THE ASSOCIATED MAP OF THE NONABELIAN GAUSS-MANIN CONNECTION

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ABSTRACT

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In ordinary Hodge theory for a compact kahler manifold, one can look at the Gauss-Manin connection as the complex structure of the manifold varies. The connection satisfies the Griffiths transversality property, and induces a map on the associated graded spaces of the Hodge filtered cohomology spaces of the manifolds. Similarly in nonabelian Hodge theory, where the manifolds are curves and nonabelian cohomology spaces are the moduli spaces of local systems on the curves, the Gauss-Manin connection will be the Isomonodromy deformation, which is a previously known structure on these moduli spaces. One can still define the Hodge filtration and calculate the map induced by the Gauss-Manin connection on the associated graded space. To do this we used deformation theory to express the tangent spaces of the moduli spaces as hypercohomologies of complexes of sheaves over the curves, and write the isomonodromy deformation as a map between such hypercohomology spaces. Under this setting the induced map can be explicitly calculated and is in fact written in an analogous form as the isomonodromy deformation. The induced map turns out to be closely related to another well-known structure called the Hitchin integrable structure, defined on the moduli spaces that correspond to the associated graded spaces. More specifically it is equal up to a factor of 2 to the quadratic Hitchin map.

Contents

1. INTRODUCTION

The variation of Hodge structures for families of complex Kähler manifolds has been a much studied subject. Let $\pi : \mathscr{X} \to S$ be a proper holomorphic submersion of connected complex manifolds. Ehresmann's Lemma says that it is a locally trivial fiber bundle with respect to its underlying differentiable structure. In particular all the fibers of π are diffeomorphic. So if $s \in S$ and X_s the fiber of π over s, $\mathscr{X} \to S$ can be viewed as a variation over S of complex structures on the underlying differentiable manifold of X_s.

Let $\pi^k : \mathscr{V} \to S$ be the corresponding vector bundle of cohomologies whose fiber at $s \in S$ is $H^k(X_s, \mathbb{C})$, $k \in \mathbb{N}$. Since $\mathscr{X} \to S$ is locally trivial differentiably (and therefore topologically), there is an induced local identification of fibers of $\mathscr{V} \to S$. In another word, there is a flat connection on the vector bundle $\mathscr{V} \to S$. This connection is called the Gauss-Manin connection for the cohomologies of the family of complex Kähler manifolds $\mathscr{X} \to S$.

From Hodge theory we know there is a natural Hodge filtration on the vector bundle $\mathscr{V} \to S$: $\mathscr{V} = F^0 \supset F^1 \supset F^2 \ldots \supset F^k$. Let ∇ be the Gauss-Manin connection, Griffiths transversality theorem says that $\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_S$, $1 \leq p \leq k$. So if $gr\mathscr{V}$ is the associated graded vector bundle of the filtered bundle $\mathscr{V}$, then the induced map $gr\nabla$ of ∇ on $gr\mathscr{V}$ will be $\mathscr{O}_S$-linear. In fact, $gr\nabla$ is equal to a certain Kodaira-Spencer map[3]. We call $gr\nabla$ the *associated map of the Gauss-Manin connection*.

The above has a nonabelian analogue. Let G be a complex algebraic group, X a smooth algebraic curve over $\mathbb{C}$ of genus g. Let $Conn_X$ be the moduli space of principal G-bundles over X equipped with a flat connection. If we denote as $H^1(X, G)$ the first Čech cohomology of X with coefficients the constant sheaf in G, then $Conn_X$ can be naturally identified with $H^1(X, G)$, by considering the gluing data of flat G-bundles. Since the group G can be nonabelian, we call $Conn_X$ the nonabelian cohomology space of X.

Let $\mathscr{M}_g$ be the moduli space of genus g complex algebraic curves. The universal curve $\mathscr{X} \to \mathscr{M}_g$ is (roughly) a variation of complex structures of the underlying real surface, and the universal moduli space of connections $\mathscr{C}onn \to \mathscr{M}_g$ is the corresponding bundle of nonabelian cohomologies. For the same reason as before there is a Gauss-Manin connection on the bundle $\mathscr{C}onn \to \mathscr{M}_g$. The local trivialization that defines it is often called the isomonodromy deformation, or the isomonodromy flow of $\mathscr{C}onn$ over $\mathscr{M}_g$.

There is also a nonabelian analogue of Hodge filtration which was determined by Carlos Simpson[8], using a generalized definition of filtration of spaces. A vector space with filtration is equivalent, by the Rees construction, to a locally free sheaf over $\mathbb{C}$ with a $\mathbb{C}^*$-action, and with the fiber over 1 being the vector space itself. To define the nonabelian Hodge "filtration" on the space $\mathscr{C}onn$ therefore, it would be reasonable to find a family of spaces over $\mathbb{C}$ whose fiber over 1 is $\mathscr{C}onn$, together with a $\mathbb{C}^*$-action on the family. The way to do it in this case is to introduce the notion of λ-connections on a principal G-bundle on X, for any $\lambda \in \mathbb{C}$. It is a generalization of the notion of connections on a G-bundle. In particular a 1-connection is an ordinary connection, and a 0-connection is a so called *Higgs field*, which is an object of much interest to people in complex geometry and high energy physics. The moduli space of principal G-bundles over X together with a Higgs field is called the Higgs moduli space over X, and denoted as $Higgs_X$. Simpson's definition of nonabelian Hodge filtration immediately implies that the associated graded space of $Conn_X$ is $Higgs_X$. Then a question arises: what is the associated map of the nonabelian Gauss-Manin connection on the associated graded space? The answer is: it is a lifting[1] of tangent vectors on the relative Higgs moduli space $\mathscr{H}iggs \to \mathscr{M}_g$. On the other hand there is a well-known Hitchin map from $Higgs_X$ to some vector spaces. The quadratic part of the Hitchin maps also induce a lifting of tangent vectors on $\mathscr{H}iggs \to \mathscr{M}_g$. The fact that the two liftings agree is the content of our theorem.

Theorem 1.1. *The lifting of tangent vectors on $\mathscr{H}iggs \to \mathscr{M}_g$ representing the associated map of the nonabelian Gauss-Manin connection is equal up to a constant multiple to the lifting of tangent vectors induced from the quadratic Hitchin map.*

Closely related results have been obtained in [1], where the authors apply localization for vertex algebras to the Segal-Sugawara construction of an internal action of the Virasoro algebra on affine Kac-Moody algebras to lift twisted differential operators from the moduli of curves to the moduli of curves with bundles. Their construction gives a uniform approach to several phenomena describing the geometry of the moduli spaces of bundles over varying curves, including a Hamiltonian description of the isomonodromy equations in terms of the quadratic part of Hitchin's system. Our result and proof are much more elementary, avoiding the need for the vertex algebra machinery.

The organization of the paper is as follows. In section 2 we give a detailed definition of all the objects concerned and a precise statement of the theorem. The rest of the sections are devoted to the proof. In section 3 we recall the definition of Atiyah bundles and some of its properties that will be useful in the proof. In section 4 we use deformation theory to write the tangent spaces to $\mathscr{C}onn$

[1] Here the word lifting has a slightly more general meaning: it means a map of tangent vectors in the opposite direction of the pushforward, without requiring its composition with pushforward being identity. In fact, this lifting here composed with pushforward is zero.

as certain hypercohomology spaces. Section 5 gives an explicit description of the lifting of tangent vector on $\mathscr{C}onn \to \mathscr{M}_g$ given by the isomonodromy flow. Section 6 extend the isomonodromy lifting to the moduli space of λ-connections for any $\lambda \neq 0$. Finally section 7 takes the limit of the lifting at $\lambda = 0$, which is precisely the associated map of the nonabelian Gauss-Manin connection, and shows that it is equal up to a constant to the quadratic Hitchin lifting of tangent vectors.

I would like to thank my advisor Ron Donagi for introducing me to the subject and for many invaluable discussions.

2. DEFINITIONS AND STATEMENT OF THE THEOREM

All objects and morphisms in this paper will be algebraic over $\mathbb{C}$, unless otherwise mentioned.

2.1. **Moduli space of connections and isomonodromy flow.** Let g be a natural number greater or equal to 2, so that a generic curve of genus g has no automorphisms. The moduli space of all genus g curves is a smooth Deligne-Mumford stack, but if we restrict to the curves that has no automorphisms, the moduli space is actually a smooth scheme. Let $\mathscr{M}_g$ be this scheme. In this paper we will ignore all the special loci of the moduli spaces (as explained below) and focus on local behaviors around generic points.

Let G be a semisimple Lie group, X a smooth curve of genus g. Let Bun_X be the coarse moduli space of regular stable G-bundles on X. Bun_X is also a smooth scheme[7]. The total space of the cotangent bundle T^*Bun_X is an open subscheme of the Higgs moduli space over X [4]. However since we are only concerned with generic situations, we will use $Higgs_X$ to denote the open subscheme T^*Bun_X.

Let $Conn_X$ be the moduli space of pairs (P, ∇), where P is a stable G-bundle on X, and ∇ is a connection on P. ∇ is necessarily flat as the dimension of X is equal to 1. $Conn_X$ is an affine bundle on Bun_X whose fiber over $P \in Bun_X$ is a torsor for $T^*_P Bun_X$. So it is also a smooth scheme.

Let $\mathscr{C}onn \to \mathscr{M}_g$ be the relative moduli space of pairs whose fiber at $X \in \mathscr{M}_g$ is $Conn_X$. Let $Irrep_X$ be the space of all irreducible group homomorphisms $\pi_1(X) \to G$, $Irrep_X$ is a smooth scheme[5]. There is also the relative space $\mathscr{I}rrep \to \mathscr{M}_g$. The Riemann-Hilbert correspondence $RH : Conn_X \to Irrep_X$ taking a flat connection to its monodromy is an analytic(and therefore differentiable) inclusion. Let $S \subset \mathscr{M}_g$ be a small neighborhood of X in analytic topology. By Ehresmann's Lemma the family of curves over S is a trivial family with respect to the differentiable structure. This implies that the restriction of $\mathscr{I}rrep$ over S is a differentiable trivial family. The trivial sections or trivial flows induce a flow on the restriction of $\mathscr{C}onn$ over S, by the Riemann-Hilbert correspondence. This flow is called the isomonodromy flow of $\mathscr{C}onn$ over $\mathscr{M}_g$.

2.2. **λ-connections and nonabelian Hodge filtration.** As explained in the last section, $Conn_X$ is the nonabelian cohomology space of X with in coefficient G, and the isomonodromy flow on $\mathscr{C}onn \to \mathscr{M}_g$ is the nonabelian Gauss-Manin connection. To define Hodge filtration on $Conn_X$ one need to generalize the definition of a filtration. A filtration on a vector space V is equivalent, by the Rees construction[4], to a locally free sheaf W on $\mathbb{C}$ whose fiber at $1 \in \mathbb{C}$ is isomorphic to V, together with a $\mathbb{C}^*$-action on W compatible with the usual $\mathbb{C}^*$-action on $\mathbb{C}$. The fiber of W at $0 \in \mathbb{C}$ will be isomorphic to the associated graded vector space of V.

This sheaf definition of filtrations can be generalized in an obvious way to define filtrations on a space that is not a vector space. In our case the space is $Conn_X$, and its Hodge filtration is constructed as follows. $Conn_X$ parametrizes pairs (P,∇). Let P also denote the sheaf of sections of P on X, adP be the adjoint bundle of P as well as the sheaf of its sections, and $\mathfrak{g}$ the Lie algebra of G. A connection ∇ is a map of sheaves

$$\nabla : P \to adP \otimes \Omega^1_X$$

that after choosing local coordinates for X and local trivialization for P can be written as

$$(\frac{\partial}{\partial x} + [A(x),\ ]) \otimes dx$$

where $A(x)$ is a $\mathfrak{g}$-valued function and the bracket means the right multiplication action of G on $\mathfrak{g}$. A λ-connection on P is defined to be a map of sheaves $\nabla_\lambda : P \to adP \otimes \Omega^1_X$ that in local coordinates can be written as $(\lambda\frac{\partial}{\partial x} + [A(x),\ ]) \otimes dx$. Let the moduli space of λ-connections be denoted as $\lambda Conn_X$. For $\lambda \neq 0$, $\nabla \leftrightarrow \lambda \cdot \nabla$ is a bijection between $Conn_X$ and $\lambda Conn_X$. For $\lambda = 0$, the definition of a 0-connection agrees with that of a Higgs field. So $0Conn_X$ is just $Higgs_X$.

Let $\mathcal{T}_X$ be the moduli space of all λ-connections for all $\lambda \in \mathbb{C}$. There is a natural map $\mathcal{T}_X \to \mathbb{C}$ taking a λ-connection to λ, whose preimage at $1 \in \mathbb{C}$ is $Conn_X$. In fact, Simpson showed that the nonabelian Hodge filtration of $Conn_X$ is precisely the sheaf of sections of this map, with the $\mathbb{C}^*$-action given by multiplication by λ for $\lambda \in \mathbb{C}^*$[4]. The $\mathbb{C}^*$-action is algebraic and induces an isomorphism of $Conn_X$ and $\lambda Conn_X$.

In the ordinary Hodge theory, if one uses the sheaf definition of filtrations, then the associated map of the Gauss-Manin connection is obtained as follows. Start with the Gauss-Manin connection on $\mathscr{V} \to S$, the local trivialization by the flat sections gives a lifting of tangent vectors

$$L : T_sS \to T_v\mathscr{V}$$

for $s \in S$ and $v \in \mathscr{V}$ s.t. $\pi^k(v) = s$. The lifting L is a spliting of π^k_*, i.e. it satisfies

$$\pi^k_* \circ L = id_{T_sS}$$

Let $\mathscr{W} \to \mathbb{C}$ be the sheaf associated to the Hodge filtration on $\mathscr{V} \to S$. The fiber of $\mathscr{W}$ at 1 is $\mathscr{V} \to S$, and denote the fiber over λ as $\pi^k_\lambda : \mathscr{V}_\lambda \to S$. The action of $\lambda \in \mathbb{C}^*$ induces an isomorphism of $\mathscr{V}$ and $\mathscr{V}_\lambda$. So the local trivialization of $\mathscr{V} \to S$ induces a local trialization of $\mathscr{V}_\lambda \to S$ via this isomorphism. Let $L_\lambda : T_sS \to T_{v_\lambda}\mathscr{V}_\lambda$ be the induced lifting on $\mathscr{V}_\lambda \to S$ *multiplied by* λ. L_λ satisfies

$$\pi^k_{\lambda *} \circ L_\lambda = \lambda \cdot id_{T_sS}$$

L_λ is defined for all $\lambda \neq 0$. For a fixed vector $\vec{t} \in T_sS$, the images of $\vec{t}$ under all the L_λ, $\lambda \neq 0$ gives a vector field on the total space of $\mathscr{W}$ away from $\mathscr{V}_0$, which is the fiber over $0 \in \mathbb{C}$. The continuous limit of that vector field on $\mathscr{V}_0$ exist, and therefore defines a lifting $L_0 : T_sS \to T_{v_0}\mathscr{V}_0$ on $\mathscr{V}_0 \to S$. L_0 satisfies

$$\pi^k_{0*} \circ L_0 = 0$$

i.e. the images of $\vec{t} \in T_sS$ under L_0 is a vectors field on the *fiber* $V_{0,s}$ of $\mathscr{V}_0$ over s. This vector field is in fact linear and defines a linear map on $V_{0,s}$. Also $\mathscr{V}_0$ is identified with $gr\mathscr{V}$. From these we see L_0 really gives a vector bundle map $gr\mathscr{V} \to gr\mathscr{V} \otimes \Omega^1_S$, and that map is the associated map of the Gauss-Manin that we started with.

So in nonabelian Hodge theory, in order to calculate the associated map of the nonabelian Gauss-Manin connection, we will start with the lifting L induced from the isomonodromy flow on $\mathscr{C}onn \to \mathscr{M}_g$(by a slight abuse of notation we will use the same notations for the liftings, the meaning should be clear from the context), and try to find the associated limit lifting L_0. Specifically, let $\mathscr{T} \to \mathscr{M}_g$ be the relative moduli space whose fiber at $X \in \mathscr{M}_g$ is $\mathcal{T}_X$. $\mathscr{T}$ maps to $\mathbb{C}$ and the fiber at λ is the relative moduli space of λ-connections, which is denoted $\lambda\mathscr{C}onn$. There is clearly also a $\mathbb{C}^*$-action on $\mathscr{T}$ compatible with the $\mathbb{C}^*$-action on $\mathbb{C}$. Let L_λ be analogously the lifting on $\lambda\mathscr{C}onn \to \mathscr{M}_g$ induced by the lifting L via the $\mathbb{C}^*$-action and multiplied by λ. Then the limit lifting L_0 will be the associated map that we want to calculate. It will again be a vertical lifting, i.e. the images of L_0 will be vectors tangent to the fibers $Higgs_X$ of $\mathscr{H}iggs \to \mathscr{M}_g$, $X \in \mathscr{M}_g$.

2.3. **Quadratic Hitchin map and statement of the theorem.** $Higgs_X$ has a symplectic structure as it is equal to T^*Bun_X. Let $< \, , \, >$ be the Killing form on the Lie algebra $\mathfrak{g}$ of G, the quadratic Hitchin map is

$$qh : Higgs_X \to H^0(X, \Omega^{\otimes 2})$$

$$(P, \theta) \mapsto < \theta, \theta >$$

where $\theta \in H^0(X, adP \otimes \Omega^1_X)$ is a 0-connection or a Higgs field. One can define a lifting of tangent vectors associated to qh

$$L_{qh} : T_X \mathscr{M}_g \to T_{(P,\theta)} Higgs_X$$

$$f \mapsto H_{qh^* f}|_{(P,\theta)}$$

where $f \in T_X \mathscr{M}_g \cong H^1(X, TX)$ is viewed as a linear function on $H^0(X, \Omega^{\otimes 2})$ by Serre duality, and $H_{qh^* f}$ is the Hamiltonian vector field of $qh^* f$ on $Higgs_X$.

The theorem can now be more precisely stated as:

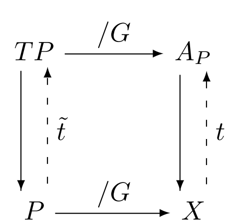
Theorem 2.1 (precise version of Theorem 1.1)**.** *The limit lifting of tangent vectors L_0 associated to the isomonodromy lifting L is equal to $\frac{1}{2} L_{qh}$.*

3. Atiyah bundles

Before starting to prove the theorem, we recall here some facts about Atiyah bundles which will be used later. As before let X be a smooth curve of genus g, G a semisimple Lie group, $p : P \to X$ a principal G-bundle over X.

3.1. **Atiyah bundle and its sections.** Let TP be the tangent bundle over P. G acts on P and has an induced action on TP. The action is free and compatible with the vector bundle structure of $TP \to P$, so the quotient will be a vector bundle $TP/G \to P/G = X$. This vector bundle over X is called the Atiyah bundle associated to P, and denoted as A_P.

In fact, TP is isomorphic to the fiber product of P and A_P over X. So any section t of A_P over X has a unique lift $\tilde{t}$ that makes the diagram commute



The lift $\tilde{t}$ can be viewed as a vector field on P which is G-invariant. Conversely, any G-invariant vector field on P defines a section t in the quotient bundle. Therefore sections of A_P over X are the same as G-invariant vector fields on P.

3.2. **Atiyah sequence.** The sequence of tangent bundles associated to $P \to X$ is:

$$0 \to T_{P/X} \to TP \to p^*TX \to 0$$

G acts on the sequence, and the quotient is

$$0 \to adP \to A_P \to TX \to 0$$

This quotient sequence is called the Atiyah sequence of A_P. We will denote the map $A_P \to TX$ also as p_*.

3.3. **Relation to connections.** If ∇ is a connection on P, then ∇ must be flat since the dimension of X is 1. So over a small open subset $U \subset X$, there is a natural trivialization of P associated to ∇

$$\tau : U \times F \longrightarrow P|_U$$

given by the flat sections of ∇. Here F denotes a torsor for G.

The local trivialization gives a local section $\tilde{s}_U : p^*TU \to TP|_U$, which is the composition

$$p^*TU \xrightarrow{\tau_*^{-1}} p_U^*TU \xrightarrow{(id,0)} p_U^*TU \oplus p_F^*TF \xrightarrow{\tau_*} TP|_U \tag{1}$$

where p_U and p_F are the projections of $U \times F$ to U and F.

Since $\tilde{s}_U$ is canonically associated to ∇, so for two such open subsets U, V, $\tilde{s}_U$ and $\tilde{s}_V$ agree on their intersection. So there is a well-defined map $\tilde{s} : p^*TU \to TP$. Since τ is G-invariant and the map $(id, 0)$ is obviously G-invariant, $\tilde{s}_U$ is G-invariant. So $\tilde{s}$ is G-invariant, and gives a map $s : TX \to A_P$. The map $(id, 0)$ in the definition of $\tilde{s}_U$ implies that s is a splitting of $p_* : A_P \to TX$, i.e. $p_* \circ s = id_{TX}$. We can also say that s is a splitting of the Atiyah sequence.

$$0 \longrightarrow adP \longrightarrow A_P \underset{s}{\overset{p_*}{\rightleftarrows}} TX \longrightarrow 0$$

To summarize, for any connection ∇ on P there is uniquely associated a splitting s of the Atiyah sequence of P. s is locally defined as the splitting $(id, 0)$ with P(and therefore A_P) locally trivialized by ∇.

4. TANGENT SPACES

Now we start to prove the theorem. In this section we will identify the tangent spaces of $\mathscr{C}onn$ and more generally $\lambda\mathscr{C}onn$ as some hypercohomology spaces, so that we may write down the isomonodromy lifting L and the extened liftings L_λ explicitly in the next two sections.

The tangent space to a moduli space at a regular point is identified with the infinitesimal deformations of the object corresponding to that point. So we are really looking at infinitesimal deformations of the objects parametrized by $\mathscr{C}onn$, which are triples (X,P,∇). We start with deformations of pairs (X,P).

4.1. **Deformation of pairs.** From Deformation Theory, the following two propositions are well-known.

Proposition 4.1.1. *The tangent space to $\mathscr{M}_g$ at a point X is naturally isomorphic to $H^1(X,TX)$.*

Proposition 4.1.2. *The tangent space to Bun_X at a point P is naturally isomorphic to $H^1(X,adP)$.*

Let $\mathscr{B}un$ be the moduli space of pairs (X,P). We expect that generically the tangent space at a point (X,P) would satisfy

$$0 \to H^1(X,adP) \to T_{(X,P)}\mathscr{B}un \to H^1(X,TX) \to 0$$

On the other hand since the Atiyah sequence of P $0 \to adP \to A_P \to TX \to 0$ induces

$$0 \to H^1(X,adP) \to H^1(X,A_P) \to H^1(X,TX) \to 0$$

It is natural to guess that

Proposition 4.1.3. *$T_{(X,P)}\mathscr{B}un$ is naturally isomorphic to $H^1(X,A_P)$.*

Proof. the proof is a combination of the usual proofs for Proposition 4.1.1 and Proposition 4.1.2. Let $\{U_i\}_{i\in I}$ be an Čech covering of X, $P_\epsilon \to X_\epsilon$ a family of principal G-bundles over $D_\epsilon = \mathbb{C}[\epsilon]/(\epsilon^2)$, which restrict to $P \to X$ over the closed point. Over each U_i, let

$$\phi_i : P|_{U_i} \times D_\epsilon \to P_\epsilon|_{U_i} \qquad (\phi_i^\vee : \mathscr{O}_{P|_{U_i}} \otimes \mathbb{C}[\epsilon]/(\epsilon^2) \leftarrow \mathscr{O}_{P_\epsilon|_{U_i}})$$

be an isomorphism of G-bundles. So it is compatible with the G-actions and descends to an isomorphism

$$\iota_i : U_i \times D_\epsilon \to X_\epsilon|_{U_i} \qquad (\iota_i^\vee : \mathscr{O}_{U_i} \otimes \mathbb{C}[\epsilon]/(\epsilon^2) \leftarrow \mathscr{O}_{X_\epsilon|_{U_i}})$$

Over $U_{ij} = U_i \cap U_j$, the transition functions are related as in the commutative diagram

$$\begin{array}{ccc} P|_{U_{ij}} \times D_\epsilon & \xrightarrow{\phi_j^{-1} \circ \phi_i} & P|_{U_{ij}} \times D_\epsilon \\ \downarrow p & & \downarrow p \\ U_{ij} \times D_\epsilon & \xrightarrow{\iota_j^{-1} \circ \iota_i} & U_{ij} \times D_\epsilon \end{array}$$

Let $\xi_{ij} \in \Gamma(U_{ij}, TX)$ be the vector field on U_{ij} such that $(\iota_j^{-1} \circ \iota_i)^\vee = Id + \epsilon \xi_{ij}$, and $\eta_{ij} \in \Gamma(P|_{U_{ij}}, TP)$ be the vector field on $P|_{U_{ij}}$ such that $(\phi_j^{-1} \circ \phi_i)^\vee = Id + \epsilon \eta_{ij}$. Because ϕ_i is G-invariant, η_{ij} is G-invariant. So one can view it as $\eta_{ij} \in \Gamma(U_{ij}, A_P)$. $(\eta_{ij})_{i,j \in I}$ form a Čech 1-cochain on X with coefficients in A_P.

$(\eta_{ij})_{i,j \in I}$ is closed because it comes from transition functions $\phi_j^{-1} \circ \phi_i$. Any closed cochain $(\eta_{ij})_{i,j \in I}$ comes from some D_ϵ family of pairs. Also for a fixed D_ϵ family of pairs, a different choice of ϕ_i's will result in a cocycle differing from $(\eta_{ij})_{i,j \in I}$ by an exact cocycle. And any exact cocycle is the result of different choices of ϕ_i's. Therefore the infinitesimal deformations of (X,P) are in natural correspondence with $H^1(X, A_P)$, which proves the proposition. □

4.2. **Deformation of triples.** Now we come to the infinitesimal deformations of a triple (X,P,∇). First a notation related to the connection ∇. As discussed in section 3.3, a connection ∇ on P is equivalent to a splitting of the Atiyah sequence

$$0 \longrightarrow adP \longrightarrow A_P \underset{s}{\overset{p_*}{\rightleftarrows}} TX \longrightarrow 0$$

Let $\hat{s} \in H^0(X, A_P \otimes \Omega_X^1)$ denote the global section associated to the splitting map s. We see that $\hat{s} \mapsto 1$ under the map $H^0(X, A_P \otimes \Omega_X^1) \to H^0(X, TX \otimes \Omega_X^1) \cong H^0(X, \mathscr{O}_X)$.

To find the deformation of the triple (X,P,∇), let $(X_\epsilon,P_\epsilon,\nabla_\epsilon)$ be a family of triples over D_ϵ starting with it. Let s_ϵ be the family of sections corresponding to ∇_ϵ. As in the proof of Proposition 4.1.3, let $\{U_i\}_{i \in I}$ again be an Čech covering of X, and ϕ_i, ι_i, $i \in I$ defined in the same way. Let $s_i : TU_i \to A_P|_{U_i}$ and $\sigma_i : TU_i \to adP|_{U_i}$ be sections such that the following diagram commutate:

$$\begin{array}{ccc} A_P|_{U_i} \times D_\epsilon & \xrightarrow{d\phi_i} & A_{P_\epsilon}|_{U_i} \\ {\scriptstyle s_i + \epsilon\sigma_i} \uparrow \;\downarrow {\scriptstyle p_*} & & {\scriptstyle p_*} \downarrow \;\uparrow {\scriptstyle s_\epsilon|_{U_i}} \\ TU_i \times D_\epsilon & \xrightarrow{d\iota_i} & TX_\epsilon|_{U_i} \end{array}$$

The target space of σ_i is adP instead of A_P, because $p_* \circ s = id$ for all s, so σ_i, being the derivative of s (locally on U_i, under the trivialization of the family ϕ_i), projects to 0 under p_*.

A deformation of the triple should contain the information about the deformation of the pair (X,P) as well as the deformation of ∇. So the data associated to the infinitesimal family $(X_\epsilon,P_\epsilon,\nabla_\epsilon)$ should be the pair:

$$(\eta_{ij})_{i,j\in I}, (\sigma_i)_{i\in I}$$

where $(\eta_{ij})_{i,j\in I}$ is defined in section 4.1.3 and shown to characterize the deformation of the pair (X,P), and $(\sigma_i)_{i\in I}$ describe the deformation of ∇.

The data $((\eta_{ij})_{i,j\in I}, (\sigma_i)_{i\in I})$ looks like a 1-cocycle in defining the hypercohomology of some complex of sheaves. Recall that the tangent space to $Higgs_X$ at a point (P,θ) is $\mathbb{H}^1(X, adP \xrightarrow{[\ ,\theta]} adP\otimes\Omega^1_X)$. We will prove an analogous result about the tangent spaces to $\mathscr{C}onn$.

On U_{ij}, the transition relations are expressed in the following diagram:

$$\begin{array}{ccc} A_P|_{U_{ij}} \times D_\epsilon & \xrightarrow{d(\phi_j^{-1}\circ\phi_i)} & A_P|_{U_{ij}} \times D_\epsilon \\ s_i+\epsilon\sigma_i \ \Big\uparrow \ \Big\downarrow\ p_* & & p_*\ \Big\downarrow \ \Big\uparrow \ s_j+\epsilon\sigma_j \\ TU_{ij}\times D_\epsilon & \xrightarrow{d(\iota_j^{-1}\circ\iota_i)} & TU_{ij}\times D_\epsilon \end{array}$$

Since $(\iota_j^{-1}\circ\iota_i)^\vee = Id+\epsilon\xi_{ij}$ and $(\phi_j^{-1}\circ\phi_i)^\vee = Id+\epsilon\eta_{ij}$, we can write down the two horizontal maps more explicitly. $\forall\ Y+\epsilon Y_1\in TU_{ij}\times D_\epsilon$, its image $Y'+\epsilon Y_1'$ under $d(\iota_j^{-1}\circ\iota_i)$ is determined by: for any function f on U_{ij},

$$(Y'+\epsilon Y_1')(f) = (I+\epsilon\xi_{ij})(Y+\epsilon Y_1)(I-\epsilon\xi_{ij})(f)$$

After simplification we get $Y'=Y, Y_1'=Y_1+[\xi_{ij},Y]$, where the bracket is the Lie bracket of vector fields on U_{ij}. Similarly $\forall\ Z+\epsilon Z_1\in A_P|_{U_{ij}}\times D_\epsilon$ (by section 3.1 it can be viewed as a G-invariant vector field on $P|_{U_{ij}}$), we get $d(\phi_j^{-1}\circ\phi_i)(Z+\epsilon Z_1) = Z+\epsilon(Z_1+[\eta_{ij},Z])$, where the bracket is the Lie bracket of (G-invariant) vector fields on $P|_{U_{ij}}$.

The diagram is commutative, i.e. $\forall\ Y+\epsilon Y_1\in TU_{ij}\times D_\epsilon$

$$d(\phi_j^{-1}\circ\phi_i)\circ(s_i+\epsilon\sigma_i)(Y+\epsilon Y_1) = (s_j+\epsilon\sigma_j)\circ d(\iota_j^{-1}\circ\iota_i)(Y+\epsilon Y_1)$$

After simplification we get

$$s_i(Y)=s_j(Y)$$

(2) $$(\sigma_j-\sigma_i)(Y) = [\eta_{ij}, s_i(Y)] - s_j([\xi_{ij},Y])$$

So if we use $\hat\sigma_i\in H^0(X, adP\otimes\Omega^1_X)$ to denote the global section associated to σ_i, the pair

$$((\eta_{ij})_{i,j\in I}, (\hat\sigma_i)_{i\in I})$$

is a hyper Čech 1-cochain on X with coefficients in

$$A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X$$

where the map $[\ ,\hat{s}]$ is defined as: if $\hat{s} = s' \otimes \omega$, where $s' \in H^0(X, A_P), \omega \in H^0(X, \Omega^1_X)$, then $[\ ,\hat{s}] := [\ ,s'] \otimes \omega - s' \otimes [p_*(\ ), \omega]$.

Proposition 4.2.1. *$T_{(X,P,\nabla)}\mathscr{C}onn$ is naturally isomorphic to $\mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X)$.*

Proof. To any D_ϵ family of triples (X_ϵ,P_ϵ,∇_ϵ) is associated a hyper 1-cochain $((\eta_{ij})_{i,j\in I}, (\hat{\sigma}_i)_{i\in I})$ by the above discussion. It is closed because of three facts: first, $(\eta_{ij})_{i,j\in I}$ is a closed Čech 1-cochain with coefficients in A_P - it's closed again because it comes from the transition function $\phi_j^{-1} \circ \phi_i$; second, because of (2); third, the complex $A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X$ has only two nonzero terms. These three facts imply that $((\eta_{ij})_{i,j\in I}, (\hat{\sigma}_i)_{i\in I})$ is closed. Any closed hyper 1-cochain comes from some D_ϵ family of triples. Also for a fixed D_ϵ family of triples, a different choice of the ϕ_i's will result in a hyper cocycle differing from $((\eta_{ij})_{i,j\in I}, (\hat{\sigma}_i)_{i\in I})$ by an exact hyper cocycle. And any exact hyper cocycle is the result of different choices of the ϕ_i's. Therefore the infinitesimal deformations of (X,P,∇) are in natural correspondence with $\mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X)$, which is what we need to prove. □

4.3. **Tangent spaces to $\lambda\mathscr{C}onn$.** Let $\lambda \in \mathbb{C}$ be a fixed complex number. For the moduli space $\lambda\mathscr{C}onn$ of triples (X,P,∇_λ) where ∇_λ is a λ-connection, the statement about its tangent spaces is completely analogous to that when $\lambda = 1$.

For a λ-connection ∇_λ on P, $\lambda \neq 0$, $\frac{1}{\lambda}\nabla_\lambda$ is an ordinary connection, therefore corresponds to a splitting $s_{\frac{1}{\lambda}\nabla_\lambda}$ of the Atiyah sequence of P. Let $s_\lambda = \lambda \cdot s_{\frac{1}{\lambda}\nabla_\lambda}$, so s_λ is a "λ-splitting" of the Atiyah sequence of P, i.e. $p_* \circ s_\lambda = \lambda \cdot id_{TX}$. Therefore to any λ-connection $\nabla_\lambda(\lambda \neq 0)$ is associated a λ-splitting of the Atiyah bundle. Notice that this is true for $\lambda = 0$ as well, as a 0-splitting of the Atiyah bundle of P is exactly a Higgs field on P.

Let $\hat{s}_\lambda \in H^0(X, A_P \otimes \Omega^1_X)$ be the global section associated to s_λ, we see $\hat{s}_\lambda \mapsto \lambda$ under the map $H^0(X, A_P \otimes \Omega^1_X) \to H^0(X, TX \otimes \Omega^1_X) \cong H^0(X, \mathscr{O}_X)$. The arguments in the last subsection can be repeated with slight changes (replace 1 by λ at appropriate places) to give the following statement.

Proposition 4.3.1. *$T_{(X,P,\nabla_\lambda)}\lambda\mathscr{C}onn$ is naturally isomorphic to $\mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}_\lambda]} adP \otimes \Omega^1_X)$, $\forall \lambda \in \mathbb{C}$*

Remark. When $\lambda = 0$, the result agrees with the previous results about tangent spaces to the Higgs moduli space.

5. ISOMONODROMY VECTOR FIELD

The nonabelian Gauss-Manin connection on $\mathscr{C}onn \to \mathscr{M}_g$ is the isomonodromy flow. The local trivialization of $\mathscr{C}onn \to \mathscr{M}_g$ given by the flow induces a lifting of tangent vectors $L : T_X\mathscr{M}_g \to T_{(X,P,\nabla)}\mathscr{C}onn$. We have identified these tangent spaces as (hyper)cohomology spaces in the last section, now we will write down the map L as a map of cohomology spaces. We start with a useful fact about an isomonodromy family of connections.

5.1. **Universal connection of an isomonodromy family.** In [5] Inaba et al. constructed the moduli space of triples (X,P,∇), and a universal G-bundle on the universal curve with a universal connection. Though they did it for a special case(rank 2 parabolic vector bundle on $\mathbb{P}^1$ with 4 points), the more general case can be done similarly. The universal connection, when restricted to an isomonodromy family of triples, has the following important property.

Proposition. *If (X_t,P_t,∇_t) is an isomonodromy family of triples over a complex line $D = Spec(\mathbb{C}[t])$, then the restriction of the universal connection on P_t(viewed as a G-bundle over the total space of X_t) is flat.*

Proof. If we only look at the underlying differentiable structure, the isomonodromy family over $D = \mathbb{C}[t]$ is a trivial family of triples. The trivial family structure gives a flat connection on P_t, which must be equal to the restriction of the universal connection on P_t since they are equal on each fiber of the family. □

5.2. **Isomonodromy lifting of tangent vectors.** For $\forall\lambda \in \mathbb{C}$, let π_λ be the projection:

$$\pi_\lambda : \lambda\mathscr{C}onn \to \mathscr{M}_g$$

$$(X, P, \nabla_\lambda) \mapsto X$$

From the proof of Proposition 4.1.3 and the discussions in front of Proposition 4.2.1 it is not hard to see that the differential of π_λ

$$\begin{array}{ccccc} T_{(X,P,\nabla_\lambda)}\lambda\mathscr{C}onn & \cong & & & \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}_\lambda]} adP \otimes \Omega^1_X) \\ \downarrow \pi_{\lambda *} & & & & \\ T_X\mathscr{M}_g & \cong & H^1(X,TX) & \cong & \mathbb{H}^1(X, TX \to 0) \end{array}$$

is induced from the map $(p_*, 0)$ of complexes of sheaves

$$\begin{array}{ccc} (A_P & \xrightarrow{[\ ,\hat{s}_\lambda]} & adP \otimes \Omega^1_X) \\ \Big\downarrow{\scriptstyle p_*} & & \Big\downarrow{\scriptstyle 0} \\ (TX & \longrightarrow & 0) \end{array}$$

The lifting of tangent vectors induced from the isomonodromy flow is a splitting of the map π_{1*}

$$\begin{array}{ccc} T_{(X,P,\nabla)}\mathscr{C}onn & \cong & \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X) \\ {\scriptstyle \pi_{1*}}\Big\downarrow \ \Big\uparrow & & \\ T_X\mathscr{M}_g & \cong & \mathbb{H}^1(X, TX \to 0) \end{array}$$

Notice that the splitting map $s : TX \to A_P$ associated to ∇ gives a map of the complexes

$$\begin{array}{ccc} (A_P & \xrightarrow{[\ ,\hat{s}]} & adP \otimes \Omega^1_X) \\ {\scriptstyle p_*}\Big\downarrow \ \Big\uparrow{\scriptstyle s} & & {\scriptstyle 0}\Big\downarrow \ \Big\uparrow{\scriptstyle 0} \\ (TX & \longrightarrow & 0) \end{array}$$

The diagram is commutative because $[\ ,\hat{s}] \circ s$ is basically bracketing $\hat{s}$ with itself and therefore equal to 0. The map of complexes $(s, 0)$ is obviously a splitting of the map $(p_*, 0)$.

The map $(s, 0)$ of the complexes of sheaves induce a map on the first hypercohomology, which we denote as $H^1(s)$.

Proposition 5.2.1. *The isomonodromy lifting L is equal to*

$$H^1(s) : H^1(X, TX) \longrightarrow \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X)$$

Proof. At a point (X,P,∇) of $\mathscr{C}onn$, let $(X_\epsilon,P_\epsilon,\nabla_\epsilon)$ be an isomonodromy family of triples over D_ϵ starting with it. Again let $\{U_i\}_{i\in I}$ be an Čech covering of X.

Over U_i, Let

$$\tau_{i,\epsilon} : X_\epsilon|_{U_i} \times F \to P_\epsilon|_{U_i}$$

be the trivialization of $P_\epsilon|_{U_i}$ over $X_\epsilon|_{U_i}$ determined by the flat universal connection (see section 5.1) on $P_\epsilon|_{U_i}$, and τ_i be its restriction at $\epsilon = 0$.

Let

$$\iota_i : U_i \times D_\epsilon \to X_\epsilon|_{U_i}$$

be an isomorphism and define

$$\phi_i : P|_{U_i} \times D_\epsilon \to P_\epsilon|_{U_i}$$

as the composition

$$P|_{U_i} \times D_\epsilon \xrightarrow{(\tau_i^{-1}, id_{D_\epsilon})} U_i \times D_\epsilon \times F \xrightarrow{(\iota_i, id_F)} X_\epsilon|_{U_i} \times F \xrightarrow{\tau_{i,\epsilon}} P_\epsilon|_{U_i}$$

Let ξ_{ij}, η_{ij}, s_ϵ, s_i and σ_i be all defined as before in the proofs of proposition 4.1.3 and section 4.2. Notice that since the local trivializations of the G-bundles are canonically given by the flat universal connection, $\tau_{i,\epsilon}$ and $\tau_{j,\epsilon}$ agree on U_{ij}, i.e. on U_{ij}

$$\tau_{i,\epsilon} = \tau_{j,\epsilon}$$

$$\tau_i = \tau_j$$

Therefore over U_{ij}, the transition map $\phi_j^{-1} \circ \phi_i$ fits in the diagram

$$\begin{array}{ccc} U_{ij} \times F \times D_\epsilon & \xrightarrow{(\iota_j^{-1} \circ \iota_i, id_F)} & U_{ij} \times F \times D_\epsilon \\ \cong \Big\uparrow (\tau_i^{-1}, id_{D_\epsilon}) & & \cong \Big\uparrow (\tau_j^{-1}, id_{D_\epsilon}) \\ P|_{U_{ij}} \times D_\epsilon & \xrightarrow{\phi_j^{-1} \circ \phi_i} & P|_{U_{ij}} \times D_\epsilon \end{array}$$

In another word with the local trivializations $(\tau_i^{-1}, id_{D_\epsilon})$ and $(\tau_j^{-1}, id_{D_\epsilon})$, the transition map $\phi_j^{-1} \circ \phi_i$ corresponds to $(\iota_j^{-1} \circ \iota_i, id_F)$. Let $(\phi_j^{-1} \circ \phi_i)'$ and η'_{ij} be $(\phi_j^{-1} \circ \phi_i)$ and η_{ij} under the local trivializations, then

$$(\phi_j^{-1} \circ \phi_i)' = (\iota_j^{-1} \circ \iota_i, id_F)$$

and therefore

$$Id + \epsilon \eta'_{ij} = (Id + \epsilon \xi_{ij}, Id_F)$$

Comparing the coefficients of ϵ we get

$$\eta'_{ij} = (\xi_{ij}, 0)$$

According to the last paragraph in section 3.3, we see this means precisely that $\eta_{ij} = s(\xi_{ij})$.

With $\phi_i : P|_{U_i} \times D_\epsilon \to P_\epsilon|_{U_i}$ defined as above, $s_\epsilon|_{U_i} : TX_\epsilon|_{U_i} \to A_{P_\epsilon}|_{U_i}$ correspond to the section $s_i : TU_i \times D_\epsilon \to A_P|_{U_i} \times D_\epsilon$ constant along D_ϵ, i.e. $\sigma_i = 0$.

Therefore $\hat{\sigma}_i = 0$, and the pair

$$(\eta_{ij})_{i,j \in I}, (\hat{\sigma}_i)_{i \in I}$$

is exactly the hyper 1-cocycle which is the image of $(\xi_{ij}, 0)$ under the map $H^1(s)$, which finishes the proof. □

6. Extended isomonodromy lifting

The associated lifting L_λ is obtained by extending the isomonodromy lifting L to $\lambda\mathscr{C}onn \to \mathscr{M}_g$ by the $\mathbb{C}^*$-action, and multiplying by λ. For a fixed λ, $\lambda \neq 0$, the $\mathbb{C}^*$-action gives an isomorphism

$$\mathscr{C}onn \leftrightarrow \lambda\mathscr{C}onn$$

$$\nabla \leftrightarrow \lambda \cdot \nabla$$

The induced lifting on $\lambda\mathscr{C}onn \to \mathscr{M}_g$ by L via the isomorphism, called the extended isomonodromy lifting, can be written very similarly as L. In the same way that the splitting map s associated to a connection ∇ induces a map $H^1(s)$ of hypercohomologies, the λ-splitting map s_λ associated to a λ-connection ∇_λ induces a map of the corresponding hypercohomology spaces, which will be denoted as $H^1(s_\lambda)$.

Proposition 6.1. *The extended isomonodromy lifting of tangent vector on $\lambda\mathscr{C}onn \to \mathscr{M}_g$ is given by:*

$$\frac{1}{\lambda}H^1(s_\lambda) : H^1(X, TX) \longrightarrow \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}_\lambda]} adP \otimes \Omega^1_X)$$

Proof. Since the map of moduli spaces is $\nabla \mapsto \lambda \cdot \nabla$ (or $s \mapsto \lambda s$, $\hat{s} \mapsto \lambda\hat{s}$), the induced map on the tangent spaces $T_{(X,P,\nabla)}\mathscr{C}onn \to T_{(X,P,\lambda\nabla)}\lambda\mathscr{C}onn$ is

$$\mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X) \xrightarrow{H^1(id,\lambda)} \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\lambda\hat{s}]} adP \otimes \Omega^1_X)$$

where (id, λ) is the map of complexes of sheaves

$$\begin{array}{ccc}
(A_P & \xrightarrow{[\ ,\hat{s}]} & adP \otimes \Omega^1_X) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \lambda} \\
(A_P & \xrightarrow{[\ ,\lambda\hat{s}]} & adP \otimes \Omega^1_X)
\end{array}$$

and $H^1(id, \lambda)$ is the induced map on hypercohomology.

So to get the corresponding lifting on $\lambda\mathscr{C}onn$, i.e. to make the following diagram commutate, the vertical map on the right must be $\frac{1}{\lambda}H^1(\lambda s)$.

$$\begin{array}{ccc}
\mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}]} adP \otimes \Omega^1_X) & \xrightarrow{H^1(id,\lambda)} & \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\lambda\hat{s}]} adP \otimes \Omega^1_X) \\
{\scriptstyle H^1(s)}\uparrow & & \uparrow{\scriptstyle \frac{1}{\lambda}H^1(\lambda s)} \\
H^1(X, TX) & \xrightarrow{id} & H^1(X, TX)
\end{array}$$

□

Since L_λ is the extended isomonodromy lifting multiplied by λ, $L_\lambda = H^1(s_\lambda)$. L_λ is a λ-lifting of tangent vectors.

7. LIMIT LIFTING AT $\lambda = 0$

The continuous limit of L_λ at $\lambda = 0$ is a 0-lifting $L_0 : T_X\mathscr{M}_g \to T_{(X,P,\nabla_0)}\mathscr{H}iggs$. Since $L_\lambda = H^1(s_\lambda)$, by continuity L_0 is equal to

$$H^1(s_0) : H^1(X,TX) \longrightarrow \mathbb{H}^1(X, A_P \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X)$$

where s_0 is the 0-splitting of the Atiyah bundle of P associated to the 0-connection(or Higgs field) ∇_0 on P. Because $\pi_{0*} \circ H^1(s_0) = 0$, so in fact $H^1(s_0)$ can be written as

$$H^1(s_0) : H^1(X,TX) \longrightarrow \mathbb{H}^1(X, adP \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X)$$

The images of a vector in $\vec{t} \in T_X\mathscr{M}_g$ under $H^1(s_0)$ form a vector field on the fiber $Higgs_X$ of π_0.

Recall that the quadratic Hitchin map on $Higgs_X$ is

$$qh : Higgs_X \to H^0(X, \Omega^{\otimes 2})$$

$$(P, s_0) \mapsto < \hat{s}_0, \hat{s}_0 >$$

and its associated lifting of tangent vectors is

$$L_{qh} : H^1(X,TX) \to \mathbb{H}^1(X, adP \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X)$$

$$f \mapsto H_{qh^* f}|_{(P,s_0)}$$

The main theorem (Theorem 2.1) is that $H^1(s_0)$ is equal to $\frac{1}{2}L_{qh}$. To prove it we need two lemmas. For the first lemma, Let $((\eta_{ij})_{i,j\in I}, (\hat{\sigma}_i)_{i\in I})$ be a representative of an arbitrary element $v \in \mathbb{H}^1(X, adP \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X)$. Because on U_{ij}, $< \hat{s}_0, \hat{\sigma}_j - \hat{\sigma}_i > = < \hat{s}_0, [\eta_{ij}, \hat{s}_0] > = - < [\hat{s}_0, \hat{s}_0], \eta_{ij} > = 0$, therefore

$$< \hat{s}_0, \hat{\sigma}_i > = < \hat{s}_0, \hat{\sigma}_j >$$

Let $< \hat{s}_0, \hat{\sigma} > \in H^0(X, \Omega^{\otimes 2})$ denote the resulting global quadratic differential form.

Lemma 7.1. *Using the above notations, the differential of the map qh is equal to:*

$$qh_* : \mathbb{H}^1(X, adP \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X) \to H^0(X, \Omega^{\otimes 2})$$

$$v \mapsto 2 < \hat{s}_0, \hat{\sigma} >$$

Proof. Let $\{U_i\}_{i\in I}$ be the Čech covering of the curve X, (P_ϵ, s_ϵ) the family of Higgs bundles over D_ϵ that correspond to v, i.e. for some $\phi_i : P|_{U_i} \times D_\epsilon \to P_\epsilon|_{U_i}$, some $s_i : TU_i \to adP|_{U_i}$ and the given $\sigma_i : TU_i \to adP|_{U_i}$, the diagram

$$\begin{array}{ccc} adP|_{U_i} \times D_\epsilon & \xrightarrow{d\phi_i} & adP_\epsilon|_{U_i} \\ s_i + \epsilon\sigma_i \, \Big\uparrow \, \Big\downarrow \, p_* & & p_* \, \Big\downarrow \, \Big\uparrow \, s_\epsilon|_{U_i} \\ TU_i \times D_\epsilon & \xrightarrow{id} & TU_i \times D_\epsilon \end{array}$$

is commutative. Because $qh : (P_\epsilon, s_\epsilon) \mapsto < \hat{s}_\epsilon, \hat{s}_\epsilon >$, and that over U_i, $< \hat{s}_\epsilon, \hat{s}_\epsilon > = < \hat{s}_i + \epsilon\hat{\sigma}_i, \hat{s}_i + \epsilon\hat{\sigma}_i > = < \hat{s}_i, \hat{s}_i > + 2 < \hat{s}_i, \hat{\sigma}_i > \epsilon$, so $qh : (P_\epsilon, s_\epsilon) \mapsto < \hat{s}_0, \hat{s}_0 > + 2 < \hat{s}_0, \hat{\sigma} > \epsilon$. Taking the coefficient of ϵ, we see that qh_* maps v to $2 < \hat{s}_0, \hat{\sigma} >$. □

For the second lemma, let ω_H be the symplectic 2-form on $Higgs_X$, $((\eta_{ij})_{i,j\in I}, (\hat{\sigma}_i)_{i\in I})$ and $((\eta'_{ij})_{i,j\in I}, (\hat{\sigma}'_i)_{i\in I})$ representatives of two vectors $v, v' \in \mathbb{H}^1(X, adP \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X)$.

Lemma 7.2. *Let* $\int : H^1(X, \Omega^1_X) \to \mathbb{C}$ *be the canonical map, then*

$$\omega_H(v, v') = \int (\eta_{ij} \sqcup \hat{\sigma}'_i + \eta'_{ij} \sqcup \hat{\sigma}_i)$$

where $\sqcup$ *means the cup product* $\cup$ *of Čech cochains composed with the Killing form* $< \ , \ >$*.*

Proof. see [6] Proposition 7.12. □

Theorem. $H^1(s_0)$ *is equal to* $\frac{1}{2}L_{qh}$.

Proof. $\forall f \in H^1(X, TX)$, we want to show that $L_{qh}(f) = 2H^1(s_0)(f)$. Let $((\eta'_{ij})_{i,j\in I}, (\hat{\sigma}'_i)_{i\in I})$ be a representative of an element $v \in \mathbb{H}^1(X, adP \xrightarrow{[\ ,\hat{s}_0]} adP \otimes \Omega^1_X)$. Using Lemma 7.1,

$$\omega_H(L_{qh}(f), v) = d(qh^* f)(v) = df(qh_* v) = df(2 < \hat{s}_0, \sigma >) = f(2 < \hat{s}_0, \sigma >)$$

Using Lemma 7.2,

$$\omega_H(H^1(s_0)(f), v) = \omega_H((s_0(f), 0), (\eta_{ij}, \sigma_i)) = f(< \hat{s}_0, \sigma >)$$

So $L_{qh}(f) = 2H^1(s_0)(f)$, $\forall f \in H^1(X, TX)$. Therefore $H^1(s_0) = \frac{1}{2}L_{qh}$. □

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