# ALGORITHMS FOR BIVARIATE SINGULARITY ANALYSIS

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# ABSTRACT

## ALGORITHMS FOR BIVARIATE SINGULARITY ANALYSIS

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An algorithm for bivariate singularity analysis is developed. For a wide class of bivariate, rational functions $F = P/Q$, this algorithm produces rigorous numerics for the asymptotic analysis of the Taylor coefficients of $F$ at the origin. The paper begins with a self-contained treatment of multivariate singularity analysis. The analysis itself relies heavily on the geometry of the pole set $\mathcal{V}_Q$ of $F$ with respect to a height function $h$. This analysis is then applied to obtain asymptotics for the number of bicolored supertrees, computed in a purely multivariate way. This example is interesting in that the asymptotics can not be computed directly from the standard formulas of multivariate singularity analysis. Motivated by the topological study required by this example, we present characterization theorems in the bivariate case that classify the geometric features salient to the analysis. These characterization theorems are then used to produce an algorithm for this analysis in the bivariate case. A full implementation of the algorithm follows.

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# Chapter 1

# Singularity Analysis Background

## 1.1 Introduction[1]

Let $\mathcal{A}$ denote a *combinatorial class*, i.e. a set of combinatorial objects. For example, $\mathcal{A}$ could be the set of all trees of a particular type, or the set of all walks on a two-dimensional grid having a particular structure, or any other manner of combinatorial object. We assume that $\mathcal{A}$ admits a natural partition into a collection of finite subsets $\mathcal{A}_{\mathbf{r}}$ indexed by $d$-tuples $\mathbf{r} = (r_1, \ldots, r_d)$ of natural numbers. For example $\mathcal{A}_{\mathbf{r}}$ could denote the set of trees with $r_1$ nodes (indexed by 1-tuples), or the set of paths on an $r_1$ by $r_2$ grid (indexed by 2-tuples), etc. The main task in enumerative combinatorics is to count the objects of $\mathcal{A}$ by obtaining formulas for the sizes $a_{\mathbf{r}} := |\mathcal{A}_{\mathbf{r}}|$ of these partitions. Often it is difficult to obtain exact formulas for the $a_{\mathbf{r}}$ directly, and so instead we shall seek asymptotic approximations for the $a_{\mathbf{r}}$ as $\mathbf{r} \to \infty$.

Our analysis begins with the construction of the *ordinary generating function* of the

[1]Portions of Chapters 1 and 2 were first published in Contemporary Mathematics in volume 520, published by the American Mathematical Society, 

class $\mathcal{A}$, which is the $d$-variable formal power series defined by

$$F(\mathbf{x}) = \sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}}\mathbf{x}^{\mathbf{r}}$$

(where $\mathbf{x}^{\mathbf{r}}$ is shorthand for $x_1^{r_1}\cdot\cdots\cdot x_d^{r_d}$). The combinatorial structure of $\mathcal{A}$ often reveals $F$ to be an analytic function in a neighborhood of the origin $\mathbf{0}\in\mathbb{C}^d$, leading to a closed-form representation for $F$. It is then hoped that analytic properties of $F$ may be used to extract information about its coefficients, freeing the problem from its discrete roots and opening it up to the techniques of analysis. This method is known as singularity analysis, due to the strong relationship between the asymptotic growth of the coefficients $a_{\mathbf{r}}$ and the singular set of the generating function $F$.

Singularity analysis in the univariate, $d = 1$ case has been studied thoroughly (see [FS09]) and is well understood, e.g. formulas exist for computing asymptotics for univariate generating functions that are rational, algebraic-logarithmic, or even of several more complicated or implicitly defined classes. The multivariate, $d \geq 2$ case is far less well understood. Recently, however, a line of research begun by Robin Pemantle and Mark Wilson (see [PW02] and [PW08]) has proved to be a fruitful generalization of singularity analysis to higher dimensions.

The singularity analysis of Pemantle and Wilson has the following basic structure: begin with Cauchy's Integral Formula, manipulate the integral/integrand, and end with saddle point integration. To be more explicit, we assume that the generating function of a particular combinatorial class takes the form $F = \eta/Q$, with $\eta : \mathbb{C}^d \to \mathbb{C}$ entire and $Q \in \mathbb{Q}[x_1, \ldots, x_d]$. Cauchy's Integral Formula then expresses the coefficients of $F$ as an integral of a particular $d$-form. By appropriately adjusting this integral, we can rewrite this as the integral of a related $(d-1)$-form defined on the variety $\mathcal{V}_Q = \{\mathbf{x} : Q(\mathbf{x}) = 0\}$ along a cycle $\mathcal{C} \subseteq \mathcal{V}_Q$. We define a height function $h$ on the variety $\mathcal{V}_Q$ related to the rate of decay of this new integrand. We then push this cycle down along $\mathcal{V}_Q$, minimizing the maximum of $h$ along $\mathcal{C}$ at critical points of the function $h$. Under the right conditions, the coefficients can finally be approximated as saddle point integrals along $\mathcal{C}$ in small neighborhoods of a finite set of these critical points, known as the *contributing points*. We shall study these techniques in more detail in the remainder of Chapter 1.

It is shown in [PW02] and [PW08] how these techniques produce automatic asymptotic formulas for many bivariate rational generating functions. Specifically when all the contributing points are *minimal* — that is, on the generating function's boundary of convergence — then an explicit algorithm exists for determining which critical points contribute and computing the saddle point integral near these points (in the bivariate case). And when the generating function is *combinatorial*, i.e. when all its coefficients are non-negative, then the contributing points will all be minimal (under the standing assumption of [PW08], Assumption 3.6).

Hoping to extend this analysis to algebraic generating functions, it was noted by Alex Raichev and Mark Wilson in [RW08] that the algebraic case reduces to the rational case, albeit in one higher dimension. This is due to Safonov's algorithm, which realizes the coefficients of any algebraic generating function as a so-called diagonal of the coefficients of a rational function in one more variable [Saf00]. It is then hoped that the results of [PW02] will be applicable to this newly-formed rational function. An analysis of this form is carried out in Chapter 2 to count the number of bicolored supertrees, but an obstacle prevents this analysis from being a straightforward application of the formulas in [PW02]. The obstacle is that the rational function produced by Safonov's algorithm is not necessarily combinatorial; only along a diagonal do the coefficients in this new generating function actually count something, and off the diagonal the coefficients are free to be (and often are) negative.

In the non-combinatorial case, [PW02] does not provide us with the locations of the contributing points. Worse than that, however, is that even once the contributing points have been found, there is no formula automatically producing the correct saddle point computation in a neighborhood of these points. This is because the structure of $\mathcal{C}$ local to the contributing points is not automatically known. (On the contrary, for minimal contributing points, an explicit construction for $\mathcal{C}$ near these points is known; see [PW02]). This is particularly bad when the contributing point is a degenerate saddle point for the height function. Since the height on $\mathcal{C}$ is locally maximized at the contributing point, it must locally approach and depart along ascent and descent paths. A greater degree of degeneracy means more ascent/descent paths, hence more possibilities for the local path followed by $\mathcal{C}$. And indeed in the case of bicolored supertrees the contributing point *is* a degenerate saddle point of the height function.

Understanding the saddle point integration near these degenerate saddles is particularly important because degenerate saddles arise frequently in combinatorial applications (despite the fact that they are nongeneric). A careful analysis of [PW02] reveals that, in the absence of such degenerate saddles, one obtains leading term asymptotics only of the form $cA^n n^{p/2}$ (for constants $c$, $A$ and integer $p$). By Safonov's algorithm, any univariate algebraic generating function can be realized as the diagonal of a bivariate rational generating function. But by univariate asymptotic methods, we know that the coefficients of such univariate functions can produce leading term asymptotics of the form $cA^n n^{p/q}$ for *arbitrary* $q \in \mathbb{N}$

(see [FS09, Section VII.7]), and so a multivariate analysis of the corresponding bivariate rational function should turn up a degenerate saddle whenever $q > 2$.

In Chapter 3 we explore the topology of $\mathcal{V}_Q$ (for bivariate $Q$) and the homology class of the cycle $\mathcal{C}$ on $\mathcal{V}_Q$. Under certain assumptions on $Q$ and the height function $h$, we will ultimately produce a topological characterization of the set of contributing points and of the structure of the cycle $\mathcal{C}$ local to these points. This characterization is particularly nice in that it is effectively computable, though this is not obvious. Finally in Chapter 4 we use the topological characterization of Chapter 3 to present a fully implemented algorithm for locating the contributing points and describing the structure of $\mathcal{C}$ local to these points. Note that portions of Chapters 3 and 4 will appear in a forthcoming work, [DPvdH11].

Chapter 2 is a case study in the techniques developed in the subsequent chapters. It shall serve as a motivating example for understanding how to apply singularity analysis to a possibly non-combinatorial rational generating function $F = P/Q$. Though the results of Chapter 2 are subsumed in the later work, the intuition behind the more general results will be better understood after seeing an example. Note that the techniques applied to describe $\mathcal{V}_Q$ in Chapter 2 are somewhat ad-hoc, and not appropriate for an automatic computation.

The next task is to explain the basic technique of singularity analysis in more detail, laying the groundwork for later chapters.

## 1.2 Coefficient representation

For the duration of this paper, let $F : \mathbb{C}^d \to \mathbb{C}$ be a function analytic in a neighborhood of the origin, having representation

$$F(\mathbf{x}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}},$$

where $\mathbf{x}^{\mathbf{r}}$ is shorthand notation for $x_1^{r_1} \dots x_d^{r_d}$. The goal is to obtain an asymptotic expansion for the coefficients $a_{\mathbf{r}}$ given $F$, and the main tool for this is Cauchy's Integral Formula.

**Theorem 1.2.1** (Cauchy's Integral Formula)**.** *Let $F$ be as above, analytic in a polydisc $D_0 = \{\mathbf{x} : |x_j| < \varepsilon_j \,\forall\, j\}$, for some positive, real $\varepsilon_j$. Assume further that $F$ is continuous on the distinguished boundary $T_0$ of $D_0$, a product of loops around the origin in each coordinate, each one positively oriented with respect to the complex orientation of its respective plane. Then*

$$a_{\mathbf{r}} = \int_{T_0} \omega_F,$$

*where*

$$\omega_F = \frac{1}{(2\pi i)^d} \cdot \frac{F(\mathbf{x})}{x_1 \cdot \ldots \cdot x_d} \mathbf{x}^{-\mathbf{r}} \, d\mathbf{x}.$$

Cauchy's Integral Formula can be found in most textbooks presenting complex analysis in a multivariable setting, and follows easily as an iterated form of the single variable formula. See, for example, [Sha92, p. 19].

We wish to use the structure of Cauchy's formula to obtain an asymptotic formula for $a_{\mathbf{r}}$ as $\mathbf{r} \to \infty$, but first we need to be more precise about what is meant by "$\mathbf{r} \to \infty$." There are many ways to send the vector $\mathbf{r}$ to infinity, but one of the most natural ways is to fix a direction in the positive $d$-hyperoctant and send $\mathbf{r}$ to infinity along this direction.

Specifically, define the $(d-1)$-simplex $\Delta^{d-1}$ by

$$\Delta^{d-1} = \left\{ (\hat{r}_1, \ldots, \hat{r}_d) : \hat{r}_j \geq 0 \, \forall j, \, \sum_{j=1}^{d} \hat{r}_j = d \right\},$$

where we choose the convention that the $\hat{r}_j$ sum to $d$ for later notational convenience. Then any $\mathbf{r}$ in the positive $d$-hyperoctant can be written uniquely as $\mathbf{r} = |\mathbf{r}|\hat{\mathbf{r}}$, where $|\mathbf{r}| \in \mathbb{R}^+$ and $\hat{\mathbf{r}} \in \Delta^{d-1}$. We examine $\mathbf{r}$ as $|\mathbf{r}| \to \infty$ and $\hat{\mathbf{r}} \to \hat{\mathbf{r}}_0$ for some fixed direction $\hat{\mathbf{r}}_0 \in \Delta^{d-1}$.

Now we turn to the structure of the integrand $\omega_F$, specifically $\mathbf{x}^{-\mathbf{r}}$ (the portion that changes as we vary $\mathbf{r}$). With an eye on the end goal of reducing our computation to a saddle integral, we use the following representation (away from the coordinate axes):

$$\mathbf{x}^{-\mathbf{r}} = \exp\left( -\sum_{j=1}^{d} r_j \log x_j \right) = \exp\left( |\mathbf{r}| H_{\hat{\mathbf{r}}}(\mathbf{x}) \right),$$

where we define the multi-valued function

$$H_{\hat{\mathbf{r}}}(\mathbf{x}) := -\sum_{j=1}^{d} \hat{r}_j \log x_j. \tag{1.2.1}$$

When no confusion exists, we will simply refer to the function $H_{\hat{\mathbf{r}}}$ as $H$. Now the overall magnitude of the integrand will be an important factor in computing an asymptotic expansion for $a_{\mathbf{r}}$, and so we next examine the magnitude of $\exp\left( |\mathbf{r}| H_{\hat{\mathbf{r}}}(\mathbf{x}) \right)$. We have

$$\left| \exp\left( |\mathbf{r}| H_{\hat{\mathbf{r}}}(\mathbf{x}) \right) \right| = \exp\left( |\mathbf{r}| \operatorname{Re} H_{\hat{\mathbf{r}}}(\mathbf{x}) \right) = \exp\left( |\mathbf{r}| h_{\hat{\mathbf{r}}}(\mathbf{x}) \right),$$

where we define the single-valued function

$$h_{\hat{\mathbf{r}}}(\mathbf{x}) := \operatorname{Re} H_{\hat{\mathbf{r}}} = -\sum_{j=1}^{d} \hat{r}_j \log |x_j|. \tag{1.2.2}$$

When no confusion exists, we will simply refer to the function $h_{\hat{\mathbf{r}}}$ as $h$. The geometry of the height function $h$ will play an important role in our analysis.

As $|\mathbf{r}| \to \infty$, the above equations show that the magnitude of the integrand grows at an exponentially slower rate along points further away from the origin (where the height function $h$ is smaller). This motivates pushing the domain of integration out towards infinity, reducing the growth rate of the integrand on the domain over which it is integrated. Of course if $F$ has poles they will present an obstruction, but we can still try push the domain of integration around these poles. In the end we obtain an integral over two domains: one near the pole set of $F$ (obtained by pushing the original domain around the poles), and one past the pole set of $F$ (far away from the origin). This idea is formalized in the theorem below.

**Theorem 1.2.2.** *Let $F = P/Q$, with $P, Q : \mathbb{C}^d \to \mathbb{C}$ entire, where the vanishing set $\mathcal{V}_Q$ of $Q$ is smooth. Let $T_0$ be a torus as in Cauchy's Integral Formula. Let $T_1 \subseteq \mathbb{C}^d$ be a torus homotopic to $T_0$ under a homotopy*

$$K : T \times [0,1] \to \mathbb{C}^d, \qquad \text{with } T_0 = T \times \{0\},\ T_1 = T \times \{1\},$$

*passing through $\mathcal{V}_Q$ transversely. Identifying $K$ with its image in $\mathbb{C}^d$, assume further that $K$ does not intersect the coordinate axes, and that $\partial K \cap \mathcal{V}_Q = \emptyset$. Define*

$$C = K \cap \mathcal{V}_Q.$$

*Then for any tubular neighborhood $\nu$ of of $C$ in $K$, we have*

$$a_{\mathbf{r}} = \int_{T_0} \omega_F = \int_{\partial\nu} \omega_F + \int_{T_1} \omega_F,$$

*given the proper orientation of $\partial\nu$.*

Note: when we say $\mathcal{V}_Q$ is smooth we mean that $\mathcal{V}_Q$ has the structure of a smooth manifold (see [Bre93, p. 68]). And when we say that $K$ passes through $\mathcal{V}_Q$ transversely we mean that the image of $K$ intersects with $\mathcal{V}_Q$ transversely as (real) submanifolds of $\mathbb{C}^d$ (see [Bre93, p. 84]).

*Proof.* Counting (real) dimensions, $\dim \mathcal{V}_Q = 2d - 2$ and $\dim K = d + 1$. Hence their transverse intersection $C$ is a $d - 1$ real-dimensional subspace of $K$.

Now take any tubular neighborhood $\nu$ of $C$ in $K$. As $\nu$ is a full-dimensional submanifold of the orientable manifold $K$, $\nu$ is orientable and hence its boundary $\partial\nu$ is orientable too. Given the proper orientation of $\partial\nu$, we have that

$$\partial(K \setminus \nu) = T_1 - T_0 + \partial\nu.$$

Note that $\omega_F$ is holomorphic on $K \setminus \nu$. By Stokes' Theorem ([Bre93, p. 267]) and the fact that $\omega_F$ is an exact form we get

$$\int_{T_1 - T_0 + \partial\nu} \omega_F = \int_{K\setminus\nu} d\omega_F = \int_{K\setminus\nu} 0 = 0,$$

leading to the equality of the theorem. $\square$

When $T_1$ is far enough away from the origin, $\int_{T_1} \omega_F$ is negligible (possibly even 0), and so the asymptotic analysis of the coefficients $a_{\mathbf{r}}$ reduces to an integral near the pole set of $F$. In the next section, we reduce this further to an integral *on* the pole set of $F$.

## 1.3 The residue theorem

In this section we present a theory generalizing the theory of residues of the complex analysis of one variable. The theory was developed by Jean Leray in 1959, and more details regarding the construction can be found in [AY83, Section 16]. The main result we obtain is Theorem

1.3.6 below, an analogue of the Cauchy Residue Theorem in one variable. Its application to coefficient analysis is found in Corollary 1.3.7.

We restrict our attention to a limited part of Leray's theory, focusing on meromorphic $d$-forms in $\mathbb{C}^d$.

**Definition 1.3.1.** Let $\eta$ be a meromorphic $d$-form, represented as

$$\eta = \frac{P}{Q}\, d\mathbf{x} \text{ on a domain } U \subseteq \mathbb{C}^d$$

where $P$ and $Q$ are holomorphic on $U$. Denote by $\mathcal{V}_Q$ the zero set of $Q$ on $U$, and assume that $\eta$ has a simple pole everywhere on $\mathcal{V}_Q$. Denote by $\iota : \mathcal{V}_Q \to U$ the inclusion map. Then we define the *residue of $\eta$ on $\mathcal{V}_Q$* by

$$\operatorname{Res}(\eta) = \iota^*\theta,$$

where $\iota^*$ denotes pullback by $\iota$ (see [Bre93, p. 263]), and where $\theta$ is any solution to

$$dQ \wedge \theta = P\, d\mathbf{x}.$$

Before delving into the existence and uniqueness of the residue, we do a few example computations.

*Example* 1.3.2. For $\eta = P/Q\, d\mathbf{x}$ as above, wherever $Q_i = \frac{\partial Q}{\partial x_i}$ does not vanish we have the representation

$$\operatorname{Res}(\eta) = (-1)^{i-1}\frac{P}{Q_i}\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_d.$$

As a special case, note that for $Q = x_1$ we obtain

$$\operatorname{Res}(\eta) = P(0, x_2, \ldots, x_d)\, dx_2 \wedge \cdots \wedge dx_d.$$

In the case where $d = 1$, this reduces to $\text{Res}(P(x)/x) = P(0)$, which is precisely the ordinary residue of $P(x)/x$ at $x = 0$. This motivates the above definition as a genuine extension of the single variable residue.

*Example* 1.3.3. As the most pertinent case of the Example 1.3.2, we examine $\text{Res}(\omega_F)$ where $F = P/Q$ is meromorphic. Away from the coordinate axes, $\omega_F$ can be written as

$$\omega_F = \frac{\frac{1}{(2\pi i)^d} \cdot \frac{P(\mathbf{x})}{x_1 \dots x_d} \exp(|\mathbf{r}| H(\mathbf{x}))}{Q(\mathbf{x})} \, d\mathbf{x},$$

where the numerator and denominator are holomorphic functions. So wherever $Q_d$ and the $x_j$ do not vanish (for all $j$), we have

$$\text{Res}(\omega_F) = \frac{(-1)^{d-1}}{(2\pi i)^d} \cdot \frac{P(\mathbf{x})}{x_1 \dots x_d Q_d(\mathbf{x})} e^{|\mathbf{r}| H(\mathbf{x})} dx_1 \wedge \cdots \wedge dx_{d-1}.$$

We now show existence and uniqueness of the residue form along the simple pole set $\mathcal{V}_Q$.

**Proposition 1.3.4.** *Let $\eta$ be as in Definition 1.3.1. Then for any point $\mathbf{p} \in \mathcal{V}_Q$, there is a neighborhood $V \subseteq U$ of $\mathbf{p}$ and a holomorphic $(d-1)$-form $\theta$ on $V$ solving the equation*

$$dQ \wedge \theta = P \, d\mathbf{x}. \tag{1.3.1}$$

*Furthermore, the restriction $\iota^* \theta$ induced by the inclusion $\iota : \mathcal{V}_Q \cap V \to V$ is unique.*

*Proof.* First, we prove the existence of a solution $\theta$ to (1.3.1) in a neighborhood of $\mathbf{p}$. As $Q$ has a simple zero at $\mathbf{p}$, the implicit function theorem implies that for some neighborhood $V$ of $\mathbf{p}$ there is a biholomorphic function $\psi : \mathbb{C}^d \to V$ such that $Q(\psi(\mathbf{x})) = x_1$. Define the form $\theta_0$ by

$$\theta_0 = (P \circ \psi) |J| \, dx_2 \wedge \cdots \wedge dx_d,$$

where $J$ is the Jacobian of the function $\psi$. The claim is that $\theta = (\psi^{-1})^*\theta_0$ is a solution to (1.3.1).

Indeed, by definition of $\theta_0$ we have that $dx_1 \wedge \theta_0 = (P \circ \psi)|J|\, d\mathbf{x}$. Pulling back both sides of this equation by $\psi^{-1}$ yields

$$d(\psi^{-1}(\mathbf{x})_1) \wedge (\psi^{-1})^*\theta_0 = P \cdot (\psi^{-1})^*(|J|\, d\mathbf{x}),$$

which simplifies to $dQ \wedge \theta = P\, d\mathbf{x}$, as desired.

To prove uniqueness, assume that we have two $(d-1)$-forms $\theta$ and $\tilde{\theta}$ such that $dQ \wedge \theta = P\, d\mathbf{x}$ and $dQ \wedge \tilde{\theta} = P\, d\mathbf{x}$. Then $dQ \wedge (\theta - \tilde{\theta}) = 0$, which implies

$$\psi^*(dQ \wedge (\theta - \tilde{\theta})) = dx_1 \wedge \psi^*(\theta - \tilde{\theta}) = 0.$$

But this means that $\psi^*(\theta - \tilde{\theta})$ is a multiple of $dx_1$. Pulling back by $(\psi^{-1})^*$, this implies that $\theta - \tilde{\theta}$ is a multiple of $dQ$. Finally, pulling back by $\iota^*$, this implies that $\iota^*(\theta - \tilde{\theta})$ is a multiple of $d(Q \circ \iota) = 0$. Thus $\iota^*(\theta - \tilde{\theta})$ vanishes, and so $\iota^*\theta = \iota^*\tilde{\theta}$. □

*Remark* 1.3.5. Let $\eta$ be as in the definition of the residue form, and let $\psi : V \to U$ be a biholomorphic function. Then

1. The residue form is natural, i.e. $\operatorname{Res}(\eta)$ does not depend on the particular $P$ and $Q$ chosen to represent $\eta$ as $(P/Q)\, d\mathbf{x}$.

2. The residue form is functorial, i.e. $\operatorname{Res}(\psi^*\eta) = \psi^* \operatorname{Res}(\eta)$ (where on the right side of the equation, $\psi$ is restricted to the domain $\psi^{-1}(\mathcal{V}_Q) = \mathcal{V}_{Q\circ\psi}$).

**Theorem 1.3.6** (Cauchy-Leray Residue Theorem)**.** *Let $\eta$ be a meromorphic $d$-form on domain $U \subseteq \mathbb{C}^d$, with pole set $\mathcal{V} \subseteq U$ along which $\eta$ has only simple poles. Let $N$ be a*

*$d$-chain in $U$, locally the product of a $(d-1)$-chain $C$ on $\mathcal{V}$ with a circle $\gamma$ in the normal slice to $\mathcal{V}$, oriented positively with respect to the complex structure of the normal slice. Then*

$$\int_N \eta = 2\pi i \int_C \operatorname{Res}(\eta).$$

*Proof.* We proceed by examining the structure of the integral locally. So fix an arbitrary $\mathbf{p} \in C$. In a neighborhood $V \subseteq \mathbb{C}^d$ of $\mathbf{p}$, the surrounding space looks like a direct product of $\mathcal{V} \cap V$ (isomorphic to $\mathbb{C}^{d-1}$ for $V$ small) and the normal space to $\mathcal{V} \cap V$ (isomorphic to $\mathbb{C}$). Hence there is a biholomorphic function

$$\begin{aligned} \varphi : V &\to \mathbb{C} \times \mathbb{C}^{d-1} \\ \mathbf{x} &\mapsto (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x})) \end{aligned}$$

where the map $\varphi_2^{-1}$ is a parametrization of $\mathcal{V} \cap V$, and

$$\begin{aligned} \varphi(\mathcal{V} \cap V) &= \{0\} \times \varphi_2(\mathcal{V} \cap V), \\ \varphi(N \cap V) &= \gamma \times \varphi_2(C \cap V), \end{aligned}$$

where $\gamma \subseteq \mathbb{C}$ is a loop around the origin, positively oriented. Furthermore, if $V$ is chosen small enough, we can guarantee that the meromorphic form $(\varphi^{-1})^*\eta$ has a *global* representation as $P/Q\, d\mathbf{x}$. Note that, by the structure of $\eta$ and definition of $\varphi$, $Q$ must vanish on the set

$$\varphi(\mathcal{V} \cap V) = \{\mathbf{x} \in \mathbb{C}^d : x_1 = 0\},$$

where it has only simple zeros.

I claim that if we can prove the equality stated in the residue theorem restricted to $V$, we will be done with the theorem. This is due to the additivity of integration and the compactness of $C$: we can split up a tubular neighborhood of $C$ (containing $N$) into finitely many such neighborhoods on which the theorem holds, then prove the theorem by breaking the integral into a sum over these pieces.

So without loss of generality, we may assume that this local structure holds globally on $C$ and that the domain of the map $\varphi$ is all of $\mathbb{C}^d$. By changing variables, we get

$$\int_N \eta = \int_{\gamma \times \varphi_2(C)} \frac{P}{Q} d\mathbf{x} = \int_{\mathbf{p} \in \varphi_2(C)} \left( \int_{\gamma \times \{\mathbf{p}\}} \frac{P}{Q}\, dx_1 \right) dx_2 \wedge \cdots \wedge dx_d. \tag{1.3.2}$$

the upshot being the ability to split the above into an iterated integral, by the product structure of $\gamma \times \varphi_2(C)$.

The next step is to compute the inner integral from (1.3.2) by the ordinary residue theorem, but doing so will require a change of variables. To that end, define the function $\psi : \mathbb{C}^d \to \mathbb{C}^d$ by

$$\psi(\mathbf{x}) = (Q(\mathbf{x}), x_2, x_3, \ldots, x_d),$$

and fix some $\mathbf{p} \in \mathbb{C}^{d-1}$. The claim is that $\psi$ is biholomorphic in a neighborhood $W \subseteq \mathbb{C}^d$ of $(0, \mathbf{p})$. By the inverse function theorem, this is true if and only if $|J(\mathbf{p})| = Q_1(\mathbf{p}) \neq 0$, where $J$ is the Jacobian of $\psi$. As $Q$ has a simple zero at $\mathbf{p}$, it can't be true that $Q_i(\mathbf{p}) = 0$ for all $i$. But $Q_i(\mathbf{p}) = 0$ for all $i \neq 1$, because $Q$ is constant (equal to 0) on the entire plane $x_1 = 0$. Thus $Q_1(\mathbf{p}) \neq 0$, as desired. Note that $\psi^{-1}$ must have the form

$$\psi^{-1}(\mathbf{x}) = (f(\mathbf{x}), x_2, x_3, \ldots, x_d)$$

for some function $f$, and that $Q \circ \psi^{-1} = x_1$.

We'd like to perform a change of variables and compute the inner integral from (1.3.2) over the domain $\psi(\gamma \times \{\mathbf{p}\})$. The only problem with this is that there is no guarantee that $\gamma \times \varphi_2(C) \subseteq W$. But we *can* make this guarantee by shrinking $N$, i.e. shrinking the loop $\gamma$

closer to the origin, and by (potentially) restricting our attention to a small portion of $C$. Note that shrinking $N$ has no effect on the original integral (the new $N$ will differ from the old $N$ by a boundary, and we are integrating a closed form), and that, as we have already stated, we need only prove the residue theorem locally. Thus we may assume without loss of generality that $\gamma \times \varphi_2(C)$ is contained entirely within the domain of $\psi$.

After the suggested change of variables, we obtain

$$\int_N \eta = \int_{\mathbf{p} \in \varphi_2(C)} \left( \int_{\psi(\gamma \times \{\mathbf{p}\})} \frac{P \circ \psi^{-1}}{x_1} \frac{\partial f}{\partial x_1} \, dx_1 \right) dx_2 \wedge \ldots \wedge dx_n.$$

By the form of $\psi$, $\psi(\gamma \times \{\mathbf{p}\})$ is simply a loop around the origin in the plane $\{x \in \mathbb{C}^d : (x_2, \ldots, x_d) = \mathbf{p}\}$. So by the ordinary residue theorem we can compute

$$\int_{\psi(\gamma \times \{\mathbf{p}\})} \frac{P \circ \psi^{-1}}{x_1} \frac{\partial f}{\partial x_1} \, dx_1 = 2\pi i \cdot P(\psi^{-1}(0, \mathbf{p})) \frac{\partial f}{\partial x_1}(0, \mathbf{p}).$$

Substituting back into (1.3.2) yields

$$\begin{aligned} \int_N \eta &= 2\pi i \int_{\mathbf{p} \in \varphi_2(C)} P(\psi^{-1}(0, \mathbf{p})) \frac{\partial f}{\partial x_1}(0, \mathbf{p}) \, dx_2 \wedge \ldots \wedge dx_n \\ &= 2\pi i \int_{\{0\} \times \varphi_2(C)} \operatorname{Res} \left( \frac{P \circ \psi^{-1} \cdot \frac{\partial f}{\partial x_1}}{x_1} \, d\mathbf{x} \right), \end{aligned}$$

where the second equality comes from the residue computation of Example 1.3.2.

But note that

$$\begin{aligned} (\psi^{-1})^* \left( \frac{P}{Q} \, d\mathbf{x} \right) &= \frac{P \circ \psi^{-1}}{x_1} \left( \sum_{j=1}^{d} \frac{\partial f}{\partial x_j} \, dx_j \right) \wedge dx_2 \wedge \cdots \wedge dx_d \\ &= \frac{P \circ \psi^{-1}}{x_1} \frac{\partial f}{\partial x_1} \, d\mathbf{x}, \end{aligned}$$

and so the integral equation becomes

$$\begin{aligned} \int_N \eta &= 2\pi i \int_{\{0\} \times \varphi_2(C)} \operatorname{Res} \left( (\psi^{-1})^* \left( \frac{P}{Q} \, d\mathbf{x} \right) \right) \\ &= 2\pi i \int_{\{0\} \times \varphi_2(C)} \operatorname{Res} \left( (\psi^{-1})^* (\varphi^{-1})^* \eta \right). \end{aligned}$$

Finally, by the functoriality of the residue form, we obtain

$$\int_N \eta = 2\pi i \int_{\{0\}\times\varphi_2(C)} (\psi^{-1})^*(\varphi^{-1})^* \operatorname{Res}(\eta) = 2\pi i \int_C \operatorname{Res}(\eta).$$

□

The residue theorem applies directly to the coefficient analysis of the previous section by the following corollary.

**Corollary 1.3.7.** *Under the assumptions and notation of Theorem 1.2.2*

$$a_{\mathbf{r}} = 2\pi i \int_C \operatorname{Res}(\omega_F) + \int_{T_1} \omega_F,$$

*given the proper orientation of* $C$.

*Proof.* By the residue theorem, $\int_{\partial\nu} \omega_F = 2\pi i \int_C \operatorname{Res}(\omega_F)$. The result follows by substituting this equality into the conclusion of Theorem 1.2.2. □

And thus the asymptotic coefficient analysis reduces to the integration of a $d-1$ form along a cycle on the pole set of the coefficient generating function. The final step is to compute this integral by means of the saddle point method.

## 1.4 Critical points of the height function

The goal is to obtain an asymptotic expansion for $2\pi i \int_C \operatorname{Res}(\omega_F)$, where $F = P/Q$ for some entire functions $P$ and $Q$, $F$ is analytic in a neighborhood of the origin, and $\mathcal{V}_Q$ is smooth. By Example 1.3.3 we can expect $\operatorname{Res}(\omega_F)$ to take the form

$$\operatorname{Res}(\omega_F) = \frac{(-1)^{d-1}}{(2\pi i)^d} \cdot \frac{P(\mathbf{x})}{x_1 \dots x_d Q_d(\mathbf{x})} e^{|\mathbf{r}|H(\mathbf{x})} dx_1 \wedge \cdots \wedge dx_{d-1}$$

(where $Q_d$ does not vanish), and as before we see that the exponential growth of this form is governed by the height function $h$. This motivates a deformation of the cycle $C$ along $\mathcal{V}_Q$, pushing $C$ down to a homologous cycle $\tilde{C}$ on which the maximum modulus of $h$ is minimized. This procedure is obstructed when the cycle gets trapped on a saddle point of $h$ on $\mathcal{V}_Q$, and the idea is to arrange $\tilde{C}$ so that the local maxima of $h$ along $\tilde{C}$ are *all* achieved at such saddle points. Away from the highest saddle points (the *contributing points*) the integral will contribute asymptotically negligible quantities, and near the contributing points the integral will be amenable to the saddle point method.

Thus the first task is to identify the location of the critical points of $h_{\hat{\mathbf{r}}}|_{\mathcal{V}_Q}$. We denote this set of points by $\Sigma_{\hat{\mathbf{r}}}$, or simply by $\Sigma$ when the direction $\hat{\mathbf{r}}$ is understood. Then the points of $\Sigma_{\hat{\mathbf{r}}}$ can be realized as the zero set of $d$ equations, as exhibited below.

**Theorem 1.4.1** (Location of Critical Points)**.** *Assume $\hat{r}_d \neq 0$. Then the set $\Sigma_{\hat{\mathbf{r}}}$ consists precisely of the points $\mathbf{p} \in \mathbb{C}^d$ satisfying the following $d$ equations:*

$$Q(\mathbf{p}) = 0,$$

$$\hat{r}_d p_j Q_j(\mathbf{p}) - \hat{r}_j p_d Q_d(\mathbf{p}) = 0 \quad \forall j \neq d.$$

*In the case $d = 2$, these critical points are actually saddle points of $h_{\hat{\mathbf{r}}}|_{\mathcal{V}_Q}$.*

For the purposes of computation it should be noted that when $Q$ is a polynomial, the above set of critical points is generically finite and can be found algorithmically by the method of Gröbner bases (see [CLO05, Section 1.3]).

*Proof.* The equation $Q(\mathbf{p}) = 0$ is clear: any critical point of $h|_{\mathcal{V}_Q}$ will have to be on $\mathcal{V}_Q$. So we turn to the remaining $d - 1$ equations.

Fix a point $\mathbf{p} \in \mathcal{V}_Q$ (not on the coordinate axes). By the Cauchy-Riemann equations, $\mathbf{p}$ is a critical point of $\operatorname{Re}\left(H|_{\mathcal{V}_Q}\right)$ if and only if it is a critical point of $\operatorname{Im}\left(H|_{\mathcal{V}_Q}\right)$. Thus $\mathbf{p}$ is a critical point of $h|_{\mathcal{V}_Q}$ exactly when

$$\nabla(H|_{\mathcal{V}_Q})(\mathbf{p}) = 0.$$

But $\nabla(H|_{\mathcal{V}_Q})(\mathbf{p})$ is simply the projection of $\nabla H(\mathbf{p})$ onto the tangent space $T_{\mathbf{p}}\mathcal{V}_Q$. Hence the previous equation is true if and only if

$$\nabla H(\mathbf{p}) \,||\, \nabla Q(\mathbf{p}),$$

as $\nabla Q(\mathbf{p})$ is a vector normal to the tangent space to $\mathcal{V}_Q$ at $\mathbf{p}$. This condition reduces to the equation

$$\left(\frac{-\hat{r}_1}{p_1}, \ldots, \frac{-\hat{r}_d}{p_d}\right) = \lambda\left(Q_1(\mathbf{p}), \ldots, Q_d(\mathbf{p})\right)$$

for some scalar $\lambda$, which is captured by the remaining $d-1$ equations of the theorem.

For the $d = 2$ case, let $\mathbf{p}$ be any critical point of $h|_{\mathcal{V}_Q}$ (hence a critical point of $H|_{\mathcal{V}_Q}$ by the above). In a chart map in a neighborhood of the origin, we can write

$$H|_{\mathcal{V}_Q}(z) = c_0 + c_k z^k(1 + O(z)),$$

for some constants $c_0$ and $c_k$ and $k \geq 2$. As $h = \operatorname{Re}(H)$, it follows that $h|_{\mathcal{V}_Q}$ has a $k^{\text{th}}$ order saddle at $\mathbf{p}$. $\square$

After deforming the domain of integration so that $h$ is locally maximized at the critical points located above, the final step is to obtain an asymptotic expansion by applying the saddle point method near these points. In the case where $d = 2$, this results in a single variable saddle integral. Specifically, we will make use of the following theorem.

**Theorem 1.4.2.** *Let $A$ and $\phi$ be holomorphic functions on a neighborhood of $0 \in \mathbb{C}$, with*

$$A(z) = \sum_{j=l}^{\infty} b_j z^j, \quad \phi(z) = \sum_{j=k}^{\infty} c_j z^j$$

*where $l \geq 0$, $k \geq 2$ and $b_l \neq 0$, $c_j \neq 0$. Let $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{C}$ be any smooth curve with $\gamma(0) = 0$, $\gamma'(0) \neq 0$ and assume that $\operatorname{Re} \phi(\gamma(t)) \geq 0$ with equality only at $t = 0$. Denote by $\gamma^+$ the image of $\gamma$ restricted to the domain $[0, \varepsilon]$. Then for some coefficients $a_j$ we have a full asymptotic expansion*

$$\int_{\gamma^+} A(z) e^{-\lambda\phi(z)}\, dz \sim \sum_{j=l}^{\infty} \frac{a_j}{k} \Gamma\left(\frac{1+j}{k}\right) (c_k \lambda)^{-(1+j)/k}$$

*as $\lambda \to \infty$, where the choice of $k^{th}$ root in $(c_k\lambda)^{-(1+j)/k}$ is made by taking the principal root of $v^{-1}(c_k \lambda v^k)^{1/k}$ where $v = \gamma'(0)$. The leading two coefficients $a_j$ are given by*

$$a_l = b_l, \quad a_{l+1} = b_{l+1} - \frac{2+l}{k} \cdot \frac{c_{k+1}}{c_k}.$$

*For the purposes of computation it should be noted that each coefficient $a_j$ can be effectively computed from the values $b_l, \ldots, b_j$ and $c_k, \ldots, c_{k+j-l}$.*

See [Pem09] for the proof, or [Hen91, Section 11.8] for a treatment from which the above may be derived. It should be noted that, while the saddle point method is a very well known and well understood technique, it is often presented only as a method for solving a general class of problems — theorems are usually only given for limited, special case applications. Theorem 1.4.2 is stated in a generality not easily found in the literature.

In the next chapter, we apply these techniques to a specific combinatorial example.

# Chapter 2

# Application to Bicolored Supertrees

## 2.1 Problem specification

We define the class $\mathcal{K}$ of bicolored supertrees as follows. First, denote by $\mathcal{G}$ the class of *Catalan trees*, i.e. rooted, unlabeled, planar trees, counted by the number of nodes. The class $\mathcal{G}$ has generating function

$$G(x) = \frac{1}{2}\left(1 - \sqrt{1-4x}\right),$$

whose coefficients are the Catalan numbers. Denote by $\tilde{\mathcal{G}}$ the class of *bicolor-planted* Catalan trees: Catalan trees having an extra red or blue node attached to the root (likewise counted by the number of nodes). The class $\tilde{\mathcal{G}}$ has generating function

$$\tilde{G}(x) = 2xG(x).$$

The class of bicolored supertrees is then defined by the combinatorial substitution $\mathcal{K} = \mathcal{G} \circ \tilde{\mathcal{G}}$. That is, the elements of $\mathcal{K}$ are Catalan trees with each node replaced by bicolor-planted Catalan trees. The class $\mathcal{K}$ has algebraic generating function $K(x) = G(\tilde{G}(x))$. More explicitly,

$$K(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x + 4x\sqrt{1-4x}} = 2x^2 + 2x^3 + 8x^4 + 18x^5 + 64x^6 + O(x^7),$$

with coefficients from [Slo09]. Denote by $k_n$ the coefficient of $x^n$ in the expansion of $K(x)$ above, i.e. the number of bicolored supertrees having $n$ nodes. An asymptotic estimate for the $k_n$ has been obtained by univariate analysis of $K(x)$ [FS09, examples VI.10 and VII.20], namely

$$k_n \sim \frac{4^n}{8\Gamma(3/4)n^{5/4}}. \tag{2.1.1}$$

The class of bicolored supertrees was constructed in [FS09] precisely to have an asymptotic growth rate of this shape, with subexponential factor of the form $n^{p/4}$. The fractional power of $-5/4$ occurs due to the manner in which the root functions are composed in the generating function $K(x)$. As mentioned in Chapter 1, a subexponential factor of the form $n^{p/q}$ with $q > 2$ is atypical of the results previously obtained by bivariate singularity analysis, and thus a bivariate analysis of the class of bicolored supertrees should serve as a good test case for the general theory.

To that end, we note that Raichev and Wilson applied Safonov's algorithm to $K(x)$ in [RW08] to produce a rational function

$$F(x,y) = P(x,y)/Q(x,y) = \sum_{r,s \geq 0} a_{r,s} x^r y^s$$

such that $a_{n,n} = k_n$ for all $n$. That is, they realized the coefficients of $K(x)$ as the diagonal coefficients of a rational generating function $F(x,y)$. Specifically, $P$ and $Q$ have the following specification:

$$P(x,y) = 2x^2y\left(2x^5y^2 - 3x^3y + x + 2x^2y - 1\right),$$
$$Q(x,y) = x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2. \tag{2.1.2}$$

We wish to use produce asymptotics on $a_{n,n}$ as $n \to \infty$, recapturing the result of equation (2.1.1). Recall that, due to the non-combinatorial nature of $F$, we can not use the formulas of [PW02] to obtain the result we desire automatically. Thus we follow the procedure outlined in Chapter 1 in full.

We carry over the notation of Chapter 1. In the case of bicolored supertrees this means

$$F = \frac{P}{Q},\ P \text{ and } Q \text{ defined as in (2.1.2)}$$

$$|\mathbf{r}| = n,\ \hat{\mathbf{r}} = \hat{\mathbf{r}}_0 = (1,1),$$

$$H(x,y) = -\log x - \log y$$

$$h(x,y) = -\log|x| - \log|y|.$$

Then as outlined in Chapter 1, the procedure will be as follows.

1. Reduce the asymptotic computation to an integral on the variety $\mathcal{V}_Q$ using Corollary 1.3.7.

2. Locate the critical points of $h|_{\mathcal{V}_Q}$ and deform the contour of integration so as to minimize the maximum of $h$ at such points.

3. Compute an asymptotic expansion for this integral by applying Theorem 1.4.2 near these maxima and bounding the order away from these maxima.

These three steps will be carried out in the sections that follow. Thanks to all the work laid out in the previous chapter, many of these steps will be automatic. The most difficult step will be step (2), finding the new saddle point contour and actually proving that it possesses the right properties (Lemma 2.4.3). The rest will be a matter of applying the theorems when appropriate.

Before jumping into computations, however, we will need to do some initial work on describing the variety $\mathcal{V}_Q$.

## 2.2 Describing the variety

Because $Q$ is quadratic in the variable $y$, we can explicitly solve $Q = 0$ for $y$ as a function of $x$. This will allow us to parametrize $\mathcal{V}_Q$ by $x$ where possible. So define

$$y_1(x) = \frac{-x^2 + x^3 - 2 + \sqrt{x^4 + 4x^2 - 4x^3 + 4}}{x^5},$$
$$y_2(x) = \frac{-x^2 + x^3 - 2 - \sqrt{x^4 + 4x^2 - 4x^3 + 4}}{x^5},$$

where in each case the principal root is chosen. Then by the quadratic formula

$$\mathcal{V}_Q = \{(x, y_j(x)) : x \in \mathbb{C} \setminus \{0\}, j = 1, 2\} \cup \{(0, 1/2)\},$$

(though note that we may write $(0, 1/2) = (0, y_1(0))$ by analytically continuing $y_1$ at $x = 0$). To parametrize $\mathcal{V}_Q$ by $x$, we define the parametrization functions

$$\iota_1(x) = (x, y_1(x)),\ \iota_2(x) = (x, y_2(x)),$$

where in the construction of $\iota_1$ we assume that the singularity at $x = 0$ has been removed. For the purposes of later computation, it will be nice to know the domain on which these parametrization functions are holomorphic.

**Lemma 2.2.1.** *The function $\iota_1$ is holomorphic on $\mathbb{C} \setminus B$, while $\iota_2$ is holomorphic on $\mathbb{C} \setminus$*

$(\{0\} \cup B)$, *where*

$$B := \left\{ x = a + ib \in \mathbb{C} : a^2 - 2a - b^2 = 0, |b| \geq \operatorname{Im}\left(1 + \sqrt{1 + 2i}\right) \right\}.$$

*Proof.* By definition of the functions $y_1$ and $y_2$, the only points where $\iota_1$ and $\iota_2$ may fail to be holomorphic are when $x = 0$ (in the case of $\iota_2$ only) or $f(x) = x^4 + 4x^2 - 4x^3 + 4 \leq 0$ (by the choice of principal square root). Thus we examine when $f(x)$ is a nonpositive real number.

Denote $a = \operatorname{Re}(x)$ and $b = \operatorname{Im}(x)$. We are interested in when $f(a + ib) \leq 0$, so we first examine the equation $\operatorname{Im} f(a + ib) = 0$, or

$$4b(a-1)(a^2 - 2a - b^2) = 0.$$

The solution set of the above equation is the union of the lines $a = 1$, $b = 0$ and the hyperbola $a^2 - 2a - b^2 = 0$. The points $x = 1 \pm \sqrt{1 \pm 2i}$ where $f(x) = 0$ partition the set $\operatorname{Im} f(x) = 0$ into 5 components on which $\operatorname{Re} f(x)$ is either all positive or all negative (by continuity of $f$ on the connected set $\operatorname{Im} f(x) = 0$). See Figure 2.1 for a depiction of these components.

By plugging sample points from each component into $f$, we can determine the components on which $f$ is negative. For example, we have

$$f(0) = 4,$$

$$f(-1 \pm i\sqrt{3}) = -44,$$

$$f(3 \pm i\sqrt{3}) = -44,$$

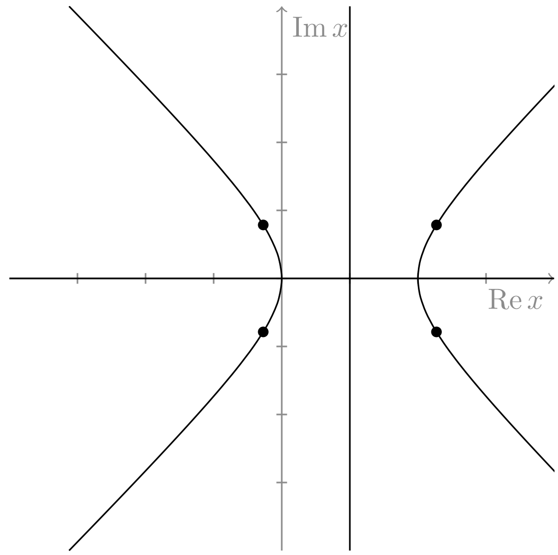


Figure 2.1: The zero sets of $\operatorname{Im} f$ and $f$.

and so we see that $f(x) < 0$ exactly on the four components contained in the set $B$, i.e. along the branches of the hyperbola that lie outside the strip

$$\operatorname{Im}\left(1 \pm \sqrt{1-2i}\right) < \operatorname{Im} x < \operatorname{Im}\left(1 \pm \sqrt{1+2i}\right).$$

□

For the purposes of constructing an appropriate cycle along which to integrate, we will only need the fact that $\iota_1$ and $\iota_2$ are holomorphic on the punctured strip

$$\left\{x \in \mathbb{C} \setminus 0 : \operatorname{Im}\left(1 \pm \sqrt{1-2i}\right) < \operatorname{Im} x < \operatorname{Im}\left(1 \pm \sqrt{1+2i}\right)\right\}.$$

For the sake of completeness, however, we note that the details of Lemma 2.2.1 enable us to understand the global topology of $\mathcal{V}_Q$ as a Riemann surface. We take a moment to sketch the construction.

Recall that the Riemann surface for $\sqrt{x}$ is constructed by taking two copies of the slit plane $\mathbb{C} \setminus \{x : x \leq 0\}$ and gluing the "top" of the branch cut on one sheet to the "bottom"

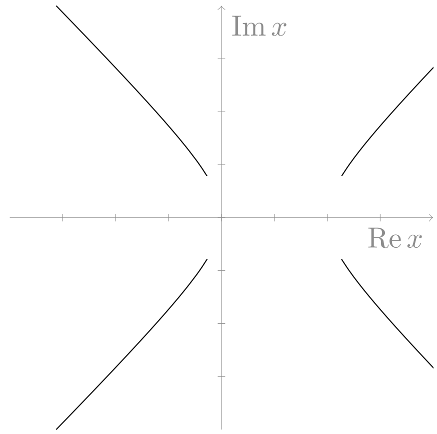


Figure 2.2: The complex plane with four branch cuts (the components of $B$).

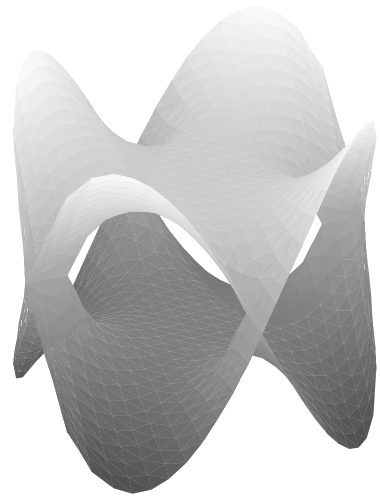

Figure 2.3: The Riemann surface for $\sqrt{f(x)}$.

of the branch cut on the other, and vice versa.

Similarly the Riemann surface for $\sqrt{x^4 + 4x^2 - 4x^3 + 4}$ can be constructed by beginning with two copies of $\mathbb{C} \setminus B$, i.e. two copies of the complex plane each with four branch cuts along which $f(x) = x^4 + 4x^2 - 4x^3 + 4 \leq 0$. Local to the points $x = 1 \pm \sqrt{1 \pm 2i}$ where $f(x) = 0$, the Riemann surface for $\sqrt{f(x)}$ should look like the Riemann surface for $\sqrt{x}$, and the branch cuts are glued together accordingly. Specifically, the tops of each of the branch cuts on one sheet are glued to the bottoms of the corresponding branch cuts on the other sheet, and vice versa. See Figure 2.2 for a representation of one of the sheets used in this construction, and Figure 2.3 for a depiction of the Riemann surface of $\sqrt{f(x)}$.

The Riemann surface representation for $\mathcal{V}_Q$ is then the same as the Riemann surface for $\sqrt{f(x)}$, however with with a puncture point added to one sheet owing to the fact that $\iota_2(x)$ is not defined at $x = 0$. The upshot of this representation is that we can use it to classify the the topology of $\mathcal{V}_Q$. Namely, the singular variety is homeomorphic to a single-holed torus with three puncture points. See Figure 2.4 on page 28 for a step-by-step construction demonstrating this fact.

Finally, as evidenced by Example 1.3.3, it will be useful for representing the residue form along $\mathcal{V}_Q$ to know where $Q_y = \frac{\partial Q}{\partial y}$ is nonzero. Computing a Gröbner basis of the ideal $\langle Q, Q_y \rangle$ in Maple ([Wat08]) via the command

```
Basis([Q,diff(Q,y)],plex(y,x));
```

we obtain the univariate polynomial $x^4 + 4x^2 - 4x^3 + 4$ as the first basis element. Hence the $x$ coordinate of any point where $Q$ and $Q_y$ simultaneously vanish must be a root of this polynomial. This justifies the following remark.

*Remark* 2.2.2. Along $\mathcal{V}_Q$, $Q_y$ is nonzero whenever $x \neq 1 \pm \sqrt{1 \pm 2i}$ (the roots of the equation $x^4 + 4x^2 - 4x^3 + 4 = 0$).

## 2.3 Representing the intersection cycle

The following lemma accounts for the first step of the analysis: using Corollary 1.3.7 to reduce the computation of $a_{n,n}$ to an integral on $\mathcal{V}_Q$.

**Lemma 2.3.1.** *For* $\varepsilon > 0$, *define*

$$C_\varepsilon = \{x \in \mathbb{C} : |x| = \varepsilon\},$$

*the circle of radius* $\varepsilon$ *about* $0 \in \mathbb{C}$, *oriented counterclockwise. Then for sufficiently small* $\varepsilon > 0$,

$$a_{n,n} = 2\pi i \int_{\iota_1(C_\varepsilon)} \operatorname{Res}(\omega_F) + 2\pi i \int_{\iota_2(C_\varepsilon)} \operatorname{Res}(\omega_F). \tag{2.3.1}$$

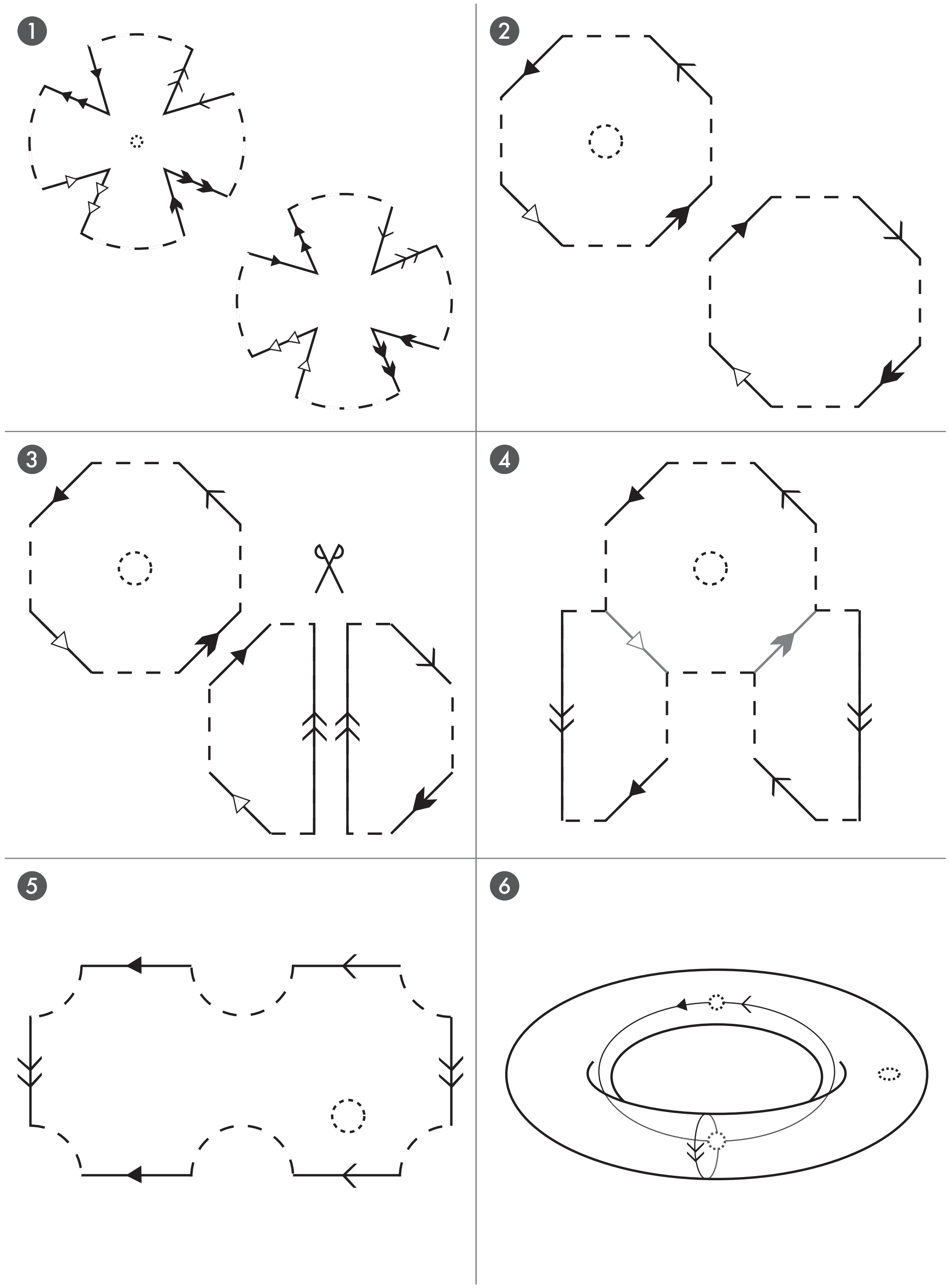


Figure 2.4: A step-by-step construction starting with the two-sheeted representation of $\mathcal{V}_Q$ and ending with a single-holed torus with three puncture points.

*Proof.* We first verify that the variety $\mathcal{V}_Q$ is smooth. This is true only if $Q$, $Q_x$ and $Q_y$ do not simultaneously vanish, which is true if and only if the variety $I = \langle Q, Q_x, Q_y \rangle$ is trivial (the whole polynomial ring). We check this algorithmically, using Gröbner bases. In Maple, we compute the Gröbner basis of $I$ with the command

```
Basis([Q,diff(Q,x),diff(Q,y)],plex(y,x));
```

Maple returns the basis `[1]` for $I$, so the ideal is indeed trivial.

Now, let $\varepsilon > 0$, $\delta > 0$ be sufficiently small so that

$$a_{n,n} = \int_{T_0} \omega_F, \quad \text{where } T_0 = \{(x,y) \in \mathbb{C}^2 : |x| = \varepsilon, |y| = \delta\}$$

by Cauchy's Integral Formula. Define the quantities

$$\begin{aligned} m_0 &= \inf\{|y_j(x)| : x \in C_\varepsilon, j = 1,2\}, \\ M_0 &= \sup\{|y_j(x)| : x \in C_\varepsilon, j = 1,2\}. \end{aligned}$$

For $\varepsilon$ sufficiently small, note that $M_0 < \infty$ (by continuity of the $y_j$; see Lemma 2.2.1) and $m_0 > 0$ (the $x$-axis intersects $\mathcal{V}_Q$ only at the point $(2,0)$).

Assume $\delta$ is chosen small enough so that $\delta < m_0$. Fix any $M > M_0$. Then define the homotopy

$$\begin{aligned} K : T_0 \times [0,1] &\to \mathbb{C}^2 \\ (x,y,t) &\mapsto \left(x, y\left(1 + t\left(\tfrac{M}{\delta} - 1\right)\right)\right), \end{aligned}$$

expanding $T_0$ in the $y$ direction past $\mathcal{V}_Q$. Then $K$ intersects $\mathcal{V}_Q$ in the set $C = \iota_1(C_\varepsilon) \cup \iota_2(C_\varepsilon)$ and avoids the coordinate axes. Furthermore, $K$ intersects $\mathcal{V}_Q$ transversely (as $K$ expands in the $y$ direction, intersecting $\mathcal{V}_Q$ where it is a graph of $x$). Thus, by Corollary 1.3.7 we obtain

$$a_{n,n} = 2\pi i \int_{\iota_1(C_\varepsilon)} \mathrm{Res}(\omega_F) + 2\pi i \int_{\iota_2(C_\varepsilon)} \mathrm{Res}(\omega_F) + \int_{T_1} \omega_F, \tag{2.3.2}$$

where $C_\varepsilon$ is oriented counterclockwise (determined by examination of Theorem 1.2.2 and the Residue Theorem).

Now fix $n$ large and let $M$ vary. As the rest of the terms in (2.3.2) have no $M$ dependence, $\int_{T_1} \omega_F$ must be a constant function of $M$. But by trivial bounds, we can show that

$$\int_{T_1} \omega_F = O(M^{1-n}) \text{ as } M \to \infty,$$

as $\frac{P}{(2\pi i)^2 xyQ} = O(1)$, $\exp(nH) = O(M^{-n})$ and the area of $T_1$ is $O(M)$. For $n > 1$, $M^{1-n} \to 0$ as $M \to \infty$. Hence the only constant $\int_{T_1} \omega_F$ can be equal to is 0. □

## 2.4 Saddle location and contour analysis

Step (2) in the analysis is to locate the saddle points of $h|_{\mathcal{V}_Q}$ and deform the contour of integration appropriately, using this information. The saddle points can be found automatically as follows.

**Lemma 2.4.1.** *$h|_{\mathcal{V}_Q}$ has three saddle points, located at*

$$\left(2, \tfrac{1}{8}\right) = \iota_1(2),$$
$$\left(1 - \sqrt{5}, \tfrac{3+\sqrt{5}}{16}\right) = \iota_1(1 - \sqrt{5}),$$
$$\left(1 + \sqrt{5}, \tfrac{3-\sqrt{5}}{16}\right) = \iota_2(1 + \sqrt{5}).$$

*Proof.* By Theorem 1.4.1, the critical points of $h|_{\mathcal{V}_Q}$ are those points where $Q$ and $xQ_x - yQ_y$ simultaneously vanish. We can compute these points algorithmically by computing the

Gröbner basis for the ideal $I = \langle Q, xQ_x - yQ_y \rangle$ . This is done in Maple with the command

```
Basis([Q,x*diff(Q,x)-y*diff(Q,y)],plex(y,x));
```

which returns a basis consisting of the following two polynomials:

$$32 - 8x^2 - 32x + 20x^3 - 8x^4 + x^5, \quad x^4 - 48 - 6x^3 + 8x^2 + 128y + 16.$$

The first polynomial factors as $(x^2 - 2x - 4)(x - 2)^3$, with roots $x = 2$ and $x = 1 \pm \sqrt{5}$. Substituting these values of $x$ into the second polynomial and solving for $y$ yields the critical points claimed in the lemma. $\square$

We note here the interesting geometry near the critical point $(2, 1/8)$, which will turn out to be the sole contributing point. Expanding $H(\iota_1(x))$ near $x = 2$, we obtain

$$H(\iota_1(x)) = H(\iota_1(2)) + \frac{1}{16}(x-2)^4 + O((x-2)^6),$$

and hence $h|_{\mathcal{V}_Q}$ has a degenerate saddle (of order 4) near this critical point, with steepest descent directions emanating from $x = 2$ at angles $\pi/4 + j(\pi/2)$ radians ($j = 1, 2, 3, 4.$). We also see that along the path $|x| = 2$, $h(\iota_1(x))$ is locally *minimized* at $x = 2$, as this path passes through the critical point along ascent directions. Hence $x = 2$ is a local maximum for $|y_1(x)|$ along this path, and so there are points $(x, y) \in \mathcal{V}_Q$ near $(2, 1/8)$ such that $|x| = 2$ and $|y| < 1/8$. Because $\mathcal{V}_Q$ cuts in toward the origin near $\iota_1(2)$, this critical point is not on the boundary of the domain of convergence of $F$. In the terminology of the introduction, this critical point is *not* minimal.

Knowing where the saddle points of $h$ are, the next task is to deform the contour of integration in (2.3.1) so as to minimize the maximum modulus of $h$ along the new contour at said saddle points. The integral over domain $\iota_2(C_\varepsilon)$ will actually be shown to vanish,

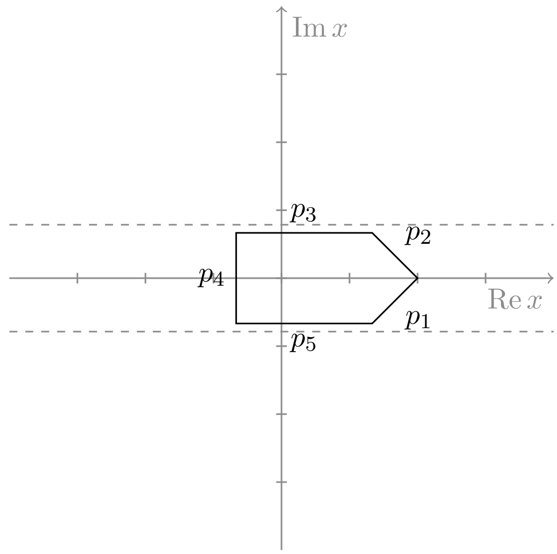


Figure 2.5: A pentagonal path $p$. The dashed lines indicate the boundary of the punctured strip along which $\iota_1$ and $\iota_2$ are holomorphic.

while the domain $\iota_1(C_\varepsilon)$ will be pushed to a "pentagonal" path through the critical point $(2, 1/8)$.

The specific path to which $\iota_1(C_\varepsilon)$ will be deformed is $\iota_1(p)$ where $p$ is the pentagonal path depicted in Figure 2.5, with vertices at the points

$$\left\{\tfrac{4}{3} - i\tfrac{2}{3}, 2, \tfrac{4}{3} + i\tfrac{2}{3}, -\tfrac{2}{3} + i\tfrac{2}{3}, -\tfrac{2}{3} - i\tfrac{2}{3}\right\}.$$

Denote by $p_1, \ldots, p_5$ the edges of $p$, as denoted in the figure.

Performing the suggested deformation results in the following lemma.

**Lemma 2.4.2.**

$$a_{n,n} = 2\pi i \int_{\iota_1(p)} \operatorname{Res}(\omega_F), \tag{2.4.1}$$

*where $p$ is oriented counterclockwise.*

*Proof.* For $\delta < \varepsilon$, let $K$ be a homotopy shrinking the circle $C_\varepsilon$ to the circle $C_\delta$. By holomorphicity of $\iota_2$ (Lemma 2.2.1), $\iota_2 \circ K$ is a homotopy from $\iota_2(C_\varepsilon)$ to $\iota_2(C_\delta)$ along $\mathcal{V}_Q$, and $\mathrm{Res}(\omega_F)$ is holomorphic along this homotopy. By Stokes' Theorem we obtain

$$\int_{\iota_2(C_\varepsilon)} \mathrm{Res}(\omega_F) = \int_{\iota_2(C_\delta)} \mathrm{Res}(\omega_F).$$

Now fix $n$ large and let $\delta$ vary. Note that as the left hand side of the above equation has no $\delta$ dependence, neither does the right.

By the fact that $y_2(x) = -4x^{-5}(1 + O(x))$ as $x \to 0$, we get that $\frac{-P}{(2\pi i)^2 xyQy} = O(\delta^{-4})$, $\exp(nH) = O(\delta^{4n})$ and the area of $\iota_2(C_\delta)$ is $O(\delta^{-4})$ as $\delta \to 0$. This implies that

$$\int_{\iota_2(C_\delta)} \mathrm{Res}(\omega_F) = \int_{\iota_2(C_\delta)} \frac{1}{(2\pi i)^2} \cdot \frac{-P}{xyQ_y} e^{nH}\, dx = O(\delta^{4n-8})$$

as $\delta \to 0$ (note that this representation of the residue is valid by Remark 2.2.2). For $n > 2$, $\delta^{4n-8} \to 0$ as $\delta \to 0$. Thus we must have that this integral is equal to 0.

As for the the integral over $\iota_1(C_\varepsilon)$ in (2.3.1), let $K$ now be a homotopy expanding the circle $C_\varepsilon$ to the pentagonal path $p$. Then by Lemma 2.2.1, $\iota_1 \circ K$ is a homotopy from $\iota_1(C_\varepsilon)$ to $\iota_2(p)$ along $\mathcal{V}_Q$, and $\mathrm{Res}(\omega_F)$ is likewise holomorphic along the image of this homotopy. Then by Stokes' Theorem,

$$\int_{\iota_1(C_\varepsilon)} \mathrm{Res}(\omega_F) = \int_{\iota_1(p)} \mathrm{Res}(\omega_F),$$

where $p$ is oriented counterclockwise. The theorem follows. □

Now we show that $h$ is indeed maximized on $\iota_1(p)$ uniquely at the point $(2, 1/8)$. That this is true local to the saddle point $(2, 1/8)$ is clear from the form of $H$ near this point, as explored following the proof of Lemma 2.4.1. To show that this is true globally will require more effort.

**Lemma 2.4.3.** $h(\iota_1(x)) < h(\iota_1(2)) = \log 4 \quad \forall x \in p \setminus \{2\}$.

*Proof.* Because $h(\iota_1(x))$ is continuous on the connected set $p$, we need only show that $h(\iota_1(x)) \neq \log 4$ for all $x \in p \setminus \{2\}$, and that $h(\iota_1(x)) < \log 4$ for some $x \in p \setminus \{2\}$. The latter condition can be easily checked by plugging some arbitrary point into $h(\iota_1(x))$. As for the former condition, the idea will be to cook up some polynomial equations that must be satisfied in order for it to be true that $h(\iota_1(x)) = \log 4$. We then use techniques from computational algebra to show that these equations can not be satisfied for any $(x, y)$ with $x \in p \setminus \{2\}$ and $y = y_1(x)$.

The conditions from which we will derive our polynomial equations are as follows:

1. $x \in p_j$ for some $j \in \{1, \ldots, 5\}$.

2. $y$ such that $(x, y) \in \mathcal{V}_Q$.

3. $h(x, y) = \log 4$, or $e^{h(x,y)} = 4$.

Each of these conditions implies a (set of) polynomial equations in the variables $\mathrm{Re}(x)$, $\mathrm{Im}(x)$, $\mathrm{Re}(y)$ and $\mathrm{Im}(y)$, as we will show shortly. Note that we are throwing away some important information in condition 2 above, namely we want $y = y_1(x)$, not $y = y_2(x)$. This will be important later in the proof.

We examine first the case where $x \in p_3$. Denote $a = \mathrm{Re}(x)$, $b = \mathrm{Im}(x)$, $c = \mathrm{Re}(y)$ and $d = \mathrm{Im}(y)$. Then condition 1 implies the polynomial constraint:

$$P_1 = b - \frac{2}{3} = 0.$$

Note: condition 1 implies the additional constraint $a \in [-2/3, 4/3]$, which we will make use of shortly.

Condition 2 implies the following two polynomial constraints:

$$P_2 = \mathrm{Re}(Q(a+ib, c+id)) = 0,$$

$$P_3 = \mathrm{Im}(Q(a+ib, c+id)) = 0.$$

Finally, condition 3 translates to $4|x||y| = 1$, or

$$P_4 = 16(a^2+b^2)(c^2+d^2) - 1 = 0.$$

We are interested in whether these four polynomial equations have a common *real-valued* solution, and we will use Gröbner bases and Sturm sequences to answer this question. Since we expect the variety generated by $I = \langle P_1, P_2, P_3, P_4 \rangle$ to be finite — $I$ is generated by four polynomials in four unknowns — we hope to use Gröbner bases to eliminate variables and produce a univariate polynomial $B(a) \in I$. Any point $(a, b, c, d)$ solving $P_j = 0$ for all $j$ will likewise solve $B = 0$. Then we try to use Sturm sequences to that such a $B$ has no real roots $a \in [-2/3, 4/3]$, proving that $h(\iota_1(x)) \neq \log 4$ for $x \in p_3$.

We compute the Gröbner basis with the command

```
Basis([P1,P2,P3,P4],plex(d,c,b,a))
```

and find that the first element $B$ of the basis *is* univariate in the variable $a$, a polynomial of degree 16. We can check that $B(-2/3) \neq 0$ and $B(4/3) \neq 0$ by direct computation in Maple. To check whether or not $B$ has any roots on the interval $(-2/3, 4/3)$ we employ Sturm's Theorem (see [BPR06, p. 52]).

To employ Sturm's Theorem, we must verify that $B$ is squarefree. This is true if and only if the ideal $\langle B, B' \rangle$ is equal to the trivial ideal $\langle 1 \rangle$ . Indeed, computing the Gröbner basis for $\langle B, B' \rangle$

```
Basis([B,diff(B,a)],plex(a));
```

returns the trivial basis `[1]`, i.e. $B$ is squarefree.

Then to count the number of roots in $(-2/3, 4/3)$ via Sturm's Theorem, we enter the command

```
sturm(sturmseq(B,a),a,-2/3,4/3)
```

and Maple returns that there are 0 real roots on the interval $(-2/3, 4/3)$.

Computations are similar for $p_4$ and $p_5$, but things are a bit more complicated along $p_1$ and $p_2$. Let's look at $p_2$. The first polynomial equation becomes

$$P_1 = a + b - 2 = 0,$$

with $a \in [4/3, 2]$, while the rest of the polynomial equations remain the same. Going through the same procedure as before, we can produce a Gröbner basis for $\langle P_1, P_2, P_3, P_4 \rangle$ with an element $B(a)$ univariate in $a$. $B(a)$ factors as

$$B(a) = (a-2)^4 \tilde{B}(a),$$

where by direct computation we see that $\tilde{B}$ is nonzero at $a = 4/3$ and $a = 2$. Note: we expected that $B$ would have a root at $a = 2$, corresponding to the fact that $h(\iota_1(2)) = \log 4$.

The next step would be to attempt to show that $\tilde{B}$ has no roots on the interval $(4/3, 2)$, but this is not true. Using Sturm sequences, one can show that $\tilde{B}$ has exactly one root $a_0 \in (4/3, 2)$, and this is because there *is* a pair $x, y$ with $x \in p_2 \setminus \{2\}$ and $h(x, y) = \log 4$. The claim is that this corresponds to a point where $y = y_2(x)$, not where $y = y_1(x)$.

To see that there must be such a pair, note that $y_2(x) \to 0$ as $x \to 2$. Hence $h(\iota_2(x)) \to \infty$ as $x \to 2$. But by direct computation we can show that $h(\iota_2(4/3)) < \log 4$. As $h(\iota_2(x))$ is continuous on $p_2 \setminus \{2\}$, there must be some $x \in p_2 \setminus \{2\}$ such that $h(\iota_2(x)) = \log 4$. This pair $x$, $y = y_2(x)$ satisfies the polynomial equations $P_j = 0$.

Now assume by way of contradiction that $h(\iota_1(x)) = \log 4$ for some $x \in p_2 \setminus \{2\}$. Because $\tilde{B}$ has just one root $a_0 \in (4/3, 2)$, it must be that this occurs at the same $x$ value for which $h(\iota_2(x)) = \log 4$, specifically $x_0 = a_0 + (2 - a_0)i$. Hence we have

$$|x_0||y_1(x_0)| = |x_0||y_2(x_0)| = \frac{1}{4},$$

which implies that $|y_1| = |y_2|$ at the point $x_0$. So at this value of $x$ we have

$$c^2 + d^2 = |y|^2 = |y_1 y_2| = \frac{|x-2|}{|x|^5}$$

The preceding equation implies that $|x|^{10}(c^2 + d^2)^2 = |x - 2|^2$, which translates into the polynomial equation

$$P_5 = (a^2 + b^2)^5 (c^2 + d^2)^2 - ((a - 2)^2 + b^2) = 0.$$

We now have a new polynomial equation that must be satisfied in order to have that $h(\iota_1(x)) = \log 4$ on $p_2 \setminus \{2\}$. But if we compute a Gröbner basis for $\langle P_1, \ldots, P_5 \rangle$, we get the trivial basis `[1]`, meaning that the polynomials have no common solution. Hence $h(\iota_1(x)) \neq \log 4$ for $x \in p_2 \setminus \{2\}$. Analogous methods can be used to handle the case of $p_1$. $\square$

## 2.5 Saddle point integration

The final step in the analysis is to use saddle point techniques and order bounds to prove (2.1.1).

**Theorem 2.5.1.**

$$k_n = a_{n,n} \sim \frac{4^n}{8\Gamma(3/4)n^{5/4}}.$$

*Proof.* We proceed from Lemma 2.4.2. The theorem will be proved in 2 steps: bounding the integral in (2.4.1) outside a neighborhood of the critical point, then applying saddle point techniques near that critical point.

For any neighborhood $N$ of $x = 2$, we look at $\int_{\iota_1(p \setminus N)} \operatorname{Res}(\omega_F)$, which can be written as

$$\int_{\iota_1(p \setminus N)} \frac{1}{(2\pi i)^2} \cdot \frac{-P}{xyQ_y} e^{nH}\, dx$$

(note that this representation is valid by Remark 2.2.2). As $h \circ \iota_1$ is continuous on the compact set $p \setminus N$, $h \circ \iota_1$ achieves an upper bound $M$ on $p \setminus N$. By Lemma 2.4.3, $M < \log 4$. Thus by trivial bounds we have

$$\int_{\iota_1(p \setminus N)} \operatorname{Res}(\omega_F) = O(e^{Mn}) = o((4-\delta)^n)$$

for sufficiently small $\delta > 0$, as $n \to \infty$. Hence

$$a_{n,n} = 2\pi i \int_{\iota_1(p \cap N)} \operatorname{Res}(\omega_F) + o((4-\delta)^n). \tag{2.5.1}$$

for *any* neighborhood $N$ of $x = 2$, provided $\delta$ is sufficiently small.

For $N$ small enough, $p \cap N = (p_1 \cap N) \cup (p_2 \cap N)$. We examine the integral over $\iota_1(p_1 \cap N)$ and $\iota_1(p_2 \cap N)$ separately, starting with $\iota_1(p_2 \cap N)$. By using the aforementioned representation of the residue form (and changing variables), we obtain

$$2\pi i \int_{\iota_1(p_2 \cap N)} \operatorname{Res}(\omega_F) = \int_{p_2 \cap N} \frac{1}{2\pi i} \cdot \frac{-P(\iota_1(x))}{xy_1(x)Q_y(\iota_1(x))} e^{nH(\iota_1(x))}\, dx.$$

After another change of variables ($x \to x + 2$) and a suitable choice of neighborhood $N$, the above integral can be rewritten as

$$4^n \int_{\gamma^+} A(x) e^{-n\phi(x)}\, dx,$$

where we have, for some fixed $\varepsilon > 0$,

$$\gamma(x) = (i-1)x;\ x \in [-\varepsilon, \varepsilon],$$

$$A(x) = \frac{1}{2\pi i} \cdot \frac{-P(\iota_1(x+2))}{(x+2)y_1(x+2)Q_y(\iota_1(x+2))},$$

$$\phi(x) = \log 4 - H(\iota_1(x+2)),$$

and we recall that $\gamma^+$ is the restriction of the image of $\gamma$ to the domain $[0, \varepsilon]$. The series expansion of $A$ and $\phi$ at $x = 0$ begin

$$A(x) = \frac{i}{16\pi}x^3 + \frac{i}{32\pi}x^4 + O(x^5),$$

$$\phi(x) = \frac{-1}{16}x^4 + O(x^6),$$

and $\operatorname{Re}\phi(x)$ is uniquely minimized on $\gamma^+$ at $x = 0$ where we have $\phi(0) = 0$, as a consequence of Lemma 2.4.3. Thus this is exactly the situation where the saddle point technique of Theorem 1.4.2 can be applied. The values of $b_j$ and $c_j$ are as in the expansions above. Then $v = \gamma'(0) = i - 1$, and we compute the principal root

$$\frac{(c_k n v^k)^{1/k}}{v} = \frac{((-1/16)n(i-1)^4)^{1/4}}{i-1} = \frac{-1-i}{2\sqrt{2}}n^{1/4}.$$

The conclusion of Theorem 1.4.2 is then

$$2\pi i \int_{\iota_1(p_2\cap N)} \operatorname{Res}(\omega_F) \sim 4^n \left( \frac{-i}{4\pi}n^{-1} + \frac{(1+i)\sqrt{2}\Gamma(5/4)}{8\pi}n^{-5/4} + O(n^{-3/2}) \right)$$

As for the integral over $\iota_1(p_1 \cap N)$, the same argument yields

$$2\pi i \int_{\iota_1(p_1\cap N)} \operatorname{Res}(\omega_F) = -4^n \int_{\gamma^+} A(x)e^{-n\phi(x)}\, dx,$$

where $A$ and $\phi$ are the same but $\gamma$ is defined by $\gamma(x) = (-i-1)x$ (and the negative sign out in front comes from a reversal of orientation). For $v = \gamma'(0) = -i-1$, we compute the principal root

$$\frac{(c_k n v^k)^{1/k}}{v} = \frac{((-1/16)n(-i-1)^4)^{1/4}}{-i-1} = \frac{-1+i}{2\sqrt{2}} n^{1/4}.$$

Then by Theorem 1.4.2 we obtain

$$2\pi i \int_{\iota_1(p_1 \cap N)} \operatorname{Res}(\omega_F) \sim 4^n \left( \frac{i}{4\pi} n^{-1} + \frac{(1-i)\sqrt{2}\Gamma(5/4)}{8\pi} n^{-5/4} + O(n^{-3/2}) \right).$$

Adding up the contribution over each piece and plugging into (2.5.1) yields

$$a_{n,n} \sim 4^n \left( \frac{\sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4} + O(n^{-3/2}) \right) + o((4-\delta)^n) \sim \frac{4^n \sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4}.$$

Using the identity $\Gamma(5/4)\Gamma(3/4) = \pi/(2\sqrt{2})$, the theorem follows. $\square$

# Chapter 3

# Homology of the Intersection Class

## 3.1 Setup and assumptions

We would like to produce an algorithm automating the analysis applied in the previous chapter. Tracing through the asymptotic analysis of bicolored supertrees, it becomes apparent that the main difficulty in realizing this goal will be achieving a sufficient understanding of the homology class of the intersection cycle. To obtain a homologous representative of the intersection cycle amenable to saddle point methods requires some global description of the singular variety – a potentially complicated space. Thus to begin, we must produce a description of the singular variety amenable to algorithmic study.

The idea will be to tackle this problem in stages, first focusing on understanding a small subset of the singular variety. Once we are done with that, we will begin revealing more and more of the surface, understanding how the topology changes along the way. The mechanism enabling this study is known as Morse Theory, in which a manifold $\mathcal{M}$ is studied with respect to some height function $h : \mathcal{M} \to \mathbb{R}$. The manifold $\mathcal{M}$ is first restricted to those points of sufficiently high (or low) value with respect to the height function, and the methods of Morse Theory reveal how the topology changes as regions of lower (or higher) height are unveiled. More details will be given in the following sections.

Portions of the following analysis will rely on topological properties of the bivariate case, and thus from here onward we will assume that $d = 2$ variables. Hence the notation and the setup of the problem will be similar to that employed in the previous chapter. We write $z = (x, y)$ rather than $\mathbf{x} = (x_1, x_2)$ to indicate points in $\mathbb{C}^2$. Similarly we write $\mathbf{r} = (r, s) = n(\hat{r}, \hat{s})$ rather than $\mathbf{r} = (r_1, r_2) = |r|\,(\hat{r}_1, \hat{r}_2)$.

We further impose the following assumptions on our analysis:

**Assumption 3.1.1.** *Assume that $\hat{r}$ and $\hat{s}$ are positive rationals.*

The preceding assumption is so that the points in $\Sigma$ (the critical points of $h$ on $\mathcal{V}_Q$) can be found algorithmically using Gröbner bases. The additional assumption that $\hat{r}$ and $\hat{s}$ both be nonzero is to guarantee that this problem does not reduce to one of univariate rational asymptotics.

**Assumption 3.1.2.** *Assume that $\Sigma$ is a finite set.*

Note that the preceding assumption is generically true, but may fail for certain directions $(\hat{r}, \hat{s})$.

**Assumption 3.1.3.** *Assume that $\mathcal{V}_Q$ is smooth.*

Our analysis will apply under the preceding assumptions, with one last technical assumption to come later. Note that the assumption that $\mathcal{V}_Q$ be smooth will be relaxed slightly at the end of this chapter.

## 3.2 Describing the variety at large height

Our Morse-theoretic analysis of $\mathcal{V}_{\mathrm{Q}}$ begins with the selection of a suitable height function. For the singular variety $\mathcal{V}_{\mathrm{Q}}$, we have the somewhat natural height function $h = h_{(\hat{r},\hat{s})}$, the function governing the exponential growth rate of the integrand from which we compute the asymptotics. Our first goal will be to describe what $\mathcal{V}_{\mathrm{Q}}$ looks like for very large values of $h$. Consequently, we develop the following notation for better describing such sets.

**Definition 3.2.1.** For each constant $M \in \mathbb{R}$ we define the set

$$\mathcal{V}^{>M} = \{z \in \mathcal{V}_{\mathrm{Q}} : M < h(z) < \infty\}.$$

We define the sets $\mathcal{V}^{\geq M}$, $\mathcal{V}^{<M}$ and $\mathcal{V}^{\leq M}$ similarly.

Note that implicit to the preceding definition is the fact that the variety $\mathcal{V}_Q$ is endowed with a specific height function. Later, when we describe $\mathcal{V}_Q$ relative to an auxiliary height function, we will indicate this by modifying our notation for the variety itself.

We next wish to develop a description for $\mathcal{V}^{>M}$ for sufficiently large $M$. By definition of $h$ we know that the height along $\mathcal{V}_Q$ is arbitrarily large only when $|x|$ or $|y|$ are sufficiently small, so we first turn to understanding the variety near such points. We have the following useful characterization of a complex variety local to any $x$ or $y$ value:

**Theorem 3.2.2.** *Let $B \subseteq \mathbb{C}$ be a circular neighborhood of $x_0 \in \mathbb{C}$ slit along a ray emanating from $x_0$. If the radius of $B$ is sufficiently small, then on $B$ every branch of $Q(x, y) = 0$ admits a representation $y = f(x)$ of the form*

$$f(x) = \sum_{j \geq j_0} c_j (x - x_0)^{j/k},$$

*for a fixed determination of* $(x - x_0)^{1/k}$*, where* $j_0 \in \mathbb{Z}$ *and* $k \in \mathbb{N}$*. The function* $f$ *is called a* Puiseux expansion *of* $y$*.*

See [FS09, Theorem VII.7] for a proof, or [BK86] for a more in-depth discussion. Note that a similar result holds for obtaining the Puiseux expansion of $x$ in terms of $y$ near any fixed value $y = y_0$.

This local representation allows us to prove the following theorem.

**Theorem 3.2.3.** *The set* $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |x| < R\}$*, for sufficiently small* $R$*, is diffeomorphic to a finite set of disjoint, punctured open disks.*

*Each such diffeomorphism takes the form*

$$G : U \longrightarrow D$$

$$z \longmapsto (z^k, g(z))$$

*for some integer* $k \geq 1$*, where* $U$ *is the punctured disk* $B_{R^{1/k}}(0) - \{0\}$ *and* $g$ *is some holomorphic function on* $U$*.*

*Proof.* By Theorem 3.2.2, for $x$ restricted to a small enough slit neighborhood of 0 in $\mathbb{C}$, any branch of $\mathcal{V}_{\mathrm{Q}}$ can be represented as $(x, f(x))$ for some fractional expansion

$$f(x) = \sum_{j \geq j_0} c_j x^{j/k}.$$

We may assume that $f$ has been represented such that $k$ is as small as possible. Removing the slit, $f$ can be extended to a neighborhood of the form

$$\{x \in \mathbb{C} : |x| < R, x \neq 0\}$$

by analytic continuation to a multiple-valued, locally holomorphic function. The goal is then to better characterize this full branch $(x, f(x))$ in $\mathcal{V}_{\mathrm{Q}}$.

Begin by defining the function

$$g(x) = \sum_{j \geq j_0} c_j x^j,$$

a function holomorphic on the punctured disk $U = B_{R^{1/k}}(0) \setminus \{0\}$. From this, we define the function

$$\begin{aligned} G : U &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto (z^k, g(z)) \end{aligned}$$

The goal is to show that $G$ is actually a diffeomorphism between the punctured disk and the previously described branch, thus completing the theorem.

The first step is to show that $G$ is one-to-one, so begin by assuming that it is not. Then there are $z_1 \neq z_2$ in $U$ satisfying

$$z_1^k = z_2^k$$

$$g(z_1) = g(z_2)$$

Denote $z_0 = z_1^k = z_2^k$. This means that for two fixed determinations $z_1$ and $z_2$ of $z_0^{1/k}$, $g(z_1) = g(z_2)$. Now we can write $z_2 = \xi z_1$ for some $k^{\text{th}}$ root of unity $\xi \neq 1$. This means that

$$g(z_1) = g(\xi z_1),$$

and so

$$z_1^m g(z_1) = z_1^m g(\xi z_1),$$

where $m = \max(0, -j_0)$. But the functions $z^m g(z)$ and $z^m g(\xi z)$ are holomorphic on the disk $B_{R^{1/k}}(0)$ (multiplying by $z^m$ was done precisely to force holomorphicity at $z = 0$), and so by the properties of complex analytic functions there are only two possibilities. Either:

1. The functions $z^m g(z)$ and $z^m g(\xi z)$ agree on $B_{R^{1/k}}(0)$, or

2. By sufficiently minimizing the radius $R$, we can assure that $z^m g(z)$ and $z^m g(\xi z)$ agree nowhere except possibly at the origin.

In the first case, it must be that the coefficients in the Taylor expansions of $z^m g(z)$ and $z^m g(\xi z)$ all agree, which means that $c_j = c_j \xi^j$ for all $j \geq j_0$. Hence $c_j = 0$ whenever $\xi^j \neq 1$. This means that $c_j \neq 0$ only when $j \in s\mathbb{Z}$ for $s \geq 2$ equal to the order of $\xi$, a divisor of $k$. But this implies that the fractional expansion $f(x)$ can be written in terms of the $(k/s)^{\text{th}}$ roots $x^{(k/s)}$, contradicting the minimality of $k$. Thus we must be in the second case.

Hence by sufficiently minimizing the radius $R$ for each possible $k^{\text{th}}$ root of unity $\xi$, we can guarantee that the function $G$ is indeed one-to-one. The inverse function $G^{-1}$ is locally smooth because the function $z \mapsto z^k$ has a locally smooth inverse away from $z = 0$. Hence $G^{-1}$ is smooth, and so we see that $G$ is indeed a diffeomorphism. $\square$

Note that a similar result holds if we restrict to sufficiently small magnitudes of $y$ rather than $x$. And furthermore, because $Q(\mathbf{0}) \neq 0$ (as $P/Q$ was assumed to be holomorphic near $\mathbf{0} \in \mathbb{C}$), we can find a sufficiently small value of $R$ so that no $(x, y) \in \mathcal{V}_Q$ satisfies both $|x| < R$ and $|y| < R$. The fact that these neighborhoods can be made disjoint will be important later on.

Now that we have a good understanding of the topology of $\mathcal{V}_{\mathrm{Q}}$ near $x = 0$ and near $y = 0$, the next thing we wish to do is to evaluate the height function on these neighborhoods. As a first step, we would like to show that the height function is eventually monotonically increasing or decreasing to $\pm\infty$ as $x$ or $y$ go to 0. Unfortunately this is not always the case, and requires a final assumption regarding the direction in which asymptotics are taken.

**Assumption 3.2.4.** *By the Puiseux expansion, we know that we can parameterize each branch of $\mathcal{V}_{\mathrm{Q}}$ local to $x = 0$ in terms of $x$ by writing*

$$y = cx^{\alpha}(1 + o(1))$$

*as $x \to 0$ for some constants $c \neq 0$ and $\alpha$. We assume that*

$$\alpha \neq \frac{-\hat{r}}{\hat{s}} \text{ for all such branches.}$$

*Similarly, we can parameterize each branch of $\mathcal{V}_{\mathrm{Q}}$ local to $y = 0$ in terms of $y$ by writing*

$$x = cy^{\beta}(1 + o(1))$$

*as $y \to 0$ for some constants $c \neq 0$ and $\beta$. We assume that*

$$\beta \neq \frac{-\hat{s}}{\hat{r}} \text{ for all such branches.}$$

This assumption precludes us from taking asymptotics in only finitely many directions, as there are only finitely many branches of $\mathcal{V}_{\mathrm{Q}}$ near $x = 0$ and $y = 0$. We also note that the finitely many possible values of $\alpha$ and $\beta$ may be read from Newton polygon of the polynomial $Q$; see section 4.3 for further details.

The idea behind this assumption is to guarantee that $h$ does not remain bounded as $x \to 0$ and, necessarily, $y \to \infty$ (or vice versa). This assumption is essential to the following lemma.

**Lemma 3.2.5.** *As in the conclusion of Theorem 3.2.3, fix a branch of $\mathcal{V}_{\mathrm{Q}}$ near $x = 0$ that is diffeomorphic to a punctured disk. By the fractional expansion $y = f(x)$ in this* *neighborhood, we can write* $y \sim cx^\alpha$ *as* $x \to 0$*, for some* $\alpha \in \mathbb{Q}$ *and some constant* $c$*. Then, for sufficiently small* $R$ *and any* $\theta \in [0, 2\pi]$*, the function*

$$\begin{aligned} h_\theta : (0, R] &\longrightarrow \mathbb{R} \\ \rho &\longmapsto h(\rho e^{i\theta}, f(\rho e^{i\theta})) \end{aligned}$$

*is monotone. That is, if the punctured disk is sufficiently small, the height function is monotone along rays emanating from the origin.*

*Furthermore, as* $\rho \to 0$*, the height* $h_\theta$ *approaches* $\infty$ *or* $-\infty$ *according to whether* $\alpha > -\hat{r}/\hat{s}$ *or* $\alpha < -\hat{r}/\hat{s}$*, respectively.*

*Proof.* Locally, thanks to the Puiseux expansion of $y$ in terms of $x$, we can consider the functions $H$ and $h$ to be functions of a single complex variable. Namely

$$H(x) = -\hat{r} \log x - \hat{s} \log f(x), \qquad \text{and} \qquad h(x) = \operatorname{Re} H(x).$$

Then $\frac{dh_\theta}{d\rho}$ is simply the derivative of $h(x)$ with respect to $\rho$, where $x = \rho e^{i\theta}$. As $h = \operatorname{Re} H$, this can be represented as

$$\frac{dh_\theta}{d\rho} = \left(\operatorname{Re} \frac{dH}{dx}\right) \cos\theta - \left(\operatorname{Im} \frac{dH}{dx}\right) \sin\theta \Big|_{x=\rho e^{i\theta}}$$

So we turn to evaluating $\frac{dH}{dx}$.

By the Puiseux expansion, we can write $f(x) = cx^\alpha(1 + g(x))$ for some fractional expansion $g(x)$ that is $o(1)$ as $x \to 0$. Then

$$\begin{aligned} H(x) &= -\hat{r} \log x - \hat{s} \log\left(cx^\alpha(1 + g(x))\right) \\ &= (-\hat{r} - \hat{s}\alpha) \log x - \hat{s} \log\left(1 + g(x)\right) - \hat{s} \log c. \end{aligned}$$

We note briefly from the above form for $H$ that $\operatorname{Re} H$ approaches $\pm\infty$ as $x \to 0$ according to the sign of $(-\hat{r} - \hat{s}\alpha)$, as claimed in the statement of the lemma. Now $\frac{dH}{dx}$ takes the form

$$\begin{aligned} H'(x) &= \frac{-\hat{r} - \hat{s}\alpha}{x} - \frac{g'(x)}{1 + g(x)} \\ &= \frac{1}{x}\left(-\hat{r} - \hat{s}\alpha - \frac{xg'(x)}{1 + g(x)}\right) \\ &\sim \frac{-\hat{r} - \hat{s}\alpha}{x} \end{aligned}$$

as $x \to 0$.

We can rewrite $1/x$ as $\cos\theta/|x| - i\sin\theta/|x|$, from which we obtain that

$$\operatorname{Re} \frac{dH}{dx} \sim \frac{(-\hat{r} - \hat{s}\alpha)\cos\theta}{|x|}$$

and

$$\operatorname{Im} \frac{dH}{dx} \sim \frac{(\hat{r} + \hat{s}\alpha)\sin\theta}{|x|}.$$

And so $\frac{dh_\theta}{d\rho} \sim \frac{-\hat{r}-\hat{s}\alpha}{|x|}$ as $x \to 0$. But $\frac{-\hat{r}-\hat{s}\alpha}{|x|} \to \pm\infty$ as $x \to 0$, and so we see that on a sufficiently small neighborhood, this derivative can never vanish. Hence we have monotonicity. $\square$

And again, note that there is nothing special about focusing our attention on branches near $x = 0$. An analogous lemma holds on branches near $y = 0$.

We now have enough information to describe the space $\mathcal{V}_{\mathrm{Q}}$ at sufficiently large height.

**Theorem 3.2.6.** *For sufficiently large $M$, the set $\mathcal{V}^{>M}$ is diffeomorphic to a finite set of disjoint, punctured open disks.*

*Proof.* Pick $R$ sufficiently small so that

1. If $Q(x, y) = 0$, then either $|x| < R$ or $|y| < R$ but not both.

2. The sets $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |x| < R\}$ and $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |y| < R\}$ are each diffeomorphic to finite sets of disjoint, punctured open disks (as in the conclusion of Theorem 3.2.3).

By the structure of $h$ we now pick $M$ large enough so that $h(x, y) \geq M$ requires either that $|x| < R$ or $|y| < R$. Note that this implies that

$$\mathcal{V}^{\geq M} \subseteq \mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |x| < R \text{ or } 0 < |y| < R\}.$$

We examine a particular punctured disk $D$, say one arising from a branch $(x, f(x))$, where

$$f(x) = x^{\alpha}(1 + o(1))$$

is a Puiseux expansion of $y$ in $x$. We look at the parametrization of $D$ that we get from Theorem 3.2.3:

$$\begin{aligned} G : U &\longrightarrow D \\ z &\longmapsto (z^k, g(z)) \end{aligned}$$

where $U = B_{R^{1/k}}(0) \setminus \{0\}$ and $g(z) = f(z^k)$ for some positive integer $k$.

By Lemma 3.2.5, $h \circ G$ is monotone on $U$ along radial paths emanating from the origin (note that $G$ sends rays to rays). As $h(x, y) < M$ along $\partial D$ by construction, we have two possibilities:

1. If $h \circ G$ is monotone decreasing to $-\infty$ as $|z| \to 0$ in $U$, then $\mathcal{V}^{\geq M} \cap D = \emptyset$.

2. Otherwise, $h \circ G$ is monotone increasing to $\infty$ as $|z| \to 0$ in $U$, and so for each $\theta \in [0, 2\pi]$ we obtain a unique $\rho(\theta) \in (0, R^{1/k})$ such that $h(G(\rho(\theta)e^{i\theta})) = M$. Note that the function $\rho(\theta)$ is continuous by the continuity of $h \circ G$.

So in this second case we obtain a simple closed curve $s$ defined by

$$s : [0, 2\pi] \longrightarrow U$$
$$\theta \longmapsto \rho(\theta) e^{i\theta}$$

which traces out the preimage of $M$ under the map $h \circ G$. By monotonicity, this implies that the set $\mathcal{V}^{>M} \cap D$ is parametrized by $G\big|_{U_0}$, where

$$U_0 = \{\rho e^{i\theta} \in \mathbb{C} : 0 < \rho < \rho(\theta)\},$$

which is homeomorphic to $U$ (hence to a punctured, open disk). By applying this argument to every such $D$, and using the fact that $\mathcal{V}^{>M}$ is contained in the union of all such disks $D$, the conclusion of the theorem follows. □

To further better describe how the components of $\mathcal{V}^{>M}$ are structured (for sufficiently large $M$), we introduce the following notation.

**Definition 3.2.7.** For $M \in \mathbb{R}$, we partition the set $\mathcal{V}^{>M}$ into its connected components as

$$\mathcal{V}^{>M} = \{R_j : j \in J\}.$$

Then we define the set $X^{>M} \subseteq \mathcal{V}^{>M}$ by

$$X^{>M} = \{R_j : \forall \varepsilon > 0, \exists (x, y) \in R_j \text{ s.t. } |x| < \varepsilon\},$$

i.e. the set of all components of $\mathcal{V}^{>M}$ containing points of arbitrarily small $x$-coordinate. Similarly we define the set $Y^{>M}$ by

$$Y^{>M} = \{R_j : \forall \varepsilon > 0, \exists (x, y) \in R_j \text{ s.t. } |y| < \varepsilon\}.$$

We call the components of $X^{>M}$ the *x-components* of $\mathcal{V}^{>M}$, and similarly the components of $Y^{>M}$ are called the *y-components* of $\mathcal{V}^{>M}$.

We define the sets $X^{\geq M}$ and $Y^{\geq M}$ analogously.

Note that it may be possible that either $X^{>M}$ or $Y^{>M}$ is empty for a given $M$. As we shall see in the next section, however, it can not be the case that $X^{>M}$ and $Y^{>M}$ are both empty unless $\mathcal{V}_Q$ is empty itself.

We also note that there may be overlap between the $x$- and $y$-components of $\mathcal{V}^{>M}$, but as evidenced in the proof of the preceding theorem this does not happen for sufficiently large $M$. For large enough $M$, the space $\mathcal{V}^{>M}$ decomposes uniquely into $x$- and $y$-components, each of which is homeomorphic to an open, punctured disk.

## 3.3 Unveiling the rest of the singular variety

Having understood $\mathcal{V}^{>M}$ for sufficiently large $M$, we begin revealing the rest of the variety by decreasing the value of $M$. We use Morse theory to examine how the topology changes as we do so.

There are two important structural theorems that will allow us to accomplish our task, which are as follows.

**Theorem 3.3.1** (First Structure Theorem)**.** *Let $m_1 < m_2 \in \mathbb{R}$ be such that $h$ has no critical values in the interval $[m_1, m_2]$. Then $\mathcal{V}^{>m_1}$ is topologically equivalent to $\mathcal{V}^{>m_2}$; in fact $\mathcal{V}^{>m_2}$ is a deformation retract of $\mathcal{V}^{>m_1}$.*

For a discussion, see [Mil63]. The only special feature of $h$ that is used in the preceding theorem is the fact that $h$ is proper on $\mathcal{V}_\mathrm{Q}$. This is guaranteed by Assumption 3.2.4, which implies that the inverse images under $h\big|_{\mathcal{V}_\mathrm{Q}}$ of bounded sets are again bounded.

The second structural theorem, which deals with when the topology changes, is as follows:

**Theorem 3.3.2** (Second Structure Theorem)**.** *Let $c$ be a critical value of the height function $h$ on $\mathcal{V}_{\mathrm{Q}}$, and define the set $\Sigma(c) \subseteq \Sigma$ by*

$$\Sigma(c) = \{\sigma \in \Sigma : h(\sigma) = c\},$$

*necessarily finite by Assumption 3.1.2. Let $m_1, m_2 \in \mathbb{R}$ be values such that $m_1 < c < m_2$ and such that $c$ is the* only *critical value of $h$ in the interval $[m_1, m_2]$. Then, topologically, $\mathcal{V}^{>m_1}$ can be obtained from $\mathcal{V}^{>m_2}$ by the attachment of $|\Sigma(c)|$ disks. Each disk is attached along $k$ disjoint subarcs of its boundary, where $k$ is the degree of degeneracy of the corresponding critical point.*

Note that in decreasing the value of $M$, no new components of $\mathcal{V}^{>M}$ are ever formed. Thus $\mathcal{V}^{>M}$ is never empty unless $\mathcal{V}_Q$ is empty itself.

Before continuing with the proof of this theorem, we note that it could be proved from standard Morse theorems by a perturbation argument, modifying $h$ slightly to assure that the critical points are non-degenerate and occur at different heights. The reason we present a full proof, however, is that the structure of the proof will be useful examining the homology class of the intersection cycle on $\mathcal{V}_Q$.

*Proof.* We first prove the theorem assuming that $\Sigma(c)$ is but a single point $\sigma_0 = (x_0, y_0)$. Then we will explain how the proof may be modified to allow for more critical points.

Our goal is to understand how the topology of $\mathcal{V}^{>m}$ changes as $m$ decreases from $m_2$ to $m_1$. The idea is to modify the height function $h$ to obtain a new proper height function $\tilde{h}$ that has all the same critical points, but such that the height at $\sigma_0$ will have been lowered to a new value $\tilde{c}$ such that $m_1 < \tilde{c} < c$. We shall denote by $\tilde{\mathcal{V}}$ the variety $\mathcal{V}$ endowed with this auxiliary height function. We will then use $\tilde{\mathcal{V}}$ as a tool in understanding $\mathcal{V}^{>m}$ for various values of $m$. The reasoning is detailed in the following steps:

1. The construction of $\tilde{h}$ will be such that $\tilde{h} \leq h$ everywhere and $\tilde{h} = h$ on $\mathcal{V}^{>m_2}$. Thus we have $\mathcal{V}^{>m_2} = \tilde{\mathcal{V}}^{>m_2}$.

2. Now let $m_0$ be any value such that $\tilde{c} < m_0 < c$. By the first structure theorem of Morse Theory, as there are no critical values of $\tilde{h}$ in $[m_0, m_2]$, we have that $\tilde{\mathcal{V}}^{>m_2}$ is topologically equivalent to $\tilde{\mathcal{V}}^{>m_0}$.

3. We now return to our original height function $h$. We will be able to understand exactly what happens topologically as we shift from $\tilde{\mathcal{V}}^{>m_0}$ to $\mathcal{V}^{>m_0}$.

4. Finally, again by the first structure theorem of Morse Theory and due to the fact that there are no critical values of $h$ in $[m_1, m_0]$, we have that $\mathcal{V}^{>m_0}$ is topologically equivalent to $\mathcal{V}^{>m_1}$.

Thus by understanding the topological changes in the third step, we will completely understand how $\mathcal{V}^{>m_2}$ and $\mathcal{V}^{>m_1}$ differ. With that outline in mind, we turn toward an examination of the height function.

We begin by examining the height function near the critical point $\sigma_0$. By smoothness of the variety at $\sigma_0$, $\mathcal{V}_{\mathrm{Q}}$ may be parameterized local to $\sigma_0$ in terms of either $x$ or $y$. Assume without loss of generality that we can parameterize in terms of $x$. Then because $h$ has a critical point at $\sigma_0$, and because $h$ is the real part of the complex analytic function $H$, we must have that $\frac{dH}{dx}$ vanishes at $x_0$ (when $H$ is parameterized by $x$). Thus locally $H$ takes the form:

$$H(x) = c + id + \sum_{j=k}^{\infty} c_j (x - x_0)^j,$$

where $k \geq 2$ and $c_k \neq 0$. Denoting $C = c + id$ and $g(x) = \sum_{j=k}^{\infty} c_j (x - x_0)^{j-k}$ we get

$$H(x) = C + (x - x_0)^k g(x),$$

where $g(x_0) \neq 0$.

Now as $g(x_0) \neq 0$, we have in a small neighborhood of $x_0$ that $g(x)^{1/k}$, the $k^{\text{th}}$ principal root of $g(x)$, is an analytic function. Thus locally the function

$$w = (x - x_0) g(x)^{1/k}$$

is analytic, and by inspection we see that the derivative $\frac{dw}{dx}$ does not vanish at $x = x_0$. So by the inverse function theorem we can parameterize $H$ locally by $w$. Note that, in terms of the variable $w$, $H$ takes on the relatively nice form:

$$H(w) = C + w^k.$$

Next we move our attention from $H$ to the actual height function $h$. Representing $w$ in polar coordinates as $w = \rho e^{i\theta}$, we get

$$H(\rho, \theta) = C + \rho^k e^{ik\theta},$$

so $h = \operatorname{Re} H$ takes the form

$$h(\rho, \theta) = c + \rho^k \cos(k\theta).$$

under this parameterization. Now let $\delta > 0$ be sufficiently small so that the ball $B_\delta(0)$ about $w = 0$ lies within the region on which this parameterization holds, and such that $\delta < (m_2 - c)^{1/k}$. The second condition guarantees that $h(w) < m_2$ on $B_\delta(0)$, which we will need later.

We now begin the construction of the modified height function $\tilde{h}$. To that end we must first construct a bump function for lowering the height of $\sigma_0$. So let $b : [0, \delta] \to \mathbb{R}$ be some smooth function such that

$$b(\rho) = \begin{cases} 1 & \text{for } \rho \leq \delta/3 \\ \\ 0 & \text{for } \rho \geq 2\delta/3 \end{cases}$$

and $0 \leq b(\rho) \leq 1$ for $\delta/3 \leq \rho \leq 2\delta/3$. Then $\tilde{h}$ is constructed as follows: on $B_\delta(0)$, in polar coordinates, we have $\tilde{h}(\rho, \theta) = h(\rho, \theta) - \varepsilon b(\rho)$ (where $\varepsilon$ is a sufficiently small positive constant). And outside of $B_\delta(0)$ we have that $\tilde{h} \equiv h$. Thus $\tilde{h}$ is obtained from $h$ by subtracting a very small bump function in a neighborhood of the critical point $\sigma_0$. Note that $\tilde{h}$ is still proper.

Assuming that $\varepsilon$ is sufficiently small, $\tilde{h}$ will have no new critical points beyond those of the original height function ($\nabla h \neq 0$ inside the modified annulus), and this will be the sought-after modified height function. Note that the height of $\sigma_0$ has been lowered: $\tilde{c} = \tilde{h}(\sigma_0) = c - \varepsilon$ (which is between $m_1$ and $c$ for sufficiently small $\varepsilon$). Note further that because $\tilde{h} < m_2$ on $B_\delta(0)$, we indeed have $\tilde{h} = h$ on $\mathcal{V}^{>m_2}$ as required.

Now let $m_0 = c - \varepsilon/2$, a value between $\tilde{c}$ and $c$. We wish to understand what the set $\tilde{\mathcal{V}}^{>m_0}$ looks like inside the neighborhood $B_\delta(0)$. To do that we first examine the constant height curves $\tilde{h} = m_0$ in this neighborhood. By the definition of $\tilde{h}$, these curves consist of

1. The constant height curve $h = m_0$ on points $w = \rho e^{i\theta}$ such that $\rho \geq 2\delta/3$.

2. The constant height curve $h = m_0 + \varepsilon = c + \varepsilon/2$ on points $w = \rho e^{i\theta}$ such that $\rho \leq \delta/3$.

3. Curves that interpolate between these segments for points $w = \rho e^{i\theta}$ such that $\delta/3 < \rho < 2\delta/3$.

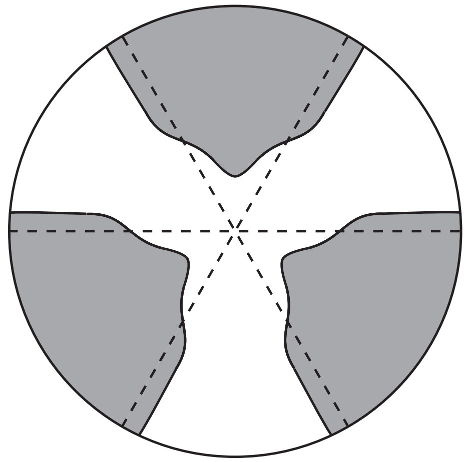

Figure 3.1: The region $\mathcal{V}_Q$ local to $\sigma_0$, where $\tilde{\mathcal{V}}^{>m_0}$ is shaded and the degeneracy $k = 3$.

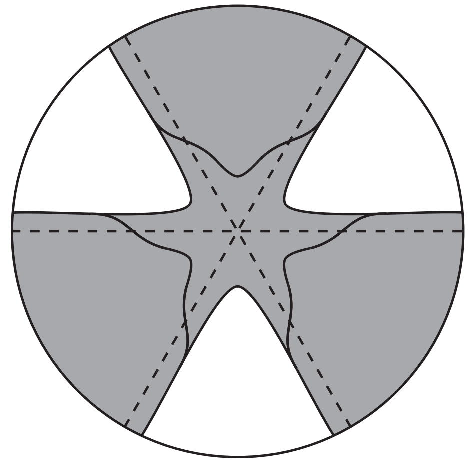

Figure 3.2: The region $\mathcal{V}_Q$ local to $\sigma_0$, where $\mathcal{V}^{>m_0}$ is shaded and the degeneracy $k = 3$.

Then we see that the set $\tilde{\mathcal{V}}^{>m_0} \cap B_\delta(0)$ is composed of $k$ connected components bound inside $B_\delta(0)$ by the curves described above. See Figure 3.1 for an example with $k = 3$.

Finally, we wish to return to our original height function, and see how the topology changes when we go from $\tilde{\mathcal{V}}^{>m_0}$ to $\mathcal{V}^{>m_0}$. Because $h$ and $\tilde{h}$ only differ on the set $B_{2\delta/3}(0)$, we see that the change is only local to $\sigma_0$. And by inspection, we see that $\mathcal{V}^{>m_0}$ is obtained from $\tilde{\mathcal{V}}^{>m_0}$ by adding a "star-shaped" region that joins the $k$ components of $\tilde{\mathcal{V}}^{>m_0} \cap B_\delta(0)$. See Figure 3.2 for a picture. This proves the theorem in the case that $|\Sigma(c)| = 1$.

If $|\Sigma(c)| > 1$ we proceed as before, however the height function $\tilde{h}$ is now obtained from $h$ by subtracting non-overlapping bump functions in a neighborhood of each of the points in $\Sigma(c)$. $\square$

Thus we have our piecewise understanding of the topology of $\mathcal{V}_Q$. Before proceeding with an investigation of the intersection cycle, we note the following corollary of the first structure theorem.

**Corollary 3.3.3.** *Let $M \in \mathbb{R}$ be such that*

$$M > \max\{h(\sigma) : \sigma \in \Sigma\}.$$

*Then $\mathcal{V}^{>M}$ is diffeomorphic to a finite set of disjoint, punctured open disks, each of which is an $x$-component or a $y$-component, but not both.*

*Proof.* By Theorem 3.2.6, $\mathcal{V}^{>M_0}$ takes the desired form for sufficiently large $M_0 > M$. But as $h$ has no critical values in the interval $[M, M_0]$, the first structure theorem tells us that the topologies of $\mathcal{V}^{>M}$ and $\mathcal{V}^{>M_0}$ are identical.

Furthermore, we noted that the components of $X^{>M_0}$ and $Y^{>M_0}$ are disjoint for sufficiently large $M_0$. As $\mathcal{V}^{>M}$ deformation retracts onto $\mathcal{V}^{>M_0}$, the same is true of their individual connected components. Hence $X^{>M}$ and $Y^{>M}$ are likewise disjoint. $\square$

## 3.4 The intersection cycle

We now have a sufficient description of the singular variety to produce an appropriate intersection cycle, and to analyze the homology of said cycle. We construct such a cycle in the following theorem/definition.

**Theorem 3.4.1.** *Let $M \in \mathbb{R}$ be larger than the maximum critical value for $h$. By Corollary 3.3.3, $\mathcal{V}^{>M}$ is diffeomorphic to a set of open, punctured disks that partition into $x$-components and $y$-components. Let the curve $\mathcal{C}$ consist of a set of disjoint cycles, one in each $x$-component of $\mathcal{V}^{>M}$ oriented positively about the puncture point. We call $\mathcal{C}$ the* intersection cycle. *Then*

$$a_{r,s} = \frac{1}{2\pi i}\int_{\mathcal{C}} \operatorname{Res}\left(\frac{P}{xyQ}e^{nH}\,dx \wedge dy\right) + O(\delta^n)$$

*as $n \to \infty$, for any arbitrarily small $\delta > 0$.*

The cycle $\mathcal{C}$ is not the exact intersection cycle as constructed in Chapter 1, but is a representative of the same homology class. Note that $\mathcal{C}$ is defined to be the empty cycle if $X^{>M}$ is empty for sufficiently large $M$.

*Proof.* Let $R$ be sufficiently small and $M$ sufficiently large so that

1. $M > \max\{h(\sigma) : \sigma \in \Sigma\}$.

2. If $Q(x, y) = 0$, then either $|x| < R$ or $|y| < R$ but not both.

3. The sets $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |x| < R\}$ and $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |y| < R\}$ are each diffeomorphic to finite sets of disjoint, punctured open disks (as in Theorem 3.2.3).

4. The set $\mathcal{V}^{>M}$ is diffeomorphic to a finite set of disjoint, punctured open disks (as in Theorem 3.2.6).

5. The set $\mathcal{V}^{>M}$ is a subset of the set $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |x| < R \text{ or } 0 < |y| < R\}$, i.e. $h(x, y) > M$ implies that either $|x| < R$ or $|y| < R$.

Now fix $r_x < R$ and $r_y$ sufficiently small so that

$$a_{r,s} = \frac{1}{(2\pi i)^2} \int_{T_0} \frac{P}{xyQ} e^{nH} \, dx \wedge dy,$$

where $T_0 = \{(x, y) \in \mathbb{C}^2 : |x| = r_x, |y| = r_y\}$. Define

$$M_0 = \sup\{|y| : (x, y) \in \mathcal{V}_{\mathrm{Q}} \text{ and } |x| = r_x\}$$

$$m_0 = \inf\{|y| : (x, y) \in \mathcal{V}_{\mathrm{Q}} \text{ and } |x| = r_x\}$$

Note that $M_0 < \infty$ because $y$ is locally a function of $x$ along the designated set, and $m_0 \geq R$ by choice of $R$: $|x|$ and $|y|$ can not simultaneously be less than $R$.

Now fix any positive $m_1 < \min\{r_y, m_0\}$ and define the torus $T = \{(x, y) \in \mathbb{C}^2 : |x| = r_x, |y| = m_1\}$. Then fix any $M_1 > M_0$ and define the homotopy

$$\begin{aligned} K : T \times [0,1] &\longrightarrow \mathbb{C}^2 \\ (x, y, t) &\longmapsto \left(x, y\left(1 + t\left(\frac{M_1}{m_1} - 1\right)\right)\right) \end{aligned}$$

$K$ intersects $\mathcal{V}_{\mathrm{Q}}$ in the set

$$C = \{(x, y) \in \mathcal{V}_{\mathrm{Q}} : |x| = r_x\},$$

and the intersection itself is transverse – the homotopy $K$ is obtained by expanding in the norm of $y$ through points on the variety where $\mathcal{V}_{\mathrm{Q}}$ is locally parameterizable in terms of $x$.

By Theorem 1.2.2, this means that

$$a_{r,s} = \frac{1}{2\pi i}\int_C \operatorname{Res}\left(\frac{P}{xyQ}e^{nH}\,dx \wedge dy\right) + \frac{1}{(2\pi i)^2}\int_{T_1} \frac{P}{xyQ}e^{nH}\,dx \wedge dy.$$

where $T_1 = K(T \times \{1\})$. By trivial bounds, we have

$$\frac{1}{(2\pi i)^2}\int_{T_1} \frac{P}{xyQ}e^{nH}\,dx \wedge dy = O\left(\left(\frac{1}{r_x^{\hat{r}} M_1^{\hat{s}}}\right)^n\right) \text{ as } n \to \infty,$$

as the only $n$ dependence in the integral comes from the $e^{nH}$ term. As $M_1$ may have been chosen to be arbitrarily large, this term is $O(\delta^n)$ for any arbitrarily small, positive $\delta$.

We now turn our attention to the cycle $C$. $C$ consists of a set of cycles, one on each punctured disk in the set $\mathcal{V}_{\mathrm{Q}} \cap \{(x, y) : 0 < |x| < R\}$, oriented positively around the puncture point. We examine such a disk $D$ and one such portion $C_0$ of the intersection cycle.

By Lemma 3.2.5, there are only two possibilities for the behavior of $h$ on the punctured disk $D$: either $h \to -\infty$, eventually and monotonically, as $x \to 0$ (in the case that $\alpha < -\hat{r}/\hat{s}$) or $h \to \infty$, eventually and monotonically, as $x \to 0$ (in the case that $\alpha > -\hat{r}/\hat{s}$).

In the former case, by shrinking $C_0$ closer to $|x| = 0$, we may assume that the maximum value of $h$ along $C_0$ is arbitrarily large negative – for instance, by properly shrinking $C_0$ to the puncture point we may assume that the maximum height along $C_0$ is less than $\log \delta$ for any small, positive $\delta$. Then again by trivial bounds, this leads to

$$\frac{1}{2\pi i} \int_{C_0} \operatorname{Res}\left(\frac{P}{xyQ} e^{nH}\, dx \wedge dy\right) = \frac{1}{2\pi i} \int_{C_0} \frac{-P}{xyQ_y} e^{nH}\, dx = O(\delta^n)$$

as $n \to \infty$. This we may drop this portion $C_0$ of the intersection cycle, and collect the difference into the $O(\delta^n)$ error term.

In the latter case we have that $h \to \infty$ near the puncture point of $D$. This implies that $D$ contains one of the $x$-components from the set $\mathcal{V}^{>M}$. By shrinking $C_0$ sufficiently close to $|x| = 0$, we may assume that $C_0$ is contained entirely within this $x$-component.

Doing this for every such $D$ and piece $C_0$ of the intersection cycle, we obtain a new cycle $\mathcal{C}$ such that

$$a_{r,s} = \frac{1}{2\pi i} \int_{\mathcal{C}} \operatorname{Res}\left(\frac{P}{xyQ} e^{nH}\, dx \wedge dy\right) + O(\delta^n),$$

where $\gamma$ consists of one cycle per $x$-component of $\mathcal{V}^{>M}$, oriented positively around the puncture points.

To finish the theorem, we have to show that this works not just for sufficiently large height $M$ but for any $M$ larger than all critical values of $h$. But this follows exactly from the first structure theorem of Morse theory (see Corollary 3.3.3). $\square$

## 3.5 First characterization theorem

We are nearly ready to characterize the homology class $[\mathcal{C}] \in H_1(\mathcal{V}_Q)$. We simply need several definitions.

**Definition 3.5.1.** We define the number $c_{xy}$ by

$$c_{xy} = \inf \{c \in \mathbb{R} : X^{>c} \cap Y^{>c} = \emptyset\} \in [-\infty, \infty).$$

Thus the value $c_{xy}$ captures the largest height $c$ at which the components of $X^{\geq c}$ and $Y^{\geq c}$ overlap. Note by the first structure theorem that this can only occur at a critical value for $h$, i.e. if $c_{xy} > -\infty$ then $c_{xy}$ is a critical value for the height function.

**Definition 3.5.2.** Define $\Xi$ to be the set of all $\sigma \in \Sigma$ such that

1. $h(\sigma) = c_{xy}$, and

2. For arbitrarily small neighborhoods $U \subseteq \mathcal{V}_Q$ of $\sigma$, $U \cap X^{>c_{xy}}$ and $U \cap Y^{>c_{xy}}$ are both non-empty.

We call $\Xi$ the set of *contributing points.*

Intuitively, the set $\Xi$ is the set of saddles at which $x$- and $y$-components first join.

Finally, we characterize the structure of $\mathcal{V}_Q$ local to any saddle point $\sigma \in \Sigma$.

**Definition 3.5.3.** For $\sigma \in \Sigma$, denote by $k \geq 2$ the degree of degeneracy of $h$ at $\sigma$. As detailed in the proof of the second structure theorem, there is a parameterization of $\mathcal{V}_Q$ local to $\sigma$ on which $h$ takes the form

$$h(w) = h(\sigma) + \operatorname{Re} w^k.$$

Thus on any neighborhood $U$ of $\sigma$ on which this parameterization holds, we see that $U \cap \mathcal{V}^{>h(\sigma)}$ consists of $k$ connected components, which we term *ascent regions.* Similarly we see that $U \cap \mathcal{V}^{<h(\sigma)}$ consists of $k$ connected components, which we term *descent regions.*

Then we have the following theorem, which characterizes the homology class $[\mathcal{C}]$ in a manner amenable to saddle point analysis.

**Theorem 3.5.4** (First Characterization Theorem)**.** *If the set $\Xi$ is non-empty, then the homology class $[\mathcal{C}]$ contains a representative cycle $\kappa$ such that*

1. *The height along $\kappa$ is maximized precisely at the points in $\Xi$, and*

2. *For a fixed $\sigma \in \Xi$ and $U \subseteq \mathcal{V}_Q$ a sufficiently small neighborhood of $\sigma$, denote the ascent regions in $U$ by $A_0, \ldots, A_{k-1}$ (ordered counterclockwise about $\sigma$). Denote by $D_j$ the descent region between $A_j$ and $A_{j+1 \bmod k}$. Then for each $j$ there is a curve $\gamma_j$ in $U$, starting at $\sigma$ and traveling down $D_j$, such that*

$$\kappa \cap U = \sum_{j=0}^{k-1} (X(j+1 \bmod k) - X(j))\gamma_j,$$

*where $X(j) := 1$ if $A_j \subseteq X^{>c_{xy}}$ while $X(j) := 0$ if $A_j \subseteq Y^{>c_{xy}}$ (well-defined by definition of $c_{xy}$).*

*If the set $\Xi$ is empty, then the homology class $[\mathcal{C}]$ contains a representative cycle $\kappa$ supported on $\mathcal{V}^{\leq m}$ for arbitrarily small $m \in \mathbb{R}$.*

We pause to note that the prescribed representative $\kappa$ is in some sense the best we can get in that there are local obstructions to lowering the height any further. Specifically, for $\sigma \in \Xi$ and for $U$ a sufficiently small neighborhood of $\sigma$, the cycle $\kappa \cap U$ is a nontrivial element of the relative homology group $H_1(U, U \cap \mathcal{V}^{\leq c_{xy} - \varepsilon})$ (for $\varepsilon > 0$ sufficiently small).

*Proof.* We begin by assuming that $\Xi$ is non-empty, and we want to use our Morse-theoretic understanding of the space $\mathcal{V}_Q$ to find a cycle $\kappa$ sought by the theorem. To begin with, let $M_0 \in \mathbb{R}$ be any value larger than all critical values of $h$. We examine $\mathcal{C}$ on $\mathcal{V}^{>M_0}$. Recall that $\mathcal{C}$ is composed of a disjoint set of cycles, one in each $x$-component of $\mathcal{V}^{>M_0}$ oriented positively about the puncture points. Note that $\mathcal{C}$ is homologous to $\partial X^{>M_0}$ (along which

$h \equiv M_0$). We wish to show that this characterization of $[\mathcal{C}]$ remains true when $M_0$ is replaced by any $M \in (c_{xy}, M_0]$. We do so inductively.

By the first structure theorem of Morse theory, the topology of $\mathcal{V}^{>M}$ (and hence the characterization of $[\mathcal{C}]$) changes only as $M$ passes through critical values of $h$. So let $c \in (c_{xy}, M_0]$ denote some critical value of $h$, and assume that $\mathcal{C}$ is homologous to $\partial X^{>M}$ in $\mathcal{V}^{>M}$ for any $M \in (c, M_0]$. Using the proof of the second structure theorem as a guide, we examine the topological changes that occur local to $\Sigma(c)$ as $M$ is decreased below $c$.

Exactly as in the the proof of the second structure theorem, we form a new height function $\tilde{h}$ that differs from our old height function only in small neighborhoods of the points in $\Sigma(c)$, on which $\tilde{h} = h - \varepsilon b$ for some sufficiently small $\varepsilon > 0$ and some bump function $b$. Denote by $\tilde{X}^{>M}$ and $\tilde{Y}^{>M}$ the $x$- and $y$-components of $\tilde{\mathcal{V}}^{>M}$ as prescribed by this new height function $\tilde{h}$. By the definition of $\tilde{h}$, we know that $\tilde{X}^{>c+\varepsilon/2}$ and $X^{>c+\varepsilon/2}$ are topologically equivalent, and hence that $\mathcal{C}$ is homologous to $\partial\tilde{X}^{c+\varepsilon/2}$. This implies that $\mathcal{C}$ is homologous to $\partial\tilde{X}^{c-\varepsilon/2}$, there being no critical values of $\tilde{h}$ in $[c - \varepsilon/2, c + \varepsilon/2]$. Finally, we wish to show that $\partial\tilde{X}^{c-\varepsilon/2}$ is homologous to $\partial X^{c-\varepsilon/2}$.

First, note that $\partial\tilde{X}^{>c-\varepsilon/2}$ and $\partial X^{>c-\varepsilon/2}$ agree outside of small neighborhoods of the points in $\Sigma(c)$. So we examine these cycles local to said critical points. As $c > c_{xy}$, there are only two possibilities for each $\sigma \in \Sigma(c)$:

1. The ascent regions local to $\sigma$ are all in $Y^{>c}$, or

2. The ascent regions local to $\sigma$ are all in $X^{>c}$.

We examine each possibility in turn.

In the first case, there is no piece of $\partial\tilde{X}^{>c-\varepsilon/2}$ or of $\partial X^{>c-\varepsilon/2}$ local to $\sigma$. So assume that we are in the second case. Denote by $k$ the degeneracy of the saddle $\sigma$. Then local

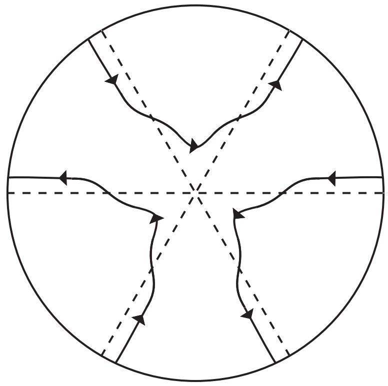

Figure 3.3: The cycle $\partial\tilde{X}^{>c-\varepsilon/2}$ local to a saddle $\sigma$ of degeneracy $k = 3$.

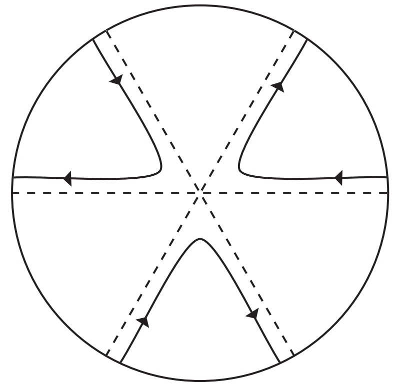

Figure 3.4: The cycle $\partial X^{>c-\varepsilon/2}$ local to a saddle $\sigma$ of degeneracy $k = 3$.

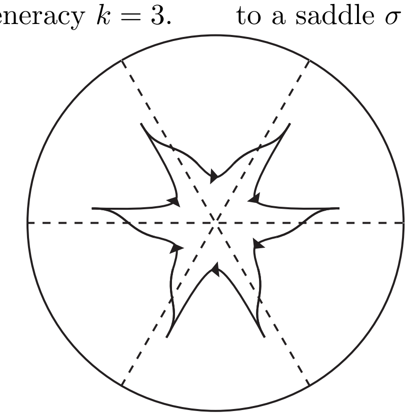

Figure 3.5: The difference between the cycles represented in Figure 3.3 and Figure 3.4.

to $\sigma$, $\partial\tilde{X}^{>c-\varepsilon/2}$ consists of $k$ curves bounding the $k$ disjoint components where $\tilde{h} > c - \varepsilon/2$ (see Figure 3.3 for an example with $k = 3$). But note that in this small neighborhood, this cycle is homologous to $\partial X^{>c-\varepsilon/2}$, as their difference is a boundary (see Figures 3.4 and 3.5). And thus globally the difference between $\partial\tilde{X}^{>c-\varepsilon/2}$ and $\partial X^{>c-\varepsilon/2}$ is a finite union of such boundaries, which shows that these cycles are homologous.

Thus we have that $\mathcal{C}$ is homologous to $\partial X^{>c-\varepsilon/2}$, and by induction we can see that $\mathcal{C}$ is homologous to $\partial X^{>M}$ for $M \in (c_{xy}, M_0)$. So what happens to $[\mathcal{C}]$ as $M$ is lowered beyond

$c_{xy}$? We examine the topological changes that occur local to the set $\Sigma(c_{xy})$. The analysis is similar to our prior analysis but with the added kink that there are points from both $X^{>c_{xy}}$ and $Y^{>c_{xy}}$ local to the contributing points.

As before we form a modified height function $\tilde{h}$ that locally pushes down the points of $\Sigma(c_{xy})$. With respect to this new function we again have that $\mathcal{C}$ is homologous to $\partial\tilde{X}^{>c_{xy}-\varepsilon/2}$ for some small $\varepsilon > 0$. So define the cycle $\kappa_0 = \partial\tilde{X}^{>c_{xy}-\varepsilon/2}$. With a few alterations, we will be able to turn $\kappa_0$ into the desired representative.

Outside of small neighborhoods of the points in $\Sigma(c_{xy})$ we have that $h \equiv \tilde{h}$, and hence the height $h$ along $\kappa_0$ is equal to $c_{xy} - \varepsilon/2$ away from these points. And by the same reasoning as in the inductive argument above, we can alter $\kappa_0$ local to the points of $\Sigma(c_{xy}) \setminus \Xi$ to obtain a homologous cycle whose height $h$ is bounded below $c_{xy}$ local to these points. What remains is to examine $\kappa_0$ local to the points of $\Xi$.

In a sufficiently small neighborhood $U$ of any $\sigma \in \Xi$, the cycle $\kappa_0 \cap U$ consists of one curve for each ascent region in $X^{>c_{xy}}$ (see Figure 3.6, with two ascent regions from $X^{>c_{xy}}$ and one ascent region from $Y^{>c_{xy}}$). Each such curve enters the neighborhood at some point $p_1$ and leaves at some point $p_2$, where $p_1$ and $p_2$ lie in separate descent regions (see Figure 3.6). And each such curve is homotopic to a curve whose height is maximized at the saddle point, namely one that travels straight from $p_1$ to $\sigma$ (contained within one descent region), and then from $\sigma$ to $p_2$ (contained within a separate descent component). See Figure 3.7. Making these adjustments to $\kappa_0$ at every such contributing point results in a cycle homologous to $\mathcal{C}$ along which $h$ is maximized precisely at the points of $\Xi$.

To complete the construction, a few more changes must be made to $\kappa_0$. Note that currently, in a sufficiently small neighborhood $U$ of any contributing point $\sigma$, $\kappa_0 \cap U$ consists

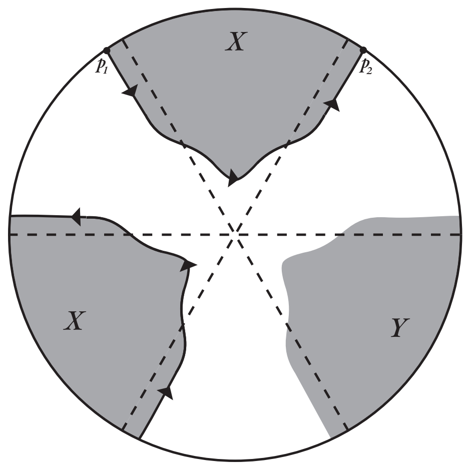


Figure 3.6: A representation of $\kappa_0$ local to $\sigma$ with two ascent regions in $X^{>c_{xy}}$ and one ascent region in $Y^{>c_{xy}}$. The set $\tilde{\mathcal{V}}^{>c_{xy}-\varepsilon/2}$ is shaded.

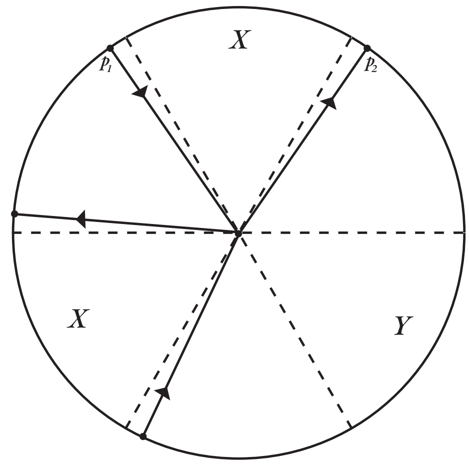


Figure 3.7: A representation of the two pieces of $\kappa_0$ after performing the alteration described in the text.

of a set of curves in the descent regions $D_j$ local to $\sigma$ as follows:

1. If $D_j$ is located between an ascent region in $X^{>c_{xy}}$ and an ascent region in $Y^{>c_{xy}}$ (in that order, counterclockwise about $\sigma$), then $\kappa_0 \cap D_j$ consists of a single curve $\gamma_j$ traveling up $D_j$ to $\sigma$.

2. If $D_j$ is located between an ascent region in $Y^{>c_{xy}}$ and an ascent region in $X^{>c_{xy}}$ (in that order, counterclockwise about $\sigma$), then $\kappa_0 \cap D_j$ consists of a single curve $\gamma_j$ traveling down $D_j$ from $\sigma$.

3. If $D_j$ is located between two ascent regions in $Y^{>c_{xy}}$, then $\kappa_0 \cap D_j = \emptyset$.

4. If $D_j$ is located between two ascent regions in $X^{>c_{xy}}$, then $\kappa_0 \cap D_j$ consists of two pieces, a curve entering $\sigma$ and a curve exiting $\sigma$.

See Figure 3.7 for an example. Note that in case (4) above, the portion of $\kappa_0$ in said descent

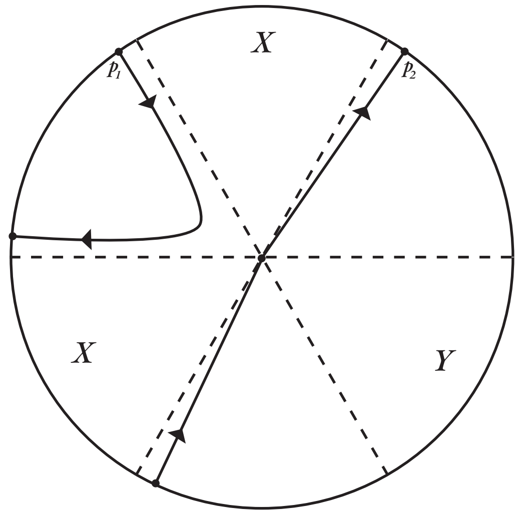


Figure 3.8: A representation of $\kappa$ local to a $\sigma \in \Xi$, after performing the final alterations.

region can be "broken off" and pushed down the descent region, so the height along this portion will be bounded strictly below $c_{xy}$. See Figure 3.8 for an example, and compare to Figure 3.7. After making the prescribed changes local to every point point of $\Xi$, we obtain the cycle $\kappa$ as prescribed in the statement of the theorem.

In the case that $\Xi$ is empty, $c_{xy} = -\infty$ and the inductive argument applied from the beginning of this proof can be extended to arbitrary large negative height. That is, $\mathcal{C}$ is homologous to $\partial X^{>m}$ (along which the $h \equiv m$) for arbitrarily small $m \in \mathbb{R}$. This yields the representative claimed in the statement of the theorem. $\square$

The representative $\kappa \in [\mathcal{C}]$ is precisely the type of cycle amenable to the technique of saddle point integration techniques. Before exploring said techniques, however, we shall weaken one of our original assumptions in order to prove a slightly stronger version of the preceding characterization theorem.

## 3.6 Generalized characterization theorem

In this this section we begin by weakening Assumption 3.1.3 to obtain the following.

**Assumption 3.6.1.** *Assume that the critical points of $\mathcal{V}_Q$ are isolated (and hence finite).*

For the purpose of our impending algorithmic study of $\mathcal{V}_Q$, we note that the preceding assumption is equivalent to the condition that the polynomial ideal $\langle Q, Q_x, Q_y \rangle$ be 0-dimensional, a fact that can be checked via Gröbner basis computation.

Under this assumption, we have the following definitions:

**Definition 3.6.2.** We define the constant $c_0$ by

$$c_0 = \max \{h(z_0) : z_0 \in \mathcal{V}_Q,\ \mathcal{V}_Q \text{ is not smooth at } z_0\},$$

and we define the constant $c_{xy}$ by

$$c_{xy} = \inf \{c > c_0 : X^{>c} \cap Y^{>c} = \emptyset\} \in [c_0, \infty)$$

Note that this definition of $c_{xy}$ generalizes the previous one if we adopt the convention that $c_0 = -\infty$ in the case that $\mathcal{V}_Q$ is smooth. We similarly generalize the definition of $\Xi$ as follows.

**Definition 3.6.3.** Define the set $\Xi$ to be the set of all $\sigma \in \Sigma$ such that

1. $h(\sigma) > c_0$,
2. $h(\sigma) = c_{xy}$, and
3. For arbitrarily small neighborhoods $U \subseteq \mathcal{V}_Q$ of $\sigma$, $U \cap X^{>c_{xy}}$ and $U \cap Y^{>c_{xy}}$ are both non-empty.

We call $\Xi$ the set of *contributing points.*

Note that $\Xi$ is empty if $c_{xy} = c_0$.

Then the key is to notice that $\mathcal{V}^{>c_0}$ is a smooth manifold, and that on this surface the analyses carried out up to this point still hold. Thus we can define the intersection cycle $\mathcal{C}$ on $\mathcal{V}^{>M}$ for sufficiently large $M$, we can use the structure theorems of Morse theory to find representatives for $[\mathcal{C}]$ as we decrease $M$, and generally implement all the methods from before – as long as we restrict our attention to $\mathcal{V}^{>c_0}$. If $\Xi$ is not empty, this enables us to find a cycle $\kappa \in [\mathcal{C}]$ exactly as before. On the other hand if $\Xi = \emptyset$, then we can find representative cycles for $[\mathcal{C}]$ along which the height is arbitrarily near, but greater than, $c_0$. This yields the following corollary to the first characterization theorem.

**Corollary 3.6.4** (Generalized Characterization Theorem)**.** *If the set $\Xi$ is non-empty, then the homology class $[\mathcal{C}]$ contains a representative cycle $\kappa$ just as in Theorem 3.5.4. On the other hand if $\Xi$ is empty, then $[\mathcal{C}]$ contains a representative cycle $\kappa$ supported on $\mathcal{V}^{\leq M}$ for any $M > c_0$.*

Applying the method of steepest descent to the cycle $\kappa$ described in the characterization theorem, we obtain the following corollary for the asymptotic analysis of the $a_{r,s}$.

**Corollary 3.6.5.** *Denote $\Xi = \{\sigma_1, \ldots, \sigma_m\}$ and let $\kappa$ be the cycle as prescribed in the characterization theorem. If $\Xi$ is not empty then we have*

$$a_{r,s} \sim \sum_{i=1}^{m} e^{nH(\sigma_i)} \sum_{j=0}^{\infty} u_{i,j} n^{-(1+j)/k_i},$$

*where $k_i \geq 2$ is the degree of degeneracy of $\sigma_i$. Moreover, the coefficients $u_{i,j}$ are computable, assuming computability of the $\sigma_i$ and of the tangent direction of the pieces of $\kappa$ local to each $\sigma_i$.*

*On the other hand if $\Xi$ is empty, then we have that*

$$a_{r,s} = o(e^{nM})$$

*for any $M > c_0$.*

*Proof.* We begin by assuming that the set $\Xi$ is non-empty. Recall that by Theorem 3.4.1 we can write

$$a_{r,s} = \frac{1}{2\pi i} \int_{\kappa} \operatorname{Res}\left(\frac{P}{xyQ} e^{nH}\, dx \wedge dy\right) + O(\delta^n) \tag{3.6.1}$$

for arbitrarily small $\delta > 0$. We will apply the method of steepest descent to obtain an asymptotic expansion for the preceding integral.

For each $i \in \{1, 2, \ldots, m\}$, let $U_i$ be a small neighborhood of $\sigma_i$ in $\mathcal{V}_{\mathrm{Q}}$. Let $U = \bigcup_{i=1}^{m} U_i$. Then we can split the cycle $\gamma_0$ into two pieces

$$\kappa = \kappa_1 + \kappa_2,$$

where $\kappa_1 = \kappa \cap U$ and $\kappa_2 = \kappa \setminus \kappa_1$. Then because $h < c_{xy}$ along the compact curve $\kappa_2$, we can find a sufficiently small $\varepsilon$ such that

$$h(z) \leq c - \varepsilon \text{ for } z \in \kappa_2$$

But the residue form $\operatorname{Res}\left(\frac{P}{xyQ} e^{nH}\, dx \wedge dy\right)$ admits the form

$$\frac{-P}{xyQ_y} e^{nH}\, dx \text{ or } \frac{P}{xyQ_x} e^{nH}\, dy$$

(according to when $Q_y \neq 0$ or $Q_x \neq 0$), and so by trivial bounds we have that

$$\frac{1}{2\pi i} \int_{\kappa_2} \operatorname{Res}\left(\frac{P}{xyQ} e^{nH}\, dx \wedge dy\right) = O(e^{n(c-\varepsilon)}).$$

Thus equation 3.6.1 reduces to

$$a_{r,s} = \frac{1}{2\pi i} \int_{\kappa_1} \operatorname{Res}\left(\frac{P}{xyQ} e^{nH}\, dx \wedge dy\right) + O(e^{n(c-\varepsilon)}),$$

and the computation is reduced to a set of integrals local the points of $\Xi$.

Local to a particular saddle point $\sigma_i$, smoothness of $\mathcal{V}_{\mathrm{Q}}$ guarantees that one of $Q_y$ or $Q_x$ does not vanish. Assume without loss of generality that $Q_y \neq 0$. Then locally we have

$$\operatorname{Res}\left(\frac{P}{xyQ}e^{nH}\,dx \wedge dy\right) = \frac{-P}{xyQ_y}e^{nH}\,dx = e^{nH(\sigma_i)}A(x)e^{-n\phi(x)},$$

where

$$A(x) = \frac{-P}{xyQ_y}$$

$$\phi(x) = H(\sigma_i) - H$$

parameterized locally by $x$.

Assuming that $U_i$ is sufficiently small, the curve $U_i \cap \kappa_1$ consists of pieces exiting $\sigma_i$ and traveling down certain descent regions and pieces entering $\sigma_i$ by traveling up certain descent regions. To finish the theorem we must integrate over these pieces and finally sum over all such $i$ from 1 to $m$.

So let $\gamma$ be an arbitrary curve segment exiting $\sigma_i$ and traveling down a descent region. We have

$$\frac{1}{2\pi i}\int_\gamma \operatorname{Res}\left(\frac{P}{xyQ}e^{nH}\,dx \wedge dy\right) = \frac{e^{nH(\sigma_i)}}{2\pi i}\int_\gamma A(x)e^{-n\phi(x)}\,dx$$

where $A$ and $\phi$ satisfy the conditions of Theorem 1.4.2. Thus we get a series expansion of the form

$$\int_\gamma A(x)e^{-n\phi(x)}\,dx \sim \sum_{j=0}^{\infty} a_j n^{-(1+j)/k_i}$$

where $k_i$ is the index of the first non-vanishing term of $\phi(x)$, also the degree of degeneracy of $h$ at the saddle point $\sigma_i$. Note that the coefficients $a_j$ are computable from the coefficients of $A$ and $\phi$ and from the tangent direction in which $\gamma$ exits $\sigma_i$. The coefficients of $A$ and

$\phi$ are in turn computable from the partials of $P$, $Q$ and $H$ at $\sigma_i$. Thus computability of the asymptotic expansion rests on computability of the saddle point $\sigma_i$ and the directions of each $\gamma$ in the description of $\kappa$ local to $\Xi$.

Integrating over $U_i \cap \kappa_1$ consists of integrating over finitely many such $\gamma$ either with positive or negative orientation, and by summing over corresponding expansions we get an expansion of the following form

$$\frac{1}{2\pi i}\int_{U_i\cap\kappa_1} \mathrm{Res}\left(\frac{P}{xyQ}e^{nH}\,dx\wedge dy\right) \sim e^{nH(\sigma_i)}\sum_{j=0}^{\infty} u_{i,j}n^{-(1+j)/k_i}.$$

Summing similar computations over all such points $\sigma_i \in \Xi$ yields the desired result.

On the other hand if $\Xi$ is empty, then by Theorem 3.6.4 we can find a cycle $\kappa \in [\mathcal{C}]$ along which $h \leq M$ for any $M > c_0$. Then again by trivial bounds we obtain

$$\frac{1}{2\pi i}\int_{\kappa} \mathrm{Res}\left(\frac{P}{xyQ}e^{nH}\,dx\wedge dy\right) = O(e^{nM}).$$

And as $a_{r,s} = O(e^{nM})$ for all $M > c_0$, we must have $a_{r,s} = o(e^{nM})$ for all $M > c_0$. $\square$

By this corollary we see that computing an asymptotic series/bound for the $a_{r,s}$ reduces to locating the points of $\Xi$ and computing the structure of $\kappa$ local to the points of $\Xi$. In the next chapter, we produce an algorithm for this task.

# Chapter 4

# Algorithmic Implementation

## 4.1 Introduction

In the following chapter we present rigorous numerics for computing the location of $\Xi$ and the directions of $\kappa \in [\mathcal{C}]$ local to the points of $\Xi$, where $\Xi$ and $\kappa$ are as characterized in Corollary 3.6.4. That is, we exhibit a rigorous algorithm for computing the preceding information to within arbitrary numerical accuracy. As indicated in Corollary 3.6.5, this information can then be used to compute asymptotic formulas/bounds for the coefficients $a_{r,s}$.

Motivated by the characterization theorem, we begin by examining the structure of $\mathcal{V}^{>c}$ local to any $\sigma \in \Sigma$, where $c = h(\sigma)$. On a sufficiently small neighborhood $U \subseteq \mathcal{V}_Q$ of $\sigma$, we have that $U \cap \mathcal{V}^{>c}$ decomposes into $k$ ascent regions, where $k$ is the degeneracy of $h$ at $\sigma$. Assuming that $c \geq c_{xy}$ and $c > c_0$, we know that each such region $A$ belongs either to $X^{>c}$ or to $Y^{>c}$, but not both. If we can determine which occurs for each region $A$ and for every such saddle point $\sigma$, we will have exactly the information necessary to construct $\Xi$ and the relevant cycle $\kappa \in [\mathcal{C}]$ local to the points in $\Xi$.

So let $\sigma \in \Sigma$ be such that $h(\sigma) \geq c_{xy}$ and $h(\sigma) > c_0$, and let $A$ be any local ascent region. Let $\gamma : [0,1) \to \mathcal{V}_Q$ be any path that begins at $\gamma(0) = \sigma$, travels out through $A$, is monotonically increasing with respect to $h$ and approaches arbitrarily large height. (Note that the existence of such a path follows directly from the Morse-theoretic decomposition of $\mathcal{V}_Q$.) Because $h(\gamma(t)) \to \infty$ as $t \to 1$, we must have that $\gamma((0,1))$ includes points arbitrarily close to $x = 0$ and/or $y = 0$. But we can't have both, as the condition $c \geq \max\{c_{xy}, c_0\}$ assures that $X^{>c}$ and $Y^{>c}$ are disjoint. Thus if it can be determined whether $\gamma$ comes arbitrarily close to $x = 0$ or $y = 0$, then it can be determined whether $A$ belongs to $X^{>c}$ or $Y^{>c}$.

This suggests the following outline for an algorithm:

1. Input $Q$, $\hat{r}$ and $\hat{s}$.

2. Use Gröbner basis computations to find the points of $\mathcal{V}_Q$ where $h$ has saddle points (the set $\Sigma$) and where the variety $\mathcal{V}_Q$ is not smooth. Terminate if any of these points are not isolated. Denote by $c_0$ the height of the highest non-smooth point ($c_0 = -\infty$ if the variety is smooth).

3. Step through each $\sigma \in \Sigma$ in decreasing order of height, while $h(\sigma) > c_0$. For each such $\sigma$, do the following:

   (a) Use Gröbner basis computations to compute the degree of degeneracy $k$ of $h$ at $\sigma$.

   (b) Compute the $k$ steepest ascent directions for $h$ at $\sigma$, i.e. the directions of the $k$ ascent regions.

(c) For each ascent region $A$, follow a strictly ascending path along $\mathcal{V}_Q$ beginning at $\sigma$ and traveling through $A$. Follow each such path until it is clear whether the path will approach arbitrarily close to $x = 0$ or $y = 0$. Keep track of this information, which determines to which of $X^{>c}$ or $Y^{>c}$ each ascent component belongs.

4. In the preceding loop: if ever a $\sigma$ is found for which some of the ascent components belong to $X^{>c}$ while others belong to $Y^{>c}$, note the height $c_{xy} = h(\sigma)$ and add $\sigma$ to $\Xi$. Continue through the rest of the saddles at height $c_{xy}$, adding them to $\Xi$ according to the same rule. Terminate the loop when done with the saddles at height $c_{xy}$.

5. The algorithm terminates with the following conditions

   (a) If $\Xi$ is empty, then there is a cycle $\kappa \in [\mathcal{C}]$ such that $\sup_\kappa h \leq M$ for any $M > c_0$. This gives a bound on the growth of the coefficients $a_{r,s}$ as prescribed in Corollary 3.6.5.

   (b) If $\Xi$ is not empty, then there is a cycle $\kappa \in [\mathcal{C}]$ such that the height along $\kappa$ is maximized precisely at the points of $\Xi$, and the structure of $\kappa$ local to $\Xi$ is as prescribed in the characterization theorem. Note that all of the information necessary to describe this local structure is contained in the steps of the preceding algorithm.

Of course it remains to be shown that an algorithm of this form can be effectively implemented. In what follows, we demonstrate this to be so by presenting a complete pseudo-code implementation. The first step is to sufficiently develop the notations/assumptions of the pseudo-language that we shall use.

## 4.2 Describing the pseudo-language

Central to the pseudo-language is a suitable means of representation for certain computable numbers, and a system whereby these numbers can be acted upon by certain computable functions and operations. Specifically, we draw upon the construct of ball numbers and ball arithmetic as developed in Mathemagix ([vdH08]). Note that the exact notation and functions developed below may not be as implemented in Mathemagix, but equivalent functions either exist or can be produced in this language.

We begin by describing the `ball` data type, which we shall use to represent certain computable numbers to within arbitrary precision. Each variable `b` of type `ball`, or *ball number*, represents some computable number $b \in \mathbb{C}$. Associated to `b` are a variable (`b.tol`) and two functions (`b.approx()` and `b.mod()`). The variable `b.tol` is a positive rational number set at the time of `b`'s initialization, but may be changed by the user at any time. It stores a computation tolerance. The function `b.approx()` then returns an element $\tilde{b} \in \mathbb{Q}[i]$ such that $|\tilde{b} - b|$ is less than the tolerance `b.tol`. The user may also pass an optional argument to `b.approx()`; if we call `b.approx(r)` with `r` a positive rational number, then this function will return an approximation of $b$ to within a tolerance of `r` (regardless of, and without changing, the value of `b.tol`). Finally, the function `b.mod()` returns an element of $\mathbb{Q}$ that approximates $|b|$ to within a tolerance of `b.tol`. And as with `b.approx()`, `b.mod()` also accepts an optional rational argument for specifying a temporary tolerance value.

The pseudo-language also includes two ball number subtypes: real ball numbers (denoted by the type `realball`), and algebraic ball numbers (denoted by the type `algball`). The `realball` data type is nearly identical to the `ball` data type, but is used only in the representation of computable real numbers. In this case, the `approx()` subfunction always returns an element of $\mathbb{Q}$. The `algball` data type is used for specifically to represent algebraic numbers. An algebraic ball number `b` representing some algebraic $b \in \mathbb{C}$ includes all the attributes associated to a generic ball number, but also includes a new subfunction: `b.poly()`. This function takes an indeterminate as its input (e.g. `x`), and outputs a rational-coefficient polynomial in that indeterminate (e.g. `P(x)`) of which $b$ is a root. Note: if it is not clear which data type is to be used in initializing an expression, the type will be expressly indicated using functional notation; e.g., `realball(expression)`.

Central to the construction of the `ball` data types is the existence of a sort of arithmetic of ball numbers. Namely many of the usual computable functions/operations that can be applied to elements of $\mathbb{C}$ can be applied to ball numbers, resulting in a ball number output representing the result of the computation. Specifically, the pseudo-language allows us to apply all arithmetic operations, roots, exponentiation, complex conjugation, complex modulus, trigonometric functions and the real logarithm to ball numbers. In the case of the complex modulus function, the output is assumed to be a real ball number. In the case of the real logarithm, the input is assumed to be a real ball number.

Having specified the structure of ball numbers, we must take some time to discuss the origin of said objects. For certain fundamental computable constants (e.g. $\pi$), we assume that a ball number representation is available in the language (e.g. `Pi`). The vast majority of ball numbers, however, will come from solving univariate polynomial equations. Specifically, we assume the existence of a function `solve()`, whose input is a univariate polynomial `P` with coefficients in $\mathbb{Q}[i]$ and whose output is an array of algebraic ball numbers corresponding to the distinct roots of `P`. Note that such a function exists and has been implemented in Mathemagix. We further assume that the roots in this array are initialized with the same tolerance, and that the tolerance has been suitably shrunk; specifically, that the tolerance is less than three times the distance between any two distinct roots of `P`. These assumptions will be useful later on, allowing us to bound subsequent root approximations away from one another (the distance between any two such approximations being greater than this initial tolerance). We will also make use of a similar function `realsolve()`, whose input is a univariate polynomial `P` with coefficients in $\mathbb{Q}$ and whose output is an array of algebraic ball numbers corresponding to the distinct *real* roots of `P`.

In addition to the `ball` data type, we also construct two special data types for the representation of saddle and singular points. These data types are nothing more than specialized arrays containing the many pieces of information that we wish to store about such points. Note that each element of the array is empty at the time of initialization, and is set in the course of the algorithm.

The simpler of the two data types is that which is used for singular points: the type `nonSmooth`. An element `p` of `nonSmooth` includes three pieces of information: algebraic ball numbers `p.x` and `p.y`, and a real ball number `p.height`. The numbers `p.x` and `p.y` represent the $x$- and $y$-coordinates of a singular point $(x, y)$, while the number `p.height` represents the exponential of its height, $e^{h(x,y)}$.

The data type for saddle points – `saddle` – includes far more information. Let `sigma` be an arbitrary saddle variable, meant to represent a saddle point $\sigma \in \Sigma$. In addition to the location and height information (`sigma.x`, `sigma.y` and `sigma.height`), `sigma` also includes the following eight pieces of information:

1. `sigma.byX` – a boolean value which is set to `True` if $\mathcal{V}_Q$ is locally parameterizable by $x$ in a neighborhood of $\sigma$. In this case we think of $x$ as an independent variable and

$y$ as a dependent variable. It is set to `False` otherwise, in which case the roles of $x$ and $y$ are reversed. Note that this boolean is only used for saddle points higher than $c_0$, where $\mathcal{V}_Q$ is smooth.

2. `sigma.Ri` – a positive rational number. This denotes the "radius of the independent variable." `sigma.Ri` is the radius of a neighborhood in the plane of the independent variable on which parameterization by the independent variable holds.

3. `sigma.Rd` – a positive rational number. This denotes the "radius of the dependent variable." When $\mathcal{V}_Q$ is parameterized by the independent variable on a disk of radius `sigma.Ri`, the corresponding value of the dependent variable remains in a disk of radius `sigma.Rd`.

4. `sigma.R` – a rational number. The number `sigma.R` $<$ `sigma.Ri` represents a radius on which parameterization by the independent variable is geometrically nice (in a sense to be described in Lemma 4.12.1 below). This radius is used in the construction of short ascent paths as parameterized by the independent variable.

5. `sigma.pathToX` – an array of boolean values. The length of `sigma.pathToX` is equal to the degeneracy of $h$ at $\sigma$, and the value of `sigma.pathToX[i]` is set to `True` if an ascent path from $\sigma$ through the $i^{\text{th}}$ ascent component local to $\sigma$ is found to approach $x = 0$. It is set to `False` otherwise.

6. `sigma.xPole` – a boolean value. `sigma.xPole` is set to `True` if all of the values in the array `sigma.pathToX` are set to `True`. It is set to `False` if all of the values in the array `sigma.pathToX` are set to `False`. This is to keep track of whether $\sigma$ is in $X^{\geq h(\sigma)}$ or $Y^{\geq h(\sigma)}$. Note that this value is not initialized for $h(\sigma) \leq c_{xy}$.

7. `sigma.out` – an array of ball numbers. If `sigma` is found to be a contributing saddle, then by the characterization theorem there is an $\kappa \in [\mathcal{C}]$ whose structure local to $\sigma$ consists of a set of short paths emanating from $\sigma$ in various directions. `sigma.out` is an array of those directions along which $\kappa$ is parameterized to flow away from $\sigma$ (as parameterized by the independent variable).

8. `sigma.in` – an array of ball numbers. This is similar to the array `sigma.out`, but stores the directions of curves along which $\kappa$ is parameterized to flow *into* $\sigma$.

Concerning functions, we assume that the pseudo-language includes a full implementation of Gröbner basis computation. We use Maple ([Wat08]) as a notational guide. Specifically we make use of a function `Basis()`, whose input consists of an array `I` of generators of a polynomial ideal, and an ordering `O` on monomials. The function then outputs a Gröbner basis for that ideal as an array of polynomials, where the first element in the array is the elimination polynomial. That is, if the ideal `I` is zero-dimensional, then the first element of `Basis(I,O)` is a univariate polynomial in the indeterminate of smallest order. We make use of a function `plex()` for producing orderings. The usage is as follows: `plex(x1,x2,...,xn)` produces the pure lexicographic ordering on monomials induced by the relation `x1 > x2 > ... > xn` on the indeterminates.

All remaining functions that we shall use are relatively simple, and the vast majority are described in the following list:

- `diff()` – This function takes two inputs: a rational function `R` (in any number of indeterminates), and an indeterminate `z`. The output is the partial derivative of `R` with respect to `z`.

- `Re()` and `Im()` – These functions take as input a polynomial (in any number of indeterminates) having coefficients in $\mathbb{Q}[i]$. Let `P` be such a polynomial. We can write `P` uniquely as `P1 + I*P2`, where `P1` and `P2` are polynomials with coefficients in $\mathbb{Q}$, and `I` represents $i \in \mathbb{C}$. Then the output of `Re(P)` is `P1` while the output of `Im(P)` is `P2`.

- `abs()` – This function takes as input a polynomial $\sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$ with rational coefficients, and outputs the polynomial $\sum_{\mathbf{j}} |a_{\mathbf{j}}| \, \mathbf{x}^{\mathbf{j}}$. It is used in upper bounding polynomials by the triangle inequality.

- `subs()` – This function takes two inputs: an expression of the form `z=n` (where `z` is an indeterminate and `n` is a number), and a polynomial `P` (in any number of indeterminates). The function returns the result of substituting `n` in for `z` in the polynomial `P`.

- `exponents()` – This function takes two variables: a polynomial `P` in any number of indeterminates, and an array `[x1,x2,...,xn]` of the indeterminates in `P`. The output is an array of lists of exponents, where `[i1,i2,...,in]` is an element of the output if and only if the monomial `x1^i1*x2^i2*...*xn^in` has nonzero coefficient in `P`.

- `dotprod()` – This function takes as input two same-length arrays (vectors) of rational numbers, and returns their dot product as a rational number.

- `BAFZ()` – This function takes as input a ball number `b` that is known to be non-zero. The function then progressively lowers the value of `b.tol` until `b.mod` is greater than `2*b.tol`. Thus a call to the function `b.approx()` will subsequently be bounded away from 0 by a distance of at least `b.tol`.

- `type()` – This function takes as input any variable, and outputs the variable's type; e.g. `integer` or `ball`, etc.

- `len()` – This function takes an array as an input, and outputs the length of the array as an integer.

- `a.append()` – To any array `a` is associated the function `a.append()`, which takes any variable as an argument and appends it to the end of the array `a`.

It is hoped that the use of any functions not explicitly detailed will be clear from context.

We make two additional notes about the structure of the pseudo-language. First, to improve readability of the algorithm, comments are included. The comment character is `%`. Consequently any text that appears after the symbol `%` on a given line of code a comment, and is immaterial to the functioning of the code. Second, note that the scope of all loops and if/then statements is indicated by level of indentation.

We are now ready to construct the functions necessary to carry out our algorithm. The supporting functions are constructed one-by-one, section-by-section, culminating in the creation of the main algorithm itself: `main()`.

## 4.3 Examining the height near infinity

### Specification of `finiteHeight()`

**Input:**

- `Q(x,y)` – a polynomial with rational coefficients in variables `x` and `y`. This is the singular polynomial $Q(x, y)$ of the generating function under investigation.

- `r` – a rational number. Together with `s`, this defines the direction `[r,s]` $= (\hat{r}, \hat{s})$ in which asymptotics are taken.

- `s` – a rational number. Together with `r`, this defines the direction `[r,s]` $= (\hat{r}, \hat{s})$ in which asymptotics are taken.

**Description:** This function examines the variety $\mathcal{V}_Q$ and searches for branches along which the height $h$ approaches a constant as $x \to \infty$ (and hence as $y \to 0$) or as $y \to \infty$ (and hence as $x \to 0$). By Assumption 3.2.4, this is exactly when there are branches of the form $y \sim cx^{-\hat{r}/\hat{s}}$ as $x \to 0$ or $x \sim cy^{-\hat{s}/\hat{r}}$ as $y \to 0$ ($c \neq 0$).

**Output:** `FHAI` – a boolean variable. `FHAI` stands for Finite Height At Infinity. It is set to `True` if the height function is ever bounded as $x \to \infty$ or as $y \to \infty$ on $\mathcal{V}_Q$. It is set to `False` otherwise.

### Supporting Theorem

The key theorem on which the implementation of this function relies is Newton's polygon method. We begin with a definition.

**Definition 4.3.1.** To any polynomial $Q(x, y) = \sum_{j \in J} c_j x^{a_j} y^{b_j}$ with non-zero coefficients $c_j$, we define the *Newton diagram* of $Q$ to be the set of points

$$\{(a_j, b_j) : j \in J\} \subseteq \mathbb{N} \times \mathbb{N}.$$

Then we have the following theorem.

**Theorem 4.3.2** (Newton Polygon Method)**.** *There exists a branch of $\mathcal{V}_Q$ with $y \sim cx^{\alpha}$ (as $x \to 0$) if and only if $\alpha$ corresponds to one of the inverse slopes of the left-most convex envelope of the Newton diagram of $Q$.*

See [FS09, section VII.7.1]. As a corollary, we obtain the following.

**Corollary 4.3.3.** *The height function $h$ approaches a constant as $x \to \infty$ or as $y \to \infty$ along $\mathcal{V}_Q$ if and only if $\hat{s}/\hat{r}$ is the slope of a line-segment on the left-most or right-most convex envelope of the Newton diagram of $Q$.*

*Proof.* By Newton's Polygon Method, there exists a branch of $\mathcal{V}_Q$ of the form $y \sim cx^{-\hat{r}/\hat{s}}$ as $x \to 0$ if and only if the left-most convex envelope of the Newton diagram of $Q$ includes a line segment of slope $\hat{s}/\hat{r}$. Similarly, there is a branch of $\mathcal{V}_Q$ of the form $x \sim cy^{-\hat{s}/\hat{r}}$ as $y \to 0$ if and only if the left-most convex envelope of the Newton diagram of $R(x, y) := Q(y, x)$ includes a line segment of slope $\hat{r}/\hat{s}$. But the Newton diagram of $R$ is simply the Newton diagram of $Q$ reflected over the line $x = y$, and hence this will be true if and only if the right-most convex envelope of the Newton diagram of $Q$ includes a line segment of slope $\hat{s}/\hat{r}$. □

## Implementation

```
finiteHeight(Q(x,y),r,s)
   FHAI = False % FHAI stands for Finite Height At Infinity
   diagram = exponents(Q,[x,y])
% Cycle through pairs of points in the Newton diagram
   j = 0
   while ((FHAI == False) AND (j < len(diagram))):
      k = 0
      onLine = False; sideA = False; sideB = False
      while(((sideA AND sideB) == False) AND (k < len(diagram)))
         if (j != k):
% Where does point k lie with respect to the line of slope s/r through
% the point j? Check with a dot product.
            position = dotprod(diagram[j] - diagram[k],[-s,r])
            if (position == 0):
               onLine = True % the line through j and k has slope s/r
            else if (position < 0):
               sideA = True % k is to one side of the line
            else:
```

```
                sideB = True % k is to the other side of the line
         k = k + 1
% Was a point found on the line of slope s/r through point j, with the
% rest of the Newton diagram either to the left or the right?
      if ((onLine == True) AND ((sideA AND sideB) == False)):
         FHAI = True
      j = j + 1
   return FHAI
```

## 4.4 Finding solutions to a polynomial system

**Specification of `solveSystem()`**

**Input:**

- `P1(x)` – a polynomial in `x` with rational coefficients.
- `xroots` – an array of algebraic ball numbers. Each of the numbers in this array is assumed to be a root of `P1(x)`.
- `P2(y)` – a polynomial in `y` with rational coefficients.
- `yroots` – an array of algebraic ball numbers. Each of the numbers in this array is assumed to be a root of `P2(y)`.
- `system` – an array of polynomials in `x` and `y` with rational coefficients.

**Description:** This function uses Gröbner basis computations to find solutions $(x, y)$ to the system of polynomials `system`, where the $x$ and $y$ come from `xroots` and `yroots`, respectively.

**Output:** `solutions` – an array of pairs of algebraic ball numbers. The array consists of the pairs `[x0,y0]` (`x0` from `xroots` and `y0` from `yroots`) such that `R(x0,y0) = 0` for every polynomial `R` in the array `system`.

### Implementation

```
solveSystem(P1,xroots,P2,yroots,system)
   solutions = empty array
   membership = len(xroots) by len(yroots) matrix of 1s
   for P(x,y) in system:
% Construct a polynomial in t whose solutions are the possible values
% of P(x0,y0) for x0 in xroots and y0 in yroots
      B = Basis([P1,P2,t-P],plex(x,y,t))
      Pt = B[0]
      troots = solve(t*Pt) % multiply by t to ensure t = 0 is a root
      for j from 0 to (len(xroots) - 1):
         for k from 0 to (len(yroots) - 1):
% Approximate the modulus of P(xroots[j],yroots[k]). If it is
% sufficiently close to 0 it must be 0.
            evaluate = P(xroots[j],yroots[k]) % a ball number
            if (evaluate.mod(troots[0].tol) > troots[0].tol):
               membership[j,k] = 0
   for j from 0 to (len(xroots) - 1):
      for k from 0 to (len(yroots) - 1):
         if membership[j,k] == 1:
            solutions.append([xroots[j],yroots[k]])
   return solutions
```

## 4.5 Finding the saddle and non-smooth points

### Specification of `findPoints()`

### Input:

- `saddleBX` – an array of polynomials with rational coefficients in variables `x` and `y`. This is a Gröbner basis defining the set $\Sigma$, whose first entry eliminates the variable `y`.
- `saddleBY` – an array of polynomials with rational coefficients in variables `x` and `y`. This is a Gröbner basis defining $\Sigma$ as well, however the first entry in this array eliminates the variable `x`.
- `smoothBX` – an array of polynomials with rational coefficients in variables `x` and `y`.

This is a Gröbner basis defining the set of singular points on $\mathcal{V}_Q$, whose first entry eliminates the variable `y`.

- `smoothBY` – an array of polynomials with rational coefficients in variables `x` and `y`. This is a Gröbner basis defining the singular points as well, however the first polynomial in this array eliminates the variable `x`.

- `isSmooth` – a boolean. This value is set to `True` if $\mathcal{V}_Q$ is a smooth manifold. It is set to `False` if it contains isolated singular points.

**Description:** This function computes the locations $(x, y)$ of all saddle and non-smooth points of $\mathcal{V}_Q$ such that $x \neq 0$ and $y \neq 0$.

**Output:** `points` – an array of numbers of type `saddle` or `nonSmooth`. This array consists of all the saddle points and singularities of $\mathcal{V}_Q$ not on the coordinate axes, initialized to include their $x$- and $y$-coordinate data.

### Implementation

```
findPoints(saddleBX,saddleBY,smoothBX,smoothBY,isSmooth)
   points = empty array
   Px = saddleBX[0] % Poly of possible x values for saddles
% Divide Px by the highest degree of x possible, so 0 isn't a soln.
   n = min(exponents(Px,[x])); Px = Px/(x^n) % now Px(0) != 0
   xroots = solve(Px)
   Py = saddleBY[0] % Poly of possible y values for saddles
   n = min(exponents(Py,[y])); Py = Py/(y^n) % now Py(0) != 0
   yroots = solve(Py)
   polyArray = saddleBX[1..(len(saddleBX) - 1)] % Cut elimination poly
% Find all points [x0,y0] solving the saddle point Groebner basis
   saddles = solveSystem(Px,xroots,Py,yroots,polyArray)
% Initialize them as saddle points and add to the point vector
   for [x0,y0] in saddles:
      b = new variable of type: saddle
      b.x = x0
      b.y = y0
```

```
      points.append(b)
   if (isSmooth == False):
      Px = smoothBX[0] % Poly of possible x values for singularities
      n = min(exponents(Px,[x])); Px = Px/(x^n) % now Px(0) != 0
      xroots = solve(Px)
      Py = smoothBY[0] % Poly of possible y values for singularities
      n = min(exponents(Py,[y])); Py = Py/(y^n) % now Py(0) != 0
      yroots = solve(Py)
      polyArray = smoothBX[1..(len(smoothBX) - 1)] % Cut elim poly
% Find all points [x0,y0] solving the singular point Groebner basis
      singular = solveSystem(Px,xroots,Py,yroots,polyArray)
% Initialize them as nonSmooth points and add them to the point vector
      for [x0,y0] in singular:
         b = new variable of type: nonSmooth
         b.x = x0
         b.y = y0
         points.append(b)
   return points
```

## 4.6 Computing possible height values

### Specification of `possHeight()`

**Input:**

- `Px` – a polynomial with rational coefficients in the variable `x`. The $x$-coordinates of every saddle and singular point of $\mathcal{V}_Q$ are assumed to be roots of this polynomial.
- `Py` – a polynomial with rational coefficients in the variable `y`. The $y$-coordinates of every saddle and singular point of $\mathcal{V}_Q$ are assumed to be roots of this polynomial.
- `r` and `s` – as in previous functions.

**Description:** This function is ultimately used to order all saddle/singular points by height. To do so, the function uses a Gröbner basis computation to construct a polynomial `Pt` whose real roots include all possible values of

$$e^{h(x,y)} = |x|^{-\hat{r}}|y|^{-\hat{s}},$$

such that $(x, y)$ is a singular or saddle point (and $x \neq 0$, $y \neq 0$). The key fact is that $e^h$ is algebraic in the real and imaginary parts of $x$ and $y$. After constructing `Pt`, the function returns its real roots.

**Output:** `troots` – an array of real ball numbers. These are possible values of $e^h$ at the saddle points and singular points of $\mathcal{V}_Q$.

### Implementation

```
possHeights(Px,Py,r,s)
   n = lcm(denominator(r),denominator(s))
% Construct a polynomial whose solutions are t = e^h(a+I*b,c+I*d)
   hPoly = (t^(2*n))*((a^2 + b^2)^(n*r))*((c^2 + d^2)^(n*s)) - 1
% Use Groebner bases to construct a polynomial in t whose real roots
% include all e^h(x,y) where x solves Px and y solves Py
   I = [Re(Px(a+I*b)),Im(Px(a+I*b)),Re(Py(c+I*d)),Im(Py(c+I*d)),hPoly]
   B = Basis(I,plex(a,b,c,d,t))
   Pt = B[0] % Pt is the desired polynomial
   troots = solvereal(Pt)
   return troots
```

## 4.7 Computing a terminal condition

### Specification of `termCriteria()`

### Input:

- `Q(x,y)`, `r` and `s` – as in previous functions.

**Description:** This function creates a set of termination criteria for the eventual path climbing algorithm. Ultimately this will guarantee the construction of each ascent path to terminate after a finite number of steps, and to determine which plane ($x = 0$ or $y = 0$) the path approaches.

**Output:** `[hStop,epsilon]` – an array consisting of two rational numbers. Their role in the termination of ascent paths is described following the subsequent lemma.

### Supporting Lemma

The utility of this function relies on the following lemma.

**Lemma 4.7.1.** *Let $Q \in \mathbb{Q}[x, y]$ such that $Q(\mathbf{0}) \neq 0$ and such that the variety $\mathcal{V}_Q$ has finitely many non-smooth points. Fix $\hat{r}, \hat{s} > 0$, and define the height function $h(x, y) = -\hat{r} \log |x| - \hat{s} \log |y|$. Let $\varepsilon > 0$ be sufficiently small so that*

$$|x| < \varepsilon \text{ and } |y| < \varepsilon \Rightarrow |Q(x, y)| > 0,$$

*and define*

$$M = -\hat{r} \log \varepsilon - \hat{s} \log \varepsilon.$$

*Then for any $(x, y) \in \mathcal{V}_Q$ such that $h(x, y) > M$ and such that $(x, y)$ is higher than any non-smooth point of $\mathcal{V}_Q$, we have either that $|x| < \varepsilon$ or $|y| < \varepsilon$, but not both. If $|x| < \varepsilon$, then $(x, y) \in X^{>M}$. Otherwise, $(x, y) \in Y^{>M}$.*

The function `termCriteria` computes such an $\varepsilon$ (`epsilon`) by the triangle inequality, producing a lower bound for $|Q(x, y)|$ when $|x|, |y| \leq \varepsilon$ and sufficiently shrinking $\varepsilon$ until this lower bound is positive. The function returns `epsilon` along with a rational upper bound for $M$ (`hStop`). These values can then be used to determine if an ascent path that climbs higher than `hStop` is in a component of $X^{>M}$ or $Y^{>M}$: simply check the values of $|x|$ and $|y|$ to determine which is less than `epsilon`.

*Proof.* Let $(x_0, y_0) \in \mathcal{V}_Q$, and assume both that $h(x_0, y_0) > M$ and that $(x_0, y_0)$ is higher than any non-smooth point of $\mathcal{V}_Q$. First note by the structure of $h$ that we cannot have $h(x_0, y_0) > M$ unless at least one of $|x_0|$ or $|y_0|$ is less than $\varepsilon$. But note that by definition of $\varepsilon$ we cannot have both: $Q(x_0, y_0) \neq 0$ for both $|x_0| < \varepsilon$ and $|y_0| < \varepsilon$.

To prove the second part of the lemma, let $\gamma : [0, 1) \to \mathcal{V}_Q$ be a curve that starts at $(x_0, y_0)$, is non-decreasing in height, and approaches arbitrarily large values of $h$. (The existence of such a curve follows from the Morse-theoretic decomposition of $\mathcal{V}^{\geq h(x_0,y_0)}$.) Denote $\gamma(t) = (x(t), y(t))$. In order for $h(\gamma(t)) \to \infty$ as $t \to 1$, we must have that $x(t)$ or $y(t)$ approaches arbitrarily close to 0. Now assume that $|x_0| < \varepsilon$. We claim that $|y(t)| \geq \varepsilon$ for all $t$, and hence that $x(t)$ must approach arbitrarily close to 0. This will prove that $(x_0, y_0)$ is in $X^{>M}$. A symmetric argument then shows that $|y_0| < \varepsilon$ implies that $(x_0, y_0)$ belongs to $Y^{>M}$.

To prove this final claim, we assume by way of contradiction that $|y(t)| < \varepsilon$ for some value of $t$. Define

$$t_0 = \inf \{t \in (0, 1) : |y(t)| < \varepsilon\}.$$

Then there exist arbitrarily small $\delta > 0$ such that $|y(t_0 + \delta)| < \varepsilon$, and hence such that $|x(t_0 + \delta)| \geq \varepsilon$ (as $h(\gamma(t_0 + \delta)) > M$). Taking the limit as $\delta \to 0$, this implies that $|x(t_0)| \geq \varepsilon$ and $y|(t_0)| \leq \varepsilon$. But $|y(t_0)| \geq \varepsilon$, as $|y(t)| \geq \varepsilon$ for all $t < t_0$, and hence $|y(t_0)| = \varepsilon$. Thus we have

$$h(\gamma(t_0)) \leq -\hat{r} \log \varepsilon - \hat{s} \log \varepsilon = M,$$

which contradicts the assumption that $\gamma$ is non-decreasing. □

### Implementation

`termCriteria(Q(x,y),r,s)`

```
   epsilon = 16 % arbitrary starting point
   Qbound = abs(Q(0,0)) - abs(Q(x,x) - Q(0,0)) % poly in x
% Shrink epsilon until until |Q(epsilon,epsilon)| > 0
   while (Qbound(epsilon) <= 0):
      epsilon = epsilon / 2
   height = realball(-r*log(epsilon) - s*log(epsilon))
% Compute an upper bound for height -- this is the terminal height
   hStop = height.approx(0.01) + 0.01 % tolerance is arbitrary
   return [hStop, epsilon]
```

## 4.8 Determining a local parameterization variable

### Specification of `paramByX`

**Input:**

- `Q(x,y)` – as in previous functions.
- `x0` – a ball number. This represents the $x$-coordinate of a point $(x_0, y_0) \in \mathcal{V}_Q$.
- `y0` – a ball number. This represents the $y$-coordinate of a point $(x_0, y_0) \in \mathcal{V}_Q$.

**Description:** Given a polynomial $Q$ and a point $(x_0, y_0)$ on $\mathcal{V}_Q$ at which $Q$ is known to be smooth, this function determines a variable (either $x$ or $y$) with respect to which $\mathcal{V}_Q$ admits a local parameterization. This is accomplished by computing $Q_y(x_0, y_0)$ and $Q_x(x_0, y_0)$ to greater and greater accuracy, until it can be determined that one of these two values is non-zero. The rest follows from the implicit function theorem.

**Output:** `byX` – a boolean. `byX` is set to `True` if the set `Q(x,y) = 0` is parameterizable local to the point `[x0,y0]` by `x`. It is set to `False` if it is found to be locally parameterizable in terms of `y`.

### Implementation

```
paramByX(Q(x,y),x0,y0)
% Compute partials of Q at the point (x0,y0)
   Qx = diff(Q,x); Qx0 = Qx(x0,y0)
   Qy = diff(Q,y); Qy0 = Qy(x0,y0)
   tolerance = 1 % ball approximation tolerance
   nonZeroX = False; nonZeroY = False
% Compute Qx0 and Qy0 to greater and greater precision until it can
% be verified that one is not zero.
   while ((nonZeroX == False) AND (nonZeroY == False)):
      if (Qx0.mod(tolerance) > tolerance): % is Qx(x0,y0) not 0?
         nonZeroX = True
      if (Qy0.mod(tolerance) > tolerance): % is Qy(x0,y0) not 0?
         nonZeroY = True
      tolerance = tolerance / 2
   if (nonZeroY == True):
      byX = True % if dQ/dy is nonzero, parameterize by x
   else:
      byX = False % otherwise, parameterize by y
   return byX
```

## 4.9 Isolating roots

### Specification of `isoRoot`

### Input:

- `Q(w,z)` – a polynomial with rational coefficients in variables `w` and `z`. This is the same singular polynomial $Q$ from before, but in a new coordinate system. The variables $x$ and $y$ have been replaced by $w$ and $z$ (either by $w = x$ and $z = y$ or vice versa). The manner in which the new variables have been selected is to assure that $Q_z$ is non-zero at the point `[w0,z0]` on $\mathcal{V}_{Q(w,z)}$. Hence locally we may assume the singular variety is parameterizable by $w$.

- `w0` – an algebraic ball number. This is the $w$-coordinate of a point on $\mathcal{V}_{Q(w,z)}$.

- `z0` – an algebraic ball number. This is the $z$-coordinate of a point on $\mathcal{V}_{Q(w,z)}$.

**Description:** This function finds a neighborhood in the $z$-plane on which the only root of $Q(w_0, z) = 0$ is $z = z_0$.

**Output:** `isolate` – a rational number. The number `isolate` satisfies the condition that if `Q(w0,z) = 0` and `|z - z0|` is less than `isolate`, then `z = z0`.

### Supporting Lemma

The construction of the number `isolate` depends principally on the following lemma.

**Lemma 4.9.1.** *Let $Q \in \mathbb{Q}[w, z]$ and $(w_0, z_0) \in \mathbb{C}^2$ such that $Q(w_0, z_0) = 0$ and $Q_z(w_0, z_0) \neq 0$. Let $\varepsilon > 0$ be such that*

$$\frac{1}{|z - z_0|} \left|Q(w_0, z) - (z - z_0)Q_z(w_0, z_0)\right| < |Q_z(w_0, z_0)| \tag{4.9.1}$$

*for $z \neq z_0$ such that $|z - z_0| < \varepsilon$. Then $Q(w_0, z) = 0$ and $|z - z_0| < \varepsilon$ implies that $z = z_0$.*

Note that the function

$$f(z) := \frac{Q(w_0, z) - (z - z_0)Q_z(w_0, z_0)}{z - z_0}$$

appearing on the left-hand side of the inequality (4.9.1) is $O(z - z_0)$ as $z \to z_0$. The function `isoRoot` thus constructs an $\varepsilon > 0$ for which the preceding inequality holds by upper bounding $|f(z)|$ given a starting bound on $|z - z_0|$ ($z \neq z_0$), then by shrinking the bound on $|z - z_0|$ until it can be shown that $|f(z)|$ must be less than $|Q_z(w_0, z_0)|$.

*Proof.* Let $\varepsilon$ be as prescribed, and let $z$ be such that $|z - z_0| < \varepsilon$ and $z \neq z_0$. Motivated by expanding $Q(w_0, z)$ in $z$ about $z = z_0$, we write

$$Q(w_0, z) = (z - z_0) \left( Q_z(w_0, z_0) + \frac{1}{z - z_0} \left(Q(w_0, z) - (z - z_0)Q_z(w_0, z_0)\right) \right).$$

From this, we obtain

$$
\begin{aligned}
|Q(w_0, z)| &\geq |z - z_0| \cdot \left( |Q_z(w_0, z_0)| - \left| \frac{1}{z - z_0} \left( Q(w_0, z) - (z - z_0) Q_z(w_0, z_0) \right) \right| \right) \\
&> |z - z_0| \cdot \left( |Q_z(w_0, z_0)| - |Q_z(w_0, z_0)| \right) = 0
\end{aligned}
$$

where the second inequality is by definition of $\varepsilon$. Thus $Q(w_0, z) \neq 0$. That is, the only root of $Q(w_0, z) = 0$ with $|z - z_0| < \varepsilon$ is $z = z_0$. □

### Implementation

```
isoRoot(Q(w,z),w0,z0)
   isolate = 16 % arbitrary starting value for isolating radius
% Compute a lower bound for diff(Q,z) at (w0,z0)
   Qz = diff(Q,z)
   Q0 = ball(Qz(w0,z0))
   BAFZ(Q0);
   Qlb = Q0.mod() - Q0.tol % lower bound for |Qz(w0,z0)|
% Compute upper bounds for |w0| and |z0|
   w0ub = w0.mod(0.1) + 0.1
   z0ub = z0.mod(0.1) + 0.1
% Construct a polynomial for the Lemma inequality; bound from above
   P = (1/D)*(Q(a,b + D) - Qz(a,b)*D - Q(a,b)) % poly in a, b and D
   P = abs(P)
   P = subs(a = w0ub, P)
   P = subs(b = z0ub, P) % P is now a poly in D
% NOTE: P has no constant term!
   while(P(isolate) >= Qlb):
      isolate = isolate / 2
   return isolate
```

## 4.10 Finding a parameterization neighborhood

### Specification of `paramNbd()`

#### Input:

- `Q(w,z)`, `w0` and `z0` – as in previous functions.

**Description:** This function produces a neighborhood of $(w_0, z_0)$ in $\mathbb{C}^2$ on which the variety $\mathcal{V}_Q$ is parameterizable by $w$.

**Output:** `[delta,epsilon]` – an array of two rational numbers. Local to the point `[w0,z0]`, the variety $\mathcal{V}_Q$ is parameterizable by `w` on the neighborhood `|w - w0| < delta`. The corresponding `z` value is in the neighborhood `|z - z0| < epsilon`.

## Supporting Lemma

The following lemma will be used to characterize the appropriate radii of a polydisc on which $\mathcal{V}_{Q(w,z)}$ can be parameterized by $w$. Note that lemma is presented in $(x, y)$ coordinates, assuming local parameterization by $x$. This is for ease of notation in later lemmas, where the height function will be invoked.

**Lemma 4.10.1.** *Let $Q \in \mathbb{Q}[x, y]$, and let $(x_0, y_0) \in \mathcal{V}_Q \subseteq \mathbb{C}^2$ such that $Q_y(x_0, y_0) \neq 0$. Let $\delta, \varepsilon$ be positive constants such that*

$$Q(x_0, y) = 0 \text{ and } |y - y_0| < \varepsilon \implies y = y_0,$$

*and such that for all $(x, y)$ with $x \in B_\delta(x_0)$ and $y \in B_{2\varepsilon}(y_0) \setminus B_\varepsilon(y_0)$, the following inequalities hold:*

$$|Q_y(x, y_0) - Q_y(x_0, y_0)| < \frac{1}{2} |Q_y(x_0, y_0)|, \tag{4.10.1}$$

$$\frac{1}{|y - y_0|} |Q(x, y) - Q(x, y_0) - (y - y_0)Q_y(x, y_0)| < \frac{1}{4} |Q_y(x_0, y_0)|, \tag{4.10.2}$$

$$|Q(x, y_0)| < \frac{\varepsilon}{4} |Q_y(x_0, y_0)|. \tag{4.10.3}$$

*Then $\mathcal{V}_Q \cap D$ is parameterizable by $x$, where $D := B_\delta(x_0) \times B_\varepsilon(y_0)$ (a polydisc about $(x_0, y_0)$).*

The function `paramNbd()` will construct such $\delta$ and $\varepsilon$. The method used is analogous to the upper bounding technique employed in the function `isoRoot()`.

*Proof.* The first task is to show that, under the assumptions of the theorem, the equation $Q(x,y) = 0$ has no solutions $(x,y) \in B_\delta(x_0) \times (B_{2\varepsilon}(y_0) \setminus B_\varepsilon(y_0))$. To that end, we fix some such $(x,y)$ and examine $Q(x,y)$.

We begin by writing $Q$ as a partial expansion

$$Q(x,y) = Q(x,y_0) + (y-y_0)Q_y(x,y_0) + R(x,y),$$

where the remainder function $R$ is defined by

$$R(x,y) = Q(x,y) - Q(x,y_0) - (y-y_0)Q_y(x,y_0).$$

Hence by the triangle inequality we have that

$$|Q(x,y)| \geq |y-y_0| \cdot \left( |Q_y(x,y_0)| - \frac{|R(x,y)|}{|y-y_0|} \right) - |Q(x,y_0)|. \tag{4.10.4}$$

From the upper bound in (4.10.1) we get the lower bound $|Q_y(x,y_0)| > \frac{1}{2}|Q_y(x_0,y_0)|$. Combined with the upper bound (4.10.2) and the fact that $|y-y_0| > \varepsilon$, this yields

$$|y-y_0| \cdot \left( |Q_y(x,y_0)| - \frac{|R(x,y)|}{|y-y_0|} \right) > \frac{\varepsilon}{4}|Q_y(x_0,y_0)|.$$

Finally, applying the preceding lower bound together with the upper bound (4.10.3) to the inequality (4.10.4) yields

$$|Q(x,y)| > \frac{\varepsilon}{4}|Q_y(x_0,y_0)| - \frac{\varepsilon}{4}|Q_y(x_0,y_0)| = 0.$$

In other words, $Q(x,y) \neq 0$, as we wished to show.

We wish to use this property to show that $\mathcal{V}_Q \cap D$ is parameterizable by $x$. We make use of the fact that, as $x$ varies over $\mathbb{C}$, the corresponding $y$-roots of $Q(x,y) = 0$ vary continuously with $x$ (assuming we include $\infty$ as a possible $y$-root in the usual way). This can be seen, for instance, by expressing each $y$-root locally as a Puiseux expansion in $x$.

Now by assumption, there is exactly one $y$-root of $Q(x_0, y) = 0$ that lies in the ball $B_\varepsilon(y_0)$. In fact there is only one such root when counted with multiplicity, as $Q_y(x_0, y_0) \neq 0$. And because $Q(x_0, y)$ has no $y$-roots in the annulus $B_{2\varepsilon}(y_0) \setminus B_\varepsilon(y_0)$, all other such $y$-roots must lie in $\mathbb{C} \setminus B_{2\varepsilon}(y_0)$. As $x$ varies over the ball $B_\delta(x_0)$, the corresponding $y$-roots of $Q(x, y) = 0$ vary continuously and never enter the annulus $B_{2\varepsilon}(y_0) \setminus B_\varepsilon(y_0)$. Thus for $x \in B_\delta(x_0)$ there is one and only one $y$-root in $B_\varepsilon(y_0)$ such that $Q(x, y) = 0$; on a neighborhood of $x_0$ we have effectively isolated one $y$-root from all others. This enables us to construct a well-defined implicit function

$$f : B_\delta(x_0) \to B_\epsilon(y_0)$$

by the rule $Q(x, f(x)) = 0$.

Assuming that $f$ is analytic, we have constructed the desired parametrization of $\mathcal{V}_Q$ on the polydisc $D$. But again, we can see this by the Puiseux expansion theorem. Representing $f$ local to any $x_1 \in B_\delta(x_0)$ by a Puiseux expansion, we see that $f$ fails to be holomorphic at $x_1$ if and only if either: the expansion has non-trivial singular part (in which case $y \to \infty$ as $x \to x_1$) or the expansion contains non-integral powers of $(x - x_1)$ (in which case there will be a coalescence of roots as $x \to x_1$). By the isolation exhibited earlier, neither of these cases is possible. Thus $f$ is analytic, and the theorem follows. $\square$

Before proceeding with with the implementation of this algorithm, we pause to draw attention to a useful property of this parameterization that was uncovered in the proof. When parameterizing $\mathcal{V}_Q$ near $(x_0, y_0)$ by $x$ on $B_\delta(x_0)$, the corresponding $y$ root is the unique root of $Q(x, y) = 0$ in $B_\varepsilon(y_0)$, and *all other roots* of $Q(x, y) = 0$ lie outside the region $B_{2\varepsilon}(y_0)$. This isolation will be particularly useful when we parameterize a path on $\mathcal{V}_Q$ by one variable and must analytically continue the other to the correct corresponding value.

### Implementation

```
paramNbd(Q(w,z),w0,z0)
% Isolate the z0 root
   isolate = isoRoot(Q(w,z),w0,z0)
   Qz = diff(Q,z)
   Q0 = ball(Qz(w0,z0))
   BAFZ(Q0); BAFZ(w0); BAFZ(z0)
   Qlb = Q0.mod() - Q0.tol % rational lower bound for |Qz(w0,z0)|
   wlb = w0.mod() - w0.tol % rational lower bound for |w0|
   zlb = z0.mod() - z0.tol % rational lower bound for |z0|
% Bound w and z away from 0
   delta = wlb/2; epsilon = min(isolate,zlb/2) % attempt at radii
% Obtaining the first inequality from the lemma:
   P1 = Qz(a + D,b) - Qz(a,b) % polynomial in a, b and D
   P1 = abs(P1)
   P1 = subs(a = w0.mod() + w0.tol,P1)
   P1 = subs(b = z0.mod() + w0.tol,P1) % P1 is now a poly in D only
% NOTE: P1 has no constant term!
   while(P1(delta) >= Qlb/2):
      delta = delta/2
% Obtaining the second inequality from the Lemma:
   P2 = (Q(a+D,b+E) - Q(a+D,b) - E*Qz(a+D,b))/E % poly in a,b,D,E
   P2 = abs(P2)
   P2 = subs(a = w0.mod() + w0.tol,P2)
   P2 = subs(b = z0.mod() + z0.tol,P2)
   P2 = subs(D = delta) % P2 is now a poly in E only
% NOTE: P2 has no constant term!
   while(P2(2*epsilon) >= Qlb/4):
      epsilon = epsilon/2
% Obtaining the third inequality from the lemma:
   P3 = Q(a + D,b) - Q(a,b) % polynomial in a, b and D
   P3 = abs(P3)
   P3 = subs(a = w0.mod() + w0.tol,P3)
   P3 = subs(b = z0.mod() + z0.tol,P3) % P3 is now a poly in D only
% NOTE: P3 has no constant term!
   while(P3(delta) >= Qlb*epsilon/4):
```

```
      delta = delta/2
return [delta, epsilon]
```

## 4.11 Calculating the degeneracy of a saddle point

### Specification of `degeneracy()`

**Input:**

- `Q(w,z)`, `w0`, `z0`, `r`, `s`, `byX` – as in previous functions.

**Description:** This function calculates the degree of degeneracy of the height function at the point represented by `[w0,z0]`, assumed to be a saddle. This is accomplished by repeated differentiation and applications of the function `solveSystem()`.

**Output:** `[n,DH]` – an array of two elements: an integer `n` and a rational function `DH` in variables `w` and `z`. The integer `n` is the degeneracy of $h$ at `[w0,z0]`, while `DH` is a formula for $\frac{d^n}{dw^n}H(w, z(w))$ local to this point (in terms of `w` and `z`).

### Implementation

```
degeneracy(Q(w,z),w0,z0,r,s,byX)
% Compute Dz, the derivative of z with respect to w on Q = 0.
   Dz = -diff(Q,w)/diff(Q,z)
% Compute DH, the first derivative of H with respect to w.
   if (byX == True):
      DH = -r/w - (s/z)*Dz
   else:
      DH = (-r/z)*Dz - (s/w)
   n = 2
% Set DH equal to the second derivative of H with respect to w.
   DH = diff(DH,w) + diff(DH,z)*Dz
   Hnum(w,z) = numerator(DH)
% Is (w0,z0) a solution of DH = 0?
   solns = solveSystem(w0.poly(x),[w0],z0.poly(y),[z0],[Hnum(x,y)])
% While (w0,z0) is such a solution, set DH to the next derivative
   while (solns is not empty):
```

```
    n = n + 1
    DH = diff(DH,w) + diff(DH,z)*Dz
    Hnum(w,z) = numerator(DH)
    solns = solveSystem(w0.poly(x),[w0],z0.poly(y),[z0],[Hnum(x,y)])
return [n,DH]
```

## 4.12 Finding a neighborhood for ascent steps

### Specification of `ascentNbd()`

**Input:**

- `w0`, `z0`, `r`, `s`, `byX` – as in previous functions.
- `Ri` – a rational number. This is the "radius of the independent variable." We assume that $\mathcal{V}_{Q(w,z)}$ is parameterizable by `w` for `|w - w0| < Ri`. This is an output of the function `paramNbd()`.
- `Rd` – a rational number. This is the "radius of the dependent variable." When parameterizing by `w` on the neighborhood described above, the corresponding `z` satisfy `|z - z0| < Rd`. This is an output of the function `paramNbd()`.
- `n` – an integer. This is the degree of degeneracy of the height function at the point `[w0,z0]`.
- `DH` – a rational function in variables `w` and `z`. This is a local formula for $\frac{d^n}{dw^n} H(w, z(w))$ near the point `[w0,z0]`.

**Description:** This function finds a neighborhood on which the parameterization of $\mathcal{V}_{Q(w,z)}$ by $w$ is particularly nice in a geometric sense (to be detailed following Lemma 4.12.1). In particular, this neighborhood is perfectly suited to constructing local ascent paths.

**Output:** `R` – a rational number. The number `R < Ri` is the radius of a neighborhood of `w0` on which parameterization by `w` is well-understood geometrically.

### Supporting Lemma

We will require significant setup/notation to construct a geometrically useful neighborhood of parameterization, which we now develop.

For a given $Q \in \mathbb{Q}[x,y]$, let $(x_0, y_0) \in \mathcal{V}_Q$ be such that $x_0 \neq 0$, $y_0 \neq 0$ and $Q_y(x_0, y_0) \neq 0$. Now let $\delta, \varepsilon$ be positive constants as in the conclusion of Lemma 4.10.1, so that $\mathcal{V}_Q$ is parameterizable by $x$ on the polydisc $D = B_\delta(x_0) \times B_\varepsilon(y_0)$. Assume further that $\delta$ and $\varepsilon$ are sufficiently small so that $0 \notin B_\delta(x_0)$ and $0 \notin B_\varepsilon(y_0)$.

Denote by

$$\iota : B_\delta(x_0) \to \mathcal{V}_Q \cap D$$

this parametrization, and define the function $\mathcal{H}$ on $B_\delta(x_0)$ by $\mathcal{H}(x) = H(\iota(x))$ (for any determination of $H$ local to $(x_0, y_0)$). Define

$$n = \inf \{k \geq 1 : \mathcal{H}^{(k)}(x_0) \neq 0\},$$

which is well-defined assuming that $H$ has isolated critical points on $\mathcal{V}_Q$. Define the constant $M$ by

$$M = \sqrt{(\hat{r} \log (|x_0| - \delta) + \hat{s} \log (|y_0| - \varepsilon))^2 + (2\pi(\hat{r} + \hat{s}))^2},$$

and let $\rho \in (0,1)$ be sufficiently small so that

$$\frac{\rho}{(1-\rho)^2}(n + 1 - n\rho) < \frac{\delta^n \left|\mathcal{H}^{(n)}(x_0)\right|}{(n-1)! M\sqrt{2}}.$$

We conclude the following:

**Lemma 4.12.1.** *For any any $\theta \in \mathbb{R}$ such that*

$$2n\theta + 2\arg\left(\mathcal{H}^{(n)}(x_0)\right) + \pi/2 \in [0, \pi] \mod 2\pi,$$

*define the curve $\gamma_\theta : [0, \rho\delta] \to \mathbb{C}$ by $\gamma_\theta(t) = x_0 + te^{i\theta}$. Then the height along $\iota \circ \gamma_\theta$ is monotone (that is, the function $\operatorname{Re}(\mathcal{H} \circ \gamma_\theta)$ is monotone).*

*Similarly for $c < \rho\delta$ and any $\theta \in \mathbb{R}$ such that*

$$2n\theta + 2\arg\left(\mathcal{H}^{(n)}(x_0)\right) = \pi/2 \mod 2\pi,$$

*define the curve $\sigma_{c,\theta} : [0, \pi/2n] \to \mathbb{C}$ by $\sigma_{c,\theta}(t) = x_0 + ce^{i(\theta+t)}$. Then the height along $\iota \circ \sigma_{c,\theta}$ is monotone.*

This lemma may seem quite technical, but the message is relatively simple. Within a wedge of angle $\pi/2n$ of the $2n$ steepest ascent/descent directions, the height along the radial paths exiting $(x_0, y_0)$ (the $\gamma_\theta$ paths) is monotone (as parameterized by $x$ on $B_{\rho\delta}(x_0)$). Similarly, within a wedge of angle $\pi/2n$ of the $2n$ constant height directions, the height along circumferential paths about $(x_0, y_0)$ (the $\sigma_{c,\theta}$ paths) is monotone (again, on a small neighborhood as parameterized by $x$). See Figure 4.1.

The function `ascentNbd()` thus produces a radius of the form $\rho\delta$ as prescribed in the lemma.

*Proof.* Let $\gamma_\theta$ be as prescribed. We wish to show that the height along $\iota \circ \gamma_\theta$ is monotone. We do so by showing that

$$\frac{d}{dt}\operatorname{Re}\mathcal{H}(\gamma_\theta(t)) = \operatorname{Re}\left(\mathcal{H}'(x_0 + te^{i\theta})e^{i\theta}\right) \neq 0$$

for $t \in (0, \rho\delta)$. Thus we simply need to show that the expression

$$\mathcal{H}'(x_0 + te^{i\theta})e^{i\theta}$$

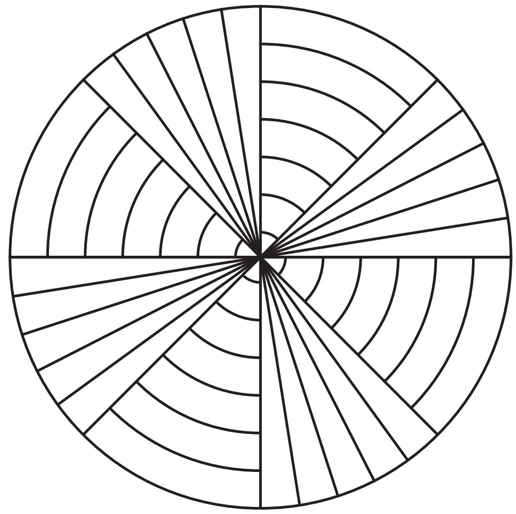

Figure 4.1: The geometric structure on a computable neighborhood local to a point $\sigma$ at which $h$ has degeneracy of degree 2. The height is monotone along indicated paths.

is not purely imaginary. We examine this expression in more detail.

By using the Taylor expansion for $\mathcal{H}$ about $x_0$, we can rewrite this as

$$\begin{aligned}\mathcal{H}'(x_0 + te^{i\theta})e^{i\theta} &= e^{i\theta} \sum_{k=n}^{\infty} \frac{\mathcal{H}^{(k)}(x_0)}{(k-1)!} \left(te^{i\theta}\right)^{k-1} \\ &= e^{in\theta} \mathcal{H}^{(n)}(x_0) \frac{t^{n-1}}{(n-1)!} \left(1 + R(t)\right),\end{aligned}$$

where

$$R(t) = \frac{(n-1)!}{\mathcal{H}^{(n)}(x_0)} \sum_{k=n+1}^{\infty} \frac{\mathcal{H}^{(k)}(x_0)}{(k-1)!} \left(te^{i\theta}\right)^{k-n}. \tag{4.12.1}$$

Then we can write

$$\arg\left(\mathcal{H}'(x_0 + te^{i\theta})e^{i\theta}\right) = \arg\left(e^{in\theta}\mathcal{H}^{(n)}(x_0)\right) + \arg\left(1 + R(t)\right).$$

But by our choice of $\theta$, we have either that

$$\arg\left(e^{in\theta}\mathcal{H}^{(n)}(x_0)\right) \in [-\pi/4, \pi/4], \text{ or}$$
$$\arg\left(e^{in\theta}\mathcal{H}^{(n)}(x_0)\right) \in [3\pi/4, 5\pi/4]$$

Thus we can assure that $\mathcal{H}'(x_0+te^{i\theta})e^{i\theta}$ is not purely imaginary by showing $\arg(1+R(t)) \in (-\pi/4, \pi/4)$. We accomplish this by showing that $|R(t)| < \sqrt{2}/2$.

To obtain this upper bound on the $|R(t)|$, we will need upper bounds on the derivatives $\mathcal{H}^{(k)}(x_0)$. We can use Cauchy's integral formula to write

$$\mathcal{H}^{(k)}(x_0) = \frac{k!}{2\pi i}\int_C \frac{\mathcal{H}(x)}{(x-x_0)^{k+1}}\,dx, \tag{4.12.2}$$

where $C$ is a circle of radius $\delta_0 < \delta$ centered at $x_0$, oriented positively. To upper bound the preceding integral we seek an upper bound on $\mathcal{H}$ on $B_\delta(x_0)$. So we examine the function

$$H(x,y) = -\hat{r}\log x - \hat{s}\log y$$

on the polydisc $D$. Choose a branch of $\log x$ holomorphic on $B_\delta(x_0)$ such that $\operatorname{Im}(\log x) \in [-2\pi, 2\pi]$ for $x \in B_\delta(x_0)$. Choose a similar branch of $\log y$ on $B_\varepsilon(y_0)$. Then on $D$,

$$\begin{aligned} |H(x,y)| &= \sqrt{(\operatorname{Re} H)^2 + (\operatorname{Im} H)^2} \\ &\leq \sqrt{(\hat{r}\log(|x_0|-\delta) + \hat{s}\log(|y_0|-\varepsilon))^2 + (2\pi(\hat{r}+\hat{s}))^2} = M. \end{aligned}$$

Returning to equation (4.12.2), we obtain

$$\left|\mathcal{H}^{(k)}(x_0)\right| \leq \frac{k!}{2\pi}2\pi\delta_0\frac{M}{(\delta_0)^{k+1}}.$$

Simplifying and letting $\delta_0 \to \delta$, we have

$$\left|\mathcal{H}^{(k)}(x_0)\right| \leq k!\frac{M}{\delta^k}.$$

Finally, applying this bound to equation (4.12.1), we get

$$\begin{aligned}
|R(t)| &= \frac{(n-1)!}{\left|\mathcal{H}^{(n)}(x_0)\right|}\left|\sum_{k=n+1}^{\infty}\frac{\mathcal{H}^{(k)}(x_0)}{(k-1)!}\left(te^{i\theta}\right)^{k-n}\right| \\
&\leq \frac{(n-1)!}{\left|\mathcal{H}^{(n)}(x_0)\right|}\sum_{k=n+1}^{\infty}\frac{k!}{(k-1)!}\frac{M}{\delta^k}(\rho\delta)^{k-n} \\
&= \frac{M\rho^{1-n}(n-1)!}{\delta^n\left|\mathcal{H}^{(n)}(x_0)\right|}\sum_{k=n+1}^{\infty}k\rho^{k-1}.
\end{aligned}$$

But note that

$$\sum_{k=n+1}^{\infty}k\rho^{k-1} = \frac{d}{d\rho}\sum_{k=n+1}^{\infty}\rho^k = \frac{d}{d\rho}\left(\frac{\rho^{n+1}}{1-\rho}\right) = \frac{(n+1)\rho^n - n\rho^{n+1}}{(1-\rho)^2},$$

which yields

$$|R(t)| \leq \frac{M(n-1)!}{\delta^n\left|\mathcal{H}^{(n)}(x_0)\right|}\cdot\frac{\rho\,(n+1-n\rho)}{(1-\rho)^2} < \frac{1}{\sqrt{2}},$$

by definition of $\rho$. This is as we wished to show.

The proof that monotonicity of the height along each $\iota \circ \sigma_{c,\theta}$ is proved by the same general method. $\square$

We note that the preceding lemma allows us to conclude the following.

**Corollary 4.12.2.** *In the notation of Lemma 4.12.1, let $(x_1, y_1) \in \mathcal{V}_Q$ such that $|x_1 - x_0| < \rho\delta$ and $|y_1 - y_0| < \varepsilon$. Define $c = \min\{h(x_0, y_0), h(x_1, y_1)\}$. Then $(x_1, y_1)$ and $(x_0, y_0)$ are in the same component of $\mathcal{V}^{\geq c}$.*

This corollary will be used to determine whether or not a point $(x_1, y_1)$ sufficiently close to a saddle point $(x_0, y_0)$ lies in a component of $X^{\geq c}$ or $Y^{\geq c}$, assuming that information to be known of the saddle point.

*Proof.* First, we begin by assuming that $c = h(x_1, y_1)$, so that $h(x_1, y_1) \leq h(x_0, y_0)$. The claim is that there is an ascent path from $(x_1, y_1)$ to $(x_0, y_0)$, which will prove the claim in this case.

By the results of Lemma 4.12.1, there is a neighborhood of $(x_0, y_0)$ containing $(x_1, y_1)$, parameterized by $x$, which has the nice geometric structure as depicted in Figure 4.1. In said neighborhood we deal with two cases: either $(x_1, y_1)$ lies in a wedge on which the radial paths to $(x_0, y_0)$ are monotone in height, or it lies in a wedge on which the circumferential paths about $(x_0, y_0)$ are monotone in height.

In the former case, the path directly from $(x_1, y_1)$ to $(x_0, y_0)$ is the desired ascent. In the latter case, we can traverse a circumferential ascent path within this wedge to some intermediate point $(x_2, y_2)$ such that $h(x_2, y_2) = h(x_0, y_0)$. From there, we can travel along the constant height path within this wedge until reaching $(x_0, y_0)$. In either case, there is a non-decreasing path from $(x_1, y_1)$ to $(x_0, y_0)$, as we wished to show.

On the other hand if $h(x_0, y_0) \leq h(x_1, y_1)$, a similar argument can be used to construct a path of non-increasing height from $(x_1, y_1)$ to $(x_0, y_0)$, in which case we also obtain the desired claim. □

### Implementation

```
ascentNbd(w0,z0,r,s,byX,Ri,Rd,n,DH)
% Compute a positive lower bound for |DH| at the point (w0,z0)
   H0 = DH(w0,z0) % a ball number
   BAFZ(H0)
   Hlb = H0.mod() - H0.tol % lower bound for |DH(w0,z0)|
% Compute lower bounds for |w0| - Ri, |z0| - Rd
   w1 = realball(|w0| - Ri); z1 = realball(|z0| - Rd)
   BAFZ(w1); BAFZ(z1)
   wlb = w1.approx() - w1.tol; zlb = z1.approx() - z1.tol()
% Compute an upper bound for M from the lemma
   if (byX == True)
      logBound = realball(r*log(wlb) + s*log(zlb))
   else
      logBound = realball(r*log(zlb) + s*log(wlb))
   M = (logBound^2 + (2*Pi*(r+s))^2)^(1/2)
   Mub = M.mod() + M.tol % upper bound for |M|
% Construct a rho satisfying the inequality for the lemma
```

```
rho = 0.8 % arbitrary starting value in (0,1)
bound = (Hlb*((Ri)^n))/((n-1)!*Mub*1.4143) % 1.4143 UB for sqrt(2)
while ((n + 1 - n*rho)*rho/(1-rho)^2 >= bound):
   rho = rho/2
R = rho*Ri
return R
```

## 4.13 Computing a single ascent step

### Specification of `pathStep`

**Input:**

- `Q(w,z)`, `w0`, `z0`, `n`, `DH`, `r`, `s` – as in previous functions.
- `dir` – a ball number. `dir` represents one of the steepest ascent directions local to `[w0,z0]` as parameterized by `w`.

**Description:** This algorithm constructs a single step along the variety $\mathcal{V}_{Q(w,z)}$ that is guaranteed to be non-decreasing in height. It does so by calling `paramNbd()` and `ascentNbd()` to find a geometrically nice neighborhood on which to parameterize by the variable $w$. Then a step is taken in $w$ within a wedge of angle $\pi/2n$ of the direction `dir`. By Lemma 4.12.1, this guarantees an ascent. The corresponding $z$-root is found according to the note following the proof of Lemma 4.10.1.

**Output:** `[w1,z1]` – an array of two ball numbers (in fact, `w1` is rational). These are the coordinates of a point that can be reached from `[w0,z0]` by an ascent path parameterized by `w`.

### Implementation

```
pathStep(Q(w,z),w0,z0,n,DH,r,s,byX,dir)
```

```
% Compute radii for parametrization and ascent neighborhoods
   [Ri,Rd] = paramNbd(Q(w,z),w0,z0)
   R = ascentNbd(w0,z0,r,s,byX,Ri,Rd,n,DH)
   m = realball(0.8*sin(Pi/(4*n)))
   BAFZ(m)
   tolerance = min(m.approx() - m.tol,0.2)
% Compute w1 such that the straight line from w0 to w1 lifts to an
% ascent, assuming that dir is a direction of steepest ascent
   w1 = w0 + 0.8*R*dir % ball number
% Compute a complex rational approximation to w1, within small enough
% tolerance to keep it in the wedge of angle Pi/2*n of dir.
   w1 = w1.approx(tolerance)
% Compute z roots corresponding to w1
   P(z) = Q(w1,z) % polynomial in z with rational complex coefficients
   zroots = solve(P(z))
% Find the z root within distance Rd of z0; this is the correct root
   j = 0
   rootFound = False
   while (rootFound == False):
      dist = z0 - zroots[j] % ball number
      if (dist.mod(Rd/3) <= 4*Rd/3):
         rootFound = True
         z1 = zroots[j]
      j = j + 1
   return [w1,z1]
```

## 4.14 Chaining the ascent steps together

### Specification of `path`

**Input:**

- `Q(x,y)`, `x0`, `y0`, `r`, `s` – as in previous functions.
- `hStop`, `radius` – two rational numbers. These are the termination criteria as output by `termCriteria()`.
- `saddles` – an array of elements of type `saddle`. These are the saddle points on $\mathcal{V}_Q$ that are higher than the height $c$ of the point `[x0,y0]`.

**Description:** This function chains together calls to `pathStep()` in order to construct a global ascent path. Each step is parameterized either by $x$ or by $y$, where the particular variable chosen is determined by a call to `paramByX()` at the beginning of each step. The function terminates when it can be determined whether or not the path lies in $X^{\geq c}$ or $Y^{\geq c}$. This occurs either when the height of the path is sufficiently large (see Lemma 4.7.1), or when the path gets sufficiently close to a saddle point (see Corollary 4.12.2).

**Output:** `xPole` – a boolean. `xPole` is set to `True` if it is determined that the path is in $X^{\geq c}$. It is set to `False` otherwise.

### Implementation

```
path(Q(x,y),x0,y0,r,s,hStop,radius,saddles)
   high = False; nearSaddle = False
   while ((high == False) AND (nearSaddle == False)):
      j = 0
% Cycle through saddles; examine distance from each saddle to (x0,y0)
      while ((nearSaddle == False) AND (j < len(saddles))):
         if (saddles[j].byX == True):
            sw = saddles[j].x
            sz = saddles[j].y
            w0 = x0; z0 = y0
         else:
            sw = saddles[j].y
            sz = saddles[j].x
            w0 = y0; z0 = x0
         R = saddles[j].R
         Rd = saddles[j].Rd
         wdist = sw - w0
% If distance is suff. small, ascent (x0,y0) and saddle have same type
         if (wdist.mod(R/4) < 3*R/4):
            zdist = sz - z0
            if (zdist.mod(Rd/2) < 3*Rd/2):
               nearSaddle = True
               xPole = saddles[j].xPole
         j = j + 1
      if (nearSaddle == False):
         height = realball(-r*log(|x0|) - s*log(|y0|))
         height = height.approx(0.01) % tolerance arbitrary
```

```
% If the height of (x0,y0) is higher than hstop, determine pole type
         if (height > hstop + 0.01):
            high = True
            tolerance = 0.1 % arbitrary pole-finding tolerance
            poleFound = False
            while (poleFound == False):
               if (x0.mod(tolerance) < radius - tolerance):
                  poleFound = True; xPole = True
               else if (y0.mod(tolerance) < radius - tolerance):
                  poleFound = True; xPole = False
               else:
                  tolerance = tolerance / 2
% If no reason to terminate has been found, take another ascent step
      if ((high == False) AND (nearSaddle == False)):
         byX = paramByX(Q(x,y),x0,y0)
         if (byX == True):
            H1old(x,y) = -r/x + (s/y)*(diff(Q,x)/diff(Q,y))
            H1(w,z) = H1old(w,z)
            dir0 = conjugate(H1(x0,y0)) % ball number
            [w1,z1] = pathStep(Q(w,z),x0,y0,1,H1,r,s,byX,dir0/|dir0|)
            x0 = w1; y0 = z1
         else:
            H1old(x,y) = -s/y + (r/x)*(diff(Q,y)/diff(Q,x))
            H1(w,z) = H1old(z,w)
            dir0 = conjugate(H1(y0,x0)) % ball number
            [w1,z1] = pathStep(Q(z,w),y0,x0,1,H1,r,s,byX,dir0/|dir0|)
            x0 = z1; y0 = w1
   return xPole
```

## 4.15 The main algorithm

We are finally ready to chain the preceding functions together to reproduce the algorithm sketched at the beginning of this chapter.

### Implementation of the main algorithm

```
main()
% Initialize the set of contributing points
   contrib = empty array
% Input the polynomial Q and the direction [r,s]
   Print "Please enter the polynomial Q(x,y): "
```

```
   Input Q % a polynomial in x and y
   Print "Please enter the direction for asymptotics, [r,s]: "
   Input [r,s] % a pair of rational numbers
   if ((r == 0) OR (s == 0)):
      Print "This is a univariate asymptotics problem."
      TERMINATE
   if (finiteHeight(Q,r,s) == True):
      Print "There are points at infinity of finite height."
      TERMINATE
% Collect Groebner basis information on where V_Q is not smooth:
   smooth = True
   smoothBX = Basis([Q,diff(Q,x),diff(Q,y)],plex(y,x))
   if smoothBX is not equal to [1]:
      smooth = False
% Check if the elimination polynomial still has a y in it:
      if (diff(smoothBX[0],y) is not equal to 0):
         Print: "Nontrivial non-smooth portion."
         TERMINATE
      smoothBY = Basis([Q,diff(Q,x),diff(Q,y)],plex(x,y))
   if (smooth == True):
      smoothBY = [1]
% Collect Groebner basis information on where H has saddles on V_Q
   saddleBX = Basis([Q,s*x*diff(Q,x) - r*y*diff(Q,y)],plex(y,x))
   if (saddleBX == [1]):
      Print "There are no saddles."
      saddleBY = [1]
      if (smooth == True)
         TERMINATE
% Check if the elimination polynomial still has a y in it:
   if (diff(saddleBX[0],y) is not equal to 0):
      print "Nontrivial saddle portion."
      TERMINATE
   if (saddleBX != [1]):
      saddleBY = Basis([Q,s*x*diff(Q,x) - r*y*diff(Q,y)],plex(x,y))
% Find points at which V_Q has saddles and/or is non-smooth
   points = findPoints(saddleBX,saddleBY,smoothBX,smoothBY,smooth)
   Px = smoothBX[0]*saddleBX[0]
   Py = smoothBY[0]*saddleBY[0]
% Initialize the exponential heights of these special points
   troots = possHieght(Px,Py,r,s)
   tolerance = troots[0].tol
   for p in points
      p.height = realball(|p.x|^(-r)*|p.y|^(-s))
   sort points: decreasing by .height.tol(tolerance)
   saddles = points
   if (smooth == False):
```

```
% Look for singularities
      j = 0
      while (type(points[j]) == saddle):
         j = j + 1
% Truncate to remove all points at or below the highest singularity
      k = 0
      distance = |points[k].height - points[j].height|
      while (distance.approx(tolerance) >= 2*tolerance):
         k = k + 1
         distance = |points[k].height - points[j].height|
      saddles = points[0..(k-1)]
      badHeight = points[k].height % e^height, highest singularity
   if (saddles is an empty array):
      Print "No saddles above highest singularity."
      TERMINATE
% Initialize an e^height c at which to stop examining saddle points
   c = badHeight
   [hStop,radius] = termCriteria(Q,r,s)
   for j from 0 to (len(saddles) - 1): % loop through the saddles
      p = saddles[j]
      addToContrib = False
% Check if the height of the saddle p is >= log(c)
      heightCheck = p.height - c
      if (heightCheck.approx(tolerance) + tolerance >= 0):
% Determine a parameterization variable and relevant neighborhoods
         p.byX = paramByX(Q(x,y),p.x,p.y)
         if (p.byX == True):
            Qnew(w,z) = Q(w,z)
            w0 = p.x; z0 = p.y
         else:
            Qnew(w,z) = Q(z,w)
            w0 = p.y; z0 = p.x
         [p.Ri,p.Rd] = paramNbd(Qnew,w0,z0)
         [n,DH] = degeneracy(Qnew,w0,z0,r,s,p.byX)
         p.pathToX = empty array of length n
         p.R = ascentNbd(w0,z0,r,s,p.byX,p.Ri,p.Rd,n,DH)
         dir0 = DH(w0,z0)
         dir0 = (conjugate(dir0))^(1/n)
         dir = dir0/|dir0| % a direction of steepest ascent
% For each ascent region: follow ascending path through that region
         for k from 0 to (n - 1):
            rotate = exp(k*2*Pi*I/n)
            direction = dir*rotate
            [w1,z1] = pathStep(Qnew,w0,z0,n,DH,r,s,p.byX,direction)
            if (p.byX == True):
               toX = path(Q,w1,z1,r,s,hStop,radius,saddles[0..(j-1)])
```

```
            else:
               toX = path(Q,z1,w1,r,s,hStop,radius,saddles[0..(j-1)])
            p.pathToX[k] = toX % where does each path go?
% Check if the saddle p has paths going to x=0 and y=0
         for k from 0 to (n - 1):
            if (p.xPole[k] != p.xPole[(k+1) mod n]):
               c = p.height % c_xy = log(c)
               addToContrib = True
               rotate = exp(k*2*Pi*I/n + Pi*I/n)
               if (p.xPole[k] == True):
                  p.in.append(dir*rotate)
               else:
                  p.out.append(dir*rotate)
         if (addToContrib == False):
            p.xPole = p.pathToX[0] % all paths go to a single pole
         else:
            contrib.append(p)
   if (contrib is an empty array):
      Print "No saddles stopped the flow of the intersection cycle."
   else:
      Print "A non-trivial contributing set was found."
   TERMINATE
```

### The main theorem

**Theorem 4.15.1.** *The program* `main()` *terminates in finitely many steps. If Assumptions 3.1.2, 3.2.4 or 3.6.1 are not satisfied, the program terminates with an error message. Otherwise, there are two possibilities for the program's final state:*

1. *$\Xi$ is empty, in which case* `contrib` *is an empty array. Or,*

2. *$\Xi$ is not empty. In this case, $\sigma \in \Xi$ if and only if there is an element* `p` *of type* `saddle` *in* `contrib` *such that $\sigma =$* `(p.x,p.y)`.

*In case (1), $\mathcal{V}_Q$ is smooth if and only if the boolean variable* `smooth` *is set to* `True`. *If $\mathcal{V}_Q$ is smooth, then there is a cycle $\kappa \in [\mathcal{C}]$ contained in $\mathcal{V}^{\leq m}$ for arbitrarily small $m$. If on the other hand $\mathcal{V}_Q$ is not smooth, then the real ball number* `log(badHeight)` *represents the* *height $c_0$ of the highest critical point. In this case, there is a cycle $\kappa \in [\mathcal{C}]$ contained in $\mathcal{V}^{\leq c_0+\varepsilon}$ for arbitrarily small $\varepsilon$.*

*In case (2), there is a cycle $\kappa \in [\mathcal{C}]$ along which the height is maximized exactly at the points in $\Xi$. Let $\sigma \in \Xi$ be such a saddle, represented by* `p` *in* `contrib`*. Then on a sufficiently small neighborhood $U \subseteq \mathcal{V}_Q$ of $\sigma$, $\kappa$ takes the form*

$$\kappa \cap U = \sum_j \gamma_j - \sum_j \tilde{\gamma}_j,$$

*where each $\gamma_j$ is a path exiting $\sigma$ in the direction of* `p.out[j]` *(as parameterized by $x$ if* `p.byX == True`*, as parameterized by $y$ otherwise), and each $\tilde{\gamma}_j$ is a path exiting $\sigma$ in the direction of* `p.in[j]` *(with similar parameterization conditions).*

*Proof.* It should be clear by the characterization theorem and the structure of the algorithm that, should the algorithm terminate, it will terminate with the prescribed conditions. Thus we will be done if we can prove that the algorithm terminates in a finite number of steps.

Tracing the algorithm, we note that there is but one loop that does not terminate simply by its structure alone: the loop in `path()`. The path algorithm loops through calls to `pathStep()`, stopping only when the height of the path is sufficiently large or when the path approaches sufficiently close to a saddle point. By way of contradiction, we assume that these terminal conditions are never obtained for some call to `path()`.

By non-termination, we must have that the height is bounded from above (and below, as the path is non-decreasing). But `path()` is only called on varieties $\mathcal{V}_Q$ along which the height approaches $\pm\infty$ as $x$ or $y$ go to 0 or $\infty$. Thus the path is strictly bounded away from 0 and $\infty$. Similarly the path must be strictly bounded away from all critical points of $h$, lest the loop terminate. Hence there is some compact set $K \subset \mathcal{V}_Q$ on which this path lies, where $K$ includes no saddle points of $h$. Note further that we may assume $\mathcal{V}_Q$ to be smooth on $K$, as `path()` is only initialized at points above the height of the highest non-smooth point.

Now we look at those steps of the path that are parameterized by $x$, hereupon called $x$-steps. We denote the $j^{\text{th}}$ such step by

$$\iota_j(\gamma_j(t)) : [0, \varepsilon_j] \to \mathcal{V}_Q$$

where $\varepsilon_j > 0$, $\gamma_j(t) = x_0 + ct$ for some $x_0, c \in \mathbb{C}$ with $|c| = 1$, and $\iota_j$ is some local parameterization of $\mathcal{V}_Q$ by $x$. First, we claim that path ascent algorithm is constructed in such a way that these $x$-steps are bounded away from the points where $Q_y$ vanish. Why is this so?

Smoothness of $\mathcal{V}_Q$ along the compact set $K$ gives us a lower bound for $\max\{|Q_x|, |Q_y|\}$ on $K$. Thus when $|Q_y|$ is sufficiently small, $|Q_x|$ will be large enough to force the variable of parameterization to be $y$ (see the function `paramByX()`). This bounds the point of initiation of any $x$-step away from the points where $|Q_y| = 0$. But note further that each $x$-step is constructed so that the value of $|Q_y|$ decreases by no more than half along the ascent segment (see `paramNbd()`). Thus the entirety of each $x$-step is bounded away from such points. Hence we have that all $x$-steps are supported on a compact set $K_0 \subseteq K$ on which $Q_y$ does not vanish.

We now turn to examining the derivative of the height function along any such $x$-step. The derivative takes the form

$$\frac{d}{dt} h(\iota_j(\gamma_j(t)) = \operatorname{Re}\left[c \cdot G\left(\iota_j(\gamma_j(t))\right)\right]$$

where the function $G$ is defined by

$$G(x, y) = \frac{\hat{s} x Q_x(x, y) - \hat{r} y Q_y(x, y)}{x y Q_y(x, y)}$$

we examine $G(x, y)$ on $K_0$.

Note that $G$ is continuous on $K_0$, because $xyQ_y$ never vanishes on $K_0$. $G$ also never vanishes on $K_0$, as $\hat{s}xQ_x - \hat{r}yQ_y$ only vanishes at saddle points. Thus, by compactness, we have uniform continuity of $G$ and a nonzero lower bound for $|G|$ on $K_0$. Consequently there exist some $m, \delta_1 > 0$ such that for any $(x, y), (x_1, y_1) \in K_0$ we have

$$m \leq |G(x, y)|, \text{ and}$$
$$|x - x_1| < \delta_1, |y - y_1| < \delta_1 \Rightarrow |G(x, y) - G(x_1, y_1)| < m/2.$$

And again by compactness, we also get uniform continuity over all of the parameterization functions $\iota_j$. That is, there is some $\delta_2 > 0$ such that for $x, x_1$ in the domain of some $\iota_j$, we have

$$|x - x_1| < \delta_2 \Rightarrow |\iota_j(x) - \iota_j(x_1)| < \delta_1.$$

Putting this together, for $t < \delta := \min\{\varepsilon_j, \delta_1, \delta_2\}$ we have

$$\arg\left(G(\iota_j(\gamma_j(t))) - G(\iota_j(x_0))\right) \in \left(\sin^{-1}\left(\frac{-m/2}{m}\right), \sin^{-1}\left(\frac{m/2}{m}\right)\right) = \left(-\frac{\pi}{6}, \frac{\pi}{6}\right),$$

and hence

$$\arg\left(c \cdot G(\iota_j(\gamma_j(t)))\right) \in \left(-\frac{\pi}{6} - \frac{\pi}{4}, \frac{\pi}{6} + \frac{\pi}{4}\right),$$

by the manner in which the direction $c$ of the $x$-step is chosen. Thus $c \cdot G(\iota_j(\gamma_j(t)))$ is not purely imaginary for such values of $t$. In other words, the derivative of the height function is bounded away from 0 on an initial segment of the $x$-step.

If we can show that the set of all $\varepsilon_j$ is bounded away from 0, i.e. that there is a nonzero lower bound on the length of any $x$-step, then the preceding will imply that there is a nonzero lower bound on the height ascended by any such $x$-step. This will prove that there can be at most finitely many $x$-steps, lest the path itself be unbounded in height.

But by inspection of the functions `paramNbd`, `isoRoot` and `ascentNbd`, we see that as long as $|x|$ and $|y|$ are bounded away from 0 and $\infty$, and as long as $|Q_y|$ and $|\hat{s}xQ_x - \hat{r}yQ_y|$ are bounded away from 0, then the length of each ascent step will be bounded away from 0. Hence there can be only finitely many $x$-steps.

But by analogous reasoning, there can be only finitely many $y$-steps. And so the algorithm does terminate after finitely many steps. $\square$

Note that the preceding proof hints at the factors that drive up the runtime of `main()`. Specifically, if calls to `pathStep()` are made in neighborhoods where $|\hat{s}xQ_x - \hat{r}yQ_y|$ is small, the lower bound on the height ascended by such a step is likewise small. This can occur, for instance, if the ascent neighborhoods around the saddle points are particularly small. Thus if the direction $(\hat{r}, \hat{s})$ is near a direction in which the saddle points of $h$ on $\mathcal{V}_Q$ coalesce – thus forcing the ascent neighborhoods of such points to be small – we would expect a rise in the number of steps before `main()` terminates.

# Future Research

The next step in this line of research is to implement the algorithm described in the previous chapter. This project is slated to begin in the coming months, in collaboration with Joris van der Hoeven. The case of bicolored supertrees, meeting the assumptions of Chapter 3 and being well-understood, will serve as a perfect test case for the algorithm.

Beyond implementation, there are many avenues along which future research may continue. Below we list just two; a short-term and long-term goal of algorithmic singularity analysis.

## Extending the bivariate algorithm

One immediate goal is to further relax the assumptions introduced in Chapter 3, thus extending the current bivariate algorithm to handle a wider class of rational functions. The assumption most suitable for attack is likely Assumption 3.2.4, which requires that the height function be unbounded on any portion of $\mathcal{V}_Q$ along which $x$ or $y$ approaches $\infty$. One of the reasons for which we would like to drop this assumption is hinted at in [RW08].

In [RW08], Raichev and Wilson attempt to use bivariate singularity analysis to produce asymptotics for the function $x/\sqrt{1-x}$. By Safonov's algorithm, they produced a bivariate rational function

$$F(x,y)=\frac{P(x,y)}{Q(x,y)}=\frac{xy(-3y+3xy^2-2+2xy+x-2y^2+2xy^3)}{-y+xy^2-2+2xy+x}=\sum_{r,s\geq 0}a_{r,s}x^ry^s$$

such that local to the origin,

$$\frac{x}{\sqrt{1-x}}=\sum_{n=0}^{\infty}a_{n,n}x^n.$$

The variety $\mathcal{V}_Q$ is smooth, but in analyzing the height function $h$ along $\mathcal{V}_Q$ in the direction $(\hat{r},\hat{s})=(1,1)$ it was found that $h$ has no critical points. A quick analysis of the Newton Diagram of $Q$ reveals a branch along which $y\sim x^{-1}$ as $x\to 0$, and thus $h(x,y)\sim 0$ as $x\to 0$ and $y\to\infty$ here. Hence $F(x,y)$ fails Assumption 3.2.4.

It turns out that by properly compactifying this branch – adding a point $(x,y)=(0,\infty)$ to $\mathcal{V}_Q$ and suitably extending the height function – it can be shown that $h$ has a critical point at the compactification point. Applying a generalization of the ascent path algorithm indicates how to obtain an asymptotic analysis of the $a_{n,n}$ by integrating over an appropriate cycle local to the point at infinity. The relevant details will appear in a forthcoming work.

One would like to generalize this technique, adding a point at infinity for each branch of $\mathcal{V}_Q$ that leads to the failure of Assumption 3.2.4. How best to do this remains an open question. One possibility (suitable for the example described above) is to rotate in the relevant point at infinity. Specifically, we view $\mathcal{V}_Q\subseteq\mathbb{C}^2\subseteq\mathbb{C}P^2$. Representing the points of $\mathbb{C}P^2$ by homogeneous coordinates in $\mathbb{C}^3$, we can then rotate in points at infinity by performing a linear transformation on $\mathbb{C}^3$. Note that after this transformation, however, the variety $\mathcal{V}_Q$ may not be smooth at the relevant point.

## Algorithms for trivariate singularity analysis

As mentioned previously, Safonov's algorithm reduces the asymptotic analysis of $n$-variate algebraic generating functions to the analysis of $(n+1)$-variate rational generating functions. As such, the bivariate asymptotics algorithm of Chapter 4 can be used to analyze certain univariate algebraic generating functions (such as the generating function for bivariate supertrees discussed in Chapter 2). There are perhaps simpler techniques for analyzing univariate algebraic functions, but the coefficients of bivariate algebraic functions are less well understood. Thus an algorithm for trivariate rational singularity analysis would be a breakthrough.

Unfortunately, the Morse-theoretic decomposition is more difficult to understand in this case. Specifically, when the singular variety is a two (or more) complex-dimensional object, we can no longer guarantee that the critical points of the height function have a saddle structure amenable to a local Morse decomposition. Still, it is hoped that the characterization theorems in the bivariate case will lead to similar theorems for three (and more) variables. This should be a main goal of any future research.

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