HIGGS BUNDLES AND OPERS

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ABSTRACT

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In this thesis we address the question of determining the Higgs bundles on a Riemann surface which correspond to opers by the non-abelian Hodge theorem. We study one specific Higgs bundle $(\mathbf{P}, \theta)$ obtained by extension of structure group from a particular rank two Higgs bundle on $K^{1/2} \oplus K^{-1/2}$. For a simple complex group we identify the tangent space to the Dolbeault moduli space at that particular point as the Hitchin base plus its dual and exhibit harmonic representatives for the hypercohomology of the corresponding deformation complex. We study the Maurer-Cartan equation for $(\mathbf{P}, \theta)$ and give an explicit, recursive (terminating after finitely many steps) method for writing a family of deformations parametrised by the tangent space which can be interpreted as a holomorphic exponential map. Next, we identify opers with their tangent space at the uniformisation oper. By quaternionic linear algebra we determine the subspace of infinitesimal deformations of $(\mathbf{P}, \theta)$ corresponding to opers and apply the exponential map to those. In this way we obtain the germ of the family of Higgs bundles corresponding to opers. In the final chapter we sketch a plan for obtaining the complete solution, as well as certain questions for further investigation.

Contents

Chapter 1

Introduction

Those who know, don't tell. Those who tell, don't know.

Zen Proverb

Non-abelian Hodge theory on a compact Kähler manifold is a correspondence between two types of data – broadly speaking, spectral and differential data. More precisely – assuming the manifold to be a smooth compact connected algebraic curve – these two types of data are degree zero Higgs bundles and flat holomorphic connections satisfying suitable stability and irreducibility conditions. The correspondence, which uses a choice of preferred hermitian metric determined by a highly non-linear PDE, induces a homeomorphism between the moduli spaces parametrising these objects. The two moduli spaces are complex algebraic varieties but the above homeomorphism is not compatible with the complex structure. In fact, the smooth locus has an underlying hyperkähler structure.

In what follows, I am going to look at one particular subvariety in the moduli space of local systems and determine the germ of its image under the above homeomorphism. This subvariety of the moduli space parametrises certain (very special) flat connections called *opers*. Knowing the Higgs bundles corresponding to opers is interesting for at least two reasons. On one hand, due to the non-explicit nature of the construction, there are very few cases when one knows explicitly what objects are matched under the correspondence. On the other hand, recent work of A.Kapustin and E.Witten suggests that geometric Langlands correspondence can be related to mirror symmetry for topological field theories associated to the moduli spaces of flat connections for Langlands dual groups, and the hyperkähler rotation described above is a key player in constructing the mirrors. The subvariety of opers is an example of what Kapustin and Witten call (A,B,A)-brane and it would be interesting if one is able to determine explicitly its mirror and compare it with the prediction coming from physics.

Chapter 2

Notation. Higgs Bundles and Local Systems.

Non-abelian Hodge theory is a vast subject which has been developing for more than twenty years and even sketching the outlines of the landscape is an arduous task which I do not plan to embark on. However, I will give some of the basic definitions mostly to set up notation and give an illusion of completeness.

The objects which we shall be dealing with – to be defined below – will satisfy the following restrictions. The base variety will always be a compact, connected, smooth curve X. Except for two examples it will be of genus at least two. The structure group will be a simple complex Lie group, except for examples with line bundles when it is $\mathbb{C}^{\times}$. Higgs bundles will always be stable of degree zero and local systems will always be irreducible; however, in this introductory section some of the objects will be defined in slightly greater generality. Boldface letters usually denote holomorphic objects. Holomorphic connections are denoted by D and their smooth counterparts by $\nabla = \overline{\partial} + D$. If K is a metric, ∇_K is the corresponding Chern connection. If $\mathbf{P}$ is a (holomorphic) principal bundle with structure group G and Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, the vector bundle with fibre $\mathfrak{g}$ associated to $\mathbf{P}$ via the adjoint representation is denoted $ad\ \mathbf{P}$, $\mathfrak{g}_{\mathbf{P}}$ or $\mathfrak{g}$. Except for one example, M_{Dol}, M_{DR}, M_B – see below – will denote the *stable*, resp. *irreducible* loci in the corresponding coarse moduli spaces, or even the smooth loci.

The main results and references on the subject can be found in [Hit87a], [Sim92], [Sim94].

If X is a compact Kähler manifold with a chosen Kähler metric, a *Higgs (vector) bundle* or *Higgs pair* is a pair $(\mathbf{E}, \theta)$, where $\mathbf{E}$ is holomorphic vector bundle and $\theta \in H^0(X, \mathscr{E}nd\ \mathbf{E} \otimes \Omega^1_X)$ satisfies – trivially if $\dim X = 1$ – the condition $\theta \wedge \theta = 0$. The sheaf of differentials on a curve will be usually denoted by K or K_X. More generally, if G is a complex connected reductive Lie group, a (principal) *G-Higgs bundle* is a pair $(\mathbf{P}, \theta)$, where $\mathbf{P}$ is a holomorphic principal G-bundle and $\theta \in H^0(ad\ \mathbf{P} \otimes \Omega^1_X)$ satisfies $\theta \wedge \theta = 0$. A metric on a Higgs bundle is a reduction to a maximal compact subgroup $K \subset G$. This data gives us a *non-metric* connection $\nabla_K + \theta + \theta^*$ and we may ask if the metric satisfies the Hermite-Yang-Mills condition. If this is the case, this new connection is flat if and only if the first and second Chern numbers of $\mathbf{P}$ vanish. For a degree zero Higgs bundle the Hermite-Yang-Mills condition gives the vanishing of the component of the curvature along the Kähler form, and on a curve this is the same as flatness, but in higher dimensions one also needs the vanishing of the second Chern number. Such a metric on a Higgs bundle is called *harmonic*. A metric P_K on the Higgs bundle $(\mathbf{P}, \theta)$ is called *irreducible* if $\ker \nabla_K \cap \ker ad(\theta) = \mathfrak{z} \subset \mathfrak{g}$; i.e., the only holomorphic sections $s \in H^0(ad\mathbf{P})$ such that $D_K(s) = 0$ and $[s, \theta] = 0$ are constant sections coming from the center of the Lie algebra. There exists a notion of stability for Higgs bundles, extending the notion of stability for bundles. For the Higgs vector bundle case this notion coincides with the usual notion of stability, but measured with respect to subbundles, invariant for the Higgs field.

A *local system* is a locally constant sheaf of vector spaces, or, equivalently, a holomorphic flat connection, or, equivalently, a representation of the fundamental group. On a curve every holomorphic connection is flat for dimensional reasons. A flat connection, D, on a principal G-bundle, $\mathbf{Q}$, is *irreducible* if $\ker D = \mathfrak{z} \subset \mathfrak{g} \subset A^0(ad\ \mathbf{Q})$, and is *semisimple* if it is a direct sum of irreducibles. In terms of the holonomy representation $\rho : \pi_1(X) \to G$, irreducibility is equivalent to the property that $im(\rho) \subset G$ is not contained in any nontrivial parabolic subgroup.

For a principal bundle with a flat connection and metric one can ask if the metric is *harmonic* in the sense of Corlette and Donaldson, i.e, if the reduction minimises Donaldson's energy functional. From the data of a flat connection and metric one can extract a Higgs field – looking at the non-metric part of the connection, as above

– provided certain integrability conditions are fulfilled; the integrability condition is the harmonicity of the metric. The theorem of Corlette and Donaldson says that on a compact Riemannian manifold a principal bundle with a real reductive structure group endowed with a flat connection admits a harmonic metric iff the Zariski closure of the monodromy representation is a reductive subgroup.

The theorem of Hitchin and Simpson (generalising the theorem of Donaldson and Uhlenbeck-Yau) states that a G Higgs bundle $(\mathbf{P}, \theta)$ admits an Hermitian-Yang-Mills metric if and only if it is poly-stable; the metric is unique and irreducible if and only if the Higgs bundle is stable. This metric is harmonic if and only if the first and second Chern numbers vanish. A flat G-bundle $(\mathbf{P}, D)$ admits a harmonic metric if and only if it is semi-simple.

One uses *harmonic bundle* for a bundle admitting a harmonic metric, equipped with the structure of a flat and Higgs bundle which can be related by a harmonic metric.

By a theorem of Simpson ([Sim92], Corollary 1.3), there is an equivalence of categories between the category of semisimple flat bundles on X and the category of polystable Higgs bundles with vanishing (first and second) Chern numbers, both being equivalent to the category of harmonic bundles.

Higgs bundles and local systems are Artin stacks, but they have open substacks consisting of stable irreducible objects which are actually schemes. For all constructions in my thesis it will be enough to deal with the coarse moduli spaces consisting of stable objects with minimum automorphisms (on a curve) which are in fact smooth manifolds. Various constructions of the moduli spaces have been given in different generality – by Hitchin ([Hit87a]), Nitsure ([Nit91]), Simpson ([Sim94]). Following Simpson, we use $M_{Dol}(G)$, $M_{DR}(G)$ and $M_B(G)$ to denote respectively the moduli spaces of Higgs bunbles of degree zero, local systems and representations of $\pi_1(X)$. In Simpson's notation these are the moduli spaces of semisimple/polystable objects, and while I will be dealing with the smooth loci M_{Dol}^{reg}, etc., I may occasionally abuse notation and suppress the superscript "reg". The equivalence of categories coming from Simpson's theorem is *not* compatible with the complex structure: it is only a homeomorphism. Since I will be dealing with the smooth locus only, I will be saying that the nonabelian Hodge theorem gives a diffeomorphism $\tau : M_{DR}^{reg} \simeq M_{Dol}^{reg}$. In fact, the two spaces, M_{DR}^{reg} and M_{Dol}^{reg} are part of a hyperkähler family.

Here are some properties of the moduli spaces for the case when X is a curve of genus at least two.

Simpson ([Sim94], Theorem 11.1, p.70) showed that for $G = GL(n, \mathbb{C})$ the moduli spaces M_B, M_{Dol} and M_{DR} are irreducible and normal algebraic varieties, of dimension $2[n^2(g-1)+1]$. More generally, for G a semisimple complex group, the dimension of these spaces is $2\dim G(g-1)$ ([Hit87a]). The "extra 1" in the general linear case is due to the center $\mathbb{C}^\times \subset GL(n, \mathbb{C})$. The moduli space $M_{Dol}(G)$ contains as a dense open the cotangent bundle, $T^\vee \mathcal{N}(G)^{reg}$ to the moduli space of stable principal G-bundles of degree zero. The moduli space M_{DR} contains a dense open which is a $T^{\vee}\mathcal{N}(G)^{reg}$-torsor.

With $(\mathbf{E}, \theta)$ one associates its Dolbeault complex $\mathbf{E} \xrightarrow{\wedge\theta} \mathbf{E} \otimes K_X$ whose hypercohomology is called the Dolbeault cohomology of $(\mathbf{E}, \theta)$. More generally, with a principal (or vector) Higgs bundle $(\mathbf{P}, \theta)$ one can also associate the Dolbeault complex of its adjoint bundle:

$$\mathscr{C}_\bullet : \qquad ad\ \mathbf{P} \xrightarrow{ad\ \theta} ad\ \mathbf{P} \otimes K\,.$$

Biswas and Ramanan in [BR94] proved that if G any algebraic group, the space of infinitesimal deformations of the pair $(\mathbf{P}, \theta)$ is $\mathbb{H}^1(\mathscr{C}_\bullet)$. There is a map of complexes $\mathscr{C}_\bullet \to ad\ \mathbf{P}$ and the long exact sequence of hypercohomology of

$$0 \longrightarrow ad\ \mathbf{P} \otimes K[-1] \longrightarrow \mathscr{C}_\bullet \longrightarrow ad\ \mathbf{P} \longrightarrow 0$$

can be used to compute the differential of the forgetful map from the Dolbeault space to the moduli of bundles. (The forgetful map is a map of stacks. If one wants to stay with the moduli spaces, this is a rational map only.) By the Lefschetz theorem for local systems ([Sim92]) $\mathbb{H}^0(\mathscr{C}_\bullet) \simeq \mathbb{H}^2(\mathscr{C}_\bullet)$. In particular, $\mathbb{H}^0(\mathscr{C}_\bullet) = \{s \in H^0(ad\ \mathbf{P}) | [\theta, s] = 0\} = \mathfrak{z}(\mathfrak{g})$ if $(\mathbf{P}, \theta)$ is stable. If G is semisimple this is zero and by Theorem 3.1 of [BR94] $(\mathbf{P}, \theta)$ is a smooth point of the moduli space (the deformation functor is pro-representable and the corresponding local algebra is regular).

The map to the moduli of bundles makes M_{Dol} into a Lagrangian fibration. The Dolbeault space admits another – generically transverse – structure of Lagrangian fibration, known as the Hitchin system, described first in [Hit87a] and ([Hit87b]). Suppose G is semisimple, and choose a basis for the invariant polynomials; then we have a map to a vector space (the Hitchin base) given by evaluating the coefficients of the characteristic polynomial of the Higgs field:

$$M_{Dol}(G) \to B_{\mathfrak{g}} = \bigoplus_{i=1}^{rk\ G} H^0(X, K^{m_i+1}),$$

where m_i are the exponents of the group G. This is an algebraic completely integrable system and the fibres are generically abelian varieties. In [Hit92] Hitchin constructed a section of this integrable system which will be briefly described in Section 6. More invariantly, the base $B_{\mathfrak{g}}$ can be described as follows. By Chevalley's theorem, if $\mathfrak{g} = \mathrm{Lie}G$, $\mathfrak{g}//G \simeq \mathfrak{t}/W = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^\vee)^G)$, where $\mathfrak{t}$ is the Cartan subalgebra and W is the Weyl group. Since $\mathrm{Sym}(\mathfrak{g}^\vee)^G$ is again a polynomial algebra, it has a $\mathbb{C}^\times$-action, and we can consider the vector bundle with fibre $\mathfrak{t}/W$ associated to the frame bundle of the canonical bundle, i.e., the vector bundle $\mathfrak{t}/W_K := \mathrm{Isom}(K, \mathcal{O}) \times_{\mathbb{C}^\times} \mathfrak{t}/W$. Then the Hitchin base is $B_{\mathfrak{g}} = H^0(X, \mathfrak{t}/W_K \otimes K)$, which we will occasionally write as $H^0(X, \mathfrak{t}/W \otimes K)$.

Given a local system $ad\ \mathbf{Q}^\nabla$, one can consider its de Rham resolution, that is, the quasi-isomorphism

$$0 \longrightarrow ad\ \mathbf{Q}^\nabla \longrightarrow \mathscr{D}_\bullet,$$

$$\mathscr{D}_\bullet : 0 \longrightarrow ad\ \mathbf{Q} \xrightarrow{D} ad\ \mathbf{Q} \otimes K \longrightarrow 0.$$

The space M_{DR} also maps – with Lagrangian fibres – to the moduli of bundles by the forgetful map, and one can compute its differential analogously to the case of Higgs bundles.

Chapter 3

Lie-algebraic Preparation

Let G be a simple complex Lie group with Lie algebra $\mathfrak{g}$ of rank l, and let x and y be two regular nilpotent elements – that is, with l-dimensional cenraliser – which, together with $h \in \mathfrak{g}$ span a subalgebra isomorphic to $\mathfrak{sl}(2)$. Such a subalgebra is called a "principal $\mathfrak{sl}(2)$" or a "principal 3-dimensional subalgebra".

Example 3.0.1 *If $\mathfrak{g} = \mathfrak{sl}(l+1)$, one choice of a principal 3-dimensional subalgebra is $\mathfrak{sl}(2) = span\{x, y, h\}$, where*

$$y = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} l & 0 & 0 & \dots & 0 \\ 0 & l-2 & 0 & \dots & 0 \\ 0 & 0 & l-4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -l \end{pmatrix},$$

$$x = \begin{pmatrix} 0 & l & 0 & 0 & \dots & 0 \\ 0 & 0 & 2(l-1) & 0 & \dots & 0 \\ 0 & 0 & 0 & 3(l-2) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & l \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Notice that this is the representation of $\mathfrak{sl}(2)$ on $Sym^l(\mathbb{C}^2) = \mathbb{C}^{l+1}$ induced by the representation $\mathfrak{sl}(2) \simeq \text{span}\{y_0, x_0, h_0\} \subset \text{End}(\mathbb{C}^2)$, $y_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Kostant's yoga of principal subalgebras ([Kos63], see also [Hit92]) is a generalisation of the embedding of $\mathfrak{sl}(2)$ in $\mathfrak{sl}(l+1)$ given by the l-th symmetric power. If $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group, then by Chevalley's theorem $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{t}]^W \simeq \mathbb{C}[p_1, \dots, p_l]$, where $\{p_1, \dots, p_l\}$ form a basis of the G-invariant polynomials. If $Z(x)$ is the centraliser of x, then, as shown by Kostant, the set $y + Z(x) \subset \mathfrak{g}$ provides a section for the Chevalley projection $\mathfrak{g} \to \mathfrak{t}/W = \mathfrak{g}//G$. Under the adjoint action of the $\mathfrak{sl}(2)$-subalgebra $\mathfrak{g}$ decomposes into l odd-dimensional irreducible representations of $\mathfrak{sl}(2)$:

$$\mathfrak{g} = \bigoplus_{i=1}^{l} W_i, \ \dim W_i = 2m_i + 1. \tag{3.0.1}$$

Here the first summand is $W_1 = \text{span}\{x, h, y\}$.

On each W_i the eigenvalues, $2m$, of h satisfy $-m_i \leq m \leq m_i$. The highest weight vectors span the centraliser of x. The integers m_i are the exponents of G: the i-th invariant polynomial p_i has degree $m_i + 1$.

For $\mathfrak{sl}(n)$ the regular nilpotent elements are conjugate to a single Jordan block with eigenvalue zero, and one can take the powers $x, x^2, \ldots, x^l$, $l = n-1$ for a basis of $Z(x)$.

If $\mathfrak{g}_m$ is the space of eigenvectors of h of eigenvalue $2m$, we have a decomposition

$$\mathfrak{g} = \bigoplus_{m=-M}^{M} \mathfrak{g}_m, \ M = \max\{m_i\}. \tag{3.0.2}$$

This is the so-called "principal grading" of $\mathfrak{g}$.

One can take the principal subalgebra to be compatible with other Lie-algebraic data. Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}_+$, with Chevalley generators $\{f_i, h_i, e_i\}$, $i = 1 \ldots l$, $f_i \in \mathfrak{n}_-$, $h_i \in \mathfrak{t}$ and $e_i \in \mathfrak{n}_+$. If $\rho^\vee$ is the dual Weyl vector (see the Appendix (10)), and if $y = \sum_{i=1}^{l} f_i \in \mathfrak{n}_-$ then there is a unique x so that $\{x, 2\rho^\vee, y\}$ is an $\mathfrak{sl}(2)$-subalgebra. In fact, it is $x = \sum_i 2\rho_i^\vee e_i$, where $\rho^\vee = \sum_i \rho_i^\vee h_i$. This will be our default choice of principal subalgebra.

The principal grading of $\mathfrak{g}$ is determined by the eigenvalues of $ad\ \rho^\vee$ but it depends only on the choice of positive roots: it is the unique Lie-algebra grading determined by $\mathfrak{g}_0 = \mathfrak{t}$, $\mathfrak{g}_1 = \oplus \mathfrak{g}_\alpha$, $\mathfrak{g}_{-1} = \oplus_\alpha \mathfrak{g}_{-\alpha}$, $\alpha \in \Delta^+$. In particular, we have that $\deg f_i = -1$, $\deg e_i = 1$, $\deg h_i = 0$. The grading of $\mathfrak{g}$ induces a grading –

and hence a filtration – of $\mathfrak{b} = \oplus \mathfrak{b}_i = \oplus \mathfrak{b} \cap \mathfrak{g}_i$. I will denote the m-th piece in the filtration by $\mathfrak{g}^{\geq m}$. The action of ady sends $\mathfrak{b}_{i+1}$ to $\mathfrak{b}_i$ injectively, and one can look for the cokernel. In fact ([Fre07]) $\mathfrak{b}_i = ady(\mathfrak{b}_{i+1}) \oplus V_i$, where $V_i \neq (0)$ only if $i = m_i$ is an exponent of G and $\dim V_i$ is the multiplicity of m_i, which is one except for the case $\mathfrak{g} = D_{2n} = \mathfrak{so}_{4n}$. In the latter case it is two and $m_{l/2} = m_l$. In this way we get a vector space $V = \oplus V_{m_i}$ (with multiplicities), which is the centraliser of x, $V = Z(x) = \ker adx = \text{coker } ady$. Except for the above-mentioned case of $\mathfrak{so}_{4n}$, the space $V_{m_i} \subset W_i$ is spanned by the highest weight vector in W_i; in the exceptional case it is spanned by the highest weight vectors in each of the two representations W_{2n}. For the case $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ the decomposition of the Lie algebra and the various subspaces can be seen in the Appendix, Example 10.0.1. The decomposition of the Lie algebra can be represented pictorially in a way which makes the proof of many statements obvious. Namely, draw a column consisting of $2m_i + 1$ boxes for each exponent m_i, so that the middle box of each column lies on the same line. Here the columns represent the irreducible $\mathfrak{sl}(2)$-representations $W_1, W_2, \ldots$, of dimension $2m_i + 1$, with $ad\, x$ and $ad\, y$ acting vertically along these; boxes on the same row represent the graded pieces $\mathfrak{g}_k$, e.g., the middle row represents the Cartan subalgebra, which, together with the ones above it gives the Borel subalgebra. The spaces $V_{m_1}, \ldots, V_{m_l}$ are represented by the top box in each column.

The choice of Chevalley generators determines a real Lie algebra $\mathfrak{g}_0$ with complexification $\mathfrak{g}$, a linear involution, η, induced by $e_i \mapsto -f_i$ (thus $f_i \mapsto -e_i$),

$h_i \mapsto -h_i$ (Chevalley involution) and, respectively, a compact real form, $\mathfrak{g}_c$, which is the fixed locus of the corresponding anti-linear involution. Namely, if $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, where $\mathfrak{g}_0$ is the real Lie algebra generated by $\{e_i, f_i, h_i\}$, then $\mathfrak{g}_0$ is invariant under η and the compact real form is the (+1)-eigenspace of the *anti-linear* extension $\overline{\eta}$, of η, where $\overline{\eta}(v \otimes z) = \eta(v) \otimes \overline{z}$. Explicitly, the (+1)-eigenspace is generated by $\{ih_j, e_j - f_j, i(e_j + f_j), j = 1 \dots l\}$. These generate the compact form of $\mathfrak{g}$. The restriction of the Killing form, $\varkappa$, to $\mathfrak{g}_c$ is negative definite. One defines an hermitian inner product on $\mathfrak{g}$ by the rule

$$(u, v) = -\varkappa(u, \overline{\eta}(v)).$$

We think of $-\overline{\eta}$ as the analogue of hermitian conjugation, so I will be using $v^* := -\overline{\eta}(v)$. One can check that if $u \in \mathfrak{g}$, then $(ad\ u)^* = ad\ u^*$, where $(ad\ u)^*$ is the adjoint with respect to this inner product. So we have

$$\mathfrak{g}_c = \{\text{fixed points of } \overline{\eta}\} = \{A \in \mathfrak{g} | A^* = -A\}.$$

Notice that $-\overline{\eta}$ is *not* a Lie algebra involution, only a vector space one. When $\mathfrak{g} = \mathfrak{sl}(n)$, $\eta(v) = -v^T$, $v^* \equiv -\overline{\eta}(v) = \overline{v}^T = v^\dagger$, where $v^\dagger$ is the hermitian adjoint induced by the standard inner product on $\mathbb{C}^n$, so $[u, v]^* = -[u^*, v^*]$, and $\mathfrak{g}_c = \mathfrak{su}(n)$, consisting of traceless skew-hermitian matrices.

However, with our default choice of $\mathfrak{sl}(2)$-subalgebra, the image of $\mathfrak{sl}(2) \hookrightarrow \mathfrak{g}$ is not preserved by the involution, i.e., $-\eta(y) \neq x$ – see for instance Example 3.0.1. This can be remedied by choosing the compact form appropriately, that is, we just have to modify η to an involution η' which is given by $e_i \mapsto -\frac{1}{2\rho_i^\vee} f_i$, $f_i \mapsto -(2\rho_i^\vee)e_i$, $h_i \mapsto -h_i$. It is straightforward to see that this extends to a Lie algebra automorphism:

$$[e_i, f_j] = \delta_{ij}h_j \mapsto [-2\rho_i^\vee f_i, -(2\rho_j)^{-1}e_j] = -\delta_{ij}h_j,$$

etc., and that the Killing form is still negative definite on the fixed locus of $\overline{\eta}'$. This real form of $\mathfrak{g}$ is generated by $\{\frac{i}{2\rho_j}h_j, e_j - \frac{1}{2\rho_j}f_j, i(e_j + \frac{1}{2\rho_j}f_j), j = 1 \ldots l\}$. In this way we endow $\mathfrak{g}$ with an hermitian metric for which the compact form acts on $\mathfrak{g}$ (via the adjoint action) by skew-adjoint operators and the $\mathfrak{sl}(2)$-subalgebra (and all the representations W_i) are all real. Also, by the very construction we have $y^* = x, x^* = y, h^* = h$.

Lemma 3.0.1 *Consider $\mathfrak{g} = \mathfrak{sl}(l+1) \subset \mathfrak{gl}(l+1, \mathbb{C})$ with the $\mathfrak{sl}(2)$-subalgebra from Example 3.0.1 and "standard" choice of Chevalley generators ($f_p = E_{p+1,p}, p = 1 \ldots l$, etc.). There is a unique hermitian metric on $\mathbb{C}^{l+1}$, such that the hermitian adjoint $A^\dagger := \overline{H}^{-1}\overline{A}^T\overline{H}$ agrees with the adjoint defined by the anti-linear involution, i.e., $A^\dagger = A^*$. With respect to the standard basis H is given by $H = diag(1, l, 2l(l-1), \ldots) = diag(H_1, \ldots, H_{l+1})$, $H_1 = 1$, $H_k = \Pi_{m=1}^{k-1} l(l-m+1), k > 1$. Notice that H has real entries. Then the compact real form is $\mathfrak{g}_c \simeq \mathfrak{su}(l+1) \subset \mathfrak{g}$,*

$$\mathfrak{g}_c = \{A \in \mathfrak{sl}(l+1) | A^* = -A\},$$

and clearly $\mathfrak{g}_c = Lie(H \cdot SU(l+1) \cdot H^{-1}) \subset SL(l+1)$. The hermitian metric given by H is determined – up to a $\mathbb{R}^\times$-multiple – by the property $y^* = x, h^* = h$.

Proof:

Both involutions, $v \mapsto -v^*$ and $v \mapsto -v^\dagger$ are determined by their action on the Chevalley generators, so the equality of the two involutions follows from the formula for H. For the last statement, a metric preserving h must be diagonal, and then $y^* = x$ determines a relation between the diagonal entries of H of the form $H_{22} = lH_{11}, H_{33} = 2(l-1)H_{22}, \ldots$, which has unique solution once H_{11} is fixed.

QED

Suppose now we have fixed a basis vector, e_{m_i} in each vector space $V_{m_i} = W_i \cap \mathfrak{g}_{m_i}$, and hence a basis of the centraliser $Z(x) = \mathrm{span}\{e_{m_1}, \ldots, e_{m_l}\}$. From it we obtain a basis for each W_i by applying successive powers of *ad* y; we are going to use $e^n_{m_i} := ad^{m_i - n}\, y(e_{m_i}) \in W_i \cap \mathfrak{g}_n$, $e_{m_i} \equiv e^{m_i}_{m_i}$, and occasionally e_{-m_i} will stand for $e^{-m_i}_{m_i}$. The map $v \to ad\ v$ embeds $\mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(W_1) \oplus \ldots \oplus \mathfrak{sl}(W_l) \subset \mathfrak{gl}(\mathfrak{g})$. The restriction, $ad\ h|_{W_i}$ is given with respect to this basis by a diagonal matrix with eigenvalues running from $2m_i = \dim W_i - 1$ downto $-2m_i = -\dim W_i + 1$; thus the matrices of $ad\ x|_{W_i}$,$ad\ y|_{W_i}$ and $ad\ h|_{W_i}$ with respect to the basis $\{e^n_{m_i}, -m_i \leq n \leq m_i\}$ are exactly like those of x, y, h from Example 3.0.1, with l replaced by $2m_i$; indeed, for *ad* y and *ad* h this is clear by construction, and the last element in the triple is uniquely determined by this.

Lemma 3.0.2 *Let $\mathfrak{g}$ be a simple Lie algebra of rank l equipped with a choice of Chevalley generators and an $\mathfrak{sl}(2)$-subalgebra as above, and assume $\mathfrak{g} \neq D_{2n}$. Then the irreducible representations W_i are mutually orthogonal with respect to the above-*

described inner product on $\mathfrak{g}$*. Moreover, with respect to the basis* $\{e^n_{m_i}, 1 \leq i \leq l, -m_i \leq n \leq m_i\}$ *the matrix of the inner product is block-diagonal and each* $(2m_i + 1) \times (2m_i + 1)$ *block is of the form given in Lemma 3.0.1 with* l *replaced by* $2m_i$*, possibly up to* $\mathbb{R}^\times$ *multiple.*

Proof:

The only statement that needs a proof is the mutual orthogonality of the various blocks W_i, since the second statement follows from the last statement in the previous Lemma – the metric on W_i is determined uniquely by the form of *ad* x,*ad* y and *ad* h. To show the orthogonality, we need the fact that the exponents are non-degenerate, and of course we are assuming them to be ordered. Alternatively, the orthogonality within each block can be deduced by modifying slightly the argument that we are going to use for showing that $W_i \perp W_j, i \neq j$. To start off, observe that the subspace V_{m_i} is orthogonal to W_j. Indeed,

$$k < m_j \Rightarrow (e_{m_i}, e^k_{m_j}) = (e^{m_i}_{m_i}, (ady)^{m_j - k} e_{m_j}) = (ad^{m_j-k}\ x\ e^{m_i}_{m_i}, e_{m_j}) = 0,$$

as $m_j - k > 0$ and e_{m_i} is the highest vector. This shows that V_{m_i} is orthogonal to each W_j/V_{m_j} – including the case $i = j$. To show that $V_{m_i} \perp V_{m_j}, i \neq j$, we observe that if $j > i$, and respectively $m_j > m_i$,

$$(e^{m_i}_{m_i}, e^{m_j}_{m_j}) = C(e^{m_i}_{m_i}, ad^{2m_j}\ x\ e^{-m_j}_{m_j}) = (ad^{2m_j}\ y\ e^{m_i}_{m_i}, e^{-m_j}_{m_j}) = 0.$$

Next, we show that $(W_i \cap \mathfrak{g}_k) \perp W_j \cap (\oplus_{s \leq k} \mathfrak{g}_s)$ by exactly the same argument:

$$(e^k_{m_i}, e^s_{m_j}) = (e^k_{m_i}, ad^{m_j - s}\ y\ e_{m_j}) = (ad^{m_j-s}\ x\ ad\ ^{m_i - k} e_{m_i}, e_{m_j}) = 0.$$

Finally, we show that $(W_i \cap \mathfrak{g}_k) \perp (W_j \cap \oplus_{s>k}\mathfrak{g}_s)$ in an entirely analogous manner:

$$(e^k_{m_i}, e^s_{m_j}) = (e^k_{m_i}, ad^{m_j+s}\ x\ e^{-m_j}_{m_j}) = (ad^{m_j+s}\ y\ e^k_{m_i}, e^{-m_j}_{m_j}) = 0,$$

since $s > k \Rightarrow m_j + s > m_i + k$.

QED

Finally, here is a technical lemma which allows us to determine the adjoints of the vectors in the various V_{m_i}.

Lemma 3.0.3 *Let $\{e_i, f_i, h_i\}$ be Chevalley generators and $\{x, y, 2\rho^\vee\}$ an $\mathfrak{sl}(2)$-triple, chosen as above. Suppose we have also chosen basis $e_{m_i} \in W_i \cap \mathfrak{g}_i = V_{m_i}, \forall i$, $\dim W_i = 2m_i + 1$. Denote $e_{-m_i} := ad^{2m_i} y(e_{m_i}) e^{-m_i}_{m_i}$. Then*

$$e^*_{m_i} = \frac{(-1)^{m_i}}{(2m_i)!} e_{-m_i}.$$

Here the hermitian conjugate is understood with respect to the compact real form preserving the $\mathfrak{sl}(2)$-triple.

Proof:

Let $e_{m_i} = ad^{m_i}x(h)$, some $h \in \mathfrak{t} = \mathfrak{g}_0$, and write $h = \sum c_n h_n$. Then, if η' is the involution whose anti-linear extension gives the compact form, we have

$$\eta'(e_{m_i}) = \sum_n c_n \eta'(ad^{m_i}x(h_n)) = (-1)^{m_i+1} \sum_n c_n ad^{m_i} y(h_n) = -(-1)^{m_i} ad^{m_i} y(h).$$

Since we have defined $e_{-m_i} := ad^{2m_i} y(e_{m_i})$, if we can find C, such that $h = C ad^{m_i}_y(e_{m_i})$, we are going to get $\eta'(e_{m_i}) = -Ce_{-m_i}$. However, the constant C is easy to determine from the theory of $\mathfrak{sl}(2)$-representations. Indeed, $e_{m_i} = ad^{m_i}x(h)$

implies $ad^{m_i}ye_{m_i} = ad^{m_i}yad^{m_i}x(h)$. But for an irreducible $\mathfrak{sl}(2)$-representation of dimension $(2m+1)$, with raising and lowering operators $\mathbf{X}$ and $\mathbf{Y}$ and highest vector v, one has $\mathbf{X}\mathbf{Y}^k v = k(2m-k+1)\mathbf{Y}^{k-1}(v)$ and hence $\mathbf{X}^m\mathbf{Y}^m v = (m!)^2 \begin{pmatrix} 2m \\ m \end{pmatrix} v$. The result follows from here, as the hermitian conjugation is $-\overline{\eta}'$.

QED

Chapter 4

Oper Basics

Opers are local systems of representation-theoretic origin which on a compact complex curve can be thought of as generalisations of projective structures; $SL(n)$-opers were defined by Teleman in [Tel59] under the name *homographic structures* (*structures homographiques*). On the other hand, $GL(n)$-opers on the punctured disk were introduced by Drinfeld and Sokolov in [DS85] in relation to the n-KdV hierarchy. Later, Beilinson and Drinfeld in [BD91] gave a coordinate-free definition of G-opers (G – a complex connected reductive group) on an algebraic curve and discussed some of their properties. One can also see [Bis00] where $SL(n)$ and $GL(n)$-opers are studied from first principles, somewhat in the spirit of Teleman. Recently there have been many wonderful reviews and expositions touching on opers, among these, [Fre07], [BZB04],[FBZ04],[Fre04] – see also the references therein for several other papers by the same set of authors.

4.1 Opers for the Groups $SL(n)$ and $GL(n)$

We start with the definition of $GL(n)$- and $SL(n)$-opers and then proceed to the general case. Everything we consider will take place on a smooth compact complex connected curve X. Eventually we shall be interested in the case $g \geq 2$, but there is no need to restrict to this case from the outset, as the examples show (see below).

A $GL(n)$-*oper* is a pair $(\mathbf{F}_\bullet \subset \mathbf{E}, D)$ where $\mathbf{E}$ is vector bundle of rank n equipped with

1. A complete flag $(0) \subset \mathbf{F}_1 \subset \ldots \subset \mathbf{F}_n = \mathbf{E}$, $rk\mathbf{F}_i = i$

2. A holomorphic connection, $D : \mathbf{E} \to \mathbf{E} \otimes K$ (necessarily flat)

such that the filtration satisfies:

1. Griffiths' transversality $D : \mathbf{F}_i \to \mathbf{F}_{i+1} \otimes K$, $i = 1..n-1$

2. Nondegeneracy (strictness):

$$\overline{D} : Gr_i\mathbf{F} \simeq Gr_{i+1}\mathbf{F} \otimes K.$$

An $SL(n)$-*oper* is a $GL(n)$-oper such that $\det \mathbf{E} = \mathcal{O}$ and D induces the trivial connection d on $\det \mathbf{E}$.

Example 4.1.1 *1. On $\mathbb{P}^1$ we have the Euler sequence*

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

and the trivial connection on $\mathcal{O} \oplus \mathcal{O}$ satisfies the oper condition.

2. *On an elliptic curve the nontrivial extension I_2 of $\mathcal{O}$ by $\mathcal{O}$ with any holomorphic connection is an oper.*

3. *On a curve X of genus $g > 1$ with a chosen theta-characteristic $K^{1/2}$ any holomorphic connection on*

$$0 \longrightarrow K^{1/2} \longrightarrow \mathbf{E} \longrightarrow K^{-1/2} \longrightarrow 0$$

   is an oper.

Notice that the last example generalises the first two. On the other hand, we shall see later that in fact it is the first example that "induces" the other two, if X is equipped with a projective structure and a choice of theta-characteristic.

One often talks about the bundle $\mathbf{E}$ ("the oper bundle") being equipped with "oper structure". It should be noticed, however, that a generic bundle does not admit such structure, so being an oper is a restriction on the bundle – rather than being an extra data.

The non-degeneracy condition (2.) is extremely restrictive: if one knows the first (or last) graded piece – or det $\mathbf{E}$, as in the SL(n)-case – one can determine all graded pieces, since the successive quotients differ by K. Then the Griffiths' transversality condition restricts the allowed extensions. It turns out that in the case of $SL(n)$-opers this determines the bundle (essentially) uniquely. In the $GL(n)$-case there remains the ambiguity of fixing the determinant of $\mathbf{E}$.

The key in extending this notion to other structure groups is to observe that in the above definition the structure group of the bundle is reduced to a Borel subgroup and the connection fails to preserve the reduction in a very controlled way.

4.2 Opers for Arbitrary Simple Groups

For the general definition, following [BD91] let G be a simple complex group (Beilinson and Drinfeld work with connected reductive G) with $B \subset G$ a fixed Borel subgroup, $N = [B, B]$ its unipotent radical, $H = B/N$ the Cartan subgroup (assuming the data of embedding $H \subset G$) and $Z = Z_G$ the centre. The corresponding Lie algebras are $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{t} = \mathfrak{b}/\mathfrak{n}$.

If $\mathbf{P}_B$ is a holomorphic principal B-bundle, $\mathbf{P}_G$ the induced G-bundle and $\mathrm{Conn}(\mathbf{P}_G)$ the sheaf of connections, one has an embedding $\mathrm{Conn}(\mathbf{P}_B) \subset \mathrm{Conn}(\mathbf{P}_G)$. Here "sheaf" means a sheaf of torsors. In order to measure the failure of a G-connection, D, to preserve the reduction, one defines its "relative position" $c(D) \in H^0(\mathfrak{g}/\mathfrak{b}_{\mathbf{P}} \otimes K)$. That is, one considers the projection $c : \mathrm{Conn}(\mathbf{P}_G) \to \mathfrak{g}/\mathfrak{b}_{\mathbf{P}} \otimes K$ defined by requiring $c^{-1}(0) = \mathrm{Conn}(\mathbf{P}_B)$ and $c(D + \nu) = c(D) + [\nu]$, where ν is a section of $\mathfrak{g}_{\mathbf{P}} \otimes K$ and $[\nu]$ is its image in $\mathfrak{g}/\mathfrak{b}_{\mathbf{P}} \otimes K$. Practically, one can take locally *any* flat B-connection D_B and use that the local sections $[D - D_B]$ glue to $c(D) \in H^0(\mathfrak{g}/\mathfrak{b}_{\mathbf{P}} \otimes K)$.

Then, ([BD91],p.61) a G-oper on X is a pair $\mathscr{P} = (\mathbf{P}_B, D)$, $D \in H^0(\mathrm{Conn}(\mathbf{P}_G))$, such that

1. $c(D) \in H^0((\mathfrak{g}_{-1})_{\mathbf{P}} \otimes K) \subset H^0(\mathfrak{g}/\mathfrak{b}_{\mathbf{P}} \otimes K)$.

2. For any simple negative root α the component $c(D)_\alpha \in H^0(X, \mathfrak{g}_{\alpha\mathbf{P}} \otimes K)$ is nowhere vanishing.

Here is an equivalent description of this condition ([Fre07],[FBZ04]). The Borel subgroup B acts on $\mathfrak{g}/\mathfrak{b}$; consider the subspace $\mathfrak{g}/\mathfrak{b}^N$ of elements fixed by N. There is an open (in $\mathfrak{g}/\mathfrak{b}^N$) B-orbit $\mathbf{O} \subset \mathfrak{g}/\mathfrak{b}^N$ consisting of vectors fixed by N which have non-zero components in the direction of any of the negative simple roots. If one chooses a (negative) nilpotent subalgebra, $\mathfrak{g} \supset \mathfrak{n}_- \simeq \mathfrak{g}/\mathfrak{b}$ with Chevalley generators $\{f_i\}$, then $\mathbf{O} = B \cdot (\sum_i f_i)$. The orbit $\mathbf{O}$ is an H-torsor. The connection D is an oper if $c(D) \in H^0(\mathbf{O}_P \otimes K)$.

Example 4.2.1 *Consider the case* $\mathfrak{g} = \mathfrak{sl}(2)$, *with*

$$\mathfrak{b} = span\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}, \ \mathfrak{g}/\mathfrak{b} = \mathbb{C} \simeq \mathfrak{n}_- = span\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}.$$

Respectively,

$$N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\} \subset B = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a \neq 0\right\} \subset SL(2).$$

One can choose an embedding $H = B/N \simeq \mathbb{C}^\times = \{diag(a, a^{-1}), a \neq 0\} \subset B$. *Then*

$$(\mathfrak{g}/\mathfrak{b})^N = \left\{\left[\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}\right], d \in \mathbb{C}\right\}$$

and $\mathbf{O} \subset \mathfrak{g}/\mathfrak{b}^N$ *is determined by the condition* $d \neq 0$. *The* $H = \mathbb{C}^\times$*-action on* $\mathbf{O}$ *is given by* $a \cdot d = a^{-2}d$.

Example 4.2.2 *For $GL(n)$ the oper condition implies ([FBZ04]) that if on an analytic (or formal) open set $U \subset X$ one chooses an isomorphism $\mathbf{E}_U \simeq \mathcal{O}^{\oplus n}$ compatible with the flag, and a local coordinate t, then one can write the flat connection in the form*

$$d + \begin{pmatrix} * & * & \dots & * & * \\ \times & * & \dots & * & * \\ 0 & \times & \dots & * & * \\ \dots & \dots & \dots & \dots & * \\ 0 & 0 & \dots & \times & * \end{pmatrix} dt$$

where $\times$ indicates non-vanishing entries.

The groupoid of opers on X is denoted by $\mathcal{O}\mathfrak{p}_G(X)$. Beilinson and Drinfeld also use the notation $\mathcal{O}\mathfrak{p}_G$ for the sheaf of groupoids on $X_{\text{ét}}$ given by the groupoids $\mathcal{O}\mathfrak{p}_G(X')$, $X' \to X$ étale. If G is the *adjoint* group with Lie algebra $\mathfrak{g}$, one uses the name $\mathfrak{g}$-opers. E.g., $\mathfrak{sl}(2)$-opers and $PGL(2)$-opers are synonyms.

Proposition 4.2.1 ([BD91], 3.1.4) *Let X be a (smooth, complete and connected) curve of genus $g > 1$ and let $(\mathbf{P}_G, D)$ be a G-local system on X that has an oper structure. Then:*

1. *The oper structure is unique: the corresponding flag $\mathbf{P}_B \subset \mathbf{P}_G$ is the Harder-Narasimhan flag.*

2. *$Aut(\mathbf{P}_G, D) = Z_G$.*

3. *The local system* $(\mathbf{P}_G, D)$ *cannot be reduced to a non-trivial parabolic subgroup of* G.

Given a principal embedding $\mathfrak{sl}(2) \hookrightarrow \mathfrak{g}$ one obtains a canonical structure of an affine space on the space of $\mathfrak{g}$-opers, which we describe below, following Beilinson and Drinfeld.

First of all, $\mathfrak{sl}(2)$-opers admit a natural action of the quadratic differentials, making $\mathcal{O}\mathfrak{p}_{\mathfrak{sl}(2)}$ into a K^2-torsor.

Indeed, if $(\mathbf{P}_B, D)$ is an $\mathfrak{sl}(2)$-oper, the "relative position" map c provides a nowhere vanishing section $c(D) \in H^0(X, \mathfrak{sl}(2)/\mathfrak{b}_{\mathbf{P}} \otimes K)$, i.e., we have a trivialisation $\mathcal{O}_X \simeq \mathfrak{sl}(2)/\mathfrak{b}_{\mathbf{P}} \otimes K$. In other words, the connection induces an isomorphism $T_X \simeq \mathfrak{sl}(2)/\mathfrak{b}_{\mathbf{P}}$. The trace gives an identification $\mathfrak{sl}(2) \supset \mathfrak{n} \simeq \mathfrak{sl}(2)/\mathfrak{b}^\vee$, so $\mathfrak{n}_{\mathbf{P}} = K$ and one obtains an embedding $K^2 = \mathfrak{n}_{\mathbf{P}} \otimes K \subset \mathfrak{sl}(2)_{\mathbf{P}} \otimes K$. In this way K^2 acts on $\mathcal{O}\mathfrak{p}_{\mathfrak{sl}(2)}$ by translating the connection D. The torsor is in fact trivial, so $\mathcal{O}\mathfrak{p}_{\mathfrak{sl}(2)}(X)$ is an affine space modelled on $H^0(X, K^2)$.

Next, suppose G is the *adjoint* group with Lie algebra $\mathfrak{g}$, and let the subscript 0 denote the various subgroups and subalgebras – Borel, unipotent,etc., – for $PSL(2)$. Let $x_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so $\mathbb{C} \simeq \mathbb{C}x_0 = \mathfrak{n}_0$ and $\mathbb{C} \simeq \mathbb{C}y_0 \simeq \mathfrak{sl}(2)/\mathfrak{b}$. The adjoint action gives an isomorphism $B_0/N_0 \to Aut(\mathfrak{n}_0) = \mathbb{C}^\times$. Using that, if $(\mathbf{P}_{B_0}, D)$ is any $\mathfrak{sl}(2)$-oper, the isomorphism $\mathfrak{n}_{0\mathbf{P}_{B_0}} \simeq K$ induces $\mathbf{P}_{B_0} \times_{Ad} B_0/N_0 \simeq \underline{Isom}(\mathcal{O}, K)$, where $\underline{Isom}(\mathcal{O}, K)$ is the frame bundle of K.

Denote by i the chosen embedding $i : \mathfrak{sl}(2) \hookrightarrow \mathfrak{g}$ and set $V := \mathfrak{g}^{N_0}$. That is,

$V = Z(i(x_0)) = \ker ad(i(x_0))$, the vector space which we considered in 3. On this vector space we define a new $\mathbb{C}^\times$-action *via* $t \cdot v := tAd(t)v$. Using this action, twist V by the canonical bundle, obtaining a vector bundle $V_{\underline{Isom}(\mathcal{O},K)}$, or V_K for short, $V_K := \underline{Isom}(\mathcal{O},K) \times_{\mathbb{C}^\times} V$. Then $\mathbb{C}x_0 = \mathfrak{n}_0 \subset V$ induces $K^2 \subset V_K$.

The isomorphism $\mathbf{P}_{B_0} \times_{Ad} B_0/N_0 \simeq \underline{Isom}(\mathcal{O},K)$ induces a canonical isomorphism $V_K = V_{\mathbf{P}_{B_0}} \otimes K$ and thus an embedding $V_K \subset \mathfrak{b}_{\mathbf{P}_{B_0}} \otimes K = \mathfrak{b}_{\mathbf{P}_B} \otimes K$.

Now given any B_0-oper $(\mathbf{P}_{B_0}, D_0)$, by extension of structure group it becomes a $\mathfrak{g}$-oper, and a section of V_K can be used to translate D.

Proposition 4.2.2 ([BD91], 3.1.10) *Let $(\mathbf{P}_{B_0}, D_0)$ be an $\mathfrak{sl}(2)$-oper, and let $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}$ be the V_K-torsor induced from the K^2-torsor $\mathcal{O}\mathfrak{p}_{\mathfrak{sl}(2)}$ by the embedding $K^2 \subset V_K$. The canonical map $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}} \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}$ is an isomorphism. In particular, for any $\mathfrak{g}$-oper $(\mathbf{P}_B, D)$ the bundle $\mathbf{P}_B$ is isomorphic to some canonical B-bundle $\mathbf{P}_B^{can}$, independent of D.*

So $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is an affine space modelled on $H^0(X, V_K)$, and the latter turns out to be the Hitchin base for the Lie algebra $\mathfrak{g}$.

Proposition 4.2.3 ([BD91],3.1.12) *There is a canonical filtration of $\mathbb{C}[\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)]$ and a canonical isomorphism of graded algebras*

$$gr\ \mathbb{C}[\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)] \simeq \mathbb{C}[B_{\mathfrak{g}}],$$

where $B_{\mathfrak{g}}$ is the Hitchin base.

This is a direct corollary of Kostant's result. First, an isomorphism $V \simeq \mathrm{Spec}(Sym[\mathfrak{g}^\vee]^G)$ is obtained by $V \ni v \mapsto v + i(y_0) \in \mathfrak{g} \to \mathfrak{g}//G \simeq \mathfrak{t}/W$, which induces $H^0(X, V_K) \simeq H^0(X, \mathfrak{t}/W_K) = B_{\mathfrak{g}}$. Second, we have $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \simeq \underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X) = \mathrm{Spec}\mathbb{C}[H^0(X, V_K)]$.

Equivalently, using the notion of "λ-connections", one can define λ-opers and consider the limit $\lambda \to \infty$. Beilinson and Drinfeld define the Rees module in the "opposite" way as compared to Deligne and Simpson, so they work with $\hbar = \frac{1}{\lambda}$ and consider $\hbar \to 0$. Notice that the limit $\lambda \to 0$ is constant, i.e., all all sections of the twistor space obtained in this way meet at a single point. Beilinson and Drinfeld in their quest for the quantisation of the Hitchin system think of this procedure as being a "classical limit". However, the Rees module procedure relies entirely on the fact that opers come with a flag – while generic local systems do not – and hence their method cannot be applied to arbitrary local systems. Indeed, in the light of [DP06] and [KW07] the "correct" classical limit is the one provided by the non-abelian Hodge theorem.

Frenkel ([FBZ04]), following [DS85] gave a local version of the above statement. Namely, given a G-oper, after trivialising it on an open disk and endowing it with a (formal) local coordinate, one can put the connection matrix in a "standard form", which is in fact the analogue of companion matrix provided by Kostant's theory.

Example 4.2.3 *Given a $GL(n, \mathbb{C})$ oper, after choosing an isomorphism $\mathbf{E}_U \simeq \mathcal{O}_U^{\oplus n}$ compatible with the flag, and a (formal or analytic) local coordinate t on the U, the*

B-gauge equivalence class has a unique representative of the form

$$d + \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} dt.$$

In the case of $\mathfrak{g} = \mathfrak{sl}(n)$ all of the above can be done from first principles and made a little bit more explicit. In particular one gets ([Bis00]) that the bundle $\mathbf{P}_{B_0}$ is the (frame bundle of) $J^{n-1}(K^{-\frac{n-1}{2}})$ with its canonical filtration.

4.3 Relation to Classical Riemann Surface Theory

As mentioned in the beginning of the section, there is a relation between opers and various objects from Riemann surface theory; in particular, $\mathfrak{sl}(2)$-opers are equivalent to projective structures, projective connections and Sturm-Liouville operators, and we review this briefly.

Recall that a (complex) *projective structure* on a smooth compact curve X is an equivalence class of (analytic) atlasses $\{U_\alpha \to X\}$, where $U_\alpha \subset \mathbb{C}$ have transition functions $\{\varphi_{\alpha\beta}\}$ which are Möbius (fractional linear) transformations, $\varphi_{\alpha\beta} \in PGL(2, \mathbb{C})$. Notice that here we are assuming that X is already endowed with a complex structure and the projective structure is compatible with it. One could start with a real curve, in which case the projective structure will induce a complex structure on X. The projective structures we are considering here are in fact a Lagrangian subvariety inside the space of all projective structures. Projective structures exist for any genus. By the very definition, a choice of projective structures determines an element of $H^1(X, PGL(2,\mathbb{C}))$, that is, a flat holomorphic $PGL(2)$-bundle on X. It is easy to see that this bundle is in fact a $PGL(2)$-oper. Indeed, the local charts $U_\alpha \hookrightarrow \mathbb{P}^1$ (i.e., the flat local coordinates) determine a nowhere horizontal section of the associated $\mathbb{P}^1 = PGL(2)/B$-bundle, which is the same thing as a nowhere horizontal B-reduction. By a choice of theta-characteristic one can lift the element of $H^1(X, PGL(2,\mathbb{C}))$ to $SL(2,\mathbb{C})$, see ([Gun66]).

Suppose z is a local coordinate on an open set in $\mathbb{C}$ and let $S_z f$ be the Schwartz derivative, $S_z f = (f''/f')^2 - \frac{1}{2}(f''/f')^2$. As observed by Tyurin (who references Schwartz) in [Tju78], Proposition 1.2.7, the space of solutions of the 3-rd order non-linear ODE $S_z(f) = q$ is the projectivisation of the space of solutions of the 2-nd order linear ODE $f'' + \frac{q}{2} f = 0$. Using that the Schwartz derivative annihilates exactly the Möbius transformations one can check that projective structures on X are in bijection with second order differential operators $\mathrm{Diff}^2(K^{-1/2}, K^{3/2})$ which are self-adjoint and have principal symbol one. The latter are often called Sturm-Liouville operators or projective connections. These are an $H^0(X, K^2)$-torsor in an obvious way – see the above linear ODE – which induces an $H^0(X, K^2)$-torsor structure on the projective connections. Since one can rewrite a (scalar) 2-nd order linear ODE as a 1-st order ODE on the jets, one can think of projective connections as holomorphic connections on $J^1(K^{-1/2})$.

One other interpretation which we are not going to use anywhere is that of Sturm-Liouville operators as kernels on the diagonal in $X \times X$, see [FBZ04], which relies on the fact that $\mathit{Diff}^n(E,F) = \frac{F \boxtimes S(E)((n+1)\Delta)}{F \boxtimes S(E)}$, $S(E) = E^\vee \otimes K$.

Finally, one can identify G-opers and $\mathfrak{g}$-opers by a choice of theta-characteristic, that is, if Z_G is the centre of G, one can choose one of the $H^1(X, Z_G)$-components of $\mathcal{O}\mathfrak{p}_G(X)$ and identify it with $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X)$. This is due to [BD91], 3.4.2, where it is shown that a theta-characteristic ζ, $\zeta^{\otimes 2} = K$, induces an equivalence

$$\Phi_\zeta : \mathcal{O}\mathfrak{p}_\mathfrak{g}(X) \times Z_G\text{-tors} \to \mathcal{O}\mathfrak{p}_G(X),$$

where Z_G-tors is the groupoid of Z_G-torsors.

To close this section, let us mention that since opers have a fixed underlying bundle, it is straightforward to check that $\mathcal{O}\mathfrak{p}(X)$ is isotropic for the holomorphic symplectic form on M_{DR} – and hence complex Lagrangian for dimension reasons.

Chapter 5

Getting Started

5.1 First Observations

Recall that a *hyperkähler manifold* is a Riemannian manifold, (M, g), with three complex structures, $I, J, K \in H^0(M, \mathrm{End}T_{M,\mathbb{R}})$ satisfying the quaternionic identities for which the corresponding fundamental forms $\omega_I = g(I\bullet, \bullet)$, ω_J, ω_K are Kähler. Linear combinations $\{xI+yJ+zK, (x, y, z) \in S^2 \subset \mathbb{R}^3\}$ of these give a $\mathbb{P}^1$-family of complex structures. The data of the family is encoded in its twistor space, $\mathcal{Z}$, which is the real manifold $S^2 \times M_{\mathbb{R}}$ endowed with the standard complex structure on the S^2 factor and the tautological complex structure on $M_{\mathbb{R}}$. This complex manifold of dimension $(\dim_{\mathbb{C}} M + 1)$ is a holomorphic fibration over $\mathbb{P}^1$; the fibre over $\{z\} \in \mathbb{P}^1$ is $(M_{\mathbb{R}}, I_z)$, where I_z is the complex structure corresponding to $z \in S^2$. The real fibration has obvious sections $\{p\} \times S^2$, $p \in M_{\mathbb{R}}$ – called *twistor lines* – which are in fact holomorphic and have normal bundles $\mathcal{O}^{\oplus 2\dim_{\mathbb{C}} M}$. It turns out that there exists a global holomorphic section of the bundle (over $\mathcal{Z}$) $\Lambda^2 T_f \otimes \mathcal{O}(2)$, where T_f is the vertical bundle of the fibration. So restricting this section to a fibre over $\{z\} \in \mathbb{P}^1$ gives a holomorphic 2-form on $(M_{\mathbb{R}}, I_z)$. Hence in each complex structure of the family one has a real-symplectic and a complex-symplectic form. For instance, in the complex structure I the Kähler form is ω_I, $\omega^c = \omega_J + i\omega_K$ is of type $(2,0)$. In the complex structure J the Kähler form is ω_J and the holomorphic one is $\omega_K + i\omega_I$. The space $\mathcal{Z}$ comes with an antilinear involution which lifts the antipodal map on S^2 and preserves all the data.

Recall that a submanifold, Y, of a symplectic manifold, (M, ω), is *Lagrangian* if $\omega|_Y = 0$ and $\dim Y = \frac{1}{2}\dim M$. If M is a complex manifod, Y a complex submanifold, ω^c is complex-symplectic and the same conditions are satisfied, then Y is called *complex-Lagrangian*. Let M be a Calabi-Yau manifold with chosen Kähler form ω and covariantly constant holomorphic volume form Ω, so $\Omega \wedge \overline{\Omega} = \omega^{\dim M}$ if $\dim_{\mathbb{C}} M$ is even (or $i\omega^{\dim M}$ otherwise). Then a submanifold $Y \subset M$ is *special Lagrangian* if $\dim_{\mathbb{C}} Y = \frac{1}{2}\dim_{\mathbb{C}} M$, $\omega|_Y = 0$ and $\mathrm{Re}\Omega|_Y = 0$. Actually, since any constant multiple of Ω is going to be again covariantly constant, one can require instead that a real linear combination $a\mathrm{Re}\Omega + b\mathrm{Im}\Omega$ vanishes upon restriction to Y.

Special Lagrangian manifolds are of much interest for differential geometers and physicists, mostly in relation to calibrated geometries, special holonomy and mirror symmetry. However, explicit examples of special Lagrangian manifolds are hard to construct, and there are three main sources of these ([Hit01]): fixed points or real structures on Calabi-Yau manifolds; non-compact Calabi-Yau manifolds; complex Lagrangian submanifolds of hyperkähler manifolds. It is the last case that is of interest to us.

To understand it better, notice first that hyperkähler manifolds are Calabi-Yau, since one can take the holomorphic volume form to be $\Omega = (\omega^c)^{\frac{\dim M}{2}}$ or a constant multiple of it. Suppose that $Y \subset (M, J)$ is complex Lagrangian, so $\omega_I + i\omega_K$ vanishes on Y. Then look at $Y \subset (M, I)$ and endow (M, I) with Kähler form ω_I and holomorphic volume form $\Omega = i(\omega_J + i\omega_K)^{\frac{\dim M}{2}}$. Upon restriction to Y the Kähler form vanishes and Ω becomes $i(\omega_J)^{\frac{\dim M}{2}}$, so $\mathrm{Re}\Omega|_Y = 0$.

The upshot of this is that the subvariety $\tau(\mathcal{O}\mathfrak{p}(X)) \subset M_{Dol}$ that we are looking for is a real, special Lagrangian submanifold of the moduli space of stable Higgs bundles. There is another property of it which can be identified easily.

Lemma 5.1.1 *If $\tau : M^{reg}_{DR}(G) \simeq M^{reg}_{Dol}(G)$ is the canonical map provided by the non-abelian Hodge theorem, then $\tau(\mathcal{O}\mathfrak{p}(X)) \subset M^{reg}_{Dol}(G) \backslash T^\vee \mathcal{N}^{reg}(G)$. That is, the Higgs bundles corresponding to opers are stable but have non-stable underlying bundles.*

Proof: Indeed, the theorem of D.Kaledin and B.Feix ([Kal99],[Fei01]) says that if X is a real-analytic Kähler manifold, there exists a unique $U(1)$-invariant hyperkähler metric on (a neighbourhood of the zero section in) $T^\vee X$. For $X = \mathcal{N}^{reg}(G)$, this metric agrees with the hyperkähler metric on M_{Dol} and produces a torsor over $T^\vee \mathcal{N}^{reg}(G)$. Hence τ maps the (unique) $T^\vee \mathcal{N}^{reg}$-torsor on $\mathcal{N}^{reg}$ to $T^\vee \mathcal{N}^{reg}$, and, respectively, preserves the complement. However, $\mathcal{O}\mathfrak{p}(X)$ is contained in the complement of the $T^\vee\mathcal{N}^{reg}$-torsor, so the result follows.

QED

In the case of $G = SL(2)$ one can say a bit more.

Lemma 5.1.2 *If $G = SL(2)$, the vector bundles that appear as underlying bundles of Higgs pairs in $\tau(\mathcal{O}\mathfrak{p}_G(X))$ and are maximally unstable form a subvariety of real dimension at most $3g-3$ containing the bundle* $\left[\left(K^{1/2}\oplus K^{-1/2}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\right]$.

Proof:

In the case $G = SL(2)$, the unstable bundles which appear in stable pairs can be distinguished by the degree of the maximal destabilising subbundle ([Hit87a]) which is at most $(g-1)$; for those of maximal degree there is an additional restriction on the cohomology, which, in the case of Higgs bundles of degree zero fixes the underlying bundle to be $\mathbf{E} = K^{1/2}\oplus K^{-1/2}$. Then $H^0(\mathscr{E}nd_0\mathbf{E}\otimes K) \simeq H^0(\mathcal{O})\oplus H^0(K)\oplus H^0(K^2)$, $\dim H^0(\mathscr{E}nd_0\mathbf{E}\otimes K) = 4g-2$. If $[(\mathbf{E},\theta)]$ is a stable pair, we must have that the (21)-entry of θ is nonzero. Also, $\mathrm{Aut}(\mathbf{E}) = \mathbb{C}^\times \times H^0(X,K)$. The quotient of $H^0(\mathscr{E}nd_0\mathbf{E}\otimes K)$ by $\mathrm{Aut}(\mathbf{E})$ is isomorphic to an affine space modelled on $H^0(X,K^2)$, which is (the image of) Hitchin's section. Thus maximally unstable bundles appearing in Higgs pairs from $\tau(\mathcal{O}\mathfrak{p}(X))$ correspond to points of the intersection of $\tau(\mathcal{O}\mathfrak{p}(X))$ with the Hitchin section. By [Hit87a] one such point is given by the Higgs pair given in the statement of the Lemma. Since the Hitchin section is a complex-

Lagrangian submanifold and $\tau(\mathcal{O}\mathfrak{p}(X))$ is special Lagrangian, the real tangent point to their intersection at a smooth point is isotropic for the Kähler form and hence of (real) dimension at most half of the (real) dimension of the Hitchin base.

QED

5.2 Strategic Remarks

Let M be a hyperkähler manifold of dimension $\dim_{\mathbb{C}} M = 2d$, $\mathcal{Z} = \mathcal{Z}(M)$ its twistor space, and let $p \in M$, so that we have

$$\begin{array}{ccc} M & \hookrightarrow & \mathcal{Z}(M) \\ \downarrow & & \downarrow \pi \\ \{0\} & \hookrightarrow & \mathbb{P}^1 \end{array}.$$

The point $p \in M$ determines a twistor line $s_p : \mathbb{P}^1 \to \mathcal{Z}(M),\ \pi s_p = 1$, and we shall also write $C_p := s_p(\mathbb{P}^1)$. We have the two exact sequences associated with π and s_p:

$$0 \longrightarrow T_f \longrightarrow T_{\mathcal{Z}} \longrightarrow \pi^* T_{\mathbb{P}^1} \longrightarrow 0$$

$$0 \longrightarrow T_{\mathbb{P}^1} \longrightarrow s_p^* T_{\mathcal{Z}} \longrightarrow \mathcal{N}_{C_p} \longrightarrow 0$$

and since the second sequence splits, $s_p^* T_f \simeq \mathcal{N}_{C_p}$.

Let $\Gamma(\pi)$ be the set of (holomorphic) sections of $\pi : \mathcal{Z} \to \mathbb{P}^1$, and let $\Gamma(\pi)^\sigma$ be the set of twistor lines, i.e., of real (with respect to the lift,σ, of the antipodal map on $\mathbb{P}^1$) sections of π. Notice that by [HKLR87] we have in fact an isomorphism of hyperkähler manifolds

$$s : M \to \Gamma(\pi)^\sigma, \ p \mapsto s_p.$$

Taking its differential, we get the isomorphism

$$ds : T_M \to s^* T_{\Gamma(\pi)^\sigma}$$

which, upon restriction to the fibre over p gives

$$ds_{(p)} : T_{M,p} \simeq T_{\Gamma(\pi)^\sigma, s_p} \subset T_{\Gamma(\pi), s_p} = H^0(\mathbb{P}^1, \mathcal{N}_{C_p}) = H^0(\mathbb{P}^1, s_p^* T_f).$$

This last map is the linear analogue of s in an obvious sense: the quaternionic vector space $T_{M,p} \simeq \mathbb{H}^d$ is a hyperkähler manifold with twistor space the vector bundle $\mathrm{tot}(s_p^* T_f) = \mathcal{Z}(T_{M,p}) \to \mathbb{P}^1$, and $\mathrm{tot}(s_p^* T_f) = \mathrm{tot}\mathcal{O}(1)^{\oplus 2d}$.

Composing $ds_{(p)}$ with the evaluation map at $1 \in \mathbb{P}^1$ which sends $H^0(\mathbb{P}^1, s_p^* T_f)$ to $T_f|_{C_p(1)}$, one obtains an $\mathbb{R}$-linear isomorphism $T_{(M,I),p} \simeq T_{(M,J),p}$.

Here is another way to think of the above maps. Since $T_{\mathcal{Z}}|_{C_p} = T_{\mathbb{P}^1} \oplus T_f|_{C_p}$ and since $\pi : \mathcal{Z} \to \mathbb{P}^1$ is trivial as a *real* fibration, it has a (relative, real) flat connection. Then ds_p and $\mathrm{ev}_1 \circ (ds)_{(p)}$ can be thought of as parallel transport maps $T_{M,p} = (T_f|_{C_p})_{(0)} \to H^0(C_p, T_f|C_p)$ and $(T_f|_{C_p})_{(0)} \to (T_f|_{C_p})_{(1)}$.

We are going to consider the case when $M = M_{Dol}^{reg}(G)$.

Hitchin in [Hit87a] determined that the harmonic metric for the Higgs bundle appearing in Lemma 5.1.2 "is" the uniformisation metric for the curve X and that the corresponding local system is an oper; i.e., after the hyperkähler rotation one obtains a holomorphic connection D_0 on the vector bundle $J^1(K^{-1/2})$; in fact, the local system is a variation of Hodge structures.

As discussed earlier, from this Higgs bundle and the corresponding oper – considered as having a structure group $SL(2)$ or $PGL(2)$ – we obtain via the principal embedding a G-Higgs bundle and a G-oper which are still related through the Hodge theorem. For the case $G = SL(n)$ this is simply the compatibility of Hodge theory with Schur functors; for the general case see [Sim94], Lemma 9.4. I am going to denote by $(\mathbf{P}, \theta)$ the Higgs bundle thus obtained (which is in fact a system of Hodge bundles) and by $(\mathbf{Q}, D_0)$ the corresponding oper, which I am going to refer to as *the uniformisation oper*.

Let τ be the diffeomorphism $\tau : M_{DR}^{reg}(G) \simeq M_{Dol}^{reg}(G)$ provided by the non-abelian Hodge theorem. I am going to sketch a plan for obtaining the germ of $\tau(\mathcal{O}\mathfrak{p}_G(X))$ at $[(\mathbf{P}, \theta)]$; that is, a plan for producing a subvariety of M_{Dol} which is tangential to $\tau(\mathcal{O}\mathfrak{p})$ at the system of Hodge bundles $[(\mathbf{P}, \theta)] = \tau([(\mathbf{Q}, D_0)])$. The result in fact will be a subvariety, Lagrangian for the real part of the complex symplectic form.

To achieve this we proceed in the following way:

1. Determine the map $d\tau^{-1} : \mathrm{ev}_1 \circ (ds)_{(p)} : (T_f|_{C_p})_{(0)} \to (T_f|_{C_p})_{(1)}$ in linear-algebraic terms.

2. Using the Dolbeault ($\mathscr{C}_\bullet$) and de Rham ($\mathscr{D}_\bullet$) complexes, identify $(T_f|_{C_p})_{(0)} = \mathbb{H}^1(\mathscr{C}_\bullet) \simeq B_\mathfrak{g} \oplus B_\mathfrak{g}^\vee$, $(T_f|_{C_p})_{(1)} = \mathbb{H}^1(\mathscr{D}_\bullet) \simeq B_\mathfrak{g} \oplus B_\mathfrak{g}^\vee$.

3. Identifying $T_f|C_p$ with $(B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee}) \otimes \mathcal{O}(1)$ and opers with the subspace $B_{\mathfrak{g}} \subset B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee} \subset (T_f|_{C_p})_{(1)}$, find its image, $d\tau(B_{\mathfrak{g}})$. This will give us the infinitesimal deformations arising from opers.

4. Describe a family of deformations of the bundle $[(\mathbf{E}, \theta)]$ parametrised by $B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee}$. Using the previous item, describe the deformations arising from infinitesimal deformations from $d\tau(B)$.

The third step is the "problematic" one; namely, it does not take into account the fact that the harmonic metric changes along $\mathcal{O}\mathfrak{p}(X)$ so every Higgs pair obtained in the above way has to be appropriately gauged, so that we really obtain $\tau(\mathcal{O}\mathfrak{p})$ and not only its germ. In the last section I will give some comments on how to attack this question.

5.3 A Remark from Linear Algebra

The following easy Lemma is well-known and essentially reduces to writing down a global trivialisation of $\mathcal{O}(1)^{\oplus 2n}$ as a real bundle.

Lemma 5.3.1 *Let $U \in Vect(\mathbb{C})$ be a quaternionic vector space, with $I = \{i\,\cdot\}$, $J \in End_{\mathbb{R}}(U)$, $J^2 = -1, K := IJ = -JI$. Then the twistor space of U is $\pi : \mathcal{Z}(U) = tot(U \otimes \mathcal{O}(1)) \to \mathbb{P}^1$, $\Gamma(\pi) = H^0(\mathbb{P}^1, U \otimes \mathcal{O}(1)) = U \otimes H^0(\mathcal{O}(1))$. If $x, y \in H^0(\mathcal{O}(1))$ are two linearly independent sections (the local coordinate on $\mathbb{P}^1$*

being $\zeta = x/y$), then the map

$$U \longrightarrow \Gamma(\pi)^{\sigma} \subset \Gamma(\pi)$$

is given by

$$u \mapsto yu - xK(u)$$

The composition of this map with the evaluation map at $\zeta = 1$ is

$$U = U(1)_{(\zeta=0)} \ni u \longmapsto w = u - K(u) \in U = U(1)_{(\zeta=1)}$$

with inverse $w \mapsto u = \frac{1}{2}(w + K(w))$.

So we have, taking $U = T_{\mathcal{M},p}$, that $(ds)_{(p)}(u) = yu - xK(u)$, $\mathrm{ev}_1 \circ (ds)_{(p)}(u) = (1-K)(u)$. Notice that we also have the "standard" isomorphism $(U_{\mathbb{R}}, I) \simeq (U_{\mathbb{R}}, J)$, $v \mapsto (1-K)^{-1}(v) = \frac{1}{2}(v + K(v))$, which is the *inverse* of the above formula. Using that would give us a twistor line which is not real and which is constant in one of the charts of $\mathbb{P}^1$ and vanishes at infinity.

Example 5.3.1 *Consider $U = B \oplus B^{\vee}$, B a* hermitian *vector space. We can endow it with an action of the quaternions*

$$J = \begin{pmatrix} 0 & A^{-1} \\ -A & 0 \end{pmatrix}, \ K = IJ = \begin{pmatrix} 0 & iA^{-1} \\ -iA & 0 \end{pmatrix} \in End_{\mathbb{R}}(B \oplus B^{\vee})$$

Here we are using the $\mathbb{C}$-antilinear isomorphism $A : B \to B^{\vee}$, $A \in Hom_{\overline{\mathbb{C}}}(B, B^{\vee}) \subset$

$Hom_{\mathbb{R}}(B, B^\vee)$ *defined by* $A(w) = h(\bullet, w)$. *Moreover,*

$$d\tau(B \subset U) = \left\{ \begin{pmatrix} w \\ \\ -i\ A(w) \end{pmatrix}, w \in B \right\}.$$

As a corollary of the above simple linear-algebraic remark we can obtain a geometric description of $\tau(\mathcal{O}\mathfrak{p}_{SL(2)}(E))$ where E is an elliptic curve.

First, let us look at the case when X is a curve of genus $g \geq 1$ and $G = \mathbb{C}^\times$. Then

$$M_{Dol}(\mathbb{C}^\times) = T^\vee Pic^0(X) = Pic^0(X) \times H^0(X, K) = H^1(X, \mathcal{O})/\mathbb{Z}^{2g} \times H^0(X, K),$$

$B = H^0(X, K)$ is the Hitchin base, pr_2 is the Hitchin map and "the" Hitchin section is $\{\zeta\} \times H^0(X, K)$, $\zeta^{\otimes 2} = K$. The De Rham space is

$$\begin{array}{ccc} H^0(X, K) & \longrightarrow & M_{DR}(\mathbb{C}^\times) \ . \\ & & \downarrow \\ & & Pic^0(X) \end{array}$$

If we endow X with a metric compatible with the complex structure, then the Hodge star gives the isomorphism $\star : H^0(X, K)^\vee \simeq \overline{H^1(X, \mathcal{O})}$ and $H^0(X, K) \oplus H^1(X, \mathcal{O}) \simeq \mathbb{H}^g$, where $\mathbb{H}$ are the quaternions (equipped with a constant metric) and so

$$\mathcal{Z}(M_{Dol}) = \mathrm{tot}((B \oplus B^\vee) \otimes \mathcal{O}(1))/\mathbb{Z}^{2g} \simeq \mathrm{tot}(\mathbb{H}^g \otimes \mathcal{O}(1))/\mathbb{Z}^{2g}.$$

That is to say, the hyperkähler metric is induced by the hyperkähler metric on $T^\vee Pic^0(X)$. If $X = E$ is an elliptic curve, we have more specifically that

$$M_{Dol}(\mathbb{C}^\times) = Pic^0(E) \times H^0(E, K) \simeq E \times \mathbb{C},\ \ M_{DR}(\mathbb{C}^\times) = \mathcal{J},$$

where $\mathcal{J} = T^{\vee}_{\omega} E$ is the algebraic group which is the unique extension of E by the additive group $\mathbb{C}$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{J} \longrightarrow E \longrightarrow 0.$$

It can be described also as $\mathbb{P}(I_2) - C_0$, where I_2 is the non-trivial extension of $\mathcal{O}$ by $\mathcal{O}$ and C_0 is the section which squares to zero. It is analytically isomorphic to $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ – this is Serre's example of a Stein variety which is not affine. Notice that $\mathbb{C}^{\times} \times \mathbb{C}^{\times} = \mathrm{Hom}(\pi_1(E), \mathbb{C}^{\times})$ is the Betti space. And, finally,

$$\mathcal{Z}(M_{Dol}(\mathbb{C}^{\times})) \simeq \mathrm{tot}(\mathbb{H} \otimes \mathcal{O}(1))/\mathbb{Z}^2.$$

For a simple complex group G the Dolbeault and De Rham space can be described in a fairly straightforward manner as the theory is "linear" in a certain sense. The description I give is from [Tha01], using some results from [DG02].

As earlier, denote by $H = \mathbb{C}^{\times} \otimes_{\mathbb{Z}} \mathbf{coweight} \subset G$ the Cartan subgroup. Letting M stand for M_{Dol} or M_{DR}, we have that the inclusion $H \to G$ induces $M(H) \to M(G)$, which descends to $\rho : M(H)/W \to M(G)$, as the Weyl group acts by inner automorphisms on $M(G)$; $M(H) = M(\mathbb{C}^{\times}) \otimes_{\mathbb{Z}} \mathbf{coweight}$. That is, $M_{Dol}(H)/W = (E \times \mathbb{C}) \otimes_{\mathbb{Z}} \mathbf{coweight}/W$ and $M_{DR}(H)/W = (\mathcal{J} \otimes_{\mathbb{Z}} \mathbf{coweight})/W$. These spaces are in general singular – they have orbifold singularities – and it is not even clear if $M_{DR}(G)$ and $M_{Dol}(G)$ are normal (Simpson's results hold for genus at least two), but they are very close to being so. Namely, as shown in [Tha01], $M_{DR,Dol}(H)/W$ are their respective normalisations and the birational morphisms

$M(H)/W \to M(G)$ are finite bijections. Notice that we have a surjective map

$$\mathcal{Z}(M_{Dol}(H)) \simeq \mathrm{tot}((\mathbb{C}^2 \otimes_{\mathbb{Z}} \mathbf{coweight}) \otimes_{\mathbb{C}} \mathcal{O}(1)) \twoheadrightarrow \mathcal{Z}(M_{Dol}(G))$$

with the hyperkähler structure coming from $B \times B^\vee \simeq H^1(E, \mathcal{O}) \times \mathbb{C} \simeq \mathbb{C}^2$.

For the sake of brevity I restrict now myself to the case $G = SL(2)$. Then **coweight** $\simeq \mathbb{Z}$, $H \simeq \mathbb{C}^\times$, and hence $(M_{Dol}(\mathbb{C}^\times) \otimes \mathbf{coweight})/W \simeq (E \times \mathbb{C})/\mathbb{Z}/2 \simeq T^\vee \mathbb{P}^1$ and $M_{DR}(\mathbb{C}^\times)/W = \mathcal{J}/W$ are the normalisations of $M_{Dol}(G)$ and $M_{DR}(G)$. Notice that the maps from these to $Pic^0(E) \simeq E$ are induced by the maps $E \times \mathbb{C} \to E$ and $\mathcal{J} \to E$, and that opers are (the images of) fibres above points of order two. So we can apply Lemma 5.3.1 to deduce

Corollary 5.3.1 *On an elliptic curve $E = H^1(X, \mathcal{O})/\pi_1(E) \simeq \mathbb{C}/\mathbb{Z}^{\oplus 2}$ the component of of $\tau(\mathcal{O}\mathfrak{p}_{SL(2)}(E))$ indexed by the point of order two $[\zeta] \in \mathbb{C}/\mathbb{Z}^2 \simeq E$ is given (set-theoretically) as*

$$\left\{\rho\left([(\zeta - i\overline{z}, z) mod\ \mathbb{Z}^{\oplus 2}]\right), z \in \mathbb{C}\right\} =$$

$$\{([\zeta] - i[\overline{z}], z) \in E \times \mathbb{C}\ mod \mathbb{Z}/W\}.$$

It is a non-holomorphic section of the Hitchin system and an infinite cover of the moduli space of S-equivalence classes of semi-stable G-bundles on E.

Chapter 6

The System of Hodge Bundles

In this section we are going to consider one very special – and somewhat popular – Higgs bundle, which happens to be a system of Hodge bundles and to be related to uniformisation, among other things like being probably the simplest stable Higgs pair with unstable underlying bundle. In the next section we are going to describe a family of deformations of it, parametrised by the tangent space to the moduli space and containing the Hitchin section. Our immediate goal is, however, to compute the tangent space to the Dolbeault moduli space (at this point) in some natural, Lie-algebraic terms.

Let G be, as usual, complex simple group with Lie algebra $\mathfrak{g}$.

The Hitchin section of the Hitchin integrable system is obtained by choosing a theta-characteristic, i.e., a line bundle, $\zeta \in Pic^{g-1}(X)$, and an isomorphism $\zeta^{\otimes 2} \simeq K$. I am going to write $K^{1/2}$ instead of ζ for the most part. Consider the frame bundle of $\zeta \oplus \zeta^{-1}$, that is, the principal $SL(2)$-bundle $\underline{Isom}(\mathcal{O}^{\oplus 2}, \zeta \oplus \zeta^{-1})$, equipped with the Higgs field $\theta_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. A choice of an $\mathfrak{sl}(2)$-triple $\{x, y, h\}$ in $\mathfrak{g}$ induces a homomorphism $SL(2) \to G$; if G is simply-connected this is an embedding $SL(2) \hookrightarrow G$, otherwise we have an embedding $PGL(2) \hookrightarrow G$. We denote by $\mathbf{P}$ the G-bundle obtained by extension of structure group. Then the Kostant section $\mathbf{k} : \mathfrak{t}/W \hookrightarrow \mathfrak{g}$ induces $B_{\mathfrak{g}} = H^0(X, \mathfrak{t}/W \otimes K) \hookrightarrow H^0(\mathfrak{g}_{\mathbf{P}} \otimes K)$, which becomes canonical after a choice of invariant polynomials is fixed. This is the Hitchin section: a fixed principal bundle, $\mathbf{P}$, endowed with a varying family of Higgs fields. We can think of it as a $\dim G(g-1)$-dimensional family of deformations of $(\mathbf{P}, \theta)$, and we shall see how to extend it to a $2\dim G(g-1)$-dimensional family.

Let $\pi : \mathfrak{g} \to \mathfrak{g}//G \simeq \mathfrak{t}/W$ be the Chevalley projection, and remember that $y + Z(x)$ gives a splitting of π. Then using the differential of the inverse, $d\mathbf{k} = d\pi|^{-1}_{y+\mathcal{Z}(x)}$ and the Dolbeault isomorphism $H^0(X, \mathfrak{t}/W \otimes K) \simeq H^{1,0}(X, \mathfrak{t}/W) \subset A^{1,0}(\mathfrak{g})$ we obtain $B_{\mathfrak{g}} = H^0(X, \mathfrak{t}/W \otimes K) \hookrightarrow A^{1,0}(\mathfrak{g}_{\mathbf{P}})$. Similarly, identifying $\mathfrak{g} \simeq \mathfrak{g}^\vee$ via the Killing form, we have $B_{\mathfrak{g}}^\vee = H^0(X, \mathfrak{t}/W \otimes K)^\vee = H^1(X, \mathfrak{t}/W^\vee) \hookrightarrow A^{0,1}(\mathfrak{g}_{\mathbf{P}})$.

Notice that the differential $d\mathbf{k}$ is just $\pi|^{-1}_{y+Z(x)} - y = \mathbf{k} - y$, and that a choice of invariant polynomials, i.e., of an isomorphism $\mathfrak{t}/W \simeq \mathbb{C}[p_1, \ldots, p_l]$ induces an isomorphism $\mathfrak{t}/W \simeq Z(x)$. In some constructions we are going to need a basis of the one-dimensional subspaces V_{m_i} from (3), $Z(x) = \oplus V_{m_i} = V$, so choosing one amounts to making a special choice of invariant polynomials. We now make the special choice of an $\mathfrak{sl}(2)$-triple which we discussed in (3). The principal grading of the Lie algebra $\mathfrak{g}$ induces decompositions

$$\mathfrak{g}_{\mathbf{P}} = \bigoplus_{m=-M}^{M} K^m \otimes_{\mathbb{C}} \mathfrak{g}_m,$$

$$\mathcal{A}^{p,q}(\mathfrak{g}_{\mathbf{P}}) = \bigoplus_{m=-M}^{M} \mathcal{A}^{p,q}(K^m) \otimes_{\mathbb{C}} \mathfrak{g}_m, \ \mathcal{A}^{p,q}(\mathfrak{g}_{\mathbf{P}} \otimes K) = \bigoplus_{m=-M}^{M} \mathcal{A}^{p,q}(K^{m+1}) \otimes_{\mathbb{C}} \mathfrak{g}_m. \tag{6.0.1}$$

Notice that the tensor products on the right hand side are over $\mathbb{C}$.

Example 6.0.2 *Let* $G = SL(2, \mathbb{C})$ *and* $\mathbf{E} = K^{1/2} \oplus K^{-1/2}$. *Then*

$$ad\ \mathbf{P} = \mathscr{E}nd_0\mathbf{E} = K^{-1} \oplus \mathcal{O} \oplus K =$$
$$K^{-1} \otimes_{\mathbb{C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \mathcal{O} \otimes_{\mathbb{C}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus K \otimes_{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Under this decomposition the natural bracket on the left side of (6.0.1) coincides with $[,]\otimes\wedge$ on the right. Certain natural isomorphisms have to be taken care of while computing commutators. We have $\mathcal{A}^{p,q}(K^m) = \mathcal{A}^{p+1,q}(K^{m-1})$ and $1 \in A^{1,0}(K^{-1}) = A^0(\mathcal{O})$. Define

$$\iota : \mathcal{A}^{p,q}(K^m) \simeq \mathcal{A}^{p+1,q}(K^{m-1}), \ \alpha \mapsto 1 \wedge \alpha$$

Explicitly, in the most important special case $p = 0, q = 1$, ι looks as

$$\iota : A^{0,1}(K^m) \simeq A^{1,1}(K^{m-1}), \ \iota(\overline{\xi} \otimes \alpha^m) = \alpha \wedge \overline{\xi} \otimes \alpha^{m-1}.$$

We have that

$$\forall \alpha \in A^{0,1}(K^m), \ \iota(\alpha) = 1 \wedge \alpha = -\alpha \wedge 1 \in A^{1,1}(K^{m-1})$$

$$\forall \alpha \in A^{0,0}(K^m),\ \iota(\alpha) = 1 \wedge \alpha = \alpha \wedge 1 \in A^{1,0}(K^{m-1}).$$

Notice that the isomorphism $\iota : A^{0,1}(K^m) \simeq A^{1,1}(K^{m-1})$ *anticommutes* with the Dolbeault operator: $\iota \circ \overline{\partial} = -\overline{\partial} \circ \iota : A^0(\mathcal{O}) \to A^{1,1}(K^{-1})$. Also notice that the Higgs field $\theta \in H^0(\mathfrak{g}_{\mathbf{P}} \otimes K) \subset A^0(\mathfrak{g}_{\mathbf{P}} \otimes K)$, and under ι we have

$$A^0(\mathfrak{g}_{\mathbf{P}} \otimes K) \ni \theta \mapsto 1 \otimes y \in A^0 \otimes_{\mathbb{C}} \mathfrak{g}_{-1} = A^{1,0}(K^{-1}) \otimes_{\mathbb{C}} \mathfrak{g}_{-1} \subset A^{1,0}(\mathfrak{g}_{\mathbf{P}}).$$

The infinitesimal deformations of our bundle are computed by the corresponding Dolbeault complex. Taking its Dolbeault resolution and passing to global sections we obtain

$$\begin{array}{ccc} A^{0,1}(\mathfrak{g}_{\mathbf{P}}) & \xrightarrow{ad\ \iota\theta} & A^{1,1}(\mathfrak{g}_{\mathbf{P}}) \\ \overline{\partial}\uparrow & & \overline{\partial}\uparrow \\ A^{0,0}(\mathfrak{g}_{\mathbf{P}}) & \xrightarrow{ad\ \iota\theta} & A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \end{array} .$$

The total complex is

$$\mathscr{C}: \qquad 0 \longrightarrow A^0(\mathfrak{g}_{\mathbf{P}}) \xrightarrow{d_0} A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}}) \xrightarrow{d_1} A^{1,1}(\mathfrak{g}_{\mathbf{P}}) \longrightarrow 0 \qquad (6.0.2)$$

with differentials $d_0 = \begin{pmatrix} ad\ \iota\theta \\ \overline{\partial} \end{pmatrix}$, $d_1 = \begin{pmatrix} \overline{\partial}, & ad\ \iota\theta \end{pmatrix}$. Our first goal is to compute the hypercohomology $\mathbb{H}^1(\mathscr{C}_\bullet)$.

Lemma 6.0.2 *Let W be an odd-dimensional vector space,* $\dim W = 2n+1$, *and $N \in EndW$ a regular nilpotent endomorphism. Let $W = \oplus W_m = W_n \oplus \ldots \oplus W_{-n}$ be the decomposition into one-dimensinal subspaces associated with N. That is,* $\ker N = W_{-n} \subset \ker N^2 = W_{-n} \oplus W_{-n+1} \subset \ldots \subset W$, *$cokerN = W_n \subset cokerN^2 =$*

$W_n \oplus W_{n-1} \subset \ldots W$ and there is a unique choice of nonzero vector from each W_m so that the matrix of N consists of a single lower-triangular Jordan block. Fix a basis of $W_n \simeq \mathbb{C}$. Consider the vector bundle $\mathscr{E} = \oplus K^m \otimes_{\mathbb{C}} W_m$ and the complex

$$C_\bullet : \qquad 0 \longrightarrow A^0(\mathscr{E}) \xrightarrow{d_0} A^{1,0}(\mathscr{E}) \oplus A^{0,1}(\mathscr{E}) \xrightarrow{d_1} A^{1,1}(\mathscr{E}) \longrightarrow 0 \qquad (6.0.3)$$

where $d_0 = \begin{pmatrix} \iota \otimes N \\ \overline{\partial} \end{pmatrix}$, $d_1 = \left(\overline{\partial}, \iota \otimes N\right)$. *Then*

$$\mathbb{H}^1(C_\bullet) = H^0(X, K^{n+1}) \otimes cokerN \oplus H^1(X, K^{-n}) \otimes \ker N.$$

Remark: For the proof we need bases of W_n and W_{-n}, but applying N^{2n} to the basis vector of the former we get a basis of the latter.

Proof:

We have

$$W = W_n \oplus \ldots \oplus W_{-n} = \bigoplus_{m>-n} W_m \oplus \ker N = \mathrm{coker} N \bigoplus_{m<n} W_m.$$

Let $\overline{N} : W/\ker N \simeq ImN$ be the natural map, $\overline{N} : W_n \oplus \ldots \oplus W_{-n+1} \to W_{n-1} \oplus \ldots \oplus W_{-n}$, $\overline{N} : W_i \simeq W_{i-1}$. Let $s = (s', s'') \in \ker d_1$. Denote by s_n, $s_{<n}$, s_{-n}, $s_{>-n}$ the projections of s onto W_n, W/W_n, W_{-n} and W/W_{-n}, respectively. The condition $d_1(s) = 0$ says that $\overline{\partial} s' = -(\iota \otimes N)s''$. Then we have

$$\overline{\partial} s'_n = 0, \; \overline{\partial} s'_{<n} + (\iota \otimes N)s'' = 0.$$

The essence of the proof is that from the second equation we can "solve" for s'' up to elements of $\ker N = W_{-n}$ and separate an exact term, the only cohomology coming from W_n (the first equation) and W_{-n}.

Indeed,

$$s'' = -(\iota \otimes \overline{N})^{-1}(\overline{\partial} s'_{<n}) \bmod A^{0,1}(K^{-n}) \otimes W_{-n}$$

$$= \overline{\partial}(\iota \otimes \overline{N})^{-1}(s'_{<n}) \bmod A^{0,1}(K^{-n}) \otimes W_{-n}. \tag{6.0.4}$$

On the other hand, for $\varepsilon \in A^0(\mathscr{E})$, $d_0(\varepsilon) = ((\iota \otimes N)\varepsilon, \overline{\partial}\varepsilon)$. So we have that

$$\ker(d_1) \ni s = (s', s'') = (s', \overline{\partial}(\iota \otimes \overline{N})^{-1}(s'_{<n})) \bmod A^{0,1}(K^{-n}) \otimes W_{-n}$$

$$= (s'_n, 0) + d_0(\varepsilon) \bmod W_{-n} \otimes A^{0,1}(K^{-n}),\ \varepsilon = (\iota \otimes \overline{N})^{-1}(s'_{<n}),\ \overline{\partial} s'_n = 0.$$

Assuming that W_n (and hence W_{-n}) are equipped with bases, say e_n and e_{-n}, respectively, there is a natural map

$$Z^{1,0}_{\overline{\partial}}(K^n) \oplus A^{0,1}(K^{-n}) \to \ker d_1,\ (\alpha', \alpha'') \mapsto \alpha' \otimes e_n + \alpha'' \otimes e_{-n}.$$

This map sends coboundaries to coboundaries:

$$(0, \overline{\partial}\alpha) \mapsto \overline{\partial}\alpha \otimes e_{-n} = d_0(\alpha \otimes e_{-n}),$$

so we have $H^{1,0}(K^n) \oplus H^{0,1}(K^{-n}) \to \mathbb{H}^1(C_\bullet)$. The above discussion (6.0.4) shows that this map has an inverse: if $s = (s', s'') \in \ker d_1$, we (roughly) take $s \mapsto (s'_n, s''_{-n})$. More precisely, if $s'_n = \sigma' \otimes e_n$, $s''_{-n} = \sigma'' \otimes e_{-n}$, we look at the map $Z^1(C_\bullet) \to Z^{1,0}(K^n) \oplus A^{0,1}(K^{-n})$ given by $s \mapsto (\sigma', \sigma'') = (e_n^\vee(s'_n), e_{-n}^\vee(s''_n))$. This map sends boundaries to boundaries:

$$d_0(\varepsilon) = ((\iota \otimes N)\varepsilon, \overline{\partial}\varepsilon) \mapsto (0, \overline{\partial} e_{-n}^\vee(\varepsilon_n))$$

and the induced map on the cohomologies inverts the previous one.

QED

The Lemma makes it straightforward to prove the following

Theorem 6.0.1 *Let* $\mathfrak{g} = Lie(G)$ *be a simple complex Lie algebra and let* $B_{\mathfrak{g}} = H^0(X, \mathfrak{t}/W \otimes K)$ *be the Hitchin base. Let* ζ *be a theta characteristic and* $\mathbf{P}$ *be the principal* G*-bundle obtained from* $\underline{Isom}(\mathcal{O}^{\oplus 2}, \zeta \oplus \zeta^{-1})$ *by extension of structure group via a choice of* $\mathfrak{sl}(2)$*-triple* $\{x, y, 2\rho^\vee\}$ *described earlier; equip* $\mathbf{P}$ *with the Higgs field* θ *mapped to* $1 \otimes y \in A^0 \otimes_{\mathbb{C}} \mathfrak{g}_{-1}$ *by the isomorphism* ι*. Then the first hypercohomology of the Dolbeault complex (6.0.2) of* $ad\mathbf{P}$ *is*

$$\mathbb{H}^1(\mathscr{C}_\bullet) \simeq B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^\vee \subset A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}}).$$

The isomorphism becomes canonical after a choice invariant polynomials and a basis of $Z(x)$ *compatible with the decomposition* $Z(x) = \oplus V_{m_i}$*, or, equivalently, after a suitable choice of invariant polynomials of* G*.*

Proof:

The choice of the $\mathfrak{sl}(2)$-subalgebra determines $\mathfrak{g} = \oplus W_i$ and the choice of Chevalley generators determines $\mathfrak{g} = \oplus \mathfrak{g}_m$, so we have

$$\oplus_i A^{1,0}(K^{m_i}) \otimes (W_i \cap \mathfrak{g}_{m_i}) \bigoplus \oplus_i A^{0,1}(K^{-m_i}) \otimes (W_i \cap \mathfrak{g}_{-m_i}) \subset A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}}),$$

and $B \oplus B^\vee$ sits inside the left hand side as the harmonic forms. The inclusion of B in the first factor on the right is *via* the differential of the Kostant section $d\mathbf{k} : \mathfrak{t}/W \simeq Z(x)$. If we have a basis vector e_{m_i} in each $V_{m_i} = W_i \cap \mathfrak{g}_{m_i}$, we can apply Lemma 6.0.2 to each summand of $A^1(\mathfrak{g}_{\mathbf{P}}) = \oplus A^1(W_{i\mathbf{P}})$. Again, since we have $d\mathbf{k} : \mathfrak{t}/W \simeq Z(x)$ and the choice of invariant polynomials $\{p_m\}$ gives a basis of $Z(x)$ – the image of the dual basis to $\{p_1, \dots, p_l\}$ under $d\mathbf{k}$. A "suitable" choice of invariant polynomials is one where each basis vector e_{m_i} is contained in the corresponding V_{m_i}.

QED

Example 6.0.3 *Consider the case* $\mathfrak{g} = A_l = \mathfrak{sl}(l+1)$.

To describe the isomorphism from the theorem, we need bases for the one-dimensional spaces V_{m_i} *and the bases of* $W_i \cap \mathfrak{g}_{-m_i}$ *induced by these.*

Let $E_{p,q}$ *denote the* $(l+1) \times (l+1)$*-matrix whose only nonzero entry is 1 at position pq. Our standard choice of* $\mathfrak{sl}(2)$*-subalgebra is the one from Example 3.0.1. In particular,*

$$Z(x) = span\{x, x^2, \dots, x^l\},\ x = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_l \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},\ a_p := p(l-p+1).$$

The exponents are $m_1 = 1, \dots, m_l = l$, *and the spaces* V_{m_i} *are spanned by* $e_{m_i} = x^{m_i}$.

One can easily compute the successive powers of x. Indeed,

$$x = a_1E_{1,2} + a_2E_{2,3} + \ldots + a_lE_{l,l+1},\ x^2 = a_1a_2E_{1,3} + \ldots a_{l-1}a_lE_{l-1,l+1},$$

and inductively

$$x^p = \sum_{k=1}^{l-p+1} x_k^p E_{k,p+k} = x_1^p E_{1,p+1} + \ldots x_{l-p+1}^p E_{l-p+1,l+1},$$

where for $m = 0 \ldots p-1$ and $k = 1 \ldots l-p+1$ we have defined

$$x_k^p = a_k^{C_0} \cdot a_{k+1}^{C_1} \cdot \ldots \cdot a_{k+p-1}^{C_{p-1}} = \prod_{s=k}^{p-k+1} a_s^{C_{s-k}},\ C_m = \begin{pmatrix} p-1 \\ \\ m \end{pmatrix}.$$

We have Lemma 3.0.3 which says that $e_{m_i}^* = \frac{(-1)^{m_i}}{(2m_i)!}$, so $e_{-m_i} = (-1)^{m_i}(2m_i)!x^{m_i*}$.

As we know from Lemma 3.0.1, the hermitian metric in our case is given by

$$H = \sum_{k=1}^{l+1} H_kE_{k,k},\ H_k = \prod_{m=1}^{k-1} a_m, k > 1, H_1 = 1$$

so the adjoint of x^p is $x^{p*} = H^{-1}\overline{x}^{pT}H$, which is explicitly

$$x^{p*} = \sum_{k=1}^{l-p+1} \left(x_k^p H_{p+k}^{-1} H_k\right) E_{p+k,k} = \prod_{m=1}^{p} a_m^{C_{s-1}-1} E_{p+1,1} + \sum_{k=2}^{l-p+1} \frac{\prod_{s=k}^{p-k+1} a_s^{C_{s-k}}}{\prod_{m=k}^{p+k-1} a_m} E_{p+k,k}.$$

Then the map $B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee} \to A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}})$ prescribed by Theorem 6.0.1 in terms of harmonic representatives looks like

$$(\alpha_1, \ldots, \alpha_l; \xi_1, \ldots, \xi_l) \mapsto \sum_{p=1}^{l} \alpha_p x^p + \sum_{p=1}^{l} \xi_p (-1)^p (2p)! x^{p*} =$$

$$\sum_{p=1}^{l} \alpha_p \sum_{k=1}^{l-p+1} \prod_{s=k}^{p-k+1} a_s^{C_{s-k}} E_{k,p+k} +$$

$$\sum_{p=1}^{l} \xi_p (-1)^p (2p)! \left(\prod_{m=1}^{p} a_m^{C_{s-1}-1} E_{p+1,1} + \sum_{k=2}^{l-p+1} \frac{\prod_{s=k}^{p-k+1} a_s^{C_{s-k}}}{\prod_{m=k}^{p+k-1} a_m} E_{p+k,k} \right),$$

where $\alpha_p \in \mathscr{H}^0(K^{p+1}), \xi_p \in \mathscr{H}^{0,1}(K^{-p})$.

Proposition 6.0.1 *If $\mathfrak{g} \neq D_{2n}$, the isomorphism*

$$\mathbb{H}^1(\mathscr{C}_\bullet) \simeq B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^\vee \subset A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}})$$

from Theorem 6.0.1 is compatible with the natural complex symplectic structre on both sides up to Lie-theoretic normalisation factors.

Proof: This is clear from the construction. The complex symplectic form on $\mathbb{H}^1(\mathscr{C}_\bullet)$ is induced by the cup product and in Dolbeault terms looks like the map

$$\left(A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}})\right)^{\times 2} \to A^{1,1} \to \mathbb{C},$$

$$((u,\alpha),(v,\beta)) \mapsto \int_X \varkappa(\beta \wedge u)) - \varkappa(\alpha \wedge v).$$

The complex symplectic form on M_{Dol} evaluated on the image of $(u_i,\alpha_i),(v_i,\beta_i) \in B \oplus B^\vee$ is $\sum_i \varkappa(e_{m_i}, e_{-m_i})(\beta_i(u_i) - \alpha_i(v_i))$, while the standard complex symplectic form on $B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^\vee$ gives the same expression but without the $\varkappa(e_{m_i}, e_{-m_i})$ factors.

QED

Proposition 6.0.2 *The isomorphism from Theorem 6.0.1 maps harmonic representatives to harmonic representatives.*

It seems useful to give a make a couple of remarks before proving this statement. Given a vector bundle, E, on a complex manifold, X, we have two operatorions on $\mathcal{A}^p(\mathscr{E}nd_0 E)$: * and $\star$. That is, given $\xi = \Phi \otimes \alpha \in A^p(\mathscr{E}nd_0 E)$, we have $\xi^* = \Phi^* \otimes \overline{\alpha}$ and $\star\xi = \Phi^* \otimes \star\alpha$. However, for X a curve the complex structure is equivalent to a conformal class of metrics so we have further relations. More specifically, $\star\alpha = i\overline{\alpha}$, if $\alpha \in A^{1,0}$, and $\star\alpha = -i\overline{\alpha}$, if $\alpha \in A^{0,1}$. Hence, if $\xi \in A^{1,0}(\mathscr{E}nd_0 E)$, $\xi^* = -i \star \xi$, and if $\xi \in A^{0,1}(\mathscr{E}nd E)$, $\xi^* = i\star\xi$. Finally, the inner product on $A^1(\mathscr{E}nd_0 V)$ is computed as $\langle u, \xi \rangle = \int_X tr(u \wedge \star\xi)$.

If G is a simple complex group and $\mathbf{P}$ a principal G-bundle, all of the above still holds, but * now stands for minus the anti-linear involution given by the compact real form (see Section 3) and tr should be replaced by the Killing form.

The bundle we are considering in this section comes with a decomposition $\oplus \mathfrak{g}_m \otimes_{\mathbb{C}} K^m$ and so the Hodge star operator can be expressed conveniently in terms of the Hodge star on each of the line bundles and the involution η' whose antilinear extension gives the compact real form.

Proof:

We have chosen a compact real form with respect to which the $\mathfrak{sl}(2)$-triple is real, and so are the irreducible representations W_i, which are mutually orthogonal, as shown in Lemma 3.0.2. Recall that the antilinear involution giving the real form (as well as the linear involution η') sends $\mathfrak{g}_k \to \mathfrak{g}_{-k}$. In view of this, it suffices to prove the proposition for each of the W_i's. That is, it suffuces to prove:

Lemma 6.0.3 *The map considered in the proof of Lemma 6.0.2,*

$$Z^{1,0}_{\overline{\partial}}(K^{m_i}) \oplus A^{0,1}(K^{-m_i}) \to \ker d_1, \;\; (\alpha', \alpha'') \mapsto \alpha' \otimes e_{m_i} + \alpha'' \otimes e_{-m_i}$$

maps harmonic representatives to harmonic representatives.

Proof:

We are going to show that the image of $\alpha = (\alpha', \alpha'')$ is L^2-orthogonal to $im(d_0)$

iff α'' is L^2-orthogonal to $im(\overline{\partial})$. Alternatively, one could show explicitly that the image is contained both in $\ker d_1$ and $\ker d_0^*$.

Indeed, if $\alpha = \alpha' \otimes e_{m_i} + \alpha'' \otimes e_{-m_i}$ and $\beta \in A^0(\oplus_{n=-m_i}^{m_i} K^n \otimes_{\mathbb{C}} W_n)$, $\beta = \sum_{n=-m_i}^{m_i} \beta_n e_{m_i}^n$, then

$$\langle d_0(\beta), \alpha \rangle = \left\langle \iota \otimes ady(\beta) + \overline{\partial}\beta, \alpha' \otimes e_{m_i} + \alpha'' \otimes e_{-m_i} \right\rangle =$$

$$\langle \iota \otimes ady(\beta), \alpha' \otimes e_{m_i} \rangle + \left\langle \overline{\partial}\beta, \alpha'' \otimes e_{-m_i} \right\rangle =$$

$$(\overline{\partial}\beta_{-m_i}, \alpha'')(e_{m_i}^{-m_i}, e_{m_i}^{-m_i}),$$

since the first term vanishes due to the fact that $ad\ y$ sends W_i to $W_i \cap \oplus_{s<m_i} \mathfrak{g}_s$, i.e., $\iota ad\ y(\beta) \in A^0(\oplus_{n=-m_i}^{m_i - 1} K^n \otimes_{\mathbb{C}} W_n)$. Now the last expression is zero iff α'' is orthogonal to coboundaries in $A^{0,1}(K^{-m_i})$.

QED

QED (The Proposition)

Chapter 7

Deforming of the System of Hodge Bundles

7.1 The Maurer-Cartan Equation

As observed by P.Deligne in a letter to J.Millson, a deformation problem (in characteristic zero) is often guided by a differential graded Lie algebra (DGLA), $(L^\bullet, d)$. Given $(L^\bullet, d)$, one considers the Maurer-Cartan equation

$$\mathcal{D}(\eta) := d\eta + \frac{1}{2}[\eta, \eta] = 0, \ \eta \in L^1, \tag{7.1.1}$$

and wants to describe the gauge-equivalence classes of solutions. In the algebraic setup this means that one is looking for solutions with coefficients in an artin local ring and wants to pro-represent the corresponding deformation functor; in the analytic setup one is dealing with a normed DGLA and takes completions – or works with differential graded Banach Lie algebras from the outset; for review of general facts in these two situations see, respectively, [GM88],[Man99], and [GM90], [Kos03], [Gri65].

Here we shall be interested in the deformations of our Higgs bundle, the system of Hodge bundles $[(\mathbf{P},\theta)] \in M_{Dol}$. The DGLA is $(A^\bullet(\mathfrak{g}_\mathbf{P}), \overline{\partial} + ad\ \iota\theta)$, with deformation complex (6.0.2). Of course, this is the deformation complex for every Higgs bundle, not only the one we are interested in; indeed, recall that the Higgs bundle is the data of a holomorphic structure operator $\overline{\partial}$ and a Higgs field θ, such that $(\overline{\partial}+\theta)^2 = 0$. If we modify these by $\overline{\partial} \mapsto \overline{\partial} + \eta''$, $\theta \mapsto \theta + \eta'$ then the result will be a Higgs bundle iff $\eta = (\eta', \eta'')$ satisfies the Maurer-Cartan equation

$$\overline{\partial}\eta' + [\theta, \eta''] + [\eta', \eta''] = 0. \tag{7.1.2}$$

In this section we are going to describe a family of deformations of $[(\mathbf{P},\theta)]$ parametrised by $B \oplus B^\vee$, the tangent space to M_{Dol} at $[(\mathbf{P},\theta)]$, which we are going to think of as a holomorphic exponential map. This will *not* be the Kuranishi family, although the procedure we use to construct it is somewhat analogous to the one used for constructing the Kuranishi family. In fact, in the case of $G = SL(2)$ one can try to compare explicitly the Kuranishi family with the family that appears below, but I will not dwell on that.

Theorem 7.1.1 *Suppose $\mathfrak{g} \neq D_{2n}$. Let $\mathcal{D} = d_1 + \frac{1}{2}[\ ,\ ] : A^1(\mathfrak{g}_\mathbf{P}) \to A^{1,1}(\mathfrak{g}_\mathbf{P})$ be the Maurer-Cartan operator, and let*

$$\mathcal{P} : A^{1,1}(\mathfrak{g}_\mathbf{P})/A^{1,1}(\oplus_{m_i} K^{m_i} \otimes V_{m_i}) \subset A^{1,1}(\mathfrak{g}_\mathbf{P}) \to A^{0,1}(\mathfrak{g}_\mathbf{P})$$

be a splitting of ad $\iota\theta$, i.e., ad $\iota\theta \circ \mathcal{P} = 1$, defined in the proof. Let $M = \max\{m_i\}$ be the largest exponent of $\mathfrak{g}$.

Then, for any $\eta_0 \in \mathscr{H}^1(\mathscr{C}_\bullet)$, $(1 - \mathcal{P}\mathcal{D})^M(\eta_0)$ is a fixed point of $(1 - P\mathcal{D})$ and $(1 - \mathcal{P}\mathcal{D})^M(\eta_0) \in \ker \mathcal{D}$. In this way we obtain a map

$$B \oplus B^\vee \simeq \mathscr{H}^1(\mathscr{C}_\bullet) = \mathbb{H}^1(\mathscr{C}_\bullet) \to \mathscr{Y} \subset \{\textit{Solutions of Eqn.7.1.2}\} \subset A^1(\mathfrak{g}_{\mathbf{P}}),$$

$$\mathscr{Y} = \left\{(1 - \mathcal{P}\mathcal{D})^M(\eta_0),\ \eta_0 \in \mathscr{H}^1(\mathscr{C}_\bullet)\right\}.$$

Equivalently,

$$\mathscr{Y} = \left\{\eta_0 - \sum_{k=1}^{M} x_k,\ x_k = -\mathcal{P}([\eta_0', x_{k-1}]), x_0 = \eta_0'',\ \eta_0 \in \mathscr{H}^1(\mathscr{C}_\bullet)\right\}.$$

The "correction terms" x_k also satisfy $x_k := \mathcal{P}\mathcal{D}(\eta_{k-1})$, $\eta_k := \eta_{k-1} - x_k$. After passing to gauge-equivalence classes this provides a holomorphic exponential map

$$\varepsilon : (B \oplus B^\vee, 0) = (T_{M_{Dol},[(\mathbf{P},\theta)]}, 0) \subset M_{Dol}.$$

It may seem a little puzzling *a priori* that something like this be possible, but the reason is that our Higgs bundle is very special. Here is one heuristic explanation. If one has a Higgs line bundle (or just a line bundle) of degree zero, the deformation complex is $\mathcal{O} \to K$ with zero differential (or just $\mathcal{O}$), the Maurer-Cartan equation involves no quadratic terms, and given a harmonic representative of an element of $H^1(X, \mathcal{O}) \times H^0(X, K)$ (or $H^1(X, \mathcal{O})$), it already satisfies the Maurer-Cartan equation and gives an "actual" deformation; namely, the image of that cohomology class in $Pic^0(X) \times H^0(X, K)$ (or $Pic^0(X)$) under the universal covering map (translated by the Higgs field of the Higgs line bundle that we are looking at). For the pair $(\mathbf{P}, \theta)$, the bundle $\mathbf{P}$ is a direct sum of line bundles $(\oplus K^{m_i} \otimes \mathfrak{g}_{m_i})$, and the only interesting contribution comes from the fact that θ is not an element of the Cartan subalgebra, which is an algebraic and not analytic complication.

Proof of Theorem 7.1.1:

The last statement, namely, that in a neighbourhood of zero in $B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee}$ one obtains a map to M_{Dol} (after passing to equivalence classes) will follow from the open nature of stability and from the fact that at zero the differential of the yet-to-be-constructed map will be the identity. More comments on how this fits in a more general framework are given after the end of the proof. For future reference we make a trivial

Claim 7.1.1 *Let $(L^{\bullet}, d)$ be a DGLA, let $\eta, x_1 \in L^1$ and let $d\eta + \frac{1}{2}[\eta, \eta] = dx_1$. Then, if $\eta_1 := \eta - x_1$,*

$$\mathcal{D}(\eta_1) = -[\eta, x_1] + \frac{1}{2}[x_1, x_1].$$

This requires only substituting into the Maurer-Cartan equation, so we omit the proof.

Next we need to define the map $\mathcal{P}$, which is straightforward, having in mind the decomposition of the Lie algebra. In each of the irreducible $\mathfrak{sl}(2)$-representations W_i we have $ad\ x \circ ad^k\ y(e_{m_i}) = k(2m_i + 1 - k) ad^{k-1}\ y(e_{m_i})$, which gives a splitting

Im $ad\ y|_{W_i} \simeq W_i \cap \oplus_{m<m_i} \mathfrak{g}_m \to W_i$ of $ad\ y$. Taking the direct sum of these maps and tensoring with ι^{-1} gives the map $\mathcal{P} : A^{1,1}(\mathfrak{g}_{\mathbf{P}})/A^{1,1}(\oplus_{m_i} K^{m_i} \otimes V_{m_i}) \to A^{1,1}(\mathfrak{g}_{\mathbf{P}})$, $\mathcal{P} : A^{1,1}(K^m) \otimes (W_i \cap \mathfrak{g}_m) \to A^{0,1}(K^{m+1}) \otimes (W_i \cap \mathfrak{g}_{m+1})$, $m \neq m_i$.

Claim 7.1.2 *On the domain of $\mathcal{P}$ one has $d_1 \circ \mathcal{P} = 1$.*

Since $d_1 = \overline{\partial} + ad(\iota\theta) = \overline{\partial} + ad\ 1 \otimes y$ and since Im$\mathcal{P} \in A^{0,1}(\mathfrak{g}_{\mathbf{P}})$, $d_1 \circ \mathcal{P} = ad\ (1 \otimes y) \circ \mathcal{P} = 1$ by construction.

Since the splitting $\mathcal{P}$ is defined only on Im $ad\ y$, the computational part of the proof will be to show that various operators are composable. Leaving this aside for the moment, here is the plan.

Starting with $\eta_0 \in \mathscr{H}^1(\mathscr{C}_\bullet)$, a harmonic representative of a cohomology class, as prescribed by Theorem (6.0.1), we are going to make iterations of the form

$$\eta_k \longmapsto \mathcal{D}(\eta_k) \longmapsto x_{k+1} := \mathcal{P}\mathcal{D}(\eta_k) \longmapsto \eta_{k+1} = (1 - \mathcal{P}\mathcal{D})(\eta_k) = \eta_k - x_{k+1} \dots$$

Since Im$\mathcal{P} \in A^{0,1}(\mathfrak{g}_{\mathbf{P}})$, $\eta'_k = \eta'_0$ for all k, as we are subtracting a $(0,1)$-term at each step. Moreover, from Claim 7.1.1 we have that for type reasons $\mathcal{D}(\eta_k) = -[\eta'_{k-1}, x_k]$, and thus

$$\mathcal{D}(\eta_0) = [\eta'_0, \eta''_0] \text{ and } \mathcal{D}(\eta_k) = -[\eta'_0, \mathcal{P}\mathcal{D}(\eta_{k-1})] = -[\eta'_0, x_k],\ \ k > 0.$$

We also have

$$x_{k+1} = -\mathcal{P}([\eta'_0, x_k]). \tag{7.1.3}$$

If for some k_0 we obtain $\mathcal{P}\mathcal{D}(\eta_{k_0}) = 0$, then $\eta_{k_0+1} = \eta_{k_0}$ and $\mathcal{D}\eta_{k_0+1} = -[\eta'_0, \mathcal{P}\mathcal{D}\eta_{k_0}]$ $= 0$. Clearly for all $n \geq k_0$ we are going to have $\eta_n = \eta_{k_0}$ and $\mathcal{D}(\eta_n) = 0$. However, such a k_0 always exists, because

$$x_k \in \mathfrak{g}_{m_0} \Longrightarrow x_{k+1} = -\mathcal{P}[\eta_0', x_k] \in \bigoplus_{m \geq m_0+2} A^{0,1}(K^m) \otimes \mathfrak{g}_m.$$

Thus the only thing that remains to be shown is that if x_k exists, so does x_{k+1}.

The proof is inductive, but the essence is an observation of Hitchin in in [Hit92], p.458 that commutators between elements of various W_i's can have nonzero projections only on certain W_j's. More precisely, if $u \in W_i \cap \mathfrak{g}_m$, $v \in W_j \cap \mathfrak{g}_n$, then $[u, v] \in \mathfrak{g}_{m+n}$ will have a nonzero projection onto $W_k \cap \mathfrak{g}_{m+n}$ only if $m_i + m_j + m_k =$ odd. Assuming that, the induction goes as follows. We start with $\eta_0 = (\eta_0', \eta_0'')$,

$$\eta_0' = \sum_{i=1}^{l} \alpha_{m_i} e_{m_i},\ \alpha_{m_i} \in A^{1,0}(K^{m_i}),\ e_{m_i} \in V_{m_i} = W_i \cap \mathfrak{g}_{m_i}$$

$$\eta_0'' = \sum_{i=1}^{l} \xi_{m_i} e_{-m_i},\ \xi_{m_i} \in A^{0,1}(K^{-m_i}),\ e_{-m_i} \in W_i \cap \mathfrak{g}_{-m_i}.$$

Then

$$\mathcal{D}(\eta_0) = [\eta_0', \eta_0''] = \sum_{i,j} \alpha_{m_i} \wedge \xi_{m_j} [e_{m_i}, e_{-m_j}] \in A^{1,1}\left(\bigoplus_{m=-M+1}^{M} \mathfrak{g}_m\right),$$

and the commutator $[e_{m_i}, e_{-m_j}]$ decomposes as a sum of its projections onto the various W_p:

$$[e_{m_i}, e_{-m_j}] = \sum_p A_p^{m_i - m_j}(1) \in \mathfrak{g}_{m_i - m_j},$$

where

$$A_p^{m_i - m_j}(1) = pr_p^{m_i - m_j}([e_{m_i}, e_{-m_j}]),\ pr_p^{m_i - m_j} : \mathfrak{g}_{m_i - m_j} \to \mathfrak{g}_{m_i - m_j} \cap W_p.$$

Now I claim that there are no terms in this decomposition for which p is an exponent. For if there were such a term, there would be an exponent m_k, such that $m_i - m_j =$

m_k and $m_i + m_j - m_k = \text{odd}$, which would mean that $2m_j = \text{odd}$. Thus we can apply $\mathcal{P}$ to $\mathcal{D}(\eta_0)$, and obtain

$$x_1 = \mathcal{P}\mathcal{D}(\eta_0) = \left(0, \sum_{i,j} \iota^{-1}(\alpha_{m_i} \wedge \xi_{m_j}) \otimes \sum_{m_p} \tilde{A}_p^{m_i - m_j + 1}\right),$$

where $\tilde{A}_p^{m_i-m_j+1}$ is obtained from $A_p^{m_i-m_j}$ by applying an appropriate multiple of $ad\ x$ and the only values of p that appear are those for which $m_i + m_j + m_p = \text{odd}$. For the inductive step, let $\eta_k = \eta_{k-1} - x_k$, with

$$x_k = (0, x_k'') = \mathcal{P}(\mathcal{D}\eta_{k-1}) \in A^{0,1}\left(\bigoplus_{m=-M+2k}^{M} \mathfrak{g}_m\right),$$

where

$$x_k'' = \sum \iota^{-1}\left(\alpha_{m_{i_k}} \wedge \iota^{-1}\left(\ldots \wedge \iota^{-1}(\alpha_{m_{i_1}} \wedge \xi_{m_j})\right)\right) \otimes \widetilde{A(k)}_{p_k}^{m_{i_1}+\ldots+m_{i_k}-m_j+k},$$

$$A(k)_{p_k}^{m_{i_1}+\ldots+m_{i_k}-m_j+k} := pr_{p_k}^{m_{i_1}+\ldots+m_{i_k}-m_j+k}([e_{m_{i_k}}, \widetilde{A(k-1)}_{p_{k-1}}^{m_{i_1}+\ldots+m_{i_{k-1}}-m_j+k}]),$$

and $\widetilde{A(k)}$ is obtained by applying a suitable multiple of $ad\ x$ to the corresponding $A(k)$. We have to show that (see Eqn 7.1.3) we can define x_{k+1} if x_k is defined, i.e., that $-[\eta_0', x_k'']$ has no nonzero projection on any V_{m_l}. Why is this always the case? Indeed, suppose we have repeated k times the procedure of applying $ad\ \eta_0'$ and then inverting ady. Applying $ad\ \eta_0'$ produces elements of level $m_{i_1} - m_j$ and projections in W_{p_1} (for various $i_1, \ldots$), and inverting $ad\ y$ changes the level to $m_i - m_j + 1$. Repeating these two steps k times we obtain an element of level $m_{i_1} + \ldots + m_{i_k} - m_j + k$ and with projection in some W_{p_k}, where

$$m_{i_1} + m_j + m_{p_1} = \text{odd},\ m_{i_2} + m_{p_1} + m_{p_2} = \text{odd},$$

$$m_{i_3} + m_{p_2} + m_{p_3} = \text{odd},\ m_{i_k} + m_{p_{k-1}} + m_{p_k} = \text{odd}.$$

Adding up these we get

$$\sum_{n=1}^{k} m_{i_n} + m_j + 2\sum_{s=1}^{k-1} m_{p_s} + m_{p_k} = \sum_{r=1}^{k} \text{odd}_r. \tag{7.1.4}$$

If now, after applying $ad\eta_0'$ we land in some V_{m_l}, it must be the case that

$$m_{i_1} + \ldots + m_{i_{k+1}} - m_j + k = m_l,\ m_{i_{k+1}} + m_{p_k} + m_l = \text{odd},$$

and combining these we get

$$\sum_{n=1}^{k} m_{i_n} + 2m_{i_{k+1}} + m_{p_k} - m_j + k = \text{odd}. \tag{7.1.5}$$

Finally, adding (7.1.4) and (7.1.5) we get

$$\text{even} + k = \sum_{r=1}^{k+1} \text{odd}_r,$$

which is impossible since k and $k+1$ are always of opposite parity.

When does this procedure terminate? Since on the k-th iteration we will have

$$x_k = \mathcal{PD}(\eta_{k-1}) \in A^{0,1}\left(\bigoplus_{m=-M+2k}^{M} \mathfrak{g}_{m_{\mathbf{P}}}\right),$$

this last (possibly) nonzero value of x_k will appear when $-M + 2k = M$, that is, $k = M$; respectively, η_M is a solution of the Maurer-Cartan equation (it involves wedge products of at most $M + 1$ terms).

QED

Before proceeding with some corollaries, let me say a few words how does this fit in a slightly broader setup. Suppose $(L^\bullet, d)$ be a DGLA, and say we pick a "Hodge" decomposition $L^i = Z^i \oplus C^i = B^i \oplus \mathscr{H}^i \oplus C^i$, where Z^i are cocycles, B^i are coboundaries and $\mathscr{H}^i \simeq H^i(L)$; again, in the Banach setup, one has to pass to the completion $\hat{L}$ and take the complement to the cocycles, C^i, to be a closed subspace. There is a map $\delta : L^{i+1} \to B^{i+1} \to C^i \to L^i$, where the second map is the inverse of the differential; δ gives a chain homotopy between the identity and the projection $h : L^\bullet \to \mathscr{H}^\bullet$ onto the harmonic elements:

$$Id - h = d\delta + \delta d.$$

In the classical examples the homotopy is actually $\delta = Gd^*$ where d^* is the adjoint of d and G is Green's operator, and it is rarely easy to describe.

Then, one can describe a model – the so called *Kuranishi space* $\mathscr{K}_L$ – for the "local moduli space", i.e., for the (base of a) semi-universal deformation as the zero set of a map

$$\mathscr{H}^1 \to \mathscr{H}^2, \ \omega \mapsto h[F^{-1}(\omega), F^{-1}(\omega)],$$

where F is the Kuranishi map $F : \hat{L}^1 \to \hat{L}^1, \ F(\omega) = \omega + \frac{1}{2}\delta[\omega, \omega]$. The base of the semi-universal deformation – the *Kuranishi family* – can be described locally as

$$Y := \left\{ \eta \ : \ d\eta + \frac{1}{2}[\eta, \eta] = 0, \delta\eta = 0 \right\} \subset \hat{L}^1.$$

I.e., there is an isomorphism of germs of complex spaces between $(Y, 0)$ and $(\mathscr{K}_L, 0)$. The deformation functor associated to the DGLA is pro-represented by the completed (at 0) local ring of the Kuranishi space, and the (germ of the) Kuranishi space is isomorphic to a finitite-dimensional slice for the gauge action (contained in the completed dgla). The space we describe in the proof is a finite-dimensional subspace of L^1 which is (locally) diffeomorphic to the slice.

As discussed earlier, the stable Higgs bundle $(\mathbf{P}, \theta)$ is a smooth point of the moduli space, the deformations are unobstructed ($\mathbb{H}^2(\mathscr{C}_\bullet) = 0$) and the Kuranishi family is in fact universal ([Fuj91]). Actually, our DGLA is formal ([Sim94]). One knows from general deformation theory that the Kuranishi space $\mathscr{K}$ is the quadratic cone in H^1, i.e., the kernel of $\eta \mapsto [\eta, \eta]$, so in this case $\mathscr{K} = H^1 \simeq \mathscr{H}^1$. Then, starting with a harmonic representative of $\mathbb{H}^1(\mathscr{C}_\bullet)$ one could try to construct the Kuranishi family by "Picard iterations" in a pretty much the same way as I did above, but using the map $\delta = Gd^*$ instead of the map $\mathcal{P}$. In a suitable analytic open neighbourhood of zero there would be a unique isomorphism relating the two families. Probably the iterative approach to the Maurer-Cartan equation is historically the oldest – it might very well date back to Kuranishi or Picard and it has been developed in great generality in [HS02]. Of course, in general one has to know whether this procedure gives a "contracting operator", i.e., whether it converges in a suitable sense – or one has to work with formal solutions.

Here are some corollaries of the proof and the construction therein.

Corollary 7.1.1 *Assume $\mathfrak{g} \neq D_{2n}$. Under the holomorphic exponential map the* *canonical holomorphic symplectic form on $A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}})$ pulls back to the canonical holomorphic symplectic form on $B_{\mathfrak{g}} \times B_{\mathfrak{g}}^{\vee}$. In particular, the images of the affine subspaces $\{p\} \times B_{\mathfrak{g}}^{\vee}$ and $B_{\mathfrak{g}} \times \{q\}$ are complex Lagrangian.*

Proof:

Let $\eta = (\eta', \eta'') \in B_{\mathfrak{g}} \times B_{\mathfrak{g}}^{\vee}$, $u, v \in T_{B_{\mathfrak{g}} \times B_{\mathfrak{g}}^{\vee}, \eta}$. Then to compute $(\varepsilon^* \omega^c)_\eta(u, v)$ observe that from our construction, $\varepsilon(\eta + u) - \varepsilon(\eta) = u + \ldots$, where the projections of the $\ldots$ terms onto the various W_i are in $A^{0,1}(\bigoplus_{m > -m_i} W_i \cap \mathfrak{g}_m)$. Respectively, $d\varepsilon_\eta(u) = u + R_1, d\varepsilon_\eta(v) = v + R_2$, $pr_{W_i}(R_j) \in A^{0,1}(\bigoplus_{m > -m_i} W_i \cap \mathfrak{g}_m), j = 1, 2$. From here

$$(\varepsilon^* \omega^c)_\eta(u, v) = \omega^c(u, v) + \omega^c(R_1, v) + \omega^c(u, R_2) + \omega^c(R_1, R_2).$$

Since R_1 has no $(1, 0)$ part, the last term vanishes. Since $\omega^c(R_1, v) = \int_X \varkappa(v', R_1'') - \varkappa(R_1', v_1'')$ and similarly for $\omega^c(u, R_2)$, again for type reasons, the second term under the integral is zero. For the term $\varkappa(v', R_1'')$, recall that the Killing form gives nonzero pairing only between root space corresponding to opposite roots, and opposite pairs of root spaces are contained in pairs $\mathfrak{g}_m$, $\mathfrak{g}_{-m}$ for various m's. We have the pairing $(a, b) = \varkappa(u, v^*)$ from Section 3 and we know that if $\mathfrak{g} \neq D_{2n}$, the various subspaces $W_i \cap \mathfrak{g}_m$ are orthogonal; but then the root space for minus the root corresponding to V_{m_i} is $W_i \cap \mathfrak{g}_{-m_i}$,so $\varkappa(v', R_1'')$ vanishes as well, giving $\varepsilon^* \omega^c = \omega^c = \omega_{can}$.

QED

Corollary 7.1.2 *The image of $B_{\mathfrak{g}} \subset B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee}$ under the holomorphic exponential map is the Hitchin section.*

Proof:

Indeed, if we start the above inductive procedure with $\eta_0 = (\eta_0', 0)$, we get that $\eta_k = \eta_0$ for all k as already x_1 is zero. Then this corresponds to Higgs pairs $(\overline{\partial}_{\mathbf{P}}, \theta + \eta_0')$.

QED

Corollary 7.1.3 *The image of $B_{\mathfrak{g}}^{\vee} \subset B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee}$ under the holomorphic exponential map consists of Higgs pairs $(\mathbf{R}, \theta)$ with ad $\mathbf{R} = \left(\mathcal{A}^0(ad\ \mathbf{P}), \overline{\partial} = \overline{\partial}_{\mathbf{P}} + \sum \xi_i e_{-m_i}\right)$, $\xi_i \in \mathscr{H}^{0,1}(K^{-m_i})$. In particular for $G = SL(n)$ one has $\mathcal{A}^0(\mathbf{R}) \simeq \mathcal{A}^0(\mathbf{P})$, $\mathbf{P} = \underline{Isom}(\mathcal{O}^n, K^{-\frac{n-1}{2}} \oplus \ldots \oplus K^{\frac{n-1}{2}})$, and*

$$\overline{\partial} = \overline{\partial}_{\mathbf{P}} + \sum_{m=1}^{n} \xi_m (-1)^m (2m)! x^{m*},$$

where x is the matrix from Example 3.0.1. In the case $G = SL(2)$ one obtains Higgs pairs of the form (ξ, ϕ), where ξ is an extension

$$0 \longrightarrow K^{-1/2} \longrightarrow \xi \longrightarrow K^{1/2} \longrightarrow 0$$

with Higgs field given by the natural map $1 \in \mathbb{C} \hookrightarrow H^0(\mathscr{E}nd_0 \xi \otimes K)$.

Proof:

Here one only has to notice that $\eta_0 = (0, \eta_0'')$ is already a solution of the Maurer-Cartan equation. The factorials appear from the relation between e_{-m_i} and $e_{m_i}^*$

shown in Lemma 3.0.3.

QED

7.2 Examples

In this subsection I give a couple of examples, namely $\mathfrak{g} = A_1$ and $\mathfrak{g} = A_2$, illustrating the above procedure for obtaining the family $\mathscr{Y}$. One could easily write more of these for the other classical groups.

Consider first that the somewhat degenerate case $\mathfrak{g} = A_1 = \mathfrak{sl}(2)$. Suppose $\eta_0 = (\alpha e_{m_1}, \xi e_{-m_1})$, where $\alpha \in \mathscr{H}^{1,0}(K) = H^0(K^2)$ and $\xi \in \mathscr{H}^{0,1}(K^{-1}) = H^1(K^{-1})$. Here $e_{m_1} = x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_{-m_1} = -2e^*_{m_1} = -2y = -2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $\frac{1}{2}[\eta_0, \eta_0] = \alpha \wedge \xi[e_{m_1}, e_{-m_1}]$, so $x_1 = \mathcal{P}(\frac{1}{2}[\eta_0, \eta_0]) = -2\iota^{-1}(\alpha \wedge \xi)e_{m_1}$. The next iteration would involve a commutator with a multiple of e_{m_1} and would be zero. Thus the solution is $\eta = \eta_1 = \eta_0 - x_1$, that is, the family is

$$\mathscr{Y} = \left\{(\alpha e_{m_1}, \xi e_{-m_1} + 2\iota^{-1}(\alpha \wedge \xi))e_{m_1}),\ \alpha \in \mathscr{H}^{1,0}(K), \xi \in \mathscr{H}^{0,1}(K^{-1})\right\}.$$

This corresponds to Higgs pairs with holomorphic structure $\overline{\partial} + \xi e_{-m_1} + 2\iota^{-1}(\alpha \wedge \xi)e_{m_1}$ and Higgs field $\theta + \alpha e_{m_1}$.

For the case of $\mathfrak{g} = A_2 = \mathfrak{sl}(3)$ we proceed similarly. We start with

$$\eta_0 = (\alpha_1 e_{m_1} + \alpha_2 e_{m_2}, \xi_1 e_{-m_1} + \xi_2 e_{-m_2}),$$

and compute

$$\frac{1}{2}[\eta_0, \eta_0] = \alpha_1 \wedge \xi_1 [e_{m_1}, e_{-m_1}] + \alpha_1 \wedge \xi_2 [e_{m_1}, e_{-m_2}] + \alpha_2 \wedge \xi_1 [e_{m_2}, e_{-m_1}] + \alpha_2 \wedge \xi_2 [e_{m_2}, e_{-m_2}].$$

Then we check that the commutators that appear are as follows:

$$[e_{m_1}, e_{-m_1}] = -2ad\ y(e_{m_1}) = -2h, \qquad [e_{m_1}, e_{-m_2}] = -4ad^3\ y(e_{m_2}),$$

$$[e_{m_2}, e_{-m_1}] = -2ad\ y(e_{m_2}), \qquad [e_{m_2}, e_{-m_2}] = 48ad\ y(e_{m_1}).$$

From here

$$\frac{1}{2}[\eta_0, \eta_0] =$$

$$-2\left\{(\alpha_1 \wedge \xi_1 - 24\alpha_2 \wedge \xi_2)ad\ y(e_{m_1}) + 2\alpha_1 \wedge \xi_2 ad^3\ y(e_{m_2}) + \alpha_2 \wedge \xi_1 ad\ y(e_{m_2})\right\},$$

so that

$$-x_1 = -\mathcal{P}\left(\frac{1}{2}[\eta_0, \eta_0]\right) =$$

$$2\iota^{-1}\left\{(\alpha_1 \wedge \xi_1 - 24\alpha_2 \wedge \xi_2)e_{m_1} + \alpha_2 \wedge \xi_1 e_{m_2} + 2\alpha_1 \wedge \xi_2 ad^2\ y(e_{m_2})\right\}$$

and $\eta_1 = \eta_0 - x_1$. For the next step we need $d\eta_1 + \frac{1}{2}[\eta_1, \eta_1] = -[\eta_0', x_1'']$. This has only one term, namely,

$$-[\eta_0', x_1''] = 4\alpha_1 \wedge \iota^{-1}(\alpha_2 \wedge \xi_2)[e_{m_1}, ad^2\ y(e_{m_2})] = -24\alpha_1 \wedge \iota^{-1}(\alpha_2 \wedge \xi_2)ad\ y(e_{m_2}) \Rightarrow$$

$$-x_2 = 24\iota^{-1}(\alpha_1 \wedge \iota^{-1}(\alpha_2 \wedge \xi_2))e_{m_2}.$$

So the solution is $\eta_1 - x_2 = \eta_0 - x_1 - x_2$, i.e., $\mathscr{Y} = \{(\eta', \eta'')\}$, where

$$\eta' = \alpha_1 e_{m_1} + \alpha_2 e_{m_2},$$

$$\eta'' = \xi_1 e_{m_1}^{-1} + \xi_2 e_{m_2}^{-2} + 2\iota^{-1}(\alpha_1 \wedge \xi_1 - 24\alpha_2 \wedge \xi_2)e_{m_1} +$$

$$2\iota^{-1}(\alpha_2 \wedge \xi_1 + 12\alpha_1 \wedge \iota^{-1}(\alpha_2 \wedge \xi_2))e_{m_2} + 2\iota^{-1}(\alpha_1 \wedge \xi_2)e_{m_2}^{0}$$

For the explicit form of the matrices appearing here, see the Appendix (10).

Chapter 8

Getting the Germ

8.1 The Uniformisation Oper

The goal of this section is to identify opers as a subspace of the tangent space to M_{DR} at a preferred point.

Let $q = [(\mathbf{Q}, D_0)] \in M_{DR}$ be the oper corresponding by the non-abelian Hodge theorem to the system of Hodge bundles $p = [(\mathbf{P}, \theta)]$ from the previous section. Denote by $\mathbf{Q}$ the canonical principal bundle (4.2.2) underlying the oper and by D_0 the holomorphic connection. The choice of D_0 turns the affine space (4.2.3) into a vector space which we identify as a subspace of $T_q M_{DR}$ using the de Rham sequence of $ad\mathbf{Q} \equiv \mathfrak{g}_{\mathbf{Q}}$. Consider the holomorphic de Rham resolution of the local system $ad\ \mathbf{Q}^{\nabla}$.

There is a short exact sequence of complexes, the tangent sequence of the morphism $M_{DR} \to Bun_G$:

$$0 \longrightarrow \mathfrak{g}_{\mathbf{Q}} \otimes K[-1] \longrightarrow \mathscr{D}_\bullet \longrightarrow \mathfrak{g}_{\mathbf{Q}} \longrightarrow 0.$$

The plan is to take the long exact sequence in hypercohomology and look at the kernel of the map $\mathbb{H}^1(\mathscr{D}_\bullet) \to H^1(\mathfrak{g}_{\mathbf{Q}})$. This gives for the long exact sequence:

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow H^0(\mathfrak{g}_{\mathbf{Q}}) \xrightarrow{D} H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K) \\ \quad \longrightarrow \mathbb{H}^1(\mathscr{D}_\bullet) \longrightarrow H^1(\mathfrak{g}_{\mathbf{Q}}) \xrightarrow{H^1(D)} H^1(\mathfrak{g}_{\mathbf{Q}} \otimes K) \longrightarrow 0 \end{array}, \tag{8.1.1}$$

since

1. $\mathbb{H}^0(\mathfrak{g}_{\mathbf{Q}} \otimes K[-1]) = 0$, $\mathbb{H}^1(\mathfrak{g}_{\mathbf{Q}} \otimes K[-1]) = H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K)$, $\mathbb{H}^2(\mathfrak{g}_{\mathbf{Q}} \otimes K[-1]) = H^1(\mathfrak{g}_{\mathbf{Q}} \otimes K)$

2. $\mathbb{H}^0(\mathscr{D}_\bullet) = ad\mathbf{Q}^\nabla = \mathfrak{z} = 0$: the local system is irreducible by 4.2.1.

3. $\mathbb{H}^0(\mathfrak{g}_{\mathbf{Q}}) = H^0(\mathfrak{g}_{\mathbf{Q}}) \neq 0$ – the bundle $\mathbf{Q}$ is unstable; $\mathbb{H}^1(\mathfrak{g}_{\mathbf{Q}}) = H^1(\mathfrak{g}_{\mathbf{Q}})$.

All morphisms in Eqn (8.1.1) have the obvious descriptions in terms of the Dolbeault resolution of $\mathscr{D}_\bullet$,

$$\begin{array}{ccc} A^{0,1}(\mathfrak{g}_{\mathbf{Q}}) & \xrightarrow{D} & A^{1,1}(\mathfrak{g}_{\mathbf{Q}}) \\ \bar{\partial} \uparrow & & \bar{\partial} \uparrow \\ A^{0,0}(\mathfrak{g}_{\mathbf{Q}}) & \xrightarrow{D} & A^{1,0}(\mathfrak{g}_{\mathbf{Q}}) \end{array}.$$

For instance the map $H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K) \to \mathbb{H}^1(\mathscr{D}_\bullet)$ is given by $[A'] \mapsto [(A', 0)]$ (suppressing the isomorphism $A^0(\mathfrak{g}_{\mathbf{Q}} \otimes K) \simeq A^{1,0}(\mathfrak{g}_{\mathbf{Q}})$) , the map $\mathbb{H}^1(\mathscr{D}_\bullet) \to H^1(\mathfrak{g}_{\mathbf{Q}})$ is given by $[(A', A'')] \mapsto [A'']$, etc.

Notice that the differentials *anti-commute*, and that the associated total complex is the de Rham complex of $\mathfrak{g}_{\mathbf{Q}}$, $(A^{\bullet}(\mathfrak{g}_{\mathbf{Q}}), \nabla = \overline{\partial}_{\mathfrak{g}_{\mathbf{Q}}} + D)$.

The long exact sequence then becomes

$$0 \longrightarrow \tfrac{H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K)}{D(H^0(\mathfrak{g}_{\mathbf{Q}}))} \longrightarrow \mathbb{H}^1(\mathscr{D}_{\bullet}) \longrightarrow H^1(\mathfrak{g}_{\mathbf{Q}}) \xrightarrow{H^1(D)} H^1(\mathfrak{g}_{\mathbf{Q}} \otimes K) \longrightarrow 0\,. \quad (8.1.2)$$

We need a couple of lemmas to identify the terms in the exact sequence.

Lemma 8.1.1 *Let $\mathscr{E}$ be a vector bundle on a compact complex curve, X. Let $D : \mathscr{E} \to \mathscr{E} \otimes K$ be a holomorphic connection on $\mathscr{E}$. The connection induces maps on cohomology, $H^0(D) : H^0(\mathscr{E}) \to H^0(\mathscr{E} \otimes K)$ and $H^1(D) : H^1(\mathscr{E}) \to H^1(\mathscr{E} \otimes K)$. Then, if $D^{\vee}$ is the induced connection on $\mathscr{E}^{\vee}$, we have that Serre duality induces*

$$H^0(D)^{\vee} = H^1(D^{\vee}) : H^1(\mathscr{E}^{\vee}) \to H^1(\mathscr{E}^{\vee} \otimes K).$$

In our case, $\mathscr{E} = \mathfrak{g}_{\mathbf{Q}}$, and by the trace we identify $\mathscr{E} = \mathscr{E}^{\vee}, D = D^{\vee}$ naturally, so

$$\ker(H^1(D)) = \left(\frac{H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K)}{D(H^0(\mathfrak{g}))}\right)^{\vee}$$

and

$$0 \longrightarrow \tfrac{H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K)}{D(H^0(\mathfrak{g}_{\mathbf{Q}}))} \longrightarrow \mathbb{H}^1(\mathscr{D}_{\bullet}) \longrightarrow \left(\tfrac{H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K)}{D(H^0(\mathfrak{g}_{\mathbf{Q}}))}\right)^{\vee} \longrightarrow 0\,. \quad (8.1.3)$$

Lemma 8.1.2 *If $B_{\mathfrak{g}}$ is the Hitchin base, we have that*

$$\frac{H^0(\mathfrak{g}_{\mathbf{Q}} \otimes K)}{D(H^0(\mathfrak{g}_{\mathbf{Q}}))} = B_{\mathfrak{g}}.$$

Proof: This is already clear from 4.2.2 and 4.2.3 and Kostant's isomorphism $V \simeq \mathrm{Spec}(Sym\ \mathfrak{g}^\vee)^G$ obtained by $V \ni v \mapsto v + i(y_0) \in \mathfrak{g}$. For a direct proof in the case $G = SL(2)$ see the next subsection. **QED**

Combining these remarks together, we obtain

Proposition 8.1.1 *Let $q = [(\mathbf{Q}, D_0)] \in M_{DR}$ be the uniformisation oper. Then*

$$0 \longrightarrow B_{\mathfrak{g}} \longrightarrow \mathbb{H}^1(\mathscr{D}_\bullet) \longrightarrow B_{\mathfrak{g}}^\vee \longrightarrow 0 \tag{8.1.4}$$

and, consequently $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) = B_{\mathfrak{g}} \subset T_{q,M_{DR}} = \mathbb{H}^1(\mathscr{D}_\bullet) \simeq B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^\vee$.

Observe that the splitting of the sequence is non-canonical but the inclusion is. In fact this feature is already present in case of line bundles.

8.2 An Example from Scratch

Here we give a proof from first principles of some of the above statements in the case $G = SL(2)$.

Let $V = J^1(K^{-1/2})$ and let $D : V \to V \otimes K$ be any $SL(2)$- oper, with $\nabla = \overline{\partial}_V + D$, and $V^\nabla = ker(V \to V \otimes K)$ being the corresponding locally constant sheaf.The bundle V sits in the unique nontrivial extension, the jet sequence of $K^{-1/2}$:

$$0 \longrightarrow K^{1/2} \xrightarrow{i} V \xrightarrow{p} K^{-1/2} \longrightarrow 0.$$

Notice that $V = V^\vee$. Also $\mathbf{Q} = \underline{Isom}(V, \mathcal{O}^{\oplus 2})$. Finally, we have

Lemma 8.2.1 *The natural map* $H^0(X, K^2) \hookrightarrow H^0(\mathscr{E}nd_0 V \otimes K)$ *has the property that for any* $SL(2)$*-oper,* D*,*

$$H^0(X, K^2) \cap D(End\ _0V) = (0).$$

Moreover, the subspace $D(End_0 V) \subset H^0(\mathscr{E}nd_0 V \otimes K)$ *is independent of* D*.*

Finally, by Hirzebruch-Riemann-Roch

$$\dim \frac{H^0(\mathscr{E}nd_0 V \otimes K)}{D(End\ _0V)} = -\chi(\mathscr{E}nd_0 V) = 3(g-1) = \dim H^0(X, K^2)$$

so we obtain a natural isomorphism $H^0(X, \mathscr{E}nd_0 V \otimes K) = H^0(X, K^2) \oplus D(End_0 V)$*.*

Proof:

The natural map $\alpha : H^0(K^2) \hookrightarrow H^0(\mathscr{E}nd_0 V \otimes K) = Hom(V, V \otimes K)$ is given by

$$\alpha : \eta \mapsto (v \mapsto (i \otimes 1)(\eta \otimes p(v))),$$

where the maps i, p are respectively the inclusion and projection maps in the 1-jet sequence of $K^{-1/2}$. Suppose $0 \neq \phi \in End\ _0V$ and $D(\phi) = \alpha(\eta), \eta \in H^0(X, K^2)$. Let $e \neq 0$ be a local section of $K^{1/2}$, $e \in \Gamma_U(K^{1/2} \subset V)$. Then $p(e) = 0$, hence $\alpha(\eta)(v) = 0$ and consequently

$$D(\phi)(e) = 0.$$

Recalling that here D is actually the induced connection on the endomorphisms, we have

$$D(\phi)(e) = D(\phi(e)) - \phi(D(e)) = 0.$$

But we know that $EndV = \mathbb{C} \oplus H^0(X, K) = \mathbb{C} \oplus \mathscr{E}nd_0 V$, where $\mathbb{C}$ sits inside as multiples of the identity and $H^0(X, K) = \mathscr{E}nd_0 V$ are nilpotent endomorphisms, $\omega : v \mapsto (i)(\omega \otimes p(v)), \omega \in H^0(K)$. Consequently, $\phi(e) = 0$ and $D(\phi(e)) = 0$, so

$$\phi(D(e)) = 0$$

and thus $D(e) \in \Gamma_U(K^{1/2})$. But $K^{1/2}$is not a horizontal subbundle. I.e., D is an oper and by strict Griffiths transversality induces an isomorphism $\overline{D} : K^{1/2} \simeq (V/K^{1/2}) \otimes K$, contradiction.

The second part is a local computation, but the idea is the following. Global endomorphisms, ϕ, are given locally by upper triangular matrices, and so is $D - D_0$, where D_0 is the oper corresponding to uniformisation. But then because of nilpotence their commutator is zero and $(D - D_0)(\phi) = 0$.

Here are the details. Locally on an open $U \subset X$, $V_U \simeq K_U^{1/2} \oplus K_U^{-1/2}$. Let $\phi \in A^0(\mathscr{E}nd_0 V), \overline{\partial}\phi = 0$, i.e., $\phi \in Z^0_{\overline{\partial}}(\mathscr{E}nd_0 V) = End_0 V$ (here $\overline{\partial} = \overline{\partial}_{\mathscr{E}nd_0 V}$). When restricted to U, ϕ looks like $\phi_U = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$, $\omega \in A^0_U(K), \overline{\partial}\omega = 0$ and for degree reasons $\partial(\omega) = 0$. Next, the connection looks locally like

$$D = \partial + \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \delta \in Z^0_{\overline{\partial}}(K^2).$$

Then

$$D(\phi) = \partial\phi + \left[\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}.$$

The answer is independent of $\delta \in H^0(X, K^2)$.

QED

8.3 The Infinitesimal Correspondence

Now we can assemble the various pieces which we have prepared so far. The hyperkähler structure on the moduli space M_{Dol} is given by $K \in \mathrm{End}_{\mathbb{R}} A^1(\mathfrak{g}_{\mathbf{P}})$, where

$$K(u, v) = (-v^*, u^*), \ u \in A^{1,0}(\mathfrak{g}_{\mathbf{P}}), \ v \in A^{0,1}(\mathfrak{g}_{\mathbf{P}}).$$

Next, the constant scalar curvature metric on X induces an hermitian metric on K^n for all n, and, consequently, on $H^{p,q}(X, K^n)$. The map $A : H^{p,q}(X, K^n) \to (H^{p,q}(X, K^n))^\vee$, is, after applying Serre duality, given by the Hodge $\star$ operator on harmonic representatives. Again, such a map exists because we have endowed the cohomology with an hermitian metric. Recalling the discussion about compact real forms in Section 3, we have that $u^* = -i \star u$ and $v^* = i \star v$ and hence

$$K([u], [v]) = (-i[\star u], -i[\star v]).$$

Theorem 8.3.1 *Let G be a simple complex Lie group. Let ζ be a theta - characteristic, let $\mathbf{P}$ be the principal G-bundle obtained from $\underline{Isom}(\mathcal{O}^{\oplus 2}, \zeta \oplus \zeta^{-1})$, by extension of structure group via a choice of an $\mathfrak{sl}(2)$-triple and endowed with a regular nilpotent Higgs field θ – both the triple and the Higgs field chosen as in the* *previous sections. Let $B = H^0(X, \mathfrak{t}/W \otimes K)$ be the Hitchin base, and suppose a choice of invariant polynomials is made. Then the infinitesimal deformations of $[(\mathbf{P}, \theta)]$ corresponding to G-opers are given by the subspace $\Gamma \subset B_{\mathfrak{g}} \oplus B_{\mathfrak{g}}^{\vee}$,*

$$\Gamma = \left\{ \left(\begin{array}{c} w \\ \\ -iA(w) \end{array} \right), w \in B_{\mathfrak{g}} \right\} = graph \left\{ [\alpha] \mapsto i[\star\alpha], \alpha \in \oplus_p \mathscr{H}^{1,0}(K^{m_p}) \right\}.$$

After a choice of basis of each $V_{m_p} \simeq \mathbb{C}$ and compatible embedding $\mathbb{H}^1(\mathscr{C}_\bullet) \subset A^1(\mathfrak{g}_{\mathbf{P}})$, Γ maps to $A^{1,0}(\mathfrak{g}_{\mathbf{P}}) \oplus A^{0,1}(\mathfrak{g}_{\mathbf{P}})$ with image

$$\left\{ \left(\sum_{p=1}^{l} \alpha_p e_{m_p}; \sum_{p=1}^{l} \frac{(-1)^{m_p}}{(2m_p)!} i \star \alpha_p e_{m_p}^{-m_p} \right), \oplus_p \alpha_p \in \oplus_p \mathscr{H}^{1,0}(K^{m_p}) \right\}.$$

Consequently, the subvariety $\varepsilon(\Gamma) \subset M_{Dol}(G)$ agrees up to first order with $\tau(\mathcal{O}\mathfrak{p}(X))$.

Proof:

By now this is already clear: it follows from Example 5.3.1 and the Lemma 3.0.3.

QED

The map $B_{\mathfrak{g}} \to \Gamma \subset \mathbb{H}^1(\mathscr{C})$ (composed with ε) "wants to be" the Kodaira-Spencer map of the "family" $\tau(\mathcal{O}\mathfrak{p}(X))$; the latter, of course, does not make sense, since this is only a "real family".

Corollary 8.3.1 *In the case of $G = SL(2, \mathbb{C})$, the subspace Γ is given by $\Gamma = \left\{ \left(\alpha e_{m_1}; -\frac{i}{2} \star \alpha e_{m_1}^{-1} \right) \right\}$ and, correspondingly, the germ of $\tau(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X))$ is given by*

$$\left\{ \left(\alpha e_{m_1}; -\frac{i}{2} \star \alpha e_{m_1}^{-1} - i\iota^{-1}(\star\alpha \wedge \alpha) e_{m_1} \right) \right\},$$

where $\alpha \in \mathscr{H}^{1,0}(K) \simeq H^0(X, K^2)$ and $\star$ is the Hodge star. This means that for each value of the "parameter", α, we have a Higgs bundle whose underlying complex *vector bundle is* $A^0(K^{1/2}\oplus K^{-1/2})$*, endowed with holomorphic structure* $\overline{\partial}-\frac{i}{2}\star\alpha e_{m_1}^{-1}-i\iota^{-1}(\alpha\wedge\star\alpha)e_{m_1}$ *(where* $\overline{\partial}$ *is the holomorphic structure of direct sum) and whose Higgs field is* $\theta+\alpha e_{m_1}$*.*

Corollary 8.3.2 *In the case of* $G=SL(3)$*, the subspace* Γ *is given by*

$$\Gamma=\left\{\left(\alpha_1 e_{m_1},\alpha_2 e_{m_2};-\frac{i}{2}\star\alpha_1 e_{m_1}^{-1},\frac{i}{24}\star\alpha_2 e_{m_2}^{-2}\right)\right\},$$

$\alpha_1\in\mathscr{H}^{1,0}(K)$*,* $\alpha_2\in\mathscr{H}^{1,0}(K^2)$ *and, correspondingly, the germ of* $\tau(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X))$ *is*

$$\eta'\alpha_1 e_{m_1}+\alpha_2 e_{m_2};$$

$$\eta''=-\frac{i}{2}\star\alpha_1 e_{m_1}^{-1}+\frac{i}{24}\star\alpha_2 e_{m_2}^{-2}--i\iota^{-1}(\alpha_1\wedge\star\alpha_1+2\alpha_2\wedge\star\alpha_2)e_{m_1}+$$
$$i\iota^{-1}(-\alpha_2\wedge\star\alpha_1+\alpha_1\wedge\iota^{-1}(\alpha_2\wedge\star\alpha_2))e_{m_2}+\frac{i}{12}\iota^{-1}(\alpha_1\wedge\star\alpha_2)e_{m_2}^{0}.$$

Chapter 9

Towards the Complete Solution

Due to its complicated analytic origin, the non-abelian Hodge theorem often turns structures of algebraic nature in M_{DR} or M_{Dol} into ones of analytic nature in M_{Dol} or M_{DR} and seems to exchange objects with simple description with more complicated ones. Even though the space of opers, $\mathcal{O}\mathfrak{p}_G(X)$ is as nice as it could be – a vector space, with a simple representation-theoretic description, its "classical limit", the subvariety $\tau(\mathcal{O}\mathfrak{p}(X))$ seems to be very transcendental in nature. However, there is a piece of analytic data luriking inside the opers – the uniformisation oper which happens to be also a variation of Hodge structure – and one would expect it to bring some simplification into the description of $\tau(\mathcal{O}\mathfrak{p})$. In order to follow that track, I plan to implement a remark of C.Simpson made in [Sim97], which I briefly recall below.

We restrict ourselves now to the case $G = GL(n)$. Let us fix a point $x \in X$

and denote by $\mathbb{C}\widehat{\pi_1} := \mathbb{C}\widehat{\pi_1(X,x)}$ the formal completion of the group algebra of the fundamental group at the augmentation ideal. One can construct from it a family (over $\mathbb{C}$) of $\mathbb{C}$-algebras with $\mathbb{C}^\times$-action. This is done by endowing $\mathbb{C}\widehat{\pi_1}$ with the Morgan-Hain Hodge filtration and taking the associated Rees module. The group algebra $\mathbb{C}\widehat{\pi_1}$ defines a functor Art $\rightarrow Sets^{op}$ which is pro-representable by the formal completion of the representation space at the trivial representation. By the Rees module construction we obtain from it the formal completion of the space of λ-connections along a section.

This construction can be modified to work in case one is looking not at the trivial representation, but at a representation $\rho : \pi_1 \rightarrow GL(n)$ which is a complex variation of Hodge structure (assume it to be in the smooth locus). In that case one looks at the completed local ring, $\hat{\mathcal{O}}_{M_{DR},\rho}$ with its Hodge structure – see a forthcoming paper of Simpson and Eyssidieux for details on the latter – and takes the Rees module to obtain a $\mathbb{C}$-family of formal schemes. The crucial thing now is the presence of polarisation, which can be used to glue the $\mathbb{C}$-family to a family over $\mathbb{P}^1$: the completion of the twistor space along a section. As mentioned earlier, the normal bundle to a twistor line is $\mathcal{O}^{\oplus 2n}$, so the nearby sections can be parametrised by the product of two fibres of the family. Now, if we take those to be antipodal fibres, we can write the involution whose fixed points are the twistor lines. On the other hand, if we identify the product with $M_{DR} \times M_{Dol}$, the fixed locus of the involution becomes the graph of the diffeomorphism $\tau : M_{DR} \rightarrow M_{Dol}$. The only data needed for this construction is the polarisation (related to the uniformisation metric in our case) and the Hodge filtration – which should be expressable in terms of iterated integrals, i.e., monodromy data.

In this light, there seems to be a natural candidate for the family $\tau(\mathcal{O}\mathfrak{p}(X))$. Namely, we constructed in the previous section the germ of the family by applying the exponential map ε to $\Gamma \subset B \oplus B^\vee$. However, by Hitchin's result from [Hit87a], $\tau^{-1}(B)$ gives a realisation of Teichmüller space, that is, each point of B provides us with unique metric of constant negative scalar curvature. We can, then, consider the Hodge star operator constructed using that metric and exponentiate the result. It would be interesting to determine if this is indeed the case.

Another question which arises from the existence of our exponential map is the following: What is the image of $\varepsilon(B^\vee)$ in the Betti space M_B?

There is a holomorphic affine bundle $P \to \mathcal{T}_g$, whose fibre over a point $[Y] \in \mathcal{T}_g$ is the space, P_Y, of (complex) projective structures on Y. Bers uniformisation identifies P with $T^\vee\mathcal{T}_g$ (after choosing a point $X \in \mathcal{T}_g$ and a marking of the smooth surface underlying X). There exists a local biholomorphic embedding $P \hookrightarrow M_B$ ([Gol04]), compatible with the complex-symplectic form. The choice of Fuchsian coordinate determines a *real* section of P ([Gun66],[Tju78]), while opers are a fibre of P over the marked point X. Our copy of $B^\vee$ will become a real subvariety of M_B, lying in the image of $T^\vee\mathcal{T}_g$. Is it a section of $T^\vee\mathcal{T}_g$ and what is its relation to uniformisation ?

Chapter 10

Appendix

I am going to give here some simple facts that we are using repeatedly, as well as some examples. In the whole discussion $\mathfrak{g}$ will be a simple complex Lie algebra.

First off, some of our Lie-algebraic constructions require a choice of Chevalley generators, so I recall what these are; in fact, the choices associated with constructing the Hitchin section can be made canonical, as shown in the recent work of Ngo [Ngo08], but we want to treat simultaneously the Hitchin section and the opers, and I have not analysed in full detail to what extent my constructions can be made choice-free.

The *Chevalley generators* are generators of the Lie algebra, $\{e_i, f_i, h_i, i = 1 \dots l\}$, which satisfy

$$[e_i, f_j] = \delta_{ij} h_j,\ [h_i, e_j] = N_{ij} e_j,\ [h_i, f_j] = -N_{ij} f_j,\ [h_i, h_j] = 0,$$

where N_{ij} is the Cartan matrix. Additionally, the generators satisfy the Serre relations,

$$(ad\ e_i)^{-N_{ij}+1}e_j = 0,\ (ad\ f_i)^{-N_{ij}+1}f_j = 0, i \neq j.$$

In Section 3 we discussed the decomposition of $\mathfrak{g}$ under the adjoint action of a principal $\mathfrak{sl}(2)$-subalgebra. Here are some examples related to this decomposition.

Example 10.0.1 *The Lie algebra* $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ *decomposes as* $\mathfrak{sl}(3,\mathbb{C}) = W_1 \oplus W_2 = \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_{-2}$, *where* W_2 *is the span of*

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

and

$$W_1 = span \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

The principal grading is

$$\mathfrak{g}_{-2} = span \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},\ \mathfrak{g}_{-1} = span \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, a, b \in \mathbb{C} \right\},$$

$$\mathfrak{g}_0 = \mathfrak{t} = span \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$\mathfrak{g}_1 = span\left\{\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, a, b \in \mathbb{C}\right\},\ \mathfrak{g}_2 = span\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}.$$

The grading of the Borel subalgebra and the spaces V_{m_i} are, respectively,

$$\mathfrak{b}_0 = \mathfrak{t}, \mathfrak{b}_1 = \mathfrak{g}_1\ , \mathfrak{b}_2 = \mathfrak{g}_2$$

$$V_1 = span\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right\}, V_2 = \mathfrak{b}_2, V = Z(x) = V_1 \oplus V_2.$$

We use several times a basis for the Lie algebra obtained – after a choice of an $\mathfrak{sl}(2)$-triple – by choosing a basis for the subspaces V_{m_i} and then applying successively the operator $ad\ y$. Here is an example of this construction.

Example 10.0.2 *Consider the Lie algebra $\mathfrak{g} = A_2 = \mathfrak{sl}(3, \mathbb{C})$. Then, if one chooses the basis vector e_{m_1} of V_{m_1} as in Example 3.0.1 and a basis vector $e_{m_2} := (e_{m_1})^2$ of V_{m_2}, one obtains the following basis for $\mathfrak{g}$:*

$$e_{m_1} \equiv e_{m_1}^{m_1} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix},\ e_{m_1}^{0} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = ad\ y(e_{m_1}),$$

$$e_{m_1}^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = ad^2\ y(e_{m_1}),\ e_{m_2} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e^1_{m_2} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix} = ad\ y(e_{m_2}),\ e^0_{m_2} = 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = ad^2\ y(e_{m_2}),$$

$$e^{-1}_{m_2} = \begin{pmatrix} 0 & 0 & 0 \\ -12 & 0 & 0 \\ 0 & 12 & 0 \end{pmatrix} = ad^3\ y(e_{m_2}),\ e^{-2}_{m_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 24 & 0 & 0 \end{pmatrix} = ad^4\ y(e_{m_2}).$$

Next, let $\varkappa$ be the Killing form. The root and weight lattices sit inside $\mathfrak{g}^\vee$ and their duals are the coweight and coroot lattices in $\mathfrak{g}$:

$$\mathbf{root} \subset \mathbf{weight} \subset \mathfrak{g}^\vee$$

$$\mathbf{coroot} \subset \mathbf{coweight} \subset \mathfrak{g}.$$

By definition, $\mathbf{coroot} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{weight}, \mathbb{Z})$ and $\mathbf{coweight} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{root}, \mathbb{Z})$. If one uses the Killing form to identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$, one can work with four lattices in a single vector space. Also, $\varkappa$ allows us to construct a coroot from every root. Indeed, recall that the identification $\mathfrak{g} \simeq \mathfrak{g}^\vee$ is given by $v \mapsto \varkappa(v, \bullet)$. Given a root $\alpha \in \mathbf{root} \subset \mathfrak{g}^\vee$, let t_α be the corresponding element of $\mathfrak{g}$, determined uniquely by $\alpha(v) = \varkappa(t_\alpha, v), \forall v \in \mathfrak{g}$. We obtain from it a coroot $h_\alpha = \frac{2t_\alpha}{\varkappa(t_\alpha, t_\alpha)} \in \mathbf{coroot} \subset \mathfrak{g}$, which we may also denote by $\alpha^\vee$. Notice that this coroot is uniquely determined by the condition $\varkappa(h_\alpha, \bullet) = \frac{2\alpha}{\varkappa(\alpha,\alpha)}$. Here we are using $\varkappa$ to denote also the Killing form on $\mathfrak{g}^\vee$ induced by the one on $\mathfrak{g}$, i.e., $\varkappa(\alpha, \beta) = \varkappa(t_\alpha, t_\beta)$. Notice that in the physics literature it is often $\frac{2\alpha}{\varkappa(\alpha,\alpha)}$ that is called "the coroot corresponding to α" (and is denoted by $\alpha^\vee$).

If $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root space corresponding to a root α, the space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t} \subset \mathfrak{g}$ is one dimensional and we can define $h_\alpha \equiv \alpha^\vee$ as the unique vector in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, satisfying the condition $\alpha(h_\alpha) = 2$. If we have chosen a set of positive roots and a basis of the root system (i.e., a collection of simple roots), say, $\{\alpha_i\}, i = 1..l$, then the fundamental (dominant) weights $\{\lambda_i\}$ are determined by the condition $\lambda_i(h_{\alpha_j}) = \delta_{ij}$, that is, $2\frac{\varkappa(\alpha_j,\lambda_i)}{\varkappa(\alpha_j,\alpha_j)} = \delta_{ij}$. Notice that one has to invert the Cartan matrix in order to obtain the weights in terms of the roots, see Humphreys [Hum78] for a table. Similarly, the fundamental coweights $\{\omega_i\}$ are detrmined by $\alpha_i(\omega_j) = \delta_{ij}$. The *Weyl vector*, $\rho \in \mathfrak{g}^\vee$ is defined as

$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha \in \mathfrak{g}.$$

It need not be a root, but it is a weight, and by [Hum78], 13.3, Lemma A, ρ equals the sum of the fundamental weights. Similarly, the *dual Weyl vector* is defined as half of the sum of the positive coroots,

$$\rho^\vee = \frac{1}{2}\sum_{\alpha^\vee>0} \alpha^\vee$$

and it equals the sum of the fundamental coweights. Sometimes $2\rho^\vee$ is called *the level vector.*

Example 10.0.3 *Consider $\mathfrak{g} = A_2 = \mathfrak{sl}(3)$, with positive roots $\alpha_{ij} = x_{ii} - x_{jj}, i < j$ and $x_{ij}(A) := A_{ij}$. The simple roots are $\alpha_1 = \alpha_{12}$ and $\alpha_2 = \alpha_{23}$. The (positive) coroots are $h_{ij} = diag(\ldots 1 \ldots - 1), i < j$ with 1 at i-th position and -1 at j-th, the other entries being zero. The simple roots are $h_1 = h_{12}$ and $h_2 = h_{23}$.*

The fundamental coweights are $\omega_1 = \frac{1}{3}(2h_1 + h_2) = \frac{1}{3}diag(2, -1, -1)$ *and* $\omega_2 = \frac{1}{3}(h_1 + 2h_2) = \frac{1}{3}(1, 1, -2)$. *Notice that* $\omega_1 + \omega_2 = \frac{1}{3}diag(3, 0, -3) = diag(1, 0, -1)$. *Then the dual Weyl vector is*

$$\rho^\vee = \frac{1}{2}(h_{12} + h_{23} + h_{13}) = diag(1, 0, -1) = \omega_1 + \omega_2 = h_1 + h_2.$$

In general, the expression of $\rho^\vee$ *in terms of simple coroots is more complicated. For instance, if* $\mathfrak{g} = A_3$,

$$\rho^\vee = \frac{1}{2}(h_{12} + h_{13} + h_{14} + h_{23} + h_{24} + h_{34})$$

$$= \frac{1}{2}diag(3, 1, -1, -3) = \omega_1 + \omega_2 + \omega_3 = \frac{3}{2}h_{12} + \frac{4}{2}h_{23} + \frac{3}{2}h_{34}.$$

For A_l, the coefficients of the expansion of $\rho^\vee$ in terms of simple coroots are $\frac{1}{2}k(l - k + 1), k = 1 \ldots l$, for C_l it is $\frac{1}{2}k(2l - k), k = 1 \ldots l$, etc. One can consult [Hum78] or [FS97] for tables of all quantities appearing here.

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