

THE BIRATIONAL GEOMETRY OF TROPICAL COMPACTIFICATIONS

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This thesis is dedicated to my father.

## ABSTRACT

### THE BIRATIONAL GEOMETRY OF TROPICAL COMPACTIFICATIONS

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We study compactifications of subvarieties of algebraic tori using methods from the still developing subject of tropical geometry. Associated to each “tropical” compactification is a polyhedral object called a tropical fan. Techniques developed by Hacking, Keel, and Tevelev [19, 45] relate the polyhedral geometry of the tropical variety to the algebraic geometry of the compactification. We compare these constructions to similar classical constructions. The main results of this thesis involve the application of methods from logarithmic geometry in the sense of Iitaka [22] to these compactifications. We derive a precise formula for the log Kodaira dimension and log irregularity in terms of polyhedral geometry. We then develop a geometrically motivated theory of tropical morphisms and discuss the induced map on tropical fans. Tropical fans with similar structure in this sense are studied, and we show that certain natural operations on a tropical fan correspond to log flops in the sense of birational geometry. These log flops are then studied via the theory of secondary polytopes developed by Gelfand, Kapranov, and Zelevinsky [16] to obtain polyhedral analogues of some results from logarithmic Mori theory.

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## Notation and Conventions

This thesis will use many notions from polyhedral geometry. We list here our conventions, and refer to [9] or [36, Appendix] for background.

- Given a finite dimensional real vector space  $V$ , a **lattice** in  $V$  is a subset  $\Lambda$  which is an abelian group under vector addition and such that  $V = \Lambda \otimes_{\mathbb{R}} \mathbb{R}$ .
- A **polyhedron** is an intersection of finitely many half-spaces in a fixed finite dimensional real vector space. The dimension of a polyhedron is equal to the smallest dimension of a linear subspace which properly contains the polyhedron. A **face** of a polyhedron is the intersection of a polytope with a subspace such that the intersection is contained in the boundary of the polyhedron. A **facet** is a face of codimension 1, a **vertex** is a face of dimension 0, and an **edge** is a face of dimension 1. A **polytope** is a compact polyhedron. A polyhedron is called **integral** with respect to a lattice if the vertices of the polytope lie on the lattice. In this thesis, typically the lattice will be fixed and clear from context, and we may omit reference to it.
- A **polyhedral cone** is a polyhedron with at most one vertex. In this thesis, such a vertex will always appear at the origin of a fixed vector space, which will be clear from context. Any polyhedral cone of dimension equal to the ambient space which appears in this thesis will have the property that all facets have rational slope with respect to a fixed lattice which will be clear from context. To keep terminology concise, the word **cone** will always mean a polyhedral cone with the above properties. The edges of a polyhedral cone are called **rays**.
- A **polyhedral complex** is a collection  $\mathcal{P} = \{P_\alpha\}$  of polyhedra in a fixed vector space such that the intersection of any two members of  $\mathcal{P}$  is also a member of  $\mathcal{P}$ , and any facet of a member of  $\mathcal{P}$  is also a member of  $\mathcal{P}$ . A polyhedral complex  $\mathcal{P}'$  is said to **refine** a polyhedral complex  $\mathcal{P}$  if every polyhedron in  $\mathcal{P}'$  is a union of polyhedra in  $\mathcal{P}$ . We will often use the word **cell** to refer to a polyhedron in a polyhedral complex, especially when thinking of  $\mathcal{P}$  as a CW-complex.
- The dimension of a polyhedral complex  $\mathcal{P}$  is defined to be the dimension of the largest polyhedron in  $\mathcal{P}$ . For  $0 \leq k \leq \dim \mathcal{P}$  we define the **k-skeleton** of  $\mathcal{P}$ , denoted  $\mathcal{P}(k)$ , to be the union of all cells of  $\mathcal{P}$  of dimension at most  $k$ .
- A morphism of polyhedral complexes  $\mathcal{P} \rightarrow \mathcal{P}'$  is a set theoretic map such that the image of every polyhedron of  $\mathcal{P}$  is equal to a polyhedron of  $\mathcal{P}'$ .
- A **fan** is a polyhedral complex  $\mathcal{P}$  where every element of  $\mathcal{P}$  is a cone, in the above sense. Fans will typically be denoted by  $\Sigma$ .
- The **support** of a polyhedral complex denotes the set theoretic union of all polyhedra in the complex. It will be denoted either  $|\mathcal{P}|$  or  $|\Sigma|$  if the complex is a fan. If  $\mathcal{P}$  and  $\mathcal{P}'$  are two polyhedral complexes in the same ambient space, we say that  $\mathcal{P}$  is **supported on**  $\mathcal{P}'$  if  $|\mathcal{P}| \subseteq |\mathcal{P}'|$ .

- A polytope of dimension  $n$  it is called a **simplex** if it has exactly  $n + 1$  vertices. A cone of dimension  $n$  is called **simplicial** if it has exactly  $n$  rays. A polyhedral complex will be called **simplicial** if either every polyhedron is a simplex, or if every polyhedron is a simplicial cone; we rely on context to distinguish these cases. If the ambient vector space is equipped with a lattice  $\Lambda$ , an integral simplex is called **smooth** if at one (equivalently any) vertex, the  $n$  vectors determined by its adjacent edges form a subset of a  $\Lambda$  basis. This condition is often called unimodular in the literature; we use the word smooth due to its meaning for toric varieties. A simplicial cone is called **smooth** if its  $n$  rays are spanned by a subset of a lattice basis.

We will also make extensive use of the theory of toric varieties. There are often conflicting notations and definitions this subject, so we collect our conventions here. Again, we refer to [9] or [36] for background.

- An **algebraic torus** is a variety isomorphic to  $(\mathbb{C}^*)^n$  for some  $n$ . Some authors refer to such a variety as a complex torus, but that phrase is now more frequently reserved for compact group varieties, so we do not use this terminology. We may be terse and simply use the word **torus** to refer to an algebraic torus.
- All toric varieties in this thesis are normal, and are thus described by a fan in the above sense. The toric variety corresponding to a fan  $\Sigma$  will be denoted  $\mathbb{P}_\Sigma$ . The reader should *not* assume that this means the toric variety is necessarily projective. We prefer this notation to the also common  $X(\Sigma)$  to prevent over-usage of the letter “X”.
- We say a fan is smooth if the corresponding toric variety is smooth. More generally, we may abuse the dictionary between fans and toric varieties say a fan has a given property if the corresponding toric variety has that property, or vice versa. We expect that this will not cause confusion.
- As above, a fan  $\Sigma$  implicitly comes with the data of a lattice, which will typically be denoted  $N$ . The dual lattice is then denoted  $M$ . The algebraic torus with character lattice  $M$  and co-character lattice  $N$  will be denoted  $T$ . We will say that  $\mathbb{P}_\Sigma$  is a toric variety for  $T$  if the character lattice of the associated  $N$  lattice is  $T$ .
- A cone  $\sigma$  of a fan  $\Sigma$  determines an orbit for the torus action on the variety  $\mathbb{P}_\Sigma$ . This orbit will be denoted  $\text{orb}(\sigma)$ . Some authors use the notation  $\mathcal{O}(\sigma)$  or  $\mathcal{O}_\sigma$ , we will not use this convention as we feel it leads to confusion with standard notation in sheaf theory.

In this thesis we work over the complex number field. All varieties and schemes should be assumed to be defined over  $\mathbb{C}$ , unless otherwise stated. We assume knowledge of basic algebraic geometry, and may introduce definitions and basic results without comment. We do recall here one definition which has varying meaning in the literature. See [10] for a nice discussion of subtleties in these definitions.

Let  $X$  be any smooth variety and  $D$  a reduced effective divisor with irreducible components  $D_1, \dots, D_k$ . We say  $D$  is a **normal crossing divisor** if for any closed point  $p \in D$  there exist regular parameters  $z_1, \dots, z_n$  such that  $D$  is given by the equation  $z_1 \cdots z_n = 0$  in  $\hat{\mathcal{O}}_{X,p}$ . We say  $D$  is a **simple normal crossing divisor** if for any closed point  $p \in D$  there exist regular parameters  $z_1, \dots, z_n$  such that  $D$  is given by the equation  $z_1 \cdots z_n = 0$  in  $\mathcal{O}_{X,p}$ . We recall that a normal crossing divisor is a simple normal crossing divisor if and only if each irreducible component of  $D$  is smooth.

# Chapter 1

## Tropical Geometry

In this chapter we review the theory of tropical compactifications, introduced in [45] and developed further in [17, 19, 30, 44]. We claim no originality in this chapter save perhaps in exposition. This is a new and specialized subject, so we make a heartfelt attempt to keep the exposition self-contained.

We speak little here of the historical foundations of the subject of tropical geometry as a whole, and instead focus only on constructions pertinent to the thesis. See [15] and [25] for general surveys of tropical geometry, or also the in progress draft of a textbook on tropical geometry by Maclagan and Sturmfels [31]. The foundational results employed in this thesis are due to Tevelev [45] and Hacking, Keel, and Tevelev [19].

### 1.1 Tropical Compactifications

In this section we discuss a method which converts certain algebraic varieties into polyhedral complexes. These polyhedral complexes will end up being closely related to the fans of toric geometry. The broad motivation of this thesis is to try and replicate the successes of the theory of toric varieties, and to interpret geometric information about a variety in terms of the polyhedral combinatorics of an associated polyhedral complex. We ask that the reader keep this theme in mind, as it underlies the entire work.

The previously mentioned polyhedral construction is as follows. Let  $K = \overline{\mathbb{C}((t))}$  be the field of Puiseux Series.  $K$  can be realized explicitly as

$$K = \cup_{n=1}^{\infty} \mathbb{C}((t^{\frac{1}{n}}))$$

and is equipped with a non-Archimedean valuation which assigns to an element of  $K$  the smallest exponent appearing in the element with non-zero coefficient. Call this valuation  $\text{val} : K^* \rightarrow \mathbb{Q}$ . For any fixed natural number  $n$  let  $\overline{\text{val}} : (K^*)^n \rightarrow \mathbb{Q}^n$  be the valuation applied coordinate wise.

**Definition 1.1.1.** If  $X$  is a closed connected subvariety of  $(K^*)^n$ , the *tropical variety* associated to  $X$  is the closure in  $\mathbb{R}^n$  of  $\overline{\text{val}}(X) \subseteq \mathbb{Q}^n$ . This set is denoted  $\text{Trop}(X)$ .



The fundamental fact about tropical varieties, recognized in the work of Bieri and Groves [2], is that they have the structure of a polyhedral complex. We defer a precise statement of this to Proposition 1.1.8. Although the theory works quite well for varieties defined over  $K$ , in this thesis we are actually concerned only with varieties which are defined over the residue field  $\mathbb{C}$ . More precisely, if  $X$  is a closed subvariety of  $(\mathbb{C}^*)^n$ , we may consider the base change  $X_K \subseteq (K^*)^n$  of  $X$  to  $K$  and define the tropical variety as above. This limits the complexity of the tropical varieties substantially: in Proposition 1.1.8 we will see that such tropical varieties can be given the structure of fans, and accordingly we call such tropical varieties *em tropical fans*. We concern ourselves with varieties defined over  $\mathbb{C}$  as we have the best control of the geometry in this situation. The fact that the valuation ring of the field of Puiseux Series is non-Noetherian makes the corresponding geometry difficult to work with, and some erroneous proofs have appeared in the literature as a result; see [38] for a discussion. On the other hand, ignoring the underlying algebraic geometry, the combinatorics of the polyhedral complexes coming from varieties not necessarily defined over the residue field are fascinating, see the thesis of David Speyer [42] for examples and conjectures. The techniques of [19] and [45], which we recall below, make no direct reference to the non-Archimedean nature of the definition of a tropical variety, and replace non-Archimedean techniques with the theory of toric varieties.

We will work repeatedly with subvarieties of algebraic tori in this thesis. We use the following terminology of [45] for brevity's sake.

**Definition 1.1.2.** A *very affine* variety is a closed subvariety of an algebraic torus.

It is clear that any very affine variety is affine. A priori the definition involves a choice of algebraic torus as an ambient space. An observation of Iitaka [22] is that there is actually a canonical choice.

**Proposition 1.1.3.** *If  $X$  is very affine then  $X$  is a closed subvariety of the algebraic torus  $T_X$  with character lattice  $M_X = \mathcal{O}^*(X)/\mathbb{C}^*$ , where the action of  $\mathbb{C}^*$  is a diagonal action on a choice of generators of  $\mathcal{O}^*(X)$ . This embedding is universal in the sense that any map  $X \rightarrow T$  from  $X$  to an algebraic torus factors as  $X \rightarrow T_X \rightarrow T$ , where the second arrow is a homomorphism of tori.*

Implicit in the proposition is that  $M_X$  is a finite rank lattice, see [33, Theorem 3.2] for a proof of this fact. The embedding of  $X$  in  $T_X$  is given explicitly by the evaluation map

$$p \mapsto (f \mapsto f(p)) \quad .$$

From the universal property it is clear that  $X$  is very affine if and only if it is a closed subvariety of  $T_X$ . The torus  $T_X$  was called the universal torus of  $X$  by Iitaka [22], we follow the language of [45] and refer to it as the intrinsic torus of  $X$ . The universal property is analogous to that of the Albanese variety, and we make later use of this analogy in Chapter 3.1. The action of  $\mathbb{C}^*$  on the group of units  $\mathcal{O}^*(X)$  is a diagonalizable action on a choice of generators, and  $T_X$  is unique up to a choice of this action. This choice has no effect on the structure of the tropical variety, and we will thus often refer to  $T_X$  as a single object. To be precise, we have the following proposition, whose proof is trivial.

**Proposition 1.1.4.** *i) Let  $T_X$  and  $T'_X$  be two intrinsic tori (i.e. corresponding to two different diagonal  $\mathbb{C}^*$  actions on  $\mathcal{O}^*(X)$ ) for a very affine variety  $X$ . Let  $N_X$  and  $N'_X$  be the associated cocharacter lattices and let  $\text{Trop}(X)$  and  $\text{Trop}'(X)$  denote the tropical varieties for the respective embeddings. Then there is an isomorphism of lattices  $\phi : N_X \rightarrow N'_X$ , and under the induced isomorphism  $\bar{\phi} : N_X \otimes \mathbb{R} \rightarrow N'_X \otimes \mathbb{R}$  we have  $\bar{\phi}(\text{Trop}(X)) = \text{Trop}'(X)$ .*

*ii) Let  $X \subset T$  be a closed subvariety of an algebraic torus with  $T_X$  be the intrinsic torus of  $X$ . Let  $\phi : T_X \rightarrow T$  be the injective homomorphism afforded by the universal property of the intrinsic torus, and  $\bar{\phi} : N_X \otimes \mathbb{R} \rightarrow N \otimes \mathbb{R}$  the induced injective map. Let  $\text{Trop}^{N_X}(X)$  and  $\text{Trop}^N(X)$  be the tropical varieties for the respective embeddings. Then  $\bar{\phi}(\text{Trop}^{N_X}(X)) = \text{Trop}^N(X)$ .*

From here on, we will thus refer to the tropical variety of a very affine variety  $X$  as a single object, unique up the above isomorphisms.

Given a very affine variety  $X \subseteq T$ , the main idea of the subject of tropical compactifications is to consider the closures of  $X$  in various toric varieties for  $T$ . The philosophy is that the tropical variety of  $X$  corresponds to toric varieties such that the closure of  $X$  has a sort of transversality at the boundary. To make this precise, we have the following fundamental result of [45].

**Proposition 1.1.5.** *Let  $X \subseteq T$  be very affine and irreducible. Let  $\mathbb{P}_\Sigma$  a toric variety for  $T$ , and  $\bar{X}$  the (Zariski) closure of  $X$  in  $\mathbb{P}_\Sigma$ . Then*

*i)  $\bar{X}$  is proper if and only if  $\text{Trop}(X) \subseteq |\Sigma|$ .*

*ii) For any cone  $\sigma \in \Sigma$ ,  $\bar{X} \cap \text{orb}(\sigma) \neq \emptyset$  if and only if  $\sigma^\circ \cap \text{Trop}(X) \neq \emptyset$  where  $\sigma^\circ$  is the relative interior of  $\sigma$ .*

That is, the above proposition says that  $\text{Trop}(X)$  records which fans have the property that  $\bar{X}$  is proper and meets every torus orbit.

It is worth emphasizing that in condition i) the containment is strictly set theoretic, as it must be since  $\text{Trop}(X)$  does not necessarily come with a preferred fan structure. This point is crucial, so we recall [44, Example 5.2] to illustrate.

*Example 1.1.6.* Using numerical methods, the authors construct a three dimensional very affine variety  $X$  of  $(\mathbb{C}^*)^6$  where  $\text{Trop}(X) \subset \mathbb{R}^6$  is shown to contain two three dimensional simplicial cones  $\sigma_1$  and  $\sigma_2$  such that  $\rho = \sigma_1 \cap \sigma_2$  is ray. If  $\Sigma$  is any fan supported on  $\text{Trop}(X)$ , the ray  $\rho$  is necessarily a cone of  $\Sigma$ . Then there must be two dimensional simplicial cones  $\tau_1 \subset \sigma_1$  and  $\tau_2 \subset \sigma_2$  such that  $\rho$  is a face of each. However, there is no canonical choice of  $\tau_1$  and  $\tau_2$ . For example, one could take  $\tau_1$  to be part of a barycentric subdivision of  $\sigma_1$  and  $\tau_2$  to be part of a subdivision of  $\sigma_2$  into two three-dimensional simplicial cones.

In general there is of course a poset of fan structures supported on  $\text{Trop}(X)$  with poset structure given by refinement, but as the above example shows there is not in general a unique coarsest element.

Perhaps the most striking observation of [45] is that there is a sort of converse to the above proposition: one can always find fans  $\Sigma$  so that the closure  $\overline{X}$  in  $\mathbb{P}_\Sigma$  has desirable behavior, with the meaning of “desirable” to be made precise shortly. From the above proposition, we expect that any such fan should necessarily have  $|\Sigma| = \text{Trop}(X)$ , but there are infinitely many fan structures supported on  $\text{Trop}(X)$  and it is a priori unclear how the corresponding compactifications of  $X$  may differ.

With notation as above, Tevelev considers the multiplication action  $m : X \times T \rightarrow T$ . Let  $\mathbb{P}_\Sigma$  be a toric variety for  $T$  and  $\overline{X}$  the (Zariski) closure of  $X$  in  $\mathbb{P}_\Sigma$ . Since  $\mathbb{P}_\Sigma$  is a toric variety, the action extends to  $\overline{m} : \overline{X} \times T \rightarrow \mathbb{P}_\Sigma$ . The image of  $\overline{m}$  is a union of torus orbits of  $\mathbb{P}_\Sigma$ , thus if  $|\Sigma| \subseteq \text{Trop}(X)$ , then  $\overline{m}$  is surjective by Proposition 1.1.5 ii). The main definition is the following.

**Definition 1.1.7** ([45, Definition 1.1]). Let  $X$  be irreducible and very affine. A fan  $\Sigma$  is a *tropical fan* for  $X$  if  $\overline{m}$  is flat and surjective. We say that  $\Sigma$  is a *schön fan* for  $X$  if  $\overline{m}$  is smooth and surjective.

Note that any schön fan is trivially a tropical fan.

The remaining propositions and theorems of this section summarize the results of [30, 44, 45] relevant for this thesis. The first result is elementary.

**Proposition 1.1.8.** *Let  $X$  be a very affine variety and  $\Sigma$  a tropical fan for  $X$ . Then  $|\Sigma| = \text{Trop}(X)$ . If  $\Sigma'$  is any refinement of  $\Sigma$  and  $\Sigma$  is tropical for  $X$ , then  $\Sigma'$  is also tropical for  $X$ .*

The second statement above allows one to frequently assume without loss of generality that a tropical fan is smooth.

We now state what is certainly the main theorem of this topic to date.

**Theorem 1.1.9.** *Let  $X$  be a very affine variety. Then there exists a fan which is tropical for  $X$ .*

This warrants some discussion. The proof of Theorem 1.1.9 is not terribly difficult, but certainly non-trivial. The published proof in [45] uses Hilbert Scheme techniques and generalizes constructions of Kapranov [24] concerning complements of hyperplane arrangements. There is an as yet unpublished proof by Tevelev which employs an equivariant form of the celebrated Raynaud-Gruson flattening theorem, and we feel it is correct to view the theorem in this light. It is important to note that not every fan structure on  $\text{Trop}(X)$  will be tropical. An obstruction preventing a fan from being tropical is the following result of [45].

**Proposition 1.1.10.** *Let  $X$  be a very affine variety and  $\Sigma$  a tropical fan for  $X$ . Suppose that  $X$  and  $\mathbb{P}_\Sigma$  is smooth. Then*

- i)  $\overline{X}$  has at worst Cohen-Macaulay singularities.*
- ii) The boundary  $\overline{X} \setminus X$  is a reduced divisor on  $\overline{X}$ . If  $D_1, \dots, D_k$  are distinct irreducible components of  $\overline{X} \setminus X$ , then  $D_1 \cap \dots \cap D_k$  is either empty or of pure codimension  $k$ .*

As noted in [44, Example 3.10], condition i) allows one to construct examples of non-tropical fans by taking sufficiently many hyperplane sections through a non Cohen-Macaulay point. See [44] for detail, although we will review a related construction due to [30] below. We are not aware of any other known obstructions to a fan being tropical besides i) and ii). Condition ii) should be viewed as a weakening of the condition that the boundary is a simple normal crossings divisor. Condition ii) can be reformulated as saying that for any cone  $\sigma \in \Sigma$

$$\dim(\text{orb}(\sigma) \cap \overline{X}) = \dim X - \dim \sigma$$

.

Using proposition 1.1.10 we have the following crude indication that the structure of  $\text{Trop}(X)$  does indeed the geometry of  $X$ .

**Corollary 1.1.11.** *Let  $X$  be an irreducible very affine variety of dimension  $n$ . Then  $\text{Trop}(X)$  has real dimension  $n$  as a polyhedral complex.*

**Proof:** It suffices to show that if  $\Sigma$  is any tropical fan supported on  $\text{Trop}(X)$ , then any maximal cone  $\sigma$  of  $\Sigma$  has dimension  $n$ . We have that  $\overline{X} \cap \text{orb}(\sigma) \neq \emptyset$ , and from the above formula the dimension of this variety is  $\dim X - \dim \sigma = n - \dim \sigma$ , so  $\dim \sigma \leq n$ . If we had  $\dim \sigma < n$ , then the boundary  $\overline{X} \setminus X$  is not a divisor, contradicting 1.1.10 ii).  $\square$

We now return to the schön condition introduced in Definition 1.1.7.

**Proposition 1.1.12.** *If  $X$  is smooth admits a schön fan, then for any fan  $\Sigma$  which is tropical for  $X$ , the corresponding compactification  $\overline{X}$  has toroidal singularities.*

2) *If  $X$  has a schön fan, then any smooth fan  $\Sigma$  supported on  $\text{Trop}(X)$  is also schön for  $X$ .*

Statement 1) is elementary to prove and is due to Tevelev [45, Theorem 2.5]. The phrase “toroidal singularities” means that at each point in the boundary  $\overline{X} \setminus X$  the variety is locally (in the analytic topology) isomorphic to a possibly singular toric variety. This in particular implies that there exists a refinement  $\Sigma'$  of  $\Sigma$  such that the boundary is a simple normal crossing divisor. Statement 2) is due to Luxton and Qu [30] and is proved using methods from the theory of toroidal embeddings which we will discuss in detail in Section 2.2. Statement 2) shows that the property of being schön is a fairly stable operation. It should be contrasted with the discussion following Theorem 1.1.9 where we see that this property fails for the condition of being just tropical. Statement 2) also shows that some varieties  $X$  will never admit schön fans. We will see elementary examples in Section 1.3. Accordingly, many of the results of this thesis assume the existence of a schön fan.

## 1.2 Geometric Tropicalization

We now turn our attention to the theory of “geometric tropicalization” which was developed in parallel with the theory of tropical compactifications. It was introduced in [19] and developed further in [44]. This technique in a sense affords a converse to the results of the above section: we have seen how the structure of  $\text{Trop}(X)$  gives information

about tropical compactifications, geometric tropicalization starts with a compactification and shows how to recover  $\text{Trop}(X)$ . This construction forms the foundation for a substantial portion of this thesis.

Let  $X$  be a smooth very affine variety, and let  $\overline{X}$  be any smooth simple normal crossing compactification. We do not necessarily need to assume that  $\overline{X}$  arises as a tropical or schön compactification, although this is often the case of interest. Let  $D = \overline{X} \setminus X$  and let  $D_1, \dots, D_n$  be the irreducible components of  $D$ . To each component  $D_i$  we associate a point  $[D_i] \in N$  by

$$m \mapsto \text{ord}_{D_i}(\chi^m) \quad \text{for any } m \in M$$

where  $\text{ord}_{D_i}(\chi^m)$  is the order of vanishing of the monomial  $\chi^m$  along  $D_i$ . More precisely, any  $m \in M$  defines the rational function  $\chi^m$  on  $T$ , and thus restricts to a rational map on  $X$ , which by abuse of notation is also called  $\chi^m$ . If  $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$  is such that  $D_{i_1} \cap \dots \cap D_{i_k} \neq \emptyset$ , define a cone  $\sigma^I \subset N \otimes \mathbb{R}$  by

$$\sigma^I = \mathbb{R}_{\geq 0}[D_{i_1}] + \dots + \mathbb{R}_{\geq 0}[D_{i_k}] .$$

Now define a fan  $\Sigma_{\overline{X}}$  as the union over all such cones  $\sigma^I$ . It is elementary to prove that this is indeed a fan, although this fact relies strongly on the assumption that the divisor was assumed simple normal crossing. The fascinating theorem, due to Hacking, Keel, Tevelev, and Qu (see [19, Theorem 2.3] and [44, Theorem 2.6] for two different proofs), is the following.

**Theorem 1.2.1.** *With notation as above,  $|\Sigma_{\overline{X}}| = \text{Trop}(X)$ .*

As mentioned above, this in principle allows one to reconstruct the complex  $\text{Trop}(X)$  if one has a reasonable compactification on hand. It is known that the theorem can fail if  $X$  is not smooth. We emphasize the following point which is critical to this thesis: although  $\text{Trop}(X)$  does not have a distinguished fan structure, a choice of compactification  $\overline{X}$  imposes a fan structure on it. It is natural to study what happens as we vary the compactifications, and we address this question in Section 3.3. In the next section we'll discuss examples where this theorem can be seen explicitly.

A simple, but very insightful observation of Luxton and Qu is that we can, in principle, generate vast amounts of examples by realizing the boundary divisors as generic hyperplane sections.

**Proposition 1.2.2** ([30]). *Let  $Y$  be a smooth projective variety and  $L$  a very ample line bundle on  $Y$ . Let  $n = \dim H^0(Y, L)$ . Let  $H_1, \dots, H_{n+2}$  be generic hyperplanes in  $\mathbb{P}H^0(Y, L)$ , and for each  $i = 1, \dots, n+2$  let  $D_i = H_i \cap Y$  (i.e.  $\{D_i\}$  are generic sections of  $L$ ). Set  $Y^\circ = Y \setminus \cup_{i=1}^{n+2} D_i$ . Then  $Y^\circ$  is very affine and  $Y = \overline{Y^\circ}$  has the structure of a schön compactification.*

That is, we can construct a schön compactification from any smooth projective variety with a very ample line bundle. At first glance, this seems problematic and perhaps the notion of a tropical compactification is much too general. In this thesis we offer the

philosophy that although the geometry of the compactifications one can produce through tropical compactifications are totally unconstrained, the data of the pair  $(\overline{X}, D)$  where  $D$  is the boundary divisor of the compactification is highly constrained. See Chapter 3.1 for our supporting results. Although the above proposition furnishes a vast number of examples of schön compactifications in principle, in practice neither the statement nor the proof give insight into the structure of  $\text{Trop}(X)$ . In the next section we study examples where the tropical fan can be computed explicitly.

### 1.3 Examples

*Example 1.3.1.* Perhaps the first class of examples which one can easily understand is the case where the very affine  $X$  is itself an algebraic torus, say of dimension  $n$ . In this case the intrinsic torus of  $X$  is the identity.  $\text{Trop}(X) = N_X \otimes \mathbb{R} \cong \mathbb{R}^n$  is just a vector space. A tropical compactification is just a complete toric variety for  $X$ , and accordingly corresponds to a choice of complete fan on  $N_X \otimes \mathbb{R}$ . Any such fan is automatically schön. Once we fix a complete fan  $\Sigma$ , the process of geometric tropicalization in Theorem 1.2.1 is tautological: given a divisor  $D_\rho$  corresponding to a ray with primitive generator  $\rho$ , we can compute  $[D_\rho] = \rho$  in the notation of 1.2.1 by the following elementary result from toric geometry [13, Chapter 3.3].

*Proposition 1.3.2.* *If  $\Sigma$  is any fan, let  $\rho$  a primitive generator for a ray of  $\Sigma$  and let  $D_\rho$  denote the corresponding divisor on  $\mathbb{P}_\Sigma$ . Then for any  $m \in M$ , one has the following equation for the order of vanishing of the character  $\chi^m$  along  $D_\rho$ :*

$$\text{ord}_{D_\rho}(\text{div}(\chi^m)) = \langle m, \rho \rangle .$$

Thus  $[D_\rho]$  is the point of  $N$  given by  $m \mapsto \langle m, \rho \rangle$  so clearly  $[D_\rho] = \rho$ . So the rays of  $\Sigma$  coincide exactly with the rays of  $\mathcal{F}_{\mathbb{P}_\Sigma}$  from Theorem 1.2.1. Indeed, by [13, Chapter 5.1] the divisors  $D_\rho$  and  $D_{\rho'}$  on a toric variety intersect non-trivially if and only if  $\rho$  and  $\rho'$  both belong to a cone of  $\Sigma$ . It follows that actually  $\Sigma = \mathcal{F}_{\mathbb{P}_\Sigma}$ , so the process of geometric tropicalization exactly recovers the fan.

*Example 1.3.3.* One of the most well-understood class of examples in the tropical literature is the case where  $X = Z_f$  is a hypersurface. Let

$$f(z_1, \dots, z_n) = \sum_{\alpha \in \Delta \cap M} c_\alpha z^\alpha \in \mathbb{C}[M]$$

be a Laurent polynomial. Here  $\Delta$  is a polytope in  $M \otimes \mathbb{R}$ . We assume that  $f$  does consist only of a single monomial, so that  $Z_f$  indeed defines a divisor on the torus  $T$ . The convex hull of the set of lattice points corresponding to exponents of  $f$  with non-zero coefficient is called the Newton Polytope of  $f$ , and we denote it  $\text{Newt}(f)$ . [8] proves the following result:

*Proposition 1.3.4.* *With notation as above, let  $\Sigma^f \subseteq N \otimes \mathbb{R}$  denote the normal fan of  $\text{Newt}(f)$ , and let  $\Sigma^f(n-1)$  denote its codimension one skeleton. Then  $\text{Trop}(Z_f) = |\Sigma^f(n-1)|$ .*

That is, the tropical fan is dual to the Newton Polytope. Note that in this class of examples there is a canonical coarsest fan structure on  $\text{Trop}(X)$ , namely the structure inherited from the normal fan. The schön condition had already been explored in this context long before the advent of tropical compactifications, and the terminology used was that the hypersurface is “non-degenerate with respect to its Newton Polytope”, see [16, Chapter 6]. [16] shows that this property is open in the space of coefficients. If the hypersurface is degenerate with respect to its Newton Polytope, i.e. does not admit a schön compactification, then I do not know of any general results explaining which fans supported on  $\text{Trop}(Z_f)$  are tropical.

The geometric tropicalization theorem is also very explicit for hypersurfaces. For simplicity assume  $\text{Newt}(f)$  has dimension equal to  $\dim(Z_f)$ . Let  $\Gamma$  be a non-vertex face of the Newton Polytope of  $f$  and let  $\sigma_\Gamma$  be the corresponding dual cone. By Proposition 1.1.10,  $\overline{Z_f} \cap \text{orb}(\sigma)$  is also a hypersurface in  $\text{orb}(\sigma)$ , and [16] show that that it is given by

$$f^\Gamma(z_1, \dots, z_n) = \sum_{\alpha \in \Gamma} c_\alpha z^\alpha \quad ,$$

i.e. the truncation of  $f$  to  $\Gamma$ . Thus the boundary divisors of  $\overline{Z_f}$  are given explicitly by  $\overline{Z_{f\Gamma}}$  as  $\Gamma$  ranges over the facets of  $\text{Newt}(f)$ . Given  $m \in M$ , it is easy to check that the order of vanishing of  $\text{div}(\chi^m)$  along  $\overline{Z_{f\Gamma}}$  is 0 if  $m \notin \Gamma$  and is greater than zero if  $m \in \Gamma$ . So, by the construction before Theorem 1.2.1, the associated point in  $N \times \mathbb{R}$  is the dual ray to  $\Gamma$ , as expected by the above proposition (indeed, this discussion actually proves the above proposition).

The above discussion fully generalizes to complete intersections, but we omit the details for the sake of brevity.

## Chapter 2

# Related Constructions

In this chapter we compare the theory of tropical compactifications with several more classical constructions. The identifications we make are mostly straightforward, but seem to have not been explicitly discussed in the literature. In several instances we'll see that making such connections is profitable and allows us to prove some easy results which are absent from the tropical literature.

### 2.1 Maps to Toric Varieties and Cox's $\Delta$ -Collections

Let  $\mathbb{P}_\Sigma$  be a toric variety with dense torus  $T$ . We discuss here a general construction which assigns some combinatorial data to a given morphism  $X \rightarrow \mathbb{P}_\Sigma$  where  $X$  is an arbitrary normal, irreducible complex algebraic variety (or integral scheme). The following theorem was first proved in unpublished work of Oda and Sankaran (see [41] for a discussion) and extended slightly in works of Kajiwara [23] and Cox [4].

**Theorem 2.1.1** (Kajiwara, Oda-Sankaran). *Let  $\mathbb{P}_\Sigma$  be a smooth (not necessarily complete) toric variety with big torus  $T$  and character lattice  $M$  and let  $X$  be any integral scheme with a morphism  $f : X \rightarrow \mathbb{P}_\Sigma$  such that  $f(X) \cap T \neq \emptyset$ . Then  $f$  uniquely determines the data of a homomorphism*

$$\phi_f : M \rightarrow \mathbb{C}(X)^*$$

and a collection  $\Sigma^f(1) \subseteq \Sigma(1)$  of rays of  $\Sigma$  such that

i) *For any distinct collection  $\rho_{i_1}, \dots, \rho_{i_k} \in \Sigma^f(1)$  of rays such that  $\{\rho_{i_1}, \dots, \rho_{i_k}\}$  do not belong to a cone of  $\Sigma$ , the intersection  $f^{-1}(D_{\rho_{i_1}}) \cap \dots \cap f^{-1}(D_{\rho_{i_k}})$  of pull-back divisors on  $Y$  is empty.*

ii) *For any  $m \in M$  one has*

$$\operatorname{div}(\phi^f(m)) = \sum_{\rho \in \Sigma^f(1)} \langle m, \rho \rangle f^{-1}(D_\rho)$$

as Weil divisors on  $Y$ . Moreover, the morphism  $f : X \rightarrow \mathbb{P}_\Sigma$  can be reconstructed from the pair  $(\phi^f, \Sigma^f(1))$



[23] proves an extension of the theorem to the case where  $\Sigma$  is only simplicial, and shows that the data of the pair  $(\phi^f, \Sigma^f(1))$  relates to Cox's  $\Delta$ -collections as defined in [4]. We will not reproduce the full proof of the above theorem here. The non-trivial part of the theorem is the statement that the morphism  $f$  can be reconstructed uniquely from  $(\phi^f, \Sigma^f(1))$ . We are primarily concerned here with the easier implication, namely the construction of  $\phi^f$  and  $\Sigma^f(1)$  from the morphism  $f$ , which we now recall. The collection of cones  $\Sigma^f(1)$  is simply the collection of rays  $\rho \in \Sigma(1)$  such that the corresponding pull-back divisor  $f^{-1}(D_\rho)$  is non-zero. It is easy to check that this collection obeys condition i). The group homomorphism  $\phi^f$  is obtained by restricting  $f$  to  $X_0 = f^{-1}(T)$  and noting that  $f|_{X_0} : X_0 \rightarrow T$  induces a pull-back on coordinate rings  $f^* : \mathbb{C}[M] \rightarrow \mathbb{C}[X_0] \subseteq \mathbb{C}(X)$  which restricts to a group homomorphism  $\phi^f : M \rightarrow \mathbb{C}(X)^*$ . Condition ii) follows readily from the more general standard fact that if  $f : X \rightarrow Y$  is a morphism with  $Y$  smooth and irreducible and  $X$  integral, and  $D = \sum D_i$  is a Weil divisor on  $Y$  whose support does not contain the image of  $f(X)$ , then  $\text{div}(f^*D) = \sum a_i \text{div} f^*(D_i)$ .

It will be useful to extend the collection of cones  $\Sigma^f(1)$  to a sub-fan of  $\Sigma$ .

**Lemma 2.1.2.** *Let  $f : X \rightarrow \mathbb{P}_\Sigma$  and  $\Sigma^f(1)$  be as in the above theorem. Then  $\Sigma^f(1)$  extends uniquely to a smooth sub-fan of  $\Sigma$  by requiring that a  $k$ -dimensional cone  $\sigma \in \Sigma$  generated by  $\rho_{i_1}, \dots, \rho_{i_k}$  is a cone of  $\Sigma^f$  if and only if  $f^{-1}(D_{\rho_{i_1}}) \cap \dots \cap f^{-1}(D_{\rho_{i_k}}) \neq \emptyset$ . We denote this fan by  $\Sigma^f$ .*

**Proof:** By construction  $\Sigma^f$  is a union of cones, so we need only show that every face of a cone of  $\Sigma^f$  is a cone, and that the intersection of two cones of  $\Sigma$  is either empty or itself a cone. But these assertions are trivial as if a collection of pull-back divisors  $f^{-1}(D_{\rho_i})$  intersects non-trivially, than so does any proper sub-collection.  $\square$ .

Theorem 2.1.1 was used in [4] and [23] to study equivariant between toric varieties, and in particular maps to projective space, and was used in [41] to study curves and abelian varieties embedded in toric varieties. Here, we consider the case where  $f$  is an inclusion of a tropical compactification into the toric variety of a corresponding tropical fan.

**Corollary 2.1.3.** *Let  $X \subset T$  be a closed subvariety and  $\Sigma$  a tropical fan for  $X$ , and let  $\iota : \overline{X} \hookrightarrow \mathbb{P}_\Sigma$  be the corresponding tropical compactification. Then with notation as above,  $\Sigma^\iota = \Sigma$  (and so  $|\Sigma^\iota| = \text{Trop}(X)$ ) and  $\phi^\iota$  is an injection.*

**Proof:** By construction, if  $\sigma \in \Sigma^\iota$  is any cone, then the intersection  $\text{orb}(\sigma) \cap \overline{X}$  is non-empty. Condition i) ensures further that if  $\tau \in \Sigma \setminus \Sigma^\iota$  then  $\text{orb}(\tau) \cap \overline{X}$  is empty, so Proposition 1.1.10 gives that  $|\Sigma^\iota| = \text{Trop}(X)$ . Since  $\Sigma^\iota$  is a sub-fan of  $\Sigma$  and  $\Sigma$  is assumed tropical, it follows that  $\Sigma^f = \Sigma$ . Lastly, note that  $\phi^\iota$  coincides with the injection from Proposition 1.1.3.  $\square$

## 2.1.1 Applications to Balancing

We note here that Theorem 2.1.1 can be used to independently recover the well-studied ‘‘balancing condition’’ of tropical fans. We briefly recall the notion of balancing for tropical

varieties, following [42]. This condition is also often called the zero-tension condition in the literature. Before stating the definition, we point out that the balancing condition will be a sort of non-trivial affine constraint on the structure of a fan. In particular, it shows that not every fan can arise as the tropicalization of some variety. This problem of determining whether a given fan occurs as the tropicalization of a variety is called the “tropical realization problem”, and is still very much open. Over the field of Puiseux Series, even the case of curves is difficult; see the work of Speyer [42] for small genus curves.

Let  $\Pi$  be any connected, polyhedral complex of pure dimension  $d$  inside a vector space  $\mathbb{R}^n$ , which we assume to be rational with respect to an integral structure  $\mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R}$ . If  $\pi \in \Pi$  is any cell of dimension  $d - 1$ , let  $\pi_1, \dots, \pi_k$  be the  $d$  dimensional cells with  $\pi$  as a facet. For each  $i$ , the cell  $\pi_i$  determines a ray in  $\mathbb{R}^n/\text{Span}(\pi_i)$ , and let  $v_i$  denote its primitive generator.

**Definition 2.1.4.** With notation as above, a function

$$m_\pi : \{1, \dots, k\} \rightarrow \mathbb{Z}_{>0}$$

is defined to be a *balancing function* for the cell  $\pi$  if

$$\sum_i^k m_\pi v_i = 0 .$$

If  $\Pi(d)$  denotes the collection of  $d$  dimensional cells of  $\Pi$ . We say that the complex  $\Pi$  is *balanced* if there exists a function

$$m_\Pi : \Pi(d) \rightarrow \mathbb{Z}_{>0}$$

which restricts to a balancing function at each  $d - 1$  cell of  $\Pi$ .

We first discuss this definition in the context of toric geometry. Recall (see, e.g. [5]) that if a fan  $\Sigma$  is simplicial, and  $\sigma \in \Sigma$  is any cone with primitive generators  $\rho_1, \dots, \rho_k$  for its generating rays, one defines

$$\text{mult}(\sigma) = [N_\sigma : \mathbb{Z}\rho_1 \oplus \dots \oplus \mathbb{Z}\rho_k]$$

where  $N_\sigma = \text{Span}(\sigma) \cap N$ . It is well-known that a fan  $\Sigma$  is smooth if and only if  $\text{mult}(\sigma) = 1$  for every cone  $\sigma \in \Sigma$  if and only if  $\text{mult}(\sigma) = 1$  for every top dimensional cone  $\sigma$ . The following result is almost obvious, but we cannot find it explicitly stated in the literature.

**Proposition 2.1.5.** *Let  $\Sigma$  be any complete simplicial fan. Then  $\Sigma$  is balanced, with balancing functions given by  $m_\Sigma(\sigma) = \text{mult}(\sigma)$  where  $\sigma$  is any top dimensional cone.*

**Proof:** Since  $\Sigma$  is simplicial there exist exactly two  $d$  dimensional cones  $\sigma_1$  and  $\sigma_2$  containing  $\tau$  as a face. Since  $\Sigma$  is complete,  $\mathbb{R}^n/\text{Span}(\tau)$  is one dimensional and the images of  $\sigma_1$  and  $\sigma_2$  span rays pointing in opposite directions. Split the lattice  $N$  as  $N = \mathbb{Z}e \oplus N_\tau$ . For each  $i = 1, 2$  there is a unique ray  $v_i$  with primitive generator  $\rho_i$  such

that  $v_i \in \sigma_i$  and  $v_i \notin \tau$ . Write  $\rho_i = a_i e + w_\tau$  uniquely where  $w_\tau \in N_\tau$  and  $a_i \in \mathbb{Z} \setminus \{0\}$ . Then  $|a_i| = \text{mult}(\sigma_i)$  and the claim follows.  $\square$

We now discuss the balancing condition for tropical fans. The main result of this section is to note that Proposition 2.1.1 allows one to unambiguously generalize proofs of the balancing condition for tropical curves given by Speyer[Section 2.5][42] and Nishinou-Siebert [35] to higher dimensions. For the sake of being self-contained, we briefly recall their arguments here.

Note first that the curve case is, in a sense, quite different from the situation in higher dimensions: if a tropical fan  $\Sigma$  is a union of rays, the unique codimension one cell is the origin, and the balancing condition becomes a linear integral constraint on the rays. In this situation, if  $\Sigma = \text{Trop}(C)$  is a tropical fan for a very affine curve  $C$ , each ray  $\rho$  determines a zero-dimensional scheme  $\overline{C} \cap \text{orb}(\rho)$ . Since the compactification is assumed tropical, this scheme is reduced, and we define  $\text{mult}(\rho)$  to be the number of points in this intersection. Now, let  $m \in M$  be any monomial, and  $\chi^m$  the associated monomial rational function on  $\mathbb{P}_\Sigma$ . By Proposition 1.3.1 the order of vanishing of  $\chi^m$  along the point  $D_\rho \in \mathbb{P}_\Sigma$  is  $\langle m, \rho \rangle$ . Let  $\iota : \overline{C} \rightarrow \mathbb{P}_\Sigma$  be the inclusion. Then the order of vanishing of  $\iota^*(\chi^m)$  is  $\text{mult}(\rho)\langle m, \rho \rangle$ . Taking the degree of the expression ii) in Proposition 1.1.10 then gives

$$0 = \sum_{\rho \in \Sigma} \text{mult}(\rho)\langle m, \rho \rangle$$

Since  $m \in M$  was arbitrary, non-degeneracy of the dual pairing  $M \times N \rightarrow \mathbb{Z}$  forces

$$\sum_{\rho \in \Sigma} \text{mult}(\rho)\rho = 0$$

which is the desired balancing condition for a one dimensional tropical fan. Speyer [42] notes that this argument easily extends to give an analogous balancing condition on any tropical curve corresponding to a family of curves defined over  $K = \mathbb{C}((t))$ , and Nishinou-Siebert [35] note that the argument extends to the case where  $\phi : \overline{C} \rightarrow \mathbb{P}_\Sigma$  is not just an inclusion, but an appropriately transverse stable map. We now note that all aspects of this argument generalize to higher dimensions by virtue of Proposition 1.1.10, so we obtain the desired result.

**Proposition 2.1.6** ([44, Corollary 3.4]). *If  $\Sigma$  is a tropical fan for some irreducible variety  $X \subset T$ , then for cone  $\sigma \in \Sigma$  of dimension equal to  $\dim X$ , define  $\text{mult}(\sigma)$  to be the number of points in the reduced zero dimensional scheme  $\text{orb}(\sigma) \cap \overline{X}$ . Then the assignment  $\sigma \mapsto \text{mult}(\sigma)$  is a balancing function for  $\Sigma$ .*

We emphasize that this result is well-known in the tropical literature, see in addition [25]. These proofs, though, rely on interpretations of the balancing condition in terms of intersection theory on toric varieties, and in particular the calculations of the Chow groups of complete toric varieties developed in [14]. We thus find it comforting that the balancing condition can be recovered in all dimensions using only simple arguments about divisors and adapting the pleasant proofs of Speyer and Nishinou-Siebert for curves. We also note that a similar technique to ours was employed in [20] in a slightly different context.

## 2.2 Toroidal Embeddings

In this section we recall the notion of a toroidal embedding introduced by Kempf, Knudsen, Mumford, and Saint-Donat [28]. This notion generalizes the construction of a toric variety to that of a variety which “locally looks like a toric variety”. Our goal is to relate their construction to schön compactifications and refine some observations of Luxton-Qu [30]. Our main result is to identify certain toroidal embeddings as schön compactifications.

### 2.2.1 Definitions

We recall the main definitions of [28, Chapter 2].

**Definition 2.2.1.** If  $Y$  is a normal variety and  $U \subset Y$  a Zariski dense subset,  $U \subset Y$  is a toroidal embedding if it is everywhere locally (in the analytic topology) formally isomorphic to a normal toric variety.

It follows from the definition that the boundary  $Y \setminus U$  is divisorial with the irreducible components each reduced. From here on we assume that the irreducible components of the boundary are normal<sup>1</sup> and denote them  $\{D_i \mid i \in I\}$ . [28] constructs a stratification of  $Y$  as follows: for any  $J \subseteq I$  such that  $\bigcap_{i \in J} D_i \neq \emptyset$  put

$$\mathcal{O}_J = \bigcap_{i \in J} D_i \setminus \bigcup_{i \notin J} D_i$$

[28] proves that  $\mathcal{O}_J$  is always smooth.

*Remark 2.2.2.* The notion for strata is troublesome as below we will often need to make reference to things such as the ring of regular functions on a stratum, which would, unfortunately, be  $\mathcal{O}(\mathcal{O}^J)$ . Unfortunately this notion is standard, even in the literature on toric varieties [13]. Other texts on toric varieties, such as [36] refer to a stratum corresponding to a cone  $\sigma$  as  $\text{orb}(\sigma)$ , the convention we have used in this thesis. This notation, of course, relies on the identification of the strata of a toric variety with torus orbits. There is no such global description of strata for a general toroidal embedding, so we fall back to the notation  $\mathcal{O}^J$ , with attempts to adjust notation as necessary to prevent confusion.

*Example 2.2.3.* Let  $Y$  be any smooth projective variety with  $D$  a simple normal crossing divisor with irreducible components  $D_1, \dots, D_k$ . Set  $U = Y \setminus D$ . Then  $U \subset Y$  has the structure of a toroidal embedding.

*Example 2.2.4.* For the case of a normal toric variety this notion corresponds exactly with the stratification by torus orbits. Here  $U = T \cong (\mathbb{C}^*)^n$  is the dense torus,  $I = \Sigma(1)$  indexes the toric boundary divisors, and for any  $J \subseteq \Sigma(1)$  as above  $\mathcal{O}_J$  is the corresponding disjoint union of open torus orbits.

The closures of strata in a toroidal embedding behave analogously to the well-known formulas for orbit closures in toric varieties [13]. To each non-empty stratum  $\mathcal{O}_J$  of a

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<sup>1</sup>“without self-intersection” in the terminology of [28]

toroidal embedding  $U \subset Y$  one defines

$$\text{Star}(\mathcal{O}_J) = \bigcup_{J': \mathcal{O}_J \subset \overline{\mathcal{O}_{J'}}} \mathcal{O}_{J'}$$

which is by construction an open set on  $Y$ ; in the case of  $Y$  a toric variety this coincides with the covering by invariant affine open sets. Now fix  $S = \mathcal{O}_J$  a non-empty stratum. One defines the following:

$$M^S = \{D \in \text{C-Div}(\text{Star}(S)) \mid \text{supp } D \subseteq \text{Star}(S) \setminus U\} \quad M_+^S = \{D \in M^S \mid D \text{ effective}\}$$

$$\sigma^S = \{\rho \in N_{\mathbb{R}}^S \mid \langle D, \rho \rangle \geq 0, \forall D \in M_+^S\} \subseteq N_{\mathbb{R}}^S$$

where  $N^S$  denotes the dual lattice of  $M^S$  as usual (it is elementary to check that  $M^S$  is always a free abelian group of finite rank and that  $\sigma^S$  is always a strongly convex rational polyhedral cone), and  $\text{C-Div}(\text{Star}(S))$  is the group of Cartier Divisors on  $\text{Star}(S)$ . In the case of  $Y$  a normal toric variety these cones are the usual cones of toric geometry. One feature of this construction is that it allows one to reproduce the fan of a toric variety without any explicit reference to the torus action: we start only from the data of the boundary divisors. To summarize, given any stratum of a toroidal embedding, we can produce a cone in a vector space where the vector space is equipped with a canonical lattice. An important point is that these lattices and vector spaces can be glued together into a generalized polyhedral complex; we use here the equivalent but more economic definition due to Payne [39].

**Definition 2.2.5.** A *conical polyhedral complex* is a triple  $\Pi = (|\Pi|, \{\sigma_\alpha\}, \{M_\alpha\})$  consisting of a topological space  $|\Pi|$  together with a finite collection of closed subsets  $\sigma_\alpha \subseteq |\Pi|$  equipped with  $M_\alpha$  a finitely generated group (lattice) of real-valued continuous function on  $\sigma_\alpha$  obeying:

1) For each index  $\alpha$ , the evaluation map  $\sigma_\alpha \rightarrow M_\alpha^\vee \otimes_{\mathbb{Z}} \mathbb{R}$  given by  $x \mapsto (f \mapsto f(x))$  maps  $\sigma_\alpha$  homeomorphically to a rational polyhedral cone.

2) For each index  $\alpha$ , each face of the cone from 1) maps homeomorphically to some  $\sigma_{\alpha'}$ , and  $V_{\alpha'}$  is exactly the restriction of elements in  $V_\alpha$  to  $\sigma_{\alpha'}$

3)  $|\Pi|$  is the disjoint union of the relative interiors of all  $\sigma_\alpha$ .

By abuse, we will use  $\sigma_\alpha$  to refer both to a cell  $\sigma_\alpha \subset |\Pi|$  and its homeomorphic polyhedral image in  $M_\alpha^\vee \otimes \mathbb{R}$ . The point of 2) is that if  $\sigma_\alpha \cap \sigma_\beta \neq \emptyset$ , then the natural restriction maps induce a boundary gluing for the associated convex polyhedral cones. The major theorem is:

**Proposition 2.2.6** ([28]). *If  $U \subset Y$  is any toroidal embedding with all irreducible boundary components normal, and taking  $\alpha$  as above to index all strata  $S$ , then the triple*

$$\left( \bigsqcup_S \sigma^S, \{\sigma^S\}, \{M^S\} \right)$$

*is a conical polyhedral complex.*

From here on we refer to the collection of lattices  $\{M^S\}$  as the integral structure of a toroidal embedding. It is worth emphasizing that the complex of cones  $|\coprod \sigma^S|$  does not necessarily come equipped with an embedding into a vector space, and in particular the lattices  $\{M^S\}$  do not necessarily arise as quotients of a single fixed lattice. See [39] for some examples of toroidal embeddings where the associated conical polyhedral complex does not embed in any vector space. However, for the case of a toric variety the fan  $\Sigma$  lies in  $N \otimes \mathbb{R}$  and the lattices  $M^\sigma$  arise as the quotients of  $M$  by the relative lattice spanned by  $\sigma^\vee$ .

## 2.2.2 Characterizations of Toric Varieties and Schön Compactifications

As noted above, it is essentially tautological that any normal toric variety is a toroidal embedding and that the toric fan coincides with the associated conical polyhedral complex. We first prove a converse that if a conical polyhedral complex is globally embedded in a vector space such that the integral structure is compatible with a lattice in the vector space, and such that the dimension of the toroidal embedding equals the dimension of the vector space, then the toroidal embedding is automatically a toric variety. We suspect that this proposition is well-known, but are unable to locate it in the literature.

**Proposition 2.2.7.** *Let  $U \subset X$  be a toroidal embedding with conical polyhedral complex  $(|\coprod_S \sigma^S|, \{\sigma^S\}, \{M^S\})$ . Suppose that the dual lattices  $\{N^S\}$  of the associated polyhedral complex form a poset under inclusion in the following sense: if two cones  $\sigma^S$  and  $\sigma^{S'}$  of  $|\Pi|$  intersect along a face  $\tau$ , then  $N^\tau \subset N^S$  and  $N^\tau \subset N^{S'}$  and suppose there is a lattice  $N$  containing every  $N^S$  as a sub-lattice respecting these inclusions. Suppose also that  $\text{rank } N = \dim U$ . Then  $X$  is a toric variety and  $U$  is the big torus for  $X$ .*

**Proof:** Since the inclusions  $\sigma^S \subset V = N \otimes \mathbb{R}$  are compatible with the restriction maps, the collection of all  $\sigma^S$  is forced to be a fan, which we denote  $\Sigma$ . From [28] we know the cones of this fan are in bijection with the strata of  $X$ , so we may label that strata by the cones of  $\Sigma$ , and we tentatively denote them by  $\mathcal{O}_\sigma$ . It remains only to show that these strata are algebraic tori, then the fan structure will give the usual toric gluing construction. Now, note that by hypothesis we have a single lattice  $N$  such that for each stratum  $\mathcal{O}_\sigma$  the corresponding dual lattice  $M_\sigma^\vee = N_\sigma$  is a sub-lattice of  $N$  of rank equal to  $\dim \sigma$ . For a stratum  $Y = \mathcal{O}_\sigma$  there is a canonical injection  $\mathcal{O}^*(Y) \hookrightarrow M_\sigma$  by mapping a unit  $f$  to its associated principal divisor, and so

$$\text{rank}(\mathcal{O}^*(Y)) \leq \text{rank } N - \dim \sigma = \dim Y$$

But we must have  $\text{rank}(\mathcal{O}^*(Y)) \geq \dim Y$  as any ray of  $\sigma$  determines a boundary divisor on the closure  $\overline{Y}$  which is supported on  $\overline{Y} \setminus Y$ . Thus  $\text{rank } \mathcal{O}^*(Y) = \dim Y$ , and it follows that  $Y$  is an algebraic torus.  $\square$

We now turn our attention to studying schön compactifications. To guide intuition, we first consider the case of a non-degenerate hypersurface.

*Example 2.2.8.* Let  $f \in \mathbb{C}[z_1^\pm, \dots, z_n^\pm]$  be a non-degenerate Laurent Polynomial. We recall some of the observations from Chapter 1.3. For simplicity assume that its Newton Polytope  $\text{Newt}(f) \subset M \otimes \mathbb{R}$  is smooth and has dimension  $n$ . If  $\Sigma$  denotes the  $n - 1$  skeleton of its normal fan, then  $\overline{Z_f} \subset \mathbb{P}_\Sigma$  is a schön compactification. A ray of  $\Sigma$  uniquely corresponds to a facet  $\Gamma$  of  $\text{Newt}(f)$ , and the boundary divisors of this compactification are given by the hypersurfaces  $\overline{Z_{f,\Gamma}}$  where  $Z_{f,\Gamma}$  denotes the truncation of the polynomial  $f$  to those monomials appearing in the facet  $\Gamma$ . Similarly, the stratification of  $\overline{Z_f}$  is given by the very affine hypersurfaces  $Z_{f,F}$  as  $F$  ranges over the facets of  $\text{Newt}(f)$  of dimension greater than 1 (i.e. non-vertices). Since  $\text{Newt}(f)$  was assumed smooth the affine variety  $\text{Star}(F)$  corresponding to a non-vertex facet is an affine space dimension  $n - \dim F$ . Accordingly  $Z_{f,F}$  defines a smooth affine hypersurface in  $\text{Star}(F)$  (after clearing denominators to remove negative exponents).

Realizing  $Z_f \subset \overline{Z_f}$  as a toroidal embedding, it is clear that the associated conical polyhedral complex  $|\Pi|$  is exactly  $\Sigma = \text{Trop}(Z_f)$  with an imposed fan structure imposed by viewing it as a sub-skeleton of the normal fan. For a stratum corresponding to a non-vertex facet  $F$  we have the corresponding lattice:

$$M^F = \{D \in \text{C-Div}(\mathbb{C}^{n-\dim F}) \mid \text{supp}(D) \subseteq \mathbb{C}^{n-\dim F} \setminus Z_{f,F}\}$$

where we have identified  $\text{Star}(F) \cong \mathbb{C}^{n-\dim F}$ . Any generic hypersurface with a monomial in  $F$  will meet  $Z_f$  by Bezout-Bernstein-Kouchnirenko's theorem [16, Chapter 6], and it follows easily that  $M^F$  is canonically isomorphic to  $M/\text{Span}(F)$ . That is, the integral structure of the toroidal embedding  $Z_f \subset \overline{Z_f}$  is given by taking the lattice of monomials  $M$ , and forming quotient lattices indexed by the non-vertex facets of  $\text{Newt}(f)$ .

The above discussion generalizes to arbitrary schön compactifications. In particular, the argument does not change if one relaxes the condition that the tropical fan  $\Sigma$  have a smooth fan structure, although the corresponding affine varieties  $\text{Star}(\sigma)$  for  $\sigma$  a cone of a schön fan  $\Sigma$  are possibly singular affine toric varieties. The only additional subtlety is in the integral structure. Note that in the above example of a non-degenerate hypersurface it is perhaps easy to identify the integral structure in the dual  $N$  lattice: the lattices  $M^F = M/\text{Span}(F)$  correspond to the dual inclusions  $N^{\sigma_F} \subseteq N$  where  $\sigma_F$  is the unique cone of the normal fan dual to a non-vertex face  $F$  of the  $\text{Newt}(f)$ . The use of Bezout-Bernstein-Kouchnirenko's theorem to establish the isomorphism  $M^F = M/\text{Span}(F)$  can be replaced by Proposition 1.1.10. We thus obtain:

**Proposition 2.2.9.** *Let  $X \subset \overline{X} \subset \mathbb{P}_\Sigma$  be a schön compactification. For each  $\sigma \in \Sigma$  let  $M^\sigma$  the quotient lattice of  $M$  dual to the inclusion  $\text{Span}(\sigma) \subset N$ . Then  $X \subset \overline{X}$  is a toroidal embedding with associated conical polyhedral complex  $(\text{Trop}(X), \{\sigma\}_{\sigma \in \Sigma}, \{M^\sigma\}_{\sigma \in \Sigma})$ .*

The above proposition allows one to carry over non-trivial results from [28] to schön compactifications. In particular, it allows one to rigorously prove some natural statements about the behavior of schön compactifications under refinements of the tropical fan. In particular, applying the difficult result [28, Chapter 2, Theorem 9\*] gives the following.

**Corollary 2.2.10.** *Let  $X \subset \overline{X} \subset \mathbb{P}_\Sigma$  be a schön compactification. Let  $\Sigma'$  be any fan refining  $\Sigma$ , and let  $\overline{X}'$  denote the closure of  $X$  in  $\mathbb{P}'_\Sigma$ . Then  $\overline{X}'$  is obtained from  $\overline{X}$  by*

the blow-up of a fractional sheaf of ideals on  $X$ . If further  $\mathbb{P}'_\Sigma$  is obtained from  $\mathbb{P}_\Sigma$  via an equivariant toric blow up along the smooth toric subvariety  $Y \subset \mathbb{P}_\Sigma$ , then  $\overline{X}'$  is the blow-up of  $\overline{X}$  along  $Y \cap \overline{X}$ .

We conclude by generalizing theorem Proposition 2.2.7 to schön compactification. A crucial step was proved by Luxton and Qu [30]; we now introduce some ideas to state their result. Recall from the previous section that if  $X \subset T \cong (\mathbb{C}^*)^n$  is very affine, then the inclusion induces a surjection on coordinate rings

$$\mathbb{C}[M] \twoheadrightarrow \mathcal{O}(X)$$

where  $M$  is the character lattice for  $T$ . If  $\overline{X} \subseteq \mathbb{P}_\Sigma$  is a schön compactification with  $\Sigma$  a smooth fan, then for any  $0 \neq \sigma \in \Sigma$  the corresponding boundary stratum  $\overline{X} \cap \text{orb}(\sigma)$  is also a very affine subset of  $\text{orb}(\sigma) \cong (\mathbb{C}^*)^k$ , and so we get a corresponding surjection

$$\mathbb{C}[M^\sigma] \twoheadrightarrow \mathcal{O}(\overline{X} \cap \text{orb}(\sigma))$$

If a cone  $\sigma$  is a face of a cone  $\tau$  then there is a natural surjection  $M^\sigma \twoheadrightarrow M^\tau$  and we have the obvious commutative diagram

$$\begin{array}{ccc} \mathbb{C}[M^\sigma] & \longrightarrow & \mathcal{O}(\overline{X} \cap \text{orb}(\sigma)) \\ \downarrow & & \downarrow \\ \mathbb{C}[M^\tau] & \longrightarrow & \mathcal{O}(\overline{X} \cap \text{orb}(\tau)) \end{array} \tag{2.2.1}$$

We will also make use of the following technical lemma.

**Lemma 2.2.11.** *Let  $X \subset \overline{X} \subset \mathbb{P}_\Sigma$  be a schön compactification and let  $\phi : \mathbb{C}[M] \rightarrow \mathcal{O}(X)$  be the induced surjection. Let  $\sigma \in \Sigma$  be a cone, and let  $\rho_1, \dots, \rho_k \in \Sigma$  denote its rays, and let  $D_1, \dots, D_k$  denote the corresponding boundary divisors of  $\overline{X}$ . Then for each  $i \in \{1, \dots, k\}$  there exists an  $m_i \in M$  such that*

$$\text{val}_{D_j} m_i = \delta_{i,j}$$

for each  $j \in \{1, \dots, k\}$ .

**Proof:** From Theorem 2.1.1 we know that  $\text{Trop}(X) \subseteq N_\mathbb{Q}$  can be computed from the collection of all  $[\text{val}_D]$  where  $D$  ranges over the irreducible boundary divisors of  $\overline{X}$ . For fixed  $i \in \{1, \dots, k\}$  and in the statement of the lemma, consider the ray spanned by  $[\text{val}_{D_i}] \in \text{Trop}(X)$ . Now let  $m_i \in M$  be the unique unit vector determined by the projection  $N_\mathbb{Q} \rightarrow N_\mathbb{Q}/\langle [\text{val}_{D_i}] \rangle$ . Then  $m_i$  has the desired properties.  $\square$

We are now ready to state the theorem of Luxton and Qu:

**Theorem 2.2.12** ([30]). *Let  $U \subset Y$  be any toroidal embedding. Suppose that  $U \hookrightarrow T$  where  $T$  has character lattice  $M$ . Suppose also that for each non-empty stratum  $\mathcal{O}^J$  the following hold:*



1) For any stratum  $S = \mathcal{O}^J$  suppose that  $X^J = \text{Star}(\mathcal{O}^J)$  is affine and such that if  $M^J \subset M$  is the set of all monomials of  $T$  which extend to regular functions on  $X^J = \text{Star}(\mathcal{O}^J)$ , then  $\mathbb{C}[M^J] \rightarrow \mathcal{O}(X^J)$  is surjective.

2) For each fixed  $i \in J$  there exists  $m_i \in M$  such that  $\text{val}_{D_j} m_i = \delta_{i,j}$  for each  $j \in J$ .

3) The collection of all cones  $\sigma^J$  as  $J$  ranges over all non-empty strata forms a fan  $\Sigma$ .

Then  $\bar{U} \hookrightarrow \mathbb{P}_\Sigma$  is schön and  $\bar{U} = Y$ .

The theorem thus affords a recognition theorem for deciding when a toroidal embedding is actually a schön compactification. We conclude this section by proving that the conclusion of their theorem can be reached with a natural condition on the integral structure of a toroidal embedding.

**Corollary 2.2.13.** *Let  $U \subset Y$  be a smooth toroidal embedding with  $Y$  proper and associated conical polyhedral complex  $(|\Pi|, \{\sigma^J\}, \{M^J\})$  and simple normal crossing boundary  $Y \setminus U$ . Suppose for any stratum  $S = \mathcal{O}^J$  we have that  $\text{Star}(S)$  is affine and if  $S' = \mathcal{O}^{J'}$  is another stratum such that  $S \subset S'$  there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{C}[M^{J'}] & \longrightarrow & \mathcal{O}(\text{Star}(S')) \\ \downarrow & & \downarrow \\ \mathbb{C}[M^J] & \longrightarrow & \mathcal{O}(\text{Star}(S)) \end{array} \quad (2.2.2)$$

with all maps surjections of abelian groups. Then  $U \subset Y$  has the structure of a schön compactification. That is,  $|\Pi|$  has the structure of a fan  $\Sigma$  and  $\Sigma$  is a schön fan for  $U$  such that  $\bar{U} = Y$ .

**Proof:** The assumptions of the corollary include condition 1) of Luxton-Qu's theorem. The commutative diagram dualizes to give a family of inclusions  $N^J \hookrightarrow N^{J'}$  whenever  $S \subset S'$ . These inclusions are mutually compatible, and so we may realize them as sublattices of a single global lattice  $N$ , which we may take to be the union over all  $N^J$  as we range over the strata. The family of cones  $\{\sigma^J\}$  are then naturally subsets of  $N$ . That they form a fan also follows from the commutative diagrams above, and compatibility ensures that  $U$  is a closed subvariety of  $\text{Spec}(\mathbb{C}[M])$  and is thus very affine. It remains only to verify condition 2) of Luxton-Qu's theorem. To see this, we recall the proof of Lemma 2.2.11, the proof of the lemma carries over if we can identify the rays corresponding to boundary divisors as rays on  $|\Pi|$ , but indeed it is known that Theorem 1.2.1 is valid for arbitrary simple normal crossing compactifications [44].  $\square$

## 2.3 Clemens' Polytopes, Intersection Complexes, Polytopes at Infinity

In this brief section we discuss the relation between tropical compactifications and some polyhedral constructions frequently employed in Hodge Theory. We claim little originality in this section (save perhaps in exposition), but do aim to draw attention to some results

which seem to have not been recognized in the tropical literature, in particular the results of Danilov [6]. We first review a central construction in Hodge Theory.

**Definition 2.3.1.** Suppose  $\pi : X \rightarrow \mathbb{C}^*$  is a smooth proper map of complex manifolds which extends to a proper flat map  $\bar{\pi} : \bar{X} \rightarrow \mathbb{C}$  with  $\bar{X}$  a complex manifold such that the central fiber  $X_0 = \pi^{-1}(0)$  of the degeneration is a simple normal crossing divisor on  $\bar{X}$  with irreducible components  $D_1, \dots, D_k$ . We assume in addition that the central fiber is reduced, i.e. that the degeneration is semi-stable. Define the Clemens Complex  $\Delta_{\bar{X}}$  to be the abstract simplicial complex obtained by introducing a vertex  $i$  for each irreducible component of  $X_0$  and including a cell  $\{i_1, \dots, i_m\} \subset \{1, \dots, k\}$  if and only if the corresponding intersection  $D_{i_1} \cap \dots \cap D_{i_m}$  is non-empty.

We note that in the context of a degeneration of algebraic curves, the complex  $\Delta_{\bar{X}}$  is often called the dual graph of the degeneration. The relevance to Hodge Theory is the following well-known result, see [34] for a self-contained discussion and larger context.

**Theorem 2.3.2.** *With notation as above, let  $\bar{\pi} : X \rightarrow \mathbb{C}$  be a semi-stable degeneration and assume that each fiber  $X_t = \pi^{-1}(t)$  for  $t \neq 0$  is Kähler. Let  $n$  be the (complex) dimension of  $X_0$ . Then  $H^n(\Delta_{\bar{X}}, \mathbb{Q}) \cong H^n(X_t, \mathbb{Q})$  for  $t$  sufficiently close to 0.*

More precisely, the cohomology group  $H^n(X_t, \mathbb{Q})$  is identified as the 0-th graded piece of the weight filtration on  $H^*(X_t, \mathbb{Q})$ , and the above theorem should be seen as a small piece of a more general theory relating  $H^*(X_t, \mathbb{Q})$  to the  $E_2$  term of an associated spectral sequence. The above theorem can be used as a crude tool to study the topology of semistable degenerations, but is particularly useful for studying degenerations of curves or surfaces, again see [34] for example calculations. There is a variant of the above theorem involving compactifications instead of degenerations. We refer now to [29, Chapter 4] for background, although the original proofs are due to Deligne [7].

**Definition 2.3.3.** Let  $X$  be a smooth algebraic variety and let  $\bar{X}$  be a smooth compactification of  $X$  with divisorial simple normal crossing boundary  $\bar{X} \setminus X$ . Label the irreducible components of the boundary as  $D_1, \dots, D_k$ . Define the associated Clemens complex  $\Delta_{\bar{X}}$  by introducing a vertex  $i$  for each irreducible component and including a cell  $\{i_1, \dots, i_m\} \subset \{1, \dots, k\}$  if and only if the corresponding intersection  $D_{i_1} \cap \dots \cap D_{i_m}$  is non-empty.

In [6] and [29] the above simplicial complex is called the polyhedron of the compactification. We prefer the above name to emphasize the relation to semistable degenerations and their study initiated by Clemens. The following observation is trivial.

**Proposition 2.3.4.** *The Clemens complex  $\Delta_{\bar{X}}$  of a simple normal crossing compactification of  $X$  is isomorphic as a polyhedral complex to a complex of polytopes.*

*Example 2.3.5.* Let  $X \cong (\mathbb{C}^*)^n$  be an algebraic torus and  $\bar{X}$  a smooth projective toric variety given by a polytope  $P$ . Then  $\Delta_{\bar{X}}$  is isomorphic to  $\partial P$  as a polyhedral complex.

An interesting question is when  $\Delta_{\bar{X}}$  is isomorphic to the boundary of a polytope, a question we will address briefly. As one should expect, there is an analogue of Proposition 2.3.2 for compactifications [29, Corollary 2.8.6].

**Proposition 2.3.6.** *Let  $\Delta_{\overline{X}}$  be the Clemens complex of a simple normal crossing compactification of  $X$ , where we assume  $X$  is a smooth algebraic variety of dimension  $m$ . Then  $H^m(\Delta_{\overline{X}}, \mathbb{Q}) \cong H^m(X, \mathbb{Q})$ .*

Hacking [17] considers the case where  $X$  is very affine. More generally, if  $X$  is an affine variety of dimension  $n$ , by the famous Andreotti-Frankel theorem  $X$  has the homotopy type of a CW-complex of dimension  $n$ , and thus  $H^k(X, \mathbb{Z}) = 0$  for  $k > n$ . By considering the remaining graded pieces of the weight filtration, [17] concludes that  $\Delta_{\overline{X}}$  is a rational homology sphere. In particular,  $\Delta_{\overline{X}}$  is a polytope. Of course, in this thesis we are interested primarily in tropical varieties and not Clemens Complexes. The geometric tropicalization constructions from the discussion preceding Theorem 1.2.1 afford a direct relation, and the below proposition is immediate.

**Corollary 2.3.7.** *Let  $X$  be a smooth very affine variety of dimension  $n$  and  $\overline{X}$  any smooth (not necessarily tropical) simple normal crossing compactification. Then  $\Delta_{\overline{X}}$  is a polytope. Assume that  $\Delta_{\overline{X}}$  has dimension at least  $n$ . Let  $\overline{\Sigma}$  denote the recession fan of  $\Delta_{\overline{X}}$  viewed as an abstract fan and let  $\overline{\Sigma}(n)$  be its  $n$ -dimensional skeleton. Then there is a morphism of polyhedral complexes  $\pi : \overline{\Sigma}(n) \rightarrow \text{Trop}(X)$ . This map is proper in the sense that if  $\sigma$  is any maximal cone of dimension  $n$  in  $\text{Trop}(X)$ , then  $\pi^{-1}(\sigma)$  is connected and a union of  $n$ -dimensional cones of  $\overline{\Sigma}(n)$ .*

Given a fixed compactification  $\overline{X}$  the image of  $\pi$  imposes a fan structure on  $\text{Trop}(X)$ , which is exactly the fan structure induced by Theorem 1.2.1. We pause to emphasize what we feel is an important point: the fan  $\overline{\Sigma}$  and the polytope  $\Delta_{\overline{X}}$  are a priori only defined as abstract polyhedral complexes, however  $\text{Trop}(X)$  is defined relative to the lattice  $N$  and thus has a canonical integral structure.  $\text{Trop}(X)$  is thus a much more canonical object than the Clemens polytope, but requires  $X$  to be very affine in order to define.

The main theorem of [6] is that the Clemens complex of a compactification is a homotopy invariant with respect to monoidal transformations. In light of the proof of the weak factorization theorem of Abramovich, Karu, Morelli, and Włodarczyk, Danilov's result can be strengthened to the following.

**Theorem 2.3.8.** *Let  $X$  be a smooth variety with  $\overline{X}$  and  $\overline{X}'$  simple normal crossing compactifications with associated Clemens complexes  $\Delta_{\overline{X}}$  and  $\Delta_{\overline{X}'}$ . Then  $\Delta_{\overline{X}}$  and  $\Delta_{\overline{X}'}$  are homotopic.*

We conclude this section by noting that this theorem can easily be combined Corollary 2.3.7 with to give an alternate proof of Theorem 1.2.1. This is, of course, a little unsatisfying as there are several known elementary proofs of Theorem 1.2.1 (see the references given after the statement of the theorem), and this approach relies on deep machinery. The author is hopeful that future investigations of the relationship between tropical varieties and the homotopy theory of Clemens complexes may yield new results. To offer a line of future investigation, we recall the following conjecture of Danilov (which is known to be false in general, see the translator's notes in [6]).

**Conjecture 2.3.9.** *Let  $X$  be an affine variety of dimension  $n$  and  $\overline{X}$  a simple normal crossing compactification. Suppose that at least one component of  $\overline{X} \setminus X$  is ample. Then  $\Delta_{\overline{X}}$  has the homotopy type of a bouquet of  $(n - 1)$ -spheres.*

A positive solution to the conjecture for the case of  $X$  very affine would imply, via the methods of Hacking mentioned above, that  $\mathrm{Trop}(X)$  also has the homotopy type of a bouquet of spheres, and as mentioned Hacking proves this holds under some restricted hypotheses [17].

## Chapter 3

# Logarithmic Birational Geometry

In this chapter we study schön compactifications from the perspective of logarithmic birational geometry, as studied in the Iitaka school. [22] is the classical reference. Using this we are able to prove some general theorems about the structure of the log canonical divisor, enriching observations of Tevelev [?]tori. We then compute some logarithmic birational invariants in terms of the tropical fan, suggesting a strong connection between tropical geometry and logarithmic birational geometry. We then study logarithmic morphisms and logarithmic birational maps, and study some specific examples which are log flops in the sense of Mori Theory. We will use standard definitions from birational geometry and Mori theory freely, and refer to [32] for background.

*Remark 3.0.10.* Recently, the phrase “log geometry” has come to mean the geometry underlying the log structures of Fontain-Illusie and Kato. This theory is a generalization of Iitaka’s theory, and is beyond the scope of the thesis. However, we will use the phrase “log geometry” throughout this chapter, and in doing so will always mean in the sense of Iitaka or log Mori theory.

### 3.1 Logarithmic Differential Forms and Intrinsic Tori

Recall the discussion of the intrinsic torus after Proposition 1.1.3; we saw that the intrinsic torus behaves analogously to the Albanese variety of a projective variety. Since  $(\mathbb{C}^*)^n$  is a group variety, it is also reasonable to expect that very affine varieties may behave analogously to subvarieties of abelian varieties. This philosophy is expressed in [19, Section 3] and has proven fruitful. Using some ideas from [18] we now make this relationship precise.

Recall that a semi-abelian<sup>1</sup> variety  $A$  is an abelian group variety which fits into an exact sequence

$$0 \rightarrow T \rightarrow A \rightarrow A_0 \rightarrow 0$$

where  $T$  is an algebraic torus and  $A_0$  is an abelian variety. Now let  $(\overline{X}, D)$  be any pair consisting of a smooth complete variety  $\overline{X}$  and  $D$  a simple normal crossing divisor on  $\overline{X}$ ,

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<sup>1</sup>Also called quasi-abelian in the literature

and set  $X = \overline{X} \setminus D$ . Following Itaka one defines the log-Albanese as

$$\mathrm{Alb}_{\log}(X) = \mathrm{coker}(H_1(X, \mathbb{Z}) \rightarrow H^0(\overline{X}, \Omega_{\overline{X}}^1(\log D))^*)$$

where the map is given by integration logarithmic 1-forms:

$$\gamma \mapsto \int_{\gamma} \omega$$

By a slight abuse, the notation does not reflect the choice of s.n.c. compactification, although at least the dimensions are invariant. The inclusion of sheaves  $\Omega_{\overline{X}}^1 \hookrightarrow \Omega_{\overline{X}}^1(\log)$  induces a surjection  $H^0(\overline{X}, \Omega_{\overline{X}}^1(\log D))^* \rightarrow H^0(\overline{X}, \Omega^1)^*$ . This surjection yields an exact sequence which relates the log-Albanese to the classical Albanese:

$$0 \rightarrow T_{\overline{X}} \rightarrow \mathrm{Alb}_{\log}(X) \rightarrow \mathrm{Alb}(\overline{X}) \rightarrow 0$$

where  $T_{\overline{X}}$  is an algebraic torus, so that  $\mathrm{Alb}_{\log}(X)$  has the structure of a semi-abelian variety. More precisely, we have

**Proposition 3.1.1** ([21, Section 2]).  *$T_{\overline{X}}$  is, up to a choice of diagonal action, the intrinsic torus of  $X$ , i.e.  $T_{\overline{X}} = T_X$ . In particular, it depends only on  $X$  and not on the choice of simple normal crossing compactification.*

It follows from the above exact sequence that  $\dim T_X = \overline{q}(X) - q(X)$  where  $\overline{q}(X) = \dim \mathrm{Alb}_{\log}(X)$  is the logarithmic irregularity of  $X$  and  $q(X) = q(\overline{X}) = \dim \mathrm{Alb}(\overline{X})$  is the usual irregularity (note that  $q(\overline{X})$  is a birational invariant and so does not depend on the choice of simple normal crossing compactification of  $X$ ; accordingly we denote it by  $q(X)$ ). By choosing a basepoint  $p \in X$  one obtains a log-Albanese morphism via integration of logarithmic forms

$$a_{\log, X} : X \rightarrow \mathrm{Alb}_{\log}(X) \quad x \mapsto (\omega \mapsto \int_0^p \omega)$$

which is easily seen to be a morphism at the smooth points of  $X$ .  $\mathrm{Alb}_X$  enjoys a universal property analogous to the universal property of the intrinsic torus and the usual Albanese: any map from  $X$  to a semi-abelian variety  $A$  factors through  $\mathrm{Alb}_{\log}(X)$ . In particular, if  $X \subset T$  is a subvariety of an irreducible torus, then  $a_{\log, X}$  embeds  $X$  into  $\mathrm{Alb}_{\log}(X)$ . Note that this embedding coincides with the embedding into the intrinsic torus in cases where  $\mathrm{Alb}(\overline{X}) = 0$ , i.e. when  $h^{1,0}(\overline{X}) = 0$ . For us the main utility in introducing the log-Albanese is to prove some effective criteria for tropical compactifications; see Section 3 below.

Again pursuing the analogy with subvarieties of abelian varieties, we also consider the Gauss map associated to  $X \subset T$ . Indeed, if  $G$  is any algebraic group and  $Y \subset G$  is an irreducible smooth subvariety of dimension  $k$ , we have a map

$$\Gamma_X : X \rightarrow \mathrm{Gr}(k, \mathfrak{g})$$

where  $\text{Gr}(k, \mathfrak{g})$  is the Grassmannian of  $k$ -planes in the Lie Algebra  $\mathfrak{g}$  of  $G$ ; the map is given by translating the tangent space at a point  $p \in X$  to the identity via the group action on  $G$ . We're interested specifically in the case  $G = T$  an algebraic torus, in which  $\mathfrak{g} = \mathfrak{t}$  is canonically identified with  $M \otimes \mathbb{C}$ . We claim that  $\Gamma_X$  interacts well with schön compactifications of  $X$ :

**Proposition 3.1.2.** *Let  $X \subset T$  be smooth of dimension  $k$  and suppose  $\Sigma$  is a quasi-projective schön fan for  $X$  with s.n.c. boundary  $D$ . Then*

- 1)  $\Gamma_X$  extends to a morphism  $\bar{\Gamma}_X : \bar{X} \rightarrow \text{Gr}(k, \mathfrak{t})$
- 2)  $\bar{\Gamma}_X^*(\Lambda_k) = K_{\bar{X}} + D$  where  $\Lambda_k$  is the tautological rank  $k$  vector bundle on  $\text{Gr}(k, \mathfrak{t})$ .
- 3) The fibers of  $\bar{\Gamma}_X$  are canonically identified with the stabilizers of sub-torus actions  $S \subseteq T$  on  $\bar{X}$ .

**Proof:** For 1) one can obtain  $\bar{\Gamma}_X$  as the regular map induced by the evaluation map

$$\Omega_{\mathbb{P}_\Sigma}^k(\log) \otimes \mathcal{O}_{\bar{X}} \rightarrow \Omega_{\bar{X}}^k(\log) \quad .$$

More precisely, this gives the composition of  $\bar{\Gamma}_X$  with the Plücker embedding) where  $\Omega_{\mathbb{P}_\Sigma}^k(\log)$  is the trivial sheaf of  $k$  log-differentials on  $\mathbb{P}_\Sigma$ , which makes sense even when  $\mathbb{P}_\Sigma$  is not complete. The inclusion of sheaves  $\Omega_{\bar{X}}^k \hookrightarrow \Omega_{\bar{X}}^k(\log)$  gives that  $\bar{\Gamma}_X$  restricts to  $\Gamma$  on  $X$ . Statements 2) and 3) follow easily from results of Ran [40].  $\square$

From statement 1) it is natural to wonder when  $\bar{\Gamma}_X$  is an embedding; by 2) this is equivalent to asking whether  $K_{\bar{X}} + D$  is very ample, a question which has been addressed by [19] and which we will recall briefly at the end of Section 3. Note that statement 2) shows that in the schön case  $K_{\bar{X}} + D$  is globally generated which also follows from more elementary considerations noticed by Tevelev [45, Theorem 1.2], which we recall below. Condition 3) should be compared with the well-known result of Griffiths and Harris that the Gauss Map associated with a variety of general type is finite; this will also be discussed in closer detail in the Section 3.

Note that there is a much more direct way to study the log canonical divisor  $K_{\bar{X}} + D$  of a schön compactification. The main point is that if  $\Sigma$  is a smooth schön fan for  $X$  then  $\bar{X}$  is regularly embedded in  $\mathbb{P}_\Sigma$ , and that the log canonical divisor can be identified as the top wedge power of the conormal bundle for the embedding. It follows from an easy calculation that  $K_{\bar{X}} + D$  is globally generated. See [45] for a full proof.

## 3.2 The Iitaka Fibration

In Mori theory one expects a fibration structure whenever  $0 < \bar{\kappa}(X) \leq \dim(X)$  and the cases  $\bar{\kappa} = 0$  and  $\bar{\kappa} = \dim(X)$  (log general type) are of particular interest. For  $\bar{\kappa} = -\infty$  one discards the Iitaka fibration and instead conjecturally one hopes to prove (log) unirationality. For the case of tropical compactifications, this isn't a concern, though, as we have:

**Theorem 3.2.1** (Kawamata [26]). *Let  $X$  be any normal algebraic variety such that there exists a finite morphism  $f : X \rightarrow A$  with  $A$  a semi-abelian variety. Then  $\bar{\kappa} \geq 0$ . If*

$\bar{\kappa}(X) = 0$  and  $X$  is in addition smooth and quasi-projective, then  $\bar{a}$  is dominant with connected fibers (so in particular  $\bar{q}(X) \leq \dim X$ ). If further  $\bar{q}(X) = \dim X$ , then  $\bar{a}$  is birational.

For the simple case of  $f$  an inclusion into  $A = T^n \cong (\mathbb{C}^*)^n$  we obtain:

**Corollary 3.2.2.** *Let  $X$  be any normal subvariety of an algebraic torus. Then  $\bar{\kappa}(X) \geq 0$*

It is worthwhile to note that [46] that for any normal irreducible variety  $X$ ,  $\bar{q}(X) = \dim(X)$  if and only if  $\bar{\kappa}(X) = 0$ . One other numerical invariant is easy to bound; from the exact sequence (loc. cit.) we have that if  $(X, \Sigma)$  be any schön compactification then  $\bar{q}(X) \geq \dim(X)$ . Combining this with Kawamata's deep result above gives

**Proposition 3.2.3.** *Let  $X$  be a normal subvariety of a torus. Then  $\bar{\kappa}(X) = 0$  if and only if  $X = S$  is an algebraic torus. In particular, the tropical compactifications of  $X$  are exactly the complete toric compactifications.*

**Proof:** It is clear that any subtorus  $S$  has  $\bar{\kappa} = 0$ . For the converse, combining the inequality  $\bar{q}(X) \geq \dim(X)$  with Kawamata's theorem we have  $\bar{a}$  is birational. Since  $X$  is affine, it follows that  $X$  is an algebraic torus (cf. [12] Corollary 2.5).  $\square$

Having disposed entirely of the  $\bar{\kappa} < 0$  case and classified the  $\bar{\kappa} = 0$  case, we now turn to positive log Kodaira dimension. The first claim is that  $\bar{\kappa}$  can be detected combinatorially from  $\text{Trop}(X)$ . In general,  $\text{Trop}(X) \subseteq N_{\mathbb{R}}$  may contain non-trivial vector subspaces. For a crude example, take  $X = S \subset T$  a sub-torus of  $T$ , then  $\text{Trop}(S) = N_{\mathbb{R}}^S \subset N_{\mathbb{R}}$  is itself a subspace. In the trivial case  $X = S$  we have  $\bar{\kappa}(S) = 0$ , so we see that  $\bar{\kappa}$  equals the dimension of  $X$  minus the dimension of the lineality space. This is a general phenomenon. First, we introduce a definition, with terminology borrowed from [43].

**Definition 3.2.4.** Given a polyhedral complex  $\mathbb{P}$ , the *lineality space*  $L_{\mathcal{P}}$  is the largest number  $n$  such that there exists a vector space  $V$  of dimension  $n$  such that  $\mathcal{P} = \mathcal{P}' \times V$  for some polyhedral complex  $\mathcal{P}'$ .

**Proposition 3.2.5.** *Let  $X \subset T$  be an irreducible subvariety of an algebraic torus, Let  $L_X$  denote the lineality space of  $\text{Trop}(X) \subseteq N_{\mathbb{R}}$ . Then  $\bar{\kappa} = \dim(X) - \dim L_X$ . If  $\bar{\kappa}(X) < \dim(X)$  then  $X$  factors as  $X = Y \times S$  where  $\bar{\kappa}(X) = \bar{\kappa}(Y) = \dim(Y)$  and  $S \subseteq T$  is a subtorus.*

**Proof:** We can construct the space  $Y$  in the statement of the theorem explicitly. The torus  $T$  acts on  $X$  by usual multiplication:  $m : T \times X \rightarrow T$ . Let  $S_X$  denote the connected component of the identity of the stabilizer of  $X$  in  $T$ . That is,

$$S_X = \{t \in T \mid t \cdot X \subseteq X\}^\circ$$

Note that  $S_X$  is a sub-torus of  $T$  which acts freely on  $X$ . So we may safely form the quotient  $Y = X/S_X \subseteq T/S_X$ . It is easy to see that  $T/S_X$  is itself an algebraic torus and that  $T \rightarrow T/S_X$  is a trivial  $S_X$  bundle. In particular,  $X$  factors as  $Y \times S_X$ . It follows



that  $L_X = \text{Trop}(S_X)$ . It remains only to show that  $\bar{\kappa}(X) = \dim(X) - \dim S_X$ , but this follows from Theorem 3 of [1].  $\square$

We emphasize that Proposition 4.4 shows that  $\bar{\kappa}$  is a purely combinatorial invariant of the tropical variety. This suggests studying how a tropical variety behaves under log morphisms or log birational maps, which we pursue in the next section.

### 3.3 Tropical Modifications

The goal of the section is to prove a general result for how tropicalization behaves under log morphisms. This will afford us some general machinery for studying the effect of log birational maps on  $\text{Trop}(X)$ . First we recall the notion of morphisms in the log category, referring again to [32] for background.

**Definition 3.3.1.** If  $(V, D)$  and  $(W, D')$  are pairs consisting of complete normal varieties  $V$  and  $W$  and Weil divisors  $D \subset X$  and  $D' \subset Y$ , we say a morphism  $f : V \rightarrow W$  is a *log morphism* if  $f(D) \subseteq D'$ , in which case we write  $f : (X, D) \rightarrow (Y, D')$ . If  $f$  is in addition birational and maps the generic point of  $D$  to the generic point of  $D'$ , then we say  $f$  is *log birational*.

Throughout, we will only consider log morphisms between log pairs coming from schön compactifications. In all cases we assume that the log morphism  $f : \bar{X} \rightarrow \bar{Y}$  does not map  $\bar{X}$  into the boundary of  $\bar{Y}$ . It is easy to see that for irreducible tropical compactifications this is equivalent to requiring that  $f|_X : X \rightarrow Y$ ; that is, that  $f$  maps interior to interior. If  $f : \bar{X} \rightarrow \bar{Y}$  is a birational log morphism and  $f|_X : X \rightarrow Y$ , we must have that any irreducible boundary divisor on  $\bar{Y}$  pulls back to some (not necessarily irreducible) boundary divisor on  $\bar{X}$ ; that is,  $f(D_{\bar{X}}) = D_{\bar{Y}}$ . The following trivial fact let's us relate this to the corresponding tropical varieties:

**Lemma 3.3.2.** *Let  $(X, \Sigma)$  and  $(Y, \Sigma')$  be irreducible schön compactifications. Let  $f : (\bar{X}, D_{\bar{X}}) \rightarrow (\bar{Y}, D_{\bar{Y}})$  be a log morphism such that  $f(D_{\bar{X}}) = D_{\bar{Y}}$ . Then  $f|_X : X \rightarrow Y$  is dominant.*

**Proof:**  $f : \bar{X} \rightarrow \bar{Y}$  is surjective so  $Y \setminus f(X) \subset D_{\bar{Y}}$ .  $\square$

*Example 3.3.3.*  $f|_X$  need not be surjective: consider  $\mathbb{P}^1 \setminus \{0, 1, \infty\} \hookrightarrow \mathbb{P}^1 \setminus \{0, \infty\}$ .

In particular, if  $f : (\bar{X}, D_{\bar{X}}) \rightarrow (\bar{Y}, D_{\bar{Y}})$  is any birational log morphism we get an induced map on tropicalization  $f_* : \text{Trop}(X) \rightarrow \text{Trop}(Y)$ . We slightly rephrase a result of Sturmfels-Tevelev [44, Corollary 2.9] describing this map divisorially:

**Lemma 3.3.4.** *Let  $(X, \Sigma)$  and  $(Y, \Sigma')$  be schön compactifications,  $f : X \rightarrow Y$  any dominant log morphism, and let  $f_{\text{trop}}$  be the induced map on tropicalization. Let  $\rho \in \Sigma$  be any ray,  $m \in \rho$  be an integral point and write  $m = [\text{val}_D]$  where  $D$  is a sum of boundary divisors of  $\bar{X}$ . Then  $f_{\text{trop}}(m) = [f_*D]$  where  $f_*D$  is the push-forward of  $D$  as a Weil divisor on  $\bar{X}$ .*

**Proof:** If  $f_*D = 0$  (e.g. the image of  $D$  is not a divisor) there is nothing to show as the valuation associated to a subvariety of codimension greater than 1 is trivial. So assume  $f_*D \neq 0$ . Let  $\chi$  be any monomial on  $Y$  (more precisely the restriction of a monomial on the intrinsic torus of  $Y$  restricted to  $Y$ ). We have  $\text{val}_{f_*D}(\chi) = \text{val}_D(\chi \circ f)$  by definition, and Sturmfels-Tevelev prove that  $[\text{val}_D(\chi \circ f)] = f_{\text{trop}}[D]$ .  $\square$

The above lemma suggests using the notation  $f_*$  for the induced map on tropical varieties a convention which we now adopt. We now prove two general statements describing the behavior of  $f_*$ . First, we fix some notation: if  $X$  is any smooth variety and  $\overline{X}$  is any s.n.c. compactification, let  $\Delta(X)$  denote its abstract simplicial intersection complex. Similarly, if  $D$  is any (reduced) s.n.c. divisor on a variety let  $\Delta(D)$  denote the simplicial intersection complex of its irreducible components. If  $\sigma \in \Delta(D)$  is a  $k$ -cell, then  $D_{\sigma_1}, \dots, D_{\sigma_k}$  will denote the divisors corresponding to its vertices. To any divisor  $D$  on a schön compactification  $\overline{X}$  we let  $[D] \in N_{\mathbb{Q}}$  denote the lattice point associated to its divisorial valuation. As above, if  $Z \subset \overline{X}$  is any proper subvariety of a schön compactification with no components of codimension 1, we say  $[Z] = 0 \in N_{\mathbb{Q}}$ . Using these observations we use Lemma 4.4 to prove:

**Proposition 3.3.5.** *Let  $(X, \Sigma)$  and  $(Y, \Sigma')$  be schön compactifications and suppose  $f : \overline{X} \rightarrow \overline{Y}$  is a log morphism such that  $f(D_{\overline{X}}) = D_{\overline{Y}}$ . Then*

$$\text{Trop}(Y) = \bigcup_{\sigma \in \Delta(\overline{X})} \mathbb{Q}_{\geq 0}[f_*D_{\sigma_1}] \oplus \dots \oplus \mathbb{Q}_{\geq 0}[f_*D_{\sigma_k}]$$

**Proof:** From [19] we know that in general the tropicalization of a schön compactification is computed as the fan over the boundary intersection complex of the boundary. Thus, it suffices to show that every boundary divisor  $D$  of  $\overline{Y}$  is of the form  $f_*E$  for some boundary divisor  $E$  on  $\overline{X}$ , but this follows from the assumption.  $\square$

We now use a stronger proposition to describe  $\text{Trop}(X)$  in terms of  $Y$ .

**Proposition 3.3.6.** *Let  $(X, \Sigma)$  and  $(Y, \Sigma')$  be schön compactifications with boundaries  $D_X$  and  $D_Y$  and suppose  $f : \overline{X} \rightarrow \overline{Y}$  is a birational log morphism with  $R_{\log} \subset D_X$  the logarithmic ramification divisor. Then*

$$\bigcup_{\sigma \in \Delta(\overline{Y})} \mathbb{Q}_{\geq 0}[f^*D_{\sigma_1}] \oplus \dots \oplus \mathbb{Q}_{\geq 0}[f^*D_{\sigma_k}] \subseteq \text{Trop}(X)$$

$$\bigcup_{\tau \in \Delta(R_{\log})} \mathbb{Q}_{\geq 0}[D_{\tau_1}] \oplus \dots \oplus \mathbb{Q}_{\geq 0}[D_{\tau_k}] \subseteq \text{Trop}(X)$$

and  $\text{Trop}(X)$  is the smallest irreducible and equidimensional polyhedral complex fan containing both of these fans.

For a discussion of the definition and properties of  $R_{\log}$  see [32, Section 2.1].

**Proof:** From above we know that the pull back of any irreducible boundary divisor  $B$  on  $\overline{Y}$  is some not necessarily irreducible boundary divisor on  $\overline{X}$ . By definition, it is clear that  $R_{\log}$  is contained in the boundary of  $\overline{X}$  (more precisely, there is a representative for  $R_{\log}$  supported on  $D_X$ ). By Lemma 1.2.1 it is clear that both of the fans given in the statement are contained in  $\text{Trop}(X)$ ; it remains to show that there is no fan properly supported on  $\text{Trop}(X)$  which also contains both cones. Refining if necessary, we can assume both  $D_X$  and  $D_Y$  are simple normal crossing. Then  $R_{\log}$  is exactly the union over all  $f$ -log exceptional boundary divisors of  $\overline{X}$  since the singularities of  $(X, \Sigma)$  and  $(Y, \Sigma')$  are log canonical. In particular, if  $A$  is any irreducible boundary divisor of  $\overline{X}$  then either  $A \subset R_{\log}$  or  $A = f^* f_* A$  so  $A$  is the pull-back of some (not necessarily irreducible) boundary divisor on  $\overline{Y}$ . Now let  $\sigma \in \Sigma$  be any top dimensional cone. We can express generators of the rays in terms of divisorial valuations coming from divisors which are either pull-backs of boundary divisors on  $\overline{Y}$  or contained in  $R_{\log}$ . In particular,  $\sigma$  must meet at least one of the above two fans along its relative interior.  $\square$

### 3.4 Birational Relations Between Schön Compactifications

We recall the following important theorem of Kawamata [27], which extends results of Birkar, Cascini, Hacon, McKernan.[3]:

**Theorem 3.4.1.** *Let  $(W, B)$  and  $(W', B')$  be log pairs consisting of projective  $\mathbb{Q}$ -factorial varieties  $W$  and  $W'$  and  $\mathbb{Q}$ -Weil boundary divisors  $B$  and  $B'$ . Suppose that the pairs are log minimal models; i.e.  $K_W + B$  and  $K_{W'} + B'$  are both nef. Suppose there exists a birational map  $\phi : W \dashrightarrow W'$  such that  $\phi_* B = B'$ . Then  $\alpha$  can be decomposed as a sequence of log-flops.*

We have already seen that log pairs coming from schön compactifications are nef (and abundant). It is natural to ask whether log birational maps between such compactifications can be decomposed into log flops of a more combinatorial nature, and without appeal to the heavy machinery of BCHM. Some evidence for this is that similar statements can be proved for toric varieties using Reid's toric Mori theory, see [11] or [32, Chapter 13] for surveys of Reid's theory. Before proving any tropical analogue of the above theorem, we first need to decide on a notion of a "tropical log flop". Our motivations come from toric Mori theory.

#### 3.4.1 Toric Constructions

We state here some results from toric geometry which are needed. Since we will be dealing frequently with incomplete toric varieties, we introduce some lemmas to reduce to the complete or projective case.

**Lemma 3.4.2.** *Let  $\mathbb{P}_{\Sigma}$  be any toric variety such that  $\Sigma(1)$  linearly spans  $N_{\mathbb{R}}$ . Then  $\mathbb{P}_{\Sigma}$  embeds in a complete toric variety  $\mathbb{P}_{\Pi}$  such that  $\Sigma(1) = \Pi(1)$ . If  $\Sigma$  is smooth  $\Pi$  may be taken smooth as well. This holds, in particular, if  $\Sigma$  is any tropical fan. We call  $\mathbb{P}_{\Pi}$  an **equivariant completion** of  $\mathbb{P}_{\Sigma}$ .*

**Proof:** Take  $\Pi$  to be the complete fan on the rays spanned by  $\Sigma(1)$ ; that is, the set of all cones spanned by they rays of  $\Sigma(1)$ . For the last statement, note that the balancing condition for tropical fans ensures that  $\Sigma(1)$  linearly spans  $N_{\mathbb{R}}$ .  $\square$

Fujino has proven that this construction can be extended to the relative case:

**Proposition 3.4.3.** *Let  $f : \mathbb{P}_{\Sigma} \rightarrow \mathbb{P}_{\Sigma'}$  be a toric morphism. Let  $\Pi$  and  $\Pi'$  be equivariant completions of  $\Sigma$  and  $\Sigma'$  as in the above lemma. Then  $f$  extends to a toric morphism  $\bar{f} : \mathbb{P}_{\Pi} \rightarrow \mathbb{P}_{\Pi'}$ . If  $f$  is a projective morphism, then  $\bar{f}$  maybe be taken to be projective.*

There are of course restrictions to  $\mathbb{P}_{\Sigma}$  being quasi-projective. This can sometimes be overcome using the following result of Oda-Park [37]

**Proposition 3.4.4.** *Let  $\mathbb{P}_{\Sigma}$  be any complete toric variety. Then there exists a simplicial projective refinement  $\Sigma'$  of  $\Sigma$  such that  $\Sigma(1) = \Sigma'(1)$ .*

We are now ready to introduce the notion of a flop between toric varieties.

**Definition 3.4.5.** Let  $\mathbb{P}_{\Sigma}$  and  $\mathbb{P}_{\Sigma'}$  be simplicial toric varieties (possibly incomplete) both of dimension  $n$  and suppose that all maximal cones of each are of the same dimension. Suppose that there exist maximal cones  $\sigma, \tau \in \Sigma$  and  $\sigma', \tau' \in \Sigma'$  such that:

- 1)  $\sigma$  and  $\tau$  share a common face
- 2)  $\sigma \cup \tau = \sigma' \cup \tau'$
- 3) If  $\sigma(1)$  and  $\tau(1)$  denote the set of primitive rays of  $\sigma$  and  $\tau$  respectively, then  $\sigma(1)$  and  $\tau(1)$  lie in the same affine hyperplane of  $N_{\mathbb{R}}$ .

Then the induced equivariant birational map  $\mathbb{P}_{\Sigma} \dashrightarrow \mathbb{P}'_{\Sigma}$  is called a **simple toric flop**. In general, a **toric flop** between toric varieties means a composition of simple toric flops.

Note that condition 2) ensures that  $\sigma'$  and  $\tau'$  also share a common face and that  $\sigma'(1)$  and  $\tau'(1)$  are also contained in the same hyperplane. By relaxing condition 3) we can state a notion of toric flip, but for now we only require flops.

It is elementary to check that if  $\mathbb{P}_{\Sigma}$  and  $\mathbb{P}'_{\Sigma}$  are projective then a toric flop in the above sense is a flop in the usual sense of birational geometry (cf. Reid). For particularly incomplete fans an interpretation is not so direct: for example in dimension  $n$  if  $\Sigma$  contains no cones of dimension  $n - 1$  then  $\mathbb{P}_{\Sigma}$  contains no  $T$ -invariant curves, so it is difficult to interpret any operation as a flop in the usual sense of birational geometry. For now we treat the above theorem purely combinatorially; when  $\Sigma, \Sigma'$  are interpreted as tropical fans we will be able to say more.

Also note that from the construction any two toric varieties related by toric flops have canonically identified  $N$  lattices and identical 1-skeleta:  $\Sigma(1) = \Sigma'(1)$ . The following result is a converse.

**Lemma 3.4.6.** *Let  $\mathbb{P}_{\Sigma}$  and  $\mathbb{P}_{\Sigma'}$  be two simplicial projective toric varieties such that  $\Sigma(1) = \Sigma'(1)$ . Then  $\mathbb{P}_{\Sigma}$  and  $\mathbb{P}_{\Sigma'}$  are related by a toric flop.*

**Proof:** This is a consequence of the Oda-Park construction of the GKZ fan. More precisely, the Chow space  $A = A_{n-1} \otimes \mathbb{R}$  of  $\mathbb{P}_\Sigma$  and  $\mathbb{P}_{\Sigma'}$  with real coefficients are canonically isomorphic and the fans  $\Sigma$  and  $\Sigma'$  each define a full dimensional cone in the GKZ decomposition of  $A$ . Oda-Park show that adjacent cones in the GKZ decomposition correspond to a simple toric flop of the corresponding toric varieties. The support of all such cones is itself a cone (the effective cone), so the fans  $\Sigma$  and  $\Sigma'$  can be connected via a composition of simple flops.  $\square$

With a little care this can be extended to incomplete toric varieties as well.

**Proposition 3.4.7.** *Let  $\Sigma$  and  $\Sigma'$  be simplicial quasi-projective toric varieties such that all maximal cones of each are of the same dimensions and such that  $|\Sigma| = |\Sigma'|$  and  $\Sigma(1) = \Sigma'(1)$ . Then  $\mathbb{P}_\Sigma$  and  $\mathbb{P}_{\Sigma'}$  are related by a toric flop.*

**Proof:** If all maximal cones are one dimensional the result is trivial, say they are all maximal of dimension  $k > 1$ . Else, let  $\sigma = (\sigma_1, \dots, \sigma_k)$  be a collection of maximal cones of  $\Sigma$  such that  $\sigma_1 \cup \dots \cup \sigma_k$  is convex and which is maximal in the sense that if  $\sigma'$  is any other cone of  $\Sigma$ , then the union of the cones with  $\sigma$  is no longer convex. Let  $N_\sigma$  be the rank  $k$  sub-lattice of  $N$  generated by the lattices spanned by the  $\sigma_i$ 's. By adjoining additional rays if necessary we can complete these to a complete dimension  $k$  toric variety for  $N_\sigma$ , call it  $\Sigma(\sigma)$ . Since  $\Sigma$  is quasi-projective we may arrange so that  $\Sigma(\sigma)$  is projective. Since  $|\Sigma| = |\Sigma'|$  there must be a corresponding collection of maximal cones  $\tau = (\tau_1, \dots, \tau_l)$  of  $\Sigma'$  such that  $\sigma_1 \cup \dots \cup \sigma_k = \tau_1 \cup \dots \cup \tau_l$ . Since  $\Sigma(1) = \Sigma'(1)$  actually  $k = l$ . Let  $\Sigma(\tau)$  be the projective toric variety obtained by the same construction. Now apply the above lemma to  $\Sigma(\sigma)$  and  $\Sigma(\tau)$  and let  $\sigma$  vary over the maximally convex collections of maximal cones of  $\Sigma$ .  $\square$

### 3.4.2 Tropical Results

We interpret the constructions of the above section in terms of schön compactification. Our main result is the following.

**Theorem 3.4.8.** *Let  $(X, \Sigma)$  and  $(Y, \Sigma')$  be two schön compactifications which are log birationally equivalent. Then  $\text{Trop}(X) = \text{Trop}(Y)$  and one has  $\Sigma(1) = \Sigma'(1)$  if and only if there exists a tropical flop  $f : \overline{X} \dashrightarrow \overline{Y}$*

The term tropical flop will be defined below. Note that the condition  $\text{Trop}(X) = \text{Trop}(Y)$  is equivalent to saying that  $|\Sigma| = |\Sigma'|$ . Implicit in the statement that  $\text{Trop}(X) = \text{Trop}(Y)$  is that the intrinsic tori  $T_X$  and  $T_Y$  are necessarily isomorphic, which will be clear from the definition. First, some easy observations.

**Proposition 3.4.9.** *Let  $\Sigma$  be a schön fan for  $X \subset T^n$ . Let  $\Sigma'$  be any fan obtained from  $\Sigma$  via a toric flop. Then  $\Sigma'$  is a schön fan for  $X$  as well.*

**Proof:** By Proposition 1.1.8 any smooth fan supported on  $\text{Trop}(X)$  is a schön fan for  $X$ . But the operation of a flop does not change the underlying support of a fan or

smoothness.  $\square$

I do not know if the closures of  $X$  in  $\Sigma$  and  $\Sigma'$  are forced to be equal, but the structure of their boundary divisors are by construction distinct, so the operation is at least logarithmically non-trivial. The following proposition relates toric flops to tropical ones and constitutes the easy part of Theorem 3.4.9.

**Proposition 3.4.10.** *Let  $\Sigma$  be a schön fan for  $X \subset T^n$  and  $\Sigma'$  a fan obtained from  $\Sigma$  via a simple toric flop. Let  $\overline{X}$  denote the closure of  $X$  in  $\mathbb{P}_\Sigma$  and  $\overline{X}'$  denote the closure of  $X$  in  $\mathbb{P}_{\Sigma'}$ . Then  $\overline{X}$  and  $\overline{X}'$  are related by a log flop.*

**Proof:** Let  $\mathcal{R}$  be the the common refinement of  $\Sigma$  and  $\Sigma'$ . It is clear that  $\mathcal{R}$  is smooth and has the same underlying support as  $\Sigma$  and  $\Sigma'$ , hence is a schön fan for  $X$ ; let  $\overline{X}''$  denote the compactification of  $X$  in  $\mathbb{P}_{\mathcal{R}}$  and  $D_{\mathcal{R}}$  its boundary divisor. Let  $p : \mathbb{P}_{\mathcal{R}} \rightarrow \mathbb{P}_\Sigma$  and  $p' : \mathbb{P}_{\mathcal{R}} \rightarrow \mathbb{P}_{\Sigma'}$  be the corresponding blow-down maps. These are proper toric morphisms, so by Tevelev  $p$  and  $p'$  are log crepant. Let  $\sigma \in \Sigma$  and  $\sigma' \in \Sigma'$  be the walls interchanged by the simple flop; by the codimension property of tropical fans the intersections  $C = V(\sigma) \cap \overline{X}$  and  $C' = V(\sigma') \cap \overline{X}'$  are complete smooth curves on  $\overline{X}$  and  $\overline{X}'$  where  $V(\sigma)$  and  $V(\sigma')$  are the corresponding orbit closures. Note that the strict transform of  $C$  under  $p_1$  is contracted by  $p_2$ , so by log crepancy we have  $(K_{\overline{X}} + D_{\overline{X}}) \cdot C = (K_{\overline{X}'} + D_{\overline{X}'}) \cdot C' = 0$  and so the birational map map  $p_1 \circ p_2^{-1}$  is seen to be a flop.  $\square$

**Definition 3.4.11.** A log flop between two schön compactifications as in the above proposition is a *tropical flop*.

**Proof of Theorem 3.4.9:** Suppose  $(X, \Sigma)$  and  $(X, \Sigma')$  are log birational. So there exists a birational map  $f : \overline{X} \dashrightarrow \overline{Y}$  such that  $f_* D_{\overline{X}} = D_{\overline{Y}}$ . Blowing up the boundary of  $\overline{X}$  if necessary, we may assume that  $f : \overline{X} \rightarrow \overline{Y}$  is a birational log morphism, so we have a dominant rational map  $f : X \rightarrow Y$  inducing a surjection  $f_* : \text{Trop}(X) \rightarrow \text{Trop}(Y)$ . Repeating the same argument for  $f^{-1}$  yields  $\text{Trop}(X) = \text{Trop}(Y)$ . It is clear that if  $\overline{X}$  and  $\overline{Y}$  are related by tropical flops then their sets of one-cones are equal. The converse follows easily from the results of Oda-Park.  $\square$

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