THE MAPPING CLASS GROUP AND SPECIAL LOCI IN MODULI OF CURVES

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ABSTRACT

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The general problem this thesis is concerned with is that of studying the subvarieties of the moduli space $\mathfrak{M}_{g,[n]}$ corresponding to curves with extra automorphisms. A typical curve with marked points has no automorphisms; but some do, depending upon the choice of curves and position of marked points. This gives us certain subvarieties in the moduli space. For Riemann surfaces, these subvarieties are characterized by specifying a finite group of mapping-classes whose action on a curve is fixed topologically. This thesis builds upon previous work by González-Díez, Harvey and Schneps. González-Díez and Harvey [GH92] considered these irreducible subvarieties for genus $g \geq 2$ curves without marked points over the complex numbers, where they have studied the coarse moduli space for surfaces with a specified automorphism group. Cornalba [Cor87] gives a complete classification of the irreducible subvarieties corresponding to the curves whose automorphism group contains a fixed cyclic subgroup of prime order, in the case $g \geq 1, n = 0$ over the complex numbers. Later Schneps [Sch02a] considered the situation of genus 0 with n marked points, and genus 1 with $n = 1$ or 2 marked points, corresponding to the

curves having a cyclic group in its automorphism group, over the complex numbers. In this thesis, we consider more general cases in higher genus and in characteristic p, with n marked points, corresponding to the marked curves whose automorphism group contains a fixed cyclic subgroup acting in a fixed way.

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Chapter 1

Introduction

1.1 Background

The purpose of this thesis is to study special loci in the moduli space $\mathfrak{M}_{g,n}$ [resp. $\mathfrak{M}_{g,[n]}$] parametrizing the smooth projective curves of genus g with n ordered [resp. unordered] marked points. Over the complex numbers, this space is closely related to the pure [resp. full] mapping class group $\Gamma_{g,n}$ [resp. $\Gamma_{g,[n]}$], which is its orbifold fundamental group. One can apply this relationship to study Galois theory, especially via the Grothendieck-Teichmüller group $\widehat{GT}$ ([Sch02b]).

The general problem this thesis is concerned with is that of studying the subvarieties of the moduli space $\mathfrak{M}_{g,[n]}$ corresponding to curves with extra automorphisms, where a typical curve with marked points has no automorphisms; but some do, depending upon the choice of curves and position of marked points. This gives us certain subvarieties in the moduli space. For Riemann surfaces, these subvarieties are characterized by specifying a finite group of mapping-classes whose action on a curve is fixed topologically. After considering this classical case, we will extend these ideas to curves over more general fields, including the characteristic p case.

We begin by recalling the situation in the classical case, before outlining what will be done later in this thesis. Let S be an orientable topological surface of genus g equipped with n distinct ordered marked points $s_1, \ldots, s_n$ (we say that S is of *type* (g, n)). A *parametrized (ordered) marked* Riemann surface of genus g is a Riemann surface X of genus g with n distinct ordered marked points $x_1, \ldots, x_n$ together with a parametrization, i.e., a diffeomorphism $\Phi : S \to X$ such that $\Phi(s_i) = x_i$ for each i. Two parametrized marked Riemann surfaces X (with marked points $x_1, \ldots, x_n$ and parameterization Φ) and X' (with marked points $x'_1, \ldots, x'_n$ and parameterization Φ') are said to be *isomorphic* if there exists an isomorphism $\alpha : X \to X'$ of Riemann surfaces with $\alpha(x_i) = x'_i$ for each i and a diffeomorphism $h : S \to S$ with $h(s_i) = s_i$, for each i, which is isotopic to the identity, via a family of diffeomorphisms $h_t : S \to S$ with $h_t(s_i) = s_i$, for $t \in [0, 1]$ and each i, such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S & \xrightarrow{\Phi'} & X' \end{array}$$

The Teichmüller space $\mathcal{T}_{g,n}$ is the set of isomorphism classes of parameterized marked Riemann surfaces of type (g, n). In fact, it is well known that the Teichmüller space forms a simply connected complex analytic space of dimension $3g-3+n$ (except in a few small cases).

The mapping class group $\Gamma_{g,[n]}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$. The unordered moduli space $\mathfrak{M}_{g,[n]}$ is realized as the quotient of the Teichmüller space $\mathcal{T}_{g,n}$ by the action of the mapping class group $\Gamma_{g,[n]}$. Similarly, the ordered moduli space $\mathfrak{M}_{g,n}$ is the quotient of $\mathcal{T}_{g,n}$ by the pure subgroup $\Gamma_{g,n}$ of $\Gamma_{g,[n]}$. The permutation group S_n acts naturally on $\mathfrak{M}_{g,n}$ by permuting the marked points on the Riemann surfaces. For any subgroup $G \in S_n$, we write $\mathfrak{M}_{g,n}(G) = \mathfrak{M}_{g,n}/G$.

Let φ be an element of finite order in the full or pure mapping class group (the group of homotopy classes of diffeomorphisms of a surface), then we consider the set of points in Teichmüller space fixed by φ. The image of this set in the quotient moduli space $\mathfrak{M}_{g,n}(G)$ is called the *special locus* of φ and denoted $\mathfrak{M}_{g,n}(G,\varphi)$. The special locus of φ is closely related to the moduli space $\mathfrak{M}(T)$ of the topological quotient $T = S/\varphi$. More explicitly, let $[\varphi]$ denote the associated permutation of the marked points and let $G \subset S_n$ be the subgroup generated by the disjoint cycles of $[\varphi]$. Then there is a natural covering map of finite degree [Sch02a]

$$\tau_\varphi : \widetilde{\mathfrak{M}}_{g,n}(G,\varphi) \to \mathfrak{M}(T) \tag{1.1.1}$$

The morphism (1.1.1) corresponds to a group homomorphism

$$Norm_{\Gamma_{g,n}(G)}(\varphi) \to \Gamma(T) \tag{1.1.2}$$

where $\Gamma_{g,n}(G)$ is the preimage of G under the canonical surjection $\Gamma_{g,[n]} \to S_n$.

These two groups are the orbifold fundamental groups of $\widetilde{\mathfrak{M}}_{g,n}(G,\varphi)$ and $\mathfrak{M}(T)$ respectively.

If the homomorphism (1.1.2) is surjective, we say that φ satisfies the *surjectivity* condition, which corresponds to the condition that the morphism (1.1.1) is a degree 1 covering. If the homomorphism (1.1.2) is split, we say that φ satisfies the *splitting condition.* This additional condition says that the natural orbifold structures on $\widetilde{\mathfrak{M}}_{g,n}(G,\varphi)$ and $\mathfrak{M}(T)$ are the same except for the automorphism φ at each point. When $g = 0$, every finite order element ϕ of $\Gamma_{g,[n]}$ satisfies both the surjectivity and splitting conditions [Sch02a].

This thesis builds upon previous work by González-Díez, Harvey and Schneps. González-Díez and Harvey [GH92] considered these irreducible subvarieties for genus $g \geq 2$ curves without marked points over the complex numbers, where they have studied the coarse moduli space for surfaces with a specified automorphism group. Cornalba [Cor87] gives a complete classification of the irreducible subvarieties corresponding to the curves whose automorphism group contains a fixed cyclic subgroup of prime order, in the case $g \geq 1, n = 0$ over the complex numbers. Later Schneps [Sch02a] considered the situation of genus 0 with n marked points, and genus 1 with $n = 1$ or 2 marked points, corresponding to the curves having a cyclic group in its automorphism group, over the complex numbers. In this thesis, we consider more general cases in higher genus and in characteristic p, with n marked points, corresponding to the marked curves whose automorphism group contains a fixed cyclic subgroup.

1.2 Outline of the thesis

This thesis continues the study of special loci of curves discussed above, and also carries over this study of more general fields, including those of characteristic p. The difficulty in generalizing the definition of "special loci" to characteristic p is that we need to give an equivalent purely algebraic definition of special loci. Schneps' original definition (Definition 2.1.9) of special loci involves the mapping class group, which is the group of homotopy classes of orientation preserving diffeomorphisms fixing [resp. permuting] in marked points of S ([Sch02a]); so doesn't make sense in characteristic p. Chapter 2 focuses on this problem.

Section 2.1 provides an equivalent definition of (differential) special loci over the complex numbers $\mathbb{C}$ without mentioning Teichmüller space and mapping class group. This definition is easier to use to prove the equivalence between the definition of special loci (in the original sense) and the definition of algebraic special loci. Section 2.2 gives two definitions of algebraic special loci, which are equivalent with the definition of special loci over the complex numbers $\mathbb{C}$. One is the "family definition" (Definition 2.2.2) which is proved equivalent to the original definition by using Hurwitz families [CH85]. Another is the "permutation definition" (Corollary 2.2.7) in the case that n is big enough, where n is the number of the marked points of curves. The permutation definition is easier to work with if we have enough marked points.

Chapter 3 describes the special loci in genus zero in all characteristics explicitly, and describes the possible permutations in S_n such that the special loci is not empty in genus 1.

If S is a sphere with n marked points, then a finite-order element of the mapping class group $\Gamma_{g,[n]}$ is the class of a diffeomorphism which is simply a rotation around an axis (Section 3.1.1). In section 3.1.1, we recall some known results of Schneps ([Sch02a]) which describe the special loci in genus zero over the complex numbers explicitly. Sections 3.1.2 and 3.1.3 give some explicit examples of special loci in characteristic 5 and 3 by considering the points having non-trivial special automorphism group in the ordered moduli space to determine the special loci in the unordered moduli space. Over the complex numbers $\mathbb{C}$, a finite-order element of the mapping class group $\Gamma_{g,[n]}$ is the class of a diffeomorphism which is simply a rotation around an axis, while in characteristic p, every finite-order automorphism of $\mathbb{P}^1_K$ with marked points is the conjugacy class of a rotation around an axis (i.e., by multiplying roots of unity) or the conjugacy class of a translation (i.e., by adding an element in K) (Proposition 3.1.8). Then we give a complete description of special loci of genus 0 in characteristic p in Section 3.1.4.

There is also a generalization of the results in genus 0 to higher genus. In particular, Proposition 3.2.1 describes the possible permutations $[\varphi]$ such that the special locus $\mathfrak{M}_{1,[n]}(\varphi)$ is not empty, where the proof is given by using the classification of automorphisms of elliptic curves and Riemann-Hurwitz formula.

Chapter 4 considers the surjectivity and splitting conditions in all characteristics in genus 0 by first generalizing the definitions of the surjectivity and splitting conditions to characteristic p.

The permutation group S_n acts naturally on $\mathfrak{M}_{g,n}$ by permuting the marked points on the Riemann surfaces. For any subgroup $G \in S_n$, we write $\mathfrak{M}_{g,n}(G) = \mathfrak{M}_{g,n}/G$. Over $\mathbb{C}$, let φ be an element of finite order in the full mapping class group and let $T = S/\varphi$. We consider the set of points in Teichmüller space fixed by φ. The image of this set in the quotient moduli space $\mathfrak{M}_{g,n}(G)$ is the special locus of φ in $\mathfrak{M}_{g,n}(G)$ and denoted $\mathfrak{M}_{g,n}(G,\varphi)$. Schneps gave splitting and surjectivity conditions such that each component of $\widetilde{\mathfrak{M}}_{g,n}(G,\varphi)$ is as close to $\mathfrak{M}(T)$ as possible ([Sch02a]).

To generalize the splitting and surjectivity conditions to characteristic p, we need to give an equivalent definition which can be applied in characteristic p. The original definition of splitting and surjectivity conditions involves the fundamental group of moduli spaces of curves, while the fundamental group of moduli spaces of curves in characteristic p is very complicated. Section 4.1 gives the geometric meaning of the surjectivity condition, which avoids the fundamental group of moduli spaces of curves (Proposition 4.1.5). It also gives an equivalent description of the splitting condition by finite group theory using Proposition 4.1.9.

Section 4.2 shows that the surjectivity and splitting conditions hold in general for genus 0 (Theorem 4.2.5 and Theorem 4.2.6); this generalizes the result of Schneps from characteristic 0 to arbitrary characteristic.

1.3 Preliminaries of moduli of curves

First I would like to give a rough idea and basic examples about moduli spaces of curves.

We let $\mathfrak{M}_g$ denote the set of isomorphism classes of smooth, complete, connected curves of genus g. The set $\mathfrak{M}_g$ can be given the structure of an algebraic variety.

Intuitively, we would like to specify the algebraic structure on $\mathfrak{M}_g$ by requiring it to be a universal parameter variety for families of curves of genus g. In other words, we would like to require that there is a flat family $\mathfrak{X} \to \mathfrak{M}_g$ of curves of genus g such that for any other flat family $X \to T$ of curves of genus g, there is a unique morphism $T \to \mathfrak{M}_g$ such that X is the pull-back of $\mathfrak{X}$. In this case we call $\mathfrak{X} \to \mathfrak{M}_g$ a universal family and say that $\mathfrak{M}_g$ is a *fine moduli space.*

Unfortunately, this can not be done in general (except for $g = 0$, when $\mathfrak{M}_g$ is a point). Nevertheless, for $g \geq 1$, there is at least a variety $\mathfrak{M}_g$ which has the following properties:

(1) the set of closed points of $\mathfrak{M}_g$ is in one-to-one correspondence with the set of isomorphism classes of curves of genus g;

(2) if $f : X \to T$ is any flat family of curves of genus g, then there is a morphism

$h : T \to \mathfrak{M}_g$ such that for each closed point $t \in T$, X_t is in the isomorphism class of curves determined by the point $h(t) \in \mathfrak{M}_g$.

This is classical for $g = 1$ (where $\mathfrak{M}_g$ is the j-line), and was shown by Mumford [Mum65, Th5.11] for $g \geq 2$. We say that $\mathfrak{M}_g$, satisfying (1) and (2), is a *coarse moduli space* for curves of genus g. Moreover, Deligne and Mumford [DM69] have shown that $\mathfrak{M}_g$ for $g \geq 2$ is an irreducible quasi-projective variety of dimension $3g - 3$ over any fixed algebraically closed field. William Fulton [Ful69] also proved the irreducibility of the moduli of curves of genus g over any field with characteristic $p > g + 1$.

The allied space $\mathfrak{M}_{g,n}$ is very important for our study. This parametrizes the isomorphism classes of objects $(C; x_1, \ldots, x_n)$, where C is a curve of genus g, and $x_1, \ldots, x_n$ are distinct ordered points of C. Thus, $\mathfrak{M}_g = \mathfrak{M}_{g,0}$.

Let us begin to look at the simplest cases:

(I) $\mathfrak{M}_{0,n} \cong [\mathbb{P}^1 - (0, 1, \infty)]^{n-3} - \{x_i = x_j \text{ for some } i \neq j\}$

In fact, if we have n distinct points $x_1, \ldots, x_n \in \mathbb{P}^1$, a unique automorphism of $\mathbb{P}^1$ takes x_1 to 0, x_2 to 1, and x_3 to ∞. The remaining $n - 3$ points are arbitrary except for being distinct and not equal to $0, 1$ or ∞.

(II) $\mathfrak{M}_{1,0} = \mathfrak{M}_{1,1} \cong \mathbb{A}^1_j$ (the affine line with coordinate j)

Because curves of genus 1 with one fixed point P_0 are groups, their automorphisms act transitively; hence $\mathfrak{M}_{1,0} = \mathfrak{M}_{1,1}$, as varieties. To determine the space, recall that each such curve is isomorphic to a plane cubic C_λ, defined by $y^2 = x(x-1)(x-\lambda)$ where $\lambda \neq 0, 1$. Equivalently, C_λ is the double cover of $\mathbb{P}^1$ ramified at $0, 1, \infty, \lambda$. One proves easily (or the reader can refer to [Har97, pages 317–320] that $C_{\lambda_1} \cong C_{\lambda_2}$ if and only if there is an automorphism of $\mathbb{P}^1$ carrying $0, 1, \infty, \lambda_1$ (unordered set) to $0, 1, \infty, \lambda_2$. This happens if and only if

$$\lambda_2 = \lambda_1, 1-\lambda_1, \frac{1}{\lambda_1}, \frac{\lambda_1 - 1}{\lambda_1}, \frac{\lambda_1}{\lambda_1 - 1} \text{ or } \frac{1}{1-\lambda_1}.$$

[E.g., note the map

$$(x, y) \longmapsto (1-x, y)$$

carries C_λ to $C_{1-\lambda}$; and the map

$$(x, y) \longmapsto (1/x, y/x^2)$$

carries C_λ to $C_{1/\lambda}$.]

One must cook up an expression in λ invariant under these substitutions and no more. It is customary to use:

$$j = j(\lambda) = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}$$

(It is readily checked that this j is invariant under these 6 substitutions and since $6 = \max(\text{degree of numerator and denominator})$, no other λ's give the same j. The coefficient 256 is thrown in to make things work in characteristic 2.)

We then get a bijection between the isomorphism classes of genus 1 curves C and the affine line $\mathbb{A}^1$ by taking C to $j(\lambda)$ if $C \cong C_\lambda$. Properties (1) and (2) above can then be checked.

Now we give the moduli problem a precise statement.

First of all, we specify a class of objects together with a notion of a family of these objects over a scheme B. Roughly speaking, such a family consists of a collection of objects X_b, for each b. Second, we choose a equivalence relation $\sim$ on the set $S(B)$ of all such families over each B. We build a functor F from the category of schemes to that of sets by the rule

$$F(B) = S(B)/\sim$$

and call F the *moduli functor* of our moduli problem. Let $\mathfrak{M}$ be a scheme such that there are natural maps

$$\Psi_{\mathfrak{M}}(B) : F(B) \to \mathrm{Hom}(B, \mathfrak{M})$$

given by $\Psi_{\mathfrak{M}}(B)(X)(b) = [X_b]$, where $b \in B$ and $[X_b]$ denotes the equivalence class of the object X_b and $\mathrm{Hom}(B, \mathfrak{M})$ denotes the set of morphisms from B to $\mathfrak{M}$. More precisely, these maps $\Psi_{\mathfrak{M}}(B)$ determine a natural transformation

$$\Psi_{\mathfrak{M}} : F \to \mathrm{Hom}(-, \mathfrak{M}).$$

Definition 1.3.1. [HM98] If F is representable by $\mathfrak{M}$ (i.e., $F \approx \mathrm{Hom}(-, \mathfrak{M})$), then we say that the scheme $\mathfrak{M}$ is a *fine moduli space* for the moduli problem F.

Definition 1.3.2. [HM98] A scheme $\mathfrak{M}$ and a natural transformation $\Psi_{\mathfrak{M}}$ from the functor F to the functor $\mathrm{Hom}_{\mathfrak{M}} = \mathrm{Hom}(-, \mathfrak{M})$ are a *coarse moduli space* for the functor F if

1) The map $\Psi_{\mathfrak{M}}(pt) : F(pt) \to \mathrm{Hom}(pt, \mathfrak{M})$ is bijective, where pt denotes any generic point.

2) Given another scheme $\mathfrak{M}'$ and a natural transformation $\Psi_{\mathfrak{M}'}$ from $F \to \mathrm{Hom}_{\mathfrak{M}'}$, there is a unique morphism $\pi : \mathfrak{M} \to \mathfrak{M}'$ such that the associated natural transformation $\Pi : \mathrm{Hom}_{\mathfrak{M}} \to \mathrm{Hom}_{\mathfrak{M}'}$ satisfies $\Psi_{\mathfrak{M}'} = \Pi \circ \Psi_{\mathfrak{M}}$.

Remark 1.3.3. (1) It is easy to check that if $\mathfrak{M}$ is a fine moduli space, then it is a coarse moduli space, but not vice versa.

(2) It can be checked that Definition 1.3.2 coincides with the definition we gave at the beginning of this section for the coarse moduli space of curves of genus g.

The reason that M_g is just a coarse moduli space, and not a fine moduli space, is that some (special) curves of a given genus have more automorphisms than the general curve of that genus. Of course this does not for $g = 0$, where the moduli space is just a point. But in genus 1 there are two curves with extra automorphisms, beyond translation and inversion. Deleting those, one obtains a fine moduli space. In genus ≥ 2, most curves have no automorphisms other than the identity, but some (e.g. hyperelliptic curves) have (finitely many) non-identity automorphisms.

A similar situation occurs for $M_{g,n}$, the moduli space of curves of genus g with n ordered marked points, and for the corresponding moduli space $M_{g,[n]}$ of curves with unordered marked points. (In the marked situation, this phenomenon can even happen in genus 0.) These special loci of (marked) curves with extra automorphisms, which prevent the moduli spaces from being fine, are the subject of this thesis.

Chapter 2

Notions of Special Loci

The focus of this chapter is to consider several notions of "special locus". We begin with the classical case of curves over the complex numbers, where Schneps's original definition (Definition 2.1.9) of special loci involves the mapping class group which is the group of homotopy classes of orientation preserving diffeomorphisms fixing [resp. permuting] in marked points of S ([Sch02a]), so doesn't make sense in characteristic p. Then we generalize this to characteristic p

Section 2.1 provides an equivalent definition of differential special loci over the complex numbers $\mathbb{C}$ without mentioning Teichmüller space and mapping class group. This definition is easier to be used to prove the equivalence between the definition of special loci (in the original sense) and the definition of algebraic special loci. Section 2.2 gives two definitions of algebraic special loci, which are equivalent with the definition of special loci over the complex numbers $\mathbb{C}$. One is the family definition (Definition 2.2.2) which is proved equivalent to the original definition by using Hurwitz families [CH85]. Another is the permutation definition (Corollary 2.2.7) provided that n is big enough, say $n > n_0(g)$, where n is the number of marked points on the curves. Let K be an algebraically closed field. For each g, there exists a number $n_0(g)$ such that every nontrivial automorphism of genus g curve over K has at most $n_0(g)$ fixed points (Proposition 2.2.4). The permutation definition is easier to work with if we have enough marked points.

2.1 Differential special loci

In this thesis, we adopt some definitions from [Sch02a]. But sometimes we use slightly different "terminology". We say differential special locus for the original term special locus.

We fix S once and for all to be an orientable topological surface of genus g equipped with n distinct ordered marked points $s_1, \ldots, s_n$. We say that S is of *type* (g, n).

Throughout this section, we only work over the complex numbers $\mathbb{C}$. In order to give the definition of special locus, we need to give the following definitions.

Definition 2.1.1. An ordered [resp. unordered] *marked* Riemann surface is a Riemann surface X of genus g together with n ordered [resp. unordered] distinct marked points $x_1, \ldots, x_n$.

Definition 2.1.2. (cf. [Sch02a, §2.1]) A *parameterized* (ordered) marked Riemann surface of genus g consists of the following data:

(1) a n ordered marked Riemann surface $(X; x_1, \ldots, x_n)$ of genus g;

(2) a *parameterization*, i.e., a diffeomorphism $\Phi : S \to X$ such that $\Phi(s_i) = x_i$ for $1 \leq i \leq n$.

We say X is of *type* (g, n)

Definition 2.1.3. ([Sch02a, §2.1]) Two parameterized marked Riemann surfaces X (with marked points $x_1, \ldots, x_n$ and parameterization Φ) and X' (with marked points $x'_1, \ldots, x'_n$ and parameterization Φ') are said to be *isomorphic* if there exists an isomorphism $\alpha : X \to X'$ of Riemann surfaces with $\alpha(x_i) = x'_i$ for $1 \leq i \leq n$ and a diffeomorphism $h : S \to S$ with $h(s_i) = s_i$, for $1 \leq i \leq n$, which is isotopic to the identity, such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S & \xrightarrow{\Phi'} & X' \end{array}$$

Remark 2.1.4. In above definition, when we say a diffeomorphism $h : S \to S$ is isotopic to the identity, we mean that it is isotopic to the identity via a family of diffeomorphisms $h_t : S \to S$ with $h_t(s_i) = s_i$, for $t \in [0, 1]$ and each i.

Remark 2.1.5. The *Teichmüller space* $\mathcal{T}_{g,n}$ is the set of isomorphism classes of parameterized marked Riemann surfaces of type (g, n). In fact, it is well known that the Teichmüller space forms a simply connected complex analytic space of dimension $3g - 3 + n$ (cf. [Nag88, Theorem 3.2.3]).

Definition 2.1.6. ([Sch02a, §2.1])

(a) We define the *full mapping class group* $\Gamma_{g,[n]}$ by setting

$$\Gamma_{g,[n]} = \mathrm{Diff}^+([S])/\mathrm{Diff}^0(S),$$

where $\mathrm{Diff}^+([S]$ denotes the group of orientation-preserving diffeomorphisms of S which fixes $\{s_1, \ldots, s_n\}$ as a set, and $\mathrm{Diff}^0(S)$ is the subgroup of those which are isotopic to the identity.

(b) We define the *pure mapping class group* (or *pure subgroup of the full mapping class group*) $\Gamma_{g,n}$, by setting

$$\Gamma_{g,n} = \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S),$$

where $\mathrm{Diff}^+(S)$ is the subgroup of $\mathrm{Diff}^+([S])$ consisting of diffeomorphisms which fix each marked point s_i.

Remark 2.1.7. For the definition of mapping class group, Schneps ([Sch02a]), Hain and Looijenga ([HL97]) use diffeomorphisms of a compact orientable surface of genus g, while González-Díez, Harvey and Maclachlan ([GH92], [MH75]) use homeomorphisms of a compact orientable surface of genus g. But these definitions of the mapping class group are equivalent because every homeomorphism of a compact orientable surface S of genus g can be approximated by a diffeomorphism of S up to homotopy (cf. ([Hir76, Chapter 5, Lemma 1.5]). So we can use all the results about mapping class group from all of above papers.

Remark 2.1.8. The mapping class group $\Gamma_{g,[n]}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$. The action is the following: if $\psi \in \Gamma_{g,[n]}$, let ψ' denote a lifting of ψ to a diffeomorphism of S; then ψ' maps the marked Riemann surface (Φ, X) to $(\Phi \circ \psi', X)$ ([Sch02a, §2.1]). The *unordered moduli space* $\mathfrak{M}_{g,[n]}$, parameterizing smooth curves of genus g together with a unordered set of n-distinct marked points, is realized as the quotient of the Teichmüller space $\mathcal{T}_{g,n}$ by the action of the mapping class group $\Gamma_{g,[n]}$. Similarly, the *ordered moduli space* $\mathfrak{M}_{g,n}$, parameterizing smooth curves of genus g together with an ordered set of n-distinct marked points, is the quotient of $\mathcal{T}_{g,n}$ by the pure subgroup $\Gamma_{g,n}$ of $\Gamma_{g,[n]}$.

Definition 2.1.9. (cf. [Sch02a, §2.1]) If φ is an element of finite order in the full or pure mapping class group, then we consider the set of points in Teichmüller space fixed by φ. The image of this set in the quotient moduli space $\mathfrak{M}_{g,n}$ or $\mathfrak{M}_{g,[n]}$ is called the *differential special locus of* φ. We denote it by $\mathfrak{M}_{\varphi}(S)$ or $\mathfrak{M}_{\varphi}[S]$ respectively.

Now we give a notation by the following definition without mentioning Teichmüller space and mapping class group.

Definition 2.1.10. (Differential equivalence) Let X (with marked points $x_1, \dots, x_n$) and X' (with marked points $x'_1, \dots, x'_n$) be two ordered marked Riemann surfaces with genus g in the ordered moduli space $\mathfrak{M}_{g,n}$. Let α be a finite order automorphism of X and let α' be a finite order automorphism of X', which fix each marked point. Then α and α' are said to be *differentially equivalent* if there exists a diffeomorphism $\Psi : X \to X'$ with $\Psi(x_i) = x'_i$ for $1 \leq i \leq n$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X'. \end{array}$$

Similarly, Let X (with marked points $x_1, \ldots, x_n$) and X' (with marked points $x'_1, \ldots, x'_n$) be two unordered marked Riemann surfaces with genus g in the unordered moduli space $\mathfrak{M}_{g,[n]}$. Let α be a finite order automorphism of X and let α' be a finite order automorphism of X', which fix the marked points as a set. Then α and α' are said to be *differentially equivalent* if there exists a diffeomorphism $\Psi : X \to X'$ which maps the set of marked points of X to the set of marked points of X' such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

Proposition 2.1.11. *Let X (with marked points $x_1, \ldots, x_n$) be an unordered [resp. ordered] marked Riemann surface in $\mathfrak{M}_{g,[n]}$ [resp. $\mathfrak{M}_{g,n}$] and α be a finite order automorphism of X. Pick a parameterization Φ of X, then (X, Φ) is a point in Teichmüller space. Let $\psi : S \to S$ be the diffeomorphism induced by α. Let φ be the equivalent class of ψ in the full [resp. pure] mapping class group.*

(a) Then (X, Φ) is fixed by φ.

(b) Let X' (with marked points $x'_1, \ldots, x'_n$) be an unordered [resp. ordered] marked Riemann surface in $\mathfrak{M}_{g,[n]}(\mathfrak{M}_{g,n})$. Then there exists a parameterized marked Riemann surface (X', Φ') (where Φ' is a parameterization) in Teichmüller

space which is also fixed by φ if and only if there exists an automorphism α' of X' which is differentially equivalent to α.

Proof. We only give a proof for unordered moduli space. For ordered moduli space, the proof is basically the same.

(a) By assumption of the proposition, we have the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ S & \xrightarrow{\Phi} & X \end{array}$$

which is equivalent to the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\alpha} \\ S & \xrightarrow{\Phi\psi} & X \end{array}$$

where $id : S \to S$ is the identity. So (X, Φ) is fixed by φ in Teichmüller space.

(b) Now suppose that there exists a parameterized marked Riemann surface (X', Φ') which is also fixed by φ. Then there exists an automorphism $\alpha' : X' \to X'$ and a diffeomorphism $h : S \to S$ with $h(s_i) = s_i$ for $1 \leq i \leq n$ which is isotopic to the identity such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi'\psi h^{-1}} & X' \end{array}$$

since ψh^{-1} is a lifting of φ. The above commutative diagram is equivalent to the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi'} & X' \end{array}$$

Let $\Psi = \Phi'\Phi^{-1}$; then we get the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

So α and α' are differentially equivalent.

Conversely, suppose there exists an automorphism α' of X' which is differentially equivalent to α. Then there exists a diffeomorphism $\Psi : X \to X'$ which maps the set of marked points to the set of marked points such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} S & \xrightarrow{\Phi} & X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi} & X & \xrightarrow{\Psi} & X' \end{array}$$

Let $\Phi' = \Phi\Psi$; then the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi'} & X' \end{array}$$

This is equivalent to the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi'\psi} & X' \end{array}$$

where $id : S \to S$ is the identity. So (X', Φ') is fixed by φ. $\square$

Corollary 2.1.12. *Let X (with marked points $x_1, \ldots, x_n$) be an unordered [resp. ordered] marked Riemann surface in $\mathfrak{M}_{g,[n]}$ [resp. $\mathfrak{M}_{g,n}$] and α be a finite order automorphism of X. The differential special locus of α is the set of points on the moduli space $\mathfrak{M}_{g,n}$ [resp. $\mathfrak{M}_{g,[n]}$] which have an automorphism differentially equivalent to α.*

Proof. By Definition 2.1.9, consider the set of points in Teichmüller space fixed by α, then the differential special locus of α is the image of this set in the quotient moduli space. By Proposition 2.1.11, let X' (with marked points $x'_1, \ldots, x'_n$) be any unordered [resp. ordered] marked Riemann surface in $\mathfrak{M}_{g,[n]}(\mathfrak{M}_{g,n})$, then there exists a parameterized marked Riemann surface (X', Φ') (where Φ' is a parameterization) in Teichmüller space which is also fixed by α if and only if there exists an automorphism α' of X' which is differentially equivalent to α. Therefore the differential special locus of α is the set of points on the moduli space $\mathfrak{M}_{g,n}$ [resp. $\mathfrak{M}_{g,[n]}$] which have an automorphism differentially equivalent to α. $\square$

2.2 Algebraic special loci

Let K be an algebraically closed field. Unlike the previous section, where we only worked over the complex numbers $\mathbb{C}$, now we generalize the definition of "special loci" to any algebraically closed field K. Wee give a definition of algebraic special loci in the following. First we give a remark about "specialization"

Remark 2.2.1. Let α be a finite order automorphism of X. Let ξ be a generic point in $\mathfrak{X}$ and ξ correspond to the curve A with an automorphism θ. And $\mathfrak{X}$ contains the point corresponding to X. Then there is an injective map

$$\mu : Aut(A) \to Aut(X).$$

We say that θ specializes α if $\mu(\theta) = \alpha$.

Moreover, let α' be a finite order automorphism of X' and $\mathfrak{X}$ contains the point corresponding to X'. Then there is an injective map

$$\mu' : Aut(A) \to Aut(X').$$

We say that θ specializes both α and α' if $\mu(\theta) = \alpha$ and $\mu'(\theta) = \alpha'$

Definition 2.2.2. Let X (with marked points $x_1, \dots, x_n$) and X' (with marked points $x'_1, \dots, x'_n$) be two unordered [resp. ordered] marked curves with genus g in $\mathfrak{M}_{g,[n]}$ [resp. $\mathfrak{M}_{g,n}$].

(a) A finite order automorphism α of X and a finite order automorphism α' of X', which fix the marked points as a set, are said to be *algebraically equivalent* if there exists an irreducible subvariety $\mathfrak{X}$ of $\mathfrak{M}_{g,[n]}$ [resp. $\mathfrak{M}_{g,n}$] and an automorphism θ of the curve A which corresponds to the generic point ξ in $\mathfrak{X}$, such that θ specializes to both α and α', where $\mathfrak{X}$ contains the points corresponding to X and X'.

(b) The *algebraic special locus* of α is the set of points in the moduli space $\mathfrak{M}_{g,[n]}$ [resp. $\mathfrak{M}_{g,n}$] which have an automorphism algebraically equivalent to α. We denote it by $\mathfrak{M}_{g,[n]}(\alpha)$ [resp. $\mathfrak{M}_{g,n}(\alpha)$].

Proposition 2.2.3. *Over the complex numbers $\mathbb{C}$, let X (with unordered [resp. ordered] marked points $x_1, \ldots, x_n$) be a marked curve with genus g in $\mathfrak{M}_{g,[n]}$ [resp. $\mathfrak{M}_{g,n}$]. Let α be a finite order automorphism of X, which fix the marked points as a set. Then the algebraic special locus of α is as same as the differential special locus of α.*

Proof. We only give a proof for unordered moduli space. For ordered moduli space, the proof is basically the same.

First we need to show that the special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$ over the complex numbers $\mathbb{C}$. By [GH92, Theorem 1, page 79], we know that the special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$ when $n = 0$, i.e., in the case of no marked points. By a similar proof, we know that the special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$.

Now let $Y = X/\langle\alpha\rangle$, then we have the branch data of the covering map $X \to Y$. By [CH85, Proposition 1.4] , we know there is a coarse moduli space for $\langle\alpha\rangle$-Galois covers of Y with description branch data. (Note: the result in [CH85, Proposition 1.4] uses the group is abelian, here it is cyclic. And the base space of the Hurwitz family is $\mathbb{P}^1$ in [CH85, Proposition 1.4], but the proof is similar for general base space Y).

Let X' be an unordered marked curve of genus g with marked points $x'_1, \ldots, x'_n$ and α' be a finite automorphism of X' which preserves the marked points as a set. Suppose α' is in the algebraic special locus of α, i.e., α' and α are algebraically equivalent. Then X and X' are in the same Hurwitz family of $\langle\alpha\rangle$-Galois covers of Y. So there exists a diffeomorphism $\Psi : X \to X'$ which maps the set of marked points of X to the set of marked points of X' such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

i.e., α and α' are differentially equivalent. Therefore α' is in the differential special locus of α.

Conversely, suppose that α' is in the differential special locus of α. Since the differential special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$, there exists a generic point ξ in $\mathfrak{M}_{g,[n]}(\alpha)$ with an an automorphism θ of the curve A which corresponds to the generic point ξ, such that θ specializes to both α and α', where $\mathfrak{M}_{g,[n]}(\alpha)$ contains the points corresponding to X and X'. Therefore α and α' are algebraically equivalent. Hence α' is in the algebraic special locus of α. □

Proposition 2.2.4. *Let K be an algebraically closed field. For each g, there exists a least integer $n_0(g)$ such that every nontrivial automorphism of genus g curve over K has at most $n_0(g)$ fixed points; i.e., if $n > n_0(g)$, then any automorphism which fixes n distinct points is an identity. In particular, $n_0(g) = 2g+2$ for all g and for all characteristics.*

Proof. First, for $g = 0$, the group of all automorphisms is isomorphic to PGL(2), we know $n_0(0) = 2$. For $g = 1$, there are only finitely many automorphisms fixing any one given point. So $n_0(1)$ exists. For $g \geq 2$, it is well-known that there are only finite many automorphisms, so $n_0(g)$ exists.

Now we show that $n_0(g) = 2g + 2$ for all g and for all characteristics. Suppose X is a genus g smooth curve with an order m automorphism and n is the number of fixed points of σ. Since the number of fixed points of σ is less or equal to the number of fixed points of a power of σ, we may assume the automorphism σ has prime order p. Let $Y = X/\langle\sigma\rangle$; then X is a branched covering space of Y. By the Riemann-Hurwitz Theorem, we have $2g - 2 \geq p(2g_Y - 2) + (p - 1)n$. (In the tamely ramified case, $2g - 2 = p(2g_Y - 2) + (p - 1)n$ and in the wildly ramified case, $2g - 2 > p(2g_Y - 2) + (p - 1)n$.) Suppose that $n > 2g + 2$; then we get $2g - 2 \geq -2p + (p - 1)(2g + 3)$ (since $g_Y \geq 0$). So we have $g(4 - 2p) + 1 - p \geq 0$, which is a contradiction since $4 - 2p \leq 0$ and $1 - p < 0$ ($p \geq 2$). Therefore, we have $n \leq 2g + 2$. If $n = 2g + 2$, then we can take $g_Y = 0$ and $p = 2$, i.e., there is a nontrivial order 2 automorphism β of a hyperelliptic curve of genus g such that β has $2g + 2$ fixed points. Hence $n_0(g) = 2g + 2$ is the least integer such that every nontrivial automorphism of genus g curve over K has at most $n_0(g)$ fixed points. □

Remark 2.2.5. For $n > n_0(g)$, if an automorphism α of a marked curve X fixes n points, then $\alpha = 1$. So if two automorphisms α and β of a marked curve X induce the same permutation, then $\alpha = \beta$. Also for X corresponding to a point in the ordered moduli space $\mathfrak{M}_{g,n}$, the only automorphism α of X which fixes each marked point is the identity. Therefore, the special locus of a non-trivial automorphism α is empty. (Note: The condition $n > n_0(g)$ is not really a restriction for $g = 0$ since $n_0(0) = 3$ and since we need $n \geq 4$ to get a non-trivial moduli space in genus 0.) But for unordered moduli spaces, there can be non-empty special loci even if n is large compared to g. This can be seen by taking a union of finitely many orbits of an finite order automorphism of the underlying curve.

Thus if $n > n_0(g)$, one can speak in terms of permutations rather than automorphisms, and these are easier to work with. This motivates the following proposition.

Let X be a marked curve with genus g in $\mathfrak{M}_{g,[n]}$. Let α be a finite order automorphism of X, which fixes the marked points as a set.We denote $[\alpha]$ for the permutation induced by α, for a point in the unordered moduli space.

Proposition 2.2.6. *For $n > n_0(g)$, let X (with unordered marked points $x_1, \ldots, x_n$) and X' (with unordered marked points $x'_1, \ldots, x'_n$) be two marked curves with genus g in $\mathfrak{M}_{g,[n]}$. Let α be a finite order automorphism of X, which fixes the marked points as a set. Then there exists a finite order automorphism α' (which fixes the marked points as a set) of X' which is differentially equivalent to α if and only if there exists a finite order automorphism α'' (which fixes the marked points as a set) of X' such that $[\alpha'']$ is conjugate to $[\alpha]$.*

Proof. First suppose that there exists a finite order automorphism α' (which fixes the marked points as a set) of X' which is differentially equivalent to α. Then there exists a diffeomorphism $\Psi : X \to X'$ which maps the set of marked points of X to the set of marked points of X' such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

Then $\alpha' = \Psi\alpha\Psi^{-1}$. So α and α' induce conjugate permutations. Let $\alpha'' = \alpha'$, then $[\alpha]$ is conjugate to α''.

Conversely, suppose that there exists a finite order automorphism α'' (which fixes the marked points as a set) of X' such that $[\alpha'']$ is conjugate to $[\alpha]$. Choose parametrizations $\Phi : S \to X$ for X and $\Phi' : S \to X'$. Let $\psi : S \to S$ be the diffeomorphism induced by α and $\psi' : S \to S$ be the diffeomorphism induced by α''. Since α and α'' induce conjugate permutations, we know that ψ and ψ' induce conjugate permutations.

Therefore, by [GP74, Corollary (of Isotopy Lemma), page 143] there exist a diffeomorphism $h : S \to S$ which is isotopic to identity and a diffeomorphism $\zeta : S \to S$ such that $\psi = \zeta^{-1}\psi'\zeta h$, i.e., the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\zeta h} & S \\ {\scriptstyle\psi h}\downarrow & & \downarrow{\scriptstyle\psi'} \\ S & \xrightarrow{\zeta h} & S \end{array}$$

which is equivalent to the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\zeta h} & S \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle\psi'} \\ S & \xrightarrow{\zeta h\psi} & S \end{array}$$

So we have the following commutative diagram:

$$\begin{array}{ccccc} S & \xrightarrow{\zeta h} & S & \xrightarrow{\Phi'} & X' \\ h\big\downarrow & & \big\downarrow\psi' & & \big\downarrow\alpha'' \\ S & \xrightarrow{\zeta h\psi} & S & \xrightarrow{\Phi'} & X' \end{array}$$

So the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'\zeta h} & X' \\ h\big\downarrow & & \big\downarrow\alpha'' \\ S & \xrightarrow{\Phi'\zeta h\psi} & X' \end{array}$$

Therefore $(X', \Phi'\zeta h)$ is fixed by the equivalence class ϕ of ψ. By Proposition 2.1.11, we know there exists an automorphism α' of X' which is differentially equivalent to α. □

Corollary 2.2.7. *For $n > n_0(g)$, let X (with unordered marked points $x_1, \ldots, x_n$) be a marked curve with genus g in $\mathfrak{M}_{g,[n]}$ and α be a finite order automorphism of X which fixes $\{x_1, \ldots, x_n\}$ as a set. The algebraic special locus of α is the set of points on the moduli space which have an automorphism whose induced permutation of the marked points is conjugate to $[\alpha]$.*

The proof follows from Corollary 2.1.12 and Proposition 2.2.3.

Chapter 3

Special loci in low genus

This chapter considers special loci in genus 0 and genus 1, in all characteristics. In genus 0 we give an explicit description of the special loci. In genus 1 we describe the possible permutations in S_n such that the special loci is not empty in genus 1.

If S is a sphere with n marked points, then a finite-order element of the mapping class group $\Gamma_{g,[n]}$ is the class of a diffeomorphism which is simply a rotation around an axis (Section 3.1.1). In section 3.1.1, we recall some known results of Schneps ([Sch02a]) which describe the special loci in genus zero over the complex numbers explicitly. Sections 3.1.2 and 3.1.3 give some explicit examples of special loci in characteristic 5 and 3 by considering the points having non-trivial special automorphism group in the ordered moduli space to determine the special loci in the unordered moduli space. Over the complex numbers $\mathbb{C}$, a finite-order element of the mapping class group $\Gamma_{g,[n]}$ is the class of a diffeomorphism which is simply a rotation around an axis; while in characteristic p, every finite-order automorphism of $\mathbb{P}^1_K$ with marked points is the conjugacy class of a rotation around an axis (i.e., by multiplying roots of unity) or the conjugacy class of a translation (i.e., by adding an element in K) (Proposition 3.1.8). Then we give a complete description of special loci of genus 0 in characteristic p in Section 3.1.4.

There is also a generalization of the results in genus 0 to higher genus. In particular, Proposition 3.2.1 describes the possible permutations $[\varphi]$ such that the special loci $\mathfrak{M}_{1,[n]}(\varphi)$ is not empty, where the proof is given by using the classification of automorphisms of elliptic curves and Riemann-Hurwitz formula.

3.1 Special loci in genus 0

3.1.1 Genus zero over the complex numbers $\mathbb{C}$

For the genus zero case, the pure mapping class group $\Gamma_{0,n}$ acts on Teichmüller space $\mathcal{T}_{g,n}$ freely [Sch02a, §2.1]. So there are no special loci in the ordered moduli space $\mathfrak{M}_{0,n}$. A permutation τ of the ordered marked points can be realized as an automorphism of the marked Riemann surface [Sch02a, §3.1.1.]. Such points are not orbifold points on the ordered moduli space, but they are preimages of orbifold points on the unordered moduli space $\mathfrak{M}_{0,[n]}$, since the τ have less than $n!$ preimages under the action of S_n. The points having non-trivial special automorphism group determine where the special loci will lie on in the unordered moduli space $\mathfrak{M}_{0,[n]}$.

Now we investigate the points having non-trivial special automorphism group and the special loci in the genus zero moduli spaces for arbitrary n. If S is a sphere with n marked points, then a finite-order element of the mapping class group $\Gamma_{g,[n]}$ is the class of a diffeomorphism which is simply a rotation around an axis. In fact, For $n \geq 5$, all finite order element in $\Gamma_{0,[n]}$ are rotations follows from [MH75, Corollary p508] and [Sch02b, §4.1]. For $n = 3$, $\Gamma_{0,[n]} = 1$, For $n = 4$, there are four conjugacy classes of finite order elements, which induces different conjugate permutations [Sch02b, Proof of Corollary 2 in §3], we can see that each conjugate class comes from a rotation.

Let φ be a finite-order element of the mapping class group $\Gamma_{g,[n]}$. We may assume that φ is a rotation, say around the axis through the north and south poles (corresponding to the points $\infty, 0$). The north and south poles of S may or may not be marked points, but they are always the only ramification points for φ. The permutation associated to a rotation φ is always of the form $c_1 \dots c_k$, where the c_i are disjoint cycles of length j such that

$$\begin{cases} jk = n & \text{if the north and south poles are not marked} \\ jk = n-1 & \text{if one of the two poles is marked} \\ jk = n-2 & \text{if both poles are marked points} \end{cases}$$

In the following theorem, we compute the points having special automorphism associated to a permutation $[\varphi]$ which is a product of k disjoint cycles of length j only in the case $jk = n-2$ (i.e., when the two fixed points of φ are marked points).

This case then gives us the result in the general case, since in the other two cases, the special locus in $\mathfrak{M}_{0,n}$ is just the image of the one we compute here in $\mathfrak{M}_{0,n+1}$ or $\mathfrak{M}_{0,n+2}$, under the morphism given by erasing the extra marked points.

Theorem 3.1.1. *[Sch02a, Theorem 3.5.1] Let S be a Riemann surface of genus zero with n marked ordered points, and let φ be a rotation of order j with $n = jk+2$ (i.e., the two fixed points of φ are marked points of S). After replacing φ by a conjugate of φ which has the same special locus as φ, we may assume that the points of S are numbered so that the permutation associated to φ is given by*

$$[\varphi] = (1 \cdots j)(j+1 \cdots 2j) \cdots (j(k-1)+1 \cdots jk)$$

Let $G_\varphi \subset S_n$ be the subgroup generated by the above disjoint j-cycles $c_1, \ldots, c_k$ of $[\varphi]$. Let T be the orbifold quotient $S/\langle\varphi\rangle$, which has k marked points with ramification index 1 and 2 marked points with ramification index j.

(i) The set of fixed points of $[\varphi]$ in the ordered moduli space $\mathfrak{M}(S)$ has $|(\mathbb{Z}/j\mathbb{Z})^*|$ disjoint connected components $\mathcal{C}_\zeta$, respectively consisting ordered markings of the form

$$(1, \zeta, \ldots, \zeta^{j-1}, a_1, a_1\zeta, \ldots a_1\zeta^{j-1}, \ldots, a_{k-1}, a_{k-1}\zeta, \ldots a_{k-1}\zeta^{j-1}, 0, \infty).$$

Here ζ runs through the primitive j-th roots of unity. Each component $\mathcal{C}_\zeta$ is isomorphic to a copy of $(\mathbb{P}^1 - \{0, 1, \zeta, \ldots \zeta^{j-1}, \infty\})^{k-1}$ minus the $j(k-1)$ lines $a_i = a_r\zeta^s$ for $r \neq i$, $0 \leq s \leq j-1$, and is thus defined over $\mathbb{Q}^{ab}$.

(ii) *The special locus of φ in the quotient space $\mathfrak{M}(S)/G_\varphi = \mathfrak{M}_{G_\varphi}(S)$ also consists of $|(\mathbb{Z}/j\mathbb{Z})^*|$ disjoint connected components C_ζ, the image of the $\mathcal{C}_\zeta$. Each C_ζ is isomorphic to*

$$(\mathbb{P}^1 - \{0, 1, \infty\})^{k-1} - \Delta \simeq \mathfrak{M}(T) \simeq \mathfrak{M}_{0,k+2},$$

where Δ denotes the multi-diagonal of points with $x_i = x_j$ for some $i \neq j$ and is thus defined over $\mathbb{Q}$; however the embeddings $\mathfrak{M}(T) \to C_\zeta \subset \mathfrak{M}(S)/G_\varphi$ are defined over $\mathbb{Q}^{ab}$.

(iii) *In the unordered moduli space $\mathfrak{M}_{S_n}(S) = \mathfrak{M}(S)/S_n$, the special locus of φ consists of a single connected component $\mathcal{C}$. It is isomorphic to the moduli space of $\mathfrak{M}_G(T)$ which is the quotient of $\mathfrak{M}(T)$ by the group G of all "admissible" permutations, i.e., permutations of marked points having the same ramification index, and the space $\mathfrak{M}_G(T)$ and the embedding $\mathfrak{M}_G(T) \to \mathcal{C} \subset \mathfrak{M}_{S_n}(S)$ are defined over $\mathbb{Q}$.*

Similarly, in characteristic p, we can think about the points having non-trivial special automorphism group in the ordered moduli space to determine the special loci in the unordered moduli space.

3.1.2 Examples in characteristic 5

Example 3.1.2. In the case $\mathfrak{M}_{0,5}$, first we calculate the points with special automorphism group. Consider the permutation $\tau = (12345)$ and a point $(\lambda, 0, 1, \infty, \mu)$ in

$\mathfrak{M}_{0,5}$ in standard representation (with three marked points fixed at $0, 1$ and ∞). The action of τ on the point takes it to $(\mu, \lambda, 0, 1, \infty)$, and then the transformation by the automorphism $x \mapsto \frac{x-\lambda}{\lambda x-\lambda}$ brings it back to $(\frac{\mu-\lambda}{\lambda\mu-\lambda}, 0, 1, \infty, \frac{1}{\lambda})$. The fixed points of τ are given by (λ, μ) with

$$\lambda = \frac{\mu - \lambda}{\lambda\mu - \lambda} \text{ and } \mu = \frac{1}{\lambda},$$

so λ is a root of $\lambda^3 - 2\lambda^2 + 1$. One root is $\lambda = 1$, but this is excluded in $\mathfrak{M}_{0,5}$. The remaining roots are $\lambda = \frac{1\pm\sqrt{5}}{2} = \frac{1}{2} = 3$ (since this is in characteristic 5); so the only fixed point of τ is $(3, 0, 1, \infty, 2)$. In fact, the point $(3, 0, 1, \infty, 2)$ is equivalent to the point $(0, 1, 2, 3, 4)$ in the moduli space $\mathfrak{M}_{0,5}$ since $(3, 0, 1, \infty, 2)$ transforms to the point $(0, 1, 2, 3, 4)$ by the linear transformation $x \mapsto \frac{3x-9}{x-9}$. Then intuitively, we can see that $(0, 1, 2, 3, 4)$ is fixed by a translation τ.

Remark 3.1.3. In Example 3.1.2, τ only fixes one point in characteristic 5, while it fixes two points in characteristic 0 (cf. Theorem 3.1.1).

Example 3.1.4. In the case $\mathfrak{M}_{0,10}$, first we calculate the points with special automorphism group. Given a permutation $\tau = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ and a point $(x_1, \ldots, x_7, 1, 0, \infty)$ in $\mathfrak{M}_{0,10}$ in standard representation (with three components fixed at $0, 1$ and ∞), then the action of τ on the point takes it to $(\infty, x_1, \ldots, x_7, 1, 0)$. But in characteristic 5, there is no transformation can bring it back to the original point $(x_1, \ldots, x_7, 1, 0, \infty)$ by the similar calculation in Example 3.1.2. So there is no fixed point of τ in $\mathfrak{M}_{0,10}$.

Remark 3.1.5. In Example 3.1.4, τ has no fixed point in characteristic 5, while it fixes two disconnected one-dimensional components in characteristic 0 (cf. Theorem 3.1.1).

3.1.3 Examples in characteristic 3

Example 3.1.6. In the case $\mathfrak{M}_{0,4}$, first we calculate the points with special automorphism group. Consider the permutation $\tau = (123)$ and a point $(\lambda, 1, 0, \infty)$ in $\mathfrak{M}_{0,4}$ in standard representation (with three components fixed at $0, 1$ and ∞). The action of τ on the point takes it to $(0, \lambda, 1, \infty)$. Then the transformation $y \mapsto y + 2$ brings the point $(0, 2, 1, \infty)$ back to the original point $(2, 1, 0, \infty)$. So the fixed point of τ in $\mathfrak{M}_{0,4}$ is $(2, 1, 0, \infty)$.

Remark 3.1.7. In Example 3.1.6, τ only fixes one point in characteristic 3, while it fixes two points in characteristic 0 (cf. Theorem 3.1.1).

3.1.4 Special loci of genus 0 in characteristic p

First, here is a result to describe the finite-order automorphism in genus zero algebraic curves with marked points.

Proposition 3.1.8. *Let K be an algebraically closed field. Every finite-order automorphism of $\mathbb{P}^1_K$ with marked points is the conjugacy class of a rotation around an axis (i.e., by multiplying roots of unity) or the conjugacy class of a translation (i.e., by adding an element in K).*

Proof. We know that the group of automorphisms of $\mathbb{P}^1_K$ is isomorphic to $\mathrm{PGL}(2, K)$. Since K is an algebraically closed field, by Jordan canonical form, every element in $\mathrm{PGL}(2, K)$ is either conjugate to

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

or conjugate to

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda, \lambda_1, \lambda_2$ are non-zero elements in K.

If it is conjugate to A, then it has one fixed point ∞ and the corresponding fractional linear transformation is $z \mapsto z + \frac{1}{\lambda}$, which is just a translation. If the automorphism is of finite order, then the translation can only happen in characteristic p.

If it is conjugate to B, then it has two fixed points 0 and ∞ and the corresponding fractional linear transformation is $z \mapsto \frac{\lambda_1}{\lambda_2} z$, which is a composition of a rotation and a dilation. If the automorphism is finite order, then it is just a rotation; this happen both in characteristic 0 and p. In characteristic 0, the rotation can have any order; in characteristic p, its order is prime to p, because there are no primitive p^{th} roots of unity. $\square$

The following result describes the special loci in characteristic p in the case $g = 0$ and $n \geq 3$ (proved by Schneps [Sch02a] in characteristic 0):

Theorem 3.1.9. *Let X be a marked curve of genus 0 with n marked points $x_1, \dots, x_n$ over an algebraically closed field K, and φ be a finite order automorphism of X which fixes $\{x_1, \dots, x_n\}$ as a set. Let $[\varphi]$ denote the permutation of marked points induced by φ. Let g' be the genus of X/φ and n' be the number of marked points coming from the marked points of X. If $\mathfrak{M}_{0,[n]}(\varphi)$ is not empty, then φ is of the form $c_1 \cdots c_k$ where the c_i are disjoint cycles of length j such that $jk = n$ or $jk = n-1$ or $jk = n-2$. Moreover:*

(a) If $p \not| j$, then $\mathfrak{M}_{0,[n]}(\varphi)$ has the same description in characteristic 0 and p.

(b) If $p|j$ and $j > p$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is empty.

(c) If $p = j$ and $jk = n-2$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is empty. If $p = j$ and $jk = n-1$ or $jk = n$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to quotient of $\{\mathbb{P}^1 - \{0, 1, \infty\}\}^{k-2} - \Delta$ by S_k, where Δ denotes the multi-diagonal of points with $x_i = x_j$ for some $i \neq j$.

Proof. By Proposition 3.1.8, we know that if $\mathfrak{M}_{0,[n]}(\varphi)$ is not empty, then φ is of the form $c_1 \cdots c_k$ where the c_i are disjoint cycles of length j such that $jk = n$ or $jk = n-1$ or $jk = n-2$.

(a) If $p \not| j$, then we have the j-th roots of unity. Since the proof for characteristic 0 (cf. Schneps [Sch02a, Theorem 3.5.1]) is only involved the pure group theory, it also works for characteristic p in this case.

(b) If $p|j$ and $j > p$, then φ has no fixed point either as a rotation or as a translation.

(c) If $p = j$, then φ is a translation.

If $jk = n - 2$, then φ has no fixed point since a translation can not fix two points pointwise.

If $jk = n - 1$, then in the ordered moduli space $\mathfrak{M}_{0,n}$, we know that φ fixes $p - 1$ disjoint connected components, each component is given by

$C_i = (0, i, \ldots, (p-1)i, a_1, \ldots, a_1 + (p-1)i, \ldots, a_{k-1}, \ldots, a_{k-1} + (p-1)i, \infty)$,

where $i = 1, \ldots, p - 1$ and $a_1, \ldots, a_{k-1}$ are any numbers in the field K such that all the marked points are distinct. In the unordered moduli space $\mathfrak{M}_{0,n}$, all the components C_i (as well as all those components corresponding to other rotations having the same cycle type as φ) become identified. So $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to one of C_i, say C_1, modulo its stabilizer in $S_n = S_{jk+1}$. We could determine its stabilizer by the similar procedure with Schneps' proof in (cf. [Sch02a, Theorem 3.5.1]) (where the proof only involved the pure group theory). In fact, the stabilizer of C_1 is generated by two natural subgroups: the first, of order $k!$, corresponding to permuting the k disjoint cycles of $[\varphi]$; the second, of order j^k, is generated by the j cycles themselves. After compute the quotient of C_1 by its stabilizer, we get $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to quotient of $\{\mathbb{P}^1 - \{0, 1, \infty\}\}^{k-2} - \Delta$ by all permutations of marked points in X/φ which comes from the marked points with the same ramification index in X, i.e., S_k, where Δ denotes the multi-diagonal of points with $x_i = x_j$.

If $jk = n$, then by a similar calculation with the case $jk = n-1$, we get $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to quotient of $\{\mathbb{P}^1 - \{0,1,\infty\}\}^{k-2} - \Delta$ by S_k, where Δ denotes the multi-diagonal of points with $x_i = x_j$.

$\square$

Theorem 3.1.9 shows that there is no automorphism of order divisible by p in characteristic p unless the order is exactly p. Also there is an automorphism of order p, viz. translation, but this automorphism behaves differently from an automorphism of the same order in characteristic 0, as the examples in sections 3.1.2 and 3.1.3 show.

3.2 Special loci in genus 1

There is also a generalization of Theorem 3.1.9 to higher genus. In particular, for $g = 1$, we give some results in the following proposition.

Proposition 3.2.1. *Let X be a marked curve of genus 1 with n marked points $x_1, \ldots, x_n$ over an algebraically closed field K of characteristic $\neq 2, 3$ and let φ be a finite order automorphism of X. Let $[\varphi] \in S_n$ be a permutation of marked points for $n \geq 5$, where $[\varphi]$ is the corresponding permutation of marked points of φ. Suppose that $\mathfrak{M}_{1,[n]}(\varphi)$ is not empty, and write $[\varphi]$ as a product of disjoint cycles $c_1 \cdots c_k$. Then either* *(i)* *the* c_i *are of the same length* j *such that* $jk = n$*; or*

(ii) *the* c_i *are of the same length* $j = 2$ *such that* $jk = n$ *or* $n-1$ *or* $n-2$ *or* $n-3$ *or* $n-4$*; or*

(iii) *the* c_i *are of the same length* $j = 3$ *such that* $jk = n$ *or* $n-1$ *or* $n-2$ *or* $n-3$*; or*

(iv) *the* c_i *are of the same length* $j = 4$ *such that* $jk = n$ *or* $n-1$ *or* $n-2$*; or*

(v) *one of the* c_i *is of length* 2 *and the others are of the same length* $j = 4$ *such that* $jk = n-2$ *or* $jk = n-3$ *or* $jk = n-4$*; or*

(vi) *the* c_i *are of the same length* $j = 6$ *such that* $jk = n$ *or* $jk = n-1$*; or*

(vii) *one of the* c_i *is of length* 2 *and the others are of the same length* $j = 6$ *such that* $jk = n-2$ *or* $jk = n-3$*; or*

(viii) *one of the* c_i *is of length* 3 *and the others are of the same length* $j = 6$ *such that* $jk = n-3$ *or* $jk = n-4$*; or*

(ix) *one* c_i *is of length* 2*, another is of length* 3*, and the remainder are all of the same length* $j = 6$ *such that* $jk = n-5$ *or* $jk = n-6$*.*

Before we give a proof of Proposition 3.2.1, let us recall the classification of automorphisms of elliptic curves [Sil86].

Theorem 3.2.2. *([Sil86, Theorem 10.1]) Let E/K be an elliptic curve. Then its automorphism group $Aut(E)$ is a finite group of order dividing 24. More precisely, the order of $Aut(E)$ is given by the following list:*

(a) 2 *if* $j(E) \neq 0, 1728$

(b) 4 *if* $j(E) = 1728$ *and* $char(K) \neq 2, 3$

(c) 6 *if* $j(E) = 0$ *and* $char(K) \neq 2, 3$

(d) 12 *if* $j(E) = 0 = 1728$ *and* $char(K) = 3$

(e) 24 *if* $j(E) = 0 = 1728$ *and* $char(K) = 2$

Now we give a proof for Proposition 3.2.1 by using the Riemann-Hurwitz formula and Theorem 3.2.2.

Proof. (of Proposition 3.2.1) Let X be a marked curve of genus 1 with n marked points $x_1, \ldots, x_n$ over an algebraically closed field K of characteristic $\neq 2, 3$ and let φ be a finite order automorphism of X. Let $[\varphi] \in S_n$ be a permutation of marked points for $n \geq 5$, where $[\varphi]$ is the corresponding permutation of marked points of φ. Let g_0 is the genus of $X/\langle\varphi\rangle$. Then the possible order m of φ and their branching data $(m_1, m_2, \ldots, m_r)$ are limited by the well-known Riemann-Hurwitz equation:

$$(2g-2)/m = (2g_0 - 2) + \sum_{i=1}^{r}(1 - \frac{1}{m_i}).$$

Here we only consider the Riemann-Hurwitz formula in the tame case since if char$(K) \neq 2, 3$, then all the branch coverings we considered are tame. Since we only consider the cyclic group $\langle\varphi\rangle$ of order m, let $M = \text{lcm}(m_1, m_2, \ldots, m_r)$, then the following conditions are satisfied ([Bre00, corollary 9.4]):

(i) $\text{lcm}(m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r) = M$ for all i;

(ii) M divides m, and if $g_0 = 0, M = m$;

(iii) $r \neq 1$, and if $g_0 = 0$, $r \geq 3$;

(iv) if M is even, the number of m_i divisible by the maximum power of 2 dividing M is even.

By Theorem 3.2.2, we know the possible numbers of m_i are $2, 3, 4, 6$. Combine the above conditions on Riemann-Hurwitz equation 3.2, we get the possible Galois coverings are:

(i) $m = 2$, $r = 4$, $m_i = 2$ for all i;

(ii) $m = 3$, $r = 3$, $m_i = 3$ for all i;

(iii) $m = 4$, $r = 3$, $(m_1, m_2, m_3) = (2, 4, 4)$;

(iv) $m = 6$, $r = 3$, $(m_1, m_2, m_3) = (2, 3, 6)$.

Since X has n marked points, according to the above possible Galois coverings, we can get the possible permutations $[\varphi]$ of marked points as in the list of this proposition. $\square$

Chapter 4

Splitting and surjectivity in genus zero

As discussed in Chapter 1 above, Schneps [Sch02a] introduced splitting and surjectivity conditions in the study of special loci for moduli of marked Riemann surfaces. She also proved that these conditions hold in the genus 0 case (i.e., for marked spheres). But her definitions used the fact that the spaces were defined over the complex numbers. This chapter considers the surjectivity and splitting conditions in all characteristics in genus 0 by first generalizing the definitions of the surjectivity and splitting conditions to characteristic p.

The permutation group S_n acts naturally on $\mathfrak{M}_{g,n}$ by permuting the marked points on the Riemann surfaces. For any subgroup $G \in S_n$, we write $\mathfrak{M}_{g,n}(G) = \mathfrak{M}_{g,n}/G$. Over $\mathbb{C}$, let φ be an element of finite order in the full mapping class group and let $T = S/\varphi$. We consider the set of points in Teichmüller space fixed by φ. The image of this set in the quotient moduli space $\mathfrak{M}_{g,n}(G)$ is the special locus of φ in $\mathfrak{M}_{g,n}(G)$ and denoted $\mathfrak{M}_{g,n}(G, \varphi)$ Schneps gave splitting and surjectivity conditions such that each component of $\widetilde{\mathfrak{M}}_{g,n}(G, \varphi)$ is as close to $\mathfrak{M}(T)$ as possible ([Sch02a]).

Here we consider splitting and surjectivity conditions in characteristic p. But to do that, we first find new characterizations of these conditions over the complex numbers which do not rely on mapping class groups or Teichmüller space. We then use those characterizations as the *definitions* in characteristic p. The original definition of splitting and surjectivity conditions involves the fundamental group of moduli spaces of curves, while the fundamental group of moduli spaces of curves in characteristic p is very complicated. Section 4.1 gives the geometric meaning of the surjectivity condition, which avoids the fundamental group of moduli spaces of curves (Proposition 4.1.5). And also gives an equivalent description of splitting condition by finite group theory using Proposition 4.1.9.

Section 4.2 shows that the surjectivity and splitting conditions hold in general for genus 0 (Theorem 4.2.5 and Theorem 4.2.6), which generalize the result of Schneps in characteristic 0.

4.1 Splitting and surjectivity over the complex numbers

Throughout this section, we only work over the complex numbers $\mathbb{C}$. Let S be a topological surface of type (g,n) and let $G \subset S_n$ be a subgroup. Let $\Gamma_G(S)$ be the preimage of G under the canonical surjection $\Gamma([S]) \to S_n$. If φ is a finite-order element of the full mapping class group $\Gamma([S])$, we let $\mathfrak{M}_G(S,\varphi)$ denote the image in $\mathfrak{M}_G(S)$ of the set of points in the Teichmüller space $\mathcal{T}(S) = \mathcal{T}_{g,n}$ which are fixed by φ under the canonical action of $\Gamma([S])$ on $\mathcal{T}(S)$.

Let $[\varphi]$ denote the permutation associated to φ, and let $G_\varphi \subset S_n$ be the group generated by the disjoint cycles of $[\varphi]$. Let us write $\mathfrak{M}_\varphi(S)$ for the quotient space $\mathfrak{M}_{G_\varphi}(S) = \mathfrak{M}(S)/G_\varphi$, and $\mathfrak{M}_\varphi(S,\varphi)$ for the whole of the special locus of φ in $\mathfrak{M}_\varphi(S)$. We also write $\Gamma_\varphi(S)$ for the group $\Gamma_{G_\varphi}(S)$, the preimage in $\Gamma([S])$ of $\langle[\varphi]\rangle$ under the surjection $\Gamma([S]) \to S_n$.

Remark 4.1.1. Since S is determined by (g,n), we also write $\Gamma(S)$ [resp. $\Gamma([S])$] as $\Gamma_{g,n}$ (resp. $\Gamma_{g,[n]}$) and $\mathfrak{M}(S)$ [resp. $\mathfrak{M}([S])$] as $\mathfrak{M}_{g,n}$ [resp. $\mathfrak{M}_{g,[n]}$].

Let $T = S/\varphi$ and we assume that all branch points of this cover (and their preimage) are marked points. Let g' denote the genus of T and n' the number of marked points; the fundamental group of T is given by generators and relations as

$$\pi_1(T) = \langle a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots c_{n'} | \prod_{i=1}^{g'}(a_i,b_i)c_1\cdots c_{n'} = 1\rangle.$$

The group of inertia-preserving automorphisms of $\pi_1(T)$, $Aut^*([\pi_1(T)])$ is defined to be the group of automorphisms

$$Aut^*([\pi_1(T)]) = \{\psi \in Aut^*(\pi_1(T)) | \exists \sigma \in S_n \text{ such that } \psi(c_i) \sim c_{\sigma(i)} \text{ for } 1 \leq i \leq n'\}$$

where $\sim$ means "is conjugate to".

Let $Aut^*([S/T])$ denote the subgroup of $Aut^*([\pi_1(T)])$ consisting of elements which preserve the subgroup $\pi_1(S) \in \pi_1(T)$. We introduce the following notations:

$$\Gamma_{[S/T]} = Aut^*([S/T])/Inn(\pi_1(T)) \subset Out^*([\pi_1(T)]) = \Gamma([T]),$$

and

$$\Gamma_{S/T} = Aut^*(S/T)/Inn(\pi_1(T)) = \Gamma_{[S/T]} \cap \Gamma(T) \subset \Gamma(T).$$

For any subgroups H and K of a group G, we denote $\mathrm{Norm}_H(K) = \mathrm{Norm}_G(K) \cap G$, where $\mathrm{Norm}_G(K)$ is the normalizer of K in G. For any element $g \in G$, we denote $\mathrm{Norm}_H(g) = \mathrm{Norm}_H(\langle g \rangle)$.

Now we give the definition of surjectivity condition of φ.

Definition 4.1.2. ([Sch02a, §4.2]) We say that a finite-order element $\varphi \in \Gamma([S])$ satisfies the *surjectivity condition* if $\Gamma_{S/T} = \Gamma(T)$, i.e., every element of $Aut^*(\pi_1(T))$ preserves the subgroup $\pi_1(S)$; in other words, the homomorphism

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle \varphi \rangle \to \Gamma(T)$$

is surjective.

Before we give the geometric meaning of the surjectivity condition, let us give some similar results of González-Díez and Harvey [GH92] in the case of type (g, n), i.e., Riemann surfaces with marked points.

Theorem 4.1.3. *Let S be a topological surface of type (g, n), and let φ be a finite-order element of $\Gamma([S])$. Assume that the quotient $T = S/\varphi$ is of genus g' with n' marked points, including all the branch points.*

(i) Denote by $\mathcal{T}(S, \varphi)$ the subset of points of the Teichmüller space $\mathcal{T}(S) = (T)_{g,n}$ fixed by φ. Then each component of $\mathcal{T}(S, \varphi)$ is isomorphic to $\mathcal{T}_{g',n'} = \mathcal{T}(T)$.

(ii) The set of elements of $\Gamma([S])$ globally preserving each component of $\mathcal{T}(S, \varphi)$ in $\mathcal{T}(S)$ is exactly the subgroup $Norm_{\Gamma([S])}(\varphi)$.

(iii) For every $G \subset S_n$ containing the permutation $[\varphi]$ associated to φ, the quotient $\widetilde{\mathfrak{M}}_G(S, \varphi) = \mathcal{T}(S, \varphi)/Norm_{\Gamma_G(S)}(\varphi)$ is isomorphic to the normalization of the special locus $\mathfrak{M}_G(S, \varphi) \subset \mathfrak{M}_G(S)$.

Remark 4.1.4 *(on the proof).* The proof can be given by following the proof of González-Díez and Harvey [GH92] in the case without marked points. But there are slight differences in the marked points case from González-Díez and Harvey's results. In the case without marked points, $\mathcal{T}(S, \varphi)$ is isomorphic to $\mathcal{T}_{g',n'} = \mathcal{T}(T)$, where $\mathcal{T}(S, \varphi)$ is irreducible ([GH92, Theorem B]). While in the marked points case, $\mathcal{T}(S, \varphi)$ might consist several irreducible components, each component is isomorphic to $\mathcal{T}_{g',n'} = \mathcal{T}(T)$. In fact, we can see that the map from $\mathcal{T}(S, \varphi)$ to $\mathcal{T}_{g',n'}$ is not bijective in the marked points case, since there could be different permutations of the marked points of S for any given $\{g', n'\}$. (Cf. [Harv71, Theorem 2 & Corollary 3])

In order to see the geometric meaning of the surjectivity condition, we have the following proposition.

Proposition 4.1.5. *(cf. [Sch02a, §4.2]) A finite-order element $\varphi \in \Gamma([S])$ satisfies the surjectivity condition if and only if the map from each component of $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$ to $\mathfrak{M}(T)$ is one-to-one, consisting only in forgetting the orbifold structure of $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$ due to the action of φ.*

Proof. Suppose that φ satisfies the surjectivity condition, then by definition of surjectivity, the homomorphism

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \to \Gamma(T)$$

is surjective. By Proposition [Sch02a, Proposition 4.1.2.], there is a canonical injective homomorphism

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \to \Gamma(T),$$

whose image is of finite index. Therefore,

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \to \Gamma(T)$$

is an isomorphism. By Theorem 4.1.3, since $\langle\varphi\rangle$ fixes every point of $\mathcal{T}(S,\varphi)$, the action of $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$ factors through the quotient group $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle$, and there is a canonical one-to-one correspondence

$$\mathcal{T}(S,\varphi)/Norm_{\Gamma_\varphi(S)}(\varphi) \leftrightarrow \mathcal{T}(S,\varphi)/(Norm_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle)$$

(the difference between the two spaces is hidden in the orbifold structure due to the action of $\langle\varphi\rangle$ fixing each point).

Now using Theorem 4.1.3, each component of $\mathcal{T}(S,\varphi)$ is isomorphic to $\mathcal{T}_{g',n'} = \mathcal{T}(T)$ and the normalization $\widetilde{\mathfrak{M}}_G(S,\varphi) \simeq \mathcal{T}(S,\varphi)/Norm_{\Gamma_{G(S)}}(\varphi)$, so each component of

$$\widetilde{\mathfrak{M}}_G(S,\varphi) \simeq \mathcal{T}(S,\varphi)/Norm_{\Gamma_{G(S)}}(\varphi) \leftrightarrow \mathcal{T}(S,\varphi)/(Norm_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle)$$

to

$$\mathcal{T}(T)/(Norm_{\Gamma_\varphi(S)}(\varphi) \leftrightarrow \mathcal{T}(T)/\Gamma(T) \simeq \mathfrak{M}(T)$$

is one-to-one, consisting only in forgetting the orbifold structure of $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$ due to the action of φ.

Conversely, suppose that the map from each component of $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$ to $\mathfrak{M}(T)$ is one-to-one, consisting only in forgetting the orbifold structure of $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$ due to the action of φ, then by above proof, we can see that this map induces a bijective homomorphism

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \to \Gamma(T).$$

Therefore, φ satisfies the surjectivity condition.

□

Remark 4.1.6. This was stated without mentioning components in [Sch02a]. Actually the geometric condition in above proposition differs slightly from the one in Schneps' paper [Sch02a], where she neglects to mention that the map from $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$

to $\mathfrak{M}(T)$ must restrict to connected component. As the reader can see the example $n = 5, j = 3$, then φ induces a permutation (123), and $\mathfrak{M}_\varphi(S,\varphi)$ consists of two disjoint points, while $\mathfrak{M}(T)$ consists of only one point. But in the case without marked points, $\widetilde{\mathfrak{M}}_\varphi(S,\varphi)$ is connected, this issue does not rise. The reader can see the paper [GH92].

The splitting condition is defined as follows.

Definition 4.1.7. ([Sch02a, §4.2]) A finite-order element $\varphi \in \Gamma([S])$ satisfies the *splitting condition* if the surjection

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \to \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \simeq \Gamma_{S/T}$$

splits; in other words, if we have a semi-direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle\varphi\rangle \rtimes \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle$$

Sometimes, the semi-direct product in Definition 4.1.7 becomes direct product, so we give the following definition.

Definition 4.1.8. A finite-order element $\varphi \in \Gamma([S])$ satisfies the *strong splitting condition* if we have a direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle\varphi\rangle \times \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle.$$

Actually, the splitting condition can be described by finite group theory (i.e., without mentioning $\Gamma_\varphi(S)$) using the following proposition.

Proposition 4.1.9. *Let φ be a finite-order element in $\Gamma([S])$ and let $[\varphi]$ denote the permutation of n marked points induced by φ. Then φ satisfies the splitting condition if and only if*

$$G_\varphi = \langle[\varphi]\rangle \rtimes G_\varphi/\langle[\varphi]\rangle.$$

Proof. First suppose that φ satisfies the splitting condition, i.e., we have a semi-direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle\varphi\rangle \rtimes \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle.$$

Then we want to show that

$$G_\varphi = \mathrm{Norm}_{G_\varphi}([\varphi]) = \langle[\varphi]\rangle \rtimes G_\varphi/\langle[\varphi]\rangle$$

holds (note: G_φ is abelian). According to the notations, it suffices to show that each $\varphi \in \Gamma([S])$ induces only one permutation $[\varphi]$. Recall that

$$\Gamma([S]) = \mathrm{Diff}^+([S])/\mathrm{Diff}^0(S),$$

where $\mathrm{Diff}^+([S]$ denotes the group of orientation-preserving diffeomorphisms of S which fixes $\{s_1, \ldots, s_n\}$ as a set, and $\mathrm{Diff}^0(S)$ is the subgroup of those which are isotopic to the identity via a family of diffeomorphisms $h_t : S \to S$ with $h_t(s_i) = s_i$, for $t \in [0, 1]$ and each i. Since there are only finite many s_i, they are discrete. So the element in $\mathrm{Diff}^0(S)$ does not change the permutation of the lifting $\varphi' \in \mathrm{Diff}^+([S])$ of φ. Therefore, φ induces only one permutation.

Now suppose that we have the pure group equality

$$G_\varphi = \langle [\varphi] \rangle \rtimes G_\varphi / \langle [\varphi] \rangle.$$

Then pick up one element φ in $\Gamma([S])$ such that it induces the permutation $[\varphi]$, then by notation, we can easily see that we have a semi-direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle \varphi \rangle \rtimes \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle \varphi \rangle.$$

Therefore, φ satisfies the splitting condition. □

Corollary 4.1.10. *In genus* 0*, let* φ *be a finite-order element in* $\Gamma([S])$ *and let* $[\varphi]$ *denote the permutation of* n *marked points induced by* φ*. let* $H_\varphi \subset G_\varphi$ *be generated by any choice of all but one of the disjoint cycles of* $[\varphi]$ *(*$H_\varphi = \{ id \}$ *if* $[\varphi]$ *is a single cycle). Then* φ *satisfies the splitting condition if and only if*

$$G_\varphi = \langle [\varphi] \rangle \times H_\varphi.$$

Proof. By [Sch02a, Theorem 4.3.1], we know in genus zero the splitting condition has the following strong form:

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle \varphi \rangle \times \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi).$$

Then by Proposition 4.1.9, we are done. □

Remark 4.1.11. In the case of the genus zero moduli spaces, the strong splitting condition ([Sch02a, Theorem 4.3.1]) and the surjectivity condition ([Sch02a, Theorem 4.3.2]) are always satisfied.

4.2 Splitting and surjectivity conditions in genus zero case in all characteristics

Let K be an algebraic closed field. We have considered the special loci of the moduli space $\mathfrak{M}_{g,n}$ or $\mathfrak{M}_{g,[n]}$ of algebraic curves over K. Now we give some notations which are suitable for all characteristics, unlike the previous section, where we used notations in terms of topological space S and T over the complex numbers.

The mapping class group $\Gamma_{g,[n]}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$. The unordered moduli space $\mathfrak{M}_{g,[n]}$ is realized as the quotient of the Teichmüller space $\mathcal{T}_{g,n}$ by the action of the mapping class group $\Gamma_{g,[n]}$. Similarly, the ordered moduli space $\mathfrak{M}_{g,n}$ is the quotient of $\mathcal{T}_{g,n}$ by the pure subgroup $\Gamma_{g,n}$ of $\Gamma_{g,[n]}$. The permutation group S_n acts naturally on $\mathfrak{M}_{g,n}$ by permuting the marked points on the Riemann surfaces. For any subgroup $G \in S_n$, we write $\mathfrak{M}_{g,n}(G) = \mathfrak{M}_{g,n}/G$.

Let φ be an element of finite order in the full or pure mapping class group (the group of homotopy classes of diffeomorphisms of a surface), then we consider the set of points in Teichmüller space fixed by φ. The image of this set in the quotient moduli space $\mathfrak{M}_{g,n}(G)$ is the special locus of φ in $\mathfrak{M}_{g,n}(G)$ and denoted $\mathfrak{M}_{g,n}(G, \varphi)$ (when $G = S_n$, we denote it $\mathfrak{M}_{g,[n]}(\varphi)$, which is the special locus in the unordered moduli space). In particular, let $[\varphi]$ denote the associated permutation of the marked points and let $G \subset S_n$ be the subgroup generated by the disjoint cycles of $[\varphi]$.

Over $\mathbb{C}$, let φ be an element of finite order in the full mapping class group and let $T = S/\varphi$. Schneps gave splitting and surjectivity conditions such that each component of $\widetilde{\mathfrak{M}}_{g,n}(G,\varphi)$ is as close to $\mathfrak{M}(T)$ as possible.

To apply it to characteristic p, we rephrase the surjectivity condition as the following:

Definition 4.2.1. Let X be a marked curve over K with n marked points $x_1, \ldots, x_n$, and α be a finite order automorphism of X which fixes $\{x_1, \ldots, x_n\}$ as a set. Let g' be the genus of X/α and n' be the number of marked points coming from the marked points of X. We say that α satisfies the *surjectivity condition* if and only if the map from each component of $\widetilde{\mathfrak{M}}_{g,n}(G,\alpha)$ to $\mathfrak{M}_{g',n'}$ is one-to-one, consisting only in forgetting the orbifold structure of $\widetilde{\mathfrak{M}}_{g,n}(G,\alpha)$ due to the action of α.

Remark 4.2.2. Definition 4.2.1 and Definition 4.1.2 are equivalent over $\mathbb{C}$ by Proposition 4.1.5.

Now we rephrase the splitting condition as the following:

Definition 4.2.3. Let X be a marked curve over K with n marked points $x_1, \ldots, x_n$, and α be a finite order automorphism of X which fixes $\{x_1, \ldots, x_n\}$ as a set. Let $[\alpha]$ denote the permutation induced by α, and let $G_\alpha \subset S_n$ be the group generated by the disjoint cycles of $[\alpha]$. We say that α satisfies the *splitting condition* if and only if

$$G_\alpha = \langle [\alpha] \rangle \rtimes G_\alpha / \langle [\alpha] \rangle.$$

Remark 4.2.4. a) Definition 4.2.3 and Definition 4.1.7 are equivalent over $\mathbb{C}$ by Proposition 4.1.9.

b) In particular, over $\mathbb{C}$, in genus 0, let $H_\alpha \subset G_\alpha$ be generated by any choice of all but one of the disjoint cycles of $[\alpha]$ ($H_\alpha = \{ \text{ id } \}$ if $[\alpha]$ is a single cycle). Then α satisfies the splitting condition if and only if

$$G_\alpha = \langle [\alpha] \rangle \times H_\alpha.$$

In fact, this followed from the Corollary 4.1.10. So we can adopt this as the definition of the splitting condition in all characteristics for genus 0.

The following theorems show that the surjectivity and splitting conditions hold in general for genus 0 (which generalize the result of Schneps in characteristic 0):

Theorem 4.2.5. *Let X be a marked curve of genus 0 over K with n marked points $x_1, \ldots, x_n$, and let α be a finite order automorphism of X which fixes $\{x_1, \ldots, x_n\}$ as a set. Let $[\alpha]$ denote the permutation of marked points induced by α. Then α satisfies the surjectivity condition in all characteristics.*

Proof. In characteristic 0, α satisfies the surjectivity condition by [Sch02a, Theorem 4.3.2]).

In characteristic p, by Theorem 3.1.9, we know that $[\alpha]$ is of the form $c_1 \cdots c_k$ where the c_i are disjoint cycles of length j such that $jk = n$ or $jk = n - 1$ or $jk = n - 2$. Then we prove the following cases by Theorem 3.1.9:

(a) If $p \not| j$, then $\mathfrak{M}_{0,[n]}(\alpha)$ has the same description in characteristic 0 and p. So α satisfies the surjectivity condition by Schneps [Sch02a, theorem 4.3.2]).

(b) If $p|j$ and $j > p$, then $\mathfrak{M}_{0,[n]}(\alpha)$ is empty. By the proof of Theorem 3.1.9, we can see that for any $G \subset S_n$, $\mathfrak{M}_{0,n}(G, \alpha)$ is empty. So it is trivial that α satisfies the surjectivity condition.

(c) If $p = j$ and $jk = n - 2$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is empty. So it is trivial that α satisfies the surjectivity condition. If $p = j$ and $jk = n - 1$ or $jk = n$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to quotient of $\{\mathbb{P}^1 - \{0, 1, \infty\}\}^{k-2} - \Delta$ by S_k, where Δ denotes the multi-diagonal of points with $x_i = x_j$ for some $i \neq j$. By the proof of Theorem 3.1.9, We can see that for any $G \subset S_n$, the map from each component of $\mathfrak{M}_{0,n}(G, \alpha)$ to $\mathfrak{M}_{0,n'}$ is one-to-one. Therefore α satisfies the surjectivity condition.

□

Theorem 4.2.6. *Let X be a marked curve of genus 0 over K with n marked points $x_1, \ldots, x_n$, and let α be a finite order automorphism of X which fixes $\{x_1, \ldots, x_n\}$ as a set. Let $[\alpha]$ denote the permutation induced by α, and let $G_\alpha \subset S_n$ be the group generated by the disjoint cycles of $[\alpha]$. Let $H_\alpha \subset G_\alpha$ be generated by any choice of all but one of the disjoint cycles of $[\alpha]$ ($H_\alpha = \{ id \}$ if $[\alpha]$ is a single cycle). Then α satisfies the splitting condition in all characteristics.*

Proof. First observe that α satisfies the splitting condition if and only if

$$G_\alpha = \langle [\alpha] \rangle \times H_\alpha$$

Now we prove this equality by group theory.

Take a subgroup $H_\alpha \subset G_\alpha$ as in the statement of the theorem. We prove the equality in the following two steps.

Step 1: we show that G_α is generated by H_α and $\langle [\alpha] \rangle$. Let $g \in G_\alpha$. Then if $g \in H_\alpha$, we are done. If g is not in H_α, then since $g \in G_\alpha$, there exists an integer m such that $g[\alpha]^m \in H_\alpha$. Then there exists an element $h \in H_\alpha$ such that $g = h[\alpha]^{-m}$.

Step 2: we show that $H_\alpha \cap \langle [\alpha] \rangle = 1$. We write $[\alpha] = c_1 \ldots c_k$ as a product of k disjoint cycles c_i each of length j, and assume that H_α is generated by $c_1, \ldots, c_{k-1}$. Then there is no non-trivial power of $[\alpha]$ lies in H_α since any non-trivial power of $[\alpha]$ has some power of c_k. So we are done. $\square$

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