A GENERALIZATION OF CONNES-KREIMER HOPF ALGEBRA

JUNGYOON BYUN

Abstract. "Bonsai" Hopf algebras, introduced here, are generalizations of Connes-Kreimer Hopf algebras, which are motivated by Feynman diagrams and renormalization. We show that we can find operad structure on the set of bonsais. We introduce a new differential on these bonsai Hopf algebras, which is inspired by the tree differential. The cohomologies of these are computed here, and the relationship of this differential with the appending operation $*$ of Connes-Kreimer Hopf algebras is investigated.

1. Motivation

In [Kr], Kreimer discovered a Hopf algebra structure on Feynman diagrams and the forest formula of perturbative quantum field theory. In [CK], Connes and Kreimer suggested the representation of Feynman graphs using rooted tree diagrams and represented the Hopf algebra structure with the notion of 'cuts' of tree diagrams. That expression is as following: let us consider a Feynman diagram in ϕ^3 theory as in Figure 1;

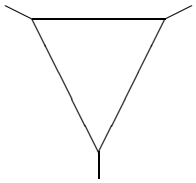


Figure 1

This is a 1-loop graph. Now let us look at another loop having subloops in Figure 2;

In Kreimer's expression of a Feynman diagram using decorated rooted trees ([CK]), if the loop of Figure 1 is labeled 1 (In Kreimer's context, this label indicates a specific shape of loop. So, later in this paper, if every loop in Feynman diagram has the same shape, we do not need this label.), the loop of Figure 2 is expressed as in Figure 3. In Figure 2, the loops labeled 2 are immediate subloops of loop 1, and the loop 3 is an immediate subloop of the lower loop 2 and not of loop 1.

In Connes and Kreimer's context, we call a connected rooted tree, which corresponds to a connected Feynman graph, a *tree* and we call a diagram of trees having more than one connected component a *forest*.

The Connes-Kreimer Hopf Algebra $\mathcal{H}_K$ is a Hopf algebra with forests of rooted trees as basis elements (See section 3 for details).

In Figure 2, the author observed that the biggest loop cannot include more than 3 immediate subloops of the shape of Figure 1. Hence, in the tree diagram, the

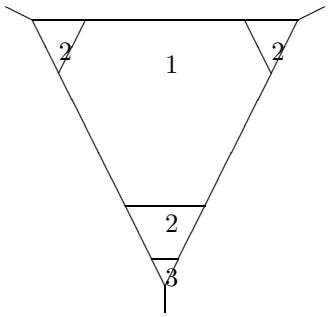


FIGURE 2

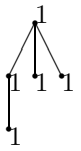


FIGURE 3

vertex labeled 1 cannot have more than 3 subsidiary vertices labeled 1, and so the rooted tree of Figure 3 cannot have a ramification number (or arity, branch number) greater than 3 at the root.

So, in the ϕ^3 theory in which the only allowed loop is that of Figure 1, the corresponding tree diagrams are forbidden to have ramification number greater than 3. The theory of such ramification number bounded trees is our main interest in this paper. We will call them *bonsais*.

For a more precise description of Feynman diagrams, let us consider the positions of subloops in a loop. For the loops having subloops like Figure 2, in the context of [CK2], sometimes we need to indicate which subloop is shrinking and what position is available for a subloop. For that, we label each corner of the loop in Figure 4 and change that loop into a tree as shown in Figure 4, by expressing a subloop as a subsidiary vertex in the tree diagram and attaching the labels representing the subloop positions to the edges.

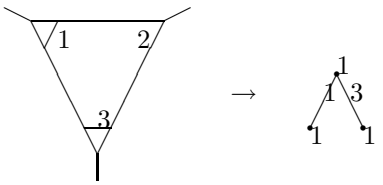


FIGURE 4

The tree diagram of Figure 4 asssigns the numbers of the occupied corners in the big loop to edges of the tree. Note that there is no edge numbered 2. This means there is no subloop on the corner 2. We easily see that, in this expression, the left tree of Figure 5 is allowed but the right tree of Figure 5 is not.

FIGURE 5

2. Main results

Definition 2.1. *We define a new Hopf algebra which has the same operations as in the Connes-Kreimer Hopf Algebra, and whose basis elements are forests of* trees having ramification numbers at each vertex smaller than or equal to m and under each vertex v, each subsidiary edge of v has labels from 1,2,...,m without duplication. *We call this Hopf algebra the* m-bonsai Hopf algebra $\mathcal{H}_{b,m}$. *In* $\mathcal{H}_{b,m}$, *each tree is called a* m-bonsai.

As in [CK], we can show that

Theorem 2.1. $\mathcal{H}_{b,m}$ *is a Hopf algebra.*

As in [CK], when we define an *appending operation*

$$T * T' = \Sigma(\text{a bonsai obtained by connecting the root of } T \text{ to a vertex } v \text{ of } T' \text{ with one edge, where the added edge has every possible label}),$$

(An example of the $*$ operation is in Figure 11)

Theorem 2.2. *The operation* $*$ *is pre-Lie, and we have* $\mathcal{H}_{b,m} = \mathcal{U}(\mathcal{L})^\vee$, *where* $V^\vee$ *is the dual of* V.

In the m-bonsai Hopf algebra, the set of m-bonsais has a structure of an operad, thus there is a natural analog of the tree differential (as in [MSS]). We call it the *vertex-appending differential* ∂ (Definition 9.1).

Then, mainly using the Künneth theorem, we can calculate the cohomology groups of ∂ as:

Theorem 2.3. *In* m*-bonsai,*

$$H^i(\mathcal{H}_{b,m}, \partial) = \begin{cases} k^{\frac{(mn)!}{((m-1)n+1)!n!}} & \text{if } i = (2m-1)n+1, n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\frac{(mn)!}{((m-1)n+1)!n!}$ is the number of rooted trees consisting of n of m-corollas, which is called the 'm-Catalan number'. When $m = 2$, this number is just the Catalan number. Representatives of H^i are Σ(a bonsai obtained by appending edges to all tips of a rooted tree every vertex of which except tips has ramification number m, one edge to each tip, with every possible label).

When we define $T_1 *_1 T_2$, which is the deviation from ∂ being a derivation of $*$ as

$$T_1 *_1 T_2 = (\partial T_1) * T_2 + T_1 * (\partial T_2) - \partial(T_1 * T_2),$$

we have

Theorem 2.4. *With coefficients mod 2,*

$$\begin{aligned} T_1 *_1 T_2 &= \Sigma(\textit{a bonsai obtained by connecting a tip } v \textit{ of } T_2 \\ &\quad \textit{and the root of } T_1 \textit{ with one-edge, and attaching} \\ &\quad \textit{an edge to } v\textit{, added edge have every admissible label}) \\ &+ \Sigma(\textit{a bonsai obtained by connecting a non-tip of } T_2 \\ &\quad \textit{and the root of } T_1 \textit{ with two-edge ladder, having} \\ &\quad \textit{every possible label}) \end{aligned}$$

and

$$\partial(T_1 *_1 T_2) = (\partial T_1) *_1 T_2 + T_1 *_1 (\partial T_2).$$

We have an example in Figure 6.

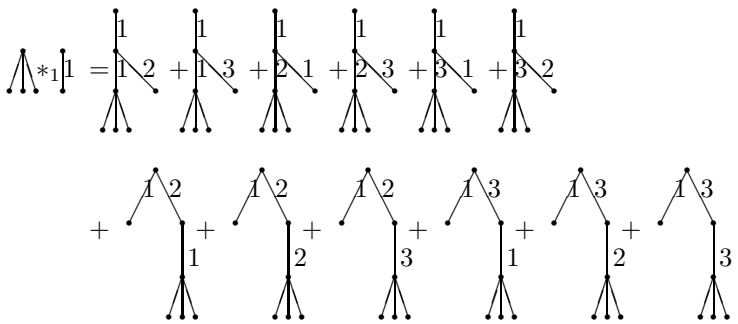


FIGURE 6

Now, let us consider another Hopf algebra, having the same operations but the trees having ramification numbers at each vertex smaller than or equal to m but no edge labels, and call it *clear-edged m-bonsai Hopf algebra* $\mathcal{H}_{c,m}$. (In other words, a clear-edged m-bonsai is an m-bonsai without edge labels.)

Clear-edged m-bonsai Hopf algebras still represent Feynman graphs, actually more physically relevant, and also appear in the tree diagrams of "open-closed homotopy algebra(OCHA)"([KS]).

Then we can define the *vertex-appending differential* similarly to the case of m-bonsai. For example, in planar clear-edged 3-bonsai, we can get an example like Figure 7.

The cohomology groups of the vertex-appending differential in clear-edged bonsai are not as easy to calculate as in m-bonsai and we have just partial results as follows:

We first define a specific form of bonsai S called "seedling" (Definition 11.3), and then we define the complexes $(\mathbf{C}^{S,*}, \partial)$ Then we can show that the cohomology of the whole bonsai complex $H^i = \bigoplus H^i(S)$, where the sum is over all seedlings.

By the definition of seedling, when S_1, S_2,..., S_n are seedlings, the new bonsai S obtained by appending the roots of each S_i's to a single new root is a seedling again. There is an example in Figure 8

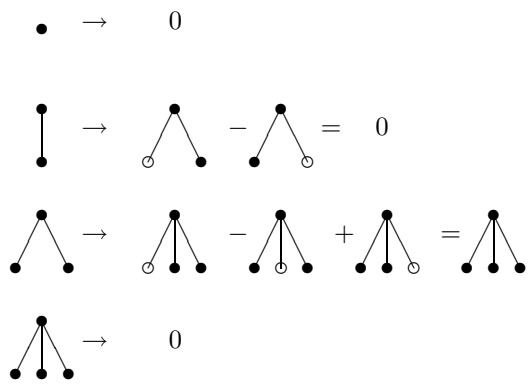


FIGURE 7

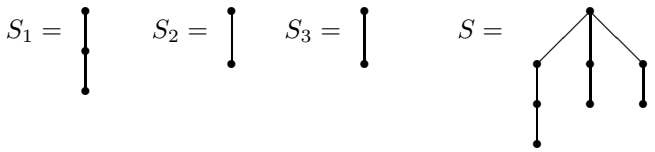


FIGURE 8

On the way to find the relationship of $H(S)$ and $H(S_1)$,...,$H(S_n)$, we have a new definition of a bonsai called *grafting seedlinggs*$(n; T_1, ..., T_{n+1}; S_1, ..., S_n)$ (Definition 11.6), a complex $\{\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))\}$ (Definition 11.8) and

Theorem 2.5. *When $H^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))$ is the i-th cohomology group of the complex $\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))$, the i-th cohomology group H^i of clear-edged m-bonsai is $H^i = \bigoplus_{S \text{ is a grafting seedling}} H^i(S)$, and*

$$H^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n)) = \bigoplus_{j_1+...+j_n=i-m} [H^{i_1}(S_1) \otimes ... \otimes H^{i_n}(S_n)]^{\oplus N}$$

where N is combinatorially all determined and

$$P = degT_1 + ... + degT_n.$$

Finally, as in the case of m-bonsai, we have again

Theorem 2.6. *For any clear-edged m-bonsai T_1 and T_2, $\partial(T_1 *_1 T_2) = (\partial T_1) *_1 T_2 + T_1 *_1 \partial T_2$ with coefficients mod 2.*

3. Bonsai Hopf Algebra

As seen in the last section, loops in Feynman diagrams of a specific theory have a maximum number of immediate subloops. In the example of the last section, the maximum number is 3 and each edge of the tree diagram corresponding to a Feynman diagram has label 1, 2 or 3.

From this motivation, we define

Definition 3.1. *A* simple cut *of rooted tree is a cut of edges such that at any vertex of T, the path between it and the root has at most one cut, $P_c(T)$ is the part of T cut off by c and $R_c(T)$ is the part of T remaining after cut c.*

Definition 3.2. *Let $\mathcal{H}_{b,m}$ be the vector space having as its basis the forests consisting of trees whose vertices have ramification numbers $\leq m$ and whose edges are labeled by numbers in $1, 2, ..., m$.*

We equip this $\mathcal{H}_{b,m}$ with operations, as in [CK],

(1) *(multiplication)* $m(T_1T_2...T_m, S_1S_2...S_n) = T_1...T_mS_1...S_n$

(T_i, S_j are trees, m is commutative)

(2) *(diagonal)* $\Delta(T) = T \otimes 1 + \sum_c P_c(T) \otimes R_c(T)$ *(T is a tree)*

$\Delta(T_1...T_n) = \Delta(T_1)...\Delta(T_n)$

(3) *(antipode)* $S(T) = -\sum_c S(P_c(T))R_c(T)$ *(T is a tree)*

$S(1) = 1, S(v) = -v,$ *(where v is the one-vertex bonsai)*

$S(T_1...T_n) = S(T_n)...S(T_1)$

where c runs over simple cuts of T including $c = \emptyset$, and a counit function

(4) $\epsilon : \mathcal{H}_{b,m} \to \mathcal{H}_{b,m}$ *such that* $\epsilon(1) = 1$ *and* $\epsilon(f) = 0$ *if a forest* $f \neq 1$.

We call the rooted tree T an m-bonsai *and $\mathcal{H}_{b,m}$ the* m-bonsai Hopf algebra.

It will be proved in the next section that this vector space $\mathcal{H}_{b,m}$ is actually a Hopf algebra.

Definition 3.3. *Sometimes we will ignore the positions of subloops in Feynman graphs and use trees without labels on edges. Then the trees in the forests corresponding to the Feynman graphs have no label on their edges. In this case, we denote the vector space having a basis consisting of forests of planar trees as $\mathcal{H}_{c,m}$, where m is the maximum of ramification number of each vertex in the trees of the forests in $\mathcal{H}_{c,m}$. We equip $\mathcal{H}_{c,m}$ with operations* (1)-(4) *in Definition 3.2. Then we call that Hopf algebra* planar clear-edged m-bonsai Hopf algebra.

4. Basic Results Related to Hopf Algebras

In order to develop a basic theorem related to Lie algebras, let us adapt [CK] to our bonsai language and get some basic results.

In order to prove that our Hopf algebras are actually Hopf algebras and derive some algebraic results, let us give another expression of bonsai Hopf algebras and their elements.

First we give

Definition 4.1. *For a bonsai T, $deg(T)$ is the number of vertices of T.*

For each p, we let Σ_p be the set of bonsai T such that $deg(T) \leq p$ with the restriction of ramification numbers by m, and let $\mathcal{H}_p$ be the polynomial commutative algebra generated by the symbols,

$$\delta_T, \qquad T \in \Sigma_p. \tag{5}$$

We define a coproduct on $\mathcal{H}_p$ by,

$$\Delta\delta_T = \delta_T \otimes 1 + \sum_c \Big(\prod_{P_c(T)} \delta_{T_i} \Big) \otimes \delta_{R_c(T)}, \tag{6}$$

where the last sum is over all simple cuts including $c = \emptyset$, while the product $\prod_{P_c(T)}$ is over the cut branches. Sometimes $\prod_{P_c(T)} \delta_{T_i}$ is written $\delta_{P_c(T)}$. The antipodal map S is given as

$$S(1) = 1 \tag{7}$$

$$S(\delta_T) = -\delta_T - \sum_{\text{simple cuts } c \neq \emptyset \text{ of } T} S(\delta_{P_c(T)})\delta_{R_c(T)}. \tag{8}$$

We let $\mathcal{H}_{b,m} = \bigcup \mathcal{H}_p$ and extend the maps on $\mathcal{H}_p$ to $\mathcal{H}_{b,m}$.
Coassociativity of Δ and $m((S \otimes id)\Delta) = \epsilon$ can be shown just by introducing the notion of double cuts of T. But in order to emphasize the algebraic aspect of the new definition, let us give another proof of the following theorem.

Theorem 4.1. Δ *is coassociative.*

Proof. It is enough to check

$$(id \otimes \Delta)\Delta\delta_T = (\Delta \otimes id)\Delta\delta_T \qquad \forall T \in \Sigma_p. \tag{9}$$

where T is a tree in $\mathcal{H}_{b,m}$. Define $L_T : \mathcal{H}_{b,m} \to \mathcal{H}_{b,m}$ as follows; Let T'_1,...,T'_n be the subsidiary branches of the root of T in T. Let T_{n_i} be a subtree of T'_{n_i} whose root is the root of T'_{n_i}. Define T' to be the tree obtained by appending T_{n_i} 's to a new root $*$ and the edge connecting $*$ and the root of T_{n_i} is labeled the same as the edge connecting the root of T and the root of T'_{n_i}. Then $L_T(\delta_{T_{n_1}}...\delta_{T_{n_p}}) = \delta_{T'}$. If some T_j is not a subsidiary branch of the root in T, $L_T(\delta_{T_1}...\delta_{T_n})$ is 0. (An example of this notation is in Figure 9.)

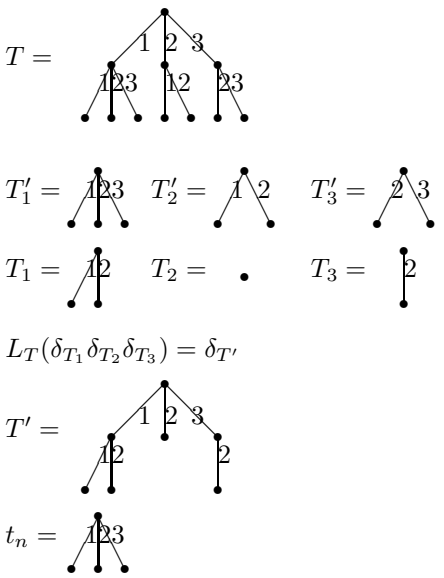


FIGURE 9

First let us show that

$$\Delta \circ L_T(a) = L_T(a) \otimes 1 + (id \otimes L_T) \circ \Delta(a) \tag{10}$$

where $a = \delta_{T_1}\delta_{T_2}...\delta_{T_n}$ and $T_1, ..., T_n$ are all subsidiary branches of the root of T in T so that $L_T(a) = \delta_T$. From (6), we get

$$\Delta(L_T(a)) - L_T(a) \otimes 1 = \sum_c (\prod_{P_c} \delta_{T'_i}) \otimes \delta_{R_c}, \tag{11}$$

where all simple cuts of T (including $c = \emptyset$) are allowed. Moreover,

$$\Delta(a) = \prod_{i=1}^{n} (\delta_{T_i} \otimes 1 + \sum_{c_i} (\prod_{P_{c_i}} \delta_{T''_{i_j}}) \otimes \delta_{R_{c_i}}), \tag{12}$$

where again all simple cuts c_i of T_i are allowed.

Let t_n be the corolla with root $*$ and n other vertices v_i labeled by $i_1, ..., i_n$, where i_j is the label of the edge in T connecting the root of T and the vertex of T'_j, all directly connected to the root $*$, as in Figure 10.

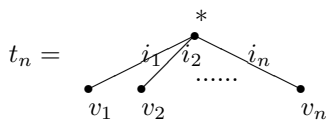


FIGURE 10

We view t_n in an obvious way as a subgraph of the tree T, where $*$ is the root of T and the vertex v_i is the root of T_i, i.e., we can get T by attaching the root of T_i to the vertex v_i of t_n. Given a simple cut c of T we get by restriction to the corolla subgraph $t_n \subset T$ a cut of t_n. It is characterized by the subset $I = \{i | (*, v_i) \in c\} \subset \{1, ..., m\}$. The simple cut c is uniquely determined by the restriction c_i of c to each subtree T'_i. Thus the simple cuts c_i of T are in one to one correspondence with the various terms of the expression (12), namely the $\prod_{k \in I}(\delta_{T_k} \otimes 1) \prod_{i \in \{1,...,m\} - I} \prod_{P_{c_i}} \delta_{T''_{i_j}} \otimes \delta_{R_{c_i}}$. So, applying $id \otimes L$ to (12) and comparing with (11), we get (10).

Now let us show (10) by induction. We have,

$$\Delta\delta_\bullet = \delta_\bullet \otimes 1 + 1 \otimes \delta_\bullet \tag{13}$$

where $\bullet$ is the one-vertex bonsai, so that $\mathcal{H}_1$ is coassociative. Let us assume that $\mathcal{H}_n$ is coassociative and prove it for $\mathcal{H}_{n+1}$. It is enough to check (9) for the generators δ_T, with $deg(T) \leq n + 1$. We have $\delta_T = L_T(\delta_{T_1}\delta_{T_2}...\delta_{T_m}) = L_T(a)$ where the degrees of all T_j are $\leq n$, i.e. $a \in \mathcal{H}_n$. Using (10) we can replace $\Delta\delta_T$ by

$$L_T(a) \otimes 1 + (id \otimes L_T)\Delta(a) \tag{14}$$

where Δ is the coassociative coproduct in $\mathcal{H}_n$.

The first term of (9) is then:

$$\begin{aligned} &(id \otimes \Delta)(L_T(a) \otimes 1 + (id \otimes L_T)\Delta(a)) \\ &= L_T(a) \otimes 1 \otimes 1 + \Sigma a_{(1)} \otimes \Delta \circ L_T a_{(2)} \end{aligned} \tag{15}$$

where $\Delta(a) = \Sigma a_{(1)} \otimes a_{(2)}$, which by (10) gives

$$L_T(a) \otimes 1 \otimes 1 + \Sigma a_{(1)} \otimes L_T a_{(2)} \otimes 1 + \Sigma a'_{(1)} \otimes a'_{(2)} \otimes L_T a'_{(3)} \tag{16}$$

where

$$(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) = \Sigma a'_{(1)} \otimes a'_{(2)} \otimes a'_{(3)}. \tag{17}$$

by induction hypothesis on n, since $a \in \mathcal{H}_n$.

The second term of (9) is $\Delta \circ L(a) \otimes 1 + \Sigma \Delta a_{(1)} \otimes L a_{(2)}$, which by (10) gives,

$$L(a) \otimes 1 \otimes 1 + \Sigma a_{(1)} \otimes L a_{(2)} \otimes 1 + \Sigma a'_{(1)} \otimes a'_{(2)} \otimes L a'_{(3)}. \tag{18}$$

Thus we conclude that Δ is coassociative. □

Theorem 4.2. *$m((S \otimes id)\Delta) = \epsilon$.*

Proof. We have $m((S \otimes id)\Delta)(1) = m(S \otimes id)(1 \otimes 1) = S(1)1 = 1 = \epsilon(1)$. And when $\delta_T \neq 1$,

$$\begin{aligned} m((S \otimes id)\Delta)(\delta_T) = \quad & m((S \otimes id)(\delta_T \otimes 1 + \sum_{\text{simple cuts } c} \delta_{P_c(T)} \otimes \delta_{R_c(T)})) \\ = \quad & S(\delta_T) + m(\sum_{\text{simple cuts } c} S(\delta_{P_c(T)}) \otimes \delta_{R_c(T)}) \\ = \quad & S(\delta_T) + \sum_{\text{simple cuts } c} S(\delta_{P_c(T)})\delta_{R_c(T)} \\ = \quad & 0 \end{aligned} \tag{19}$$

where the last equality is by the definition of the antipodal map S. □

5. Lie Algebra $\mathcal{L}^1$

Let $\mathcal{L}^1 \subset \mathcal{H}_{b,m}^{\vee}$ be the linear space having basis $\{Z_T | T \in \mathcal{H}_{b,m} \text{ is a tree}\}$, where δ_T is defined as

$$\langle Z_T, \delta_T \rangle = 1 \tag{20}$$

and

$$\langle Z_T, P(\delta_{T_i}) \rangle = (\partial / \partial \delta_T P)(0) \tag{21}$$

for each rooted tree T.

We introduce an operation on $\mathcal{L}^1$ by

$$Z_{T_1} * Z_{T_2} = \sum_T n(T_1, T_2; T) Z_T, \tag{22}$$

where the integer $n(T_1, T_2; T)$ is determined as the number of simple cuts c with cardinality $|c| = 1$ of bonsai T such that the cut branch is T_1 while the remaining trunk is T_2.

With a notational abuse such as $T = Z_T$, we have an example of $*$ in Figure 11.

In this section, we will show that $\mathcal{L}^1$ is a Lie algebra and the Hopf algebra $\mathcal{H}_{b,m}$ is the dual of the enveloping algebra of $\mathcal{L}^1$.

Theorem 5.1. *$deg(T)$ defines a grading of the Lie algebra $\mathcal{L}^1$.*

Proof. If we write $Z_{T_1} * Z_{T_2} = \Sigma\ Z_T$,then the number of vertices in T is the sum of numbers of vertices in T_1 and T_2. □

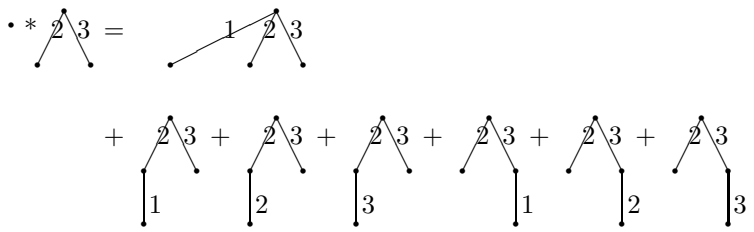


FIGURE 11

Definition 5.1. *We define the bracket* $[Z_{T_1}, Z_{T_2}] = Z_{T_1} * Z_{T_2} - Z_{T_2} * Z_{T_1}$.

Theorem 5.2. *a) The bracket of the previous definition makes* $\mathcal{L}^1$ *a Lie algebra.*

b) The Hopf algebra $\mathcal{H}_{b,m}$ *is the dual of the enveloping algebra of the Lie algebra* $\mathcal{L}^1$.

First we define the associator

$$A(T_1, T_2, T_3) := Z_{T_1} * (Z_{T_2} * Z_{T_3}) - (Z_{T_1} * Z_{T_2}) * Z_{T_3}. \tag{23}$$

and see

Lemma 1. $A(T_1, T_2, T_3) = \Sigma n(T_1, T_2, T_3; T) Z_T$, *where the integer* $n(T_1, T_2, T_3; T)$ *is the number of simple cuts* c *of* T *such that the number of elements* $|c|$ *of* c *is 2 and the two branches cut out from* T_3 *by* c *are* T_1, T_2 *while the remaining trunk* $R_c(T) = T_3$.

Proof. When we evaluate (23) against Z_T we get the coefficient,

$$\sum_{T'} n(T_1, T'; T) n(T_2, T_3; T') - \sum_{T''} n(T_1, T_2; T'') n(T'', T_3; T). \tag{24}$$

The first term corresponds to pairs of cuts, c, c' of T with $|c| = |c'| = 1$ and where c' is a cut of $R_c(T)$. These pairs of cuts fall into two classes either $c \cup c'$ is an admissible cut or it is not. The second sum corresponds to pairs of cuts c_1, c_1' of T such that $|c_1| = |c_1'| = 1$, $R_{c_1}(T) = T_3$ and c_1' is a cut of $P_{c_1}(T)$. In such a case $c_1 \cup c_1'$ is never an admissible cut so the difference (24) amounts to subtracting from the first sum the pairs c, c' such that $c \cup c'$ is not an admissible cut. This gives,

$$A(T_1, T_2, T_3) = \sum_T n(T_1, T_2, T_3; T) Z_T \tag{25}$$

where $n(T_1, T_2, T_3; T)$ is the number of admissible cuts c of T of cardinality 2 such that the two cut branches are T_1 and T_2 and T_3 is the remaining trunk. □

Now for the theorem, we have

Proof. a) By the lemma, it is clear that

$$A(T_1, T_2, T_3) = A(T_2, T_1, T_3). \tag{26}$$

Now compute $[[Z_{T_1}, Z_{T_2}], Z_{T_3}] + [[Z_{T_2}, Z_{T_3}], Z_{T_1}] + [[Z_{T_3}, Z_{T_1}], Z_{T_2}]$. We can write it as a sum of 12 terms,

$$\begin{aligned}(27)\qquad &(T_1 * T_2) * T_3 - (T_2 * T_1) * T_3 - T_3 * (T_1 * T_2) + T_3 * (T_2 * T_1)\\ &+(T_2 * T_3) * T_1 - (T_3 * T_2) * T_1 - T_1 * (T_2 * T_3) + T_1 * (T_3 * T_2)\\ &+(T_3 * T_1) * T_2 - (T_1 * T_3) * T_2 - T_2 * (T_3 * T_1) + T_2 * (T_1 * T_3)\\ \\ &= - A(T_1, T_2, T_3) + A(T_2, T_1, T_3) - A(T_3, T_1, T_2) + A(T_3, T_2, T_1)\\ &\quad - A(T_2, T_3, T_1) + A(T_1, T_3, T_2) = 0\end{aligned}$$

b) Let P and Q be polynomials of δ_T's. By the definition of Z_T, Z_T vanishes when paired with any monomial $\delta_{T_1}^{n_1}...\delta_{T_k}^{n_k}$ except when this monomial is δ_T. Since $P \to P(0)$ is the counit ϵ of $\mathcal{H}_{b,m}$ and since Z_T satisfies

$$\begin{aligned}(28)\qquad \langle Z_T, PQ\rangle &= (\partial/\partial\delta_T PQ)(0)\\ &= (\partial/\partial\delta_T P)(0)Q(0) + P(0)(\partial/\partial\delta_T Q)(0)\\ &= \langle Z_T, P\rangle\epsilon(Q) + \epsilon(P)\langle Z_T, Q\rangle,\end{aligned}$$

it follows that the coproduct of Z_T is,

$$(29)\qquad \Delta Z_T = Z_T \otimes 1 + 1 \otimes Z_T.$$

The product of two elements of $\mathcal{H}_{b,m}^\vee$ is defined by

$$(30)\qquad \langle Z_1 Z_2, P\rangle = \langle Z_1 \otimes Z_2, \Delta P\rangle.$$

Since the commutator of two derivations is still a derivation, the subspace of $\mathcal{H}_{b,m}^\vee$ satisfying (36) is stable under bracket. What remains is to show that

$$(31)\qquad Z_{T_1} Z_{T_2} - Z_{T_2} Z_{T_1} = [Z_{T_1}, Z_{T_2}],$$

where $[Z_{T_1}, Z_{T_2}] = Z_{T_1} * Z_{T_2} - Z_{T_2} * Z_{T_1}$ by definition.
Let $\mathcal{H}_0 = \text{Ker } \epsilon$ be the augmentation ideal of $\mathcal{H}_{b,m}$. By definition of Δ,

$$(32)\qquad \Delta\delta_T = \delta_T \otimes 1 + 1 \otimes \delta_T + R_T$$

where $R_T \in \mathcal{H}_0 \otimes \mathcal{H}_0$. In fact, we have

$$(33)\qquad R_T = \sum_c \delta_{T_c'} \otimes \delta_{T_c}$$

modulo $(\mathcal{H}_0)^2 \otimes \mathcal{H}_0$, where c varies among single cuts of the bonsai tree T, where T_c is the trunk of T that contains the root, and T_c' is the tree which remains. When we compute

$$(34)\qquad \langle Z_{T_1} Z_{T_2}, \delta_T\rangle = \langle Z_{T_1} \otimes Z_{T_2}, \Delta\delta_T\rangle,$$

the only part which contributes comes from R_T and it counts the number of ways of obtaining T from T_1 and T_2, which gives (31).

If a map f satisfies

$$(35)\qquad \langle f, PQ\rangle = \langle f, P\rangle\epsilon(Q) + \epsilon(P)\langle f, Q\rangle,$$

f is determined by $f(\delta_T) = \langle f, \delta_T\rangle$'s and each of them is a scalar. Since $f(\delta_T) = \Sigma_{T'} f(\delta_{T'}) Z_{T'}(\delta_T)$, f has the form $\Sigma f(\delta_T) Z_T$. Hence $\{Z_T\}$ is a basis of the subspace of $\mathcal{H}_{b,m}^\vee$ consisting of the vectors f satisfying (35).

Since every Z_T satisfies (35) by (28) and $f \in \mathcal{H}_{b,m}^{\vee}$ satisfies (35) if and only if f satisfies

$$\Delta f = f \otimes 1 + 1 \otimes f, \tag{36}$$

we have $\mathcal{L}^1 = Prim(\mathcal{H}_{b,m}^{\vee})$ and they are isomorphic as Lie algebras. Since $\mathcal{H}_{b,m}^{\vee}$ is connected and cocommutative, by the Milnor-Moore theorem, $\mathcal{H}_{b,m}^{\vee} = \mathcal{U}(Prim(\mathcal{H}_{b,m}^{\vee})) = \mathcal{U}(\mathcal{L}^1)$ and so $\mathcal{H}_{b,m} = \mathcal{U}(\mathcal{L}^1)^{\vee}$. □

6. Operad of m-Bonsai

Now let us consider operad theory with respect to the m-bonsai Hopf algebra structure. As seen in the last section, for trees $T, T' \in \mathcal{H}_{b,m}$ we can define $T * T'$ and this is a (left) pre-Lie operation. The map $T \mapsto Z_T$ is a pre-Lie isomorphism from the space spanned by trees to $\mathcal{L}^1$. In $\mathcal{H}_{b,m}$, we denote this $\mathcal{L}^1$ as $\mathcal{L}_{b,m}$. We will sometimes allow a notational abuse such as $T = Z_T$ from now on.

Let us start from a rudimentary idea. Every bonsai in $\mathcal{H}_{b,m}$ has a unique form in which for each vertex, its subsidiary edges are arranged so that lower edge-label is on the left of higher edge-label as in Figure 12.

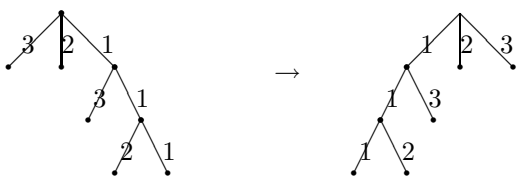


Figure 12

We can number the possible positions in the bonsai of Figure 12 to append other bonsais as in Figure 13, the example in $\mathcal{L}_{b,3}$ (the orders of possible appending positions are underlined). When the labeling of Figure 13 is changed into that of Figure 14, then the numbering of possible appending positions is also changed.

Then, by taking the standard form of bonsai and ordering the possible positions of appending, we can get the transform of a bonsai into the broomstick diagram used in [MSS] like Figure 15, again in $\mathcal{L}_{b,3}$.

So, we can define $T_1 \circ_i T_2$ as appending T_1 to T_2 at the i-th appending position of T_2, and for T_2 and T_1 in Figure 15, $T_1 \circ_4 T_2$ is given as in Figure 16.

Then obviously, this $\circ_i$ satisfies the definition of *pre-Lie system* of [G] (It is called *nonsymmetric pseudo-operad* in [MSS], but it has a difference in the convention of grading). When we use the pseudo-operad later, we will give an extra definition, which we give here;

Definition 6.1. *When $\{V_i\}$ is a graded module over a field k and $\circ_i = \circ_i(m,n) : V_m \otimes V_n \to V_{m+n}$ is an operation satisfying; when f^m, g^n and h^p are in V_m, V_n and V_p respectively,*

$$h^p \circ_j (g^n \circ_i f^m) = \begin{cases} g^n \circ_{i+p} (h^p \circ_j f^m) & \text{if } 0 \le j \le i-1 \\ (h^p \circ_{j-i} g^n) \circ_i f^m & \text{if } i \le j \le n+i \end{cases} \tag{37}$$

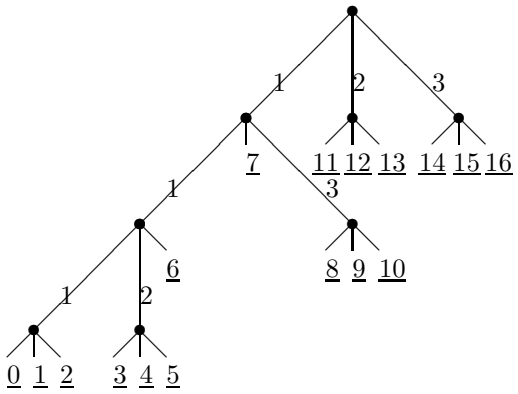


FIGURE 13

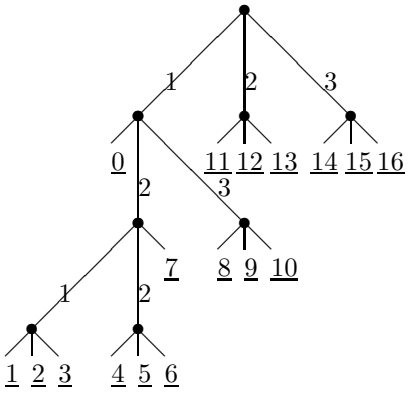


FIGURE 14

then $\{\{V_i\}, \circ_i\}$ *is called a* (left) pre-Lie system.

(In [G] the right pre-Lie system is defined, but we define and use the left pre-lie system. This is mainly intended for the theory related to Hopf algebra we will argue later.) By the broomstick diagrams shown in Figure 15-16, we have

Definition 6.2. *When* $W_{m,n}$ *is the vector subspace of* $\mathcal{L}_{b,m}$ *generated by the trees having the number* n *of possible appending positions, and* $T_1 \circ_i T_2$ *is appending* T_1 *to* T_2 *at the appending position* i *of* T_2*, then* $\{\{W_{m,n}\}, \circ_i\}$ *is a left pre-Lie system. It is called* m-bonsai pre-Lie system. *For trees* T *which are basis elements of* $W_{m,n}$*,* n *is called the appending degree of* T*, and denoted* $deg_{ap}(T)$*.*

(Graphically, a basis element of $W_{m,n}$ has the broomstick representation like Figure 17.)

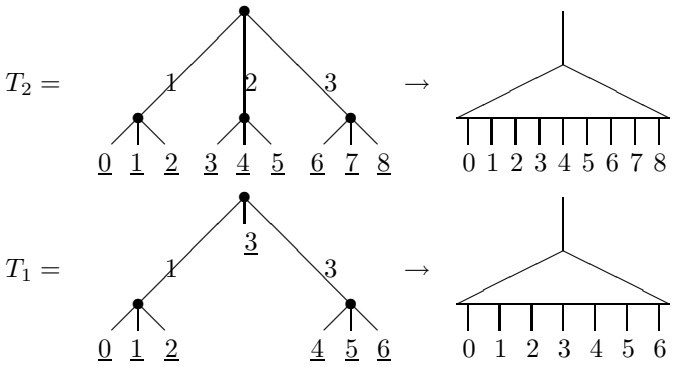


FIGURE 15

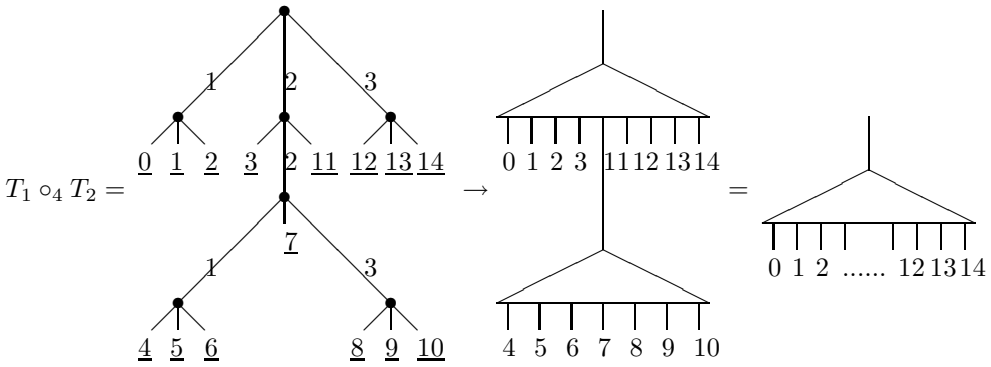


FIGURE 16

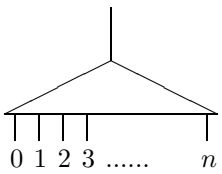


FIGURE 17

7. Branch-fixed Differential

In the next several sections, following the oracle of [MSS], we will define some complexes related to bonsais. To get the analogy of the *cobar complex* and the *tree differential* in Section 3.1 of [MSS], first let us give an order of edges of a bonsai as in

Figure 18, i.e., starting from the root, sweeping around the bonsai counterclockwise and numbering the edges. We call this order the *traversing order*. In the traversing

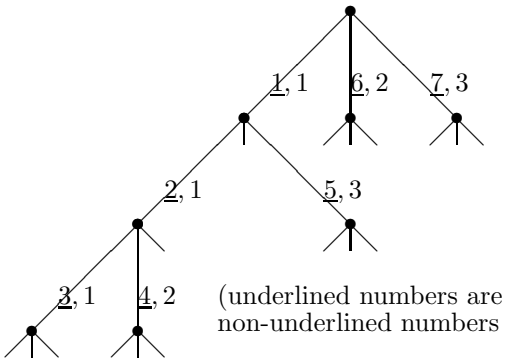


FIGURE 18

order, e_{k,l_k} is a vector representing the k-th edge of a tree T, such that $1 \leq l_k \leq m$ is the edge-label of the k- th edge.

Second, let us define a vector space $\mathbf{C}^n$ having basis $T \otimes e_{1,l_1} \wedge ... \wedge e_{k,l_k}$, where T is a m-bonsai (not forest) having n edges and the pairs k, l_k run over the labels of edges of T. (If T is a vertex, i.e. a connected bonsai without any edge, then $e_{1,l_1} \wedge ... \wedge e_{k,l_k}$ is the constant unit 1.) For later use, we denote this $e_{1,l_1} \wedge ... \wedge e_{k,l_k}$ as $det(T)$ and call it the *determinant term* of T, and call $T \otimes det(T)$ a *determinanted bonsai*. So the basis element corresponding to the bonsai T of Figure 18 is $T \otimes e_{1,1} \wedge e_{2,1} \wedge e_{3,1} \wedge e_{4,2} \wedge e_{5,3} \wedge e_{6,2} \wedge e_{7,3} = T \otimes det(T)$. Let us denote $\mathbf{C} = \oplus \mathbf{C}^m$. Once we define d, we call this complex $(\mathbf{C}, d)$ the *bonsai cobar complex* after the *cobar complex* of [MSS].

Third, let us define a map $d^i : \mathbf{C}^i \to \mathbf{C}^{i+1}$ and show that $d^{i+1} \circ d^i = 0$ as follows;

Definition 7.1. *Let T be an m-bonsai. Let T' be a bonsai such that we can obtain T by contracting an edge e' from T' and the following conditions are satisfied;*

i) T' does not have more branching vertices (i.e., vertices which have the ramification numbers >1) than T,

ii) e' is not attatched to a branching vertex of T so that e' becomes a subsidiary edge of that branching vertex.

We call this T' a branch-fixed extension *of T.*

For example, for the 3-bonsai T in Figure 19, T_1, T_3, T_5 and T_6 are all branch-fixed extension of T, but T_2(violating i)) and T_4(violating ii)) are not.

Then we define $d^i : C^i \to C^{i+1}$ as following; when $T \in \mathbf{C}^i$,

$$d^i(T) = \sum T' \otimes e'_{j,l_j} \wedge e'_{1,l_1} \wedge ... \wedge \widehat{e'_{j,l_j}} \wedge ... \wedge e'_{i+1,l_{i+1}} \tag{38}$$

where the sum runs over T', which is a branch-fixed extension of T having an edge e added to T and that e is denoted e'_{j,l_j} in the edge-ordering of T'.

Theorem 7.1. $d^{i+1} \circ d^i = 0$.

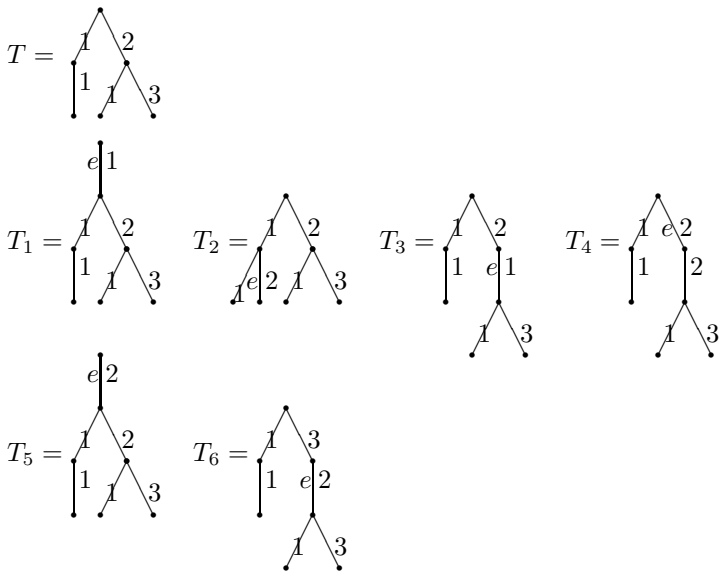


FIGURE 19

Proof. Suppose T'' is a branch-fixed extension of a bonsai T' with added edge e'', which is a branch-fixed extension of T with added edge e'. Then, when e'' is e''_{j,l_j} and e' is e''_{k,l_k} in the edge-ordering of T'' and $d^{i+1} \circ d^i(T)$ is wrote $\sum S \otimes f_{1,p_1} \wedge ... \wedge f_{i+2,p_{i+2}}$ where S runs over the bonsais obtained by attaching two edges as given in i) and ii) in Definition 7.1 and f_{i,p_i}'s are the edges of S, $T'' \otimes e''_{1,l_1} \wedge ... \wedge e''_{i+2,l_{i+2}}$ can be obtained only in two ways;

i) adding e' first to T: then the component of $T'' \otimes e''_{1,l_1} \wedge ... \wedge e''_{i+2,l_{1+2}}$ is $T'' \otimes e''_{j,l_j} \wedge e''_{k,l_k} \wedge e''_{1,l_1} \wedge ... \wedge \widehat{e''_{j,l_j}} \wedge ... \wedge \widehat{e''_{k,l_k}} \wedge ... \wedge e''_{i+2,l_{i+2}}$.

ii) adding e'' first to T: then the component of $T'' \otimes e''_{1,l_1} \wedge ... \wedge e''_{i+2,l_{1+2}}$ is $T'' \otimes e''_{k,l_k} \wedge e''_{j,l_j} \wedge e''_{1,l_1} \wedge ... \wedge \widehat{e''_{j,l_j}} \wedge ... \wedge \widehat{e''_{k,l_k}} \wedge ... \wedge e''_{i+2,l_{i+2}}$.

Since the orders of e''_{k,l_k} and e''_{j,l_j} are different in the wedge products, the sum of two terms in i) and ii) is 0, and this is true for all components of $d^{i+1} \circ d^i(T)$. Hence $d^{i+1} \circ d^i = 0$. □

We call this boundary map d^i the *branch-fixed differential*. A simple example is given in Figure 20. We will study the cohomology of this $\{d^i\}$, but before that, following [MSS], let us see an important property of this bonsai complex in the next section.

8. Cohomology of Branch-fixed Differential

In this section, we study the cohomology theory of the cochain complex $\{\mathcal{C}^i, d^i\}$, where $\mathcal{C}^i$ is the bonsai cobar complex and d^i is the branch-fixed differential. We will define a new kind of bonsai called *seedling* and a new complex $\{\mathbf{C}^{S,j}, d^{i+j}\}_{j\geq 0}$ called *thread* and show that the cohomology groups of $\{\mathcal{C}^i, d^i\}$ are the direct sum of cohomology groups of threads $\{\mathbf{C}^{S,j}, d^{i+j}\}_{j\geq 0}$.

First, let us give some definitions:

$$d^0(\ \bullet \otimes 1) = e' \overset{\bullet}{\underset{\bullet}{|}} 1 \otimes e_{1,1} \quad + \quad e' \overset{\bullet}{\underset{\bullet}{|}} 2 \otimes e_{1,2}$$

$$d^1(\ e' \overset{\bullet}{\underset{\bullet}{|}} 1 \otimes e_{1,1}) = \begin{matrix} e'' \\ e' \end{matrix} \overset{\bullet}{\underset{\bullet}{|}} \begin{matrix} 1 \\ 1 \end{matrix} \otimes e_{1,1} \wedge e_{2,1} \ + \ \begin{matrix} e' \\ e'' \end{matrix} \overset{\bullet}{\underset{\bullet}{|}} \begin{matrix} 1 \\ 1 \end{matrix} \otimes e_{2,1} \wedge e_{1,1}$$

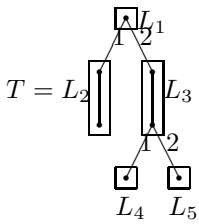
$$+ \ \begin{matrix} e'' \\ e' \end{matrix} \overset{\bullet}{\underset{\bullet}{|}} \begin{matrix} 2 \\ 1 \end{matrix} \otimes e_{1,2} \wedge e_{2,1} \ + \ \begin{matrix} e' \\ e'' \end{matrix} \overset{\bullet}{\underset{\bullet}{|}} \begin{matrix} 1 \\ 2 \end{matrix} \otimes e_{2,2} \wedge e_{1,1}$$

FIGURE 20

Definition 8.1. *A bonsai every vertex of which has the ramification number 0 or 1 is called a* ladder. *In other words, a ladder is a bonsai which has no branching vertex.*

Definition 8.2. *If a bonsai T has an edge, a vertex v which is an end of only one edge and is not the root, is called a* tip. *If a bonsai T is a one-vertex bonsai, the root v is a tip.*

By the definition of d^i, all terms in $d^i(T)$ are of the form $\pm T' \otimes e \wedge det(T)$, where T' runs over bonsais obtained by adding a new edge e to T so that i) and ii) of Definition 7.1 hold. So T' has the form of extending a subladder of T which does not contain the subsidiary edges of branching vertices, as in the example of Figure 21, boxed subladders of which are denoted $L_1, L_2, ..., L_5$.



L_1 only grows upward by ii) of Definition 7.1.

FIGURE 21

So, the action of d^i on $T \otimes det(T)$ is by extending a ladder of T, getting a new edge e and changing $det(T)$ into $e \wedge det(T)$. Acting by d^i's on $T \otimes det(T)$, the possible bonsais appearing in the $d^i(T)$ are obtained by extending a subladder of T as in the example of Figure 21.

keeping this intuitive fact in mind, we have some definitions;

Definition 8.3. *A* seedling *is an m-bonsai all of whose vertices other than tips are branching vertices. For example, in 2-bonsai, S_1 of Figure 22 is a seedling, but S_2 is not, because the root vertex is not a branching vertex. In other words, a seedling is a bonsai which cannot be obtained from another bonsai by adding an edge so that i) and ii) of Definition 7.1 are satisfied.*

Definition 8.4. *Let $\mathbf{C}^{S,0}$ be the submodule of $\mathcal{C}^i$, where S is a seedling and i is the number of edges of S, generated by $S \otimes det(S)$. Let $\mathbf{C}^{S,j}$ $(j \geq 0)$ be the submodule of $\mathcal{C}^{i+j}$ generated by $T \otimes det(T)$, where T is an m-bonsai obtained by adding j edges to S so that i) and ii) of Definition 7.1 are satisfied.*

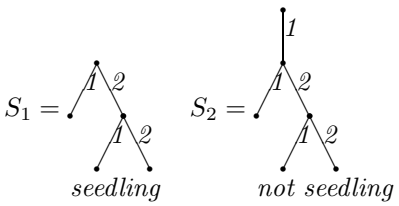


FIGURE 22

Then, every $\mathcal{C}^i$ is the direct sum of some $\mathbf{C}^{S,j}$'s and $d(\mathbf{C}^{S,j}) \subset \mathbf{C}^{S,j+1}$. For example, in 2-bonsai, when S_0, S_1, S_2 and S_3 are as given in Figure 23, we have

$$
\begin{aligned}
(39) \qquad \mathcal{C}^0 &= \mathbf{C}^{S_0,0}, \\
\mathcal{C}^1 &= \mathbf{C}^{S_0,1}, \\
\mathcal{C}^2 &= \mathbf{C}^{S_0,2} \oplus \mathbf{C}^{S_1,0}, \\
\mathcal{C}^3 &= \mathbf{C}^{S_0,3} \oplus \mathbf{C}^{S_1,1}, \\
\mathcal{C}^4 &= \mathbf{C}^{S_0,4} \oplus \mathbf{C}^{S_1,2} \oplus \mathbf{C}^{S_0,3} \oplus \mathbf{C}^{S_1,1}, \\
\mathcal{C}^5 &= \mathbf{C}^{S_0,5} \oplus \mathbf{C}^{S_1,3} \oplus \mathbf{C}^{S_0,3} \oplus \mathbf{C}^{S_1,1}, \\
&\cdots\cdots
\end{aligned}
$$

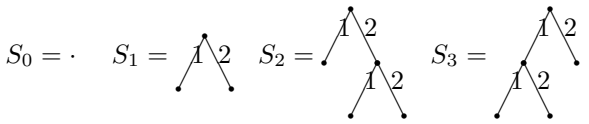


FIGURE 23

Definition 8.5. *For a given seedling S, when i is the number of edges of S, $\{\mathbf{C}^{S,j}, d^{i+j}\}_{j\geq 0}$ is called a* thread *of S.*

So the cohomology groups of $\{\mathcal{C}^i, d^i\}$ are the direct sum of cohomology groups of threads $\{\mathbf{C}^{S,j}, d^{i+j}\}_{j\geq 0}$.

Let us look into each of these threads. For $\mathbf{C}^{S_0,0}$, where S_0 is a vertex, $\mathbf{C}^{S_0,i}$ is the module with the basis $\{T \otimes det(T)\}$, where T is a ladder with i edges and the boundary maps extend the ladders by adding an edge e and replacing $det(T)$ with $e \wedge det(T)$. Let us consider a chain complex which is isomorphic to the thread $\mathbf{C}^{S_0,0}$ of the ladder S_0. For any m-bonsai, consider a vector space V which has a basis $\{v_1, ..., v_m\}$, and let $V_n = V^{\otimes n} (n \geq 1)$. Then we define a map $\delta^n : V^n \to V^{n+1}$ as

$$
\begin{aligned}
(40) \qquad v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n} \mapsto \quad & \sum_{k=1}^{m} v_k \otimes v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n} \\
+ \quad & \sum_{k=1}^{m} (-1)^1 v_{i_1} \otimes v_k \otimes v_{i_2} \otimes ... \otimes v_{i_n} \\
+ \quad & ... \\
+ \quad & \sum_{k=1}^{m} (-1)^n v_{i_1} \otimes ... \otimes v_{i_{n-1}} \otimes v_{i_n} \otimes v_k,
\end{aligned}
$$

and it is easily seen that this δ^n is a boundary map, so we have made $\{V^n, \delta^n\}$ a cochain complex. By the cochain map f as in Figure 24, the cochain complexes $\{\mathbf{C}^{S_{0,n}}, d^n\}$ and $\{V^n, \delta^n\}$ are isomorphic, since d^n acts as in Figure 25, that is, we have $f \circ \delta^n = d^n \circ f$.

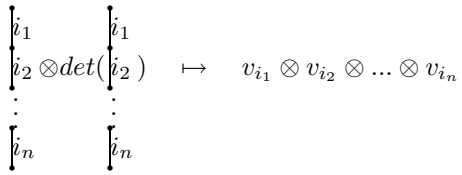


FIGURE 24

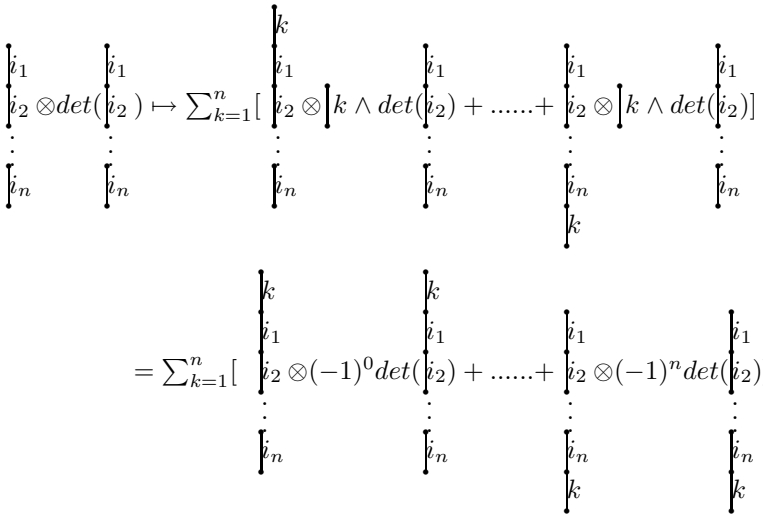


FIGURE 25

In $\{V^n, \delta^n\}_{n\geq 1}$, by the definition of δ^n, inductively we have

(41)

$$\delta((v_1 \otimes ... \otimes v_i) \otimes v) = \delta(v_1 \otimes ... \otimes v_i) \otimes v + (-1)^{i+1}(v_1 \otimes ... \otimes v_i) \otimes v \otimes \sum_{k=1}^{m} v_k$$

where $v_1 \otimes ... \otimes v_i \in V^i$ and $v \in V$. Suppose $\delta(\sum_{k=1}^m v'_k \otimes v_k) = 0$ where $v'_k \in V^i$, then we have

$$\begin{aligned} &\sum_{k=1}^{m} \delta(v'_k) \otimes v_k + (-1)^{i+1}(\sum_{k=1}^{m} v'_k \otimes v_k) \otimes \sum_{l=1}^{m} v_l \\ &= \sum_{l=1}^{m} \{\delta(v'_l) - (-1)^k (\sum_{k=1}^{m} v'_k \otimes v_k)\} \otimes v_l = 0. \end{aligned} \tag{42}$$

Therefore, we have $(\sum_{k=1}^m v'_k \otimes v_k) = (-1)^k \delta(v'_l)$ and it is a coboundary. So, $\{V^n, \delta^n\}$ is acyclic, and so is $\{\mathbf{C}^{S_{0,n}}, d^n\}$.

Here, we can directly calculate $H^0(\mathbf{C}^{S_0,*}) = 0$, since the boundary map image of a one-vertex bonsai is the sum of one-edge bonsais over all labels 1,2,...,n. So $\{\mathbf{C}^{S_0,n}, d^n\}$ is acyclic with $H^0 = 0$.

Now for an arbitrary seedling S, when S has n edges, there are $n+1$ vertices and each vertex other than the root has one and only one edge whose branch-end is that vertex. When we order the edges of a bonsai T with the shape S as in Figure 13 and denote them as e_l's ($l = 1, 2, ..., n$), we can denote the branch-end vertex of e_l as v_l and denote the root v_0. Then the bonsais which appear in the basis of $\mathbf{C}^{S,j}$ are obtained by extending the vertices of T into upward-growing ladders, and each ladder grown from v_l is denoted as L_l, as in the example of Figure 26.

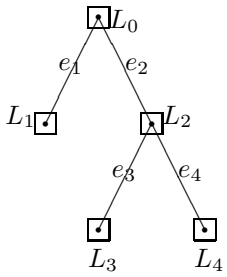


FIGURE 26

To get the cohomology of $\mathbf{C}^{S,j}$, let us consider the complexes $\mathbf{C}_k^{S_0,p}$, where $k = 0, ..., n$ and each of $\mathbf{C}_k^{S_0,p}$ is a copy of $\mathbf{C}^{S_0,p}$, i.e., each of $\mathbf{C}_k^{S_0,p}$ has the basis $\{L_k^p \otimes det(L_k^p)\}$, where L_k^p is a ladder with p edges. Then we have an isomorphism F between

$$\mathbf{D}^l = \bigoplus_{p_0+...+p_n=l} \mathbf{C}_0^{S_0,p_0} \otimes ... \otimes \mathbf{C}_n^{S_0,p_n} \tag{43}$$

and $\mathbf{C}^{S,l}$ given by

$$\begin{gathered} L_0^{p_0} \otimes det(L_0^{p_0}) \otimes ... \otimes L_n^{p_n} \otimes det(L_n^{p_n}) \mapsto \\ \Sigma(\text{The bonsai obtained by putting } L_i^{p_0} \text{ into the place of vertex } v_i) \\ \otimes det(L_0^{p_0}) \wedge e_1 \wedge det(L_1^{p_1}) \wedge ... \wedge e_n \wedge det(L_n^{p_n}) \end{gathered} \tag{44}$$

as in the example of Figure 27, for the seedling of Figure 26.

From now on, we write $L_i^{p_i}$ just as L_i for convenience of writing.

In $T \otimes det(T) \in \mathbf{C}^{S,l}$, $det(T)$ is

$$det(L_0) \wedge e_{1,k_1} \wedge det(L_1) \wedge ... \wedge e_{n,k_n} \wedge det(L_n) \tag{45}$$

where k_l is the label of the edge e_l, and $d^n(T \otimes det(T))$ is

$$\Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to one of the } L_i) \otimes f \wedge det(T)$$

and this is

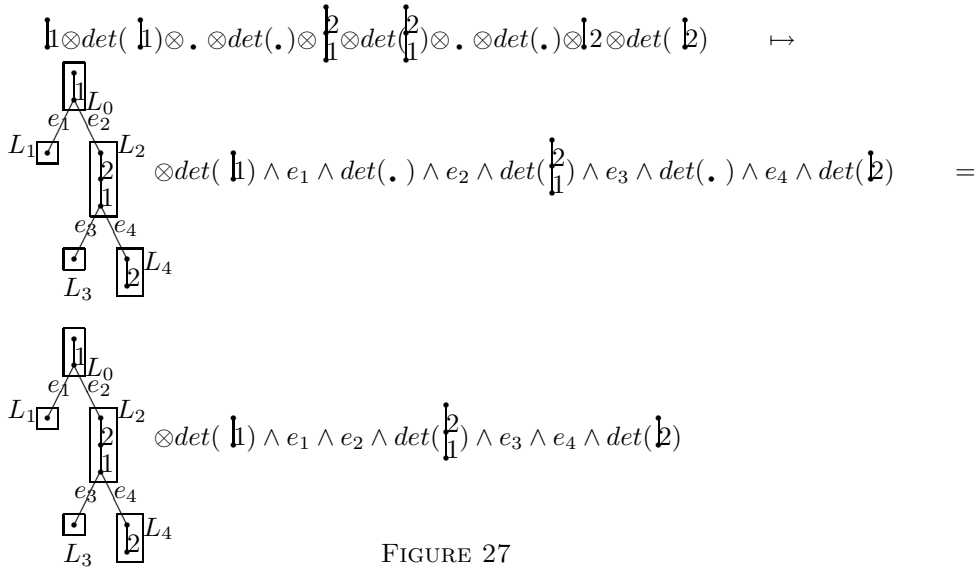


FIGURE 27

$$\Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to one of } L_i)$$
$$\otimes f \wedge det(L_0) \wedge e_{1,k_1} \wedge det(L_1) \wedge ... \wedge e_{n,k_n} \wedge det(L_n)$$

$$= \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to one of } L_i)$$
$$\otimes(-1)^{\beta} det(L_0) \wedge e_{1,k_1} \wedge det(L_1) \wedge ... \wedge e_{i,k_i} \wedge f \wedge det(L_i) \wedge ... \wedge e_{n,k_n} \wedge det(L_n)$$

where $\beta = (deg(L_0)+1) + (deg(L_1)+1) + ... + (deg(L_{i-1})+1)$.

Here, $det(T')$ is obtained by replacing $det(L_i)$ with $f \wedge det(L_i)$ in (45), and we get the sign $(-1)^{\beta}$ since when f is added to L_i, in the ordering of edges of T', the edges of $L_j (j = 0, 1, ..., i-1)$ and edges e_1, e_2,...,e_i are prior to the edges of L_i. So we can conclude that $d(T \otimes det(T))$ is the sum of $(-1)^{deg(L_0)+1+...+deg(L_{i-1})+1} T' \otimes det(T')$, where T' is the bonsai obtained by adding a new edge in the subladder L_i.

Thus, when we construct a coboundary map ∂^l for $\{\mathbf{D}^l\}$, acting on $\mathbf{C}_1^{S_0,p_1} \otimes ... \otimes \mathbf{C}_n^{S_0,p_n}$ by

$$\begin{aligned}
& \partial^l(L_0 \otimes det(L_0) \otimes ... \otimes L_n \otimes det(L_n)) \\
= \;& d(L_0 \otimes det(L_0)) \otimes L_1 \otimes det(L_1) \otimes ... \otimes L_n \otimes det(L_n) \\
+ \;& (-1)^{deg(L_0)+1} L_0 \otimes det(L_0) \otimes d(L_1 \otimes det(L_1)) \otimes ... \otimes L_n \otimes det(L_n) \\
+ \;& ... \\
+ \;& (-1)^{(deg(L_0)+1)+...+(deg(L_{n-1})+1)} L_0 \otimes det(L_0) \otimes L_1 \otimes det(L_1) \otimes ... \otimes d(L_n \otimes det(L_n)),
\end{aligned}$$

the isomorphism F becomes a cochain isomorphism between $\{\mathbf{D}^l\}$ and $\{\mathbf{C}^{S,l}\}$, and since each $\{\mathbf{C}_k^{S_0,p_k}\}$ is acyclic and $H^0 = 0$, by the Künneth theorem, $\{\mathbf{D}^l\}$ is acyclic and $H^0 = 0$, and so is $\{\mathbf{C}^{S,l}\}$.

Then, by a Künneth argument, every thread of $\{\mathbf{C}^{S,j}, d^{i+j}\}$ (i is the number of edges of S) is acyclic with $H^0 = 0$, and so we have the

Theorem 8.1. *$\{\mathcal{C}^n, d^n\}$ is acyclic.*

9. Vertex-appending Differential

In this section, we consider a new differential, different from the ladder-extension. Again, all bonsais of this section are m-bonsais.

Definition 9.1. *We define the* vertex-appending differential *∂ as follows; Consider a determinanted bonsai $T \otimes det(T)$. Then $\partial(T \otimes det(T))$ is the sum of $T' \otimes e \wedge det(T)$, where T' is a bonsai obtained by*

i) appending a vertex to T

ii) except to tips of T,

and so, getting a new edge e.

If there is no available appending position an a bonsai, the map ∂ assigns 0 to that bonsai.

For example, in 3-bonsai, we have an example like Figure 28 (in bonsais, determinanted terms are omitted. Note that one vertex in the first example is also a tip and the fourth bonsai is a cocycle).

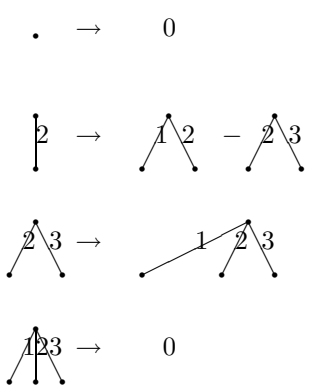


Figure 28

First, we have

Theorem 9.1. *$\partial^{i+1} \circ \partial^i = 0$. That is, ∂ is actually a differential.*

Proof. This proof is almost the same as that of Theorem 7.1. Suppose T'' is obtained by appending a vertex to T' with added edge e'' so that i) and ii) of Definition 9.1 are satisfied, where T' is obtained by appending a vertex to T with added edge e' so that i) and ii) of Definition 9.1 are satisfied. Here, if T' has no available position to append a vertex, then $\partial(T') = 0$, so $T'' = 0$. Otherwise, when e'' is e''_{j,l_j} and e' is e''_{k,l_k} in the edge-ordering of T'', in $\partial^{i+1} \circ \partial^i(T)$ hits the component $T'' \otimes e''_{1,l_1} \wedge ... \wedge e''_{i+2,l_{i+2}}$ just by adding the edge e' and e'' to T, and it can be done only in two ways;

i) adding e' first to T: then the component of $T'' \otimes e''_{1,l_1} \wedge ... \wedge e''_{i+2,l_{1+2}}$ is $T'' \otimes e''_{j,l_j} \wedge e''_{k,l_k} \wedge e''_{1,l_1} \wedge ... \wedge \widehat{e''_{j,l_j}} \wedge ... \wedge \widehat{e''_{k,l_k}} \wedge ... \wedge e''_{i+2,l_{i+2}}$.

ii) adding e'' first to T: then the component of $T'' \otimes e''_{1,l_1} \wedge ... \wedge e''_{i+2,l_{1+2}}$ is $T'' \otimes e''_{k,l_k} \wedge e''_{j,l_j} \wedge e''_{1,l_1} \wedge ... \wedge \widehat{e''_{j,l_j}} \wedge ... \wedge \widehat{e''_{k,l_k}} \wedge ... \wedge e''_{i+2,l_{i+2}}$.

Since the order of e''_{k,l_k} and e''_{j,l_j} are different in the wedge products, the sum of the two terms in i) and ii) is 0. This is true for all components of $\partial^{i+1} \circ \partial^i(T)$, and so $\partial^{i+1} \circ \partial^i = 0$. □

9.1. **Definition of seedling.** By the definition of ∂^i, all terms in $\partial^i(T)$ are of the form $\pm T' \otimes e \wedge det(T)$, where T' runs over bonsais obtained by adding a new edge e to T so that i) and ii) of Definition 9.1 hold. So T' has the form of appending a vertex to a vertex of T which is not a tip. Having this intuitive fact in mind, let us present some new definitions and reorganize the cochain complex of bonsais.

Definition 9.2. *For a bonsai T, an edge e of T is called* twiggy *if it is at the end of a branch and the opposite end of the tip is a branching vertex. In Figure 29, e is twiggy in T and e' is not, and f is not a twiggy edge of T'.*

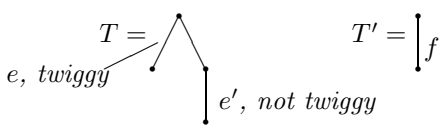


FIGURE 29

Definition 9.3. *A bonsai which has no twiggy edge is called a* vertex-appending seedling. *In this section, we will call this just a* seedling. *The left bonsai in Figure 35 is not a seedling, and the ones in Figure 30 are all seedlings in 2-bonsai. Note that the one-vertex bonsai is a seedling. Intuitively, a seedling is a bonsai which cannot be obtained by adding edges like i) and ii) of Definition 9.1.*

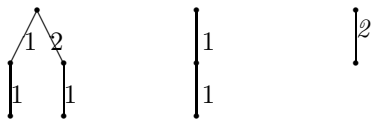


FIGURE 30

Definition 9.4. *For two seedlings S and S', we define an equivalence relation $S \sim S'$ if S is obtained by changing labels of branch-end edges of S'. For example, all four seedlings in Figure 31 are equivalent, and so are the first and second seedlings of Figure 32, but the first and third seedlings of Figure 32 are not equivalent.*

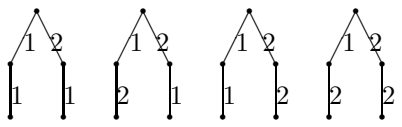


FIGURE 31

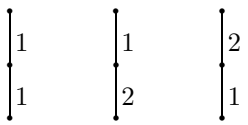


FIGURE 32

Let us try the same trick as in the proof of acyclicity of the branch-fixed differential. When S is a seedling, let $\mathbf{C}^{[S],0}$ be the subspace of the determinanted m-bonsai space having $\{T \otimes det(T)\}$ as the basis, where T is in the equivalence class $[S]$ of S by $\sim$. And let $\mathbf{C}^{[S],i+1}$ be the space with the basis $\{T' \otimes det(T')\}$, where T' is obtained by adding an edge to T, where $\{T \otimes det(T)\}$ is the basis of $\mathbf{C}^{S,i}$, as i) and ii) of Definition 9.1. Every bonsai is obtained by adding some edges to a seedling as given in Definition 9.1 and if S and S' are not equivalent seedlings, then the bonsais obtained by adding edges to S and S' as given in Definition 9.1 are not equivalent; the space of determinanted bonsais is the direct sum of $\mathbf{C}^{[S],i}$. Then, since $\partial(\mathbf{C}^{[S],i}) \subset \mathbf{C}^{[S],i+1}$, the cohomology groups of determinanted bonsais by the differential ∂ is the direct sum of cohomology groups of the threads $\{\mathbf{C}^{[S],i}\}$.

9.2. **The cohomology groups of the vertex-appending differential.** First let us consider a coboundary complex $\{\mathbf{D}^{m,i}\}$ consisting of corollas, such that each corolla is an m-bonsai, and the boundary map is the vertex-appending differential, but in this complex, appending to the one-vertex bonsai is allowed, so we have to be careful not to be confused with the definition of the above vertex-appending differential. By the definition of $\partial^i : \mathbf{D}^{m,i} \to \mathbf{D}^{m,i+1}$, all terms in $\partial^i(T)$ are of the form $\pm T' \otimes e \wedge det(T)$, where T' runs over bonsais obtained by adding one vertex as defined in Definition 9.1. Since appending to a tip is forbidden, every ∂^i is appending a vertex to the root of a corolla. Now let us show some boundary map sequences of the thread starting from one vertex. For one vertex, ∂^0 acts as in Figure 33.

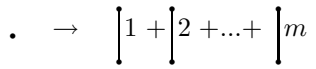


FIGURE 33

For the one-edge corolla, ∂^1 acts as in Figure 34, where $1 \leq n \leq m$, and for the two-edge corolla, ∂^2 acts as in Figure 35, where $1 \leq n_1 < n_2 \leq m$, and so on.

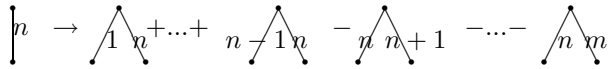


FIGURE 34

This sequence of coboundary maps is the same as that of the reduced cohomology of the $(m-1)$-simplex with vertices $v_1, v_2, ..., v_m$, once we identify the corolla with labels $i_1, i_2, ..., i_k$ with the simplex generated by vertices $v_{i_1}, v_{i_2}, ..., v_{i_k}$. So the cohomology groups of this thread of boundary maps is acyclic, and the lowest

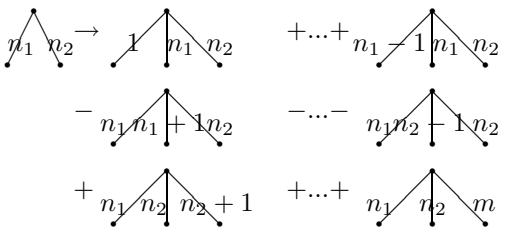


FIGURE 35

degree group is trivial. Let us denote the module having the basis consisting of one vertex having m available positions of vertex appending as $\mathbf{D}^{m,0}$, and the module having the basis consisting of corollae with n edges as shown above $\mathbf{D}^{m,n}$. Also, let $\{\mathbf{B}^{m,n}\}$ be a cochain complex defined by $\mathbf{B}^{m,n} = \mathbf{D}^{m,n+1}$ for later convenience. Then $\{\mathbf{B}^{m,n}\}$ is acyclic and $H^0 = k$, where k is the base field, since the cohomology of $\{\mathbf{B}^{m,n}\}$ is isomorphic to the cohomology (not the reduced cohomology) of the $(m-1)$-simplex.

9.3. **The case of general seedlings.** Now let us show through an example

Lemma 2. *Any complex $\{\mathbf{C}^{[S],i}\}$ is isomorphic to it, which is represented as a direct sum of tensor products of $\{\mathbf{D}^{m,i}\}$'s and $\{\mathbf{B}^{m,i}\}$'s (as in the proof of acyclicity of the branch-fixed differential).*

In 5-bonsai, the seedling S in Figure 36 can get twiggy edges at the positions of the twigs shown in the picture which are grouped as surrounded by squares.

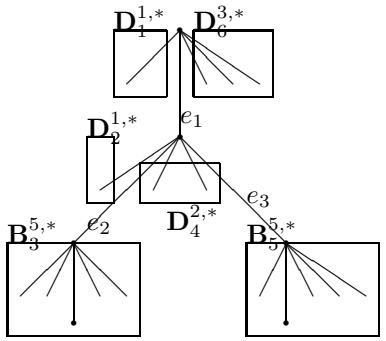


FIGURE 36

Note that, in Figure 36 adding an edge to each square is the same as attaching an edge to the corolla at the vertex at which the square is appended, each corolla corresponds to the module that is written on each square (In Figure 36, $\mathbf{D}_i^{m,*}$ is isomorphic to $\mathbf{D}^{m,*}$ and $\mathbf{B}_i^{m,*}$ is isomorphic to $\mathbf{B}^{m,*}$).

Keeping this in mind, we can define new modules $\{\mathbf{D}^l\}$ and an isomorphism F of them with $\{\mathbf{C}^{[S],i}\}$ like the following;

$$\mathbf{D}^l = \bigoplus_{p_1+\ldots+p_6=l} \mathbf{D}_1^{1,p_1} \otimes \mathbf{D}_2^{1,p_2} \otimes \mathbf{B}_3^{5,p_3} \otimes \mathbf{D}_4^{2,p_4} \otimes \mathbf{B}_5^{5,p_5} \otimes \mathbf{D}_6^{3,p_6} \tag{46}$$

and when $\mathbf{M}_1 = \mathbf{D}_1^{1,p_1}$, $\mathbf{M}_2 = \mathbf{D}_2^{1,p_2}$, $\mathbf{M}_3 = \mathbf{B}_3^{5,p_3}$, $\mathbf{M}_4 = \mathbf{D}_4^{2,p_4}$, $\mathbf{M}_5 = \mathbf{B}_5^{5,p_5}$ and $\mathbf{M}_6 = \mathbf{D}_6^{3,p_6}$, the map $F : \mathbf{D}^l \to \mathbf{C}^{[S],l}$ is defined as, when $c_i \in \mathbf{M}_i$ is a corolla,

(47)
$$c_1 \otimes det(c_1) \otimes ... \otimes c_6 \otimes det(c_6) \mapsto$$
$$\Sigma(\text{The bonsai obtained by attaching } c_i \text{ to the square corresponding to } \mathbf{M}_i)$$
$$\otimes det(c_1) \wedge e_1 \wedge det(c_2) \wedge e_2 \wedge det(c_3) \wedge det(c_4) \wedge e_3 \wedge det(c_5) \wedge det(c_6)$$

as in the example of Figure 37.

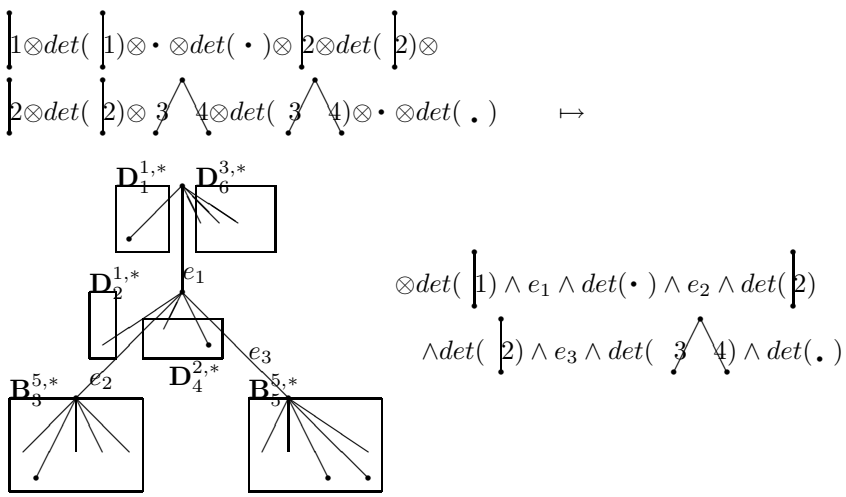


FIGURE 37

Then in $T \otimes det(T) \in \mathbf{C}^{S,l}$, $det(T)$ is

(48)
$$det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge det(c_6)$$

where k_l is the label of the edge e_l, and $\partial^n(T \otimes det(T))$ is

(49)
$$\Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to one of } c_i) \otimes f \wedge det(T)$$

$$= \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to one of } c_i)$$
$$\otimes f \wedge det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge det(c_6)$$

$$= \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_1)$$
$$\otimes f \wedge det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge det(c_6)$$

$$+ \quad \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_2) \otimes(-1)^{deg(c_1)+1} det(c_1) \wedge e_{1,k_1} \wedge f \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge det(c_6)$$

$$+ \quad \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_3) \otimes(-1)^{(deg(c_1)+1)+(deg(c_2)+1)} det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge f \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge det(c_6)$$

$$+ \quad \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_4) \otimes(-1)^{(deg(c_1)+1)+(deg(c_2)+1)+(deg(c_3)+0)} det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge f \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge det(c_6)$$

$$+ \quad \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_5) \otimes(-1)^{(deg(c_1)+1)+(deg(c_2)+1)+(deg(c_3)+0)+(deg(c_4)+1)} det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge f \wedge det(c_5) \wedge det(c_6)$$

$$+ \quad \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_6) \otimes(-1)^{(deg(c_1)+1)+(deg(c_2)+1)+(deg(c_3)+0)+(deg(c_4)+1)+(deg(c_5)+0)} det(c_1) \wedge e_{1,k_1} \wedge det(c_2) \wedge e_{2,k_2} \wedge det(c_3) \wedge det(c_4) \wedge e_{3,k_3} \wedge det(c_5) \wedge f \wedge det(c_6)$$

So, when $m_1 = deg(c_1)+1$, $m_2 = deg(c_2)+1$, $m_3 = deg(c_3)+0$, $m_4 = deg(c_4)+1$ and $m_5 = deg(c_5)+0$, we can write $\partial(T \otimes det(T))$ as

$$+ \quad \Sigma(\text{A bonsai } T' \text{ obtained by adding a new edge } f \text{ to } c_i) \otimes(-1)^{m_1+...+m_{i-1}} det(c_1) \wedge e_{1,k_1} \wedge ... \wedge (f \wedge det(c_i)) \wedge ... \wedge det(c_6).$$

Hence, when we define the coboundary map on $\{\mathbf{D}^l\}$ as

$$c_1 \otimes ... \otimes c_6 \mapsto \Sigma(-1)^{\beta_i} c_1 \otimes ... \otimes \partial(c_i) \otimes ... \otimes c_6 \tag{50}$$

where $\beta_i = m_1+...+m_{i-1}$ and $\beta_1 = 0$, the map F defined in (47) becomes a cochain isomorphism of $\{\mathbf{D}^l\}$ and $\{\mathbf{C}^{[S],l}\}$. Then by the Künneth theorem, the cohomology of the cochain complex $\{\mathbf{D}^l\}$ is expressed as the sum of $H^{q_1}(\mathbf{M}_1) \otimes ... \otimes H^{q_6}(\mathbf{M}_6)$ for some q_i's, and since $H^i(\mathbf{M}_1) = H^i(\mathbf{D}^{1,*}) = 0$ for any i, the cohomology of $\mathbf{C}^{[S],l}$ for S of Figure 36 is acyclic.

Definition 9.5. *As shown for the example of Figure 36, for any seedling S, we have a cochain complex as in* (46) *and an isomorphism F of it with $\mathbf{C}^{[S],l}$ as in* (47)*. We call this cochain complex as in* (46) *the* tensor product representation *of $\mathbf{C}^{[S],l}$.*

Then, whether $\{\mathbf{C}^{[S],l}\}$ is acyclic or not depends on whether its tensor product representation contains $\mathbf{D}^{m,*}$. As shown in Figure 36, $\mathbf{B}^{m,*}$ appears only at the branch-end edges of a seedling, and $\mathbf{D}^{m,*}$ appears only at the available positions of vertex-appending other than at branch-end edges. So, the only case where $\{\mathbf{C}^{[S],l}\}$ is not acyclic is that the bonsai obtained by deleting all branch-end edges of S is a cocycle, i.e., that bonsai has no available position of vertex appending and so there is no room for $\mathbf{D}^{m,*}$ on S, like the bonsais of 38, in 2-bonsai. The only nontrivial cohomology group of $\{\mathbf{C}^{[S],l}\}$ is $H^0 = k$ by Künneth Theorem, where k is the base field, since all $H^0(\mathbf{B}^{m,i}) = k$

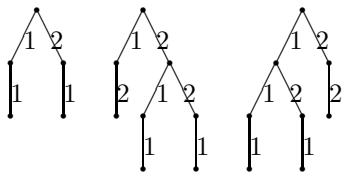


FIGURE 38

In m-bonsai, every cocycle C is a planar tree, all of whose vertices which are not branch-ends have ramification number m, so if C contains n corollas with m edges, it has $mn + 1$ vertices, n of those vertices have m successors and $(mn + 1) - n$ of those vertices have 0 successors, i.e., are the endpoints of edges, in the language of [S]. Then, by Theorem 5.3.10 of [S], the number of such C is

$$\frac{1}{mn+1}\begin{pmatrix} mn+1 \\ (mn+1)-n, ..., n \end{pmatrix} \tag{51}$$

which is $\frac{(mn)!}{((m-1)n+1)!n!}$. A seedling S is obtained by adding one edge to every branch-end vertex of C and C has $(mn + 1) - n$ branch-end vertices. So S has $mn + ((mn + 1) - n) = (2m - 1)n + 1$ vertices. Thus we have

Theorem 9.2. *The cohomology groups H^i of m-bonsai Hopf algebra by the vertex-appending differential is,*

$$H^i = \begin{cases} k^{\frac{(mn)!}{((m-1)n+1)!n!}} & \text{if } i = (2m-1)n+1, n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In 2-bonsai, the representatives of H^i are as in Figure 39.

10. Appending Operation $*$ and Its Deviation

Over a general base field k for a bonsai Hopf algebra, it is not so easy to find a good algebraic relationship between $\partial(T_1 * T_2)$ and $(\partial T_1) * T_2 \pm T_1 * (\partial T_2)$ where ∂ is the vertex-appending differential, mainly because of the signs of determinanted bonsais. But if k has characteristic 2, we don't need to consider signs. Moreover, the vertex-appending differential becomes just appending of a vertex, taking no consideration of determinanted terms, but it is still a boundary map by the same argument as in the previous section. Then the relation of $\partial(T_1 * T_2)$ and $(\partial T_1) * T_2 + T_1 * (\partial T_2)$ becomes much simpler. In this section, we consider only the case where the characteristic of k is 2.

Let us define the binary operation $*_1$ by

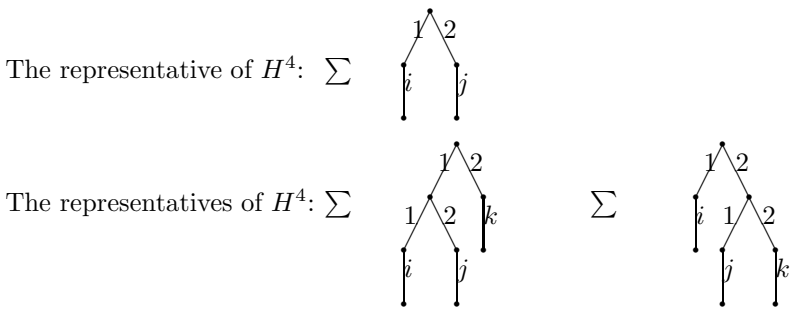


FIGURE 39

$$\partial(T_1 * T_2) = (\partial T_1) * T_2 + T_1 * (\partial T_2) + T_1 *_1 T_2 \tag{52}$$

and call this operation $*_1$ the *first deviation* of the operation $*$. We let $*_2$ be defined by

$$\partial(T_1 *_1 T_2) = (\partial T_1) *_1 T_2 + T_1 *_1 (\partial T_2) + T_1 *_2 T_2 \tag{53}$$

and call this operation the *second deviation* of the operation $*$. We define 3rd, 4th,... deviations iteratively. Note that $*_2$ is the first deviation of $*_1$.

First, for the appending operation $*$, we have

Theorem 10.1. *In bonsai Hopf algebra, with its base field of characteristic 2, $T_1 *_1 T_2$ is the sum of all T''s, where T' is any bonsai obtained by connecting a tip v of T_2 and the root of T_1 with one edge and attatch another edge to that vetrex of T_2, or by connecting a non-tip of T_2 and the root of T_1 with a length-2 ladder. In both cases, the edges added to T_1 and T_2 are allowed to have all possible labels, as in the example in 3-bonsai of Figure 40.*

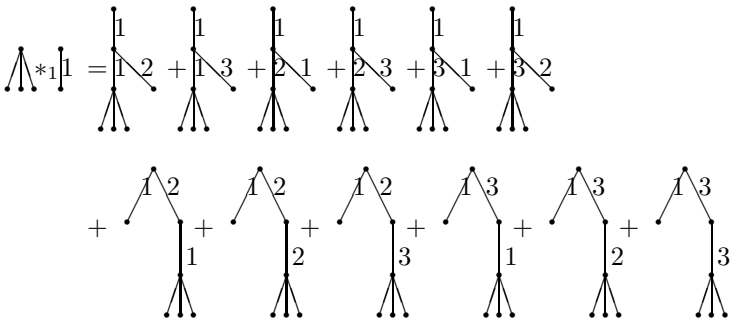


FIGURE 40

Proof. Let us use a graphical illustration. Bonsais which are summands of $\partial(T_1 * T_2)$ are as in Figure 41, where T_1 and T_2 and the appended vertex are drawn as broomsticks, and i,j and k are indices of twigs and $T_1 * T_2$ is the sum of these two kinds of bonsai over i, j and k. In this proof, the black circles in the pictures

represent the twigs at non-tips of bonsais, and the white circles represent the twigs at tips of bonsais. Here, i is on the vertices of T_2 which are not tips, and k is on the vertices of T_1 which are not tips and i' is on the vertices of T_2 which are tips, but not tips in $\partial(T_1 * T_2)$ since the connecting edge is attached.

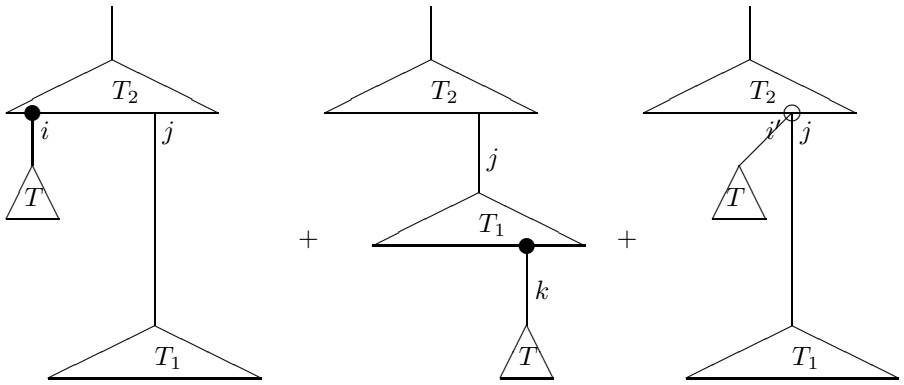


FIGURE 41

The bonsais which are summands of $T_1 * \partial T_2$ are as in Figure 42. In the right bonsai of Figure 42, the label of the edge connecting T and T_1 can be anything out of $1, 2, ...m$.

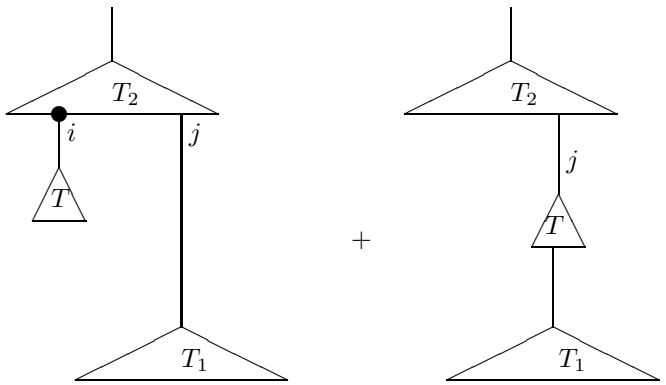


FIGURE 42

The bonsais which are summands of $\partial T_1 * T_2$ are as in Figure 43. Then the discrepancy between $\partial(T_1 * T_2)$ and $\partial T_1 * T_2 + T_1 * \partial T_2$ is the sum of the third bonsai of Figure 41 and the second bonsai of Figure 42, which gives the wanted formula.

□

Theorem 10.2. *Mod 2, for any m-bonsais T_1 and T_2 , we have $T_1 *_2 T_2 = 0$. In other words, $\partial(T_1 *_1 T_2) = \partial T_1 *_1 T_2 + T_1 *_1 \partial T_2$.*

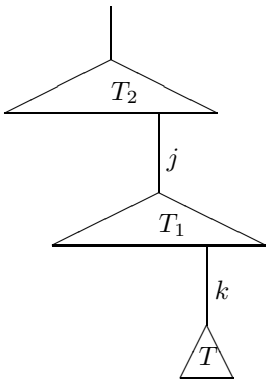


FIGURE 43

Proof. Bonsais which are summands of $T_1 *_1 T_2$ are those in Figure 44, which represent the bonsais obtained by connecting T_1 and T_2 with one edge and attatching another edge and obtained by connecting T_1 and T_2 with a length-2 ladder. As in the proof of the previous theorem, a black circle represents a non-tip of a bonsai and a white circle represents a tip of a bonsai.

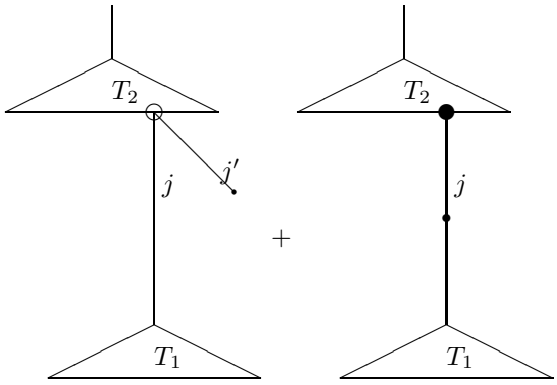


FIGURE 44

The bonsais in $\partial(T_1 *_1 T_2)$ obtained by ∂ acting on the left bonsai in Figure 44 are those in Figure 45, and the bonsais obtained by ∂ acting on the right bonsai in Figure 44 are those in Figure 46. In both pictures, the broomstick lettered T is a one-vertex bonsai which is appended by the vertex-appending differential ∂.

The bonsais in $\partial(T_1 *_1 T_2)$ corresponding to the left bonsai in Figure 44 are like Figure 45, and the ones corresponding to the right one are like Figure 46. For later use, we denote those bonsais A, B,...,G as assigned in the Figures. In both pictures, the broomstick lettered T is a one-vertex bonsai which is appended by the vertex-appending differential ∂.

For the bonsai F, actually we have a term like 47 also in $\partial(T_1 *_1 T_2)$, and since we are working mod 2, the bonsais looking like F are all canceled. So we have F=0.

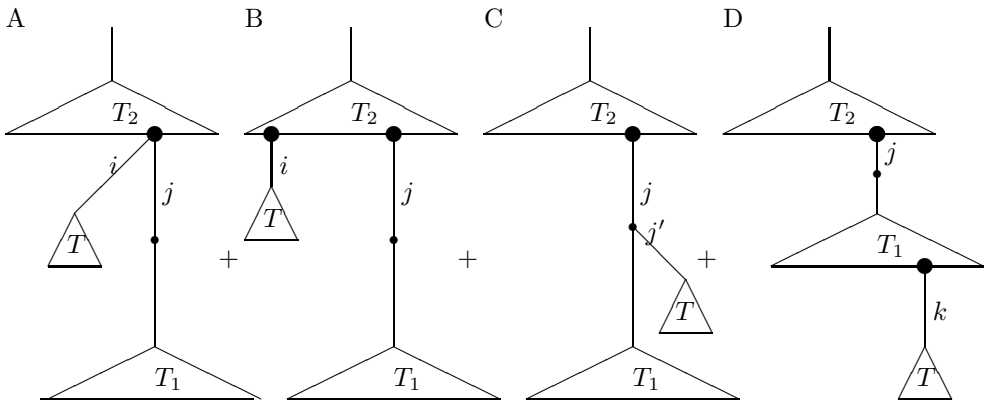


FIGURE 45

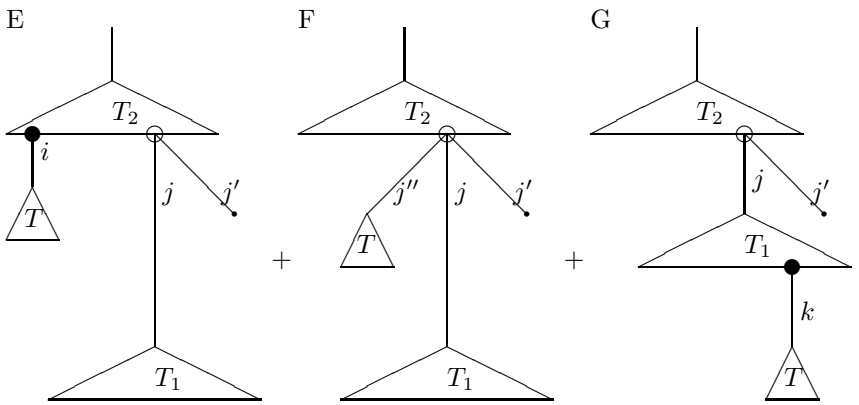


FIGURE 46

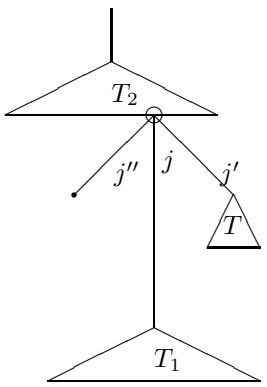


FIGURE 47

Now, the bonsais in $\partial(T_1) *_1 T_2$ are as in Figure 48. Let us denote them as a and b.

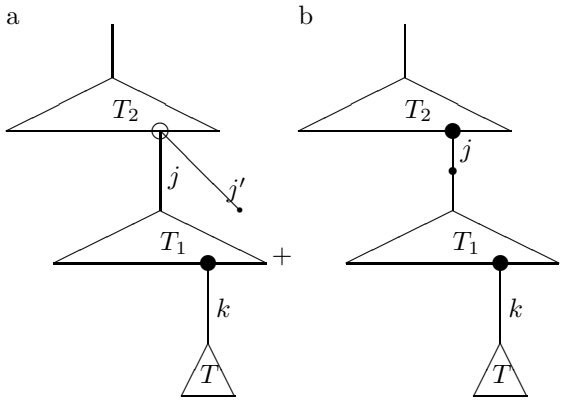


FIGURE 48

The bonsais in $T_1 *_1 \partial(T_2)$ are like those in Figure 49, where we denote the bonsais as c, d, e and f.

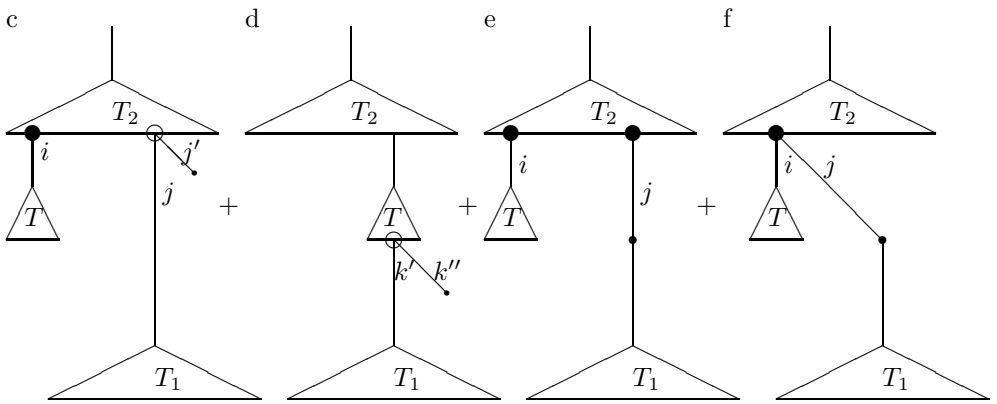


FIGURE 49

In Figures 45-49, we have a=G, b=D, c=E, d=C, e=B and f=A. We already showed that F=0, so we have $\partial(T_1 *_1 T_2) = \partial(T_1) *_1 T_2 + T_1 *_1 \partial(T_2)$.

□

11. Planar Clear-edged m-bonsai and its Derivative

In planar clear-edged m-bonsai, we can define a differential and calculate some cohomologies as for m-bonsai.

Definition 11.1. *For the planar clear-edged m-bonsai, we define the* vertex-appending differential ∂ *as follows; Consider a determinated planar clear-edged m-bonsai $T \otimes det(T)$. Then $\partial(T \otimes det(T))$ is the sum of $T' \otimes \wedge det(T)$, where T' is a planar clear-edged m-bonsai obtained by*

i) appending a vertex to T

ii) except to tips of T,

and so, getting a new edge e.

If there is no available appending position an a bonsai, the map assigns 0 to that bonsai.

For example, in planar clear-edged 3-bonsai, we can get an example like Figure 50 (in bonsais of Figure 50, determinanted terms are omitted. Note that one vertex in the first example is also a tip and the third bonsai is a cocycle. In the picture, newly appended vertices are drawn as open vertices, not indicating colors).

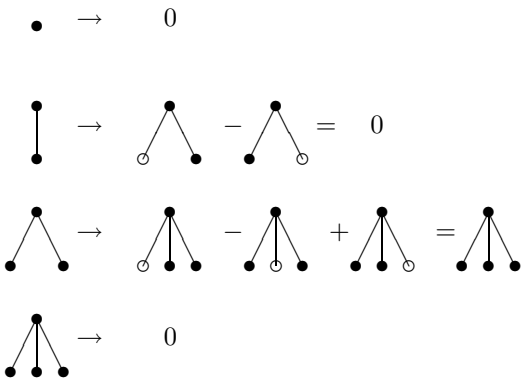


FIGURE 50

And by the totally same argument as for edge-numbered bonsai, we get

Theorem 11.1. $\partial^{i+1} \circ \partial^i = 0$. *That is, ∂ is a differential.*

Now let us consider the cohomology groups of this differential. First let us consider a cochain complex $\{\mathbf{D}^i\}$ consisting of corollae, with the boundary map $\bar{\partial}$ being the vertex-appending differential, but in this complex, appending to the one-vertex bonsai is allowed, Here, let us denote as $\mathbf{D}^0$ the module having the basis consisting of one vertex, and as $\mathbf{D}^n$ the one-dimensional module having the basis consisting of the corolla with n edges as shown above. By the definition of ∂^i, all terms in $\partial^i(T)$ are of the from $\pm T' \otimes e \wedge det(T)$, where T' runs over bonsais obtained by adding one vertex as defined in Definition 11.1. Since appending to an edge-end is forbidden, every ∂^i is appending a vertex to the root of a corolla. Now let us show some boundary map sequences of the thread starting from one vertex. For one vertex, ∂^0 acts as in Figure 51.

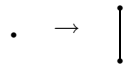


FIGURE 51

For the one-edge corolla, ∂^1 acts as in the second map of Figure 50, for the two-edge corolla, ∂^2 acts as in the third map of Figure 50, and so on.

This sequence of coboundary maps is

$$k \overset{id}{\to} k \overset{0}{\to} k \overset{id}{\to} k \overset{0}{\to} ... \tag{54}$$

where k is the base field.

So the cohomology groups of this thread of boundary maps is acyclic, and the lowest degree group is trivial.

Also, let $\{\mathbf{B}^n\}$ be a cochain complex defined by $\mathbf{B}^n = \mathbf{D}^{n+1}$ for later convenience. Then $\{\mathbf{B}^n\}$ is acyclic and $H^0 = k$, where k is the base field.

Now let us consider the general case. By the definition of ∂^i, all terms in $\partial^i(T)$ are of the form $\pm T' \otimes e \wedge det(T)$, where T' runs over bonsais obtained by adding a new edge e to T so that i) and ii) of Definition 11.1 hold. So T' has the form of appending a vertex to a vertex of T other than a tip. Having this intuitive fact in mind, let us present some new definitions and reorganize the cochain complex of bonsais.

Definition 11.2. *For a bonsai T, an edge e of T is called* twiggy *if it is at the end of a branch and the opposite end of the tip is a branching vertex. In Figure 52, e is twiggy in T and e' is not, and f is not a twiggy edge of T'.*

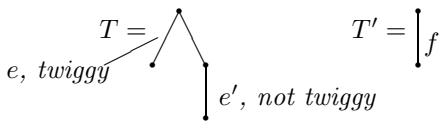


FIGURE 52

Definition 11.3. *A bonsai which has no twiggy edge is called a* vertex-appending seedling. *In this section, we will just call this* seedling. *The bonsais in Figure 53 are all seedlings. Note that the one-vertex bonsai is a seedling. Intuitively, a seedling is a bonsai which cannot be obtained by adding edges like i) and ii) of Definition 9.1.*

FIGURE 53

11.1. **Cohomology Groups of ∂ when $m = \infty$.** Let us try the same trick as in the proof of acyclicity of the branch-fixed differential.

Definition 11.4. *When S is a seedling, let $\mathbf{C}^{S,0}$ be the subspace of the determinanted planar clear-edged ∞-bonsai space having $\{S \otimes det(S)\}$ as the basis. And let $\mathbf{C}^{S,i+1}$ be the space with the basis $\{T' \otimes det(T')\}$, where T' is obtained by adding an edge to T, where $\{T \otimes det(T)\}$ is the basis of $\mathbf{C}^{S,i}$, as i) and ii) of Definition 11.1. Then since every bonsai is obtained by adding some edges to a seedling as given in Definition 11.1 and if S and S' are different seedlings, then the bonsais obtained by adding edges to S and S' as given in Definition 11.1 are different, the space of determinanted bonsais is the direct sum of the $\mathbf{C}^{S,i}$. We call this complex $\{\mathbf{C}^{S,i}\}$ a* thread *starting from S.*

Then, since $\partial(\mathbf{C}^{S,i}) \subset \mathbf{C}^{S,i+1}$, the cohomology groups of determinanted bonsais by the differential ∂ are the direct sum of cohomology groups of the threads $\{\mathbf{C}^{S,i}\}$.

Now let us show through an example that, for any thread $\{\mathbf{C}^{S,i}\}$, we can get a cochain complex which is isomorphic to it, obtained from the direct sums of tensor products of $\{\mathbf{D}^i\}$'s and $\{\mathbf{B}^i\}$'s as in the proof of acyclicity of branch-fixed differential. In 5-bonsai, the seedling S in Figure 54 can have twiggy edges at the positions of the twigs shown in the picture, and those twigs are grouped as surrounded by squares.

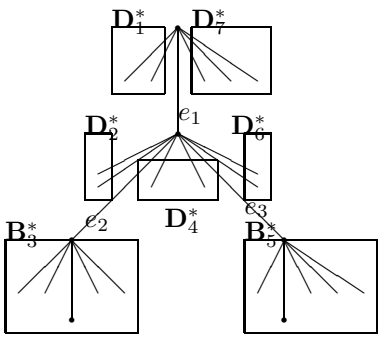


FIGURE 54

Note that, in Figure 54, adding edges to the corolla in each square is the same as attaching edges to the vertex at which the square is appended, each corolla corresponds to the module that is written on each square (In Figure 54, $\mathbf{D}_i^{m,*}$ is isomorphic to $\mathbf{D}^{m,*}$ and $\mathbf{B}_i^{m,*}$ is isomorphic to $\mathbf{B}^{m,*}$).

Then, as in Section 11, we can get the cochain isomorphism of $\{\mathbf{C}^{S,i}\}$ and $\mathbf{D}_1^* \otimes \mathbf{D}_2^* \otimes \mathbf{B}_3^* \otimes \mathbf{D}_4^* \otimes \mathbf{B}_5^* \otimes \mathbf{D}_6^* \otimes \mathbf{D}_7^*$. and use Künneth's theorem. Since the cohomology of $\{\mathbf{D}^*\}$ is acyclic with trivial base degree cohomology, it is clear that any $\{\mathbf{C}^{S,*}\}$ whose tensor product representation has $\{\mathbf{D}^*\}$ is acyclic with trivial base degree cohomology. The only seedlings having no $\{\mathbf{D}^*\}$ are the first two bonsais of Figure 50, i.e., the one-vertex bonsai v and the one-edge bonsai e. Since v is a cocycle, we have $H^0 = k$ and since $\mathbf{C}^{e,*} = \mathbf{B}^*$, we have $H^1 = H^0(\mathbf{B}^*) = k$. This illustrates a general argument. Thus we have

Theorem 11.2. *The cohomology groups H^i of the planar clear-edged m-bonsai Hopf algebra by vertex-appending differential are,*

$$H^i = \begin{cases} k & \text{if } i = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

11.2. **Cohomology Groups of ∂ when $m < \infty$: Terminology.** In this subsection, let us consider the cohomology groups in case $m < \infty$. It is not as easy as in the previous subsection to get a simple tensor product representation of a thread $\{\mathbf{C}^{S,i}\}$ when each vertex has the upper bound m of ramification number. So we need to change our strategy for the case of $m < \infty$. Since every $\{\mathbf{C}^{S,i}\}$ is finite-dimensional, we can calculate the cohomology groups by considering a finite number of bonsais. So from now on, we develop an "inductive strategy" for calculating the cohomology groups of the thread $\{\mathbf{C}^{S,i}\}$ for any given S.

Let us illustrate the basic idea of the " inductive strategy" using an example. In 2-bonsai, let S_i be the ladder of length i. Then the cohomology $H^j(S_1)$ of $\{\mathbf{C}^{S_1,j}\}$ is, by a sequence as in Figure 55, $H^1(S_1) = H^2(S_2) = k$, the base field.

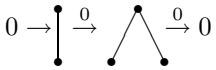


FIGURE 55

Let us find an inductive step to get $H^i(S_{j+1})$'s from $H^i(S_j)$'s, so that we can get the cohomology group of every $\{\mathbf{C}^{S_i,j}\}$.

As in Figure 56, when T is a linear combination of m-bonsais, the other two expressions of Figure 56 represent linear combinations of bonsais obtained by attaching a bonsai to the roots of bonsais which are the components of T.

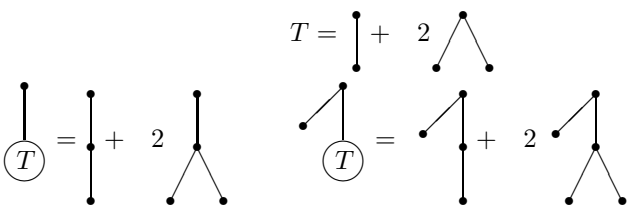


FIGURE 56

Then, when T is a linear combination of bonsais in $\{\mathbf{C}^{S_i,j}\}$ in 2-bonsai, the map ∂ on $\{\mathbf{C}^{S_{i+1},j}\}$ is expressed in Figure 57.

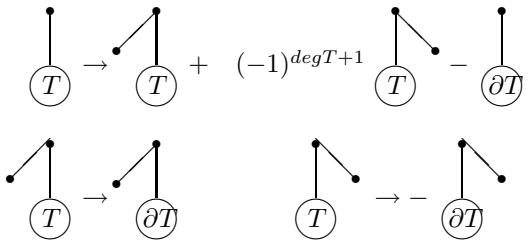


FIGURE 57

From Figure 57, we can find that the kernel of ∂ on $\{\mathbf{C}^{S_{i+1},j}\}$ is generated by the linear combinations of bonsais shown in Figure 58 in which $T \in ker\partial$ and $\partial T' = \partial T'' = T$, and that the image of ∂ on $\{\mathcal{C}^{S^{i+1},j}\}$ is generated by the linear combinations of bonsais shown in Figure 59.

So we can write

$$\frac{ker\partial}{im\partial} = \frac{\langle A, B, C\rangle}{\langle a, b, c\rangle}. \tag{55}$$

In A, we have $\partial T = 0$, $\partial T' = T$ and $\partial T'' = T$, so we have A − a $= (-1)^{degT''+1}$C. Hence we have C is generated by A and a. Also, in c, when $T' = \partial T$, we have $a - b = (-1)^{degT} c$. So c is generated by a and b. So we have

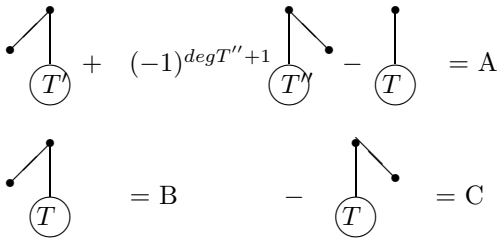


FIGURE 58

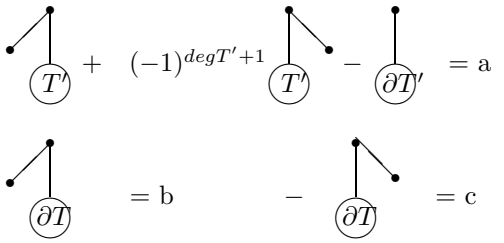


FIGURE 59

$$\frac{ker\partial}{im\partial} = \frac{\langle A, B\rangle}{\langle a, b\rangle}. \tag{56}$$

In Figure 58, $\partial T'' = T$. So A in Figure 58 can be redrawn as Figure 60.

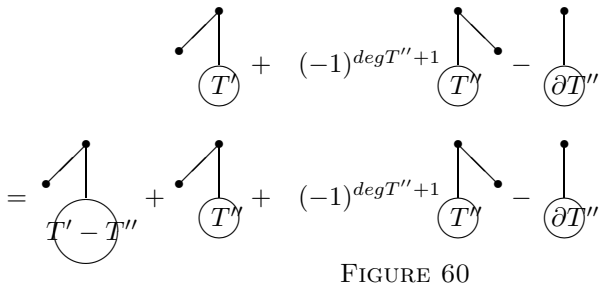


FIGURE 60

Since $\partial T' = \partial T'' = T$, we have $\partial(T' - T'') = 0$ and so A can be rewritten as B+a. Hence we have

$$\frac{ker\partial}{im\partial} = \frac{\langle B + a, B\rangle}{\langle a, b\rangle} = \frac{\langle a, B\rangle}{\langle a, b\rangle} = \frac{\langle a\rangle \oplus \langle B\rangle}{\langle a\rangle \oplus \langle b\rangle} = \frac{\langle B\rangle}{\langle b\rangle}. \tag{57}$$

Obviously, $\frac{\langle B\rangle}{\langle b\rangle}$ is isomorphic to the cohomology group of $\mathbf{C}^{S_i,j}$. Since B and b have two more edges than bonsais in $\mathbf{C}^{S_i,j}$, we can see

Theorem 11.3. *When $H^j(S_i)$ is the j-th cohomology group of the thread $\mathbf{C}^{S_i,j}$, $H^{j+2}(S_{i+1}) = H^j(S_i)$.*

As in the previous theorem, we can generalize the process of getting the cohomology of the thread starting from S' which is obtained by attaching the root of a seedling S to a tip of a corolla, when the cohomology of the thread starting from S is already known. Let us define some new terminology. We first define a new kind of "seedling".

Definition 11.5. *We define a* grafting seedling *as a bonsai defined as one of the following;*

1) a seedling

2) a bonsai obtained by attaching even-arity corollas to vertices of a seedling S which are more than two edges from any tip, and by replacing branch-end edges of S to even-arity corollas.

In Figure 61, the first three bonsais are grafting seedlings and the last is not.

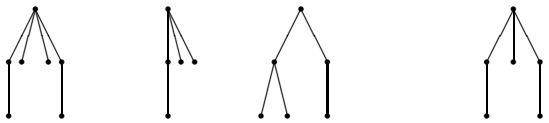


FIGURE 61

Definition 11.6. *A grafting seedling $gs(n; T_1, T_2, ...T_{n+1}; S_1, S_2, ..., S_n)$, which we call a* grafting seedling *is constructed like the following; In the corolla C with arity n, the corollae T_1, T_2, ... , T_{n+1} of* even *arities are attached so that the root of T_1 is attached to the root of C on the left of the leftmost edge of C, the root of T_2 is attached to the root of C between the first leftmost edge and the second leftmost edge of C,..., and so on, and the grafting seedlings S_1, S_2, ... ,S_n are attached so that the root of S_1 is attached to the tip of the first leftmost edge of C, the root of S_2 is attached to the tip of the second leftmost edge of C, ... , and so on, as in Figure 62 in 6-bonsai.*

$gs(3; T_1, T_2, T_3, T_4; S_1, S_2, S_3) =$

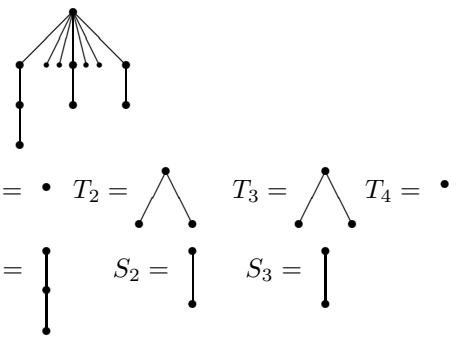


FIGURE 62

Note that the grafting seedling defined here is different from the seedling we used until now.

Definition 11.7. *We define the relation $T_1 \to T_2$ of clear-edged m-bonsais T_1 and T_2 as follows;*

i) T_2 is a nonzero component of ∂T_1 and T_2 is obtained by attaching one edge to the root *of T_1*

or

ii) T_2 is obtained by attaching an edge to a non-root vertex of T_1.

In 3-bonsai, we have examples as in Figure 63.

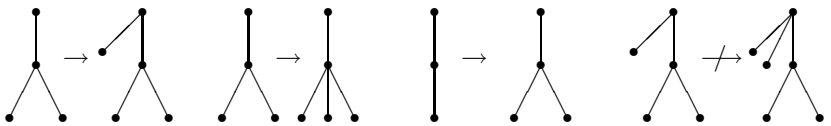


FIGURE 63

Definition 11.8. *We define the relation $T_1 \Rightarrow T_1'$ if there is a sequence of m-bonsais such that $T_1 \to T_2 \to ... \to T_n = T_1'$ or $T_1 = T_1'$.*

Let $\mathbf{K}(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))$ be the vector space generated by the bonsais T' such that $gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n) \Rightarrow T'$. Also, when S is a grafting seedling and $\mathbf{C}$ is the cochain complex of m-bonsai, we define $\mathbf{K}^i(S) := \mathbf{C}^i \cap \mathbf{K}(S)$.

Theorem 11.4. *When S is a grafting seedling and $H^i(S)$ is the i-th cohomology group of the thread $\{\mathbf{K}^i(S)\}$, the i-th cohomology group H^i of m-bonsai is $H^i = \bigoplus_{S \text{ is a grafting seedling}} H^i(S)$.*

Proof. We have $\mathbf{C} = \bigoplus_{S \text{ is a grafting seedling}} \mathbf{K}(S)$ since i)there is no T such that $T \Rightarrow T'$, $T \neq T'$ and T' is a grafting seedling and ii)if two grafting seedlings S and S' are not equal, then $\mathbf{K}(S) \cap \mathbf{K}(S') = \emptyset$. Also we have $\partial \mathbf{K}^i(S) \subset \partial \mathbf{K}^{i+1}(S)$. So we get the wanted result. □

11.3. **The Cohomology Groups for each $\{\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))\}$.** Now we have to calculate the cohomology groups for $\{\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))\}$. First let us define some notation.

Definition 11.9. *For any integer $n \geq 0$, the cochain complex $\{\mathbf{D}_{2n}^j\}$ is defined as follows; when $j = 2n$ or $2n+1$, $\mathbf{D}_{2n}^j$ is a one-dimensional vector space with the basis $\{C_j\}$, where C_j is the corolla of arity j (when $j = 0$, C_j is the one-vertex bonsai), and otherwise, $\mathbf{D}_{2n}^j = 0$. And when $j = 2n$, the boundary map $\partial : \mathbf{D}_{2n}^j \to \mathbf{D}_{2n}^{j+1}$ is given by $C_j \mapsto C_{j+1}$ and otherwise, $\partial = 0$.*

To calculate the cohomology groups of $\{\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))\}$, we use a similar type of tensor product representation as in Section 11.

Definition 11.10. *Let $B(n; U_1, ..., U_{n+1}; V_1, ..., V_n)$ be the bonsai obtained by replacing T_i by U_i and S_i by V_i in $gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n)$, where U_i is the corolla C_k of arity $k = degT_i$ or $degT_i + 1$ (i.e., U_i is a basis element of $\{\mathbf{D}_{degT_i}\}$) and V_i is a bonsai in the thread $\{\mathbf{C}^{S_i,i}\}$ starting from S_i (see Subsection 11.2).*

Let us define the cochain complexes $\{\mathbf{D}_i\}$ and $\{\mathbf{E}_i\}$ as $\{\mathbf{D}_{degT_i}\}$ and $\{\mathbf{C}(S_i)\}$. Then we can define an isomophism P from $\mathbf{K}^i(S)$ to $\mathbf{L}^i = \mathbf{D}_1^{i_1} \otimes \mathbf{E}_1^{j_1} \otimes ... \otimes \mathbf{D}_n^{i_n} \otimes \mathbf{E}_n^{j_n} \otimes \mathbf{D}_{n+1}^{i_{n+1}}$ where $i_1 + ... + i_{n+1} + j_1 + ... + j_n + n = i$ and $i_1 + ... + i_{n+1} + n \leq m$, by sending $B(n; U_1, ...U_{n+1}; V_1, ..., V_n)$ to $U_1 \otimes V_1 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}$. the differential d in $\mathbf{L}$ is defined as, when $degU_1 + ... + degU_{n+1} < m - n$,

$$\begin{aligned} & d(U_1 \otimes V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}) \\ = \quad & \partial U_1 \otimes V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1} \\ + \quad & U_1 \otimes (-1)^{p_1} \partial V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}) \\ + \quad & U_1 \otimes V_1 \otimes (-1)^{q_1} \partial U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}) \\ + \quad & U_1 \otimes V_1 \otimes U_2 \otimes (-1)^{p_2} \partial V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}) \\ & +... + \\ + \quad & U_1 \otimes V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes (-1)^{q_n} \partial U_{n+1}), \end{aligned}$$

where $p_i = degU_1 + degV_1 + ... + degU_{i-1} + degV_{i-1} + degU_i + i$ and $q_i = degU_1 + degV_1 + ... + degU_i + degV_i + i$, and if $degU_1 + ... + degU_{n+1} = m - n$,

$$\begin{aligned} & d(U_1 \otimes V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}) \\ = \quad & \partial U_1 \otimes V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1} \\ + \quad & U_1 \otimes V_1 \otimes (-1)^{q_1} \partial U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1}) \\ & +... + \\ + \quad & U_1 \otimes V_1 \otimes U_2 \otimes V_2 \otimes ... \otimes U_n \otimes V_n \otimes (-1)^{q_n} \partial U_{n+1}). \end{aligned}$$

Then by the definition of ∂ in $\mathbf{C}$, the isomorphism P is a cochain complex isomorphism from $(\mathbf{K}^i(S), \partial)$ to $(\mathbf{L}^i, d)$.

Now let us define a double complex $\{\mathbf{M}^i \otimes \mathbf{N}^j\}^{i,j}$ where $\mathbf{M}^i = \sum \mathbf{D}_1^{i_1} \otimes ... \otimes \mathbf{D}_{n+1}^{i_{n+1}}$ where $i_1, ..., i_{n+1}$ satisfy $i_1 + ... + i_{n+1} + n = i$ and $i \leq m$, and $\mathbf{N}^j = \sum_{j_1 + ... + j_n = j} \mathbf{E}_1^{j_1} \otimes ... \otimes \mathbf{E}_n^{j_n}$ where $j_1, ..., j_n$ satisfy $j_1 + ... + j_n = j$, and differentials $\bar{\partial_1}$ and $\bar{\partial_2}$ are

$$\begin{aligned} & \bar{\partial_1}(U_1 \otimes U_2 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes ... \otimes V_n) \\ = \quad & \partial U_1 \otimes U_2 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes ... \otimes V_n \\ + \quad & U_1 \otimes (-1)^{q_1} \partial U_2 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes ... \otimes V_n \\ & +... + \\ + \quad & U_1 \otimes U_2 \otimes ... \otimes (-1)^{q_n} \partial U_{n+1} \otimes V_1 \otimes ... \otimes V_n \end{aligned}$$

where $U_1 \otimes ... \otimes U_{n+1} \in \mathbf{M}^j (j < m)$ (if $j \geq m$, since $\mathbf{M}^j = 0$, $\bar{\partial_1} = 0$), and

$$\begin{aligned} & \bar{\partial_2}(U_1 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes V_2 \otimes ... \otimes V_n) \\ = \quad & U_1 \otimes ... \otimes U_{n+1} \otimes (-1)^{p_1} \partial V_1 \otimes V_2 \otimes ... \otimes V_n \\ + \quad & U_1 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes (-1)^{p_2} \partial V_2 \otimes ... \otimes V_n) \\ & +... + \\ + \quad & U_1 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes V_2 \otimes ... \otimes (-1)^{p_n} \partial V_n). \end{aligned}$$

Then each of $\bar{\partial_1}$ and $\bar{\partial_2}$ is a differential of Künneth products of $\{\mathbf{D}_{2n}^i\}$'s and $\{\mathbf{C}^{S,j}\}$'s, respectively. Let $\bar{\partial} = \bar{\partial_1} + \bar{\partial_2}$. When the bijection $Q : \{\mathbf{L}^i\} \to \{\mathbf{M}^i \otimes \mathbf{N}^j\}$

is given by $U_1 \otimes V_1 \otimes ... \otimes U_n \otimes V_n \otimes U_{n+1} \mapsto U_1 \otimes ... \otimes U_{n+1} \otimes V_1 \otimes ... \otimes V_n$, immediately by the definitions of d and $\bar{\partial}$, we have

$$\bar{\partial} \circ Q = Q \circ d. \tag{58}$$

Hence $\bar{\partial}$ satisfies $\bar{\partial}\circ\bar{\partial} = 0$. So we have $0 = \bar{\partial}\circ\bar{\partial} = \bar{\partial_1}\circ\bar{\partial_1}+\bar{\partial_1}\circ\bar{\partial_2}+\bar{\partial_2}\circ\bar{\partial_1}+\bar{\partial_2}\circ\bar{\partial_2} = \bar{\partial_1}\circ\bar{\partial_2}+\bar{\partial_2}\circ\bar{\partial_1}$, therefore $\{\mathbf{M}^i \otimes \mathbf{N}^j\}$ is a double complex. Also, by (58), Q induces the cochain complex isomorphism $(\{\mathbf{L}^i\}, d) \to (\{\mathbf{M}^i \otimes \mathbf{N}^j\}, \bar{\partial})$. So by the isomorphism $Q \circ P$, $(\{\mathbf{K}(gs(n; T_1, ...T_{n+1}; S_1, ..., S_n))\}, \partial)$ and $(\{\mathbf{M} \otimes \mathbf{N}\}, \bar{\partial})$ are isomorphic. In order to calculate the cohomology of $(\{\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))\})$, we can use the cohomology groups $(\{\mathbf{M} \otimes \mathbf{N}\}, \bar{\partial})$. Let us use the spectral sequence starting with $\bar{\partial_1}$.

Since $\bar{\partial_1}$ acts only on $\{\mathbf{M}^i\}$, we can write $E_1^{i,j} = H_{\bar{\partial_1}}(\{\mathbf{M}^i \otimes \mathbf{N}^j\}) = H^i_{\bar{\partial_1}}(\{\mathbf{M}\}) \otimes \mathbf{N}^j$, and similarly, since $\bar{\partial_2}$ acts only on $\{\mathbf{N}^j\}$, we can write $E_2^{i,j} = H_{\bar{\partial_2}}(E_1) = H^i_{\bar{\partial_1}}(\{\mathbf{M}\}) \otimes H^j_{\bar{\partial_2}}(\{\mathbf{N}\})$. Since $\bar{\partial_2}$ is a Künneth product of $\mathbf{E}_i$'s, $H_{\bar{\partial_2}}(\{\mathbf{N}\})$ is the Künneth product $H(\mathbf{E}_1) \otimes ... \otimes H(\mathbf{E}_n) = H(S_1) \otimes ... \otimes H(S_n)$. But as we can see in the definition of $\bar{\partial_1}$, it is not exactly the canonical differential of Künneth product, so the calculation of $H(\{\mathbf{M}\})$ takes some more consideration.

When $i < m$, $\mathbf{M}^i = \bigoplus_{i_1+...+i_{n+1}=i-n} \mathbf{D}_1^{i_1} \otimes ... \otimes \mathbf{D}_{n+1}^{i_{n+1}}$ and $\bar{\partial_1}$ is a Künneth differential. So when $i < m$, $H^i_{\bar{\partial_1}}(\{\mathbf{M}\})$ is a Künneth product of $H(\{\mathbf{D}_j\})$'s, and so it is 0, since each $\{\mathbf{D}_j\}$ is acyclic.

When $i > m$, $\mathbf{M}^j = 0$. So when $i \geq m$, $\bar{\partial_1} = 0$ on $\mathbf{M}^i$ and $H^i_{\bar{\partial_1}}(\{\mathbf{M}\}) = 0 (i > m)$.

Let us calculate $H^m_{\bar{\partial_1}}(\{\mathbf{M}\}) = ker\bar{\partial_1}|_{\mathbf{M}^m} / im\bar{\partial_1}|_{\mathbf{M}^{m-1}}$.

First, if $(i_1 + 1) + .. + (i_{n+1} + 1) + n < m$, $\mathbf{M}^m = 0$. So, $H^m_{\bar{\partial_1}}(\{\mathbf{M}\}) = 0$.

Suppose that $(i_1 + 1) + .. + (i_{n+1} + 1) + n \geq m$. Since the clear-edged m-bonsai is a vector space over the field k, we just need to calculate the dimension of H^m. We have

$$dim(H^m_{\bar{\partial_1}}(\{\mathbf{M}\})) = dim(ker\bar{\partial_1}|_{\mathbf{M}^m}) - dim(im\bar{\partial_1}|_{\mathbf{M}^{m-1}}). \tag{59}$$

$$dim(ker\bar{\partial_1}|_{\mathbf{M}^m}) = dim(\mathbf{M}^m) \tag{60}$$

and

$$dim(im\bar{\partial_1}|_{\mathbf{M}^{m-1}}) = dim(\mathbf{M}^{m-1}) - dim(ker\bar{\partial_1}|_{\mathbf{M}^{m-1}}). \tag{61}$$

Since

$$H^j_{\bar{\partial_1}}(\{\mathbf{M}\}) = 0 \text{ when } j < m, \tag{62}$$

we have

(63)

$$dim(ker\bar{\partial_1}|_{\mathbf{M}^{m-1}}) = dim(im\bar{\partial_1}|_{\mathbf{M}^{m-2}}) = dim(\mathbf{M}^{m-2}) - dim(ker\bar{\partial_1}|_{\mathbf{M}^{m-2}}),$$

and so, by (59)-(63),

$$\begin{aligned} & dim(H^m_{\bar{\partial_1}}(\{\mathbf{M}\})) \\ = \quad & dim(\mathbf{M}^m) - (dim(\mathbf{M}^{m-1}) - dim(ker\bar{\partial_1}|_{\mathbf{M}^{m-1}})) \\ = \quad & dim(\mathbf{M}^m) - dim(\mathbf{M}^{m-1}) + dim(\mathbf{M}^{m-2}) - dim(ker\bar{\partial_1}|_{\mathbf{M}^{m-2}}). \end{aligned} \tag{64}$$

Continuing like this, we have

$$dim(H^m_{\overline{\partial}_1}(\{\mathbf{M}\})) \tag{65}$$
$$= \quad dim(\mathbf{M}^m) - dim(\mathbf{M}^{m-1}) + dim(\mathbf{M}^{m-2}) - dim(\mathbf{M}^{m-3}) + ...$$

and since $\mathbf{M}^i = 0$ if $i < n + P$ where

$$P = degT_1 + ... + degT_{n+1}, \tag{66}$$

we have

$$dim(H^m_{\overline{\partial}_1}(\{\mathbf{M}\})) = dim(\mathbf{M}^m) - dim(\mathbf{M}^{m-1}) \tag{67}$$
$$+... + (-1)^{m-(n+P)} dim(\mathbf{M}^{n+P}).$$

Let N be this number and let us calculate it. By the definition of $\mathbf{M}^i$, its dimension is that of

$$\bigoplus_{i_1+...+i_{n+1}+n=i} \mathbf{D}_1^{i_1} \otimes ... \otimes \mathbf{D}_n^{i_n}. \tag{68}$$

Every $\mathbf{D}_k^{i_k}$ is one-dimensional when $p_k = i_k - degT_k$ is 0 or 1, and 0 otherwise. So the above direct sum is,

$$\bigoplus_{p_1+...+p_{n+1}+P+n=i} \mathbf{D}_1^{p_1+degT_1} \otimes ... \otimes \mathbf{D}_n^{p_n+degT_n}. \tag{69}$$

Hence, $dim(\mathbf{M}^i)$ is the number of $(p_1, ..., p_{n+1})$'s satisfying $p_1+...+p_{n+1}+P+n = i$ and each p_k is 0 or 1. Hence

$$dim(\mathbf{M}^i) = \binom{n+1}{i-P-n} \tag{70}$$

and

$$N = \binom{n+1}{m-P-n} - \binom{n+1}{(m-1)-P-n} + ... + (-1)^{m-P-n} \binom{n+1}{0}.$$

Now we have

$$E_2^{i,j} = \begin{cases} \bigoplus_{j_1+...+j_n=j} k^N \otimes H^{j_1}(S_1) \otimes ... \otimes H^{j_n}(S_n) & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases}$$

Since $E_2^{i,j} = 0$ except when $j = m$, every "knight's move map" on $E^{i,j}$'s is trivial. So the spectral sequence collapses and we have $H^n_{\overline{\partial}} = \bigoplus_{i+j=n} E^{i,j} = \bigoplus_{j_1+...+j_n=n-m} k^N \otimes H^{j_1}(S_1) \otimes ... \otimes H^{j_n}(S_n)$, and since

$$k^N \otimes H^{j_1}(S_1) \otimes ... \otimes H^{j_n}(S_n) = (H^{j_1}(S_1) \otimes ... \otimes H^{j_n}(S_n))^{\oplus N}, \tag{71}$$

we finally have

Theorem 11.5. *When $H^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))$ is the i-th cohomology group of the thread $\mathbf{K}^i(gs(n; T_1, ..., T_{n+1}; S_1, ..., S_n))$, the i-th cohomology group H^i of clear-edged m-bonsai is $H^i = \bigoplus_{S \text{ is a grafting seedling}} H^i(S)$.*

And, if $P = degT_1+...+degT_n < m-2n+1$, then $H^i(gr(n; T_1, ..., T_{n+1}; S_1, ..., S_n)) = 0$. Otherwise,

$$H^i(gr(n; T_1, ..., T_{n+1}; S_1, ..., S_n)) = \bigoplus_{j_1+...+j_n=n-m} [H^{j_1}(S_1) \otimes ... \otimes H^{j_n}(S_n)]^{\oplus N}$$

where

$$N = \binom{n+1}{m-P-n} - \binom{n+1}{(m-1)-P-n} + ... + (-1)^{m-(n+P)} \binom{n+1}{0}.$$

12. Differential and Appending of Clear-edged Bonsai

Let us consider the relationship between the appending operation $*$ on the clear-edged m-bonsai Hopf algebra $\mathcal{H}_{c,m}$ and the vertex appending differential ∂. We work mod 2 again. First, the operation $T_1 * T_2$ is the sum of all m-bonsais obtained by connecting the root of T_1 and a vertex of T_2 with an edge, as illustrated for 3-bonsai in Figure 64.

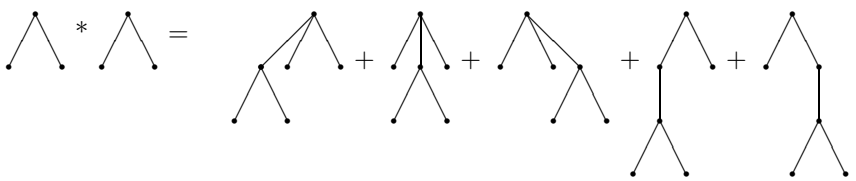


Figure 64

In this section, we will show that $T_1 *_2 T_2 = 0$ for every T_1 and T_2 as in Section 10.

Temporarily in this section, we use a differential $\bar{\partial}$ of corollas which is the same as ∂ except $\bar{\partial}(v) = e$, where v is the one-vertex bonsai and e is the one-edge bonsai.

12.1. **Brief Table of Contents.** In this section, first we will describe $T_1 *_1 T_2$ for each of the following cases when T_2 is a corolla, by dividing the cases as follows;

i) When $\partial T_2 \neq 0$ and $deg(T_2) \leq m-2$
ii) When $\partial T_2 \neq 0$ and $deg(T_2) = m-1$
iii) When $\partial T_2 = 0$ and $deg(T_2) \leq m-2$
iv) When $\partial T_2 = 0$ and $deg(T_2) = m-1$
v) When $deg(T_2) = m$

Second, we will show that $T_1 *_2 T_2$ for each of the following cases when T_2 is a corolla.

i) When $\partial T_2 \neq 0$ and $deg(T_2) \leq m-3$
ii) When $\partial T_2 \neq 0$ and $deg(T_2) = m-2$
iii) When $\partial T_2 \neq 0$ and $deg(T_2) = m-1$
iv) When $\partial T_2 = 0$ and $deg(T_2) \leq m-2$
v) When $\partial T_2 = 0$ and $deg(T_2) = m-1$
vi) When $deg(T_2) = m$

Finally, we show that $T_1 *_2 T_2$ for a general T_2.

12.2. $T_1 *_1 T_2$ **when** T_2 **is a corolla.** For clear-edged bonsai, since it is not an operad, we cannot use broomstick diagrams for graphical proof. Let us look into $T_1 *_1 T_2$ by dividing the cases of ∂T_2 and $deg(T_2)$.

12.2.1. *When* $\partial T_2 \neq 0$ *and* $deg(T_2) \leq m-2$. If T_1 is not the one-vertex bonsai, then $(\partial T_1) * T_2$ is the sum of terms in $\partial(T_1 * T_2)$ obtained by attaching edges to T_1. So $T_1 *_1 T_2 = (\partial T_1) * T_2 + T_1 * (\partial T_2) - \partial(T_1 * T_2)$ ("$-$" in this equation is in fact "$+$", since we are working mod 2),

$T_1 * (\partial T_2)$

$+ \sum$(a term in $\partial(T_1 * T_2)$ which is obtained by attaching an edge to a vertex of $T_1 * T_2$ not in T_1 so that i) and ii) of Definition 11.1 is satisfied)

Then the first summand is the sum of i) bonsais A_1 obtained by connecting a non-root vertex of ∂T_2 and the root of T_1 with an edge, which is depicted as in the first equation of Figure 65 in 3-bonsai and ii) bonsais A_2 obtained by connecting the root of ∂T_2 and the root of T_1 with an edge, which is depicted as in the second equation in Figure 65 in 3-bonsai.

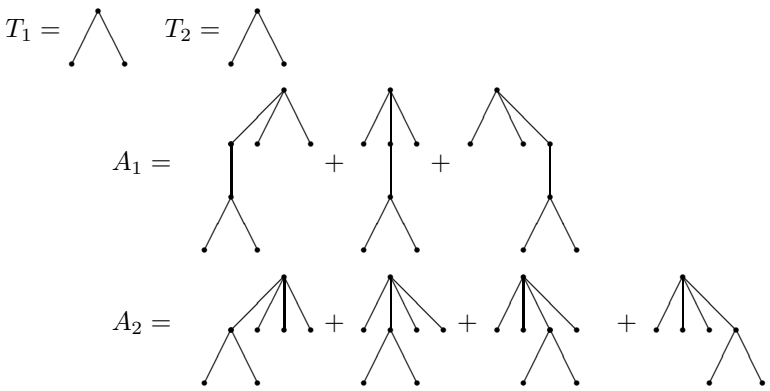


FIGURE 65

The second summand is the sum of A_3, A_4 and A_5, where A_3, A_4 and A_5 are as follows:

A_3 is the sum of bonsais obtained by connecting a vertex v of T_2 and the root of T_1 with one edge and attaching an edge to v, as in Figure 66 in 3-bonsai.

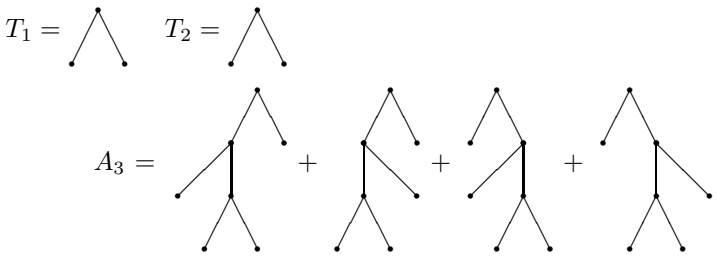


FIGURE 66

A_4: bonsais B obtained as follows; suppose that T_2 is constructed by attaching the roots of corollae X_1 and X_2 so that X_1 is on the left and X_2 is on the right.

Then B is obtained by attaching the roots of $\bar{\partial}X_1$, V and X_2 from the left or X_1, V and $\bar{\partial}X_2$ from the left, where V is obtained by attaching the root of T_1 to the lower vertex of the one-edge clear-edged bonsai, illustrated in Figure 67. In Figure 67, the first term of A_4 is constructed by $\bar{\partial}X_1$, V and X_2, and the second term is constructed by X_1, V and $\bar{\partial}X_2$.

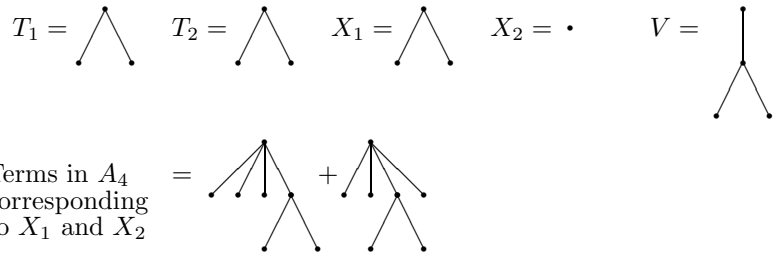


FIGURE 67

A_5: bonsais B obtained as follows; Suppose T_2 is constructed by attaching the roots of bonsais Y_1, E and Y_2 from the left, where Y_1 and Y_2 are corollae and E is the one-edge bonsai. Then B is constructed by attaching the roots of $\bar{\partial}Y_1$, V and Y_2 from the left or Y_1, V and $\bar{\partial}Y_2$ from the left, where V is obtained by connecting the root of T_1 to the lower vertex of E with one edge. This is illustrated in Figure 68.

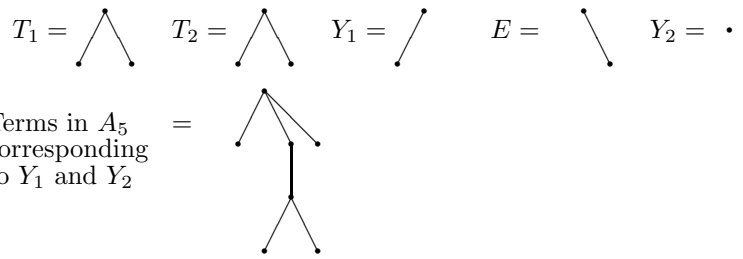


FIGURE 68

If T_1 is the one-vertex bonsai, A_2 becomes 0, since it is equal to $\partial\partial T_2$. And $A_3 = 0$, since it is twice a multiple of the bonsais obtained by attaching the root of the two-edge corolla to a tip of T_2 (note that we are working mod 2). Also, $A_4 = \partial\partial T_2 = 0$. So $T_1 *_1 T_2 = A_1 + A_5$.

12.2.2. *When* $\partial T_2 \neq 0$ *and* $deg(T_2) = m-1$. When T_1 is not the one-vertex bonsai: This is almost the same as the previous case, but we cannot add more than one edge to the root of T_2. So $T_1 *_1 T_2$ is the sum of the terms A_1, A_3 and A_5.

When T_1 is the one-vertex bonsai: $A_3 = 0$, since it is twice a multiple of the bonsais obtained by attaching the root of the two-edge corolla to a tip of T_2(note that we are working mod 2). So $T_1 *_1 T_2 = A_1 + A_5$.

12.2.3. *When* $\partial T_2 = 0$ *and* $deg(T_2) \leq m-2$. When T_1 is not the one-vertex bonsai: As in the first case, $T_1 *_1 T_2 = (\partial T_1) * T_2 + T_1 * (\partial T_2) - \partial(T_1 * T_2)$ is

$T_1 * (\partial T_2)$

$+ \sum$(a term in $\partial(T_1 * T_2)$ that is obtained by attaching an edge to a vertex of $T_1 * T_2$ not in T_1 so that i) and ii) of Definition 11.1 is satisfied)

Here $\partial T_2 = 0$. So we just have the latter summand in $T_1 *_1 T_2$. As in the first case again, we have $T_1 *_1 T_2 = A_3 + A_4 + A_5$.

When T_1 is the one-vertex bonsai: $A_3 = 0$, since it is a twice multiple of the bonsais obtained by attaching the root of the two-edge corolla to a tip of T_2 (note that we are working mod 2). Also, $A_4 = \partial\partial T_2 = 0$. Hence, $T_1 *_1 T_2 = A_5$.

12.2.4. *When* $\partial T_2 = 0$ *and* $deg(T_2) = m-1$. When T_1 is not the one-vertex bonsai: This is almost the same as the previous case, but we cannot add more than one edge to the root of T_2. So $T_1 *_1 T_2$ is the sum of the terms A_3 and A_5.

When T_1 is the one-vertex bonsai: $A_3 = 0$, since it is a twice multiple of the bonsais obtained by attaching the root of the two-edge corolla to a tip of T_2(note that we are working mod 2). Hence, $T_1 *_1 T_2 = A_5$.

12.2.5. *When* $deg(T_2) = m$. When T_1 is not the one-vertex bonsai: This is almost the same as the previous case, but we cannot add any more edges to the root of T_2. So $T_1 *_1 T_2$ is A_3.

When T_1 is the one-vertex bonsai: $A_3 = 0$, since it is twice a multiple of the bonsais obtained by attaching the root of the two-edge corolla to a tip of T_2(note that we are working in mod 2). Hence, $T_1 *_1 T_2 = 0$.

12.3. $T_1 *_2 T_2$ **when** T_2 **is a corolla.** For each case in the last subsection, let us show that $T_1 *_2 T_2 = 0$.

12.3.1. *When* $\partial T_2 \neq 0$ *and* $deg(T_2) \leq m-3$. In this case, we have $deg(\partial T_2 \leq m-2)$ and $\partial(\partial T_2) = 0$. And as in the first case of the last subsection, $T_1 *_2 T_2$ is

$T_1 *_1 (\partial T_2)$

$+ \sum$(a term in $\partial(T_1 *_1 T_2)$ obtained by attaching an edge to a vertex of $T_1 * T_2$ not in T_1, so that i) and ii) of Definition 11.1 are satisfied)

When T_1 is not the one-vertex bonsai: By the third case of the last subsection, $T_1 *_1 (\partial T_2)$ is $A_3 + A_4 + A_5$, and by the first case of the last subsection, the sum of the terms in $\partial(T_1 *_1 T_2)$ obtained by attaching an edge to a vertex not in T_1 is $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}A_1 + \hat{\partial}A_2 + \hat{\partial}A_3 + \hat{\partial}A_4 + \hat{\partial}A_5$, where $\hat{\partial}X$, when X is a sum of terms in $T_1 *_1 T_2$, is the sum of the bonsais in ∂X obtained by attaching an edge to a vertex which is not originally in T_1.

In order to show that $T_1 *_2 T_2 = T_1 *_1 \partial T_2 + \hat{\partial}(T_1 *_1 T_2) = 0$ pictorially, let us introduce a new picture convention. In Figure 69, where T_2 is a corolla with 4 edges, each triangle represents a corolla (including the one-vertex bonsai).

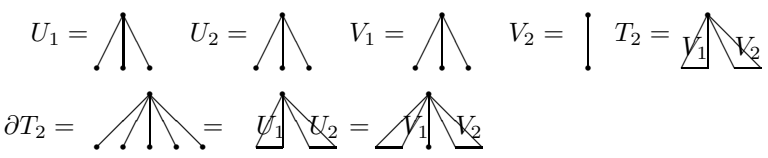


FIGURE 69

With this pictorial convention, we can draw $T_1 *_1 \partial T_2 = A_3 + A_4 + A_5$ as in Figure 70, like A+B+C+D+E+F.

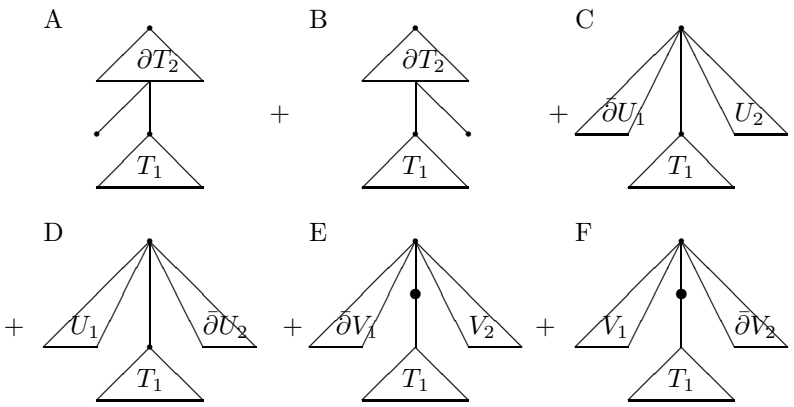


FIGURE 70

In $\hat{\partial}(T_1 *_1 T_2)$, $\hat{\partial} A_1$ can be drawn as in Figure 71.

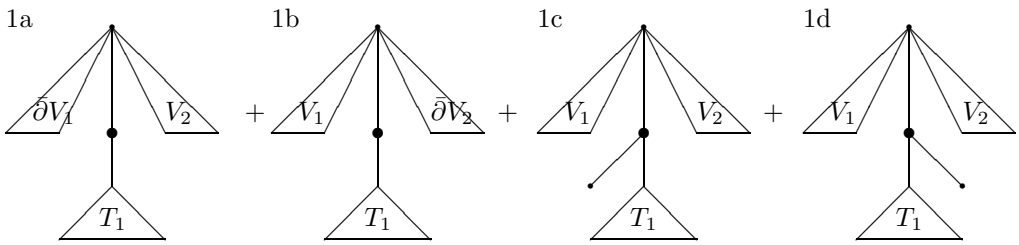


FIGURE 71

$\hat{\partial} A_2$ is as in Figure 72.

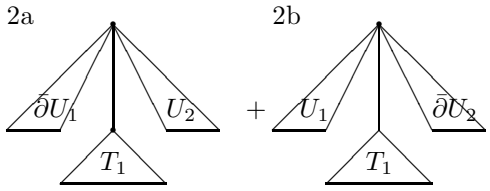


FIGURE 72

$\hat{\partial} A_3$ is as in Figure 73, where W_1 and W_2 are as in Figure 74.

$\hat{\partial} A_4$ can be drawn as in Figure 75.

And $\hat{\partial} A_5$ can be drawn as in Figure 76.

Then we can get the pairs which are canceled as follows; A and 1c, B and 1d, C and 2a, D and 2b, E and 1a, F and 1b, 3a and 3b, 3c and 5e, 3d and 5f, 3e and 5g, 3f and 5h, 4b and 4c, 5b and 5c. 4a, 4d, 5a and 5d are 0, because they have $\bar{\partial}^2$ in it. (Since 4b and 4c cancel each other, $\hat{\partial} A_4 = 0$.)

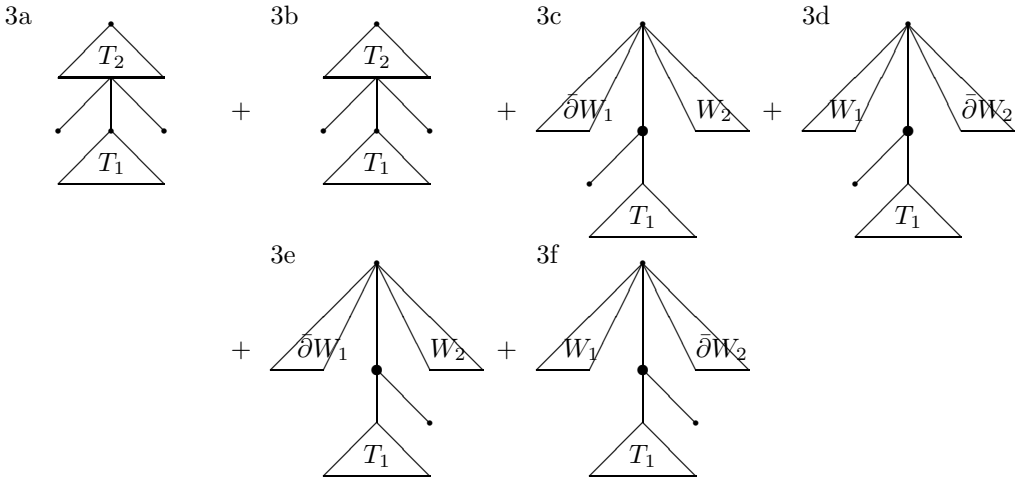


FIGURE 73

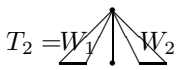


FIGURE 74

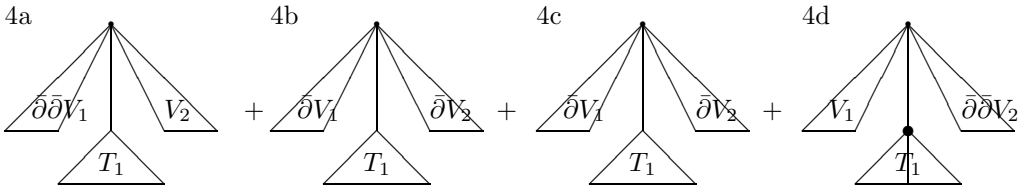


FIGURE 75

And every bonsai in Figures 70 - 76 ais in one of those pairs. So $T_1 *_2 T_2 = 0$.

When T_1 is the one-vertex bonsai: Since $T_1 *_1 T_2 = A_3 + A_5$, it is the sum of A, B, E and F. And since $T_1 *_1 T_2 = A_1 + A_5$, we have $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}A_1 + \hat{\partial}A_5$, and it is the sum of the bonsais in Figure 71 and Figure 76. So as above, we have $T_1 *_1 T_2 = 0$.

12.3.2. *When* $\partial T_2 \neq 0$ *and* $deg(T_2) = m-2$. When T_1 is not the one-vertex bonsai: We have $\partial(\partial T_2) = 0$ and $deg(\partial T_2) = m - 1$. So we have $T_1 *_1 \partial T_2 = A_3 + A_5$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_1 + A_2 + A_3 + A_4 + A_5)$. So the $T_1 *_2 T_2$ is the sum of the bonsais A, B, E and F of Figure 70 and the bonsais in Figures 71 - 73 and 75 - 76(2a and 2b are 0, since the valences at the roots are $m + 1$). So as in the above case, $T_1 *_2 T_2 = 0$.

When T_1 is not the one-vertex bonsai: $T_1 *_1 \partial T_2 = A_5$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_1 + A_5)$. So $T_1 *_2 T_2$ is the sum of the bonsais E and F of the Figure 70 and the bonsais in Figures 71 and 76.

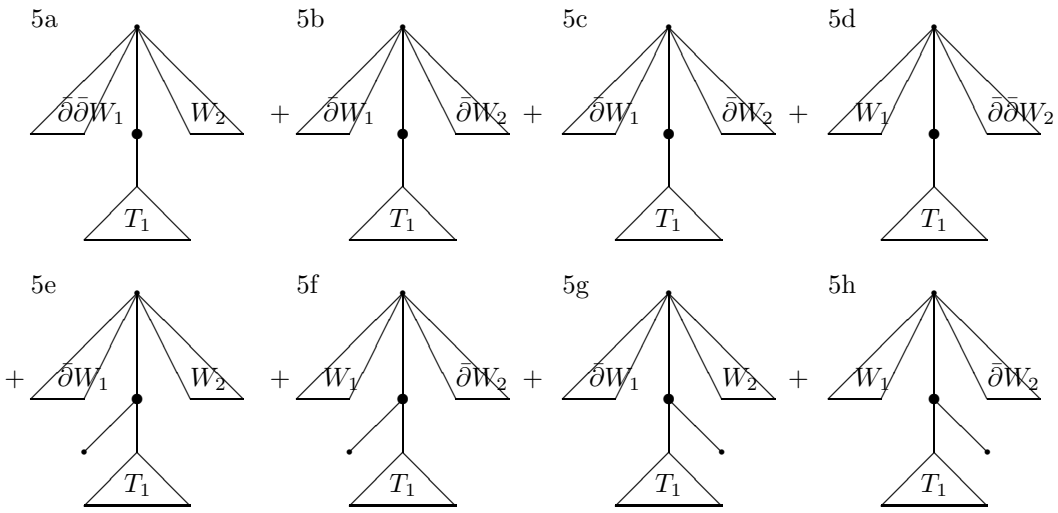


FIGURE 76

12.3.3. *When* $\partial T_2 \neq 0$ *and* $deg(T_2) = m-1$. In this case and others, we will just write down what $T_1 *_1 \partial T_2$ and $\hat{\partial}(T_1 *_1 T_2)$ are. In each case, as above, the bonsais in $T_1 *_2 T_2$ all cancel out.

In this case, $\partial(\partial T_2) = 0$ and $deg(\partial T_2) = m$.

When T_1 is not the one-vertex bonsai: $T_1 *_1 \partial T_2 = A_3 + A_5$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_1 + A_3 + A_5)$.

When T_1 ia the one-vertex bonsai: $T_1 *_1 \partial T_2 = A_5$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_1 + A_5)$.

12.3.4. *When* $\partial T_2 = 0$ *and* $deg(T_2) \leq m-2$. In this case, $\partial T_2 = 0$ and $deg(T_2) \leq m-2$.

When T_1 is not the one-vertex bonsai: $T_1 *_1 \partial T_2 = 0$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_3 + A_4 + A_5)$.

When T_1 ia the one-vertex bonsai: $T_1 *_1 \partial T_2 = 0$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_5)$.

12.3.5. *When* $\partial T_2 = 0$ *and* $deg(T_2) = m-1$. In this case, $\partial T_2 = 0$ and $deg(T_2) = m-1$.

When T_1 is not the one-vertex bonsai: $T_1 *_1 \partial T_2 = A_3 + A_5$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_3 + A_5)$.

When T_1 ia the one-vertex bonsai: $T_1 *_1 \partial T_2 = 0$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_5)$.

12.3.6. *When* $deg(T_2) = m$. In this case, $\partial T_2 = 0$ and $deg(T_2) = m$.

When T_1 is not the one-vertex bonsai: $T_1 *_1 \partial T_2 = 0$ and $\hat{\partial}(T_1 *_1 T_2) = \hat{\partial}(A_3 + A_5)$.

When T_1 ia the one-vertex bonsai: $T_1 *_1 \partial T_2 = 0$ and $\hat{\partial}(T_1 *_1 T_2) = 0$.

12.4. $T_1 *_2 T_2$ **for general** T_2**.** Now let us consider the case where T_2 is a general m-bonsai, not only a corolla. First, let us give an expression of a general m-bonsai as a concatenation of corollas. When we have a general clear-edged m-bonsai T as in Figure 77, we first enumerate the non-tip vertices of T in traversing order (cf. Section 7), as in Figure 78, and for each non-tip vertex numbered i, denote the corollas attached to that vertex as $T_{i1}, T_{i2}, ..., T_{im_i}$ from the left, and denote the

given bonsai as $T(T_{11},...,T_{1m_1};...;T_{k1},...,T_{km_k})$, where k is the number of non-tip vertices of T.

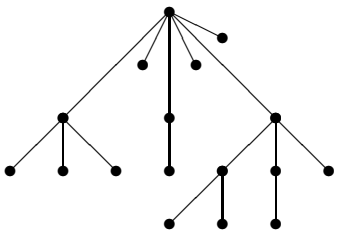


FIGURE 77

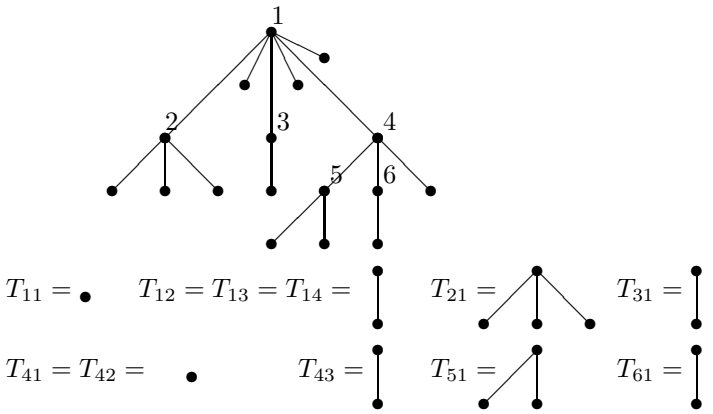


FIGURE 78

In Figure 78, the given bonsai T is

$$T(T_{11},T_{12},T_{13},T_{14};T_{21};T_{31};T_{41},T_{42},T_{43};T_{51};T_{61}). \tag{72}$$

Now let $T = T(T_{11},...,T_{1m_1};...;T_{k1},...,T_{km_k})$ and just for convenience of algebra, let us donote T as $T(T_1,T_2,...,T_m)$, where $T_1 = T_{11}, T_2 = T_{12}, ...T_k = T_{km_k}$. Then we have $\partial T = \sum_i T(T_1,...,\bar{\partial}T_i,...,T_m)$ and $S * T = \sum_i T(T_1,...,S * T_i,...,T_m)$. So we have, by the definitions of $\hat{\partial}$, $\bar{\partial}$ and $*$,

$$\begin{aligned} S *_1 T &= S * \partial T + \hat{\partial}(S * T) \\ &= \textstyle\sum_i T(T_1,...,S * \bar{\partial}T_i,...,T_m) \\ &\quad + \textstyle\sum_{i\neq j} T(T_1,...,\bar{\partial}T_i,...,S * T_j,...,T_m) \\ &\quad + \textstyle\sum_i T(T_1,...,\hat{\partial}(S * T_i),...,T-m) \\ &\quad + \textstyle\sum_{i\neq j} T(T_1,...,\bar{\partial}T_i,...,S * T_j,...,T_m) \\ &= \textstyle\sum_i T(T_1,...,S * \bar{\partial}T_i,...,T_m) \\ &\quad + \textstyle\sum_i T(T_1,...,\hat{\partial}(S * T_i),...,T-m) \\ &= \textstyle\sum_i T(T_1,...,S *_1 T_i,...,T_m). \end{aligned} \tag{73}$$

Similarly, we have

$$
\begin{aligned}
(74) \qquad & S *_2 T = S *_1 \partial T + \hat{\partial}(S *_1 T) \\
= \quad & \textstyle\sum_i T(T_1, ..., S *_1 \bar{\partial} T_i, ..., T_m) \\
& + \textstyle\sum_{i \neq j} T(T_1, ..., \bar{\partial} T_i, ..., S *_1 T_j, ..., T_m) \\
& + \textstyle\sum_i T(T_1, ..., \hat{\partial}(S *_1 T_i), ..., T - m) \\
& + \textstyle\sum_{i \neq j} T(T_1, ..., \bar{\partial} T_i, ..., S *_1 T_j, ..., T_m) \\
= \quad & \textstyle\sum_i T(T_1, ..., S *_1 \bar{\partial} T_i, ..., T_m) \\
& + \textstyle\sum_i T(T_1, ..., \hat{\partial}(S *_1 T_i), ..., T - m) \\
= \quad & \textstyle\sum_i T(T_1, ..., S *_2 T_i, ..., T_m) \\
(75) \qquad = \quad & 0.
\end{aligned}
$$

So we have

Theorem 12.1. *Mod 2, for any clear-edged m-bonsai T_1 and T_2, we have $T_1 *_2 T_2 = 0$.*

13. Further Direction

In the next paper of the author, we will investigate a generalization of the m-bonsai Hopf algebra, its differentials and cohomology groups. Also we will investigate some possibility of generalization of the appending operation $*$.

References

[BK] C. Bergbauer, D. Kreimer, The Hopf algebra of rooted trees in Epstein-Glaser renormalization, hep-th/0403207

[CK] A. Connes and D. Kreimer, Hopf Algebras, Renormalization and Noncommutative Geometry, *Eur. Phys. J. C7(1999) 697-708*, hep-th/9808042

[CK2] A. Connes and D. Kreimer, Insertion and Elimination: the doubly infinite Lie algebra and Feynman graphs, hep-th/0201157

[G] M. Gerstenhaber, The Cohomology Structure of an Associative Ring, *The Annals of Mathematics, Second Series, Vol. 78, Issue 2 (Sep.,1963) 267-288*

[Ha] D. Harrivel, Planar Binary Trees and Perturbative Calculus of Observables in Classical Field Theory, AP/0410050

[Kr] D. Kreimer, On the Hopf algebra structure of pertubative quantum field theories,*Adv. Theor. Math. Phys. 2(1998) 303-334* , q-alg/9707029

[KS] H. Kajiura, J. Stasheff, Homotopy algebras inspired by classical open-closed string field theory, QA/0410291

[MSS] M. Markl, S. Shnider and J. Stasheff, *Operads in Algebra, Topology and Physics*, American Mathematical Society, 2002

[PS] L. Pachter, B. Sturmfels, The Mathematics of Phylogenomics, math.ST/0409132

[S] R. P. Stanley, *Enumerative Combinatorics Volume 2*, Cambridge University Press, 1999