2 belong to $\{0,1\}$. □

Lemma 2.3.4. *The entries b_1 and b_2 are symmetric.*

Proof. Note that $g_1 = (1,2,0;0,0,\delta_3)$ with $\delta_3 \in \{0,1\}$ and $g_2 = (2,0,1;b_2,b_3,0)$. The group spanned by g_1, g_2 is the same as the group spanned by g_1^2, g_1g_2, that is $(2,1,0;0,0,\delta_3)$ and $(0,2,1;b_1,b_2,\delta_3)$. We now rearrange the order of the tori : $(1\ 2\ 3) \rightsquigarrow (2\ 1\ 3)$ so that the new generators read $(1,2,0;0,0,\delta_3)$ and $(2,0,1;b_2,b_1,\delta_3)$. By conjugating by an appropriate element of the third torus we get as second generator the required $(2,0,1;b_2,b_1,0)$. □

Corollary 2.3.5. *There are at most 6 isomorphism classes. They are :*

- $g_1 = (1,2,0;0,0,1)$ $g_2 = (2,0,1;0,0,0)$
- $g_1 = (1,2,0;0,0,1)$ $g_2 = (2,0,1;1,0,0)$
- $g_1 = (1,2,0;0,0,1)$ $g_2 = (2,0,1;1,1,0)$
- $g_1 = (1,2,0;0,0,0)$ $g_2 = (2,0,1;0,0,0)$
- $g_1 = (1,2,0;0,0,0)$ $g_2 = (2,0,1;1,0,0)$
- $g_1 = (1,2,0;0,0,0)$ $g_2 = (2,0,1;1,1,0)$

We will now show that among the 6 classes that we have so far, 2 are redundant. We will also simplify the notation further by writing $\underline{x}$ for (x_1, x_2, x_3). For example, $\underline{b} = (1, 1, 0)$ denotes the element $g_2 = (2, 0, 1; 1, 1, 0)$

Lemma 2.3.6. *The group of generator* $\underline{a} = (0, 0, 1)$ *and* $\underline{b} = (0, 0, 0)$ *is isomorphic to the group of generators* $\underline{a} = (0, 0, 0)$*,* $\underline{b} = (0, 1, 0)$*.*

Proof. We can replace the generators g_1 and g_2 by their squares : $(2, 1, 0; 0, 0, 2)$ and $(1, 0, 2; 0, 0, 0)$. If we rearrange the terms in the order $(1\ 2\ 3) \rightsquigarrow (1\ 3\ 2)$ and we permute the generators we get $(1, 2, 0; 0, 0, 0)$ and $(2, 0, 1; 0, 2, 0)$. Which is what we want up to relabelling. $\square$

Lemma 2.3.7. *The group* $\underline{a} = (0, 0, 1)$ *and* $\underline{b} = (1, 0, 0)$ *is isomorphic to the group to* $\underline{a} = (0, 0, 0)$*,* $\underline{b} = (1, 1, 0)$ *:*

Proof. The basis $g_2, g_1^2 g_2^2$ is equivalent to g_1, g_2. It is made out of the vectors $(2, 0, 1; 1, 0, 0)$ and $(0, 1, 2; 2, 0, 2)$. We now rearrange the tori using the permutation $(1\ 2\ 3) \rightsquigarrow (3\ 1\ 2)$ to get the basis $(1, 2, 0; 0, 1, 0)$,$(2, 0, 1; 2, 2, 0)$. We now conjugate with an appropriate translation on the second tori to get $(1, 2, 0; 0, 0, 0)$,$(2, 0, 1; 2, 2, 0)$.

$\square$

Proposition 2.3.8. *There are four isomorphism classes of groups of rank* 0 *in* $\mathcal{H}_3$*.*

We have written a representative of each class in the following table:

#	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(h_{11}, h_{12})	π_1
$III.1$	$(0, 0, 0)$	$(0, 0, 0)$	$(84, 0)$	1
$III.2$	$(0, 0, 0)$	$(0, 1, 0)$	$(24, 12)$	1
$III.3$	$(0, 0, 0)$	$(1, 1, 0)$	$(18, 6)$	1
$III.4$	$(0, 0, 1)$	$(1, 1, 0)$	$(12, 0)$	$\mathbb{Z}/3$

Proof. We have seen through the previous lemmas that there are at most 4 types of isomorphism. By computing the Hodge diamond of the associated Calabi-Yau threefold (see last section of the first chapter) we deduce that they are all four different. □

rank 1

We label the 3^{rd} generator $g_3 = (0, 0, 0; c_1, c_2, c_3)$. We will extend the list of rank 0 groups using the following rules :

Lemma 2.3.9. *(Donagi & Faraggi [11]) (Reduction principle) All groups of rank* 1 *are isomorphic to a a group at most least* 2 c_i*'s are not equal to* 0*.*

The idea is that if there is an element which consists in translation on a unique curve, we can first take the quotient by this element and have another three torus on which the rest of the group acts.

Lemma 2.3.10. *If* $c_k \neq 0$ *and* $a_k = b_k = 0$ *we can assume* $c_k = 1$*.*

Proof. It follows from the symmetry between t_3 and $2t_3$. □

Lemma 2.3.11. *We can assume $\underline{c}$ is not a non-zero multiple of $\underline{a}$ or of $\underline{b}$.*

Proof. Assume that $\underline{c}$ is a multiple of $\underline{a}$, then, we can substitute g_1 by $g_1 g_3^k$ (where $\underline{ca}^k = (0,0,0)$) to get a new first generator without translation. In other words, we have reduced the group to a previous case. □

We will now try to discern the groups :

1. $\underline{a} = (0,0,0); \underline{b} = (0,0,0)$

   We can choose $\underline{c}$ to be either $(1,1,0)$ or $(1,1,1)$. All other cases resume to these two using the previous points and S_3 symmetry.

2. $\underline{a} = (0,0,0); \underline{b} = (0,1,0)$

   We can choose, using the previous points, a $\underline{c}$ of the form $(\delta_1, c_2, \delta_3)$ where the $\delta_i \in \{0,1\}$.

   However, we can also assume that $c_2 \in \{0,1\}$:

   Proof. The generator g_3 is equivalent to g_3^2, so since we can relabel the first and third translations without loss of generality, we see that we can assume c_2 to be in $\{0,1\}$. □

   It now seems that we have 4 possible choices for $\underline{c}$, namely $(1,1,0)$, $(0,1,1)$, $(1,0,1)$ and $(1,1,1)$. We will show that the second is actually redundant :

Proof. The group generated by g_1, g_2, g_3 is the same as the group spanned by $g_1, g_2g_3^2, g_3$. The element $g_2g_3^2 = (2,0,1;0,0,2)$. If we conjugate these new elements by an adequate translation on the third torus, the first and last generator are unchanged while $g_2g_3^2 \rightsquigarrow (2,0,1;0,0,0)$. $\square$

The computation of the Hodge numbers will assure us that the 3 remaining cases are independent.

3. $\underline{a} = (0,0,0); \underline{b} = (1,1,0)$

We can choose $\underline{c}$ to be of the form (c_1, c_2, δ_3) where $\delta_3 \in \{0,1\}$ (same argument as previously). Now, we could as well replace g_3 by its square and, since we can change the last component of $\underline{c}$ freely it means we can choose to replace (c_1, c_2, δ_3) to (c_1^2, c_2^2, δ_3). We will do this so have a minimal number of entries equal to 2 in (c_1, c_2).

Using the above rules, the possible $\underline{c}$'s are $(1,2,0)$,$(0,1,1)$,$(1,0,1)$,$(1,1,1)$ or $(1,2,1)$.

Actually, none of these cases is new :

- $(1,2,0)$: We have $(g_1, g_2, g_3) = (g_1, g_2g_3^2, g_3)$. The element $g_2g_3^2 = (2,0,1;0,2,0)$. After conjugation, and up to transforming the third generator g_3, we can let $g_2g_3^2 \rightsquigarrow (2,0,1;0,0,0)$.
- $(0,1,1)$: We have $(g_1, g_2, g_3) = (g_1^2, g_1g_2g_3^2, g_3)$, where $g_1^2 = (2,1,0;0,0,0)$ and $g_1g_2g_3^2 = (0,2,1;1,0,2)$. Conjugating by some appropriate transla-

tion on the third torus, the second generator becomes $g_1g_2g_3^2 \rightsquigarrow (0,2,1;1,0,0)$. We now reorder the tori $(1,2,3) \rightsquigarrow (2,1,3)$ and we get a previous case.

- $(1,0,1)$: We have $(g_1,g_2,g_3) = (g_1,g_2g_3^2,g_3)$. The element $g_2g_3^2 = (2,0,1;0,1,2)$. After conjugating with an element of translation on the third torus, the second generator can be taken to be $(2,0,1;0,1,0)$.

- $(1,1,1)$: We have $(g_1,g_2,g_3) = (g_1,g_2g_3^2,g_3)$. The element $g_2g_3^2 = (2,0,1;0,0,1)$. Again, we can conjugate by an element of translation on the third torus, so to have our second generator $(2,0,1;0,0,0)$.

- $(1,2,1)$: We have $(g_1,g_2,g_3) = (g_1,g_2g_3^2,g_3)$. The element $g_2g_3^2 = (2,0,1;0,2,2)$. We can conjugate by an element of translation on the third torus, so to have our second generator $\rightsquigarrow (2,0,1;0,2,0)$. Up to renaming, we again reduced to a previous case.

So we conclude that there are no interesting extensions in this case.

4. $\underline{a} = (0,0,1); \underline{b} = (1,1,0)$

We can choose $\underline{c}$ to be of the form (c_1,c_2,c_3). Let us first undercover some symmetry : we have $(g_1,g_2,g_3) = (g_1^2,g_1g_2,g_3)$ and if we permute the order of the tori $(1,2,3) \rightsquigarrow (2,1,3)$ we get the generators $(1,2,0;0,0,2)$,$(2,0,1;1,1,1)$ and $(0,0,0;c_2,c_1,c_3)$. Now we can conjugate by a translation on the third torus to let the second generator become $(2,0,1;1,1,0)$ and the other two remain unchanged. Now we can relabel the new a_3 into a 1 and we must therefore substitute the new c_3 by its square. Finally we get the generators $(1,2,0;0,0,1)$, $(2,0,1;1,1,0)$ and $(0,0,0;c_2,c_1,c_3^2)$.

So we can let c_3 be 0 or 1 up to a permutation of the c_1, c_2.

Using the above symmetry, we have the following possibilities which we show to be reducible to previous cases :

- $(1,2,0):$ We have $(g_1,g_2,g_3) = (g_1,g_2g_3,g_3)$ where $g_2g_3 = (2,0,1;2,0,0)$. Using the symmetry of the first translation entries of the second generator (up to variation of g_3) we have reduced to a previous case.

- $(1,0,1):$ We have $(g_1,g_2,g_3) = (g_1g_3^2,g_2,g_3)$ where $g_1g_3^2 = (1,2,0;2,0,0)$. Up to changing g_2 we can conjugate by a translation element on the first torus to get the first generator to become $(1,2,0;0,0,0)$. Se we reduced to a previous case.

- $(0,1,1):$ We have $(g_1,g_2,g_3) = (g_1g_3^2,g_2,g_3)$ where $g_1g_3^2 = (1,2,0;0,2,0)$. We can conjugate by a translation element on the second torus to get the first generator to become $(1,2,0;0,0,0)$. Se we reduced to a previous case.

- $(1,1,1):$ We have $(g_1,g_2,g_3) = (g_1,g_2g_3^2,g_3)$ where $g_2g_3^2 = (2,0,1;0,0,2)$. We can conjugate by a translation element on the third torus to get the second generator to become $(2,0,1;0,0,0)$.

- $(2,1,1):$ We have $(g_1,g_2,g_3)=(g_1g_3^2,g_2,g_3)$ where $g_1g_3^2=(1,2,0;1,2,0)$. Up to changing g_2 we can conjugate by a translation element on the first and second torus to get the first generator to become $(1,2,0;0,0,0)$. Se we reduced to a previous case.
- $(1,2,1):$ We have $(g_1,g_2,g_3)=(g_1g_3^2,g_2,g_3)$ where $g_1g_3^2=(1,2,0;1,2,0)$. Up to changing g_2 we can conjugate by a translation element on the first and second torus to get the first generator to become $(1,2,0;0,0,0)$. Se we reduced to a previous case.

From the above discussion we conclude:

Proposition 2.3.12. *There are five isomorphism classes of groups of rank* 1 *in* $\mathcal{H}_3$. *We have written a representative of each class in the following table:*

#	(a_1,a_2,a_3)	(b_1,b_2,b_3)	(c_1,c_2,c_3)	(h_{11},h_{12})	π_1
$III.5$	$(0,0,0)$	$(0,0,0)$	$(0,1,1)$	$(40,4)$	1
$III.6$			$(1,1,1)$	$(36,0)$	1
$III.7$	$(0,0,0)$	$(0,1,0)$	$(1,1,0)$	$(40,4)$	1
$III.8$			$(1,0,1)$	$(16,4)$	1
$III.9$			$(1,1,1)$	$(18,6)$	1

Proof. It is sufficient to show that types III.5 and III.7 are different but this follows from the fact that the groups have a different number of elements which do not fix any element. □

rank 2

Since we do not consider rank two group which we can reduce to a lower rank, we have restrictrions on the third and fourth generator : by the reduction principle they can't generate an element with only one non-zero entry in the shift part. We can consider the third and fourth generators as elements of $\mathbb{F}_3^3$ as they just have shift parts.

Lemma 2.3.13. *There are exactly 4 linear planes in $\mathbb{F}_3^3$ which do not intersect the coordinate axes outside the origin.*

Proof. Let H be a plane which does not intersect the coordinate axes outside the origin. Since it contains the origin, it intersects the three coordinate planes in a line. For each coordinate plane P_i, the only two possible lines are the diagonal Δ_i and, the only other line which is not a coordinate axis, l_i. The choice of any two lines. not in the same coordinate plane, out of $\{l_1, l_2, \Delta_1, \Delta_2\}$ gives an adequate plane. In particular, there are 4 of them. If we include Δ_3 and l_3 in the picture we look at the coplanarity of the lines, which is readily checked, and can be visualized in figure 2.2. Each plane is represented by either an edge of the triangle or the inscribed circle. □

Remarks :

- The three planes represented by the edges of the triangle in figure 2.2 are clearly S_3 symmetric. Therefore, it is enough to consider the adjunction of

Figure 2.2: Co-Incidence in $\mathbb{F}_3^3$

any one of those three to the first case of rank 0 subgroups.

- Since all rank 1 cases associated to $\underline{a} = (0,0,0), \underline{b} = (1,1,0)$ and $\underline{a} = (0,0,1), \underline{b} = (1,1,0)$ were reduced to previous cases, all rank 2 extensions will also be reducible, so we just need to deal with $\underline{a} = (0,0,0), \underline{b} = (0,0,0)$ and $\underline{a} = (0,0,0), \underline{b} = (0,1,0)$.

We will now work out the rank 2 cases :

- Using S_3 symmetry, for $\underline{a} = \underline{b} = (0,0,0)$, we have as possible extension $\underline{c} = (0,1,1), \underline{d} = (1,1,0)$ and $\underline{c} = (0,1,2), \underline{d} = (1,2,0)$. Now, we can replace $\underline{d}$ by its square and since c_3 and d_1 are freely choosen we see that this case is equivalent to the previous one.

- All 4 possible extensions of $\underline{a} = (0,0,0), \underline{b} = (0,1,0)$ contain either the element $(0,2,2)$ or $(0,2,1)$. Using this element it is easy to see that we can reduce g_2 to $(2,0,1;0,0,0)$: multiply g_2 by this element and conjugate by an appropriate translation on the third coordinate. We show hereby that there are no further rank 2 cases.

The are thus only rank 2 cases:

Proposition 2.3.14. *There is one isomorphism class of groups of rank* 2 *in* $\mathcal{H}_3$. *We have written a representative of each class in the following table:*

#	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(c_1, c_2, c_3)	(d_1, d_2, d_3)	(h_{11}, h_{12})	π_1
$III.10$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 1, 1)$	$(1, 1, 0)$	$(84, 0)$	1

rank 3

The only case is the whole group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times T_3$. It reduces to the $(84, 0)$ case.

2.3.3 Order 4

Since there are only two fixed points per torus, the structure of the translation locus is simpler. A translation element will be of the form $(\delta_1, \delta_2, \delta_3)$ with $\delta_i \in \{0, 1\}$.

rank 0

Using the same argument as for $\mathbb{Z}/3\mathbb{Z}$ (being careful to replace square by inverse) we quickly get the following list :

Proposition 2.3.15. *There are four isomorphism classes of groups of rank* 0 *in*

$\mathcal{H}_3$. *We have written a representative of each class in the following table:*

#	(a,b,c)	(a',b',c')	(h_{11},h_{12})
$IV.1$	$(0,0,0)$	$(0,0,0)$	$(90,0)$
$IV.2$	$(0,0,0)$	$(0,1,0)$	$(54,0)$
$IV.3$	$(0,0,0)$	$(1,1,0)$	$(42,0)$
$IV.4$	$(0,0,1)$	$(1,1,0)$	$(30,0)$

rank 1

Again, we will use the same arguments as for $n=3$.

Lemma 2.3.16. *If* $\underline{a}=(0,0,0)$, $\underline{b}=(0,0,0)$ *we can assume* $\underline{c}$ *to be of the form*

- $(0,1,1)$.
- $(1,1,1)$.

Proof. Using S_3 symmetry, all other $\underline{c}$'s with two non-zero entries or are equivalent to the one liste here. Also, the reduction principle excludes all $\underline{c}$'s with a single non-zero entry. □

Lemma 2.3.17. *If* $\underline{a}=(0,0,0)$, $\underline{b}=(0,1,0)$ *then we can assume that* $\underline{c}$ *is one of the following:*

- $(1,1,0)$.
- $(1,0,1)$.

- $(1,1,1)$.

Proof. The case where $\underline{c}$ is $(0,1,1)$ is reducible : replace g_2 by $g_2g_3 = (3,0,1;0,0,1)$ and conjugate by a translation element on the third torus to get a second generator which is translation free. □

Lemma 2.3.18. *If* $\underline{a} = (0,0,0)$, $\underline{b} = (1,1,0)$ *then there are no new cases.*

Proof. If $\underline{c}$ is

- $(0,1,1)$. This case is reducible: We replace g_1 by g_1^{-1} and $g-2$ by $g_2g_3 = (0,3,1;1,0,1)$. Secondly we conjugate by a translation element on the third torus to let the second generator become $(0,3,1;1,0,0)$. Finally, we arrange the torus in the order $(1,2,3) \rightsquigarrow (2,1,3)$. We reduced to an extension of the previous type of rank 0.

- $(1,0,1)$. This case is reducible: replace g_2 by $g_2g_3 = (3,0,1;0,1,1)$ and conjugate by a translation element on the third torus to get a second generator of the form $(3,0,1;0,1,0)$. We reduced to an extension of the previous type of rank 0.

- $(1,1,1)$. This case is reducible: replace g_2 by $g_2g_3 = (3,0,1;0,0,1)$ and conjugate by a translation element on the third torus to get a second generator which is translation free.

□

Lemma 2.3.19. *If* $\underline{a} = (0,0,1)$*,* $\underline{b} = (1,1,0)$ *then there are no new cases.*

Proof. If $\underline{c}$ is

- $(0,1,1)$. We have $(g_1, g_2, g_3) = (g_1g_3, g_2, g_3)$ where $g_1g_3 = (1,3,0;0,1,0)$. We can conjugate by a translation element on the second torus to get the first generator to become $(1,3,0;0,0,0)$. Se we reduced to a previous case.

- $(1,0,1)$. We have $(g_1, g_2, g_3) = (g_1g_3, g_2, g_3)$ where $g_1g_3 = (1,3,0;1,0,0)$. Up to changing g_2 we can conjugate by a translation element on the first torus to get the first generator to become $(1,3,0;0,0,0)$. Se we reduced to a previous case.

- $(1,1,1)$. We have $(g_1, g_2, g_3) = (g_1g_3, g_2, g_3)$ where $g_1g_3 = (1,3,0;1,1,0)$. Up to changing g_2 we can conjugate by a translation element on the first and second torus to get the first generator to become $(1,3,0;0,0,0)$. Se we reduced to a previous case.

□

Using those lemmas we have

Proposition 2.3.20. *There are four isomorphism classes of groups of rank* 1 *in*

$\mathcal{H}_3$. *We have written a representative of each class in the following table:*

#	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(c_1, c_2, c_3)	(h_{11}, h_{12})
$IV.5$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 1, 1)$	$(61, 1)$
$IV.6$			$(1, 1, 1)$	$(54, 0)$
$IV.7$	$(0, 0, 0)$	$(0, 1, 0)$	$(1, 1, 0)$	$(61, 1)$
$IV.8$			$(1, 0, 1)$	$(38, 0)$
$IV.9$			$(1, 1, 1)$	$(42, 0)$

Proof. It is sufficient to show that types IV.5 and IV.7 are different but this follows from the fact that the groups have a different number of elements which do not fix any element. □

rank 2

Since we do not want cases which reduce to lower ranks, we need the 3rd and 4th generator to span a subgroup which does not contain elements where only one of the entries is non-zero. In geometric language. There is actually a unique possibility :

Lemma 2.3.21. *There is a unique* 2 *dimensional vector subspaces of* $\mathbb{F}_2^3$ *which does not intersect the coordinate axes outside the origin.*

Proof. Let H be a plane verifying the above conditions. Since H passes through the origin, it must intersect each coordinate plane in at least a line. Since H does not

Figure 2.3: $\mathbb{F}_2^3$

intersect the coordinate axes, the intersection lines must the first diagonal. Therefore $\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\} \subset H$ and since a plane over $\mathbb{F}_2$ contains 4 elements, we are done. $\square$

Since the group of translations is isomorphic to $\mathbb{F}_3^3$ we get the direct result that

Corollary 2.3.22. *There is at most one rank 2 group associated to each group of rank 0 – which we classified earlier.*

Corollary 2.3.23. *There is no non-reducible rank 2 case associated to the last three types of rank 0.*

Proof. In the last two types of rank 0, one generator has a shift with 2 non-zero entries. Since this shift belongs to the plane H, the case can be reduced to a previous one.

In the second case of rank 0, $\underline{b} = (0,1,0)$, so replacing g_2 with its composition with $(0,0,0;0,1,1)$ we get $(3,0,1;0,0,1)$. We can now conjugate with an adequate translation on the third torus to get the second generator in the form $(3,0,1;0,0,0)$, i.e. we are back to the first type of rank 0. $\square$

So we conclude that there is only one type of rank 2 subgroups, namely : Therefore,

Proposition 2.3.24. *There is a unique type of rank* 2 *subgroups, namely :*

#	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(c_1, c_2, c_3)	(d_1, d_2, d_3)	(h_{11}, h_{12})
$IV.10$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 1, 1)$	$(1, 1, 0)$	$(90, 0)$

rank 3

The only case is the whole group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times T_4$. It is reducible to the $(90, 0)$ case.

2.3.4 Order 6

There is a unique case as there are no translations commuting with the multiplicative $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ action. Its Hodge numbers are $(80, 0)$.

2.4 The Hodge Structure

In this section we will compute the Hodge numbers of the orbifolds which we have classifed above. The Hodge numbers were written in the tables spread along last section. Let $G \in \mathcal{H}_n$, the Hodge structure of X_n/G is given by

$$H^{i,j}(X_n/G) = \bigoplus_{[g]\in G} H^{i-\kappa(g),j-\kappa(g)}(X_n^g)^{C(g)}$$

where we sum up over the conjugacy classes of G and $C(g)$ denotes the centralizer of g in G. The set X_n^g is the fixed locus of any element in the conjugacy class of g, i.e. $\{x \in X_n | g.x = x\}$. Since the elements of $\mathcal{H}_n$ are abelian, the formula simplifies to

$$H^{i,j}(X_n/G) = \bigoplus_{g \in G} H^{i-\kappa(g),j-\kappa(g)}(X_n^g)^G$$

If the action of g sends $[z_i]$ to $[e^{2\pi i\theta_i} z_i]$ with $0 \leq \theta_i < 1$ then the we define the shift function by $\kappa(g) = \sum_{i=1}^{3} \theta_i$. Note that by definition of the Vafa-Witten groups, the function κ will take its values in $\{0, 1, 2\}$.

The action of an element of G on the cohomology is depends uniquely on its projection in the Vafa-Wiffen group : the action of ζ_i is $\zeta_i.dz_i = \zeta_i dz_i$ and $\zeta_i.d\bar{z}_i = \bar{\zeta}_i d\bar{z}_i$ while the action of T_i is trivial.

Example 2.4.1. Take $g = (a_1, a_2, a_3; \tau_1, \tau_2, \tau_3)$, we have

$$dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3 \overset{g}{\longmapsto} \zeta_i^{(a_1+a_2)-(a_1+a_3)} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3 = \zeta_i^{(a_2-a_3)} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3$$

In order to compute the Hodge structure of the X_n/G we first need to classify the possible fixed loci of an element of $g \in G$ on X_n. There are four possibilities depending on the element $g = (m_1, m_2, m_3; a_1, a_2, a_3)$.

1. The fixed locus of the identity element, $(0, 0, 0; 0, 0, 0)$, is X_n.

2. If for a certain index $k \in \{1, 2, 3\}$ we have that $m_k = 0$ and $a_k \neq 0$ then the fixed locus of g is the empty set. Indeed, translation acts fixed point freely

on an elliptic curve and therefore the action of g on the whole of X_n is also fixed point free.

3. The fixed locus of g where all a_k's are different from 0 is a collection of points: it is well known that the action can be linearized on each elliptic curve.

4. The fixed locus of g where for one exactly one of the k's $m_k = a_k = 0$ (and excluding case 2) is made out of elliptic curves.

Note that if two of the a_k's are zero then the third one also has to be zero. Therefore all cases have been exhausted in the above list.

The G invariant cohomology from these fixed loci can also be computed :

Lemma 2.4.2. *For all $n \in \{3, 4, 6\}$, the G-invariant part of the cohomology of X_n is*[1]

$$H^*(X_n)^G = \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

Proof. The Hodge diamond of the 3-torus is

[1]We have tilted the Hodge diamonds 45^o to the left to facilitate typesetting.

$$\begin{matrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{matrix}$$

The invariance of $H^{0,0}$ and $H^{3,0}$ is straightforward.

For the other components, note that the action of G on $H^{1,0}$ and $H^{0,1}$, and thus on the whole of $H(X_n)$, is diagonal with respect to the standard basis. Therefore, it will be enough to check the behaviour of the basis elements to find the G-invariant part. Let dz_k be a generator of $H^{1,0}$, it is not fixed by the element $(1, 1, i-2; *, *, *)$. Therefore we have ${h^{1,0}}^G = 0$.

Similarly, $dz_k \wedge dz_l$, a generator of $H^{2,0}$, is not fixed by the element $(1, 1, i-2; *, *, *)$ and thus ${h^{2,0}}^G = 0$.

Only the generators of $H^{1,1}$ of the form $dz_i \wedge d\bar{z}_i$ are invariant, the others are killed by the element $(1, i-1, 0; *, *, *)$. Therefore the dimension of the G-invariant part of have $H^{1,1}$ is 3.

Consider now $H^{2,1}$; by symmetry we can restrict ourselves to the generators $dz_1 \wedge dz_2 \wedge d\bar{z}_3$ and $dz_1 \wedge dz_2 \wedge \ d\bar{z}_1$. The former is not fixed by $(1, 0, i-1; *, *, *)$, while the latter is not fixed by $(1, 1, i-2; *, *, *)$. We conclude that ${h^{2,1}}^G = 0$. Finally, the diamond is completed using $dz_k \leftrightarrow d\bar{z}_k$ and Hodge symmetry.

□

Lemma 2.4.3. *Assume that after identification via G-action, $g \in G$ has as fixed locus a collection of n points. The contribution to the cohomology of g and g^{-1}, $H^*(X^g)^G \oplus H^*(X^{g^{-1}})^G$ is equivalent to the contribution of n projective lines: $H^*(n\mathbb{P}^1)$.*

Proof. The elements g and g^{-1} have the same fixed locus, whose cohomology is exclusively $H^{0,0}$. Given that the fixed locus of g is made of points, we now that $\kappa(g)$ is non-zero and that none of the θ_i is 0. We claim that $\{\kappa(g), \kappa(g^{-1})\}$ is exactly $\{1, 2\}$. Indeed, if we denote by θ'_i the linearized action of g^{-1} on the i^{th} component then we have the relation $\theta'_i = 1 - \theta_i$. Therefore $\kappa(g^{-1}) = 3 - \kappa(g)$.

□

Lemma 2.4.4. *The G-invariant part of the cohomology of a fixed elliptic curve will be either the cohomology of the projective line $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ or the one of an elliptic curve $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$. In either case κ will be 1.*

Proof. Since $H^{0,0}$ is obviously invariant, only $H^{1,0}$ which is of dimension 1 might not preserved, in which case the invariant cohomology corresponds to that of a $\mathbb{P}^1$. Since we do not deal with the fixed locus of the identity, κ is not 0. Moreover, we know that the action is trivial on one component. Therefore we have that κ, which is the sum $\theta_1 + \theta_2 + \theta_3$ where one $\theta_i = 0$ and the two others of norm less than one, can only be one.

□

We are now ready to compute the Hodge numbers of each orbifold.

Each group will be represented, as previously, by the shift part of its generators. We will list the group elements g which have a nonempty fixed locus X^g and their contribution, i.e. for each group we will have a collection $\{(g, (h^{1,1}, h^{1,2}))\}$. To lighten the notation we will

1. Count twice the contribution of non-trivial elements which are not involutions but have a union of curves as fixed locus. In compensation we will not be writing their inverse.

2. If a non involutive element fixes a union of points, then we will count its contribution together with the one of its inverse. We will not write down its inverse.

2.4.1 Order 3

We have nine cases to compute:

- $(0,0,0)(0,0,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(18,0)),\ ((2,0,1;0,0,0),(18,0)),$

  $((0,1,2;0,0,0),(18,0)),\ ((1,1,1;0,0,0),(27,0))\ \}$

  Total $= (84,0)$

- $(0,0,0)(0,1,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,6)),\ ((0,2,1;0,1,0),(6,6)),$

$((1,1,1;0,1,0),(9,0))\}$

Total $=(24,12)$

- $(0,0,0)(1,1,0)$

  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,6)),\ ((1,1,1;1,1,0),(9,0))\}$

  Total $=(18,6)$

- $(0,0,1)(1,1,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,1,1;1,1,2),(9,0))\}$

  Total $=(12,0)$

- $(0,0,0)(0,0,0)(0,1,1)$

  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,0)),\ ((2,0,1;0,0,0),(6,0)),$

  $((0,1,2;0,0,0),(6,0)),\ ((0,1,2;0,1,1),(2,2)),\ ((0,1,2;0,2,2),(2,2)),$

  $((1,1,1;0,0,0),(9,0)),\ ((1,1,1;0,1,1),(3,0)),\ ((1,1,1;0,2,2),(3,0))\}$

  Total $=(40,4)$

- $(0,0,0)(0,0,0)(1,1,1)$

  $\{(\ (0,0,0;0,0,0),(3,0)),(\ (1,2,0;0,0,0),(6,0)),(\ (2,0,1;0,0,0),(6,0)),$

  $(\ (0,1,2;0,0,0),(6,0)),(\ (1,1,1;0,0,0),(9,0)),(\ (1,1,1;1,1,1),(3,0)),$

  $(\ (1,1,1;2,2,2),(3,0))\}$

  Total $=(36,0)$

- $(0,0,0)(0,1,0)(1,1,0)$

$\{(\ (0,0,0;0,0,0),(3,0)),(\ (1,2,0;0,0,0),(2,2)),(\ (1,2,0;1,1,0),(6,0)),$

$(\ (1,2,0;2,2,0),(6,0)),(\ (2,0,1;2,0,0),(2,2)),(\ (0,2,1;0,1,0),(6,0)),$

$(\ (1,1,1;0,1,0),(3,0)),(\ (1,1,1;1,2,0),(9,0)),(\ (1,1,1;2,0,0),(3,0))\}$

Total $=(40,4)$

- $(0,0,0,)(0,1,0)(1,0,1)$

  $\{\ ((0,0,0;0,0,0),(3,0)),(\ (1,2,0;0,0,0),(2,2)),(\ (0,2,1;0,1,0),(2,2)),$

  $(\ (1,1,1;0,1,0),(3,0)),(\ (1,1,1;1,1,1),(3,0)),(\ (1,1,1;2,1,2),(3,0))\ \}$

  Total $=(16,4)$

- $(0,0,0)(0,1,0)(1,1,1)$

  $\{\ ((0,0,0;0,0,0),(3,0)),(\ (1,2,0;0,0,0),(2,2)),(\ (2,0,1;2,0,2),(2,2)),$

  $(\ (0,2,1;0,1,0),(2,2)),(\ (1,1,1;0,1,0),(3,0)),(\ (1,1,1;1,2,1),(3,0)),$

  $(\ (1,1,1;2,0,2),(3,0))\ \}$

  Total $=(18,6)$

To compute the Hodge numbers of the rank 2 case we have used the fact that we can get the rank 3 case by adjoining $(0,0,0;0,0,1)$. The orbifold associated the rank 2 groups will have the same Hodge numbers as the rank 3 case, namely $(84,0)$.

2.4.2 Order 4

For this family, we also have nine cohomoly rings to compute:

- $(0,0,0)(0,0,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),(\ (1,3,0;0,0,0),(8,0)),(\ (3,0,1;0,0,0),(8,0)),$

  $(\ (0,3,1;0,0,0),(8,0)),(\ (1,1,2;0,0,0),(12,0)),(\ (1,2,1;0,0,0),(12,0)),$

  $(\ (2,1,1;0,0,0),(12,0)),(\ (2,2,0;0,0,0),(9,0)),(\ (2,0,2;0,0,0),(9,0)),$

  $(\ (0,2,2;0,0,0),(9,0))\ \}$

  Total $=(90,0)$

- $(0,0,0)(1,1,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(4,0)),\ ((1,1,2;0,0,0),(8,0))$

  $,\ ((1,2,1;1,1,0),(4,0)),\ ((2,1,1;1,1,0),(4,0)),\ ((2,2,0;0,0,0),(5,0))$

  $,\ ((2,0,2;0,0,0),(7,0)),\ ((0,2,2;0,0,0),(7,0)),\ \}$

  Total $=(42,0)$

- $(0,0,0)(0,1,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(4,0)),\ ((0,1,3;0,1,0),(4,0))$

  $,\ ((1,1,2;0,0,0),(8,0)),\ ((1,2,1;0,1,0),(4,0)),\ ((2,1,1;0,1,,[3],0),(8,0))$

  $,\ ((2,2,0;0,0,0),(7,0)),\ ((2,0,2;0,0,0),(9,0)),\ ((0,2,2;0,0,0),(7,0))$

  $,\ \}$

  Total $=(54,0)$

- $(0,0,1)(1,1,0)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,1,2;0,0,0),(4,0)),\ ((1,2,1;0,1,0),(4,0))$

  $,\ ((2,1,1;0,1,0),(4,0)),\ ((2,2,0;0,0,0),(5,0)),\ ((2,0,2;0,0,0),(5,0))$

$, ((0,2,2;0,0,0),(5,0)), \}$

Total $= (30,0)$

- $(0,0,0)(0,0,0)(0,1,1)$

  $\{$,[3], $((0,0,0;0,0,0),(3,0))$ $((1,3,0;0,0,0),(4,0)),$ $((3,0,1;0,0,0),(4,0))$

  $, ((0,3,1;0,0,0),(4,0)),$ $((1,1,2;0,0,0),(6,0)),$ $((1,2,1;0,0,0),(6,0))$

  $, ((2,1,1;0,0,0),(6,0)),$ $((2,2,0;0,0,0),(6,0)),$ $((2,0,2;0,0,0),(6,0))$

  $, ((0,2,2;0,0,0),(5,0)),$ $((0,3,1;0,1,1),(2,0)),$ $((1,1,2;0,1,1),(2,0))$

  $, ((1,2,1;0,1,1),(2,0)),$ $((2,1,1;0,1,1),(4,0)),$ $((0,2,2;0,1,1),(1,1))$ $\}$

  Total $= (61,1)$

- $(0,0,0)(0,0,0)(1,1,1)$

  $\{$ $((0,0,0;0,0,0)$,$(3,0)),$ $((1,3,0;0,0,0),(4,0)),$ $((3,0,1;0,0,0),(4,0)),$

  $((0,3,1;0,0,0),(4,0)),$ $((1,1,2;0,0,0),(6,0)),$ $((1,2,1;0,0,0),(6,0)),$

  $((2,1,1;0,0,0),(6,0)),$ $((2,2,0;0,0,0),(5,0)),$ $((2,0,2;0,0,0),(5,0)),$

  $((0,2,2;0,0,0),(5,0)),$ $((1,1,2;1,1,1),(2,0)),$ $((1,2,1;1,1,1),(2,0)),$

  $((2,1,1;1,1,1),(2,0))$ $\}$

  Total $= (54,0)$

- $(0,0,0)(0,1,0)(1,1,0)$

  $\{$ $((0,0,0;0,0,0),(3,0)),$ $(1,3,0;0,0,0),(2,0)),$($(3,0,1;1,0,1),(2,0)),$

  ($(0,1,3;0,1,0),(2,0)),$($(1,1,2;0,0,0),(4,0)),$($(1,1,2;1,1,1),(2,0)),$

  ($(1,2,1;0,1,0),(2,0)),$($(1,2,1;1,0,1),(4,0)),$($(2,1,1;0,1,0),(4,0)),$

$(\ (2,1,1;1,0,1),(2,0)),(\ (2,2,0;0,0,0),(5,0)),(\ (2,0,2;0,0,0),(5,0)),$

$(\ (0,2,2;0,0,0),(5,0))\ \}$

Total $=(42,0)$

- $(0,0,0)(0,1,0)(1,0,1)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(2,0)),\ ((0,1,3;0,1,0),(2,0)),$

  $((1,1,2;0,0,0),(4,0)),\ ((1,1,2;1,0,1),(2,0)),\ ((1,2,1;0,1,0),(2,0)),$

  $((1,2,1;1,1,1),(2,0)),\ ((2,1,1;0,1,0),(4,0)),\ ((2,1,1;1,1,1),(2,0)),$

  $((2,2,0;0,0,0),(4,0)),\ ((2,0,2;0,0,0),(5,0)),\ ((2,0,2;1,0,1),(1,0)),$

  $((0,2,2;0,0,0),(4,0))\ \}$

  Total $=(37,0)$

- $(0,0,0)(0,1,0)(1,1,1)$

  $\{\ ((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(2,0)),\ ((1,3,0;1,1,0),(4,0)),$

  $((3,0,1;1,0,0),(4,0)),\ ((0,1,3;0,1,0),(4,0)),\ ((1,1,2;0,0,0),(4,0)),$

  $((1,1,2;1,1,0),(6,0)),\ ((1,2,1;0,1,0),(2,0)),\ ((1,2,1;1,0,0),(6,0)),$

  $((2,1,1;0,1,0),(6,0)),\ ((2,1,1;1,0,0),(2,0)),\ ((2,2,0;0,0,0),(5,0)),$

  $((2,2,0;1,1,0),(1,1)),\ ((2,0,2;0,0,0),(6,0)),\ ((0,2,2;0,0,0),(6,0))\ \}$

  Total $=(61,1)$

2.4.3 Order 6

For $n = 6$, we have a single orbifold. In order to further shorten the notation, we will use S_3 symmetry in $\mathbb{Z}/6^3$: we will write one element per S_3 orbit (the size of the orbit is is written between square brackets):

$\{ ((0,0,0;0,0,0),(3,0))[1]$, $((1,5,0;0,0,0),(6,0))[6]$, $((1,4,1;0,0,0),(6,0))[3]$, $((1,3,2;0,0,0),(24,0))[6]$, $((2,4,0;0,0,0),(24,0))[6]$, $((2,2,2;0,0,0),(5,0))[1]$, $((3,3,0;0,0,0),(12,0))[3] \}$

Total $= (80,0)$.

2.5 The fundamental Group

We will compute π_1 of our orbifolds using the fact that they are the quotients of a simply connected space ($\mathbb{C}^3$).

Consider $E_n \times E_n \times E_n/G$ as the quotient of $\mathbb{C}^3$ by $\tilde{G}$, extension of G by the lattice group Λ_n. Let $F = \{g \in \tilde{G} : \exists x \in \mathbb{C}^3 \mid g.x = x\}$ and $N(F)$ be its normal closure in $\tilde{G}$.

Theorem 2.5.1. *The fundamental group of $E_n \times E_n \times E_n/G$ is $\tilde{G}/N(F)$.*

Proof. Since $\mathbb{C}^3$ is semi-locally simply connected, [5] tells us that the fundamental groupoid $\Pi(\mathbb{C}^3/\tilde{G})$ is equal to the quotient groupoid $\Pi(\mathbb{C}^3)//\tilde{G}$. In particular the object group $Hom_{\Pi(\mathbb{C}^3/\tilde{G})}(x,x) \simeq \pi_1(\mathbb{C}^3/\tilde{G},x) \simeq \pi_1(E_n \times E_n \times E_n/G)$ is isomorphic to $\tilde{G}/N(F)$.

A more heuristic argument is given in [9]. □

2.5.1 Order 4 and 6

Proposition 2.5.2. *Let G be an extension of the Vafa-Witten Group with $n = 4$ or 6. The closure $N(F)$ of F is the whole of $\tilde{G}$.*

Proof. We will generalize slightly the notation used up to now, we will write $g = (a_1, a_2, a_3; \tau_1 t_i, \tau_2 t_i, \tau_3 t_i)$ where before we would have omitted the t_i. This permits to write the elements of the lattice group Λ in the form $(0, 0, 0; \alpha_1+\beta_1\omega, \alpha_2+\beta_2\omega, \alpha_3+\beta_3\omega)$. Let g be an elements of $\tilde{G} - F$ it is of the form $(a_1, a_2, a_3; v_1, v_2, v_3)$ where, for at least one k, v_k is not zero and $a_k = 0$.

The key is that $\tilde{G}$ always contains an element of the form $h = (b_1, b_2, b_3; *, *, *)$ such that $a_k + b_k \neq 0$ and $b_k \neq 0$ for all $k \in \{1, 2, 3\}$.

Case 1: only one of the $a_k = 0$. We can assume without loss of generality that $k = 1$. Pick $\epsilon, \delta \in \mathbb{Z}/i$ such that $\epsilon, \delta \neq 0$,$\delta \neq -\epsilon$,$\delta \neq a_2$ and $\delta \neq a_2 - \epsilon$. For n large enough (that is $n > 3$) we see that there exists $n^2 - 5n + 6 > 0$ such possible pairs. Indeed the previous conditions take away (in the right order) n, $n-1$, $n-1$, $n-2$ and $n-2$ possibities from the n^2 possible pairs of $\mathbb{Z}/n \times \mathbb{Z}/n$. (See figure 2.4) Now, we just pick $h = (\epsilon, \delta - a_2, a_2 - \delta + \epsilon; *, *, *)$.

Case 2: all $a_k = 0$. We can pick h to be $(1, 1, n-2; *, *, *)$. Since $n > 3$, $n - 2 \neq 0$.

Figure 2.4: “Good pairs” of indices.

In both cases h and $h^{-1}.g$ belong to F so from the trivial equality $g = h.(h^{-1}.g)$ we deduce that $g \in N(F)$.

□

Corollary 2.5.3. *Let G be an extension of the Vafa-Witten Group with $n = 4$ or 6, $\pi_1(X(G), x)$ is trivial.*

2.5.2 Order 3

- The set F always contains $g_1 + 2g_2$ as it is of the form $(2, 2, 2; *, *, *)$. So, if the group spanned by the first two generators intersects F outside the group spanned by $g_1 + 2g_2$, e.g. in $(1, 2, 0; *, *, 0)$, then the group spanned by the first two generators lies in $N(F)$.

- The group spanned by the elements which are pure translations lie in $N(F)$. In particular the group spanned by the lattice group lies in $N(F)$

  Proof. Pure elements of translations are of the form $(0, 0, 0; v_1, v_2, v_3)$. Since $g_1 + 2g_2 = (2, 2, 2; *, *, *)$ and $j - g_1 - 2g_2 = (1, 1, 1; *, *, *)$ act with fixed points, $j \in N(F)$ as well. ∎

Proposition 2.5.4. *The fundamental group of orbifold for the case $n = 3$ is either trivial or $\mathbb{Z}/3$.*

Proof. By the above remark, the span of the latice group and the $g_{i>2}$ lies in $N(F)$, therefore, the information regarding the fundamental group lies in the first two generators.

Since the span of $g_1 + 2g_2$ also lies in F, the fundamental group is either trivial of $\mathbb{Z}/3$. ∎

Using the above remarks, it is easy to see that only extensions of the the fourth type of rank 0 can have a non trivial fundamental group. Indeed, in the first three cases, $g_1 = (1, 2, 0; 0, 0, 0)$ is an element of F.

Chapter 3

The Borcea-Voisin Construction

3.1 Preamble

In the following chapter we will attempt to generalize the construction of Calabi-Yau threefolds introduced by Borcea and Voisin ([2],[20]) in the nineties. Their construction is remarkable in two aspects: it is simple and provides many examples of threefolds and, more importantly, they provide a family of Calabi-Yau threefolds and mirror pairs. Voisin was even able to compute the Yukawa coupling on these varieties.

Hoping to get other interesting varieties and possibly other mirror pairs, we allow non-symplectic automorphism of $K3$ surfaces of order 3. The geometry of the $K3$ and the automorphism will determine the structure of the Calabi-Yau threefolds. In the following chapter we classify the possible fixed loci containing a curve of genus 1 or 2 by using the geometry induced by these curves. Our main result is a complete classification when there is a curve of genus 1 or 2. For other genera, we just give some constraints but we do not give any existence result.

3.2 General Borcea-Voisin construction

Let S be a nonsingular projective surface whose canonical divisor is trivial ($K_S = 0$) and having zero irregularity ($q = H^1(\mathcal{O}_S) = 0$); we say that S is a $K3$ *surface*. The complex structure of S gives a Hodge decomposition of the second cohomology :

$$H(S, \mathbb{C}) = H^{2,0}(S) + H^{1,1}(S) + H^{0,2}(S),$$

or numerically $22 = 1 + 20 + 1$. We will denote by ω, a holomorphic 2 form generating $H^{2,0}$. Suppose S has an automorphism j of order 3, which acts on $H^{2,0}$ by multiplying ω by ζ_3, a primitive cubic root of unity. Call G the group generated by j.

Let E be the elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\zeta_3)$ which has a $\mathbb{Z}/3\mathbb{Z}$ automorphism k generated by multiplication by ζ_3. Consider the variety $X = E \times S$; it is acted upon by $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ via the generators $k\times\mathrm{id}$ and $\mathrm{id}\times j$. There is an induced action, ϕ, of $\mathbb{Z}/3\mathbb{Z}$, the kernel of the addition map:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/3\mathbb{Z} & \hookrightarrow & (\mathbb{Z}/3\mathbb{Z})^2 & \xrightarrow{+} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\ & & & \overset{\phi}{\searrow} & \downarrow & & & & \\ & & & & Aut(X) & & & & \end{array}$$

We define the variety V as the quotient of X by ϕ. The geometry of V will depend on the action of $\mathbb{Z}/3\mathbb{Z}$ on the surface S and its geometry.

3.3 Non-symplectic automorphisms of K3's

3.3.1 General Properties

An automorphism of a $K3$ which, like j in the previous section, does not preserve the $(2,0)$ forms is called *non-symplectic.* We will study the fixed loci of non-symplectic automorphisms on $K3$'s which have order 3.

Lemma 3.3.1. *The fixed locus, S^j, consists of a disjoint union of smooth curves and points.*

Proof. Around a fixed point P, we can linearize the action ([6]).

Since j sends ω to $\zeta_3 \cdot \omega$, the determinant of the action around P is ζ_3, moreover j is of order 3, so the only possibilities are, up to change of coordinates:

1. $\begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$, in which case P lies on a smooth fixed curve.

2. $\begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$, in which case P is an isolated singularity.

□

Using the Hodge Index theorem, we can constrain the genus of the fixed curves in S^j.

Theorem 3.3.2. *(Hodge Index) Let C, D be divisors with rational coefficients on an algebraic surface X. If $C^2 > 0$ and $C.D = 0$ then $D^2 \leq 0$. Moreover, D is homologous to 0 (in rational cohomology) if and only if $D^2 = 0$.*

Indeed, we have the following theorem

Corollary 3.3.3. *(Voisin) Let S be an algebraic $K3$ containing N disjoint smooth curves, $C_1, \cdots, C_N$. At most one curve has genus strictly bigger than 1, and if there is such a curve, all the others are rational.*

Proof. Without loss of generality we can assume that $g_1 > 1$. Since we are on a $K3$, adjunction tells us that $C_i^2 = 2g_i - 2$, and therefore $C_1^2 > 0$. Since the curves are disjoint, the Hodge Index Theorem implies that $2g_i - 2 = C_i^2 \leq 0$ for all i's not equal to 1. Since the chern map on a K3 is injective, and that C is effective the inequality is strict, i.e. for the other curves $g_i < 1$. □

When one of the curves is of genus 1 we can say:

Lemma 3.3.4. *If C_1 and C_2 are both of genus 1 then, there are neither other fixed curves nor other isolated fixed points.*

Proof. It is well known that an elliptic curve C_1 on a $K3$ surface, S, gives an elliptic fibration with rational base: $\pi : S \to \mathbb{P}^1$. Any connected curve, with 0

Figure 3.1: Genus 1 case

intersection with the elliptic curve, is a component of the fibration and forms a whole fiber if it smooth. Therefore, j descends to an automorphism of $\mathbb{P}^1$. The $\mathbb{P}^1$ automorphism must have *two* fixed points. We might assume that they are 0 and ∞. The fixed locus of the $K3$ automorphism lies therefore in two fibres of the fibration. If $C_1 = \pi^{-1}(0)$ then all other singularities are located above ∞, in particular if the genus of C_2 is 1, then $\pi^{-1}(\infty) = C_2$ and the fixed locus consists solely of these two curves. □

An analogue result is proved by Voisin in ([20]) by using local coefficients and comparing the cohomology of S and S/j. Our proof is more geometric and gives insight on why a genus one fixed curve strongly constrains the fixed locus.

3.3.2 C_1 is elliptic.

To know what are the possible fixed loci, it is necessary to see how j acts on the fiber at infinity. We will start with the assumption that $\pi^{-1}(\infty)$ is singular. Kodaira has classified the possible singular fibers in ([14]). Not all these fibers interest

Figure 3.2: Chain of $\mathbb{P}^1$'s

us. In the next paragraph we will restrict the list to the plausible cases.

Since S is a $K3$ surface, its Euler characteristic is 24. Also, we can compute the Euler characteristic of S using the fact that is is an elliptic fibration:

$$24 = \chi_T(S) = \sum \chi(F_i),$$

where the sum is taken over all singular fibers $F_i = \pi^{-1}(i)$. As j descends to an automorphism of $\mathbb{P}^1$, all these fibers, except the one at infinity, come in triples. Therefore $\chi(F_\infty)$ must be congruent to 0 modulo 3 (and less than 24).

Also, we know that the fixed locus of j does not contain singular curves, and we can therefore exclude from the possible fibers at infinity, those with irreducible singular components (e.g. a Neil parabola). We see that the possible fibers above ∞ are

$$\mathcal{F} = \{I_0, I_{3k}, III, I_{3k}^*, III^*\}$$

The following lemmas will help us determine the possible non-symplectic $\mathbb{Z}/3\mathbb{Z}$ actions on the above configurations of rational curves :

Lemma 3.3.5. *Let l_1,l_2 and l_3 be a chain of projective lines on S, which intersect transversally as in figure 3.2. Denote by P_i the intersection point of l_i and l_{i+1}. Assume j acts on this configuration of lines.*
If l_1 is pointwise fixed, then j acts non-trivially on l_2 and l_3. Conversely, if j acts nontrivially on l_2 then either l_1 or l_3 is fixed pointwise and j acts nontrivially on the other.

Proof. If we consider the action of j around P_1 we see that the l_1 direction corresponds to the eigenvalue 1. Therefore, l_2 corresponds to the eigendirection multiplied by ζ_3. We can choose our coordinates on l_2 so that $P_1 = 0$ and $P_2 = \infty$. The action of j corresponds to $(x : y) \mapsto (\zeta_3 x : y)$. Taking local coordinates around P_2, this is multiplication by $\frac{1}{\zeta_3} = \zeta_3^2$. This means that around P_2, l_2 corresponds to the eigenvalue ζ_3^2 and l_3 therefore as well. As a consequence, j also acts non trivially on l_3.
Similarly, if j acts non trivially on l_2, either around P_1 or P_2 will it correspond to the eigenvalue ζ_3 thereby forcing the corresponding transversal line to be fixed pointwise. □

Corollary 3.3.6. *Let $U = \{l_i\}_{i=1\ldots n}$ be a chain of $\mathbb{P}^1$'s interesecting transversally as in the previous lemma. If j acts on U and fixes l_1 pointwise then j also fixes l_i pointwise for all $i \equiv 1 \pmod 3$ and acts nontrivially on all the other lines.*

Lemma 3.3.7. *Assume a line $l = \mathbb{P}^1$ is intersected transversally by three chains of* *projective lines, not all three of equal length. If j acts on these lines then, l is fixed pointwise.*

Proof. If all 3 branches are different then j can obviously not permute them. If 2 branches are the same, they cannot be permuted as we would have an order 2 subgroup of $\mathbb{Z}/3\mathbb{Z}$. This implies that the three intersection points must be fixed. However, a non-trivial linear automorphism of $\mathbb{P}^1$ fixes at most two points so the whole line must be fixed pointwise. □

Although the surfaces we consider do not necessarily have a section, we can from now on assume that they do. Indeed, we can substitute a surface S by the surface $\hat{S}$ where each curve has been substituted by its Jacobian. This new surface has the same singularities, has a section and has an action induced by j. ([8]) We are now ready to study the specific fibers :

Theorem 3.3.8. *The possible actions of j on the fiber at infinity are uniquely determined by the type of fiber unless the fiber is of type I_0^* where there are a priori two possible actions.*

Proof.

1. I_{3k} is a closed chain of $3k$ transversal projective lines.

   The section of π must intersect one of the links which is thus preserved and in turn forces all other lines to be preserved. It follows now from lemma 3.3.5 that one line out of three will be fixed together with the intersection point of the other pairs of lines.

Figure 3.3: The III^* fiber

2. III is made of two rational curves intersecting at point with multiplicity 2.

   Since the two curves have the same tangential direction at the intersection point, either both are fixed pointwise, or neither is fixed pointwise. Since the fixed locus is made out of disjoint rational curves, the only possibility is thus to have 3 fixed points. One of the points is the intersection point, there is 1 fixed point along each of the remaining two $\mathbb{P}^1$'s.

3. The dual graph of III^* is $\tilde{E}_7$

   By lemma 3.3.7 the central component must be fixed. It follows then, from lemma 3.3.5, that the two extreme lines are also fixed, as well as the other points as marked on figure 3.3 *(full lines and thick points are fixed pointwise).*

4. I^*_{3k} : the dual graph is $\tilde{D}_{3k+4}$

   Using similar arguments it is easy to see that the there is a sole possibility if $k > 0$ (see figure 3.4) *(full lines and thick points are fixed pointwise).* If $k = 0$ there is another possibility consisting in permuting three of the free ends and preserving the fourth. [1].

□

[1]This possibility will be excluded by looking at the quotient S/j. (see 3.3.12)

Figure 3.4: The I_{3k}^* fiber

For simplicity we have copied the possible cases in table 3.1.

Table 3.1: Fixed locus above ∞

type of fiber	fixed $\mathbb{P}^1$'s	fixed points
I_{3k}	k	k
III	0	3
I_{3k}^*	$k+1$	$k+4$
III^*	3	3

Although we have found numerical constraints on the possible fibers and fixed loci, we do not have yet any information concerning their existence. To do this, we will quotient S by j and observe the resulting surface.

Lemma 3.3.9. *The surface S/j is rational elliptic.*

Proof. Since j induces an automorphism of order 3 on $\mathbb{P}^1$ and preserves the fibration, we get the following commutative diagram :

$$\begin{array}{ccc} S & \xrightarrow{j} & S/j \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{P}^1 & \xrightarrow[3:1]{} & \mathbb{P}^1 \end{array}$$

The automorphism j fixes only two of the fibers and therefore S/j is rational elliptic.

□

Table 3.2: Fibers above ∞

$Z' = \phi^{-1}(\infty)$	$Z = \pi^{-1}(\infty)$	$\chi(Z')$	$\chi(Z)$	$3\chi(Z') - \chi(Z)$
I_0	I_0	0	0	0
I_k	I_3k	k	$3k$	0
II	I_0^*	2	6	0
III	III^*	3	9	0
IV	I_0	4	0	12
I_k^*	I_{3k}^*	$6+3k$	$6+9k$	12
II^*	I_0^*	10	6	24
III^*	III	9	3	24
IV^*	I_0	8	0	24

To generate the singular fibers on S it is sufficient to pick appropriate singular fibers of a rational elliptic surface and to use a triple base change to pull them back to S. We know that the fiber resulting from the triple base change should be in $\mathcal{F}$. We have placed the singular fibers which yield, after triple base change, a fiber of $\mathcal{F}$ in table 3.2 (for details see ([15])). Again not all these fibers do interest us as there are some topological restrictions :

Theorem 3.3.10. *The fiber $\phi^{-1}(\infty) = \pi^{-1}(\infty)/j$ is either of type IV or I_k^*.*

Proof. Let Z' be the ϕ-fiber at infinity and Z its preimage under j. Since the Euler characteristic of a rational surface (respectively $K3$) is 12 (respectively 24) we get the following equations :

$$\begin{cases} 12 &= \chi(Z') + \chi(A) \\ 24 &= \chi(Z) + 3\chi(A) \end{cases}$$

where A represents the other singular fibers on the rational surface. We deduce that $12 = 3\chi(Z') - \chi(Z)$. Using table 3.2 we see that this restricts the interesting fibers to $Z' = IV$ or $Z' = I_k^*$. □

We will now use the results of Ulf Persson ([17]) who has classified the singular fibers of rational elliptic surfaces. All the cases listed by Persson which contain a fiber of type IV or I_k^* will give an elliptic $K3$ with $\mathbb{Z}/3\mathbb{Z}$ automorphism. There are 88 types of fibrations containing at least one fiber of the type $\{I_2^*, I_1^*, I_0^*, IV\}$. Hence,

Proposition 3.3.11. *Let X be $K3$ surface with a non-symplectic involution of order 3 such that the fixed locus contains an elliptic curve. The remain of the fixed locus is contained in a fiber of type I_6^*, I_3^*, I_0^*, or I_0.*

Corollary 3.3.12. *There exists a unique action on the I_0^* fiber. This action fixes 4 points and 1 line.*

Moreover we know exactly the nature of the fixed locus:

Corollary 3.3.13. *In the conditions of the preceding proposition, the singular fiber consists in :*

- 3 *fixed lines and* 6 *fixed points. (within an* I_6^* *fiber)*
- 2 *fixed lines and* 5 *fixed points. (within an* I_3^* *fiber)*
- 1 *fixed line and* 4 *fixed points (within an* I_0^* *fiber)*

3.3.3 C_1 is of genus 2.

As in the previous section we will use the linear system $|C_1|$ to study the action of j on S. We will show that the only possible fixed locus consists in a curve of genus 2 together with two isolated fixed points or one fixed line or 2 fixed lines and 4 fixed points. The Hodge numbers appear in the last section.

Proposition 3.3.14. *The linear system of* C_1 *yields a morphism of degree* 2 *onto the projective plane:* $S \xrightarrow{\phi_{|C_1|}} \mathbb{P}^2$. *Moreover, the automorphism* j *descends to an automorphism* $\hat{\jmath}$ *of of the projective plane.*

Proof. Since C_1 is smooth of genus 2, the linear system $|C_1|$ does not have a base locus and gives a map $\phi_{|C_1|}$ which is the usual double cover of $\mathbb{P}^2$ ramified above a sextic (see for example ([18]). The sextic is smooth as singularities of the branch locus correspond to singularities of the cover – in our case S – which is nonsingular. The curve C_1 is fixed by j, therefore, j descends to an automorphism $\hat{\jmath}$ of $\mathbb{P}^2$. We have the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{j} & S \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{P}^2 & \xrightarrow[\hat{j}]{} & \mathbb{P}^2 \end{array}$$

□

Corollary 3.3.15. *The fixed locus of j consists in C_1 together with a number of components contained in a unique fiber of ϕ.*

Proof. The planar automorphism $\hat{j}$ fixes the line $L = \phi_{|C_1|}(C_1)$, image of C_1. If we choose L to be the line at infinity , $(x : y : 0)$, then, up to a change of coordinates, $\hat{j}$ is given by

$$\begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Outside the line at infinity, there is a unique other fixed point, the origin $\mathbf{o} = (0 : 0 : 1)$. As the j-ϕ diagram commutes, the fixed locus of j consists of some components of the preimage of L together with some components of the fiber above $\mathbf{o}$. Since, C_1 is a genus 2 curve, by Riemann-Hurwitz, $\phi|_{C_1}$ must be ramified in 6 points. This implies that L and the sextic intersect in six distinct point, i.e. L does not pass through singularities of the sextic. This means that $\phi^{-1}(L)$ is exactly C_1. □

The nature of the fixed locus will therefore depend essentially on the respective locations of the origin and of the branch locus, Γ.

The isolated point lies on the branch locus

When the isolated point lies on Γ, then the branch locus has a constrained equation:

Lemma 3.3.16. *The equation of the branch locus is of the following form:*

$$F_6 + F_3 z^3 = 0$$

whenever it contains the isolated fixed point (i.e. the origin). The symbol F_i denotes a homogeneous polynomial of degree i in the variables x and y.

Proof. Since Γ is invariant under the action of $(x, y) \mapsto (\zeta_3^2 x, \zeta_3^2 y)$, all monomials making up the equation must be of same xy-degree modulo 3. Therefore the equation must be of one of the three following forms:

- $F_4 z^2 + F z^5 = 0$
- $F_5 z + F_2 z^4 = 0$
- $F_6 + F_3 z^3 + \alpha . z^6 = 0$

It is well-known that the branch locus must not contain multiple components ([18]) and therefore we can eliminate the first type. Since the preimage of the line at infinity must be a genus 2 curve, it cannot be part of the ramification locus : we can eliminate the second equation. Finally, since the origin belongs to Γ, α has to be zero in the last choice. □

It follows therefore that

Corollary 3.3.17. *Under the conditions of the previous lemma,* Γ *is singular at the origin.*

It is now necessary to see the type of singularities that could occur at the origin.

Proposition 3.3.18. *The singularity of the origin is of type* D_4 *or* D_7

Proof. It is well-known that the singularity must be of ADE type ([21]) without which our original $K3$ would have been singular.
The affine form of the equation is $F_6(x, y) + F_3(x, y)$. So either the singularity is of multiplicity 6 (which we exclude as this would not be an ADE singularity) or it is of multiplicity 3 with 1,2 or 3 tangent directions. Let us analyze the cases one by one.

- 3 tangent directions. From the classification of simple singularities ([1]) it is known that all singularities of multiplicity 3 with 3 tangent directions are of ADE type. It is easy to see that is in fact a singularity of type D_4. We can find a minimal resolution by taking the canonical resolution ([1]). After taking the resolution of the curve, we get four exceptional divisors as in figure 3.5 and the double cover is branched over the strict transform of the sextic and the exceptional divisor not intersecting it.

- 2 tangent directions. Again, it follows from the classification of simple singularities that all singularities of multiplicity 3 with 2 tangent directions are

Figure 3.5: Minimal resolution of $D4$ singularity.

Figure 3.6: Minimal resolution of $D7$ singularity.

of ADE type. By taking some simple transforms, we can see that is a singularity of type D_7. Using canonical resolution we get an exceptional divisor consisting in a chain of 6 rational lines and a double cover branched along the thick lines as in figure 3.6.

- 1 tangent direction. Without loss of generality, the equation is of the form $x^3 + \sum_{i=0}^{6} a_i x^i y^{6-i}$ which we can rewrite as

$$x^3(1 + a_6 x^3) + x^2y^2 \underbrace{(a_4 x^2 y^2 + a_3 xy + a_2 y^2)}_{\phi_1} + xy^3 \underbrace{(a_1)}_{\phi_2} + y^4 \underbrace{(a_0 y^2)}_{\phi_3}$$

following [1], section $II.8$. In our case, $\phi_3(0) = \phi_2(0) = \phi_3'(0) = 0$ hence we do not have a simple singularity. (Another possibility would have been to blow-up the singularity once and see higher order multiplicities appear).

□

Corollary 3.3.19. *The fixed locus of j when the isolated fixed point lies on the branch locus consists in C_1 together with* 1 *line or* 2 *lines and* 4 *isolated points.*

Proof. We have to see what are the possible non-symplectic $\mathbb{Z}/3\mathbb{Z}$ actions on the configurations obtained in the previous proposition. Using 3.3.5 and 3.3.7 it follows that in the D_4 case, there is only one fixed line outside the genus curve and that in the D_7 case there are 2 lines and 4 fixed points. □

The isolated point lies outside the branch locus

The case is straightforward as the preimage of the isolated point is made of two points.

Proposition 3.3.20. *The fixed locus of j within the fiber above the isolated fixed point consists in two isolated fixed points.*

Proof. The fiber above the isolated point consists in two points. Since the fiber is preserved, the only possible action of $\mathbb{Z}/3\mathbb{Z}$ is to fix both points. □

3.4 Computation of the cohomology of X

To compute the orbifold cohomology of X we will use the orbifold cohomology of Ruan & al. ([7]):

$$H^{i,j}(X) = \bigoplus_{\substack{[g] \in G \\ U_i \in (S \times E)^g}} H^{i-\kappa(g),j-\kappa(g)}(U_i)^{C(g)}$$

where we sum up over the conjugacy classes of G and the components of the fixed locus of each element. $C(g)$ denotes the centralizer of g in G. The set $S \times E^g$ is the fixed locus of any element in the conjugacy class of g, i.e. $\{x \in S \times E | g.x = x\}$, U_i denotes one of its components. Since we are dealing with an Abelian group, the formula simplifies to

$$H^{i,j}(X) = \bigoplus_{\substack{g \in G \\ U_i \in (S \times E)^g}} H^{i-\kappa(g),j-\kappa(g)}(U_i)^{G}.$$

If the action of g sends $[z_i]$ to $[e^{2\pi i\theta_i} z_i]$ with $0 \leq \theta_i < 1$ then the we define the shift function [2] by $\kappa(g) = \sum_{i=1}^{3} \theta_i$. Moreover, since our group is cyclic of prime order, we have $H^*((S \times E)^g)^G = H^*((S \times E)^g)$ as soon as g is not the identity.

[2]Also known as *fermionic shift*, *degree shifting number* or *age*

To perform our computation, we will need two main ingredients : the cohomology of the U_i and $H^*(S \times E)^G$.

Computation of $H^*(S \times E)^G$

In order to find the G-invariant part of the cohomology of $S \times E$ we need to analyze the action of j (resp. k) on the cohomology of S (resp. E).

The map j induces a linear map $\tilde{\jmath}$ on the cohomology of S and we can thus decompose the cohomology into eigenspaces of $\tilde{\jmath}$. Since j is of order 3, the eigenvalues of $\tilde{\jmath}$ will belong to $\{1, \zeta_3, \zeta_3^{-1}\}$. Let H^0 (resp. H^+,H^-) be the eigenspace of 1 (resp. ζ_3,ζ_3^{-1}).

Lemma 3.4.1. *The $\tilde{\jmath}$-eigenspaces of $H^*(S)$ are*

$$H^0(S) = \begin{matrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{matrix}, \quad H^+(S) = \begin{matrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{matrix} \quad \text{and } H^-(S) = \begin{matrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{matrix},$$

where $a + 2b = 20$.

Proof. By definition, j sends ω to $\zeta_3.\omega$. This corresponds to the class of ω being mapped to $\zeta_3.[\omega]$, i.e. $H^{2,0} \subset H^+$. Let b be the dimension of $H^+ \cap H^{1,1}$. By Hodge symmetry, $H^{0,2}$ is a subset of H^- and $\dim(H^- \cap H^{1,1})$ is also b. Finally, the $(0,0)$ forms are clearly invariant under $\tilde{\jmath}$ and the $(2,2)$ forms as well (the class $[\omega \wedge \bar{\omega}]$ is sent to $\zeta_3.\bar{\zeta}_3 = 1$ times itself.) □

Similarly, the action k on E induces a decomposition of the cohomology:

Lemma 3.4.2. *The cohomology of the elliptic curve E decomposes into the following $\tilde{k}$-eigenstates*

$$H^0(E) = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \; H^+(E) = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \; \textit{and } H^-(E) = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}.$$

We are now ready to compute invariant part of the cohomology of the product, $H^*(X)^G$. It is given by

$$(H^0(E) \otimes H^0(S)) \oplus (H^+(E) \otimes H^+(S)) \oplus (H^-(E) \otimes H^-(S)).$$

Using the two above lemmas we get :

$$H^*(X)^G = \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & a+1 & b & 0 \\ 0 & b & a+1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

.

Cohomology of the fixed loci

Let U_i be a component of the fixed locus. The local action of an element g of G at a point of U_i is given by a three by three matrix with determinant one. If each diagonal entry is written as $e^{\frac{2i\pi k}{3}}$ then the sum of the k's is the shift. The contribution of U_i to the cohomology of the quotient is given by the invariant part of the cohomology of U_i under the action of G. We have written the data down in table 3.3: the first column represents the element, the second column shows the local action on U_i, for U_i a smooth curve or an isolated point, and the last column is the G invariant part of the cohomology of U_i

Cohomology of X

To patch together the above information, it is useful to define the following constants:

Definition 3.4.3. We call

- g the genus of C_1.
- n the number of rational curves fixed by j.
- p the number of points fixed by j.

Remember that a is the dimension of j invariant part of $H^{1,1}(S)$ and that $a+2b = 20$. Since k fixes three points on E, (j,k) fixes 3 (resp. $3n$, $3p$) curves of genus g (resp. $\mathbb{P}^1$'s, points) on $S \times E$. From the orbifold cohomology formula follows then that

Proposition 3.4.4. *The Hodge diamond of X is*

$$H^*(X) = \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & a+7+6n+3p & b+6g & 0 \\ 0 & b+6g & a+7+6n+3p & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

Proof. There are 3 curves of genus g, each contributing to 2 dimensions to $H^{1,1}$ and $H^{2,2}$ (one for ζ_3 and one for ζ_3^2). Moreover they each contribute to $2g$ dimensions to $H^{1,2}$ and $H^{2,1}$.

The $3n$ rational curves contribute each to 2 dimensions for $H^{1,1}$ and $H^{2,2}$. Finally, the $3p$ points contribute each to 1 dimension for $H^{1,1}$ and $H^{2,2}$.

All together we have that the dimension of $H^{1,1}$ (which is also the dimension of $H^{2,2}$) is $a+1+3\times 2+3n\times 2+3p\times 1$ and the dimension of $H^{1,2}$ or $H^{2,1}$ is $b+3\times 2g$. $\square$

The only undetermined value in the cohomology of X is a. We will determine a by looking at the rational surface obtained by quotienting the $K3$ by j.

Lemma 3.4.5. *The dimension of the j invariant part of $H^{1,1}$, a, is $\frac{1}{3}(22+2p-4g+4l)$.*

Proof. The surface S/j is singular at the p fixed points. We will therefore work with a slightly modified surface. Let $\tilde{S} \xrightarrow{\pi} S$ be the blow-up of S along the p isolated fixed points. Around p, j acts as $\begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$ so the action extends to $\tilde{S}$ - we will call $\breve{j}$ the lift of j. If $(x:y)$ are a set of coordinates on one of the exceptional divisors. The map $\breve{j}$ sends it to $(\zeta_3^2 x : \zeta_3^2 y) = (x:y)$ so $\breve{j}$, restricted to the exceptional locus, acts trivially.

The quotient surface $\tilde{S}/\breve{j}$ is smooth and we will compute its Euler characteristic in two ways. First note that $h^{1,1}(\tilde{S}) = 20+p$ as we have blown up at p points, this yields an Euler characteristic of $24+p$ for $\tilde{S}$. Now, using the Riemann-Hurwitz formula we get:

$$24 + p = \chi(\tilde{S}) = 3\chi(\tilde{S}/\breve{\jmath}) - 2\chi(\tilde{S}^{\breve{\jmath}}).$$

The fixed locus of $\breve{\jmath}$ on $\tilde{S}$ consists of the preimage under π of the fixed locus of j, i.e. it consists in a curve of genus g, n projective lines coming from the fixed $\mathbb{P}^1$'s on S and p other projective lines, exceptional divisors above the isolated fixed points of j. We have thus

$$\chi(\tilde{S}^{\breve{\jmath}}) = \chi(C_g) + (n+p)\chi(\mathbb{P}^1) = 2 - 2g + 2n + 2p.$$

From the above formulas we deduce that:

$$\chi(\tilde{S}/\breve{\jmath}) = \frac{28 - 4g + 4n + 5p}{3}$$

On the other hand, the $\breve{\jmath}$ invariant part of the cohomology of $\tilde{S}$ is $\begin{matrix} 1 & 0 & 0 \\ 0 & a+p & 0 \\ 0 & 0 & 1 \end{matrix}$.

Therefore, we compute that $\chi(\tilde{S}/\breve{\jmath}) = a + p$. Putting together the two results

$$a = \frac{22 - 4g + 4n + 2p}{3}$$

□

We can now express the cohomology of X as a function of the fixed locus of j:

Proposition 3.4.6. *The space X is a Calabi-Yau threefold with $h^{1,1} = \frac{1}{3}(43 - 4g + 22n + 11p)$ and $h^{1,2} = \frac{1}{3}(19 + 20g - 2n - p)$.*

Proof. Just substitute the value obtained of a in the preceding proposition. $\square$

This gives us an elegant corollary :

Corollary 3.4.7. *The Euler characteristic of X is eight times the Euler characteristic of the fixed locus of j.*

Proof. The Euler characteristic for a Calabi-Yau threefold is $2(h^{1,1} - h^{1,2})$. In our case, we get $\frac{2}{3}(43 - 4g + 22n + 11p - 19 - 20g + 2n) = 8(2 - 2g + 2n + p)$ which is nothing but $8\chi(S^j)$.
Equivalently, we could use Riemann-Hurwits for the triple covers $\tilde{S} \to \tilde{S}/\breve{\jmath}$ and $E \times \tilde{S} \to E \times \tilde{S}/G$ and get the same result. $\square$

Numerical values

Using the above formula, we have tabulated the numerical values in tables 3.4 and 3.5.

3.4.1 Topology

Proposition 3.4.8. *The fundamental group of all the preceding Calabi-Yau threefolds is trivial.*

Proof. We can write $X = S \times E/(j,k)$ or $X = S \times \mathbb{C}/\mathcal{G}$ where $\mathcal{G}$ is the extension of G by the lattice group $\mathbb{Z} \oplus \zeta_3\mathbb{Z}$. The space $S \times \mathbb{C}$ is simply connected and we can use the same result of Brown and Higgins ([5]) as in the previous chapter :

$$\pi_1(X) = \mathcal{G}/N$$

where N is the normal closure of the subgroup generated by elements of $\mathcal{G}$ fixing at least one point. An element of $\mathcal{G}$ is of the form $(x, y) \mapsto (\zeta_3^i x, \zeta_3^i y + \tau)$ for some τ in $\mathbb{Z} \oplus \zeta_3\mathbb{Z}$. Such an element does not have fixed points if and only if it is of the form $(x, y) \mapsto (x, y + \tau)$ where τ is not zero. However, we can decompose this element into

$$(x, y) \overset{(\zeta_3 x, \zeta_3 y)}{\longrightarrow} (\zeta_3 x, \zeta_3 y) \overset{(\zeta_3^2 x, \zeta_3^2 y + \tau)}{\longrightarrow} (x, y + \tau)$$

where the two intermediary elements have fixed points. Therefore, the elements having fixed points span the whole group and $\pi_1(X)$ is trivial. □

◇

Table 3.3: Action of G on the cohomology of the fixed locus

	local action on U_i	shift	$H^*(U_i)^G$
1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\kappa = 0 + 0 + 0 = 0$	see previous section
ζ	on fixed point : $\begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$\kappa = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$	(1)
	on fixed curve : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$\kappa = 0 + \frac{1}{3} + \frac{2}{3} = 1$	$\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}$
ζ^2	on fixed point : $\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$\kappa = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$	(1)
	on fixed curve : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$\kappa = 0 + \frac{2}{3} + \frac{1}{3} = 1$	$\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}$

Table 3.4: genus of C_1 is 1

type of $\pi^{-1}(\infty)$	fixed points	fixed lines	(h_{11}, h_{12})
$(I_{3n}^*)_{0\leq n\leq 2}$	$n+4$	$n+1$	(46+11n,10-n)

Table 3.5: genus of C_1 is 2

type of $\mathbf{o}$	fixed points	fixed lines	(h_{11}, h_{12})
isolated pt	2	0	$(19, 19)$
D_4	0	1	$(19, 19)$
D_7	4	2	$(41, 17)$

Chapter 4

The Future

4.1 Vafa-Witten Construction

As was unveiled in the first chapter, the threefolds which were produced so far do not have, except in one occasion, a non-trivial fundamental group. Physicists believe that the fundamental group to break the symmetry of the GUT when passing to the standard model. A natural choice would be to allow translations which do not commute with the $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ action. In this case, computer systems such as CARAT (a program specialized in computation with crystallographic groups) might be useful.

Eventually, it will also be interesting to build vector bundles on the threefolds that we obtained. Constructions could be inspired by those of Donagi & al. ([12] e.g.) or the one of Donagi and Bouchard ([3]).

4.2 Borcea-Voisin Manifolds

The first project is to complete the case with the automorphism of order 3 by analyzing the actions fixing a curve of genus larger than 2 or only rational curves and points. In a second time, using the same canvas, we will focus on automorphisms of order 4 or 6.

With the classification in hand the aim will be to find a pattern which would give us a way to find mirror pairs. Up to now, we do not have enough varieties to see mirror pairs. As the physics is simpler with flat orbifolds, it would finally be interesting to see which Borcea-Voisin Calabi-Yaus degenerate to Vafa-Witten orbifolds...

"Vaste programme!"

Général de Gaulle

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