Non-negatively Curved Cohomogeneity One Manifolds

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ABSTRACT

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A Riemannian manifold M is called cohomogeneity one if it admits an isometric action by a compact Lie group G and the orbit space is one dimension. Many new examples of non-negatively curved manifolds were discovered recently in this category. However not every cohomogeneity one manifold carries an invariant metric with non-negative sectional curvature. We show a large family of cohomogeneity one manifolds is obstructed to have a non-negatively curved metric. It generalizes the first examples obtained by K. Grove, B. Wilking, L. Veridiani and W. Ziller.

The cohomogeneity one manifolds which have a small family of invariant metrics are also studied. If the principal isotropy action has three or less summands and G is a simple Lie group, then the manifold is equivariantly diffeomorphic to a double or a symmetric space.

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Chapter 1

Introduction

Riemannian manifolds with non-negative sectional curvature have been of interest since the beginning of the global Riemannian geometry. The general structure theorems in this subject are few. By Gromov's betti number theorem, the total betti number is bounded above by an explicit constant which only depends on the dimension, see [Gr]. The celebrated soul theorem of J.Cheeger and D.Gromoll in [CG] says that the open(noncompact and complete) manifold M has a totally geodesic submanifold S, called the soul, such that M is diffeomorphic to the normal bundle of S.

The understanding of the fundamental group is relatively well understood. A theorem of Synge asserts that an even dimensional orientable manifold with positive curvature is simply connected. An odd dimensional positively curved manifold is orientable by Synge and the fundamental group is finite by a classical theorem of

Bonnet and Myers. For a non-negatively curved manifold, M.Gromov showed that the number of the elements in the generating set is bounded above by a universal constant that only depends on the dimension. J.Cheeger and D.Gromoll proved that the fundamental group is virtually abelian, i.e, it contains an abelian subgroup with finite index. Recently V.Kapovitch, A.Petrunin and W.Tuschmann proved that the fundamental group contains a nilpotent subgroup and the index is bounded from above by a constant which only depends on the dimension, see [KPT].

There are some conjectures concerning the general structure of non-negatively curved manifolds.

- (Hopf) There exists no metric with positive sectional curvature on $\mathbb{S}^2 \times \mathbb{S}^2$, or more generally on products of two compact manifolds.
- (Hopf) A manifold with non-negative curvature has non-negative Euler characteristic. An even dimensional manifold with positive curvature has positive Euler characteristic.
- (Bott) A non-negatively curved manifold is rationally elliptic, i.e., has finite dimensional rational homotopy.

Though there are many examples of such manifolds, methods of constructing them are few. Apart from taking products, most examples come from compact Lie groups and their quotients. These include all homogeneous spaces and biquotients. Using a gluing method, J.Cheeger constructed non-negatively curved metric on the connect sum of any two rank one symmetric spaces, see [Ch].

A breakthrough came with K.Grove and W.Ziller's generalization of this gluing method to the cohomogeneity one manifolds in [GZ1].

Definition 1.0.1. A manifold M is called a *cohomogeneity one manifold* if there exists a compact Lie group G acting on M by isometries and the cohomogeneity of the action, defined as $\mathrm{cohom}(M,G) = \dim(M/G)$, is equal to 1.

Since the orbit space is one dimensional, it is either a circle or a closed interval I. In the former case, M always carries a G invariant metric with non-negative sectional curvature. In the latter, there are precisely two singular orbits $B_\pm$ with isotropy groups $K^\pm$ corresponding to the endpoints of I and principal orbits corresponding to the interior points with isotropy group H. By the slice theorem, $K^\pm/H$ are spheres $\mathbb{S}^{l_\pm - 1}$ ($l_\pm \geq 1$), and M can be reconstructed by gluing two disk bundles along a principal orbit as follows

$$M = G \times_{K^-} \mathbb{D}^{l_-} \cup_{G/H} G \times_{K^+} \mathbb{D}^{l_+}, \tag{1.0.1}$$

where $\mathbb{D}^{l_\pm}$ is the normal disk to $B_\pm$. Therefore we can identify M with the groups $H \subset \{K^-, K^+\} \subset G$ by the gluing construction (1.0.1).

Theorem (Grove-Ziller)**.** *A compact cohomogeneity one manifold with* $l_\pm \leq 2$ *has an invariant metric with non-negative sectional curvature.*

They showed that a surprisingly rich class in cohomogeneity one manifolds satisfies the condition $l_\pm = 2$. It includes 10 of the 14(unoriented) exotic spheres in dimension 7, the 4 (oriented) diffeomorphism types which are homotopy equivalent to $\mathbb{R}P^5$, the total space of every vector bundle and every sphere bundle over $\mathbb{S}^4$, all principal $SO(k)$ bundle which are not spin over $\mathbb{C}P^2$ and the associated sphere bundles, and vector bundles, see [GZ1] and [GZ3].

In the same paper, it was conjectured that any cohomogeneity one manifold admits a non-negatively curved metric. This turns out to be false. The first examples of an obstruction were discovered in [GVWZ]:

Theorem (Grove-Verdiani-Wilking-Ziller)**.** *For each pair* (l_-, l_+) *with* $(l_-, l_+) \neq (2,2)$ *and* $l_\pm \geq 2$ *there exist infinitely many cohomogeneity one manifolds that do not carry an invariant metric with non-negative sectional curvature.*

The most interesting ones in this Theorem are the higher dimensional Kervaire spheres (of dimension 9 and up) which are known to be the only exotic spheres that can carry a cohomogeneity one action.

In light of the construction of examples with non-negative sectional curvature and the examples of obstructions to non-negatively curved metrics, it is important to answer the question raised by W.Ziller in [Zi1]:

"How large is the class of cohomogeneity one manifolds that admit a non-negatively curved metric?"

We concentrate here on finding further obstructions. We will generalize the examples in [GVWZ] to a larger family:

Theorem 1. *Let $K'/H' = \mathbb{S}^k$ with $k \geq 2$ and $\rho : K' \longrightarrow SO(m)$ be an irreducible faithful representation such that $m \geq k+2$ and $\rho(H') \subset SO(m-1)$. For any integer $n > m+1$, set $G = SO(n)$ and*

$$\begin{aligned} K^- &= \rho(K') \cdot SO(n-m) \subset SO(m)SO(n-m) \subset SO(n) \\ K^+ &= \rho(H') \cdot SO(n-m+1) \subset SO(m-1)SO(n-m+1) \subset SO(n) \\ H &= \rho(H') \cdot SO(n-m) \subset SO(n), \end{aligned} \tag{1.0.2}$$

then the cohomogeneity one manifold M defined by the groups $H \subset \{K^-, K^+\} \subset G$ does not admit a G invariant metric with non-negative sectional curvature.

In [GVWZ], this Theorem was proved under the additional assumption that the slice representation of K' is not in the symmetric square $\mathrm{Sym}^2\rho$ and $\rho(K')$ does not act transitively on the sphere $\mathbb{S}^{m-1} = SO(m)/SO(m-1)$.

Theorem 1 is optimal in the sense that if $m = k+1$, the manifold M does admit an invariant non-negatively curved metric, since it is diffeomorphic to the homogeneous space $SO(n+1)/(\rho(K')\cdot SO(n-m+1))$ endowed with the cohomogeneity one action of $SO(n) \subset SO(n+1)$.

The requirement of faithfulness on ρ is very restrictive and excludes many representations of K'. As a next generalization, Theorem 1 can be extended to the case where the representation ρ is neither irreducible nor faithful.

Theorem 2. *Let K', H' as before and $\rho : K' \longrightarrow SO(m)$ be an almost faithful representation such that $\ker\rho \subset H'$ and there exists a nonzero vector v_0 in the* *representation space V which is fixed by $\rho(H')$ and* $\dim \operatorname{Span}\{\rho(K').v_0\} \geq k+2$. *For any integer $n > m+1$, the cohomogeneity one manifold M defined by the groups $H \subset \{K^-, K^+\} \subset G$ in (1.0.2) does not admit a G invariant metric with non-negative sectional curvature.*

In this Theorem the condition that $\ker \rho \subset H'$ is necessary to ensure that K^-/H is a sphere. The existence of the particular vector v_0 can be interpreted as a condition on ρ that it contains a *class one* representation whose degree is bigger or equal to $k+2$. Class one representations are well studied, see e.g. [VK] and [Wa]. The definition and some properties will be given in section 2.1 and Appendix A.

If the representation ρ is over the complex or the quaternions, i.e., the image of K' is in the unitary or the symplectic group, similar results still hold.

Let $G(m)$ be the unitary group $U(m)$ in the complex case and the symplectic group $Sp(m)$ in the quaternionic case, then we have

Theorem 3. *Let $K'/H' = \mathbb{S}^k$ with $k \geq 2$ and $\rho : K' \longrightarrow G(m)$ be an almost faithful complex or quaternionic representation with* $\ker \rho \subset H'$. *Suppose there exists a nonzero vector v_0 in the representation space V which is fixed by $\rho(H')$ and* $\dim \operatorname{Span}\{\rho(K').v_0\} \geq a_0(k+2)$. *$a_0$ equals to $\frac{1}{2}$ in the complex case and $\frac{1}{4}$ in the quaternionic case. For any integer $n \geq m+2$, set $G = G(n)$ and*

$$\begin{aligned} K^- &= \rho(K') \times G(n-m) \subset G(m) \times G(n-m) \subset G(n) \\ K^+ &= \rho(H') \times G(n-m+1) \subset G(m-1) \times G(n-m+1) \subset G(n) \\ H &= \rho(H') \times G(n-m) \subset G(n), \end{aligned} \tag{1.0.3}$$

then the cohomogeneity one manifold M defined by the groups $H \subset \{K^-, K^+\} \subset G$ does not admit a G invariant metric with non-negative sectional curvature.

Now we turn to the question of constructions of non-negatively curved metric on cohomogeneity one manifolds.

The simplest example is the so called *double*. A cohomogeneity one manifold M is called a double if M admits a cohomogeneity one action by G with $K^+ = K^-$. Then M is a union of two identical disk bundles. On each disk bundle, we put a non-negatively curved invariant metric with totally geodesic boundary. On the boundaries of the two disk bundles, the metrics on G/H are the same so that we can glue them together. So a double carries a non-negatively curved metric.

Another example is one which has a totally geodesic principal orbit with normal homogeneous metric. The normal homogenous metric on G/H is the one which is induced by the bi-invariant metric on G. Then M can be viewed as a union of two homogeneous disk bundles which are glued along the common totally geodesic boundary. So the classification of such manifolds reduces to the classification of non-negatively curved *cohomogeneity one homogeneous disk bundles* $E = G \times_K V \longrightarrow G/K$ with *normal homogeneous collar*. The bundle E is called cohomogeneity one if K acts transitively on the unit sphere $\mathbb{S}^{l-1} \subset V$ with isotropy, henceforth denoted by $H \subset K$. Normal homogenous collar means that outside a compact set, the metric is G-equivariantly isometric to the Riemannian product of an interval and G/H with a normal homogeneous metric. So we can identify the bundle E with the group triple $(H \subset K \subset G)$ with $K/H = \mathbb{S}^{l-1}$. If two different bundles E and E' share the common pair (H, G), then the union M has the group diagram $H \subset \{K^- = K, K^+ = K'\} \subset G$.

The rank of the bundle is the dimension of V. The bundle E is called *essentially trivial* if it has the form $E = (G \times L)_{K \times L} V \longrightarrow G/K$, where the action of $\{1\} \times L \subset K \times L$ on V is transitive on the unit sphere in V. In this case, the existence of such a metric follows from a construction in [STu]. In the proof of Grove-Ziller's Theorem, they showed that any cohomogeneity one homogeneous vector bundle with rank at most 2 carries a non-negatively curved invariant metric with normal homogeneous collar. L.Schwachhöfer and K.Tapp considered the general case in [STa] and they prove

Theorem (Schwachhöfer-Tapp). *Let $E := G \times_K V \longrightarrow G/K$ be a cohomogeneity one homogeneous vector bundle which is essentially non-trivial. If E admits a G-invariant metric with nonnegative curvature and normal homogeneous collar, then the rank of this bundle must be one of* $2, 3, 4, 6,$ *or* 8.

They obtained a complete classification when the rank is equal to 3, 6 or 8. In this case, the cohomogeneity one manifold that arise from the union of two such bundles, other than a double, is the one that has group diagram $G_2 \subset \{Spin_+(7), Spin_-(7)\} \subset Spin(8)$ where $Spin_\pm(7)$ are two different embeddings of $Spin(7)$ in $Spin(8)$. However this manifold is equivariantly diffeomorphic to the sphere $\mathbb{S}^{15}$, see, for examples, [GWZ]. They also exhibited some new examples when the rank is 4

One different approach which has potential applications both to the existence and non-existence question is to study cohomogeneity one manifolds with a small family of invariant metrics. The simplicity of the metric will enable us to construct new non-negatively curved manifolds or find more examples which are obstructed. Since the union of the principal orbits in M are open and dense, the metric is determined by its restriction to the principal orbits. On a principal orbit G/H, the metric is G left invariant. On the tangent space $T_{[\mathrm{id}]}(G/H)$ at the left coset of the identity element it is Ad_H invariant, so it is also Ad_{H_0} invariant where H_0 is the identity component of H. $\mathrm{Ad}(H_0)$ sends X to $\mathrm{Ad}(h)X = hXh^{-1}$ for any $h \in H_0$ and $X \in T_{[\mathrm{id}]}(G/H)$. When the number of the irreducible summands of $\mathrm{Ad}(H_0)$ is small, we have the following rigidity result.

Theorem 4. *Suppose G be a simple compact Lie group, M be a compact simply connected Riemannian manifold, and the action of G on M be cohomogeneity one with principal isotropy subgroup H. Let H_0 be the identity component of H. If the adjoint action $\mathrm{Ad}(H_0)$ on the principal orbit G/H has 3 or less irreducible summands, then M is G equivariantly diffeomorphic to a symmetric space or a double.*

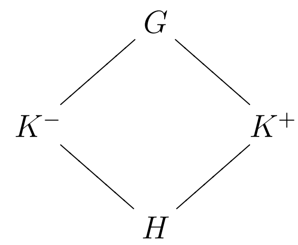
Chapter 2

Obstructions to the Non-negative Curved Metrics

2.1 Preliminaries

In this section, we recall some basic and well-known facts about cohomogeneity one manifolds. For more detail, we refer to, for example, [AA] and [GWZ].

As we have seen, there are precisely two non-principal orbits $B_\pm$ in a cohomogeneity one manifold. Suppose M is endowed with an invariant metric g and the distance between the two non-principal orbits is L. Let $c(t)$, $t \in \mathbb{R}$ be a geodesic minimizing the distance with $c(0) = p_- \in B_-$ and $c(L) = p_+ \in B_+$. The isotropy subgroups at $p_\pm$ are denoted by $K^\pm$ and the principal isotropy subgroup at any point $c(t)$, $t \in (0, L)$, is denoted by H. We can draw the following group diagram for the manifold M:



The group diagram $H \subset \{K^-, K^+\} \subset G$ is not uniquely determined by the manifold since one can switch K^- with K^+, change g to another invariant metric, or choose another minimal geodesic $c(t)$.

Definition 2.1.1. Two group diagrams are called *equivalent* if they determine the same cohomogeneity one manifold up to equivariant diffeomorphism.

The following characterizes which two group diagrams are equivalent, see [GWZ].

Lemma 2.1.2. *Two group diagrams $H \subset \{K^-, K^+\} \subset G$ and $\tilde{H} \subset \left\{\tilde{K}^-, \tilde{K}^+\right\} \subset G$ are equivalent if and only if after possibly switching the roles of K^- and K^+, the following holds: There is an element $b \in G$ and $a \in N(H)_0$ where $N(H)_0$ is the identity component of the normalizer of H, with $\tilde{K}^- = bK^-b^{-1}$, $\tilde{H} = bHb^{-1}$, and $\tilde{K}^+ = abK^+b^{-1}a^{-1}$.*

Let $C \subset M$ be the image of the minimal geodesic $c(t)$. Then the Weyl group $\mathcal{W}$ of the $G-$action on M is by definition the stablizer of C modulo its kernel H. $\mathcal{W}$ is characterized by the following proposition.

Proposition 2.1.3. *The Weyl group $\mathcal{W}$ of a cohomogeneity one manifold is a dihedral subgroup of $N(H)/H$. It is generated by involutions $w_\pm \in (N(H) \cap K^\pm)/H$ and*

$C/\mathcal{W} = M/G = [0, L]$. *Each of these involutions can be represented as an element* $a \in K^{\pm} \cap N(H)$ *such that* a^2 *but not* a *lies in* H.

From the group action, the invariant metric is determined by its restriction to the minimal geodesic $c(t)$. Suppose $c(t)$ is parameterized by the arc length, i.e., $T = \frac{\mathrm{d}}{\mathrm{d}t}$ has length 1, then we can write g as

$$g = \mathrm{d}t^2 + g_t,$$

and $\{g_t\}_{t \in [0,L]}$ is a one-parameter family of homogeneous metrics on the orbits $M_t = G.c(t)$.

Fix a bi-invariant inner product Q on the Lie algebra $\mathfrak{g}$ of G and let $\mathfrak{p} = \mathfrak{h}^{\perp}$ be the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H. For each $X \in \mathfrak{p}$, let X^* be the Killing vector field generated by X along $c(t)$, i.e.,$X^*(t) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} \exp(sX).c(t)$. For each $t \in (0, L)$, M_t is diffeomorphic to the homogeneous space G/H, and hence $T_{c(t)}M_t$ can be identified with $\mathfrak{p}$ by means of Killing vector fields as $X \mapsto X^*(t)$. Then g_t defines an inner product on $\mathfrak{p}$ which is invariant under the isotropy action of Ad_H. We set

$$g_t(X, Y) = g_t(X^*(t), Y^*(t)) = Q(P_t X, Y) \text{ for } X, Y \in \mathfrak{p}, \tag{2.1.1}$$

where $P_t : \mathfrak{p} \longrightarrow \mathfrak{p}$ is a Q-symmetric $\mathrm{Ad}(H)$-equivariant endomorphism. So the metric g is completely determined by the one parameter family $\{P_t\}$, $t \in [0, L]$, and at $t = 0$ and L, P_t should satisfy further conditions to guarantee smoothness of g.

On the other hand, each principal orbit M_t is a hypersurface in M with normal vector T. If $S_tX = S_tX^*(t) = -\nabla_{X^*}T$ is the shape operator of M_t at $c(t)$, we have

$$S_t = -\frac{1}{2}P_t^{-1}P_t' \tag{2.1.2}$$

in terms of P.

In the rest of this section, we give a short introduction to class one representations with more details in Appendix A. First we recall:

Definition 2.1.4. A representation (μ, W) of a compact Lie group K is called a *real (complex or quaternionic) representation* if W is a vector space over $\mathbb{R}$ ($\mathbb{C}$ or $\mathbb{H}$).

Definition 2.1.5. A pair (K, H) of compact Lie groups with $H \subset K$ and $K/H = \mathbb{S}^k$ ($k \geq 2$) is called a *spherical group pair*.

If we assume that K is connected, the image of μ will be a closed subgroup of $SO(l)$, $U(l)$ or $Sp(l)$ if μ is over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

For each group pair (K, H) with $H \subset K$ a closed subgroup, we have

Definition 2.1.6. A non-trivial irreducible representation (μ, W) of K is called *a class one representation of the pair (K, H)* if $\mu(H)$ fixes a nonzero vector $w_0 \in W$.

In the case where $(K, H) = (SO(k+1), SO(k))$, let ϖ_1 be the highest weight of the standard representation ϱ_{k+1} on $\mathbb{R}^{k+1}$, then the class one representations over $\mathbb{R}$ are precisely those with the highest weights as $m\varpi_1$, $m = 1, 2, \ldots$. These representation spaces can also be realized as the space of homogeneous harmonic polynomials. Let $\{x_1, \ldots, x_{k+1}\}$ be the basis of $\mathbb{R}^{k+1}$ and $SO(k+1)$ acts by the matrix multiplication.

Then an element $A \in SO(k+1)$ acts on a polynomial $f(x_1, \ldots, x_{k+1})$ through the action on the variables, i.e., $(A.f)(x_1, \ldots, x_{k+1}) = f(A^{-1}.x_1, \ldots, A^{-1}.x_{k+1})$.

A polynomial f is called harmonic if $\Delta f = 0$ where $\Delta = \sum_{i=1}^{k+1} \frac{\mathrm{d}^2}{\mathrm{d}x_i^2}$. Let H_m be the space of the homogeneous harmonic polynomials in $x_1, \ldots, x_{k+1}$ of degree m, then the representation of $SO(k+1)$ on H_m has the highest weight $m\varpi_1$.

Real, complex and quaternionic class one representations of all spherical group pairs are classified in Appendix A. More properties are discussed there as well.

2.2 Group Components and the Weyl Group of the New Examples

First let us restate the conditions in Theorem 2 using the concept of class one representation :

Theorem 2.2.1. *Let $K'/H' = \mathbb{S}^k$ with $k \geq 2$ and $\rho : K' \longrightarrow SO(m)$ be an almost faithful representation such that $\ker \rho \subset H'$ and it contains a class one representation μ of the pair (K', H') with $l = \deg \mu \geq k+2$. For any integer $n > m+1$, the cohomogeneity one manifold M defined by the groups $H \subset \{K^-, K^+\} \subset G$ in (1.0.2) does not admit a G invariant metric with non-negative sectional curvature.*

Remark 2.2.2. Proposition A.5.2 lists all non-faithful class one representations. We see that many class one representations are excluded by the faithfulness restriction in [GVWZ].

Remark 2.2.3. Proposition A.5.5 lists all real class one representations which have less dimensions than $k+2$. We see that only the defining representation of K', the 9 dimensional representation ϱ_9 of the pair $(Spin(9), Spin(7))$, the 5 dimensional representation of $(Sp(2), Sp(1))$ and the 3 dimensional representations of $(SU(2), \{\text{Id}\})$ and $(U(2), U(1))$ are excluded in the above theorem.

The following lemma shows that the condition $\ker \rho \subset H'$ is necessary for the cohomogeneity one construction.

Lemma 2.2.4. *The groups $H \subset \{K^-, K^+\} \subset G = SO(n)$ given in (1.0.2) define a compact cohomogeneity one manifold M if and only if* $\ker \rho \subset H'$.

Proof : We only need to check that both K^-/H and K^+/H are homeomorphic to certain spheres. It is clear for K^+/H since $K^+/H \cong SO(n-m+1)/SO(n-m) = \mathbb{S}^{n-m}$. For K^-/H, we have the following induced map:

$$\begin{aligned} \pi : \mathbb{S}^k = K'/H' &\longrightarrow \rho(K')/\rho(H') \\ xH' &\mapsto \rho(x)\rho(H'). \end{aligned}$$

Since ρ is almost faithful, π is a finite covering. For two points xH' and yH' in K'/H', $\pi(xH') = \pi(yH')$ implies that $\rho(xy^{-1}) \in \rho(H')$. Therefore π is injective if and only if $xy^{-1} \in H'$, i.e., $\ker \rho \subset H'$. □

We look at some interesting examples of the above theorem. Take $(K', H') = (SO(3), SO(2))$ with $K'/H' = \mathbb{S}^2$, then the lowest class one representation with $\deg \geq 4$ is the unique $5-$dimensional representation of $SO(3)$ which is also faithful.

Let μ be that representation, $m = 5$ and $n = 7$. The manifold M has the groups as

$$\mu(SO(2)) \times SO(2) \subset \{\mu(SO(3)) \times SO(2), \mu(SO(2)) \times SO(2)\} \subset SO(7),$$

which has dimension 20. It turns out to be the lowest dimension of the new examples.

Since we do not assume that ρ is irreducible, the non-faithful class one representation is allowed in this construction. Suppose μ is a class one representation with $\deg \mu \geq k+2$ and $\ker \mu$ is possibly not inside H'. Then take an arbitrary representation τ of K' with $\ker \tau \cap \ker \mu \subset H'$. Let $\rho = \tau \oplus \mu$ and then ρ satisfies the conditions in the theorem. For example, take $(K', H') = (SO(6), SO(5))$ with $k = 5$ and μ be the adjoint representation of $SO(6)$ which is 15 dimensional and $\ker \mu = \mathbb{Z}_2$ not inside $SO(5)$. To construct a cohomogeneity one manifold, we can choose $\tau = \varrho_6$ as the standard representation of $SO(6)$ which is faithful and let $\rho = \tau \oplus \mu$.

In the next, we work out the explicit embedding of the groups. $SO(m) \times SO(n-m)$ is embedded in $G = SO(n)$ block-wise, i.e., $SO(m)$ sits in the upper-left $m \times m$-block and $SO(n-m)$ is in the lower-right block. From the assumption, ρ is an almost faithful representation of K' with representation space V. Choose an inner product on V and let $W_1, \ldots, W_\alpha$ be all invariant subspaces of V such that they are pairwisely orthogonal, $v_0 \in W_1$ and the restrictions of $\rho(K')$ on them are equivalent and irreducible. Let U be the orthogonal complement of $W = W_1 \oplus \cdots \oplus W_\alpha$. According to the decomposition of V into invariant spaces, we can write ρ as

$$\rho = \tau \oplus \mu \oplus \cdots \oplus \mu, \tag{2.2.1}$$

where τ is restriction of ρ to U and μ is the class one representation of (K', H'). Suppose $r = \dim U$, $l = \dim W_i = \deg \mu$, then $m = r + \alpha l$. By choosing a suitable basis of V, for each element $x \in K'$, $\rho(x)$ is a block diagonal matrix in $SO(m)$ as

$$\rho(x) = \begin{pmatrix} \tau(x) & & & \\ & \mu(x) & & \\ & & \ddots & \\ & & & \mu(x) \end{pmatrix} \in SO(r) \times SO(l) \times \cdots \times SO(l). \qquad (2.2.2)$$

When μ is restricted to the subgroup $H' \subset K'$, it is not irreducible any more. Let l_0 be the multiplicity of the trivial representation in the restriction $\mathrm{Res}(\mu)$, then from the existence of v_0, $l_0 \geq 1$. Hence for any element $y \in H'$, $\mu(y) \in SO(l - l_0) \subset SO(l)$ and $SO(l - l_0)$ is embedded as the upper left $(l - l_0) \times (l - l_0)$ block in $SO(l)$.

With the explicit description of the embeddings, we can show the following proposition on the Weyl group of the new examples.

Proposition 2.2.5. *$\mathcal{W}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof : For any element $x \in K'$ and $A \in SO(n - m)$, let $M(x, A)$ denote the block diagonal matrix $\mathrm{diag}(\rho(x), A)$ with $\rho(x) \in SO(m)$. If $x \in H'$, A can be a matrix in $SO(n - m + 1)$ since $\rho(x) \in SO(m - 1)$.

First notice that w_+ can be represented by the unique element $a \in K^+ = \rho(H') \times SO(n - m + 1)$ which is not in H, but $a^2 \in H$. Choose x as the identity element id and A be a diagonal matrix with -1 at the $(1, 1)$-entry, i.e., $M(\mathrm{id}, A)$ has the following matrix form:

$$a = \begin{pmatrix} I_{m-1} & & & & \\ & -1 & & & \\ & & \varepsilon_1 & & \\ & & & \ddots & \\ & & & & \varepsilon_{n-m} \end{pmatrix}, \tag{2.2.3}$$

where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n-m$ and $\varepsilon_1 \cdots \varepsilon_{n-m} = -1$. Then $a = M(\mathrm{id}, A)$ is a representative of w_+.

Suppose $b = M(x, A)$ be a representative of w_-, i.e., $x \in K' - H'$, $A \in SO(n-m)$ and b^2 but not $b \in H$. In each $W = W_i (i = 1, \ldots, \alpha)$, let Y be the subspace fixed by $\mu(H')$(or equivalently by H) and X be its orthogonal complement. Then $\dim X = l - l_0$ and $\dim Y = l_0$. Since $b \in N(H)$, for any vector $v \in Y$, $b.v$(the matrix multiplication) is also fixed by H, i.e., $b.v \in Y$ as $b.v \in W$. Since b is an orthogonal transformation, b maps any vector in X into X itself, i.e., $b = \left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right) \in S(O(l - l_0) \times O(l_0))$. Then $b^2 \in H$ implies that b_2 is a symmetric matrix. Now we choose an element γ from the identity component of $N(H)$, such that $\gamma.b.\gamma^{-1}$ is a diagonal matrix in the subspace Y. Or we can replace K^- in (1.0.2) by its conjugation by γ and the new manifold is G-equivariant to the old one. Therefore we can assume that w_- has a representative b with the following matrix form:

$$b = \begin{pmatrix} A_1 & & & & & & \\ & A_2 & & & & & \\ & & \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_{l_0}) & & & & \\ & & & \ddots & & & \\ & & & & A_2 & & \\ & & & & & \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_{l_0}) & \\ & & & & & & I_{n-m} \end{pmatrix}, \qquad (2.2.4)$$

where $A_1 \in SO(r)$, $A_2 \in O(l - l_0)$ and there are α copies of $\begin{pmatrix} A_2 & \\ & \mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_{l_0}) \end{pmatrix}$. Given the representatives of $w_\pm$, it is easy to check that $w_+ w_- = w_- w_+$ which is not an element in H. So $\mathcal{W} =< w_-, w_+ >\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

Remark 2.2.6. In the proof of Proposition 2.2.5, we chose an equivalent group diagram such that the representative b has the simple matrix form and then the Weyl group $\mathcal{W}$ is small. In the following of the paper, we will assume that the matrix in (2.2.4) is an element of K^-.

Using the explicit representatives of the generators of the Weyl group, we have the following smoothness condition of the invariant metric on M which will be used in the proof of the Theorem 2.2.1. We use $E_{i,j}$, $i \neq j$, to denote the skew-symmetric matrix having 1 at $i, j-$entry, -1 at $j, i-$entry and zero otherwise.

Corollary 2.2.7. *For any $G-$invariant metric g on M, let $h(t)$ be the length of the Killing vector field generated by $E_{m,m+1}$ along the geodesic $c(t)$, then $h(t)$ is an even* *function with* $h(0) \neq 0$ *and* $h(L) = 0$.

Proof : The fact that $E_{m,m+1}$ lies in the Lie algebra of K^+ but not K^- implies that $h(0) \neq 0$ and $h(L) = 0$. The generator w_- is a reflection of $c(t)$ at the point p_- and maps $c(t)$ to $c(-t)$. The induced map $\mathrm{d}w_-$ takes $T_{c(t)}M$ to the tangent space $T_{c(-t)}M$. From the matrix form (2.2.4) of a representative of w_-, we have $\mathrm{d}w_-(E^*_{m,m+1}(t)) = \varepsilon_{l_0} E^*_{m,m+1}(-t)$. Therefore $h(t) = h(-t)$, i.e., $h(t)$ is an even function. □

2.3 Killing Vector Fields along the Geodesic $c(t)$

In this section, we begin the study of the invariant metrics on the examples. In general, the family of the G-invariant metrics on M is very large. Though there are relatively more rigidity results in the positively curved metrics, for example, L. Verdiani classified all positively curved cohomogeneity one manifolds in even dimensions, see [Ve2] and [Ve3], K.Grove, B.Wilking and W.Ziller obtained a short list of cohomogeneity one manifolds which possibly have invariant positively curved metrics in [GVWZ] and recently K.Grove, L.Verdiani and W.Ziller have succeeded in constructing positively metric on one of them in [GVZ], the rigidity results in the non-negatively metrics are very few. Recently, B.Wilking considered the transversal Jacobi field and proved some fundamental rigidity theorems in the non-negatively curved metrics, see [Wi2]. His results work for a very general setting.

Let $\mathfrak{g} = \mathfrak{so}(n)$ and $\mathfrak{h}$ be the Lie algebras of $G = SO(n)$ and $H = \rho(K') \times SO(n-m)$

respectively. Choose the bi-invariant inner product as $Q = -\frac{1}{2}\mathrm{Tr}$ on $\mathfrak{g}$ and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$. First we identify some subspaces of $\mathfrak{p}$ and then study the metric properties of them.

Let $\{E_{i,j}\}_{1<i,j<n}$ be the $n \times n$ square matrix with 1 at (i,j)-entry, -1 at (j,i)-entry and zero elsewhere and let $Q = -\frac{1}{2}\mathrm{Tr}$, then $\{E_{i,j}\}$ is an orthonormal basis of $\mathfrak{so}(n)$. Let

$$\begin{aligned} \mathfrak{q}_0 &= \mathrm{span}\left\{E_{i,j} | 1 \leq i \leq r, m+1 \leq j \leq n\right\} \\ \mathfrak{q}_1 &= \mathrm{span}\left\{E_{i,j} | r+1 \leq i \leq r+l, m+1 \leq j \leq n\right\} \qquad (2.3.1) \\ &\cdots \\ \mathfrak{q}_\alpha &= \mathrm{span}\left\{E_{i,j} | m+1-l \leq i \leq m, m+1 \leq j \leq n\right\}, \end{aligned}$$

and $\mathfrak{q} = \mathfrak{q}_0 + \mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha$. We write the last subspace $\mathfrak{q}_\alpha$ as a sum of two subspaces as follows:

$$\begin{aligned} \mathfrak{n}_1 &= \mathrm{span}\left\{E_{i,j} | m+1-l \leq i \leq m-1, m+1 \leq j \leq n\right\}, \qquad (2.3.2) \\ \mathfrak{n}_2 &= \mathrm{span}\left\{E_{m,j} | m+1 \leq j \leq n\right\}. \end{aligned}$$

Let $\mathfrak{q}^\perp$ be the Q-orthogonal complement of $\mathfrak{q}$ in $\mathfrak{p}$, then $\mathfrak{q}^\perp$ is fixed by the isotropy action of the subgroup $SO(n-m) \subset H = \rho(H') \times SO(n-m)$, so the Killing vector field X^*, any $X \in \mathfrak{q}^\perp$, is orthogonal to Y^*, any $Y \in \mathfrak{p}$, along $c(t)$ from Schur's lemma.

Terminology. In the rest of the paper, for any two subspace $\mathfrak{p}_1$, $\mathfrak{p}_2 \subset \mathfrak{p}$, the term that $\mathfrak{p}_1^*$ *is orthogonal to* $\mathfrak{p}_2^*$ *along* $c(t)$ and the notation $\mathfrak{p}_1^* \perp \mathfrak{p}_2^*$ mean that any Killing vector field generated by an element in $\mathfrak{p}_1$ is orthogonal to any Killing vector field generated by an element in $\mathfrak{p}_2$ along $c(t)$.

We apply Wilking's rigidity results to our examples and obtain the following theorem:

Theorem 2.3.1 (Wilking's Rigidity Theorem)**.** *Suppose that the cohomogeneity one manifold (M, g) has non-negative sectional curvatures, then the family of the Killing vector fields generated by $\mathfrak{q}$ along $c(t)$, $t \in \mathbb{R}$, denoted by Λ, have the following orthogonal decomposition:*

$$\Lambda = \Upsilon \oplus \{J \in \Lambda | J \textit{ is parallel }\}, \tag{2.3.3}$$

where

$$\Upsilon = \operatorname{span}\{J \in \Lambda | J(t) = 0 \textit{ for some } t \in \mathbb{R}\}.$$

Furthermore, let

$$\Upsilon(t) = \{J(t) | J \in \Upsilon\} \oplus \{J'(t) | J \in \Upsilon, J(t) = 0\},$$

and t_0 is said to be generic if $J(t_0) \neq 0$ for any $J \in \Upsilon$. Then if $J \in \Lambda$ and $J(t_0) \perp \Upsilon(t_0)$ at a generic t_0, then J is parallel along $c(t)$, $t \in \mathbb{R}$.

Definition 2.3.2. A point $t_0 \in \mathbb{R}$ or $c(t_0)$ is said to be *a singular point* if for some $J \in \Lambda$, $J(t_0) = 0$.

We determine the component Υ in the splitting (2.3.3). $p_- = c(0)$ and $p_+ = c(L)$ are two singular points. w_+ fixes p_+ and reflects $c(t)$ about p_+. Let $q_- = c(2L) = w_+(p_-) \in B_-$, then the isotropy subgroup at q_- is $K_1^- = \mathrm{Ad}_{w_+} K^-$ with

Lie algebra $\mathfrak{k}_1^- = \mathrm{Ad}_{w_+}\mathfrak{k}^-$. Similarly, w_- fixes p_- and reflects $c(t)$ about p_-. Let $q_+ = c(-L) = w_-(p_+) \in B_+$, then q_+ has isotropy subgroup $K_1^+ = \mathrm{Ad}_{w_-}K^+$ with Lie algebra $\mathfrak{k}_1^+ = \mathrm{Ad}_{w_-}\mathfrak{k}^+$. Since $w_-.w_+ = w_+.w_-$, the image of q_- under the reflection w_- about p_- is $w_-(q_-) = w_-.w_+(p_-) = w_+.w_-(p_-) = w_+(p_-) = q_-$, i.e., $c(2L) = c(-2L)$. Therefore $c(t)$ is a closed geodesic with period $4L$ and the singular points are $p_- = c(0)$, $p_+ = c(L)$, $q_- = c(2L)$ and $q_+ = c(3L)$. Then the vanishing Killing vector fields are those generated by the vectors in the Lie algebras of the isotropy subgroups at singular points. In conclusion, we have the following lemma:

Lemma 2.3.3. *In the family Λ, we have $\Upsilon = \{X^*(t) | X \in \mathfrak{n}_2\}$, where $\mathfrak{n}_2$ is defined in (2.3.2). And for any $X^* \in \Upsilon$, $X^*(t)$ vanishes simultaneously at p_+ and q_+.*

In the following, we prove some properties of the invariant metrics g on M under the non-negative sectional curvatures assumption. We use $E_{a,i,\xi}$ to denote $E_{r+(i-1)l+a,m+\xi} \in \mathfrak{q}_i$, where $a = 1, \ldots, l$; $i = 1, \ldots, \alpha$ and $\xi = 1, \ldots, n-m$.

Proposition 2.3.4. *Suppose (M, g) is non-negatively curved, then we have*

1. *$\mathfrak{q}_0^*$ is orthogonal to $(\mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha)^*$ along $c(t)$;*

2. *$E^*_{a,i,\xi}$ is orthogonal to $E^*_{b,j,\zeta}$ along $c(t)$ if $a \neq b$ or $\xi \neq \zeta$;*

3. *$E^*_{a,i,\xi}$ and $E^*_{a,i,\zeta}$ have the same length along $c(t)$;*

4. *$g(E^*_{a,i,\xi}, E^*_{a,j,\xi})_{c(0)} = g(E^*_{b,i,\zeta}, E^*_{b,j,\zeta})_{c(0)}$.*

Proof : At $p_- = c(0)$, the metric g restricted on the singular orbit $B_- \cong G/K^-$ is Ad_{K^-} invariant. The actions of Ad_{K^-} on $\mathfrak{q}_0$ and $\mathfrak{q}_i$ $(i > 0)$, are $\tau \otimes \varrho_{n-m}$ and

$\mu \otimes \varrho_{n-m}$ respectively, where ϱ_{n-m} is the standard representation of $SO(n-m)$ on $\mathbb{R}^{n-m}$. Since τ and μ are non-equivalent, $\mathfrak{q}_0^*$ is orthogonal to $\mathfrak{q}_i^*$ at p_-. In particular $\mathfrak{q}_0^*$ is orthogonal to $\mathfrak{n}_2^*$, so any Killing vector field generated by a vector in $\mathfrak{q}_0$ is parallel along $c(t)$. Hence $\mathfrak{q}_0^*$ is orthogonal to $(\mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha)^*$ along $c(t)$ and (1) is proven.

On each principal orbit $M_t \cong G/H$, Ad_H acts on each $\mathfrak{q}_i$ $(i > 0)$, by the representation $Res_{H'}^{K'}(\mu) \otimes \varrho_{n-m}$. By Schur's lemma, $E_{a,i,\xi}^*$ is orthogonal to $E_{b,j,\zeta}^*$ along $c(t)$ if $\xi \neq \zeta$ and $E_{a,i,\xi}^*$ has the same length as $E_{a,i,\zeta}^*$ which proves (3) and one case of (2) where $\xi \neq \zeta$.

Now assume that $\zeta = \xi$ and $a \neq b$. If none of a or b is equal to l, then $E_{a,i,\xi}^*(0)$ and $E_{b,j,\xi}^*(0)$ are orthogonal to each other and both are orthogonal to $\Upsilon(0)$ since the Ad_{K^-} actions on $\mathfrak{q}_i$ and $\mathfrak{q}_j$ are equivalent and given by the representation $\mu \otimes \varrho_{n-m}$, so they are parallel and then orthogonal to each other along $c(t)$. If one of a, b, say b, is equal to l, then $E_{a,i,\xi}^*$ is a parallel vector field. Write $E_{l,j,\xi}^*(0) = (E_{l,j,\xi} - \lambda E_{l,\alpha,\xi})^*(0) + \lambda E_{l,\alpha,\xi}^*(0)$, where the constant λ is determined by the following equation:

$$g(E_{l,j,\xi}^*, E_{l,\alpha,\xi}^*)_{c(0)} = \lambda g(E_{l,\alpha,\xi}^*, E_{l,\alpha,\xi}^*)_{c(0)}.$$

Then $(E_{l,j,\xi} - \lambda E_{l,\alpha,\xi})^*(0) \perp \Upsilon(0)$ and hence $(E_{l,j,\xi} - \lambda E_{l,\alpha,\xi})^*$ is a parallel vector field. And $E_{a,i,\xi}^*$ is orthogonal to $(E_{l,j,\xi} - \lambda E_{l,\alpha,\xi})^*$ at $c(0)$, so they are orthogonal to each other along $c(t)$. So $E_{a,i,\xi}^*$ is orthogonal to $E_{l,j,\xi}^*$ along $c(t)$ which completes the proof of (2).

At $B_- \cong G/K^-$, from the irreducibility of $\mu \otimes \varrho_{n-m}$ and Schur's lemma, we have

$$g(E_{a,i,\xi}^*, E_{a,j,\xi}^*)_{c(0)} = g(E_{b,i,\xi}^*, E_{b,j,\xi}^*)_{c(0)} = g(E_{b,i,\zeta}^*, E_{b,j,\zeta}^*)_{c(0)}$$

which proves (4). □

From the above Proposition 2.3.4, the restriction of the endomorphism P on $(\mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha)^*$ has the initial data at $t = 0$ which are given by the following $\alpha \times \alpha$-matrix:

$$P_0 = \begin{pmatrix} f_{1,1} & \cdots & f_{1,\alpha} \\ \vdots & \ddots & \vdots \\ f_{\alpha,1} & \cdots & f_{\alpha,\alpha} \end{pmatrix} \tag{2.3.4}$$

where $f_{i,j} = f_{j,i} = g(E^*_{1,i,1}, E^*_{1,j,1})_{c(0)}$.

We have seen that there are plenty of parallel Killing vector fields in $(\mathfrak{q}_1+\cdots+\mathfrak{q}_\alpha)^*$. Using these parallel vector fields, we can solve the restriction of P_t on $(\mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha)^*$ in the following proposition.

Proposition 2.3.5. *The non-negatively curved metric g along $c(t)$ satisfies the following conditions:*

1. $g_t(E^*_{a,i,\xi}, E^*_{a,j,\xi}) = f_{i,j}$, *if* $a = 1, \ldots, l-1$;

2. $g_t(E^*_{l,i,\xi}, E^*_{l,j,\xi}) = p_{i,j}(t) = a_i a_j h^2(t) + f_{i,j} - a_i a_j f_{\alpha,\alpha}$,

where $h(t) = \left|\left|E^*_{l,\alpha,\xi}(t)\right|\right|$ *and* $f_{i,\alpha} = a_i f_{\alpha,\alpha}, \quad i = 1, \ldots, \alpha$.

Proof : Part (1) is obvious since both $E^*_{a,i,\xi}$ and $E^*_{a,j,\xi}$ are parallel vector fields along $c(t)$ if $a \leq l-1$.

For part (2), let

$$X_i = E_{l,i,\xi} - a_i E_{l,\alpha,\xi}, \quad i = 1, ..., \alpha. \tag{2.3.5}$$

From the define equation of a_i, we have

$$f_{i,\alpha} - a_i f_{\alpha,\alpha} = 0. \tag{2.3.6}$$

Note that if $i = \alpha$, $a_i = 1$ and if $i \neq \alpha$, $a_i \neq 1$ since $E_{l,i,\xi} - E_{l,\alpha,\xi}$ does not belong to $\mathfrak{k}^-$. Each $X_i \in \mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha$ and generates the Killing vector field X_i^* along $c(t)$. By (2.3.5), (2.3.6), we have

$$g(X_i^*, E_{l,\alpha,\xi}^*)_{c(0)} = 0,$$

or equivalently, $X_i^*(0) \perp \Upsilon(0)$. So X_i^* is parallel, i.e.,

$$\nabla_T X_i^*(t) = 0.$$

By the expression of the shape operator (2.1.2) and that P_t is nonsingular for X_i^*, we have

$$P_t'(X_i) = 0 \quad \forall t \in \mathbb{R}.$$

Substitute X_i by $E_{l,i,\xi} - a_i E_{l,\alpha,\xi}$, we have

$$\begin{aligned} 0 &= P_t'(E_{l,i,\xi} - a_i E_{l,\alpha,\xi}) = P_t'(E_{l,i,\xi}) - a_i P_t'(E_{l,\alpha,\xi}) \\ &= \sum_{j=1}^{\alpha} p_{i,j}'(t) E_{l,j,\xi} - a_i \sum_{j=1}^{\alpha} p_{\alpha,j}'(t) E_{l,j,\xi}. \end{aligned} \tag{2.3.7}$$

Therefore from the identity (2.3.7), we have the following ordinary differential equations system on $p_{i,j}(t)$:

$$p_{i,j}'(t) - a_i p_{\alpha,j}'(t) = 0. \quad \forall i, j = 1, ..., \alpha. \tag{2.3.8}$$

By (4) in Proposition 2.3.4 and initial data of P_t in (2.3.4), $p_{i,j}(t)$ has the initial conditions as $p_{i,j}(0) = f_{i,j}$. If $i = \alpha$, the differential equation (2.3.8) is an identity since $a_\alpha = 1$.

Let $j = \alpha$ and $h^2(t) = p_{\alpha,\alpha}(t)$, the equation (2.3.8) becomes

$$p'_{i,\alpha}(t) - a_i(h^2(t))' = 0.$$

It has the solution as

$$p_{i,\alpha}(t) = a_i h^2(t) + (f_{i,\alpha} - a_i f_{\alpha,\alpha}) = a_i h^2(t). \tag{2.3.9}$$

Substitute $p_{\alpha,j}(t)$ in (2.3.8) by the solution (2.3.9), we have

$$p'_{i,j}(t) - a_i a_j (h^2(t))' = 0.$$

It has the solution as

$$p_{i,j}(t) = a_i a_j h^2(t) + f_{i,j} - a_i a_j f_{\alpha,\alpha},$$

or equivalently

$$p_{i,j}(t) = a_i a_j h^2(t) + f_{i,j} - a_i f_{j,\alpha}. \tag{2.3.10}$$

□

2.4 Non-negative Sectional Curvature Obstructions

In this section, we will develop some contradiction from the assumption that the manifold (M, g) is non-negatively curved. First we list some lemmas which will be used in the proof of the main theorem in the introduction.

Lemma 2.4.1. *The image $\mu(K') \subset SO(l)$ does not act transitively on the sphere $\mathbb{S}^{l-1} = SO(l)/SO(l-1)$.*

Proof : We already knew there is a non-zero vector $v_0 \in W_i \cong \mathbb{R}^l$ fixed by $\mu(H')$ such that $\dim \operatorname{Span}\{\mu(K').v_0\} \geq k+2$, in other words, we have $l \geq k+2$. If $\mu(K')$ could act transitively on $\mathbb{S}^{l-1}$, then K' would act transitively on both $\mathbb{S}^k$ and $\mathbb{S}^{l-1}$ and $l-1 \geq k+1$. By the classification of the transitive action on the spheres, we have that (K', H') is either $(SO(7), SO(6))$ with $\mu(SO(7)) = Spin(7) \subset SO(8)$ or $(SO(9), SO(8))$ with $\mu(SO(9)) = Spin(9) \subset SO(16)$. But in both cases, the restriction of μ on H' has no fixed non-zero vector v_0 which is a contradiction. □

The following is a result in calculus which was already used in [GVWZ] and we state it as a lemma without proof.

Lemma 2.4.2. *Suppose $f(t)$ is a C^2 non constant even function on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ with $f(0) = 0$, then there is no such $\gamma \geq 0$ that satisfies the following inequality:*

$$\gamma^2 (f(t))^2 - (f'(t))^2 \geq 0 \tag{2.4.1}$$

We will compute the sectional curvatures for some 2-planes in our examples. The formula in terms of P_t is well established in [GZ2] and we quote it for the convenience of the reader as the following theorem.

Theorem 2.4.3 (Grove-Ziller)**.** *If $X, Y \in \mathfrak{p}$, the sectional curvatures of M at $c(t)$*

are determined by

$$
\begin{array}{llcl}
(a) & g(R(X,Y)X,Y) & = & Q(A_-(X,Y),[X,Y]) - \frac{3}{4}Q(P[X,Y]_{\mathfrak{p}},[X,Y]_{\mathfrak{p}}) \\
 & & & +Q(A_+(X,Y),P^{-1}A_+(X,Y)) \\
 & & & -Q(A_+(X,X),P^{-1}A_+(Y,Y)) \\
 & & & +\frac{1}{4}Q(P'X,Y)^2 - \frac{1}{4}Q(P'X,X)Q(P'Y,Y) \\
(b) & g(R(X,Y)T,Y) & = & -\frac{1}{2}Q(P'X,P^{-1}A_+(Y,Y)) + \frac{1}{2}Q(P'Y,P^{-1}A_+(X,Y)) \\
 & & & +\frac{3}{4}Q([X,Y],P'Y) \\
(c) & g(R(X,T)X,T) & = & Q((-\frac{1}{2}P'' + \frac{1}{4}P'P^{-1}P')X,X).
\end{array}
$$

Remark 2.4.4. In the formulas above, $[X,Y]_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of $[X,Y]$ and $A_{\pm}$: $\mathfrak{p} \times \mathfrak{p} \longrightarrow \mathfrak{g}$ are defined as

$$A_{\pm}(X,Y) = \frac{1}{2}([X,P_tY] \mp [P_tX,Y]).$$

Lemma 2.4.5. *$h(t)$ is an even function with $h(L) = 0$, $h'(0) = 0$ and $h''(t) \leq 0, \forall t \in [0,L)$.*

Proof : The facts that $h(t)$ is an even function, $h(L) = 0$ and $h'(0) = 0$ were proved in Corollary 2.2.7.

The second derivative of $h(t)$ follows from the nonnegativity of the sectional curvature of the 2-plane spanned by T and $Z^* = E^*_{l,\alpha,1}$. By Proposition 2.3.5,

$$P_t(Z) = h^2(t)\sum_{i=1}^{\alpha} a_i E_{l,i,1},$$

so

$$P_t'(Z) = 2h(t)h'(t)\sum_{i=1}^{\alpha} a_i E_{l,i,1},$$

and

$$P_t''(Z) = (2h(t)h''(t) + 2(h'(t))^2)\sum_{i=1}^{\alpha} a_i E_{l,i,1}.$$

Hence by using the formula in Theorem 2.4.3, we have

$$\sec(Z,T) = -h''(t)h(t).$$

Therefore $h''(t) \leq 0$ since $h(t) \geq 0$. □

From Lemma 2.4.5, $f(t) = h^2(0) - h^2(t)$ satisfies the conditions in Lemma 2.4.2. We will show the inequality (2.4.1) holds for some constant $\gamma \geq 0$ by looking at the sectional curvature of a carefully chosen 2-plane. Then the main theorem follows from Lemma 2.4.2.

Proof of the Main Theorem : Since P_0 defined in (2.3.4) is symmetric and positive definite, we can write $P_0 = ADA^\top$, where A is an orthogonal $(\alpha \times \alpha)$-matrix and D is a diagonal matrix with positive entries as $D = \mathrm{diag}(d_1, \cdots, d_\alpha)$. Let

$$A = (A_{i,j})_{\alpha\times\alpha},$$

and define the following vectors in $\mathfrak{q}_1 + \cdots + \mathfrak{q}_\alpha$:

$$X^u = \sum_{i=1}^{l-1} b_i E_{i,u,1} + E_{l,u,2} \qquad (2.4.2)$$

$$Y^u = \sum_{i=1}^{l-1} b_i E_{i,u,2} + E_{l,u,1}, \qquad (2.4.3)$$

where $u = 1, ..., \alpha$ and $\sum_{i=1}^{l-1} b_i^2 = 1$. The further condition of b_i's will be determined later.

In the matrix A, there is a column, say i_0-th column with $A_{\alpha,i_0} \neq 0$. Then we denote A_{u,i_0} by A_u, $u = 1, \dots, \alpha$, and define the following two vectors X, Y in $\mathfrak{p}$:

$$X = \sum_{u=1}^{\alpha} A_u X^u, \quad Y = \sum_{u=1}^{\alpha} A_u Y^u. \tag{2.4.4}$$

From the definitions of X^u and Y^v, it is easy to see that $[X^u, Y^u] = 0$, and if $u \neq v$, then

$$[X^u, Y^v] = \sum_{i=1}^{l-1} b_i (E_{vl,i+(u-1)l} + E_{i+(v-1)l,ul}).$$

Hence

$$\begin{aligned} [X, Y] &= \sum_{u,v=1}^{\alpha} A_u A_v [X^u, Y^v] = \sum_{i=1}^{l-1} \sum_{u \neq v} A_u A_v b_i (E_{vl,i+(u-1)l} + E_{i+(v-1)l,ul}) \\ &= 0. \end{aligned}$$

The commutativity of X and Y make the computation of the sectional curvature of the 2−plane spanned by X^* and Y^* a little easier and the first two terms in the curvature formula (a) in Theorem 2.4.3 vanish. The other four terms are computed in the Proposition 2.4.6 below. Plug in the result of each term into the formula (a) in Theorem 2.4.3, then we have the following simple expression of the sectional curvature:

$$\begin{aligned} \sec(X^*, Y^*)_{c(t)} &= (h^2(t) - h^2(0))^2 Q(X_0, P_t^{-1}(X_0)) - \left(\frac{d_{i_0} A_\alpha}{f_{\alpha,\alpha}}\right)^4 h^2(t)(h'(t))^2 \\ &- Q(A_+(X, X), P_t^{-1} A_+(X, X)). \end{aligned}$$

Here $X_0 \in \mathfrak{p}$ and it is orthogonal to $\mathfrak{k}^-$ with respect to Q by choosing proper values of b_i's.

Since P_t is positive definite as well as P_t^{-1}, we have $Q(A_+(X,X), P_t^{-1}A_+(X,X)) \geq 0$. Therefore $\sec(X^*, Y^*) \geq 0$ implies that

$$(h^2(t) - h^2(0))^2 Q(X_0, P_t^{-1}(X_0)) - \left(\frac{d_{i_0} A_\alpha}{f_{\alpha,\alpha}}\right)^4 h^2(t)(h'(t))^2 \geq 0.$$

The existence of the constant $\gamma \geq 0$ follows from the facts that $A_\alpha \neq 0$ and $Q(X_0, P_t^{-1}(X_0))$ is bounded from above near $t = 0$. □

Proposition 2.4.6. *For the vectors X and Y defined in (2.4.4), by choosing proper values of b_i's, we have*

1. *There exists some $X_0 \in \mathfrak{p}$ which is orthogonal to $\mathfrak{k}^-$ with respect to Q such that $A_+(X,Y) = (h^2(t) - h^2(0))X_0$;*
2. *$A_+(X,X) = A_+(Y,Y)$;*
3. *$Q(P_t'(X), Y) = 0$;*
4. *$-\frac{1}{4}Q(P_t'(X), X)Q(P_t'(Y), Y) = -\left(\frac{d_{i_0} A_\alpha}{f_{\alpha,\alpha}}\right)^4 h^2(t)(h'(t))^2$.*

Proof : First we compute the endomorphism P_t on X and Y. From the defining equations (2.4.2) of X^u and (2.4.3) of Y^u, we have

$$P_t(X^u) = \sum_{i=1}^{l-1}\sum_{r=1}^{\alpha} b_i f_{u,r} E_{i,r,1} + \sum_{r=1}^{\alpha} p_{u,r}(t) E_{l,r,2} \tag{2.4.5}$$

and

$$P_t(Y^v) = \sum_{j=1}^{l-1}\sum_{s=1}^{\alpha} b_j f_{v,s} E_{j,s,2} + \sum_{s=1}^{\alpha} p_{v,s}(t) E_{l,s,1}. \tag{2.4.6}$$

So

$$
\begin{aligned}
& [X^u, P_t(Y^v)] \\
= \ & [\sum_{i=1}^{l-1} b_i E_{i+(u-1)l,\alpha l+1} + E_{ul,\alpha l+2}, \quad \sum_{j=1}^{l-1}\sum_{s=1}^{\alpha} b_j f_{v,s} E_{j+(s-1)l,\alpha l+2} + \sum_{s=1}^{\alpha} p_{v,s}(t) E_{sl,\alpha l+1}] \\
= \ & \left(\sum_{i=1}^{l-1} b_i^2 f_{v,u} - p_{v,u}(t)\right) E_{\alpha l+2,\alpha l+1} + \sum_{i=1}^{l-1}\sum_{s=1}^{\alpha} b_i (f_{v,s} E_{i+(s-1)l,ul} - p_{v,s}(t) E_{i+(u-1)l,sl}),
\end{aligned}
$$

and

$$
\begin{aligned}
& [P_t(X^u), Y^v] \\
= \ & [\sum_{i=1}^{l-1}\sum_{r=1}^{\alpha} b_i f_{u,r} E_{i+(r-1)l,\alpha l+1} + \sum_{r=1}^{\alpha} p_{u,r}(t) E_{rl,\alpha l+2}, \quad \sum_{j=1}^{l-1} b_j E_{j+(v-1)l,\alpha l+2} + E_{vl,\alpha l+1}] \\
= \ & \left(\sum_{i=1}^{l-1} b_i^2 f_{u,v} - p_{u,v}(t)\right) E_{\alpha l+2,\alpha l+1} + \sum_{i=1}^{l-1}\sum_{r=1}^{\alpha} b_i (f_{u,r} E_{vl,i+(r-1)l} - p_{u,r}(t) E_{rl,i+(v-1)l}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& A_+(X^u, Y^v) \qquad (2.4.7) \\
= \ & \frac{1}{2}\sum_{i=1}^{l-1}\sum_{s=1}^{\alpha} b_i (f_{v,s} E_{i+(s-1)l,ul} + f_{u,s} E_{i+(s-1)l,vl} - p_{v,s}(t) E_{i+(u-1)l,sl} - p_{u,s}(t) E_{i+(v-1)l,sl}).
\end{aligned}
$$

So only these terms as $E_{j+(r-1)l,wl}$'s have nonzero coefficients in $A_+(X,Y)$ and it is denoted by $c_{j,r,w}$. From the formula (2.4.7) and the bi-linearity of A_+, we have

$$
\begin{aligned}
c_{j,r,w} &= \frac{1}{2} b_j \left(\sum_{v=1}^{\alpha} (f_{v,r} A_w A_v - p_{v,w}(t) A_r A_v) + \sum_{u=1}^{\alpha} (f_{u,r} A_u A_w - p_{u,w} A_u A_r) \right) \\
&= b_j \left(A_w \sum_{v=1}^{\alpha} f_{v,r} A_v - A_r \sum_{v=1}^{\alpha} p_{v,w}(t) A_v \right). \qquad (2.4.8)
\end{aligned}
$$

We can compute the terms in (2.4.8) explicitly as follows,

$$
\sum_{v=1}^{\alpha} f_{v,r} A_v = \sum_{v=1}^{\alpha}\sum_{i=1}^{\alpha} A_{v,i} d_i A_{r,i} A_{v,i_0} = (A^\tau A D A^\tau)_{i_0,r} = d_{i_0} A_{r,i_0} = d_{i_0} A_r \qquad (2.4.9)
$$

and

$$
\begin{aligned}
\sum_{v=1}^{\alpha} p_{v,w}(t)A_v &= \sum_{v=1}^{\alpha}(a_v a_w h^2(t)A_v + f_{v,w}A_v - a_v f_{w,\alpha}A_v) \\
&= d_{i_0}A_w + (a_w h^2(t) - f_{w,\alpha})\sum_{v=1}^{\alpha} a_v A_v \\
&= d_{i_0}A_w + a_w(h^2(t) - f_{\alpha,\alpha})\sum_{v=1}^{\alpha}\frac{f_{v,\alpha}A_v}{f_{\alpha,\alpha}} \\
&= d_{i_0}A_w + \frac{d_{i_0}a_w A_\alpha}{f_{\alpha,\alpha}}(h^2(t) - h^2(0)), \qquad (2.4.10)
\end{aligned}
$$

where the first equality follows the solution (2.3.10) of p_{ij} proved in Proposition 2.3.5.

By substituting the new expressions (2.4.9) and (2.4.10) back in the expression (2.4.8) of $c_{j,r,w}$, we have

$$
c_{j,r,w} = -\frac{d_{i_0}A_r A_\alpha a_w b_j}{f_{\alpha,\alpha}}(h^2(t) - h^2(0)). \qquad (2.4.11)
$$

If $r \neq w$, then $E_{j+(r-1)l,wl}$ is orthogonal to $\mathfrak{k}^-$ with respect to Q. If $r = w$, by the formula (2.4.11) of $c_{j,r,w}$ and the fact that the representation ρ is the direct sum of μ, we know that if for some r, the vector $v_r = \sum_{j=1}^{l-1} c_{j,r,r}E_{j+(r-1)l,rl}$ is orthogonal to $\mathfrak{k}^-$, then all the vectors v_q's for the other q's are orthogonal to $\mathfrak{k}^-$. So by Lemma 2.4.1 after choosing the proper values of b_i's, v_r is orthogonal to $\mathfrak{k}^-$. So we can conclude that $A_+(X,Y) = (h^2(t) - h^2(0))X_0$ for some $X_0 \in \mathfrak{p}$ which is orthogonal to $\mathfrak{k}^-$ with respect to Q and therefore (1) is proved.

By (2.4.2) and (2.4.5), we have

$$\begin{aligned}
&[X^u, P_t(X^v)]\\
= & [\sum_{i=1}^{l-1} b_i E_{i+(u-1)l,\alpha l+1} + E_{ul,\alpha l+2}, \quad \sum_{j=1}^{l-1}\sum_{s=1}^{\alpha} b_j f_{v,s} E_{j+(s-1)l,\alpha l+1} + \sum_{s=1}^{\alpha} p_{v,s}(t) E_{sl,\alpha l+2}]\\
= & \sum_{i\neq j, \text{ or } u\neq s} b_i b_j f_{v,s} E_{j+(s-1)l,i+(u-1)l} + \sum_{s\neq u} p_{v,s}(t) E_{sl,ul}.
\end{aligned}$$

By (2.4.3) and (2.4.6), we have the same result for $[Y^u, P_t(Y^v)]$, therefore $A_+(X,X) = [X, P_t(X)] = [Y, P_t(Y)] = A_+(Y,Y)$ which proves the formula in (2).

Next we will prove the formulas in (3) and (4) which will finish the proof of the proposition.

By (2.4.5) and the differential equation (2.3.8) of p_{ij} we have

$$P_t'(X^u) = \sum_{r=1}^{\alpha} p_{r,u}'(t) E_{l,r,2} = 2h(t)h'(t)\sum_{r=1}^{\alpha} a_u a_r E_{l,r,2},$$

so

$$\begin{aligned}
Q(P_t'(X^u), Y^v) &= 2a_u h(t)h'(t)\sum_{r=1}^{\alpha} a_r \left(\sum_{j=1}^{l-1} b_j Q(E_{l,r,2}, E_{j,v,2}) + Q(E_{l,r,2}, E_{l,v,1})\right)\\
&= 0.
\end{aligned}$$

Therefore $Q(P_t'(X), Y) = 0$.

By taking the inner product with X^v instead of Y^v, we have

$$\begin{aligned}
Q(P_t'(X^u), X^v) &= 2a_u h(t)h'(t)\sum_{r=1}^{\alpha} a_r \left(\sum_{j=1}^{l-1} b_j Q(E_{l,r,2}, E_{j,v,1}) + Q(E_{l,r,2}, E_{l,v,2})\right)\\
&= 2a_u a_v h(t)h'(t).
\end{aligned}$$

Therefore

$$\begin{aligned} Q(P_t'(X), X) &= 2\left(\sum_{u,v=1}^{\alpha} A_u A_v a_u a_v\right) h(t)h'(t) = 2\left(\sum_{u=1}^{\alpha} A_u a_u\right)^2 h(t)h'(t) \\ &= 2\left(\frac{d_{i_0} A_\alpha}{f_{\alpha,\alpha}}\right)^2 h(t)h'(t), \end{aligned}$$

where the last equality follows either from (2.4.9) or (2.4.10). Similarly we have the same result of $Q(P_t'(Y), Y)$ as of $Q(P_t'(X), X)$. Then the result in (4) follows. □

Remark 2.4.7. In the introduction, we pointed out that there are unknown but interesting cases when $m \geq k+2$ and $n = m+1$. The minimal dimension of these manifolds is 15 as $k = 2$, $m = 5$ and $n = 6$. This manifold has different cohomology group from the two 15 dimensional symmetric spaces $\mathbb{S}^{15}$ and $SO(8)/(SO(5) \times SO(3))$. The geometry of these examples will be studied in another paper.

2.5 The Cases where $G = U(n)$ and $Sp(n)$

In this section, we will prove the theorems in the complex and quaternionic cases . First let us restate the theorem in each case using class one representation.

Theorem 2.5.1. *Let $K'/H' = \mathbb{S}^k$ with $k \geq 2$ and $\rho : K' \longrightarrow U(m)$ be an almost faithful representation with $\ker \rho \subset H'$. Suppose ρ contain a class one representation $\mu : K' \longrightarrow U(l)$ with $2l \geq k+2$. For any integer $n \geq m+2$, the cohomogeneity one manifold M defined by the groups $H \subset \{K^-, K^+\} \subset G$ in (1.0.3) does not admit a G invariant metric with non-negative sectional curvature.*

Remark 2.5.2. Proposition A.5.6 lists the complex class one representations which have less dimension than $\frac{1}{2}(k+2)$. It shows that only the defining representations of $SU(n)$, $U(n)$, $Sp(n)$ and $Sp(n) \times U(1)$ are excluded by the above theorem.

Similar to the Lemma 2.4.1 in the orthogonal case, we have the following lemma:

Lemma 2.5.3. *Assume that K', H' and μ as in Theorem 2.5.1 with $2l \geq k+2$, then $\mu(K')$ does not act transitively on $\mathbb{C}\mathrm{P}^{l-1} = U(l)/(U(l-1) \times U(1))$.*

Proof : From the classification of the transitive actions on complex projective spaces, we only need to look at the pair $(SU(2), U(1))$ for $\mathbb{C}\mathrm{P}^1$, $(U(n), U(n-1))$ for $\mathbb{C}\mathrm{P}^{n-1}$ and $(Sp(n), Sp(n-1))$ for $\mathbb{C}\mathrm{P}^{2n-1}$. In the first case, the subgroup $U(1) \subset SU(2)$ does not fix any vector in $\mathbb{C}^2$. In the last two cases, we have $2l = k+1$ which contradicts the assumption on l. So the action of $\mu(K')$ on $\mathbb{C}\mathrm{P}^{l-1}$ is non-transitive.

□

Sketch of the Proof of Theorem 2.5.1: Suppose ρ has the decomposition as (2.2.1) in the orthogonal case. Let $c(t)$ be the normal geodesic connecting the two singular orbits $B_\pm = G/K^\pm$ with $c(0) = p_- \in B_-$ and $c(L) = p_+ \in B_+$. Up to equivalence of the group diagram, the Weyl group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the generators have the following representatives as follows:

$$w_+ = \begin{pmatrix} I_{m-1} & & \\ & -1 & \\ & & I_{n-m} \end{pmatrix},$$

and

$$w_- = \begin{pmatrix} A_1 & & & & & & \\ & A_2 & & & & & \\ & & \mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_{l_0}) & & & & \\ & & & \ddots & & & \\ & & & & A_2 & & \\ & & & & & \mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_{l_0}) & \\ & & & & & & I_{n-m} \end{pmatrix},$$

where $A_1 \in U(r)$ and $A_2 \in U(l - l_0)$.

In addition to the matrices $\{E_{i,j}\}_{1\leq i\neq j\leq n}$, let $F_{i,j}$ be the symmetric matrix with $\imath(\imath = \sqrt{-1})$ at the (i,j) and (j,i)-entries if $i \neq j$ and $\sqrt{2}\imath$ at (i,i)-entry if $i = j$. Then $\{E_{i,j}\}$ and $\{F_{i,j}\}$ consist the orthonormal basis of the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ with $Q = -\frac{1}{2}\Re\mathrm{Tr}$, where $\Re$ takes the real part of a complex number. Without loss of generality, we may assume that $r = 0$, i.e., ρ is the α copies of the class one representation μ. In addition to the notation $E_{a,i,\xi}$, we denote $F_{(i-1)l+a,m+\xi}$ by $F_{a,i,\xi}$. Let $\mathfrak{p}$ be the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H in the Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$ of G and $\mathfrak{q}$ be the subspace of $\mathfrak{p}$ spanned by the vectors $\{E_{a,i,\xi}\}$ and $\{F_{a,i,\xi}\}$. By using Schur's Lemma and Wilking's Rigidity Theorem, we have the following proposition for the non-negatively curved metric g on M:

Proposition 2.5.4. *Suppose that the metric g is non-negatively curved, then the endomorphism P restricted to $\mathfrak{q}$ satisfies the following conditions:*

1. *$P(E_{a,i,\xi}) = \sum_{j=1}^{\alpha} f_{i,j} E_{a,j,\xi}$ and $P(F_{a,i,\xi}) = \sum_{j=1}^{\alpha} f_{i,j} F_{a,j,\xi}$, if $a = 1, \ldots, l-1$;*

2. *$P(E_{l,i,\xi}) = \sum_{j=1}^{\alpha} p_{i,j}(t) E_{l,j,\xi}$ and $P(F_{l,i,\xi}) = \sum_{j=1}^{\alpha} p_{i,j}(t) F_{l,j,\xi}$,*

where $p_{i,j}(t) = a_i a_j h^2(t) + f_{i,j} - a_i a_j f_{\alpha,\alpha}$, $h(t) = \left|\left|E^*_{l,\alpha,\xi}\right|\right| = \left|\left|F^*_{l,\alpha,\xi}\right|\right|$ and $f_{i,\alpha} = a_i f_{\alpha,\alpha}$.

The non-negativity of the sectional curvature of the plane spanned by $E^*_{l,\alpha,\xi}$ and T implies that $h(t)$ is an even function, $h(L) = 0$ and $h''(t) \leq 0$ for $t \in [0, L)$. Let $P_0 = (f_{i,j})_{\alpha\times\alpha}$, then from the decomposition of P_0 as in the orthogonal case, we have the constants A_u for $u = 1, \ldots, \alpha$ with $A_\alpha \neq 0$. Let

$$X^u = \sum_{i=1}^{l-1} b_i E_{i,u,1} + E_{l,u,2} + \sum_{i=1}^{l-1} c_i F_{i,u,1} + F_{l,u,2} \tag{2.5.1}$$

$$Y^u = \sum_{j=1}^{l-1} b_j E_{j,u,2} + E_{l,u,1} + \sum_{j=1}^{l-1} c_j F_{j,u,2} + F_{l,u,1}, \tag{2.5.2}$$

where $u = 1, \ldots, \alpha$ and $\sum_{i=1}^{l-1}(b_i^2 + c_i^2) = 2$. Then we define the following two vectors:

$$X = \sum_{u=1}^{\alpha} A_u X^u, \quad Y = \sum_{v=1}^{\alpha} A_v Y^v. \tag{2.5.3}$$

A computation shows that $[X, Y] = 0$ and results in the following claim:

Claim. *For properly chosen values of b_i's and c_i's, we have*

1. *There exists some $X_0 \in \mathfrak{p}$ which orthogonal to $\mathfrak{k}^-$, the Lie algebra of K^-, with respect to Q such that $A_+(X, Y) = (h^2(t) - h^2(0))X_0$;*

2. *$A_+(X, X) = A_+(Y, Y)$;*

3. *$Q(P_t'(X), Y) = 0$;*

4. $-\frac{1}{4}Q(P_t'(X),X)Q(P_t'(Y),Y) = -4\left(\frac{d_{i_0}A_\alpha}{f_{\alpha,\alpha}}\right)^4 (h(t)h'(t))^2.$

The existence of X_0 follows from the non-transitive action of $\mu(K')$ on $\mathbb{C}\mathrm{P}^{l-1} = U(l)/(U(l-1)\times U(1))$ proved in Lemma 2.5.3. The same argument as in the orthogonal case shows that the non-negativity of the sectional curvature of the 2−plane spanned by X^* and Y^* gives the desired contradiction. This completes the proof in the unitary case. □

Finally we are in the case where $G = Sp(n)$.

Theorem 2.5.5. *Let $K'/H' = \mathbb{S}^k$ with $k \geq 2$ and $\rho : K' \longrightarrow Sp(m)$ be an almost faithful representation with $\ker\rho \subset H'$. Suppose ρ contain a class one representation $\mu : K' \longrightarrow Sp(l)$ with $4l \geq k+2$. For any integer $n \geq m+2$, the cohomogeneity one manifold M defined by the groups $H \subset \{K^-, K^+\} \subset G$ in (1.0.3) does not admit a G invariant metric with non-negative sectional curvature.*

Remark 2.5.6. Proposition A.5.7 lists the quaternionic class one representations which have less dimension than $\frac{1}{4}(k+2)$. It show that only the standard representation of $Sp(l)$ for the pair $(Sp(l), Sp(l-1))$ is excluded by the above theorem.

We have the following lemma on the non-transitive action of $\mu(K')$ on the quaternionic projective spaces which is analogue to the Lemma 2.4.1 and Lemma 2.5.3:

Lemma 2.5.7. *Assume that K', H' and μ as in Theorem 2.5.5 with $4l \geq k+2$, then $\mu(K')$ does not act transitively on $\mathbb{H}\mathrm{P}^{l-1} = Sp(l)/(Sp(l-1)\times Sp(1))$.*

Proof : From the transitive actions on $\mathbb{HP}^{l-1}$, we have $\mu(K') = Sp(l)$ and then $H' = Sp(l-1)$, μ is the standard representation of $K' = Sp(l)$. However in this case, $k = 4l - 1$ which contradicts with the assumption $4l \geq k + 2$. $\square$

Sketch of the Proof of Theorem 2.5.5: The proof follows the complex case where $G = U(n)$ step by step. Let $G_{i,j}$ denote the symmetric matrix with 1 at the $i, j-$ and $j, i-$entries. Let $G_{a,i,\xi} = G_{a+i(l-1)+r,m+\xi}$ and $\{1, \imath, \jmath, \kappa\}$ be the basis of $\mathbb{H}$ over $\mathbb{R}$.. For the endomorphism P_t, one can show the following proposition which is similar to the orthogonal and complex cases.

Proposition 2.5.8. *Suppose that the metric g is non-negatively curved, then the endomorphism P on $\mathfrak{p}$ satisfies the following conditions:*

1. $P(E_{a,i,\xi}) = \sum_{j=1}^{\alpha} f_{i,j} E_{a,j,\xi}$ *and* $P(\theta G_{a,i,\xi}) = \sum_{j=1}^{\alpha} f_{i,j} \theta G_{a,j,\xi}$*, if* $a = 1, \ldots, l-1$*;*

2. $P(E_{l,i,\xi}) = \sum_{j=1}^{\alpha} p_{i,j}(t) E_{l,j,\xi}$ *and* $P(\theta G_{l,i,\xi}) = \sum_{j=1}^{\alpha} p_{i,j}(t) \theta G_{l,j,\xi}$*,*

where $p_{i,j}(t) = a_i a_j h^2(t) + f_{i,j} - a_i a_j f_{\alpha,\alpha}$*,* $h(t) = \left|\left|E^*_{l,\alpha,\xi}\right|\right| = \left|\left|\theta G^*_{l,\alpha,\xi}\right|\right|$*,* $f_{i,\alpha} = a_i f_{\alpha,\alpha}$ *and* θ *can be* $\imath$*,* $\jmath$ *and* κ*.*

Furthermore $h(t)$ is an even function with $h(0) \neq 0$ and $h(L) = 0$. To get the desired contradiction we choose the vectors X^u and Y^v as follows:

$$
\begin{aligned}
X^u &= \sum_{i=1}^{l-1} b_i E_{(u-1)l+i,m+1} + E_{ul,m+2} + \sum_{i=1}^{l-1} c_i G_{(u-1)l+i,m+1} + c G_{ul,m+2}, \\
Y^u &= \sum_{j=1}^{l-1} b_j E_{(u-1)l+j,m+2} + E_{ul,m+1} + \sum_{j=1}^{l-1} c_j G_{(u-1)l+j,m+2} + c G_{ul,m+1},
\end{aligned}
$$

where $c = \imath + \jmath + \kappa$, b_i's are real numbers and c_i's are pure quaternionic numbers, i.e. the real part is zero. The constants satisfy the equation $1 - c^2 - \sum_{i=1}^{l-1}(b_i^2 - c_i^2) = 0$.

Let $X = \sum_{u=1}^{\alpha} A_u X^u$ and $Y = \sum_{v=1}^{\alpha} A_v Y^v$, then a computation shows that $[X, Y] = 0$ and the following claim:

Claim. *For properly chosen values of b_i's and c_i's, we have*

1. *There exists some $X_0 \in \mathfrak{p}$ which orthogonal to $\mathfrak{k}^-$, the Lie algebra of K^-, with respect to Q such that $A_+(X, Y) = (h^2(t) - h^2(0))X_0$;*

2. $A_+(X, X) = A_+(Y, Y)$;

3. $Q(P_t'(X), Y) = 0$;

4. $-\dfrac{1}{4}Q(P_t'(X), X)Q(P_t'(Y), Y) = -16\left(\dfrac{d_{i_0}A_\alpha}{f_{\alpha,\alpha}}\right)^4 (h(t)h'(t))^2.$

The existence of X_0 follows from the non-transitive action of $\mu(K')$ on $\mathbb{H}\mathrm{P}^{l-1}$ proved in Lemma 2.5.7. Then the contradiction follows as we showed in the orthogonal case. □

Chapter 3

Manifolds with Small Family of Invariant Metrics

In this chapter, we will study the cohomogeneity one manifolds which have a small family of invariant metrics.

3.1 General Restrictions on Group Diagrams

Suppose M is a compact simply-connected cohomogeneity one manifold which has group diagram $H \subset \{K^-, K^+\} \subset G$. In this chapter, we assume that G is a simple Lie group, i.e., G is $SO(n)$ or its universal cover $Spin(n) (n \geq 3)$, $SU(n) (n \geq 2)$, $Sp(n) (n \geq 1)$, $E_i (i = 6, 7, 8)$, F_4 or G_2.

The union of principal orbits is an open and dense set M_r in M and the metric g is determined by its restriction to M_r. For $t \in (0, L)$, $M_t = G.c(t)$ is a principal orbit which can be identified with the homogeneous space G/H and its tangent space $T_{c(t)}M_t$ is identified with $\mathfrak{p}$, the Q-orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$ where Q is an fixed bi-invariant inner product on $\mathfrak{g}$. Let H_0 be the identity component of H. As a representation of Ad_{H_0}, $\mathfrak{p}$ decomposes into irreducible subrepresentations as $\mathfrak{p} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_l$. The larger the value of l, the more complicated the invariant metrics on M. We will study the case where $l \leq 3$. When $l = 1$, i.e., the isotropy action is strongly irreducible, the cohomogeneity one manifold is a sphere and the action has two fixed points. When $l = 2$, the manifold is either a double which will be defined later or a sphere and the action is a sum action. Therefore in the rest of this chapter, we assume that $l = 3$.

Our goal is to find more examples of cohomogeneity one manifold which will be new examples either of non-negatively curved manifold or of obstructions to such metric. In dimension $3, 4, 5, 6$ and 7, all simply connected cohomogeneity one manifolds are classified in [Ho]. Except in dimension 7, the non-negatively curved ones are classified too. In dimension 7, there are some unknown examples on which the principal isotropy representation has more than 4 irreducible summands. Hence we will look at examples which are 8 dimensional and up.

Some group diagrams give us known examples, for examples, the double we will define later and the actions with non-empty fixed point set. Some diagrams define examples on which the existence of non-negatively curved metric relies on another cohomogeneity one manifold with simpler group diagram, for example, non-primitive group diagrams. There are some group diagrams called *reducible*, by which the manifold is defined has a simpler group diagram. In our classification, we will put restrictions on the group diagram to avoid the above situations.

Definition 3.1.1. A manifold is called *a double* if it admits a cohomogeneity one action and $K^- = K^+$ in the group diagram.

One can put a non-negatively curved invariant metric on the disk bundle $G \times_K \mathbb{D}^l$ making the boundary totally geodesic. On a double M, if we glue the two identical disk bundles along the totally geodesic boundary, then M has an invariant metric with non-negative sectional curvatures.

Definition 3.1.2. A cohomogeneity one group diagram is *non-primitive* if there is a proper connected subgroup $L \subset G$ with $K^{\pm} \subset L$. A group diagram is called *primitive* if it is not non-primitive.

If M has a non-primitive group diagram $H \subset \{K^{\pm}\} \subset G$, then $K^{\pm} \subset L$ for some proper subgroup L of G. Let N be the cohomogeneity one manifold defined by the diagram $H \subset \{K^{\pm}\} \subset L$, then M is G equivariantly diffeomorphic to $G \times_L N$. So M has an invariant metric with non-negative sectional curvatures if N admits such a metric.

Definition 3.1.3. A cohomogeneity one group diagram is *nonreducible* if H does not project onto any factor of G.

If the group diagram is not nonreducible, or reducible, then the cohomogeneity one manifold defined by the diagram has a simple group diagram by the following proposition proved in [Ho].

Proposition 3.1.4. *Let M be the cohomogeneity one manifold given by the group diagram $H \subset \{K^-, K^+\} \subset G$ and suppose $G = G_1 \times G_2$ with $\mathrm{Proj}_2(H) = G_2$. Then the sub-action of $G_1 \times \{1\}$ on M is also by cohomogeneity one, with the same orbits, and with isotropy groups $K_1^\pm = K^\pm \cap (G_1 \times \{1\})$ and $H_1 = H \cap G_1 \times \{1\}$.*

Another case that is excluded by our classification is the fixed point action.

Proposition 3.1.5. *A cohomogeneity one action on a compact simply-connected manifold with a fixed point is equivalent to an isometric action on a compact rank one symmetric space.*

The above proposition is proved in [Ho] as Proposition 2.1.28 and he also determined all possible group diagrams in this case in the same paper.

In the rest of this chapter, we will assume that the group diagram is primitive, nonreducible and fixed point free.

Not every group diagram gives a simply connected cohomogeneity one manifold and the necessary conditions are given in [GWZ] as

Lemma 3.1.6. *Assume that G acts on M by cohomogeneity one with M connected and G connected. Then*

1. *There are no exceptional orbits, i.e., $l_\pm \geq 1$.*

2. If both $l_\pm \geq 2$, then $K^\pm$ and H are all connected.

3. If one of $l_\pm$, say $l_- = 1$, and $l_+ \geq 2$, then $K^- = H \cdot \mathbb{S}^1 = H_0 \cdot \mathbb{S}^1$, $H = H_0 \cdot \mathbb{Z}_k$ and $K^+ = K_0^+ \cdot \mathbb{Z}_k$.

In the last case where $l_- = 1$, we have K^- is connected, $\mathbb{Z}_k \subset N_G(K_0^+)/K_0^+$ and $\mathbb{S}^1$ normalizes H_0.

3.2 Two Singular Orbits are Strongly Isotropy Irreducible

First we recall

Definition 3.2.1. A homogeneous space G/H is *isotropy irreducible* if the isotropy action on the tangent space by H is irreducible. If the isotropy action by the identity component of H is irreducible, then it is a *strongly isotropy irreducible* homogeneous space.

Any left-invariant metric on an isotropy irreducible homogeneous space is homothetic(i.e., multiplication by a positive constant) to the one which is induced by a bi-invariant inner product on $\mathfrak{g}$. Other than the irreducible symmetric spaces, J.Wolf classified all strongly isotropy irreducible homogeneous spaces in [Wo1]. M.Wang and W.Ziller later classified all isotropy irreducible homogeneous spaces in [WZ].

In this section, we will show that the two singular orbits $B_\pm = G/K^\pm$ are strongly isotropy irreducible homogeneous spaces unless the cohomogeneity one manifold M is equivariantly diffeomorphic to some sphere.

We look at the case where $\mathfrak{m}_i$'s are pairwisely non-equivalent as representations of Ad_{H_0}.

Theorem 3.2.2. *If* $\mathfrak{m}_1$*,* $\mathfrak{m}_2$ *and* $\mathfrak{m}_3$ *are pairwisely non-equivalent as the* Ad_{H_0} *representations, then* $G/K^{\pm}$ *are strongly isotropy unless* M *is* G *equivariantly diffeomorphic to a sphere.*

Proof : There are three different types of the group diagram. In the first type, none of $K^{\pm}/H$ are strongly isotropy irreducible, then both $G/K^{\pm}$ are strongly isotropy irreducible.

In the second type, only one of $K^{\pm}/H$, say K^{-}/H, is strongly isotropy irreducible. We will show the manifold is equivariantly diffeomorphic to some sphere. For any X, $Y \in \mathfrak{p}$, we denote $Q(X,Y)$ by $< X,Y >$. Without loss of generality, we may assume that $\mathfrak{k}^{-} = \mathfrak{h} \oplus \mathfrak{m}_1$ and $\mathfrak{k}^{+} = \mathfrak{h} \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ since the group diagram is primitive and G-action is fix-point free. Let $\mathfrak{p}_1 = \mathfrak{m}_1$ and $\mathfrak{p}_2 = \mathfrak{m}_2 \oplus \mathfrak{m}_3$. For any $X_1, Y_1 \in \mathfrak{p}_1$, $X_2, Y_2 \in \mathfrak{p}_2$ and $Y_0 \in \mathfrak{h}$, since $[X_2, Y_0] \in \mathfrak{k}^{+} = \mathfrak{h} \oplus \mathfrak{p}_2$, $[Y_1, X_1] \in \mathfrak{k}^{-} = \mathfrak{h} \oplus \mathfrak{p}_1$ and $[X_2, Y_2] \in \mathfrak{k}^{+} = \mathfrak{h} \oplus \mathfrak{p}_2$, we have

$$\begin{aligned}
< [X_1, X_2], Y_0 > &= < X_1, [X_2, Y_0] > = 0, \\
< [X_1, X_2], Y_1 > &= < Y_1, [X_1, X_2] > = < [Y_1, X_1], X_2 > = 0, \\
< [X_1, X_2], Y_2 > &= < X_1, [X_2, Y_2] > = 0.
\end{aligned}$$

Therefore $[X_1, X_2]$ is orthogonal to any vector in $\mathfrak{g}$, i.e., $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$.

For any subset $\mathfrak{q}$ of $\mathfrak{p}$, denote the annihilator of $\mathfrak{q}$ by $\mathrm{Ann}(\mathfrak{q})$ which is the collection of the element X in $\mathfrak{p}$ such that $[X,Y]=0$ for any $Y\in\mathfrak{q}$. Let

$$\mathfrak{h}_0=\mathrm{Ann}(\mathfrak{p}_1\oplus\mathfrak{p}_2)\cap\mathfrak{h}=\{X\in\mathfrak{h}|[X,Y]=0,\ \text{for any } Y\in\mathfrak{p}_1\oplus\mathfrak{p}_2\},$$

and

$$\mathfrak{h}_i=\mathrm{Ann}(\mathfrak{p}_i)\cap\mathfrak{h}_0^{\perp}\cap\mathfrak{h},\quad (i=1,2),\quad \mathfrak{h}_3=(\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\mathfrak{h}_2)^{\perp}\cap\mathfrak{h},$$

where $\perp$ is the orthogonal complement with respect to the inner product Q.

Since $[\mathfrak{p}_1,\mathfrak{p}_1]\subset\mathfrak{k}^-=\mathfrak{h}\oplus\mathfrak{p}_1$, $[\mathfrak{p}_1,\mathfrak{h}]\subset\mathfrak{p}_1$ and $[[\mathfrak{p}_1,\mathfrak{p}_1],\mathfrak{p}_2]=-[[\mathfrak{p}_1,\mathfrak{p}_2],\mathfrak{p}_1]-[[\mathfrak{p}_2,\mathfrak{p}_1],\mathfrak{p}_1]=-[0,\mathfrak{p}_1]-[0,\mathfrak{p}_1]=0$, we have $[\mathfrak{p}_1,\mathfrak{p}_1]\subset\mathfrak{p}_1\oplus\mathfrak{h}_0\oplus\mathfrak{h}_2$. Moreover $<[\mathfrak{p}_1,\mathfrak{p}_1],\mathfrak{h}_1>=<\mathfrak{p}_1,[\mathfrak{p}_1,\mathfrak{h}_1]>=<\mathfrak{p}_1,0>=0$, we have $[\mathfrak{p}_1,\mathfrak{p}_1]\subset\mathfrak{p}_1\oplus\mathfrak{h}_2$. Let $\mathrm{Lie}(\mathfrak{p}_1)$ be the Lie algebra generated by $\mathfrak{p}_1$, then $\mathrm{Lie}(\mathfrak{p}_1)\subset\mathfrak{p}_1\oplus\mathfrak{h}_2$. Suppose $\mathrm{Lie}(\mathfrak{p}_1)\subsetneq\mathfrak{p}_1\oplus\mathfrak{h}_2$, i.e., there is a vector $X\in\mathfrak{p}_1\oplus\mathfrak{h}_2$ such that $X\perp\mathrm{Lie}(\mathfrak{p}_1)$. This implies that $X\in\mathfrak{h}_2$, so $[X,\mathfrak{p}_1]\subset\mathfrak{p}_1$. Then for any vector $Y\in\mathfrak{p}_1$, we have $<[X,\mathfrak{p}_1],Y>=<X,[\mathfrak{p}_1,Y]>=0$ since $[\mathfrak{p}_1,Y]\subset L(\mathfrak{p}_1)$. Therefore $[X,\mathfrak{p}_1]=0$ which implies that $X\in\mathrm{Ann}(\mathfrak{p}_1)\cap\mathfrak{h}_2=0$. So we have $\mathrm{Lie}(\mathfrak{p}_1)=\mathfrak{p}_1\oplus\mathfrak{h}_2$. Similarly, we have $\mathrm{Lie}(\mathfrak{p}_2)=\mathfrak{p}_2\oplus\mathfrak{h}_1$.

Consequently we have $<[\mathfrak{h}_3,\mathfrak{p}_1],\mathfrak{p}_1>=<\mathfrak{h}_3,[\mathfrak{p}_1,\mathfrak{p}_1]>=0$ since $[\mathfrak{p}_1,\mathfrak{p}_1]\subset\mathrm{Lie}(\mathfrak{p}_1)=\mathfrak{p}_1\oplus\mathfrak{h}_2$. And from the fact that $\mathfrak{k}^-=\mathfrak{h}\oplus\mathfrak{p}_1$ we have $[\mathfrak{h}_3,\mathfrak{p}_1]\subset\mathfrak{p}_1$, therefore we have $[\mathfrak{h}_3,\mathfrak{p}_1]=0$, i.e., $\mathfrak{h}_3\subset\mathfrak{h}_1$, which implies that $\mathfrak{h}_3=0$ by the definition of $\mathfrak{h}_3$.

By the Jacobi identity, we have

$$[[\mathfrak{h}_0,\mathfrak{h}],\mathfrak{p}_1\oplus\mathfrak{p}_2]+[[\mathfrak{h},\mathfrak{p}_1\oplus\mathfrak{p}_2],\mathfrak{h}_0]+[[\mathfrak{p}_1\oplus\mathfrak{p}_2,\mathfrak{h}_0],\mathfrak{h}]=0.$$

Since $[\mathfrak{h},\mathfrak{p}_1\oplus\mathfrak{p}_2]\subset\mathfrak{p}_1\oplus\mathfrak{p}_2$ and $\mathfrak{h}_0=\mathrm{Ann}(\mathfrak{p}_1\oplus\mathfrak{p}_2)\cap\mathfrak{h}$, we have $[[\mathfrak{h}_0,\mathfrak{h}],\mathfrak{p}_1\oplus\mathfrak{p}_2]=0$, which implies that $[\mathfrak{h}_0,\mathfrak{h}]\subset\mathfrak{h}_0$, i.e., $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$. Similarly, we have that

$[\mathfrak{h}_1, \mathfrak{h}] \subset \mathrm{Ann}(\mathfrak{p}_1) \cap \mathfrak{h}$. Furthermore $< [\mathfrak{h}_1, \mathfrak{h}], \mathfrak{h}_0 >=< \mathfrak{h}_1, [\mathfrak{h}, \mathfrak{h}_0] >= 0$ since $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$, i.e., $[\mathfrak{h}_1, \mathfrak{h}] \subset \mathfrak{h}_1$ which implies that $\mathfrak{h}_1$ is also an ideal of $\mathfrak{h}$. Using the same argument we can show that $\mathfrak{h}_2$ is an ideal of $\mathfrak{h}$. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ and $\mathfrak{h}_0$ annihilates $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, $\mathfrak{h}_0$ is an ideal of $\mathfrak{g}$. By the assumption that the group diagram is non-reducible, we have $\mathfrak{h}_0 = 0$. So we have

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2, \quad \text{and} \quad \mathfrak{k}^- = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{p}_1, \quad \mathfrak{k}^+ = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{p}_2.$$

We claim that $\mathrm{Lie}(\mathfrak{p}_1) = \mathfrak{h}_2 \oplus \mathfrak{p}_1$ is an ideal in $\mathfrak{g}$. In fact, $[\mathfrak{h}_2 \oplus \mathfrak{p}_1, \mathfrak{h}_1] = [\mathfrak{h}_2, \mathfrak{h}_1] = 0$ and $[\mathfrak{h}_2 \oplus \mathfrak{p}_1, \mathfrak{p}_2] = 0$ imply that $[\mathrm{Lie}(\mathfrak{p}_1), \mathfrak{g}] \subset \mathrm{Lie}(\mathfrak{p}_1)$. Similarly $L(\mathfrak{p}_2)$ is also a ideal in $\mathfrak{g}$. Therefore

$$G = L_1 \times L_2, \quad K^- = H_1 \times L_1, \quad K^+ = H_2 \times L_2, \quad \text{and} \quad H = H_1 \times H_2,$$

where $\mathfrak{h}_i$ is the Lie algebra of H_i, $\mathrm{Lie}(\mathfrak{p}_i)$ is the Lie algebra of L_i for $i = 1, 2$ and L_1/H_2, L_2/H_1 are spheres. Hence the G-action is a sum action and the manifold M is G-equivariant to a sphere.

Now we consider the last type where both $K^{\pm}/H$ are strongly isotropy irreducible. Since the group diagram is primitive, without loss of generality, we may assume that $\mathfrak{k}^- = \mathfrak{h} \oplus \mathfrak{m}_1$ and $\mathfrak{k}^+ = \mathfrak{h} \oplus \mathfrak{m}_2$. By the similar argument as we did at the beginning of the proof of the second type, we have that $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_3$ and $[\mathfrak{m}_1, \mathfrak{m}_2]$ is an invariant space under the action Ad_{H_0}. By the irreducibility of $\mathfrak{m}_3$, $[\mathfrak{m}_1, \mathfrak{m}_2]$ is either equal to 0 to $\mathfrak{m}_3$. In the first case, $\mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is an ideal of $\mathfrak{g}$, so the group generated by K^- and K^+ is a proper normal subgroup of G which contradicts the primitivity assumption. Therefore $[\mathfrak{m}_1, \mathfrak{m}_2] = \mathfrak{m}_3$ which implies that both $G/K^{\pm}$ are strongly isotropy irreducible. □

Next we look at the general case where we may have two or three equivalent representations in $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$.

All complex irreducible representations of simple Lie algebras are highest weight representations, so we can identify the representation with its highest weight. Each highest weight is the linear combination of the so called fundamental weights with non-negative integer coefficients. If the Lie algebra $\mathfrak{g}$ has rank$= n$, then there are n fundamental weights $\varpi_1, \cdots, \varpi_n$. In the Appendix A, we list the fundamental weights for all classical Lie algebras and the exceptional Lie algebra $\mathfrak{g}_2$.

For the Lie algebras $\mathfrak{so}(4)$ and Lie algebra $\mathfrak{so}(6)$, we specify some representations and their highest weights. For $\mathfrak{so}(4)$, the representations with highest weights $\varpi_1 = \frac{1}{2}(e_1 - e_2)$ and $\varpi_2 = \frac{1}{2}(e_1 + e_2)$ are the two spin representations. The standard representation ϱ_4 of $SO(4)$ on $\mathbb{C}^4$ has the highest weight $\varpi_1 + \varpi_2 = e_1$. For $\mathfrak{so}(6)$, the representation with highest weight $\varpi_1 = e_1$ is the standard representation ϱ_6 of $SO(6)$ on $\mathbb{C}^6$. However the representation of $\mathfrak{su}(4)$ with the highest weight ϖ_1 is the standard representation μ_4 of $SU(4)$ on $\mathbb{C}^4$ though $\mathfrak{so}(6)$ is isomorphic to $\mathfrak{su}(4)$.

Suppose K/H is a sphere and the action of K on the sphere is effective and transitive, and Ad_H action on the tangent space is irreducible. For any other compact Lie group L, we have $(K \times L)/(H \times L)$ as the same sphere. Let $\varphi \otimes \psi$ be an irreducible real representation of $K \times L$, i.e. both φ and ψ are real or quaternionic representations. Suppose that the restriction of φ to H is $\varphi_1 \oplus \varphi_2$. In order that $\varphi \otimes \psi$ remains as an real irreducible representation under the restriction from $K \times L$ to $H \times L$, φ_i must be of different type from ψ for $i = 1, 2$ and φ_1 is the complex conjugation of φ_2. Moreover if we exam each pair of (K, H), we will have the following proposition:

Proposition 3.2.3. *Let K/H be a sphere with irreducible Ad_H action on the tangent space and φ be an irreducible complex representation of K. Suppose that $\mathrm{Res}_H^K(\varphi) = \varphi_1 \oplus \varphi_2$ and φ_1, $\varphi_2 = \varphi_1^*$ are two irreducible complex representations of H with different type from φ. Then (K, H, φ) is one of the following triples:*

1. *$(SO(2n+1)$,$SO(2n)$,$\varpi_n)$($n \geq 3$ an odd integer) and $\mathrm{Res}(\varphi) = \varpi_{n-1} \oplus \varpi_n$,*
2. *$(Sp(1)$, $U(1)$, $\varpi_1)$ and $\mathrm{Res}(\varphi) = \phi \oplus \phi^*$.*

Proof : Since K/H is isotropy irreducible, we have that (K, H) is one the pairs: $(SO(2n), SO(2n-1))$(for $n \geq 1$), $(SO(2n+1), SO(2n))$(for $n \geq 1$), $(Spin(7), G_2)$, $(G_2, SU(3))$ and $(Sp(1), U(1))$.

If $(K, H) = (SO(2n), SO(2n-1))$ for $n \geq 2$, then φ will be a representation with highest weight $a_1e_1 + \cdots + a_ne_n$ where $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0$ are all integers or half integers. From the classical branching rule for the pair $(SO(2n), SO(2n-1))$, φ_1 is the representation with highest weight $a_1e_1 + \cdots + a_{n-1}e_{n-1}$. So φ_1 cannot be of complex type since any representation of $SO(2n-1)$ is either real or quaternionic.

It is easy to see that (K, H) cannot be the pair $(SO(2), SO(1))$ or $(SO(3), SO(2))$.

If $(K, H) = (SO(2n+1), SO(2n))$ for $n \geq 2$ then φ is a representation with highest weight $a_1 e_1 + \cdots + a_n e_n$ where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ are all integers or half integers. From the classical branching rule for the pair $(SO(2n+1), SO(2n))$ and the same argument as in the pair $(SO(2n), SO(2n-1))$, φ_1 which is the highest weight representation with highest weight $a_1 e_1 + \cdots + a_n e_n$ is a complex representation. So n is an odd integer and $a_n \neq 0$. From the branching rule, φ_2 is the representation with highest weight $a_1 e_1 + \cdots + a_{n-1} e_{n-1} - a_n e_n$ and $a_1 = \cdots = a_{n-1} = a_n = \frac{1}{2}$, i.e., φ is the spin representation of $SO(2n+1)$ and φ_1, φ_2 are two half-spin representations of $SO(2n)$.

If $(K, H) = (Spin(7), G_2)$, then all representations of G_2 are real, so φ_i cannot be a complex representation.

If $(K, H) = (G_2, SU(3))$, then φ is a real representation. Suppose φ is a representation with the highest weight $a_1 \varpi_1 + a_2 \varpi_2$ with $a_1, a_2 \geq 0$. Let π_1 and π_2 be the fundamental weights of $\mathfrak{su}(3)$. Then the representation φ_1 with highest weight $(a_1 + a_2)\pi_1 + a_2 \pi_2$ appears in $\mathrm{Res}(\varphi)$ with multiplicity 1. Since $\varphi_2 = \varphi_1^*$ and φ_2 is not equivalent to φ_1, we know that φ_1 is a complex representation, i.e. $a_1 > 0$, and φ_2 has the highest weight $a_2 \pi_1 + (a_1 + a_2)\pi_2$. From the branching rule for the pair $(G_2, SU(3))$, the representation with the highest weight $a_2 \pi_1 + a_2 \pi_2$ appears in $\mathrm{Res}(\varphi)$ with multiplicity $a_1 + 1 - \max\{0, a_1 - a_2\} \geq 1$, i.e. $\mathrm{Res}(\varphi)$ has more representations than φ_1 and φ_2.

For the last pair $(K, H) = (Sp(1), U(1))$, only representation φ with the highest weight e_1, or the standard representation of $Sp(1)$, decomposes to $\phi \oplus \phi^*$ when it is restricted $U(1)$. This finishes the proof of the proposition. $\square$

The next proposition gives the classification of the triples (K, H, φ) such that $\mathrm{Res}^K_H(\varphi)$ remains irreducible as a complex representation and K/H is a sphere.

Proposition 3.2.4. *Suppose that* (K, H) *is a sphere and* Ad_H *action on the tangent space is irreducible. Then* (K, H, φ) *such that* $\varphi \neq \mathrm{Id}$ *and* $\mathrm{Res}(\varphi)$ *is irreducible is one following triples:*

1. $(SO(2n),\ SO(2n-1),\ k\varpi_{n-1})(n \geq 2,\ k \geq 1)$ *and* $\mathrm{Res}(\varphi) = k\varpi_{n-1}$,

2. $(SO(2n),\ SO(2n-1),\ k\varpi_n)(n \geq 2,\ k \geq 1)$ *and* $\mathrm{Res}(\varphi) = k\varpi_{n-1}$,

3. $(Spin(7),\ G_2,\ k\varpi_1)(k \geq 1)$ *and* $\mathrm{Res}(\varphi) = k\varpi_1$,

4. $(U(1),\ \{1\},\ \phi^k)(k \in \mathbb{Z})$ *and* $\mathrm{Res}(\varphi) = \mathrm{Id}$.

Proof : From the Dynkin's more general classification or the branching rule for the possible pairs (K, H). $\square$

Now we consider the group-triple $H \subset K \subset G$ such that K/H is a sphere and the Ad_H action on the tangent space is irreducible and the isotropy representation Ad_K on the tangent space of G/K has two non-equivalent irreducible summands $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. In order that the Ad_H action on the tangent space of G/H has only three irreducible summands, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ should remain irreducible when restricted to H. With the Proposition 3.2.3 and Proposition 3.2.4, we can prove:

Theorem 3.2.5. *Let G be a simple Lie group and $H \subset K \subset G$ be three compact connected Lie groups such that K/H is a sphere and Ad_H is irreducible on the tangent space $\mathfrak{m}_3$ of K/H. Suppose the Ad_K action on the tangent space of G/K has only two irreducible summands $\mathfrak{m}_1$ and $\mathfrak{m}_2$. If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ remain irreducible when they are viewed as the representations of Ad_H, then $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are pairwisely non-equivalent.*

Proof : The classification of compact homogeneous spaces with two irreducible isotropy summands has been done by W. Dickinson and M. Kerr in [DM]. In their paper, they have listed all the triples (G, K, χ) where $\mathrm{Ad}_K = \chi = \chi_1 \oplus \chi_2$ and G is a compact simple Lie group. One observation they made in the section of preliminaries is that χ_1 and χ_2 are non-equivalent except for the pair $(SO(8), G_2)$ listed as **I.16**. For this pair, the isotropy representation Ad_K is $\varpi_1 \oplus \varpi_1$. Since K/H is a sphere, we have $H = SU(3)$. Then $\mathrm{Res}(\varpi_1) = [\varpi_1]_{\mathbb{R}} \oplus \mathrm{Id}$, i.e., the isotropy representation of G/H has more than 3 irreducible summands.

Now take a triple $(G, K, \chi = \chi_1 \oplus \chi_2)$ in their classification for which K can act transitively on some sphere. For each possible H such that K/H is a sphere and $\mathrm{Res}^K_H(\chi_i)$ remains irreducible, we claim that $\mathrm{Res}(\chi_1)$ and $\mathrm{Res}(\chi_2)$ are non-equivalent. In most triples, it can be shown by the fact that χ_1 and χ_2 have different dimensions. If χ_1 and χ_2 have the same dimension which are **I.1, I.2, I.3, I.4, I.5, I.14, I.18, II.5, III.6, V.1**, then we prove it by the following argument. We copy the examples from their list in Table 3.1:

<table>
<tr><th></th><th>K</th><th>χ_1</th><th>χ_2</th></tr>
<tr><td>I.1</td><td>$SO(m) \times SU(k), (k \geq 3)$</td><td>$\mathrm{Id} \otimes [\varpi_2 \varpi_{k-1}^2]_{\mathbb{R}}$</td><td>$\varrho_m \otimes \varpi_1 \varpi_{k-1}$</td></tr>
<tr><td>I.2</td><td>$SO(m) \times SO(k), (k \geq 7)$</td><td>$\mathrm{Id} \otimes \varpi_1 \varpi_3$</td><td>$\varrho_m \otimes \varpi_2$</td></tr>
<tr><td>I.3</td><td>$SO(m) \times SO(k), (k \geq 5)$</td><td>$\mathrm{Id} \otimes \varpi_1^2 \varpi_2$</td><td>$\varrho_m \otimes \varpi_1^2$</td></tr>
<tr><td>I.4</td><td>$SO(m) \times Sp(k), (k \geq 3)$</td><td>$\mathrm{Id} \otimes \varpi_1 \varpi_3$</td><td>$\varrho_m \otimes \varpi_2$</td></tr>
<tr><td>I.5</td><td>$SO(m) \times Sp(k), (k \geq 2)$</td><td>$\mathrm{Id} \otimes \varpi_1^2 \varpi_2$</td><td>$\varrho_m \otimes \varpi_1^2$</td></tr>
<tr><td>I.14</td><td>$SO(65) \times E_7$</td><td>$\mathrm{Id} \otimes \varpi_2$</td><td>$\varrho_{65} \otimes \varpi_1$</td></tr>
<tr><td>I.18</td><td>$SO(m) \times U(k), (k \geq 2)$</td><td>$\mathrm{Id} \otimes [\varpi_2 \otimes \phi]_{\mathbb{R}}$</td><td>$\varrho_m \otimes [\varpi_1 \otimes \phi]_{\mathbb{R}}$</td></tr>
<tr><td rowspan="2">II.5</td><td>$SU(p) \times SU(q) \times S(U(1) \times U(m))$</td><td colspan="2">$\varpi_1 \varpi_{p-1} \otimes \varpi_1 \varpi_{q-1} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id}$</td></tr>
<tr><td>$(p, q \geq 2, m \geq 1)$</td><td colspan="2">$[\varpi_1 \otimes \varpi_1 \otimes \phi \otimes \phi^* \otimes \varpi_{m-1}]_{\mathbb{R}}$</td></tr>
<tr><td>III.6</td><td>$Sp(1) \times U(3)$</td><td>$\mathrm{Id} \otimes [\varpi_1^2 \otimes \phi^2]_{\mathbb{R}}$</td><td>$[\varpi_1 \otimes \varpi_1 \otimes \phi]_{\mathbb{R}}$</td></tr>
<tr><td>V.1</td><td>$SO(m) \times SO(n), (m, n \geq 3, m, n \neq 4)$</td><td>$\varpi_2 \otimes \varpi_1^2$</td><td>$\varpi_1^2 \otimes \varpi_2$</td></tr>
</table>

Table 3.1: (K, χ_1, χ_2) with $\dim \chi_1$ may equal to $\dim \chi_2$

Suppose $K = K_1 \times L$ and $H = H_1 \times L$ such that the K_1 action on the sphere K/H is effective and transitive. Since χ_i is an irreducible real representation, so χ_i is either an irreducible representation ρ over $\mathbb{C}$ and ρ is of real type, or the sum of an irreducible representation ρ with its complex conjugation ρ^* and ρ is not of real type. In the first case we write $\chi_i = \rho$ and in the second case we write $\chi_i = [\rho]_{\mathbb{R}}$. Under the restriction from K to H, if $\rho = \rho_1 \oplus \rho_2$, then $[\rho]_{\mathbb{R}}$ decomposes as $[\rho_1]_{\mathbb{R}} \oplus [\rho_2]_{\mathbb{R}}$, i.e., χ_i is not irreducible any more. In the case where $\chi_i = \rho$, let $\rho = \tau \otimes \varphi$ where

τ and φ are irreducible representations of K_1 and L respectively. If $\mathrm{Res}(\tau)$ remains irreducible, then ρ will be irreducible as a representation of H. If $\mathrm{Res}(\tau) = \tau_1 \oplus \tau_2$, then $\mathrm{Res}(\rho) = \tau_1 \otimes \varphi \oplus \tau_2 \otimes \varphi$. Hence if both τ_1 and τ_2 are of complex type and conjugate to each other then $\mathrm{Res}(\rho) = [\tau_1 \otimes \varphi]_{\mathbb{R}}$ is an irreducible real representation.

First we consider the case where $\chi_i = \tau_i \otimes \varphi_i$ for $i = 1, 2$ and τ_i, φ_i are irreducible representations of K_1, L respectively. Then τ_i's and φ_i's are self-conjugated. Since $\mathrm{Res}(\chi_1) = \mathrm{Res}(\chi_2)$, we have $\varphi_1 = \varphi_2$, $\tau_1 \neq \tau_2$ and $\mathrm{Res}(\tau_1) = \mathrm{Res}(\tau_2)$. From Proposition 3.2.4, we have $K_1 = SO(2n)$, $H_1 = SO(2n-1), (n \geq 2)$ and $\tau_1 = k\varpi_{n-1}$, $\tau_2 = k\varpi_n$ for $k \geq 1$, $\varphi_1 = \varphi_2$. There is no example which satisfies these conditions.

Next without loss of generality, we assume that $\chi_1 = [\tau_1 \otimes \varphi_1]_{\mathbb{R}}$ and $\chi_2 = \tau_2 \otimes \varphi_2$. So τ_2 and φ_2 are self-conjugated and $\mathrm{Res}(\tau_1)$ is irreducible. In order to have $\mathrm{Res}(\chi_1) = \mathrm{Res}(\chi_2)$, we obtain that $\mathrm{Res}(\tau_2) = \alpha \oplus \alpha^*$ and $\tau_1 = \alpha$, $\varphi_1 = \varphi_1^* = \varphi_2$. There is no pair of (K_1, H_1) which satisfies these conditions.

Finally, we assume that $\chi_1 = [\tau_1 \otimes \varphi_1]_{\mathbb{R}}$ and $\chi_2 = [\tau_2 \otimes \varphi_2]_{\mathbb{R}}$. So $\mathrm{Res}(\tau_1)$ and $\mathrm{Res}(\tau_2)$ are irreducible and $\mathrm{Res}(\tau_1) \otimes \varphi_1 \oplus \mathrm{Res}(\tau_1)^* \otimes \varphi_1^* = \mathrm{Res}(\tau_2) \otimes \varphi_2 \oplus \mathrm{Res}(\tau_2)^* \otimes \varphi_2^*$. Since $\chi_1 \neq \chi_2$, from Proposition 3.2.4 we obtain that $K_1 = SO(2n)$, $H_1 = SO(2n-1)$ and $\tau_1 = k\varpi_{n-1}$, $\tau_2 = k\varpi_n$, $\varphi_1 = \varphi_2$ for $n \geq 2$ and $k \geq 1$. There is no example satisfies these conditions.

There are examples where the K_1 action is not effective on the sphere K/H. These are the examples **I.18, II.5, III.6** where $K_1 = S(U(m) \times U(n))$ and $H_1 = SU(m+n)$, or $K_1 = U(m)$ and $H_1 = SU(m)$. For these examples, the restriction makes the representation of the center of K_1 to be the trivial representation. By checking each of them, we obtain that $\mathrm{Res}(\chi_1)$ is non-equivalent to $\mathrm{Res}(\chi_2)$.

Next we are going to prove that the isotropy representation Ad_H of K/H is non-equivalent to $\mathrm{Res}(\chi_1)$ or $\mathrm{Res}(\chi_2)$ which will finish the proof.

We denote the representation of Ad_H on the tangent space of K/H by χ_3. For each possible pair of (K_1, H_1), the χ_3 is given as :

K_1	H_1	χ_3
$SO(n+1)$	$SO(n)(n \geq 2)$	$\varrho_n \otimes \mathrm{Id}$
$SU(2)$	$U(1)$	$[\phi]_{\mathbb{R}} \otimes \mathrm{Id}$
G_2	$SU(3)$	$[\varpi_1]_{\mathbb{R}} \otimes \mathrm{Id}$
$Spin(7)$	G_2	$\varpi_1 \otimes \mathrm{Id}$
$S(U(m) \times U(n))$	$SU(m) \times SU(n)$	$\mathrm{Id} \otimes \mathrm{Id}$
$U(n)$	$SU(n)$	$\mathrm{Id} \otimes \mathrm{Id}$

Table 3.2: Isotropy irreducible sphere K_1/H_1

Here χ_3 is written as the outer tensor product of one representation of H_1 and one of L. From Proposition 3.2.3 and Proposition 3.2.4, for $H_1 = SO(n)$, ϱ_n cannot be the restriction of some representation of $SO(n+1)$ and the same is true for the pair $(G_2, SU(3))$. For the pair $(SU(2), U(1))$, only the restriction of the representation $\varpi_1 \otimes \mathrm{Id}$ of $SU(2) \times L$ is $[\phi]_{\mathbb{R}} \otimes \mathrm{Id}$. However $\varpi_1 \otimes \mathrm{Id}$ is a quaternionic representation which cannot be χ_1 or χ_2. For the other possible pairs, we list all possible choices of

χ_1 or χ_2 such that the restrictions are χ_3:

K_1	H_1	χ_1 or χ_2
$Spin(7)$	G_2	$\varpi_1 \otimes \mathrm{Id}$
$S(U(m) \times U(n))$	$SU(m) \times SU(n)$	$\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id}$
$U(n)$	$SU(n)$	$\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id}$

Here the last Id in each pair is the trivial representation of L. The conditions for $(S(U(m) \times U(n)), SU(m) \times SU(n))$ and $(U(n), SU(n))$ are very restrictive and no example in their list satisfies those requirement. For the pair $(Spin(7), G_2)$, three examples which are **I.30**: $(Spin(9), Spin(7))$, **II.13**: $(SU(8), Spin(7))$ and **III.10**: $(Sp(8), Spin(7) \times Sp(1))$ satisfy the condition that one of χ_1 and χ_2 is $\varpi_1 \otimes \mathrm{id}$. But in each examples the other χ_i decompose as restricted to H. Therefore there is no pair (G, K) such that $\mathrm{Res}(\chi_i) = \chi_3$, $i = 1, 2$. □

From the results in Theorem 3.2.2 and the above Theorem 3.2.5, we have

Corollary 3.2.6. *Suppose the cohomogeneity one manifold M has the group diagram $H \subset \{K^-, K^+\} \subset G$. If one of the homogeneous spaces G/K^- and G/K^+ is not strongly isotropy irreducible, then M is G-equivariant to a sphere.*

3.3 Group Triple $H \subset K \subset G$ with G/K Strongly Isotropy Irreducible

Suppose that G/K is a strongly isotropy irreducible homogeneous space and H is a closed subgroup of K with K/H being a sphere. In the following proposition we list all triples with Ad_H has only three irreducible summands:

Proposition 3.3.1. *All triples $H \subset K \subset G$ with Ad_H having three irreducible summands are listed in Table 3.3 and Table 3.4.*

The proof is straightforward. We use the classification of strongly isotropy irreducible homogeneous space G/K in [Wo1]. For each pair (G, K), we list the possible H's such that K/H is a sphere and then compute the isotropy representation Ad_H of G/H. If Ad_H has precisely three irreducible summands, then we include the triple $H \subset K \subset G$ in our list.

Remark 3.3.2. The triples $\{1\} \subset SO(2) \subset SU(2)$, $U(1) \subset \Delta SU(2) \subset SU(2) \times SU(2)$ and $\Delta Sp(1) \subset Sp(1) \times Sp(1) \subset Sp(2)$ can be viewed as the triple $SO(p) \subset SO(p+1) \subset SO(p+2)$ when $p = 1$, 2 and 3.

G	K	H	
$SU(2) \times SU(2)$	$\Delta SU(2)$	$U(1)$	
$Spin(n) \times Spin(n)$	$\Delta Spin(n)$	$Spin(n-1)$	$n \geq 6$
$Spin(7) \times Spin(7)$	$\Delta Spin(7)$	G_2	
$G_2 \times G_2$	ΔG_2	$SU(3)$	
$SU(4p)$	$SU(4) \times SU(p)$	$Sp(2) \times SU(p)$	$p \geq 2$
	$SU(p) \times SU(4)$	$SU(p) \times Sp(2)$	
$SU(2p)$	$SU(p) \times SU(2)$	$SU(p) \times U(1)$	$p \geq 3$
$SU(16)$	$Spin(10)$	$Spin(9)$	
$SU(4)$	$SU(2) \times SU(2)$	$U(1) \times SU(2)$	
		$SU(2) \times U(1)$	
$SU(2)$	$SO(2)$	$\{1\}$	
$SO(p+2)$	$SO(p+1)$	$SO(p)$	$p \geq 4$
$SO(p+q+2)$	$SO(p+1) \times SO(q+1)$	$SO(p) \times SO(q+1)$	$p, q \geq 1$
		$SO(p+1) \times SO(q)$	
$Spin(6+2p)$	$Spin(6) \times SO(2p)$	$SU(3) \times SO(2p)$	$p \geq 1$
	$SO(2p) \times Spin(6)$	$SO(2p) \times SU(3)$	

Table 3.3: $H \subset K \subset G$ with Ad_H having 3 summands

$Spin(7+2p)$	$Spin(6) \times SO(2p+1)$	$SU(3) \times SO(2p+1)$	$p \geq 1$
$Spin(128)$	$Spin(16)$	$Spin(15)$	
$Spin(16)$	$Spin(9)$	$Spin(8)$	
$SO(8)$	$U(4)$	$SU(4)$	
$Spin(7)$	$Spin(6)$	$SU(3)$	
$Spin(7)$	G_2	$SU(3)$	
$SO(4)$	$SO(2) \times SO(2)$	$\Delta SO(2)$	
$Sp(2)$	$Sp(1) \times Sp(1)$	$\Delta Sp(1)$	
E_6	$SU(6) \cdot SU(2)$	$SU(5) \cdot SU(2)$	
E_6	$SU(3) \times G_2$	$SU(3) \times SU(3)$	
F_4	$Spin(9)$	$Spin(8)$	
F_4	$SO(3) \times G_2$	$SO(3) \times SU(3)$	
G_2	$SO(4)$	$SO(3)$	

Table 3.4: $H \subset K \subset G$ with Ad_H having 3 summands, continued.

3.4 Construction of Cohomogeneity One Group Diagrams

In this section, we will construct the cohomogeneity one group diagram from the group triples.

First let us describe the methods to produce other group diagrams from a given one $H \subset \{K^-, K^+\} \subset G$. Take an automorphism τ of G and apply to one side, say $H \subset K^- \subset G$. If $\tau(H)$ happens to be still inside K^+, then we have the new group diagram $\tau(H) \subset \{\tau(K^-), K^+\} \subset G$ which is equivalent to the diagram $H \subset \{K^-, \tau^{-1}(K^+)\} \subset G$. Hence at the beginning we may assume that τ leaves H invariant and changes both of $K^{\pm}$. If $\tau(K^-)$ can be obtained by a conjugation by an element $g \in G$, then g cannot be inside $N(H)_0$ otherwise the new diagram is equivalent to the original one. So the number of the possible non-equivalent group diagrams equals to the number of the connected components of the double coset space $\mathrm{Aut}(G, K^-)\backslash\mathrm{Aut}(G, H)/\mathrm{Aut}(G, K^+)$ where $\mathrm{Aut}(G, L)$ is the group of automorphisms of G leaving the subgroup L invariant.

Definition 3.4.1. A group diagram $H \subset \left\{\widetilde{K}^-, \widetilde{K}^+\right\} \subset G$ is called *a variation* of $H \subset \{K^-, K^+\} \subset G$ if after possible switching of K^- and K^+, $\widetilde{K}^- = \tau(K^-)$, $\widetilde{K}^+ = K^+$ for some $\tau \in \mathrm{Aut}(G, H)$.

First we look at the case where H is not connected. We have

Theorem 3.4.2. *The cohomogeneity one manifold defined by the group diagram $H \subset \{K^{\pm}\} \subset G$ with G simple, H disconnected and the isotropy representation Ad_{H_0} has three irreducible summands is either a double or a complex project space.*

Proof : By Lemma 3.1.6, we may assume that $l_- = 1$, i.e., K^- is connected and

$K^-/H_0 = \mathbb{S}^1$. From the classification in Proposition 3.3.1, $H_0 \subset K \subset G$ is one of

$$SO(p) \subset SO(2) \times SO(p) \subset SO(2+p), \quad (p \geq 3), \tag{3.4.1}$$

$$SU(4) \subset U(4) \subset SO(8). \tag{3.4.2}$$

Suppose that $H_0 \subset K \subset G$ is the triple in (3.4.1) and $p \neq 6$. One possible K_0^+ is $SO(2) \times SO(p)$ and then both $l_\pm$ equal to 1. Since $N(H_0) = S(O(2) \times O(p)) = SO(2) \times SO(p) \cup (SO(2) \times SO(p)) \cdot A$ with $A = \mathrm{diag}(1, -1, -1, I_{p-1})$, the possible diagrams are

$$SO(p) \cdot \{1, A\} \subset \{S(O(2) \times O(p)), S(O(2) \times O(p))\} \subset SO(p+2),$$

$$SO(p) \cdot \mathbb{Z}_k \subset \{SO(2) \times SO(p), SO(2) \times SO(p)\} \subset SO(p+2),$$

where $\mathbb{Z}_k$ is the cyclic group inside $SO(2)$ for $k \geq 2$. The cohomogeneity one manifolds defined by the above diagrams are doubles. They are not simply connected and finitely covered by the manifold defined by the diagram $SO(6) \subset \{SO(2) \times SO(6), SO(2) \times SO(6)\} \subset SO(8)$.

Another possible K_0^+ is $SO(p+1)$ and then $N(K_0^+)/K_0^+ = \mathbb{Z}_2$ generated by the matrix $\mathrm{diag}(-I_2, I_p)$. So $H = SO(p) \cdot \mathbb{Z}_2$, $K^+ = O(p+1)$ and $K^- = SO(2) \times SO(p)$. The manifold corresponded to this diagram is $\mathbb{C}\mathrm{P}^{1+p}$.

Next we consider the variation of these diagrams. If p is odd, then $\mathrm{Aut}(G, H) = S(O(2) \times O(p))$ which is the same as $\mathrm{Aut}(G, K^-)$. If p is even, then G has an outer automorphism which is conjugation by $\mathrm{diag}(-1, I_{p+1})$. It is clear that this automorphism leaves K^- invariant. Therefore the variation does not give rise to another new diagram.

If $p = 6$, in addition to the group diagrams followed by the above constructions in the case where $p \neq 6$, there are a few new possible constructions. Let us lift $G = SO(8)$ to its universal cover $Spin(8)$, then the triple is lifted to $Spin(6) \subset (SO(2) \times Spin(6))/\mathbb{Z}_2 \subset Spin(8)$ where $\mathbb{Z}_2$ is generated by $-\mathrm{id} \in Spin(8)$. $Spin(8)$ has another order 3 outer automorphism denoted by σ. There are three different embeddings of $Spin(7)$ into $Spin(8)$ and σ permutes them. If σ leaves some $Spin(6)$ invariant, then it would be contained in the intersection of the three $Spin(7)$'s which would imply this $Spin(6)$ is contained in G_2. Therefore there is no such $Spin(6)$ invariant by the automorphism σ. On the other hand there is another intermediate subgroup $\widetilde{U}$, the image of $U(4) \subset SO(8)$ by the lifting, between $Spin(6) = SU(4)$ and $Spin(8)$. Since $Spin(8)/\widetilde{U} = SO(8)/U(4)$ is simply-connected, $\widetilde{U}$ is connected. Both $(SO(2) \times Spin(6))/\mathbb{Z}_2$ and $\widetilde{U}$ contain $Spin(6)$ and the isotropy representation of the space $Spin(8)/Spin(6)$ contains only one trivial representation Id, so they are the same subgroup in $Spin(8)$. Divided by the ineffective kernel, the diagram

$$Spin(6) \subset \left\{(SO(2) \times Spin(6))/\mathbb{Z}_2, \widetilde{U}\right\} \subset Spin(8)$$

reduces to

$$SO(6) \subset \{SO(2) \times SO(6), SO(2) \times SO(6)\} \subset SO(8)$$

which does not give rise to a new diagram.

If $H_0 \subset K \subset G$ is the one in (3.4.2), then we may assume that both $K_0^{\pm} = K = U(4)$. We have $N_G(K)/K = \mathbb{Z}_2$ and it is generated by the diagonal matrix

$A = \mathrm{diag}(I_4, -I_4)$. Since $N_G(H_0) = N_G(K)$ and there is no circle group inside $N_G(H_0)/H_0$ containing A, this triple does not give rise to any cohomogeneity one diagram with H disconnected. □

Next we consider the cases where H is connected. Since there is no exceptional orbit, both $K^{\pm}$ are connected. In the classification in Proposition 3.3.1, a lot of pairs (H, G) contain only one intermediate subgroup. The following observation will be very useful in this case to show that the triple (H, K, G) gives only a double.

Definition 3.4.3. Two irreducible representations φ and ψ of H are *automorphically equivalent* if $\varphi = \tau(\psi)$ by an automorphism of H.

Automorphical equivalence is a generalization of the equivalence in the ordinary sense. Two representations are equivalent if and only if the automorphism is an inner one.

Recall that χ_1, χ_2 and χ_3 are the three irreducible summands of the isotropy representation Ad_H on G/H. Let $\mathrm{Ad}_H(G/K)$ and $\mathrm{Ad}_H(K/H)$ be the restrictions of Ad_H to the tangent spaces of G/K and K/H respectively.

Lemma 3.4.4. *Suppose that any irreducible summand of* $\mathrm{Ad}_H(K/H)$ *is not automorphically equivalent to the summand in* $\mathrm{Ad}_H(G/K)$*, then the cohomogeneity one manifold defined by any variation of the diagram* $H \subset \{K, K\} \subset G$ *is a double.*

Proof : We give a proof when $\mathrm{Ad}_H(K/H) = \chi_3$ is irreducible. The other case where $\mathrm{Ad}_H(K/H)$ follows easily. Let $\tau \in \mathrm{Aut}(G, H)$, then τ is an automorphism of H

and it permutes the three summands. By assumption, $\tau(\chi_3)$ is not automorphically equivalent to χ_1 or χ_2, so $\tau(\chi_3) = \chi_3$ which implies the Lie algebra of K and hence K itself is invariant by τ. Therefore the manifold defined by $H \subset \{K, \tau(K)\} \subset G$ is a double. □

We list all triples which satisfy the condition in Lemma 3.4.4.

Proposition 3.4.5. *The following Table 3.5 includes all triples* $H \subset K \subset G$ *in Table 3.3 and Table 3.4 such that any irreducible summands of* $\mathrm{Ad}_H(K/H)$ *is not automorphically equivalent to the summand of* $\mathrm{Ad}_H(G/K)$.

In the Table 3.5, when $G = SO(p+q+2)$ the two factors of H should be of different sizes, i.e., $p \neq q+1$ or $p+1 \neq q$. The representation which is denoted by, for examples, $\varpi_1 + \varpi_{p-1}$, has the highest weight $\varpi_1 + \varpi_{p-1}$.

Now We can state the classification result when H is connected:

Theorem 3.4.6. *The cohomogeneity one manifold defined by the group diagram* $H \subset \{K^{\pm}\} \subset G$ *with* G *simple,* H *connected and the isotropy representation* Ad_H *has three irreducible summands is either a double or a Grassmannian.*

Here we view the sphere as a special case of Grassmanianns.

Proof : There are two main steps in the proof. In Step 1, we consider the pairs (H, G) for which there are at least two intermediate groups. In Step 2, we consider the variations of doubles. We fix the notations for the outer automorphisms of $SO(2m)$(or $Spin(2m)$): λ is the degree 2 outer automorphism and σ is the degree 3 outer automorphism of $Spin(8)$.

G	H	$\mathrm{Ad}_H(K/H)$	$\mathrm{Ad}_H(G/K)$
$SU(4p)$	$Sp(2)\times SU(p)$	$\varpi_1\otimes \mathrm{Id}$	$(\varpi_2\oplus\varpi_1)\otimes(\varpi_1+\varpi_{p-1})$
	$SU(p)\times Sp(2)$	$\mathrm{Id}\otimes\varpi_1$	$(\varpi_1+\varpi_{p-1})\otimes(\varpi_2\oplus\varpi_1)$
$SU(2p)$	$SU(p)\times U(1)$	$\mathrm{Id}\otimes[\phi]_{\mathbb{R}}$	$(\varpi_1+\varpi_{p-1})\otimes(\mathrm{Id}\oplus[\phi]_{\mathbb{R}})$
$SU(16)$	$Spin(9)$	ϖ_1	$2\varpi_4\oplus\varpi_3$
$SU(4)$	$U(1)\times SU(2)$	$[\phi]_{\mathbb{R}}\otimes\mathrm{Id}$	$(\mathrm{Id}\oplus[\phi]_{\mathbb{R}})\otimes 2\varpi_1$
	$SU(2)\times U(1)$	$\mathrm{Id}\otimes[\phi]_{\mathbb{R}}$	$2\varpi_1\otimes(\mathrm{Id}\oplus[\phi]_{\mathbb{R}})$
$SO(p+q+2)$	$SO(p)\times SO(q+1)$	$\varpi_1\otimes\mathrm{Id}$	$(\varpi_1\oplus\mathrm{Id})\otimes\varpi_1$
	$SO(p+1)\times SO(q)$	$\mathrm{Id}\otimes\varpi_1$	$\varpi_1\otimes(\varpi_1\oplus\mathrm{Id})$
$Spin(7+2p)$	$SU(3)\times SO(2p+1)$	$(\mathrm{Id}\oplus[\varpi_1]_{\mathbb{R}})\otimes\mathrm{Id}$	$[\varpi_1]_{\mathbb{R}}\otimes\varpi_1$
$Spin(6+2p)$	$SU(3)\times SO(2p)$	$(\mathrm{Id}\oplus[\varpi_1]_{\mathbb{R}})\otimes\mathrm{Id}$	$[\varpi_1]_{\mathbb{R}}\otimes\varpi_1$
	$SO(2p)\times SU(3)$	$\mathrm{Id}\otimes(\mathrm{Id}\oplus[\varpi_1]_{\mathbb{R}})$	$\varpi_1\otimes[\varpi_1]_{\mathbb{R}}$
$Spin(128)$	$Spin(15)$	ϖ_1	$\varpi_5\oplus\varpi_6$
$Spin(16)$	$Spin(8)$	ϖ_1	$(\varpi_3+\varpi_4)\oplus\varpi_2$
$SO(8)$	$SU(4)$	Id	$\varpi_2\oplus\varpi_2$
E_6	$SU(5)\cdot SU(2)$	$(\mathrm{Id}\oplus[\varpi_1]_{\mathbb{R}})\otimes\mathrm{Id}$	$[\varpi_2]_{\mathbb{R}}\otimes\varpi_1$
E_6	$SU(3)\times SU(3)$	$\mathrm{Id}\otimes[\varpi_1]_{\mathbb{R}}$	$(\varpi_1+\varpi_2)\otimes([\varpi_1]_{\mathbb{R}}\oplus\mathrm{Id})$
F_4	$SO(3)\times SU(3)$	$\mathrm{Id}\otimes[\varpi_1]_{\mathbb{R}}$	$4\varpi_1\otimes([\varpi_1]_{\mathbb{R}}\oplus\mathrm{Id})$

Table 3.5: $H\subset G$ such that $\mathrm{Ad}_H(K/H)$ has no summand automorphically equivalent to one summand of $\mathrm{Ad}_H(G/K)$.

Step 1: From the classification of the triples, between the following two pairs of (H, G), there are more than one intermediate subgroups K. They are

$$SU(3) \subset \{Spin(6), G_2\} \subset Spin(7), \tag{3.4.3}$$

and

$$SO(p)\times SO(q) \subset \{SO(p) \times SO(q+1), SO(p+1) \times SO(q)\} \subset SO(p+q+1), \quad (p, q \geq 1). \tag{3.4.4}$$

One manifold defined by the diagram (3.4.3) is the sphere $\mathbb{S}^{14}$ and the embedding $Spin(7) \hookrightarrow SO(15)$ is given by $\varrho_7 \oplus \Delta_7$ where Δ_7 is the spin representation of $Spin(7)$. We know that $N_{Spin(7)}(SU(3))/SU(3) = \mathbb{Z}_2$ and the generator can be represented as, for example, $A = \operatorname{diag}(I_3, -I_4)$. Both G_2 and $Spin(6)$ are invariant under the conjugation of A. Hence any variation of the diagram gives the same cohomogeneity one manifold .

One manifold defined by the diagram (3.4.4) is the Grassmannian $SO(p+q+2)/(SO(p+1)\times SO(q+1))$ and the embedding $SO(p+q+1) \hookrightarrow SO(p+q+2)$ is given by $\varrho_{p+q+1} \oplus \mathrm{id}$. Let K^- and K^+ denote $SO(p) \times SO(q+1)$ and $SO(p+1) \times SO(q)$ respectively and assume that one of p, q is bigger than 1. If $p \neq q$, by Proposition 3.4.5, any $\tau \in \mathrm{Aut}(G, H)$ leaves both $K^{\pm}$ invariant. So we only need to consider the case $p = q$. In this case, there is one automorphism of H given by the conjugation of the matrix

$$J = \begin{pmatrix} & & I_p \\ & 1 & \\ I_p & & \end{pmatrix}, \tag{3.4.5}$$

where the entries without specifying values have zeros. But $K^{\pm}$ switch each other by the conjugation of J. Therefore there is no new manifold from the variation.

Step 2: Combining the results in Proposition 3.3.1 and Proposition 3.4.5, there are a few triples $H \subset K \subset G$ which need to be considered. In the following, we analyze each of them.

Example 1. $SO(p) \times SO(p) \subset SO(p+1) \times SO(p) \subset SO(2p+1)(p \geq 2)$.

The conjugation by J defined in (3.4.5) maps $SO(p+1)\times SO(p)$ to $SO(p)\times SO(p+1)$, so the variation gives the Grassmannian $SO(2p+2)/(SO(p+1) \times SO(p+1))$ which already appeared in Step 1.

Example 2. $SO(p) \subset SO(p+1) \subset SO(p+2)$.

If p is odd, then $\mathrm{Aut}(G, H) = N_G(H) = S(O(2) \times O(p))$ is connected and hence any variation gives the double. If p is even then the automorphism λ leaves K invariant too. If $p = 6$, then σ does not leave any $SO(6)$ invariant. So this triple only gives rise to a double.

Example 3. $SU(3) \subset Spin(6) \subset Spin(7)$.

Let $i : SU(3) \hookrightarrow Spin(6)$ and $j : Spin(6) \hookrightarrow Spin(7)$ be the embeddings. Since $SU(3)$ is simply-connected, we have the following commutative diagram:

$$\begin{array}{ccccc} SU(3) & \xrightarrow{i} & Spin(6) & \xrightarrow{j} & Spin(7) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ SU(3) & \xrightarrow[\gamma]{} & SO(6) & \longrightarrow & SO(7). \end{array}$$

The embedding γ is given by the representation $[\varpi_1]_{\mathbb{R}}$ of $SU(3)$. The outer automorphism(the complex conjugation) of $SU(3)$ is given by an inner automorphism of $SO(7)$, the conjugation by the matrix $\mathrm{diag}(I_3, -I_4)$, and $SO(6)$ is invariant by the conjugation. So every element in $N_{Spin(7)}(SU(3))$ leaves $Spin(6)$ invariant and the variation gives only a double.

Example 4. $SU(3) \subset G_2 \subset SO(7)$.

As seen in the previous example, conjugation by the matrix $\mathrm{diag}(I_3, -I_4)$ represents the outer automorphism of $SU(3)$. From the embedding of the Lie algebras $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ given in (A.4.7), it is easy to check that $\mathfrak{g}_2$ is invariant by the conjugation and hence G_2 is also invariant. So only the double can be obtained from this triple.

Example 5. $Spin(8) \subset Spin(9) \subset F_4$.

The pair $(Spin(8), F_4)$ appeared in the classification of isotropy irreducible Riemannian manifolds in [WZ]. There are three different embeddings of $Spin(9)$ in F_4 which are denoted by $K_i (i = 1, 2, 3)$ and every outer automorphism of $Spin(8)$ lifts to an inner automorphism of F_4. We use the same notations as λ and σ for their images in $\mathrm{Aut}(F_4)$. Then λ exchanges K_1, K_2 and fixes K_3, and σ permutes

K_i cyclically. Other than the diagram $Spin(8) \subset \{K_1, K_1\} \subset F_4$ which defines the double, we have the following three group diagrams:

$$Spin(8) \subset \{K_1, K_2\} \subset F_4, \quad Spin(8) \subset \{K_2, K_3\} \subset F_4, \quad Spin(8) \subset \{K_1, K_3\} \subset F_4. \tag{3.4.6}$$

If we apply σ to the first diagram, then we get the second one. Then we apply λ to the second one, we obtain the last one. So the three group diagrams above are equivalent. The cohomogeneity one manifold defined by them is the round sphere $\mathbb{S}^{25}$ and F_4 is embedded into $SO(26)$ by its unique 26 dimensional representation.

Example 6. $SO(3) \subset SO(4) \subset G_2$.

All three groups are embedded in $SO(7)$ which acts on the Cayley numbers $\mathbb{O}$ fixing the identity element 1 and G_2 is the automorphism group of $\mathbb{O}$.

Let $\{1, \imath, \jmath, \kappa, e, \imath e, \jmath e, \kappa e\}$ be the basis of $\mathbb{O}$ over the reals, then $\mathbb{O}$ can be written as $\mathbb{H} \oplus \mathbb{H}e$. For every element $(q_1, q_2) \in Sp(1) \times Sp(1)$, it acts on $a + be \in \mathbb{O}$ by $(q_1 a \bar{q}_1) + (q_2 b \bar{q}_1)e$. The kernel of the action is $\{(1,1), (-1,-1)\}$, so it induces an action by $SO(4)$. If we choose $(q_1, q_2) \in \Delta Sp(1)$, then it induces the $SO(3)$ action on the Cayley numbers. It is clear from the action that $SO(3)$ fixes the elements 1, e and its normalizer in $SO(7)$ consists of the reflection about the real line and the rotation $R(t)$ as follows:

$$\imath \mapsto \imath \cos t + \imath e \sin t, \quad \jmath \mapsto \jmath \cos t + \jmath e \sin t, \quad \kappa \mapsto \kappa \cos t + \kappa e \sin t,$$

$$\imath e \mapsto -\imath \sin t + \imath e \cos t, \quad \jmath e \mapsto -\jmath \sin t + \jmath e \cos t, \quad \kappa e \mapsto -\kappa \sin t + \kappa e \cos t.$$

The reflection is not an automorphism of $\mathbb{O}$ and a computation shows that $R(t) \in G_2$ if and only if t equals to $0, \frac{2}{3}\pi$ or $\frac{4}{3}\pi$. Therefore $N_G(H)/H = \mathbb{Z}_3$ and it is generated by $\theta = R(\frac{2}{3}\pi)$. From the action of $SO(4)$ on $\mathbb{O}$, we know that θ does not leave $SO(4)$ invariant. So except the double, we obtained another group diagram: $SO(3) \subset \{SO(4), \mathrm{Ad}_\theta(SO(4))\} \subset G_2$. The corresponding cohomogeneity one manifold is the Grassmannian $SO(7)/(SO(3) \times SO(4))$ and G_2 acts on it via the embedding $G_2 \hookrightarrow SO(7)$ by its unique 7 dimensional representation. $\square$

Appendix A

Class One Representations of Spherical Group Pairs

In this appendix, we shall classify all class one representations of the spherical group pairs. Except the last section, all representations are considered over complex numbers.

A.1 Introduction

First we reduce our classification to the case where the group action on the sphere is almost effective. Suppose G, H are compact Lie groups, G is connected and $G/H = \mathbb{S}^{n-1}$ with $n \geq 3$. If the G action is not almost effective, then let C be the ineffective kernel and it is the maximal normal subgroup of G contained in H. So we can write $G = C \times G_1$ and $H = C \times H_1$. Suppose $\mu \otimes \tau$ is a class one representation of the pair (G, H), i.e., μ and τ are irreducible representations of C and G_1 respectively and $\mathrm{Res}(\mu \otimes \tau)$ fixes a non-zero vector, i.e., the trivial representation of $C \times H_1$ appears in the decomposition of $\mu \otimes \tau$. Therefore μ is the trivial representation of C and τ is a class one representation of the pair (G_1, H_1). So in the rest, we only consider the almost effective action on the spheres. In this case, we have the following classification of the transitive and almost effective action on the spheres:

- $SO(n)/SO(n-1) = \mathbb{S}^{n-1}$,
- $SU(n)/SU(n-1) = \mathbb{S}^{2n-1}$,
- $U(n)/U(n-1) = \mathbb{S}^{2n-1}$,
- $Sp(n)/Sp(n-1) = \mathbb{S}^{4n-1}$,
- $Sp(n) \times Sp(1)/(Sp(n-1) \times Sp(1)) = \mathbb{S}^{4n-1}$,
- $Sp(n) \times U(1)/(Sp(n-1) \times U(1)) = \mathbb{S}^{4n-1}$,
- $G_2/SU(3) = \mathbb{S}^6$, $\quad Spin(7)/G_2 = \mathbb{S}^7$, $\quad Spin(9)/Spin(7) = \mathbb{S}^{15}$.

For each spherical group pair (G, H) with $G/H = \mathbb{S}^{n-1}$, the defining representation $\Phi : G \longrightarrow SO(n)$ is of class one. If the pair is $(SO(n), SO(n-1))$, then the class one representations are well known. They are consisted of the irreducible representations on the space of the homogeneous harmonic polynomials. In fact the class one representations of the spherical group pairs are closely related to the representations of harmonic polynomials as stated in the following theorem.

Theorem A.1.1. *The representation μ is a class one representation of the spherical group pair (G, H) if and only if μ is in the decomposition of $\mathrm{Res}_G^{SO(n)}\rho$, where ρ is a class one representation of the pair $(SO(n), SO(n-1))$ and G is viewed as a subgroup of $SO(n)$ via the representation Φ.*

For a compact Lie group, the irreducible representations are highest weight representations and each highest weight is a linear combination of the fundamental weights with nonnegative integers as coefficients. We list the fundamental weights for classical groups as follows.

$$\begin{aligned}
SO(n) \ &: \ \varpi_1 = e_1, \cdots, \varpi_{k-1} = e_1 + e_2 + \cdots + e_{k-1}, \varpi_k = \tfrac{1}{2}(e_1 + \cdots + e_k), \ \text{n} = \text{2k+1}, \\
&\quad\ \varpi_1 = e_1, \cdots, \varpi_{k-1} = \tfrac{1}{2}(e_1 + \cdots + e_{k-1} - e_k), \varpi_k = \tfrac{1}{2}(e_1 + \cdots + e_k), \ \text{n} = \text{2k}, \\
SU(n) \ &: \ \varpi_1 = e_1, \cdots, \varpi_{n-1} = e_1 + \cdots + e_{n-1}, \ \text{with } e_1 + \cdots + e_n = 0, \\
Sp(n) \ &: \ \varpi_1 = e_1, \cdots, \varpi_n = e_1 + \cdots + e_n.
\end{aligned}$$

The exceptional Lie group G_2 has two fundamental weights: ϖ_1 which is the highest weight of the 7 dimensional representation and ϖ_2 which is 14 dimensional. The group $U(n)$ has finite cover as $SU(n) \times U(1)$ and hence its irreducible representation can be written as $\mu \otimes \phi^k$ where μ is an irreducible representation of $SU(n)$ and $\phi^k (k \in \mathbb{Z})$ is the 1 dimensional representation of $U(1)$ as $z : \mathbb{C} \longrightarrow \mathbb{C}$, $v \mapsto z^k v$ for any $z \in U(1)$. And any irreducible representation ρ of $U(n)$ with highest weight $a_1 e_1 + \ldots + a_n e_n (a_1 \geq \ldots \geq a_n)$ is the tensor product of the irreducible representation

μ of $SU(n)$ and ϕ^k of $U(1)$ where μ has highest weight as $(a_1 - a_n)e_1 + \ldots + (a_{n-1} - a_n)e_{n-1}$ and $k = a_1 + \ldots + a_n$.

To look for the class one representations for each pair (G, H), we use the branching rule for this pair, i.e. the rule that how an irreducible representation of G decomposes under the Res_H^G functor. If the trivial representation of H appears in the decomposition, then this representation is of class one. We have the following classification result:

Theorem A.1.2. *For each pair (G, H), the following table gives the classification of the class one representation ρ and the multiplicity of the trivial representation in $\mathrm{Res}_H^G \rho$. In the table a, b are non-negative integers and k is an integer which satisfy further specified restrictions. The last column is the range of n.*

In the table A.1, if $n = 1$ for the $Sp(n)$ factor, then there does not exist $b\varpi_2$ in ρ, or equivalently let $b = 0$.

The proof of the theorem is divided into three parts. The results of the first four group pairs in Table A.1 are the direct consequences of the classical branching rules for those pairs. The second part includes the pairs $(Sp(n) \times Sp(1), Sp(n-1) \times Sp(1))$ and $(Sp(n) \times U(1), Sp(n-1) \times U(1))$. They will be proved in the section A.3 by using a branching rule of Lepowsky. The last group pairs are covered in section A.4. We will apply Kostant's Branching Theorem to each pair to obtain the class one representations.

G	H	ρ	mul.	
$SO(n)$	$SO(n-1)$	$a\varpi_1$	1	$a \geq 1$
$SU(n)(n \geq 3)$	$SU(n-1)$	$a\varpi_1 + b\varpi_{n-1}$	1	$a + b \geq 1$
$U(n)$	$U(n-1)$	$ae_1 - be_n$	1	$a + b \geq 1$
$Sp(n)$	$Sp(n-1)$	$a\varpi_1 + b\varpi_2$	$a + 1$	$a + b \geq 1$
$Sp(n) \times Sp(1)$	$Sp(n-1) \times Sp(1)$	$(a\varpi_1 + b\varpi_2) \otimes a\varpi_1$	1	$a \geq 1$
$Sp(n) \times U(1)$	$Sp(n-1) \times U(1)$	$(a\varpi_1 + b\varpi_2) \otimes \phi^k$	1	$a + b \geq 1$ $k \neq 0$
G_2	$SU(3)$	$a\varpi_1$	1	$a \geq 1$
$Spin(7)$	G_2	$a\varpi_3$	1	$a \geq 1$
$Spin(9)$	$Spin(7)$	$a\varpi_1 + b\varpi_4$	1	$a + b \geq 1$

Table A.1: Class one representations of spherical group pairs

In the last section, section A.5, we will study some properties of these representations, for example, the type, the kernel and the dimension.

A.2 The Class One Representations and Harmonic Polynomials

In this section we will prove Theorem A.1.1. To do this, we consider the separable Hilbert space L of all the square-integrable complex valued functions defined on $\mathbb{S}^{n-1}$

under the following inner-product

$$< f_1, f_2 >= \int_{\mathbb{S}^{n-1}} f_1 \bar{f}_2 d\sigma, \quad \forall f_1, f_2 \in L,$$

where $d\sigma$ is an $SO(n)$ bi-invariant measure on $\mathbb{S}^{n-1}$ with total measure $= 1$.

From the $SO(n)$ action on $\mathbb{S}^{n-1}$, $SO(n)$ acts on L by $l(g)f(x) = f(g^{-1}x)$ for any $g \in SO(n)$. This representation is called the *left-regular representation* of $SO(n)$ on L and it is a unitary representation.

Since $SO(n)/SO(n-1) = \mathbb{S}^{n-1}$ as a homogeneous space, L is also the induced representation $L = \mathrm{Ind}_{SO(n-1)}^{SO(n)}\mathrm{Id}$, where Id is the trivial representation of $SO(n-1)$ on $\mathbb{C}$. By the Frobenius reciprocity, for any irreducible representation ρ of $SO(n)$, we have the following multiplicities equality

$$[\mathrm{Ind}_{SO(n-1)}^{SO(n)}\mathrm{Id} : \rho] = [\mathrm{Res}_{SO(n-1)}^{SO(n)}\rho : \mathrm{Id}], \quad \text{or} \quad [L : \rho] = [\mathrm{Res}_{SO(n-1)}^{SO(n)}\rho : \mathrm{Id}],$$

where $[\rho_1 : \rho_2]$ denotes the multiplicity of the irreducible representation ρ_2 in ρ_1. By the classical branching rule for the pair $(SO(n), SO(n-1))$, $[\mathrm{Res}_{SO(n-1)}^{SO(n)}\rho : \mathrm{Id}] \neq 0$ if and only if ρ has the highest weight ae_1 where a is a nonnegative integer. On the other hand, for any class one representation ρ, $[\mathrm{Res}_{SO(n-1)}^{SO(n)}\rho : \mathrm{Id}] = 1$, or equivalently $[L : \rho] = 1$.

The representation with highest weight ae_1 can be realized as the the representation of the complex valued homogeneous harmonic polynomials with degree a on $\mathbb{S}^{n-1}$ which is denoted by $\mathcal{H}^a$. And $\mathcal{H}^a$ is the complexification of the real valued homogeneous harmonic polynomials with degree a. We have the following orthogonal decomposition

$$L = \bigoplus_{a=0}^{\infty} \mathcal{H}^a.$$

Since $G/H = \mathbb{S}^{n-1}$, G acts on the Hilbert space L and $L = \mathrm{Ind}_H^G \mathrm{Id}$, where Id is the trivial representation of H on $\mathbb{C}$. As a subgroup of $SO(n)$, G acts invariantly on $\mathcal{H}^a$. In general $\mathcal{H}^a$ is not an irreducible representation of G and it is decomposed orthogonally into irreducible summands as follows:

$$\mathrm{Res}_G^{SO(n)} \mathcal{H}^a = \bigoplus_b \mathcal{H}^{a,b},$$

and there are only finite many b's for each value of a. Hence we have the following orthogonal decomposition of L into irreducible representations of G:

$$L = \bigoplus_{a,b} \mathcal{H}^{a,b}.$$

Suppose μ is an irreducible representation of G, then μ is of class one of the pair (G, H) if and only if $[L : \mu] \neq 0$ by the Frobenius reciprocity applied to the pair (G, H). Then from the above decomposition and the uniqueness of this decomposition, μ is equivalent to one of $\mathcal{H}^{a,b}$'s. □

Remark A.2.1. The existence and uniqueness of the decomposition of L follow from Theorem 9.4 and Corollary 9.6 in [Kn].

A.3 Pairs $(Sp(n)\times Sp(1), Sp(n-1)\times Sp(1))$ and $(Sp(n)\times U(1), Sp(n-1)\times U(1))$

For each pair (G, H), there is an intermediate group K and we will apply the branching rule successively as $\mathrm{Res}^G_H\rho = \mathrm{Res}^K_H(\mathrm{Res}^G_K\rho)$ for any irreducible representation ρ of G.

The branching rule we will use is the one for the pair $(Sp(n), Sp(n-1)\times Sp(1))$ which is established by J.Lepowsky in [Le]. For each irreducible representation $\mu_1\otimes\mu_2$ of $Sp(n-1)\times Sp(1)$, let $\mu = b_1e_1 + \cdots + b_{n-1}e_{n-1} + b_ne_n$ be the highest weight, then $b_1 \geq \cdots \geq b_{n-1} \geq 0$ and $b_n \geq 0$.

Definition A.3.1. Let $l, m \in \mathbb{Z}$, $m \geq 1$ and let $q_1, \cdots, q_m$ be positive integers. We define $F_m(l; q_1, \cdots, q_m)$ to be the number of ways of putting l indistinguishable balls into m distinguishable boxes with capacities $q_1, \cdots, q_m$.

Theorem A.3.2 (Lepowsky)**.** *Let ρ, μ be irreducible representations of $Sp(n), Sp(n-1)\times Sp(1)$ with highest weights $\rho = a_1e_1 + \cdots a_ne_n$, $\mu = b_1e_1 + \cdots b_{n-1}e_{n-1} + b_ne_n$ respectively. Define*

$$\begin{aligned} A_1 &= a_1 - \max(a_2, b_1), \\ A_i &= \min(a_i, b_{i-1}) - \max(a_{i+1}, b_i) \qquad (2 \leq i \leq n-1), \\ A_n &= \min(a_n, b_{n-1}) \end{aligned}$$

Then the multiplicity $[\mathrm{Res}\rho : \mu] = 0$ *unless* $\sum_{i=1}^n (a_i + b_i) \in 2\mathbb{Z}$ *and* $A_1, \cdots, A_{n-1} \geq 0$.

Under these conditions,

$$\begin{aligned}[\mathrm{Res}\rho : \mu] &= F_{n-1}\left(\frac{1}{2}(b_n - A_1 + \sum_{i=1}^{n} A_i); A_2, \cdots, A_n\right) \\ &- F_{n-1}\left(\frac{1}{2}(-b_n - A_1 + \sum_{i=2}^{n} A_i) - 1; A_2, \cdots, A_n\right). \qquad \text{(A.3.1)}\end{aligned}$$

Proof of the 2nd part of Theorem A.1.2 : Let $G = Sp(n) \times Sp(1)$ and $H = Sp(n-1) \times Sp(1)$, then $K = Sp(n-1) \times Sp(1) \times Sp(1)$ lies between them and the $Sp(1)$ factor in H is diagonally embedded in the $Sp(1) \times Sp(1)$ inside K. Suppose $\rho = \rho_1 \otimes \rho_2$ is a class one representation of the pair and ρ_1 has the highest weight $a_1 e_1 + \cdots + a_n e_n$ and ρ_2 has the highest weight be_1. Then $\mathrm{Res}^G_K \rho = (\mathrm{Res}^{Sp(n)}_{Sp(n-1)\times Sp(1)} \rho_1) \otimes \rho_2$. Since the trivial representation appears in $\mathrm{Res}^G_H \rho$, it appears in $(\mathrm{Res}^K_H \mu) \otimes \rho_2$ for some irreducible representation $\mu = \mu_1 \otimes \mu_2$ in $\mathrm{Res}^{Sp(n)}_{Sp(n-1)\times Sp(1)} \rho_1$.

Under the restriction from $Sp(n-1) \times Sp(1) \times Sp(1)$ to $Sp(n-1) \times \Delta Sp(1)$, $\mu_1 \otimes \mu_2 \otimes \rho_2$ splits as $\mu_1 \otimes \mathrm{Res}^{Sp(1)\times Sp(1)}_{\Delta Sp(1)} (\mu_2 \otimes \rho_2)$. Therefore μ_1 is a trivial representation of $Sp(n-1)$ and μ_2, ρ_2 have the same dimension, i.e., $\mathrm{Res}^{Sp(n)}_{Sp(n-1)\times Sp(1)} \rho_1$ contains the representation $\mathrm{Id} \otimes \mu_2$ which has the highest weight be_n. From Theorem A.3.2, we have $A_1 = a_1 - a_2 \geq 0$, $A_2 = -a_3 \geq 0$, …, $A_{n-1} = -a_n \geq 0$, $b + a_1 + a_2$ is an even number and the multiplicity is equal to

$$m = F_{n-1}\left(\frac{1}{2}(b - (a_1 - a_2)); 0, \cdots, 0\right) - F_{n-1}\left(\frac{1}{2}(-b - (a_1 - a_2)) - 1; 0, \cdots, 0\right).$$

Hence $m \neq 0$ if and only if $b = a_1 - a_2$ and in this case we have $m = 1$.

For the second part, let $G = Sp(n) \times U(1)$, $H = Sp(n-1) \times U(1)$, $K_1 = Sp(n-1) \times Sp(1) \times U(1)$ and $K_2 = Sp(n-1) \times U(1) \times U(1)$, then we have the embedding $H \subset K_1 \subset K_2 \subset G$ and $U(1)$ in H lies diagonally in $U(1) \times U(1) \subset K_1$. Suppose $\rho = \mu \otimes \phi^k$ is a class one representation of the pair (G, H) and μ has the highest weight $\mu = a_1 e_1 + \cdots + a_n e_n$. From the first part of the proof, we have $a_3 = \cdots = a_n = 0$ and the multiplicity of the trivial representation in $\mathrm{Res}^G_H \rho$ is equal to the multiplicity of the trivial representation in $(\mathrm{Res}^{Sp(1)}_{U(1)}(a_1 - a_2)e_n) \otimes \phi^k$. Therefore k, $a_1 - a_2$ have the same parity and $|k| \leq a_1 - a_2$. And in this case the multiplicity is 1. □

A.4 Pairs $(G_2, SU(3))$, $(Spin(7), G_2)$ and $(Spin(9), Spin(7))$

In this section, we will develop the branching rule for each group pair and then prove the corresponding result in Theorem A.1.2. The main tool is Kostant's Branching Theorem. We quote the statement of the theorem from [Kn].

First we establish the notation we will use. Let G be a connected compact Lie group and let H be a connected closed subgroup. Choose a maximal torus $S \subset H$. The special assumption is that the centralizer of S in G is a maximal torus T of G. Let Δ_G be the set of the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, let Δ_H be the set of roots of $(\mathfrak{h}^{\mathbb{C}}, \mathfrak{s}^{\mathbb{C}})$, and let W_G be the Weyl group of Δ_G. Introduce compatible positive systems Δ_G^+ and Δ_H^+, let bar denote restriction from the dual $(\mathfrak{t}^{\mathbb{C}})^*$ to the dual $(\mathfrak{s}^{\mathbb{C}})^*$, and let δ_G be half the sum of the members in Δ_G^+. The restrictions to $\mathfrak{s}^{\mathbb{C}}$ of the members of Δ_G^+, repeated according to their multiplicities, are the nonzero positive weights of $\mathfrak{s}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$. Deleting from this set the members of Δ_H^+, each with multiplicity 1, we obtain the set Σ of positive weights of $\mathfrak{s}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}}$, repeated according to multiplicities. The associated Kostant partition function is defined as follows: $\mathcal{P}(\nu)$ is the number of ways that a member of $(\mathfrak{s}^{\mathbb{C}})^*$ can be written as a sum of members of Σ, with the multiple versions of a member of Σ being regarded as distinct.

Theorem A.4.1 (Kostant's Branching Theorem)**.** *Let G be a compact connected Lie group, let H be a closed connected subgroup, suppose that the centralizer in G of a maximal torus S of H is abelian and is therefore a maximal torus T of G, and let other notation be as above. Let ρ be an irreducible representation of G with the highest weight $\rho \in (\mathfrak{t}^{\mathbb{C}})^*$, and let μ be an irreducible representation of H with the highest weight $\mu \in (\mathfrak{s}^{\mathbb{C}})^*$. Then the multiplicity of μ in the restriction of ρ to H is given by*

$$[\rho : \mu] = \sum_{w \in W_G} \varepsilon(w)\mathcal{P}(\overline{w(\rho + \delta_G) - \delta_G} - \mu).$$

Remark A.4.2. To apply this theorem to our special examples, we will use an equivalent assumption on the Cartan subalgebras instead of the maximal tori: the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$ is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. In our three group pairs, this assumption is verified.

To apply Kostant Branching Theorem, we need to work out the explicit embeddings of the Lie algebras. These groups and Lie algebras are well studied, see, for examples, [GlWZ] and [Mu] and references given there.

We start with the pair $(Spin(9), Spin(7))$. Let $\mathbb{O}$ be the set of the Cayley numbers which is isomorphic to $\mathbb{R}^8$ as vector spaces. $SO(8)$ acts on $\mathbb{O}$ by left multiplication and we have the *Principle of Triality*:

Proposition A.4.3. *For each* $\theta_1 \in SO(8)$*, there exists* θ_2*,* $\theta_3 \in SO(8)$ *such that*

$$\theta_1(x)\theta_2(y) = \theta_3(xy), \quad \textit{for any } x, y \in \mathbb{O} \tag{A.4.1}$$

Moreover if θ_2'*,* θ_3' *satisfy the above equation as* θ_2*,* θ_3*, then* $(\theta_2', \theta_3') = \pm(\theta_2, \theta_3)$*.*

And the infinitesimal version of the above principle is given as follows:

Proposition A.4.4. *For any* $X \in \mathfrak{so}(8)$*, there exist* Y*,* $Z \in \mathfrak{so}(8)$ *such that*

$$(Xx)y + x(Yy) = Z(xy) \quad \textit{for any } x, y \in \mathbb{O}. \tag{A.4.2}$$

Moreover Y*,* Z *is uniquely determined by* X *with* $Y = \lambda(X), Z = \lambda\kappa(X)$ *and* λ*,* κ *are two outer automorphisms of* $\mathfrak{so}(8)$ *with* $\lambda^3 = 1$*,* $\kappa^2 = 1$ *and* $\kappa\lambda^2 = \lambda\kappa$*.*

We identify $\mathbb{O} \oplus \mathbb{O}$ with $\mathbb{R}^{16}$ and then $Spin(9) \subset SO(16)$ acts transitively on $\mathbb{S}^{15} \subset \mathbb{O} \oplus \mathbb{O}$. Let $v_0 \in \mathbb{R} \subset \mathbb{O}$ be the unit length vector, then $Spin(7)$ is the isotropy subgroup at $(v_0, 0)$. Consider the following Hopf fibration:

$$\begin{array}{ccccc} \mathbb{S}^7 & \longrightarrow & \mathbb{S}^{15} & \longrightarrow & \mathbb{S}^8 = \mathbb{O} \cup \{\infty\}, \\ & & (x, y) & \mapsto & y^{-1}\bar{x}. \end{array}$$

Then the isotropy subgroup of the fiber $\{(x, 0) | x \in \mathbb{O}\}$ is given as follows:

$$Spin(8) = \left\{(\theta_1, \theta_2) \in SO(8) \times SO(8) | \exists \theta_3 \in SO(8) \text{ such that } \theta_1(x)\theta_2(y) = \overline{\theta_3(\overline{xy})}\right\}, \tag{A.4.3}$$

and the embedding of $Spin(7) \subset Spin(8)$ is

$$Spin(7) = \left\{(\theta_1, \theta_2) \in Spin(8) | \theta_1 \in SO(7) \subset SO(8) \text{ or } \theta_1(x)\theta_2(y) = \overline{\theta_2(\overline{xy})}\right\}. \tag{A.4.4}$$

The automorphism $\theta \mapsto (x \mapsto \overline{\theta(\bar{x})})$ of $SO(8)$ induces the automorphism κ. Let $\theta_1 = \exp(X)$, $\theta_2 = \exp(Y)$ and $\theta_3 = \exp(Z)$, from Proposition A.4.4, we have $X = \lambda(\kappa Z)$ and $Y = \lambda^2(\kappa Z)$. So we can write the Lie algebra of $Spin(8)$ as

$$\text{Lie}(Spin(8)) = \left\{(\lambda(X), \lambda^2(X)) | X \in \mathfrak{so}(8)\right\}.$$

From the embedding of $Spin(7) \subset Spin(8)$ in (A.4.4), since $\theta_1 \in SO(7)$ we have the Lie algebra of $Spin(7)$ as

$$\text{Lie}(Spin(7)) = \{(Y, \lambda(Y)) | Y \in \mathfrak{so}(7)\} \subset \text{Lie}(Spin(8)).$$

Therefore if we identify Lie$(Spin(8))$ with $\mathfrak{so}(8)$ by its first component, then we have

$$\text{Lie}(Spin(7)) = \left\{\lambda^2(X) | X \in \mathfrak{so}(7)\right\}. \tag{A.4.5}$$

Let $\{e_i \pm e_j | 1 \leq i < j \leq 4\}$ be the positive root system of $\mathfrak{so}(8)$ with vanishing e_4 on $\mathfrak{so}(7)$, then the automorphism λ^2 induces the following transformation of e_i.

$$\lambda^2 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{A.4.6}$$

In (A.4.4) the θ_2-component acts transitively on $\mathbb{S}^7 \subset \mathbb{O}$ and the isotropy subgroup at v_0 is G_2. Since κ is the identity map on $SO(7)$, $\theta_1(x)\theta_2(y) = \theta_2(xy)$ for $\theta_2 \in G_2 \subset SO(7)$ which implies $\theta_1 = \theta_2$. Hence we have

$$G_2 = \{(\theta, \theta) | \theta \in SO(7) \text{ and } \theta(x)\theta(y) = \theta(xy), \text{ for any } x, y \in \mathbb{O}\},$$

and the Lie algebra is

$$\mathfrak{g}_2 = \{(X, X) | X \in \mathfrak{so}(7), X = \lambda(X)\}.$$

If we identify Lie$(Spin(7))$ with $\mathfrak{so}(7)$, then we have

$$\mathfrak{g}_2 = \{X | X \in \mathfrak{so}(7), X = \lambda(X)\}.$$

Choose a basis of $\mathbb{O}$ over $\mathbb{R}$, then we can write down the explicit embedding of $\mathfrak{g}_2$, see also in [Mu]. The typical element of $\mathfrak{g}_2$ has the following form:

$$X = \begin{pmatrix} 0 & x_1 - y_1 & x_3 + y_3 & -x_2 + y_2 & -x_4 - y_4 & x_6 + y_6 & x_5 - y_5 \\ -x_1 + y_1 & 0 & a & y_5 & y_6 & y_4 & y_2 \\ -x_3 - y_3 & -a & 0 & x_6 & x_5 & x_2 & x_4 \\ x_2 - y_2 & -y_5 & -x_6 & 0 & b & y_3 & y_1 \\ x_4 + y_4 & -y_6 & -x_5 & -b & 0 & x_1 & x_3 \\ -x_6 - y_6 & -y_4 & -x_2 & -y_3 & -x_1 & 0 & a + b \\ -x_5 + y_5 & -y_2 & -x_4 & -y_1 & -x_3 & -a - b & 0 \end{pmatrix}, \tag{A.4.7}$$

where $a, b, x_1, \cdots, x_6, y_1, \cdots, y_6 \in \mathbb{R}$.

So the linear functionals $\{e_1, e_2, e_3\}$ of the Cartan subalgebra of $\mathfrak{so}(7)$ satisfy the relation $e_1 = e_2 + e_3$ when restricted to the Cartan subalgebra of $\mathfrak{g}_2$. Suppose $\{\alpha_1, \alpha_2\}$ be the set of the positive simple roots of G_2 where α_1 is the short one, then under the restriction, $\overline{e_3} = \alpha_1$ and $\overline{e_2 - e_3} = \alpha_2$.

We consider the last pair $(G_2, SU(3))$. G_2 acts transitively on the unit sphere $\mathbb{S}^6 = \{x \in \mathbb{O} | \, \|x\| = 1, < x, v_0 >= 0\}$. Let v_1 be a unit element which is orthogonal to v_0, then the isotropy group at v_1 is isomorphic to $SU(3) \subset SO(6)$. Since G_2 and $SU(3)$ share the same maximal torus, the restriction of the roots of $\mathfrak{g}_2$ is the identity map.

In the proof we will use the classical branching rules for the special orthogonal groups which are well-known and the proof can be found, for examples, in [Kn].

Theorem A.4.5 (Branching Rule for $(\mathfrak{so}(2k+1), \mathfrak{so}(2k))$)**.** *The irreducible representation with highest weight $a_1e_1 + a_2e_2 + \cdots + a_ke_k$ of $\mathfrak{so}(2k+1)$ decomposes with multiplicity 1 under $\mathfrak{so}(2k)$, and the representations of $\mathfrak{so}(2k)$ that appear are exactly those with highest weights $\mu = c_1e_1 + c_2e_2 + \cdots + c_ke_k)$ such that*

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_k \geq |c_k| \, ,$$

where $a_i (i = 1, 2, \cdots, k)$ are integers or all half integers and $c_j (j = 1, 2, \cdots, k)$ are all integers or all half integers.

Theorem A.4.6 (Branching Rule for $(\mathfrak{so}(2k), \mathfrak{so}(2k-1))$)**.** *The irreducible representation with highest weight $a_1e_1 + a_2e_2 + \cdots + a_ke_k$ of $\mathfrak{so}(2k)$ decomposes with* *multiplicity* 1 *under* $\mathfrak{so}(2k-1)$*, and the representations of* $\mathfrak{so}(2k-1)$ *that appear are exactly those with highest weights* $\mu = c_1e_1 + c_2e_2 + \cdots + c_{k-1}e_{k-1})$ *such that*

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq c_{k-1} \geq |a_k| \,,$$

where $a_i (i = 1, 2, \cdots, k)$ *are integers or all half integers and* $c_j (j = 1, 2, \cdots, k-1)$ *are all integers or all half integers.*

Proof of the 3rd Part of Theorem A.1.2: First let $G = Spin(9)$, $H = Spin(7)$ and $\mathfrak{g} = \mathrm{Lie}(Spin(9))$, $\mathfrak{h} = \mathrm{Lie}(Spin(7))$ be their Lie algebras. Then $K = Spin(8)$ lies between them and denote its Lie algebra $\mathrm{Lie}(Spin(8))$ by $\mathfrak{k}$. Let

$$\Delta_G^+ = \{e_i \pm e_j | 1 \leq i < j \leq 4\} \cup \{e_i | 1 \leq i \leq 4\} \,,$$

be the positive root system of $\mathfrak{g}$, then $\mathfrak{k}$ has the positive roots system

$$\Delta_K^+ = \{e_i \pm e_j | 1 \leq i < j \leq 4\} \,.$$

Since $\mathfrak{g}$ and $\mathfrak{k}$ share the same Cartan subalgebra, $\Sigma = \{e_1, e_2, e_3, e_4\}$. So the branching rule for the pair $(\mathfrak{g}, \mathfrak{k})$ is the same as the classical one for $(\mathfrak{so}(9), \mathfrak{so}(8))$.

Let $f_i = \lambda^2(e_i)$ for $i = 1, 2, 3, 4$. Since λ is an automorphism of $\mathfrak{so}(8)$, we can write

$$\Delta_K^+ = \{f_i \pm f_j | 1 \leq i < j \leq 4\} \,,$$

and $\mathfrak{h}$ has the positive root system

$$\Delta_H^+ = \{f_i \pm f_j | 1 \leq i < j \leq 3\} \cup \{f_i | 1 \leq i \leq 3\} \,.$$

Therefore $\Sigma = \{f_1, f_2, f_3\}$. The Weyl group W_K also acts on f_i's and $\delta_G = 3e_1 + 2e_2 + e_3 = 3f_1 + 2f_2 + f_3$. Suppose μ be an irreducible representation of K with the highest weight $\mu = b_1 f_1 + b_2 f_2 + b_3 f_3 + b_4 f_4$, then by the classical branching rule for $(\mathfrak{so}(8), \mathfrak{so}(7))$, μ decomposes with multiplicity 1 when restricted to $\mathfrak{h}$, and the representations of $\mathfrak{h}$ that appear are exactly those with highest weights $\tau = c_1 f_1 + c_2 f_2 + c_3 f_3$ such that

$$b_1 \geq c_1 \geq b_2 \geq c_2 \geq b_3 \geq c_3 \geq |b_4|,$$

where $b_i (i = 1, 2, 3, 4)$ and $c_j (j = 1, 2, 3)$ are all integers or half integers. So the class one representations of the pair (K, H) are exactly those with highest weights bf_1 with $b \in \mathbb{Z}$, or $\frac{1}{2}b(e_1 + \cdots + e_4)$ and the multiplicity of trivial representation is 1.

Now suppose that μ is a class one representation of the pair (K, H) with the highest weight $\rho = a_1 e_1 + \cdots + a_4 e_4$, then $\mathrm{Res}^G_K \rho$ contains at least one representation with highest weight $\frac{1}{2}b(e_1 + \cdots + e_4)$. From the branching rule of $(\mathfrak{so}(9), \mathfrak{so}(8))$, we have

$$a_1 \geq \frac{1}{2}b \geq a_2 \geq \frac{1}{2}b \geq a_3 \geq \frac{1}{2}b \geq a_4 \geq \frac{1}{2}b.$$

Therefore $a_1 \geq a_2 = a_3 = a_4 \geq 0$ and all are integers or half integers. If a_i's satisfy the conditions, then ρ contains only one representation with the highest weight μ which implies that $[\mathrm{Res}^G_H \rho : Id] = 1$.

Next we look at the pair $(G, H) = (Spin(7), G_2)$. We identify $\mathfrak{g}$ with $\mathfrak{so}(7)$ so that

$$\Delta^+_G = \{e_i \pm e_j | 1 \leq i < j \leq 3\} \cup \{e_i | 1 \leq i \leq 3\}.$$

By the relation $e_1 = e_2 + e_3$ and the restriction $\overline{e_3} = \alpha_1$, $\overline{e_2 - e_3} = \alpha_2$, we have

$$\overline{\Delta_G^+} = \{\alpha_1, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, \alpha_2, 2\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1\}$$

under the restriction. In terms of α_1, α_2, $\mathfrak{h}$ has the following positive root system

$$\Delta_H^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Therefore $\Sigma = \{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. The Weyl group W_G permutes $\{e_1, e_2, e_3\}$ and changes the sign of each e_i.

Let ρ be a class one representation with the highest weight $\rho = a_1e_1 + a_2e_2 + a_3e_3$ $(a_1 \geq a_2 \geq a_3 \geq 0)$. Then from the Kostant's Branching Theorem A.4.1, by a computation, the multiplicity of the trivial representation Id of G_2 in Resρ is

$$m = (a_1+a_3+1)-(a_1-a_3)-(a_1+a_3)+\max\{0, a_1-a_3-1\}-(a_1-a_3-1)+\max\{0, a_1-a_3-1\}.$$

Therefore $m \neq 0$ if and only if $a_1 = a_3$, i.e., $a_1 = a_2 = a_3$. And in this case we have $m = 1$.

Finally we look at the pair $(G, H) = (G_2, SU(2))$. $\mathfrak{g} = \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{su}(3)$ share the same Cartan subalgebra and hence the restriction of the roots is the identity map. $\mathfrak{g}$ has the following positive root system

$$\Delta_G^+ = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\},$$

where $\alpha_1 = e_3$, $\alpha_2 = e_2 - e_3$ and $e_2 + e_3 - e_1 = 0$. Among the 12 roots of $\mathfrak{g}$, the six long roots consist the root system of $\mathfrak{h}$, i.e.

$$\Delta_H^+ = \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\Sigma = \{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. The Weyl group W_G is the symmetry group of the regular hexagon $D_6 =< \sigma, \tau | \sigma^6 = 1, \tau^2 = 1 >$ with the following operations on α_1, α_2:

$$\begin{aligned} \sigma &: \quad \alpha_1 \mapsto -\alpha_1 - \alpha_2, \quad \alpha_2 \mapsto 3\alpha_1 + 2\alpha_2, \\ \tau &: \quad \alpha_1 \mapsto -\alpha_1 - \alpha_2, \quad \alpha_2 \mapsto \alpha_2. \end{aligned}$$

Suppose ρ is a class one representation with the highest weight $\rho = a_1\varpi_1 + a_2\varpi_2 = (2a_1+3a_2)\alpha_1 + (a_1+2a_2)\alpha_2$, where a_1, a_2 are two nonnegative integers. A computation using Kostant Branching Theorem shows that $a_2 = 0$ and the multiplicity $[\mathrm{Res}\rho : \mathrm{Id}]$ is 1. □

A.5 Properties of Class One Representations

We recall the definition of the type of a representation first.

Definition A.5.1. Suppose V is an irreducible representation over the complex numbers, then V is called a *real* representation or of *real type* if it comes from a representation over reals by extension of scalars. It is of *quaternionic type* if it comes from a representation over quaternions by restriction of scalars. It is of *complex type* if it is neither real or quaternionic.

The types and kernels of class one representations are given as follows:

Proposition A.5.2. *The type and non-trivial kernel K of each class one representation for each spherical group pair is listed in Table A.2 and Table A.3:*

G	ρ	type
$SO(n)$	$a\varpi_1$	real
$SU(n)$	$a\varpi_1 + b\varpi_{n-1}$	real: if $a = b$; complex: otherwise
$U(n)$	$ae_1 - be_n$	real: if $a = b$; complex: otherwise
$Sp(n)$	$a\varpi_1 + b\varpi_2$	real: a is even; quaternionic: otherwise
$Sp(n) \times Sp(1)$	$(a\varpi_1 + b\varpi_2) \otimes a\varpi_1$	real
$Sp(n) \times U(1)$	$(a\varpi_1 + b\varpi_2) \otimes \phi^k$	complex
G_2	$a\varpi_1$	real
$Spin(7)$	$a\varpi_3$	real
$Spin(9)$	$a\varpi_1 + b\varpi_4$	real

Table A.2: Type of class one representations

In the table A.3, $\mathbb{Z}_2 \subset G$ is generated by $-\text{id}$ and $\mathbb{Z}_l \subset SU(n)$(or $U(n)$) is generated by $\varphi_l \cdot \text{Id}$ with $\varphi_l = \exp(\frac{2\pi\sqrt{-1}}{l})$. $\text{lcm}(p,q)$ stands for the least common multiple of p, q and $\gcd(p,q)$ means the greatest common divisor of p, q.

Proof : They are direct applications of Proposition 23.13 and propositions in section 26.3 in [FH]. □

From the Table A.3, we have

Corollary A.5.3. *Suppose ρ is a class one representation of the pair (G,H) with non-trivial kernel K, then K is not contained in H.*

Next we compute the dimensions of class one representations. They are listed as

G	ρ	non-trivial kernel
$SO(n)$	$a\varpi_1$	$\mathbb{Z}_2$: both n and a are even
$SU(n)$	$a\varpi_1 + b\varpi_{n-1}$	$\mathbb{Z}_m$: $m = \gcd(a+b, n)$
$U(n)$	$ae_1 - be_n (a \neq b)$	$\mathbb{Z}_k$: $k = \mathrm{lcm}(a-b, m)$, $m = \gcd(a+b, n)$
$Sp(n)$	$a\varpi_1 + b\varpi_2$	$\mathbb{Z}_2$: a is even
$Sp(n) \times Sp(1)$	$(a\varpi_1 + b\varpi_2) \otimes a\varpi_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2$: a is even
$Sp(n) \times U(1)$	$(a\varpi_1 + b\varpi_2) \otimes \phi^k$	$\{1\} \times \mathbb{Z}_{\lvert k\rvert}$: a is odd; $\mathbb{Z}_2 \times \mathbb{Z}_{\lvert k\rvert}$: a is even
G_2	$a\varpi_1$	———————
$Spin(7)$	$a\varpi_3$	$\mathbb{Z}_2$: a is even
$Spin(9)$	$a\varpi_1 + b\varpi_4$	$\mathbb{Z}_2$: b is even

Table A.3: Kernel of class one representations

follows:

Proposition A.5.4. *The dimension of each class one representation for the pair (G, H) is listed as follows:*

- $(SO(n), SO(n-1))$: $\rho = a\varpi_1$,

  $\dim \rho = \frac{2a+n-2}{a+n-2}\binom{a+n-2}{a}$

- $(SU(n), SU(n-1))$: $\rho = a\varpi_1 + b\varpi_{n-1}$,

  $\dim \rho = \frac{a+b+n-1}{n-1}\binom{a+n-2}{a}\binom{b+n-2}{b}$

- $(U(n), U(n-1))$: $\rho = ae_1 - be_n$,

  $\dim \rho = \frac{a+b+n-1}{n-1}\binom{a+n-2}{a}\binom{b+n-2}{b}$

- $(Sp(n), Sp(n-1))$: $\rho = a\varpi_1 + b\varpi_2$,

  $\dim \rho = \frac{(a+1)(a+2b+2n-1)}{(a+b+1)(a+b+2n-1)}\binom{a+b+2n-1}{a+b}\binom{b+2n-3}{b}$

- $(Sp(n) \times Sp(1), Sp(n-1) \times Sp(1))$: $\rho = (a\varpi_1 + b\varpi_2) \otimes a\varpi_1$,

  $\dim \rho = \frac{(a+1)^2(a+2b+2n-1)}{(a+b+1)(a+b+2n-1)}\binom{a+b+2n-1}{a+b}\binom{b+2n-3}{b}$

- $(Sp(n) \times U(1), Sp(n-1) \times U(1))$: $\rho = (a\varpi_1 + b\varpi_2) \otimes \phi^k$,

  $\dim \rho = \frac{(a+1)(a+2b+2n-1)}{(a+b+1)(a+b+2n-1)}\binom{a+b+2n-1}{a+b}\binom{b+2n-3}{b}$

- $(G_2, SU(3))$: $\rho = a\varpi_1$,

  $\dim \rho = \frac{1}{120}(a+1)(a+2)(a+3)(a+4)(2a+5)$

- $(Spin(7), G_2)$: $\rho = a\varpi_3$,

  $\dim \rho = \frac{1}{360}(a+1)(a+2)(a+3)^2(a+4)(a+5)$

- $(Spin(9), Spin(7))$: $\rho = a\varpi_1 + b\varpi_4$,

  $\dim \rho = \frac{1}{1814400}(a+1)(a+2)(a+3)(b+1)(b+2)(b+3)^2(b+4)(b+5)(a+b+$

$4)(a+b+5)(a+b+6)(2a+b+7)$.

Finally for each spherical group pair (G, H), we consider the embedding $\rho : G \longrightarrow SO(l)$, $U(l)$ and $Sp(l)$. We compare the dimension of the sphere G/H and l.

If ρ is not a real representation, the $[\rho]_{\mathbb{R}} = \rho \oplus \rho^*$ is an irreducible real representation where ρ^* is the complex conjugate of ρ and $\deg[\rho]_{\mathbb{R}} = 2 \deg \rho$.

Proposition A.5.5. *Table A.4 is the list of the dimension k of the sphere G/H and the embeddings $\rho : G \longrightarrow SO(l)$ with $l \leq k+1$.*

For the complex representations, we have

Proposition A.5.6. *Table A.5 is the list of the dimension k of the sphere G/H and the embeddings $\rho : G \longrightarrow U(l)$ with $2l \leq k+1$.*

And the similar result for the quaternionic representations is as follows:

Proposition A.5.7. *The quaternionic class one representations $\rho : G \longrightarrow Sp(l)$ with $4l \leq \dim(G/H)+1$ are exactly the representation $\rho = \varpi_1$ for the pair $(Sp(n), Sp(n-1))$ $(n \geq 1)$.*

G	H	ρ	l	$k+1$
$SO(n)$	$SO(n-1)$	ϖ_1	n	n
$SU(n)$	$SU(n-1)$	$[\varpi_1]_{\mathbb{R}}$, $[\varpi_{n-1}]_{\mathbb{R}}$	$2n$, $2n$	$2n$
$U(n)$	$U(n-1)$	$[e_1]_{\mathbb{R}}$, $[-e_n]_{\mathbb{R}}$	$2n$, $2n$	$2n$
$Sp(n)$	$Sp(n-1)$	$[\varpi_1]_{\mathbb{R}}$	$4n$	$4n$
$Sp(n) \times Sp(1)$	$Sp(n-1) \times Sp(1)$	$\varpi_1 \otimes \varpi_1$	$4n$	$4n$
$Sp(n) \times U(1)$	$Sp(n-1) \times U(1)$	$[\varpi_1 \otimes \phi^k]_{\mathbb{R}}$	$4n$	$4n$
G_2	$SU(3)$	ϖ_1	7	7
$Spin(7)$	G_2	ϖ_3	8	8
$Spin(9)$	$Spin(7)$	ϖ_1, ϖ_4	9, 16	16
$Sp(2)$	$Sp(1)$	ϖ_2	5	8
$SU(2)$, $Sp(1)$	$\{\mathrm{Id}\}$	$2\varpi_1$	3	4
$U(2)$	$U(1)$	$e_1 - e_2$	3	4

Table A.4: Orthogonal representation with small dimension

G	H	ρ	l	$k+1$
$SU(n)$	$SU(n-1)$	$\varpi_1,\ \varpi_{n-1}$	$n,\ n$	$2n$
$U(n)$	$U(n-1)$	$e_1,\ -e_n$	$n,\ n$	$2n$
$Sp(n)$	$Sp(n-1)$	ϖ_1	$2n$	$4n$
$Sp(n)\times U(1)$	$Sp(n-1)\times U(1)$	$\varpi_1 \otimes \phi^k$	$2n$	$4n$

Table A.5: Complex representation with small dimension

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