CYCLE-COUNTING Q-ROOK THEORY AND OTHER GENERALIZATIONS OF CLASSICAL ROOK THEORY

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To Melanie, who inspires me.

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ABSTRACT

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James Haglund

In this dissertation, we will study some generalizations of classical rook theory. The main focus is what we call *cycle-counting q-rook theory*, a model which incorporates a weighted count of both the number of cycles and the number of inversions of a permutation. In Chapter 1 we discuss background material in rook theory. In Chapter 2, we prove some results about algebraic properties of a generalization of the cycle-counting q-hit numbers. The main result of the dissertation is presented in Chapter 3, a statistic for combinatorially generating the cycle-counting q-hit numbers, which were previously defined only algebraically. We will then apply this result to prove some new theorems about permutation statistics involving cycle-counting in Chapter 4. Finally, in Chapter 5 we explore two other models which generalize classical rook theory and prove some basic results about these models. The first is a refinement of cycle-counting q-rook theory, obtained by adding a new parameter p to the weight of each rook placement. The second is a theory in which more than one rook is allowed in a row, but a row containing more than one rook is weighted by a polynomial in α.

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Chapter 1

Introduction

1.1 Classical rook theory

We will use the notation SQ_n for the $n \times n$ square board depicted in Figure 1.1. In classical rook theory, a *board* is any subset of SQ_n for some $n \in \mathbb{N}$. We will let $B(b_1, \ldots, b_n)$ denote the board $B \subseteq SQ_n$ consisting of all squares $\{(i,j) \mid j \leq b_i\}$. For example, $B(2,1,3)$ is pictured in Figure 1.2. When we also have $b_1 \leq b_2 \leq \cdots \leq b_{n-1} \leq b_n$, we call $B(b_1, \ldots, b_n)$ a *Ferrers board*. Another way to specify a Ferrers board, which we will use frequently here, is to give the step heights and depths of the board. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$ is shown in Figure 1.3.

A *rook placement* on a board $B \subseteq SQ_n$ is a subset of squares of B such that no two of these squares lie in the same row or the same column. As the name suggests, these squares represent positions on an $n \times n$ chess board where non-attacking rooks can be placed. We will write $\mathcal{R}_k(B)$ for the set of all placements of k non-attacking rooks on B, and we will denote $|\mathcal{R}_k(B)|$, called the *kth rook number* of B, by $r_k(B)$. The set of all placements of n rooks on SQ_n such that exactly k of these rooks lie on B is denoted $\mathcal{H}_{n,k}(B)$. We will write $h_{n,k}(B)$ for $|\mathcal{H}_{n,k}(B)|$, called the *kth hit number* of B. For $B = B(b_1, \ldots, b_n)$ a Ferrers board, the rook numbers satisfy the following equation from [9]:

$$\sum_{k=0}^{n} r_{n-k}(B) z(z-1)\cdots(z-k+1) = \prod_{i=1}^{n}(z + b_i - i + 1). \tag{1.1}$$

The rook numbers and hit numbers of any board B (not necessarily Ferrers) are also related by the classical equation

$$\sum_{k=0}^{n} r_{n-k}(B) k!(z-1)^{n-k} = \sum_{k=0}^{n} h_{n,k}(B) z^k, \tag{1.2}$$

first proven by Kaplansky and Riordan in [14].

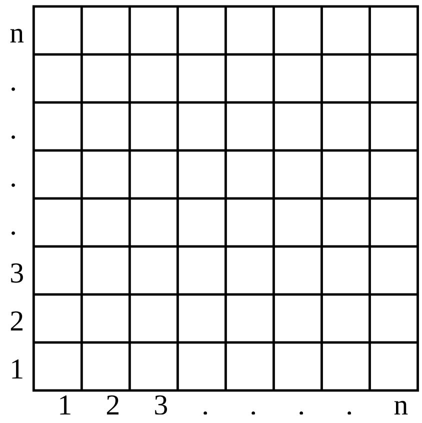

Figure 1.1: The $n \times n$ square board SQ_n.

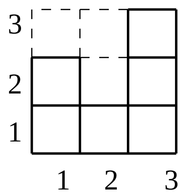

Figure 1.2: The board $B(2,1,3) \subseteq SQ_3$.

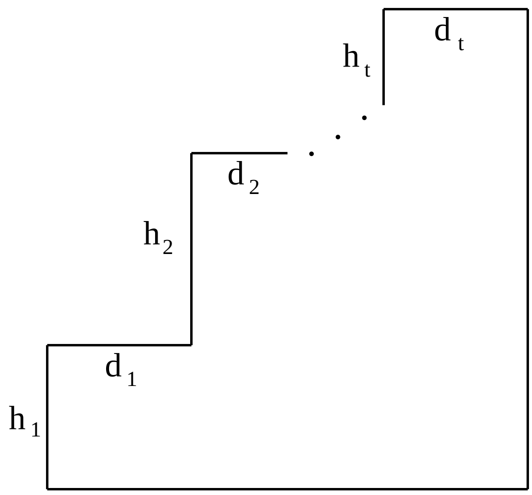

Figure 1.3: The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$.

1.2 q-Rook theory

Recently, the study of *q-analogs* has become popular in combinatorial research. A q-analog of a combinatorial quantity is an expression in the variable q (often a polynomial) such that when we let $q = 1$ we get the classical quantity. An additional desired property of a q-analog is that so-called q-versions of theorems known for the original quantity hold. For example, we might have an equation in which some or all of the quantities are q-analogs, and letting $q = 1$ gives the original known equation.

A common q-analog of the real number x, denoted $[x]$, is given by the equation

$$[x] = \frac{1-q^x}{1-q}.$$

Note here that when $n \in \mathbb{N}$,

$$[n] = \frac{1-q^n}{1-q} = \frac{(1+q+\cdots+q^{n-1})(1-q)}{1-q} = 1+q+\cdots+q^{n-1}$$

is a polynomial in q. A q-analog of $n!$ is denoted $[n]!$, and given by the equation

$$[n]! = [n][n-1]\cdots[2][1].$$

A q-analog of the binomial coefficient $\binom{n}{k}$ for $n, k \in \mathbb{N}$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{[n][n-1]\cdots[n-k+1]}{[k]!}.$$

More generally, we can define

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x][x-1]\cdots[x-k+1]}{[k]!}$$

for $k \in \mathbb{N}$ and $x \in \mathbb{C}$.

A q-analog of rook theory was introduced in [6], defined only for Ferrers boards. Given a placement P of rooks on a Ferrers board $B \subseteq SQ_n$, if we let each rook cancel all squares to the right in its row and below in its column, then we can define $inv_B(P)$ to be the number squares of B which neither contain a rook from P nor are cancelled. For example, for the rook placement P on the Ferrers board B pictured in Figure 1.4, $inv_B(P) = 7$. Garsia and Remmel defined the *kth q-rook number* of a Ferrers board $B = B(b_1, \ldots, b_n) \subseteq SQ_n$ by

$$R_k(q, B) = \sum_{P \in \mathcal{R}_k(B)} q^{inv_B(P)}.$$

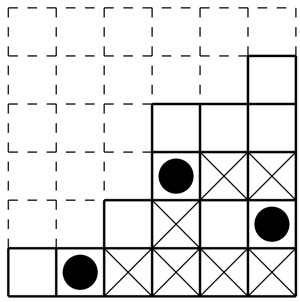

Figure 1.4: The placement P on B, with $inv_B(P) = 7$.

Note that $R_k(1,B) = r_k(B)$. They also proved the following q-analog of (1.1):

$$\sum_{k=0}^{n} R_{n-k}(q,B)[z][z-1]\cdots[z-k+1] = \prod_{i=1}^{n}[z+b_i-i+1]. \qquad (1.3)$$

Garsia and Remmel then went on in [6] to define the *q-hit numbers* of a Ferrers board $B \subseteq SQ_n$ algebraically by the equation

$$\sum_{k=0}^{n} R_{n-k}(q,B)[k]!z^k \prod_{i=k+1}^{n}(1-zq^i) = \sum_{k=0}^{n} A_{n,k}(q,B)z^k. \qquad (1.4)$$

Note that if we make the substitution $z = z^{-1}$ in (1.4), the resulting equation is

$$\sum_{k=0}^{n} R_{n-k}(q,B)[k]! \prod_{i=k+1}^{n}(z-q^i) = \sum_{k=0}^{n} A_{n,n-k}(q,B)z^k. \qquad (1.5)$$

If we then let $q = 1$ we see that the left side of (1.5) equals the left side of (1.2). Thus we must also have that the right side of (1.5) when $q = 1$ equals the right side of (1.2). In particular we must have that

$$A_{n,k}(1,B) = h_{n,n-k}(B).$$

Garsia and Remmel further proved that for a Ferrers board $B \subseteq SQ_n$ there exists a statistic $stat_{n,B}$ such that

$$A_{n,k}(q,B) = \sum_{P \in \mathcal{H}_{n,n-k}(B)} q^{stat_{n,B}(P)}, \qquad (1.6)$$

but they did not give a direct description of $stat_{n,B}$. Both Dworkin [4] and Haglund [12] independently found different statistics satisfying (1.6). Let P be a

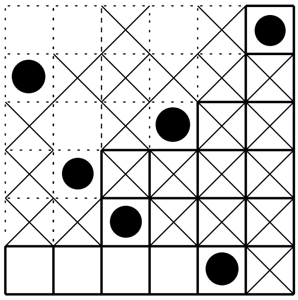

Figure 1.5: The placement P on $B \subseteq SQ_6$, with $h_{6,B}(P) = 8$.

placement of n rooks on SQ_n (with some number of rooks on the board $B \subseteq SQ_n$). Haglund's statistic, which we will denote $h_{n,B}(P)$, is given by the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme:

1. Each rook cancels all squares to the right in its row;

2. each rook on B cancels all squares above it in its column;

3. each rook off B cancels all squares below it but off B in its column.

For the placement P on the Ferrers board $B \subseteq SQ_6$ shown in Figure 1.5, $h_{n,B}(P) = 8$.

1.3 Cycle-counting q-rook theory

This theory was first introduced by Ehrenborg, Haglund, and Readdy in [5], and is defined only for Ferrers boards (like q-rook theory, which it generalizes). In order to describe the rook numbers in this model, we need to define two statistics. First we note that given a rook placement P on a board $B \subseteq SQ_n$ (here B is not necessarily a Ferrers board), it is possible to associate to P a simple directed graph G_P on n vertices. This fact was first noted in [7] (see also [2] and [3]). There will be an edge from i to j in G_P if and only if there is a rook from P on the square (i, j). We can then define $cyc(P)$ to be the number of cycles in G_P. For example, a placement P and its corresponding digraph G_P are shown in Figure 1.6. Note that $cyc(P) = 2$.

The second statistic (which makes sense only for Ferrers boards) depends on the following fact. Given any placement P of j non-attacking rooks in columns 1 through $i-1$ of a Ferrers board B (where $j \leq i-1$), it is an easy exercise to see that if $b_i \geq i$ then there is exactly one square in column i where placement of a

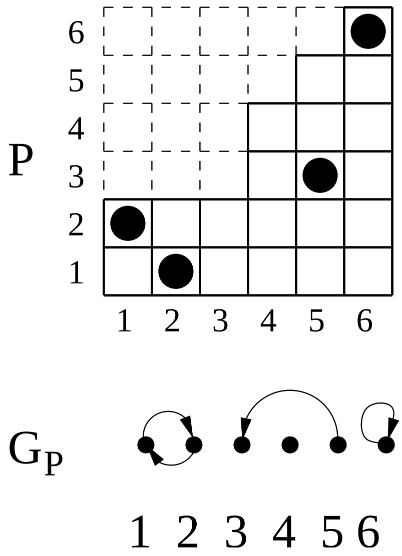

Figure 1.6: The placement P on B and the associated digraph G_P.

rook will complete a new cycle in G_P. If $b_i < i$ then there is no such square. Now we can define $s_i(P)$ to be (for i with $b_i \geq i$) the unique square which, considering only rooks from P in columns 1 through $i-1$ of P, completes a cycle. Then let $E(P)$ be the number of i such that $b_i \geq i$ and there is no rook from P in column i on or above square $s_i(P)$. For the placement P in Figure 1.6 $E(P) = 2$, corresponding to $i = 4$ and $i = 5$.

The *kth cycle-counting q-rook number* of a Ferrers board B is then given by the equation

$$R_k(y, q, B) = \sum_{P \in \mathcal{R}_k(B)} [y]^{cyc(P)} q^{inv_B(P)+(y-1)E(P)},$$

where $inv_B(P)$ is as defined in Section 1.2. Note here that $R_k(1, q, B) = R_k(q, B)$. The following generalization of (1.1) and (1.3) holds:

$$\sum_{k=0}^{n} R_{n-k}(y, q, B)[z][z-1]\cdots[z-k+1] =$$

$$\prod_{i \text{ with } b_i \geq i} [z + b_i - i + y] \prod_{i \text{ with } b_i < i} [z + b_i - i + 1]. \tag{1.7}$$

Haglund [11] then defined the *q, x, y-hit numbers* algebraically by the equation

$$\sum_{k=0}^{n} R_{n-k}(y, q, B)[x][x+1]\cdots[x+k-1]z^k \prod_{i=k+1}^{n} (1 - zq^{x+i-1}) =$$

$$\sum_{k=0}^{n} A_{n,k}(x,y,q,B)z^k. \tag{1.8}$$

If we let $x = y$ in $A_{n,k}(x,y,q,B)$, we obtain what we shall call the *kth cycle-counting q-hit number* of B, given algebraically by the equation

$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[y][y+1]\cdots[y+k-1]z^k \prod_{i=k+1}^{n}(1-zq^{y+i-1}) =$$

$$\sum_{k=0}^{n} A_{n,k}(y,q,B)z^k. \tag{1.9}$$

Note that here we are using $A_{n,k}(y,q,B)$ as shorthand notation for $A_{n,k}(y,y,q,B)$, and we will continue to use this notation throughout. If we let $y = 1$, the left side of (1.9) equals the left side of (1.4). This implies that $A_{n,k}(1,q,B) = A_{n,k}(q,B)$. The $A_{n,k}(y,q,B)$ also generalize the cycle-counting hit numbers of Chung and Graham [2], obtained by letting $q \to 1$.

The remainder of this dissertation is organized in the following way. In Chapter 2 we prove some results about algebraic properties of the $A_{n,k}(x,y,q,B)$. In Chapter 3 we state and prove the main result of the dissertation, a statistic for combinatorially generating the $A_{n,k}(y,q,B)$, and apply this result to obtain a generalization of the notion of a Mahonian permutation statistic. In Chapter 4 we again apply the statistic we defined in Chapter 3 to develop a generalization of Euler-Mahonian permutation statistics. Finally, in Chapter 5 we define the cycle-counting p,q-rook numbers and prove some results about them, and define the α-hit numbers and prove a version of (1.2).

Chapter 2

Properties of $A_{n,k}(x, y, q, B)$

In this chapter we prove some results about algebraic properties of the q, x, y-hit numbers, generalizing those proved by Haglund for the q-hit numbers in [12].

2.1 Symmetry and unimodality of $A_{n,k}(x, y, q, B)$

If $B = B(h_1, d_1; \ldots ; h_t, d_t) \subseteq SQ_n$ is a Ferrers board, let us denote the Ferrers board $B(h_1, d_1; \ldots ; h_p - 1, d_p - 1; \ldots h_t, d_t) \subseteq SQ_{n-1}$ by $B - h_p - d_p$. We also let $H_i = h_1 + \cdots + h_i$, and $D_i = d_1 + \cdots + d_i$. We will call the number of squares in the board B $Area(B)$. If B is a Ferrers board with column heights $b_1, b_2, \ldots , b_n$, then $Area(B) = b_1 + b_2 + \cdots + b_n$. Finally, we will call $B = B(b_1, \ldots , b_n) = B(h_1, d_1; \ldots ; h_t, d_t)$ a *regular Ferrers board* if $b_i \geq i$ for $1 \leq i \leq n$, or equivalently if $H_i \geq D_i$ for $1 \leq i \leq t$ as was defined in [11].

Suppose

$$f(q) = \sum_{i=M}^{N} a_i q^i,$$

with $a_M, a_N \neq 0$. We call $M + N$ the *darga* of f. We will say the polynomial $f(q)$ is $zsu(d)$ if either

1. $f(q)$ is identically zero, or
2. $f(q)$ is in $\mathbb{N}[q]$, symmetric, and unimodal with darga d.

Note that for $s \in \mathbb{N}$, q^s is $zsu(2s)$ and $[s]$ is $zsu(s - 1)$. We have the following lemmas. The proof of Lemma 2.1.1 is trivial, and a proof of Lemma 2.1.2 can be found in [12].

Lemma 2.1.1. *If f and g are polynomials which are both $zsu(d)$, then $f + g$ is $zsu(d)$.*

Lemma 2.1.2. *If f is $zsu(d)$ and g is $zsu(e)$, then fg is $zsu(d+e)$.*

Now we can prove the following.

Lemma 2.1.3. *Let $a, b \in \mathbb{N}$. For any regular Ferrers board $B = B(b_1, \ldots, b_n) \subseteq SQ_n$, $A_{n,0}(a, b, q, B)$ is $zsu(Area(B) + n(b-1) - \binom{n+1}{2})$.*

Proof. Letting $z = 0$ in (1.8), we see that $A_{n,0}(x, y, q, B) = R_n(y, q, B)$. Now letting $z = 0$ in (1.7) and using the fact that B is regular (so there is no i with $b_i < i$), we get that

$$R_n(y, q, B) = \prod_{i=1}^{n} [b_i - i + y],$$

hence

$$A_{n,0}(a, b, q, B) = \prod_{i=1}^{n} [b_i - i + b].$$

Since each $b_i \geq i$, we get that $[b_i - i + b]$ is $zsu(b_i - i + b - 1)$. Finally, by Lemma 2.1.2 we see that $A_{n,0}(a, b, q, B)$ is $zsu(\sum_{i=1}^{n}(b_i - i + b - 1)) = zsu(\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} i + n(b-1)) = zsu(Area(B) + n(b-1) - \binom{n+1}{2})$. □

Lemma 2.1.4. *Let $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $B - h_t - d_t \subseteq SQ_{n-1}$ as described earlier. Then*

$$A_{n,k}(x, y, q, B) = [k + y + d_t - 1] A_{n-1,k}(x, y, q, B - h_t - d_t) +$$

$$q^{k+y+d_t-2} [n + x - y - d_t - k + 1] A_{n-1,k-1}(x, y, q, B - h_t - d_t)$$

for any $0 \leq k \leq n$.

Proof. Let $p = t$ in Lemma 5.7 of [11]. □

Theorem 2.1.5. *Let $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $a, b \in \mathbb{N}$. If $n + a + 1 \geq b + d_t + k$, then $A_{n,k}(a, b, q, B)$ is $zsu(Area(B) + n(b + k - 1) + k(a-1) - \binom{n+1}{2})$ for $0 \leq k \leq n$.*

Proof. The proof is by induction on $Area(B)$. When $Area(B) = 1$ the only regular Ferrers board is the 1×1 square SQ_1. A quick calculation from the definition shows that $A_{1,0}(x, y, q, SQ_1) = [y]$, $A_{1,1}(x, y, q, SQ_1) = q^y [x - y]$, and $A_{1,k}(x, y, q, SQ_1) = 0$ for $k > 1$. Thus $A_{1,0}(a, b, q, B) = [b]$ and $A_{1,1}(a, b, q, B) = q^b [a - b]$ are $zsu(b-1)$ and $zsu(a + b - 1)$ respectively, and the result holds for the case $Area(B) = 1$.

Now assume the result holds for all regular Ferrers boards with $Area < A$, and let B be such a board with $Area(B) = A$. We know by Lemma 2.1.3 that the result holds for $A_{n,0}(a,b,q,B)$, so assume $k > 0$. Then by Lemma 2.1.4 with $x = a$ and $y = b$, we have that

$$A_{n,k}(a,b,q,B) = [k+b+d_t-1]A_{n-1,k}(a,b,q,B-h_t-d_t)+$$

$$q^{k+b+d_t-2}[n+a-b-d_t-k+1]A_{n-1,k-1}(a,b,q,B-h_t-d_t). \tag{2.1}$$

Now we know that $[k+b+d_t-1]$ is $zsu(k+b+d_t-2)$, and by the induction hypothesis, $A_{n-1,k}(a,b,q,B-h_t-d_t)$ is $zsu(Area(B-h_t-d_t)+(n-1)(b+k-1)+k(a-1)-\binom{n}{2})$. Note here that $Area(B-h_t-d_t) = Area(B)-n-d_t+1$. Then by Lemma 2.1.2, the first term on the right side of (2.1) is $zsu(Area(B)+n(b+k-1)+k(a-1)-\binom{n+1}{2})$.

For the second term on the right side of (2.1), we know that q^{k+b+d_t-2} is $zsu(2k+2b+2d_t-4)$, $[n+a-b-d_t-k+1]$ is $zsu(n+a-b-d_t-k)$ (since we have assumed $n+a+1 \geq b+d_t+k$), and by the induction hypothesis $A_{n-1,k-1}(a,b,q,B-h_t-d_t)$ is $zsu(Area(B-h_t-d_t)+(n-1)(b+k-2)+(k-1)(a-1)-\binom{n}{2})$. Finally, applying Lemma 2.1.2 one last time we get that the second term on the right side of (2.1) is $zsu(Area(B)+n(b+k-1)+k(a-1)-\binom{n+1}{2})$, and Lemma 2.1.1 gives us the result for $A_{n,k}(a,b,q,B)$ as well. □

2.2 A more general theorem

We prove a more general result for a certain class of boards in this section.

Lemma 2.2.1. *Let $B = B(b_1,\ldots,b_n) = B(h_1,d_1;\ldots;h_t,d_t)$ be a regular Ferrers board, $j \in \mathbb{N}$. Then*

$$\frac{\prod_{i=1}^{n}[j+b_i-i+y]}{\prod_{i=1}^{t}[d_i]!} = \prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}.$$

Proof. We see that

$$\prod_{i=1}^{n}[j+b_i-i+y] =$$

$$\prod_{i=1}^{t}[j+H_i-D_{i-1}+y-1][(j+H_i-D_{i-1}+y-1)-1]\cdots[(j+H_i-D_{i-1}+y-1)-d_i+1],$$

which is the same as

$$\prod_{i=1}^{t}[j+H_i-D_{i-1}+y-1]_{d_i}.$$

Thus

$$\frac{\prod_{i=1}^{n}[j+b_i-i+y]}{\prod_{i=1}^{t}[d_i]!}=\prod_{i=1}^{t}\frac{[j+H_i-D_{i-1}+y-1]_{d_i}}{[d_i]!},$$

which is

$$\prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}$$

by definition. □

Lemma 2.2.2. *Let* $B=B(b_1,\ldots,b_n)=B(h_1,d_1;\ldots;h_t,d_t)\subseteq SQ_n$ *be a regular Ferrers board. Then* $A_{n,k}(x,y,q,B)/\prod_{i=1}^{t}[d_i]!=$

$$\sum_{j=0}^{k}\begin{bmatrix} n+x \\ k-j \end{bmatrix}\begin{bmatrix} x+j-1 \\ j \end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}\prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}.$$

Proof. By Lemma 5.1 of [11], we have

$$A_{n,k}(x,y,q,B)=\sum_{j=0}^{k}\begin{bmatrix} n+x \\ k-j \end{bmatrix}\begin{bmatrix} x+j-1 \\ j \end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}\prod_{i=1}^{n}[j+b_i-i+y].$$

The lemma now follows trivially from Lemma 2.2.1. □

Now suppose we have integers $h_1,\ldots,h_t$, $d_1,\ldots,d_t$, and $e_1,\ldots,e_t$ with $d_i\in\mathbb{P}$, $h_i\in\mathbb{N}$, and $0\le e_i\le d_i$. We will denote the vector $(e_1,e_2,\ldots,e_t)$ by $\vec{e}$. We will continue to denote the partial sum $h_1+\cdots+h_i$ by H_i, $d_1+\cdots+d_i$ by D_i, and we will also let $E_i=e_1+\cdots+e_i$. We make the convention that $H_0=D_0=E_0=0$. We can define

$$P(\vec{e},x,y)=\prod_{i=1}^{t}\begin{bmatrix} H_i-D_{i-1}+E_{i-1}+y-1 \\ d_i-e_i \end{bmatrix}\begin{bmatrix} D_i+D_{i-1}-H_i-E_{i-1}+x-y \\ e_i \end{bmatrix}$$

and prove the following lemmas.

Lemma 2.2.3. *Let* $B=B(h_1,d_1;\ldots h_{t-1},d_{t-1};h_t,d_t)\subseteq SQ_n$ *be a regular Ferrers board,* $B'=B(h_1,d_1;\ldots;h_{t-1},d_{t-1})\subseteq SQ_{H_{t-1}}$. *Then*

$$A_{n,k}(x,y,q,B)=[d_t]!\sum_{s=k-d_t}^{k}A_{H_{t-1},s}(x,y,q,B')\begin{bmatrix} y+d_t+s-1 \\ d_t-k+s \end{bmatrix}$$

$$\times\begin{bmatrix} n-y-d_t+x-s \\ k-s \end{bmatrix}q^{(k-s)(y+k-1)}.$$

Proof. Let $p = t$ in Corollary 5.10 of [11] and note that because B is a regular Ferrers board, $H_t = D_t = n$. □

Lemma 2.2.4. *Let $B = B(h_1, d_1; \ldots ; h_t, d_t)$ be a regular Ferrers board. Then*

$$A_{n,k}(x, y, q, B) =$$

$$\prod_{i=1}^{t}[d_i]! \sum_{e_1+\cdots+e_t=k,\ 0\leq e_i\leq d_i} P(\vec{e}, x, y) \prod_{i=1}^{t} q^{e_i(H_i-D_i+E_i+y-1)}. \tag{2.2}$$

Proof. By induction on t. When $t = 1$ we have that $d_1 = n$, and Lemma 2.2.3 gives us

$$A_{n,k}(x, y, q, B) = [d_1]! \sum_{s=k-n}^{k} A_{0,s}(x, y, q, \emptyset) \begin{bmatrix} y+n+s-1 \\ d_1-k+s \end{bmatrix} \tag{2.3}$$

$$\times \begin{bmatrix} n-y-n+x-s \\ k-s \end{bmatrix} \times q^{(k-s)(y+k-1)}.$$

When $t = 1$ we have that $H_1 = D_1 = d_1 = n$ and $D_0 = H_0 = 0$, so we get that the $s = 0$ term in (2.3) is equal to

$$[d_1]! \begin{bmatrix} H_1-D_0+y-1 \\ d_1-k \end{bmatrix} \begin{bmatrix} D_1+D_0-H_1+x-y \\ k \end{bmatrix} \times q^{k(H_1-D_1+k+y-1)}. \tag{2.4}$$

Note that by definition

$$A_{0,s}(x, y, q, \emptyset) = \delta_{s,0},$$

so the only nonzero summand in (2.3) occurs when $s = 0$ and hence (2.4) is actually equal to (2.3). Finally if we recall that $E_1 = e_1$ and $E_0 = 0$, we can rewrite (2.4) as

$$[d_1]! \sum_{e_1=k,\ 0\leq e_1\leq d_1} \begin{bmatrix} H_1-D_0+E_0+y-1 \\ d_1-e_1 \end{bmatrix}$$

$$\times \begin{bmatrix} D_1+D_0-H_1-E_0+x-y \\ e_1 \end{bmatrix} \times q^{e_1(H_1-D_1+E_1+y-1)},$$

which is exactly of the form of (2.2).

For $t > 1$, Lemma 2.2.3 gives that

$$A_{n,k}(x, y, q, B) = [d_t]! \sum_{E_{t-1}=E_t-d_t}^{E_t} A_{H_{t-1},E_{t-1}}(x, y, q, B') \begin{bmatrix} y+d_t+E_{t-1}-1 \\ d_t-e_t \end{bmatrix}$$

$$\times \begin{bmatrix} n-y-d_t+x-E_{t-1} \\ e_t \end{bmatrix} \times q^{e_t(y+E_t-1)}. \tag{2.5}$$

Here we are letting $E_{t-1} = s$ and defining $e_t = k - s$ and $E_t = E_{t-1} + e_t = k$. Since B is regular $H_t = D_t = n$, so $H_t - D_{t-1} = D_t - D_{t-1} = d_t$ and (2.5) can be rewritten as

$$A_{n,k}(x,y,q,B) = [d_t]! \sum_{e_t=0}^{d_t} A_{H_{t-1},E_{t-1}}(x,y,q,B') \begin{bmatrix} H_t - D_{t-1} + E_{t-1} + y - 1 \\ d_t - e_t \end{bmatrix}$$

$$\times \begin{bmatrix} D_t + D_{t-1} - H_t - E_{t-1} + x - y \\ e_t \end{bmatrix} \times q^{e_t(H_t - D_t + E_t + y - 1)}.$$

By the inductive hypothesis, the above is equal to

$$[d_t]! \sum_{e_t=0}^{d_t} \Bigg\{ \prod_{i=1}^{t-1} [d_i]! \sum_{e_1+\cdots+e_{t-1}=E_{t-1},\ 0 \le e_i \le d_i} \prod_{i=1}^{t-1} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + y - 1 \\ d_i - e_i \end{bmatrix}$$

$$\times \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + x - y \\ e_i \end{bmatrix} q^{e_i(H_i - D_i + E_i + y - 1)} \Bigg\}$$

$$\times \begin{bmatrix} H_t - D_{t-1} + E_{t-1} + y - 1 \\ d_t - e_t \end{bmatrix} \begin{bmatrix} D_t + D_{t-1} - H_t - E_{t-1} + x - y \\ e_t \end{bmatrix} q^{e_t(H_t - D_t + E_t + y - 1)}$$

which is

$$\prod_{i=1}^{t} [d_i]! \sum_{e_1+\cdots+e_t=k,\ 0 \le e_i \le d_t} P(\vec{e}, x, y) \prod_{i+1}^{t} q^{e_i(H_i - D_i + E_i + y - 1)}$$

as desired. □

Lemma 2.2.5. *Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board, $a, b \in \mathbb{N}$ with $a \ge b \ge 1$. Let e_i, d_i, h_i, E_i, D_i, and H_i be as in the definition of $P(\vec{e}, x, y)$. Assume that B is such that $d_{i-1} + d_i \ge h_i$ for $1 \le i \le t$ (where $d_0 := 0$). If any of the numerators of the q-binomial coefficients in*

$$P(\vec{e}, a, b) = \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + b - 1 \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + a - b \\ e_i \end{bmatrix}$$

are negative, then $P(\vec{e}, a, b) = 0$.

Proof. First note that $H_i - D_{i-1} + E_{i-1} + b - 1 \ge 0$ for $1 \le i \le t$, since $H_i \ge D_i \ge D_{i-1}$ and $b \ge 1$, so none of the numerators in the first q-binomial coefficient of the product are ever negative.

Now suppose that $D_k + D_{k-1} - H_k - E_{k-1} + a - b < 0$ for some k with $0 \le k \le t$. Note $D_1 + D_0 - H_1 - E_0 + a - b = d_1 - h_1 + a - b$, and since we assume $d_{i-1} + d_i \ge h_i$

(and in particular $d_1 \geq h_1$) and $a \geq b$, we have that $d_1 - h_1 + a - b \geq 0$. Thus we see that such a $k \geq 2$.

Now choose j such that $D_i + D_{i-1} - H_i - E_{i-1} + a - b \geq 0$ for $1 \leq i < j$, but $D_j + D_{j-1} - H_j - E_{j-1} + a - b < 0$ (such a j exists because of the remarks in the previous paragraph). Then $D_j + D_{j-1} - H_j - E_{j-1} + a - b < 0$ implies $D_j + D_{j-1} - H_j - E_{j-2} + a - b < e_{j-1}$, which is equivalent to $d_j + d_{j-1} - h_j + D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b < e_{j-1}$, which implies $D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b < e_{j-1}$ (since $d_j + d_{j-1} \geq h_j$). Hence

$$\begin{bmatrix} D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b \\ e_{j-1} \end{bmatrix} = 0$$

since the numerator is non-negative by definition of j but less than the denominator, so the product $P(\vec{e}, a, b) = 0$ as well. □

Theorem 2.2.6. *Let $B = B(h_1, d_1; \ldots ; h_t, d_t)$ be a regular Ferrers board (so $H_i \geq D_i$ for $1 \leq i \leq t$). Let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$, and set*

$$L_k^{a,b}(B) = Area(B) + n(b-1) + k(n+a-1) - \sum_{i=1}^{t} d_i D_i.$$

Then $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is either zero or symmetric with darga $L_k^{a,b}(B)$. If in addition $d_{i-1} + d_i \geq h_i$ for $1 \leq i \leq t$, then $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is $zsu(L_k^{a,b}(B))$.

Proof. By Lemma 2.2.2, $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]! =$

$$\sum_{j=0}^{k} \begin{bmatrix} n+a \\ k-j \end{bmatrix} \begin{bmatrix} a+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{t} \begin{bmatrix} j + H_i - D_{i-1} + b - 1 \\ d_i \end{bmatrix},$$

which is a polynomial in q (the first two q-binomial coefficients in the sum are clearly polynomials, and the third is since $H_i \geq D_{i-1}$ and $b \geq 1$). Using the fact that $\begin{bmatrix} r \\ s \end{bmatrix}$ is $zsu(s(r-s))$ (see [10, 16] for a proof) and Lemma 2.1.2, each term on the right side above has darga $(k-j)(n+a-k+j) + j(a-1) + (k-j)(k-j-1) + \sum_{i=1}^{t} d_i(j + H_i - D_i + b - 1)$, which is exactly $L_k^{a,b}(B)$. Since the sign alternates, we can only conclude that $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is symmetric with darga $L_k^{a,b}(B)$.

However if $d_{i-1} + d_i \geq h_i$ for $1 \leq i \leq t$, by Lemma 2.2.4

$$\frac{A_{n,k}(a, b, q, B)}{\prod_{i=1}^{t}[d_i]!} = \sum_{e_1 + \cdots + e_t = k,\ 0 \leq e_i \leq d_i} P(\vec{e}, a, b) \prod_{i=1}^{t} q^{e_i(H_i - D_i + E_i + b - 1)},$$

and all of the terms on the right hand side above are in $\mathbb{N}[q]$ by Lemma 2.2.5. Each term is $zsu(\sum_{i=1}^{t}\{(d_i - e_i)(H_i - D_i + E_i + b - 1) + e_i(D_i + D_{i-1} - H_i - E_i + a - b) + 2e_i(H_i - D_i + E_i + b - 1)\})$, which a simple calculation shows is the same $zsu(L_k^{a,b}(B))$. Thus by Lemma 2.1.1, $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is $zsu(L_k^{a,b}(B))$ as well. $\square$

Chapter 3

A Statistic for Generating $A_{n,k}(y, q, B)$

In this chapter we will prove the main result of this dissertation, a statistic for combinatorially generating the $A_{n,k}(y, q, B)$. The proof is broken into three steps. In Section 3.1 we show that when $m \in \mathbb{N}$, $A_{n,k}(m, q, B)$ is related to the kth q-hit number of a much larger board B_m. In Section 3.2, we use Haglund's statistic for combinatorially generating $A_{n,k}(q, B_m)$ to find a combinatorial expression for $A_{n,k}(m, q, B)$ when $m \in \mathbb{N}$. In Section 3.3, we use the fact that two polynomials with infinitely many common values must be equal to extend our combinatorial expression to $A_{n,k}(y, q, B)$ for any $y \in \mathbb{R}$. We finish by using results from this chapter to define a generalization of the notion of a Mahonian permutation statistic.

3.1 The $A_{n,k}(m, q, B)$ when $m \in \mathbb{N}$

When $B = B(h_1, d_1; \ldots; h_t, d_t) = B(b_1, \ldots, b_n)$ is a Ferrers board let us define, for $m \in \mathbb{N}$, the board

$$B_m = B(h_1 + m - 1, d_1; \ldots; h_t, d_t + m - 1).$$

If B is a regular Ferrers board (and hence $b_n = n$), then B_m is also a regular Ferrers board, with $n + m - 1$ columns, of heights

$$b_1 + m - 1,\ b_2 + m - 1, \ldots, b_n + m - 1, \underbrace{n + m - 1, \ldots n + m - 1}_{m-1}.$$

Note since at least the last m columns of $B_m \subseteq SQ_{n+m-1}$ for any regular Ferrers board B have height $n+m-1$, any rooks in the last m columns of SQ_{n+m-1} must be on B_m. Thus in particular any placement of $n+m-1$ rooks on SQ_{m+n-1} must have at least m rooks on B_m, so $\mathcal{H}_{n+m-1,k}(B_m) = \emptyset$ for $0 \leq k \leq m-1$.

We use the following lemmas to prove the main proposition of this section.

Lemma 3.1.1. *For $B = B(b_1, \ldots, b_n)$ a regular Ferrers board, $m \in \mathbb{N}$ and B_m as defined above,*

$$A_{n,0}(m, q, B) = A_{n+m-1,0}(q, B_m)/[m-1]!.$$

Proof. Suppose $F = F(f_1, \ldots, f_d) \subseteq SQ_d$ is a regular Ferrers board. Letting $z = 0$ in (1.9) we get that $A_{d,0}(y, q, F) = R_d(y, q, F)$, and letting $z = 0$ in (1.7) gives that $R_d(y, q, F) = \prod_{j=1}^{n} [f_j - j + y] = \prod_{j=1}^{n} [(f_j + y - 1) - j + 1]$. Hence letting $F = B$, $d = n$, and $y = m \in \mathbb{N}$ yields

$$A_{n,0}(m, q, B) = \prod_{i=1}^{n} [(b_i + m - 1) - i + 1].$$

If we then let $F = B_m$, $d = n + m - 1$ and $y = 1$ we get that

$$A_{n+m-1,0}(q, B_m) = [b_1 + m - 1][(b_2 + m - 1) - 1] \cdots [(b_n + m - 1) - n + 1]$$

$$\times [(n + m - 1) - n][(n + m - 1) - n - 1] \cdots [(n + m - 1) - n - m + 2] =$$

$$\prod_{i=1}^{n} [(b_i + m - 1) - i + 1] \times [m-1]!,$$

and the lemma follows. □

Lemma 3.1.2. *For any regular Ferrers board $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ we have that*

$$A_{n,k}(y, q, B) = [y + k + d_t - 1] A_{n-1,k}(y, q, B - h_t - d_t) +$$

$$q^{y+k+d_t-2} [n - k - d_t + 1] A_{n-1,k-1}(y, q, B - h_t - d_t)$$

for $0 \leq k \leq n$.

Proof. Let $x = y$ in Lemma 2.1.4. □

Lemma 3.1.3. *For any regular Ferrers board $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$, we have that*

$$A_{n,k}(q, B) = [k + d_t] A_{n-1,k}(q, B - h_t - d_t) +$$

$$q^{k+d_t-1} [n - k - d_t + 1] A_{n-1,k-1}(q, B - h_t - d_t)$$

for $0 \leq k \leq n$.

Proof. Let $y = 1$ in Lemma 3.1.2. □

The next proposition is integral to proving the main result of the dissertation in Section 3.3.

Proposition 3.1.4. *For any regular Ferrers board B and $m \in \mathbb{N}$, we have that*

$$A_{n,k}(m, q, B) = \frac{A_{n+m-1,k}(q, B_m)}{[m-1]!}$$

for $0 \leq k \leq n$.

Proof. We will prove this proposition by induction on $Area(B)$. When $Area(B) = 1$ the only regular Ferrers board is the 1×1 square SQ_1, and an easy calculation shows that $A_{1,0}(m, q, SQ_1) = [m]$ and $A_{1,1}(m, q, SQ_1) = 0$. By the definition given in Chapter 1 of $h_{n+m-1,B_m}(P)$, $A_{1+m-1,0}(q, B_m) = A_{m,0}(q, SQ_m) = [m]!$ and $A_{m,1}(q, SQ_m) = 0$, so the proposition holds in this case.

Now assume the proposition holds for all regular Ferrers boards of $Area < A$, and suppose $B = B(h_1, d_1; \ldots; h_t, d_t) = B(b_1, \ldots, b_n)$ is such that $Area(B) = A$. By Lemma 3.1.1, we have that $A_{n,0}(m, q, B) = A_{n+m-1,0}(q, B_m)/[m-1]!$. Then by Lemma 3.1.2 when $y = m$, we have for $k > 0$ that

$$A_{n,k}(m, q, B) = [m + k + d_t - 1]A_{n-1,k}(m, q, B - h_t - d_t) +$$
$$q^{m+k+d_t-2}[n - k - d_t + 1]A_{n-1,k-1}(m, q, B - h_t - d_t),$$

which is

$$[k + (d_t + m - 1)]A_{n-1,k}(m, q, B - h_t - d_t) + q^{k+(d_t+m-1)-1}\times$$
$$[(n + m - 1) - (d_t + m - 1) - k + 1]A_{n-1,k-1}(m, q, B - h_t - d_t). \tag{3.1}$$

By induction, $A_{n-1,k}(m, q, B-h_t-d_t) = A_{(n-1)+m-1,k}(q, (B-h_t-d_t)_m)/[m-1]!$, which equals

$$A_{(n-1)+m-1,k}(q, B(h_1 + m - 1, d_1; \ldots; h_t - 1, d_t - 1 + m - 1))/[m - 1]!$$

and $A_{n-1,k-1}(m, q, B - h_t - d_t)$ is

$$A_{(n-1)+m-1,k-1}(q, B(h_1 + m - 1, d_1; \ldots; h_t - 1, d_t - 1 + m - 1)/[m - 1]!.$$

Thus (3.1) is equal to

$$[k+(d_t+m-1)]A_{(n-1)+m-1,k}(q, B(h_1+m-1, d_1; \ldots; h_t-1, d_t-1+m-1))/[m-1]!$$
$$+q^{k+(d_t+m-1)-1}[(n + m - 1) - (d_t + m - 1) - k + 1]\times$$

$$A_{(n-1)+m-1,k-1}(q, B(h_1+m-1, d_1; \ldots; h_t-1, d_t-1+m-1)/[m-1]!,$$

which is

$$\frac{1}{[m-1]!}\Big\{[k+(d_t+m-1)]A_{(n-1)+m-1,k}(q, B(h_1+m-1, d_1; \ldots; h_t-1, d_t-1+m-1))$$

$$+q^{k+(d_t+m-1)-1}[(n+m-1)-(d_t+m-1)-k+1]\times$$

$$A_{(n-1)+m-1,k-1}(q, B(h_1+m-1, d_1; \ldots; h_t-1, d_t-1+m-1)\Big\}.$$

Now by Lemma 3.1.3, the above is equal to

$$\frac{1}{[m-1]!}A_{n+m-1,k}(q, B(h_1+m-1, d_1; \ldots; (h_t-1)+1, (d_t-1+m-1)+1)$$

$$= \frac{1}{[m-1]!}A_{n+m-1,k}(q, B(h_1+m-1, d_1; \ldots; h_t, d_t+m-1)),$$

which is

$$\frac{1}{[m-1]!}A_{n+m-1,k}(q, B_m)$$

and the proposition follows. □

3.2 The map $\phi_{n,B,m}$ and its properties

For any Ferrers board $F \subseteq SQ_d$, let us denote $\cup_{i=0}^{d}\mathcal{H}_{d,i}(F)$ by $\mathcal{P}_d(F)$. Throughout this section let $B \subseteq SQ_n$ be some fixed regular Ferrers board, $B_m \subseteq SQ_{n+m-1}$ as previously defined for some fixed $m \in \mathbb{N}$. If $P \in \mathcal{P}_{n+m-1}(B_m)$, let $r_i(P)$ denote the rook from P in the ith column of SQ_{n+m-1}, and analogously for $Q \in \mathcal{P}_n(B)$ and $r_i(Q)$.

We define a mapping $\phi_{n,B,m} : \mathcal{P}_{n+m-1}(B_m) \to \mathcal{P}_n(B)$ as follows. Suppose $P \in \mathcal{P}_{n+m-1}(B_m)$. Beginning in column 1 and proceeding from left to right one column at a time, the following occurs.

1. $r_i(P)$ is on one of the m lowest squares in column i not attacked by a rook to the left if and only if $r_i(P)$ maps to the unique square $s_i(\phi_{n,B,m}(P))$ which completes a cycle in the image of P so far. (That is, you consider the placement of rooks on $SQ_n \supseteq B$ in columns 1 through $i-1$ given by $\phi_{n,B,m}(r_1(P)), \phi_{n,B,m}(r_2(P)), \ldots, \phi_{n,B,m}(r_{i-1}(P))$, and $s_i(\phi_{n,B,m}(P))$ is the unique square in column i which would complete a cycle in this placement.)

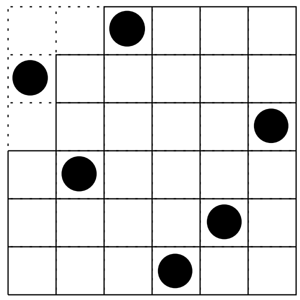

Figure 3.1: The placement P on $B_m \subseteq SQ_{n+m-1}$ for $B = B(1,3,4,4)$, $m = 3$, and $n = 4$.

2. Otherwise, $r_i(P)$ is on the $(m+a_i)$th square ($a_i > 0$) in column i not attacked by a rook to the left if and only if $r_i(P)$ maps to the a_ith available square in column i of B so far *which does not complete a cycle* (that is, the a_ith available square in column i of B, not counting the square $s_i(\phi_{n,B,m}(P))$ described above).

The best way to understand this mapping is to do an example in detail. Consider the placement P of 6 rooks on the board $SQ_6 \supseteq B_3$, where $B = B(1,3,4,4) \subseteq SQ_4$. This board and placement are depicted in Figure 3.1. The leftmost rook $r_1(P)$ is in the fifth available position in its column, which is also the fifth square in this column not attacked by a rook to the left (because there are no rooks to the left). Since $m = 3$ in this case (so $5 = m + 2$), $\phi_{4,B,3}(r_1(P))$ is on the second available square in column 1 of SQ_4 which does not complete a cycle. Since the square $(1,1)$ is always the cycle square in the first column, $r_1(P)$ maps to square $(1,3)$.

Now the cycle square in column 2 of B is $(2,2)$. Since $r_2(P)$ is on one of the 3 lowest squares in column 2 of SQ_6 not attacked by a rook to the left, $\phi_{4,B,3}(r_2(P))$ is on the cycle square $(2,2)$.

At this point the cycle square is $(3,1)$. Here $r_3(P)$ is on the fourth square not attacked by a rook to the left (and $4 = m + 1$), so $\phi_{4,B,3}(r_3(P))$ is on the first available square of SQ_4 which does not complete a cycle. In this case square $(3,1)$ is the cycle square, and squares $(3,2)$ and $(3,3)$ are attacked by the rooks in columns 1 and 2 of SQ_4, so the first available non-cycle square is $(3,4)$.

Finally, the cycle square in column 4 of SQ_4 is $(4,1)$. Since $r_4(P)$ in on the lowest square in its column (and hence one of the 3 lowest not attacked by a rook to the left), $\phi_{4,B,3}(r_4(P))$ is on the cycle square. The image $\phi_{4,B,3}(P)$ is depicted in Figure 3.2.

The general principle behind $\phi_{n,B,m}$ is the following. Suppose you want to map a rook in column i of a placement P on $SQ_{n+m-1} \supseteq B_m$. Imagine covering

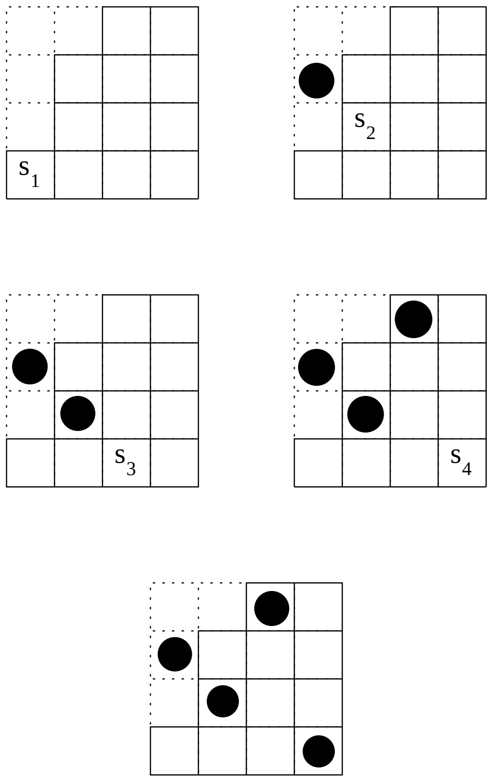

Figure 3.2: The image of P from Figure 3.1 under $\phi_{4,B,3}$ at each step.

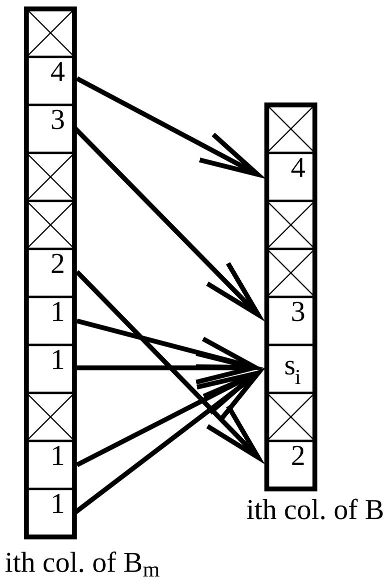

Figure 3.3: The idea behind the map $\phi_{n,B,m}$ in the ith column.

columns $i+1$ through $n+m-1$ of SQ_{n+m-1}, so that only columns 1 through i can be seen. If $r_i(P)$ is on one of the m lowest available squares in column i of this "covered" board, then $r_i(P)$ maps to the square of $SQ_n \supseteq B$ which completes a cycle in the image so far. The remaining $(n+m-1)-(i-1)-m = n-i$ squares in column i of SQ_{n+m-1} are then mapped in order to the $n-(i-1)-1 = n-i$ available non-cycle squares in column i of SQ_n. Figure 3.3 illustrates this idea further.

Note that in the definition of $\phi_{n,B,m}$ we ignore the rooks from a placement $P \in \mathcal{P}_{n+m-1}(B_m)$ in columns $n+1$ through $n+m-1$ of SQ_{n+m-1}. Thus for a fixed arrangement of n rooks in columns 1 through n of SQ_{n+m-1}, we see there will be $(m-1)!$ total ways to arrange the rooks in the last $m-1$ columns of SQ_{n+m-1}. Hence these $(m-1)!$ placements will all map to the same placement of n rooks on SQ_n.

We have the following lemmas.

Lemma 3.2.1. *$\phi_{n,B,m}$ is surjective.*

Proof. Given a placement $Q \in \mathcal{P}_n(B)$, we build a placement $P \in \mathcal{P}_{n+m-1}(B)$ from left to right. If the rook from Q in the ith column is on the square which completes a cycle, then we choose $r_i(P)$ to be on one of the m lowest available squares of SQ_{n+m-1} (so for each rook on a cycle square from Q, we will have m choices for the rook from P in the same column). If $r_i(Q)$ is on the a_ith square in its column not attacked by a rook to the left and which does not complete a cycle, then $r_i(P)$

must be on the $(m + a_i)$th available square in column i of SQ_{n+m-1}. Once the rooks in columns 1 through n are determined, we choose any arrangement of rooks in columns $n+1$ through $n+m-1$ which results in a non-attacking placement. It is clear that this procedure will result in a placement $P \in \mathcal{P}_{n+m-1}(B)$, and each rook from P was chosen to ensure that $Q = \phi_{n,B,m}(P)$. □

Lemma 3.2.2. *Let $P \in \mathcal{P}_{n+m-1}(B_m)$, and $Q = \phi_{n,B,m}(P)$. For $1 \le i \le n$, $r_i(P)$ is on B_m if and only if $r_i(Q)$ is on B, and $r_i(P)$ is off B_m on square $(i, j_i + m - 1)$ if and only if $r_i(Q)$ is off B on square (i, j_i).*

Proof. Fix n, B and m; the proof is by induction on i. If $i = 1$, then by definition of $\phi_{n,B,m}$ any rook on one of the m lowest squares in column 1 maps to the unique square in column 1 of B which completes a cycle, namely $(1, 1)$, and a rook on square $(1, j + m - 1)$ (for $j > 1$) maps to square $(1, j)$. Thus $r_1(P)$ is on B_m if and only if $r_1(Q)$ is on B, and $r_1(P)$ is off B_m on square $(1, j + m - 1)$ if and only if $r_1(Q)$ is off B on square $(1, j)$ as desired.

Now consider the rook $r_i(P)$ in column i of P for $i > 1$. Let k_i denote the number of rooks from P in columns 1 through $i - 1$ which can attack a square on B_m in column i; that is, k_i is the number of rooks in columns 1 through $i - 1$ of SQ_{n+m-1} which are in rows 1 through $b_i + m - 1$, where b_i denotes the height of column i of B. Then we see that there are $b_i + m - 1 - k_i$ available squares in column i of SQ_{n+m-1} which are on B_m.

By induction, any rook from P in columns 1 through $i-1$ is on B_m if and only if this rook maps to a rook on B, and any rook is off B_m on square $(s, j_s + m - 1)$ if and only if this rook maps to a rook off B on square (s, j_s). These two facts imply that a rook in columns 1 through $i - 1$ in a row between 1 and $b_i + m - 1$ of SQ_{n+m-1} maps to a rook in columns 1 through $i - 1$ of SQ_n in a row between 1 and b_i. Thus the number of rooks in columns 1 through $i - 1$ of SQ_n from Q which can attack a square on B in column i is also k_i, and hence there are $b_i - k_i$ available squares in column i of SQ_n which are on B.

A rook on one of the lowest m available squares in column i of SQ_{n+m-1} will map to the unique square in column i of SQ_n which completes a cycle in Q. Since B is a regular Ferrers board, this square will lie on B. Thus there are $(b_i + m - 1) - k_i - m = b_i - k_i - 1$ available squares on B_m in column i of SQ_{n+m-1} which do not map to $s_i(Q)$. On SQ_n we see that there is one square which completes a cycle in Q, and $b_i - k_i - 1$ squares which do not complete a cycle. Hence by the definition of $\phi_{n,B,m}$ the $b_i - k_i - 1$ squares on B_m which do not map to $s_i(Q)$ are in one to one correspondence with the $b_i - k_i - 1$ available squares on B in column i, so $r_i(P)$ is on B_m if and only if $r_i(Q)$ is on B.

Finally, the remaining $(n+m-1)-(b_i+m-1)-(i-1-k_i) = n-b_i-i+1+k_i$ available squares in column i of SQ_{n+m-1} off B_m are in one-to-one correspondence with the $n - b_i - (i - 1 - k_i) = n - b_i - i + 1 + k_i$ available squares in column i

of SQ_n off B. By induction a rook on square $(s, j_s + m - 1)$ for $1 \leq s \leq i - 1$ which is off B_m maps to a rook on square (s, j_s) which is off B. Thus we see that in column i a square $(i, j_i + m - 1)$ off B_m is available if and only if the square (i, j_i) (which is off B) is available. Thus by definition of $\phi_{n,B,m}$ we see that $r_i(P)$ is off B_m on square $(i, j_i + m - 1)$ if and only if $r_i(Q)$ is off B on square (i, j_i). □

Note that a corollary of Lemma 3.2.2 is that $\phi_{n,B,m}|_{\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)}$ is actually a map from $\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)$ to $\mathcal{H}_{n,n-k}(B)$.

Now let us weight a placement $Q \in \mathcal{H}_{n,k}(B)$ by

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{h_{n+m-1,B_m}(P)}, \tag{3.2}$$

where $h_{n+m-1,B_m}(P)$ is as described in Chapter 1. As was earlier discussed, the rooks from some $P \in \mathcal{P}_{n+m-1}(B_m)$ in columns $n+1$ through $n+m-1$ will all lie on B_m. Thus by the definition of Haglund's statistic, if we fix the rooks in the first n columns and sum over all the possible $(m-1)!$ placements of non-attacking rooks in the last $m-1$ columns, we will generate $[m-1]!$.

Given a statistic *stat* which can be calculated for any rook placement R on a board $SQ_d \supseteq F$, we will denote by $stat(R)_i$ the contribution to $stat(R)$ coming from the ith column of SQ_d. Thus for $Q \in \mathcal{H}_{n,k}(B)$, we see that

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{h_{n+m-1,B_m}(P)} = [m-1]! \sum_{P'} \prod_{i=1}^{n} q^{h_{n+m-1,B_m}(P)_i}$$

where the second sum is over all placements P' of rooks in columns 1 through n of $SQ_{n+m-1} \supseteq B_m$ which extend to a placement $P \in \phi^{-1}_{n,B,m}(Q)$ and P is any one of these extensions of P'.

We have the following lemmas about this weighting.

Lemma 3.2.3. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is on the square $s_i(Q)$. Then*

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{h_{n+m-1,B_m}(P)_i} = [m].$$

Proof. If $r_i(Q)$ is on $s_i(Q)$, then by definition for $P \in \phi^{-1}_{n,B,m}(Q)$ $r_i(P)$ is on one of the m lowest squares in column i not attacked by a rook to the left. The lowest square gives a contribution from column i of 1, the second lowest a contribution of q, $\ldots$, the mth lowest a contribution of q^{m-1}. Thus we see that $\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{h_{n+m-1,B_m}(P)_i} = [m]$. □

Lemma 3.2.4. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is below the square $s_i(Q)$ on the a_ith square not attacked by a rook to the left. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of q^{m-1+a_i} to each summand of (3.2).*

Proof. $r_i(Q)$ is on the a_ith square not attacked by a rook to the left, which is also (since $r_i(Q)$ is below $s_i(Q)$) the a_ith square not attacked by a rook to the left which does not complete a cycle. Thus we see by the definition of $\phi_{n,B,m}$ that $r_i(P)$ must be on the $(m + a_i)$th square in column i of SQ_{n+m-1} not attacked by a rook to the left. Since $r_i(Q)$ is below $s_i(Q)$ it must be on B, so by Lemma 3.2.2 $r_i(P)$ is on B_m. Thus $r_i(P)$ has $m - 1 + a_i$ uncancelled squares below it, so it contributes $m - 1 + a_i$ to $h_{n+m-1,B_m}(P)$ and hence a factor of q^{m-1+a_i} to each summand of (3.2). □

Lemma 3.2.5. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ on B is above the square $s_i(Q)$, and on the a_ith square not attacked by a rook to the left. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of q^{m-1+a_i-1} to each summand of (3.2).*

Proof. $r_i(Q)$ is on the a_ith square not attacked by a rook to the left, which is (since $r_i(Q)$ is above $s_i(Q)$) the $(a_i - 1)$th square not attacked by a rook to the left which does not complete a cycle. Thus we see by the definition of $\phi_{n,B,m}$ that $r_i(P)$ must be on the $(m + a_i - 1)$th square in column i of SQ_{n+m-1} not attacked by a rook to the left. Again by Lemma 3.2.2, since $r_i(Q)$ is on B $r_i(P)$ must be on B_m. Thus $r_i(P)$ has $m-1+a_i-1$ uncancelled squares below it, so it contributes $m-1+a_i-1$ to $h_{n+m-1,B_m}(P)$ and hence a factor of q^{m-1+a_i-1} to each summand of (3.2). □

Lemma 3.2.6. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is off B. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of $q^{m-1+h_{n,B}(Q)_i}$ to each summand of (3.2).*

Proof. By Lemma 3.2.2 and its proof, we see that if $r_i(Q)$ is on (i, j) then $r_i(P)$ is on $(i, j+m-1)$ and the number of rooks below and to the left of $r_i(Q)$ is equal to the number of rooks below and to the left of $r_i(P)$. Thus the number of squares coming from column i when calculating $h_{n+m-1,B_m}(P)$ is the same as the number of squares from column i when calculating $m - 1 + h_{n,B}(Q)$, hence such a rook contributes a factor of $q^{m-1+h_{n,B}(Q)_i}$ to each summand of (3.2). □

Note that for $Q \in \mathcal{P}_n(B)$ and $r_i(Q)$ not on the cycle square but on the a_ith square not attacked by a rook to the left, $a_i = h_{n,B}(Q)_i + 1$. Thus for a rook below the cycle square in column i we have that $q^{m-1+a_i} = q^{m-1+h_{n,B}(Q)_i+1}$, and for a rook on B above the cycle square in column i, $q^{m-1+a_i-1} = q^{m-1+h_{n,B}(Q)_i}$. Now we see that

$$A_{n,k}(m,q,B) = \frac{1}{[m-1]!} A_{n+m-1,k}(q,B_m) =$$

$$\frac{1}{[m-1]!} \sum_{P \in \mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)} q^{h_{n+m-1,B_m}(P)} =$$

$$\frac{1}{[m-1]!} \sum_{Q \in \mathcal{H}_{n,n-k}(B)} \left\{ \sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{h_{n+m-1,B_m}(P)} \right\} =$$

$$\frac{1}{[m-1]!}[m-1]! \sum_{Q \in \mathcal{H}_{n,n-k}(B)} \left\{ \sum_{P' \text{ which extend to some } P \in \phi^{-1}_{n,B,m}(Q)} \prod_{i=1}^{n} q^{h_{n+m-1,B_m}(P)_i} \right\} =$$

$$\sum_{Q \in \mathcal{H}_{n,n-k}(B)} [m]^{cyc(Q)} \prod_{r_i(Q) \text{ below } s_i(Q)} q^{m-1+a_i(Q)} \times$$

$$\prod_{r_i(Q) \text{ above } s_i(Q) \text{ on } B} q^{m-1+a_i(Q)-1} \prod_{r_i(Q) \text{ above } s_i(Q) \text{ off } B} q^{m-1+h_{n,B}(Q)_i} =$$

$$\sum_{Q \in \mathcal{H}_{n,n-k}(B)} [m]^{cyc(Q)} \prod_{r_i(Q) \text{ below } s_i(Q)} q^{(m-1)+h_{n,B}(Q)_i+1} \prod_{r_i(Q) \text{ above } s_i(Q)} q^{(m-1)+h_{n,B}(Q)_i} =$$

$$\sum_{Q \in \mathcal{H}_{n,n-k}(B)} [m]^{cyc(Q)} q^{(n-cyc(Q))(m-1)+b_{n,B}(Q)+E(Q)}, \tag{3.3}$$

where $b_{n,B}(Q)$ is defined as the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme:

1. Each rook cancels all squares to the right in its row;
2. each rook on B cancels all squares above it in its column (squares both on B and strictly above B);
3. each rook on B on a cycle square cancels all squares below it in its column as well;
4. each rook off B cancels all squares below it but above B.

Note that if we let $m = 1$ in (3.3), then we obtain a statistic to generate the q-hit numbers. That is,

$$A_{n,k}(q,B) = \sum_{Q \in \mathcal{H}_{n,n-k}(B)} q^{b_{n,B}(Q)+E(Q)}.$$

While this new statistic is equal to neither that of Haglund [12] nor Dworkin [4], it is a member of the family of statistics discussed by Haglund and Remmel [13, p. 479].

3.3 The main theorem and a corollary

We can now define

$$\tilde{A}_{n,k}(y,q,B) = \sum_{P \in \mathcal{H}_{n,n-k}(B)} [y]^{cyc(P)} q^{(n-cyc(P))(y-1)+b_{n,B}(P)+E(P)}$$

and prove the following.

Theorem 3.3.1. *For any regular Ferrers board B,*

$$A_{n,k}(y,q,B) = \tilde{A}_{n,k}(y,q,B)$$

for $0 \le k \le n$.

Proof. Both of the above expressions are polynomials in the variable q^y over the field $\mathbb{Q}(q)$ of fixed degree. By the previous section, $A_{n,k}(m,q,B) = \tilde{A}_{n,k}(m,q,B)$ for any $m \in \mathbb{N}$. Thus these two polynomials have infinitely many common values, hence must be equal for all y. □

A permutation statistic s is called *Mahonian* if

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = [n]!.$$

We shall say that a pair (s_1, s_2) of statistics is *cycle-Mahonian* if

$$\sum_{\sigma \in S_n} [y]^{s_1(\sigma)} q^{s_2(\sigma,y)} = [y][y+1]\cdots[y+n-1].$$

Note that the statistic s_2 may depend on both σ and y. This notion generalizes that of a Mahonian statistic, since letting $y = 1$ in the definition of cycle-Mahonian gives

$$\sum_{\sigma \in S_n} q^{s_2(\sigma,1)} = [1][2]\cdots[n] = [n]!.$$

We can associate to a permutation $\sigma \in S_n$ the placement P_σ of n rooks on SQ_n consisting of the squares $\{(i,j) \mid \sigma(i) = j\}$. We can then make any statistic *stat* defined for placements of n rooks on SQ_n into a permutation statistic by letting

$$stat(\sigma) = stat(P_\sigma).$$

In light of this definition, we have the following.

Corollary 3.3.2. *The pair $(cyc(-), (n - cyc(-))(y-1) + b_{n,B}(-) + E(-))$ is cycle-Mahonian for any regular Ferrers board $B \subseteq SQ_n$.*

Proof. By definition,

$$\sum_{\sigma\in S_n}[y]^{cyc(\sigma)}q^{(n-cyc(\sigma))(y-1)+b_{n,B}(\sigma)+E(\sigma)} = \sum_{\sigma\in S_n}[y]^{cyc(P_\sigma)}q^{(n-cyc(P_\sigma))(y-1)+b_{n,B}(P_\sigma)+E(P_\sigma)}.$$

By Theorem 3.3.1 we know that

$$\sum_{\sigma\in S_n}[y]^{cyc(P_\sigma)}q^{(n-cyc(P_\sigma))(y-1)+b_{n,B}(P_\sigma)+E(P_\sigma)} = \sum_{k=0}^{n}A_{n,k}(y,q,B).$$

Finally, it is known [11] that for any regular Ferrers board $B\subseteq SQ_n$,

$$\sum_{k=0}^{n}A_{n,k}(y,q,B) = [y][y+1]\cdots[y+n-1]. \tag{3.4}$$

□

Chapter 4

The Cycle-Counting q-Eulerian Numbers

In this chapter, we apply the statistic from Chapter 3 for the triangular board $\mathbb{T}_n$ to study a cycle-counting version of the q-Eulerian numbers. We finish the chapter by defining a generalization of the notion of an Euler-Mahonian permutation statistic, and providing an example.

4.1 The algebraic q, y-Eulerian numbers

Let $\mathbb{T}_n$ denote the triangular board $B(1, 2, \ldots, n)$ with column heights $1, 2, \ldots, n$. Recall the following permutation statistics for a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$

$$des(\sigma) = \big|\{i|\ \sigma_i > \sigma_{i+1}\}\big| \quad \text{and} \quad maj(\sigma) = \sum_{\sigma_i > \sigma_{i+1}} i,$$

called the *number of descents* and the *major index*, respectively, of the permutation σ. The *q-Eulerian numbers* are then defined by the equation

$$E_{n,k}(q) = \sum_{\sigma \in S_n,\ des(\sigma)=k-1} q^{maj(\sigma)}. \tag{4.1}$$

It is known [12] that $E_{n,k}(q) = A_{n,k-1}(q, \mathbb{T}_n)$, hence we obtain a q, y-version of the Eulerian numbers, which we call the *algebraic q, y-Eulerian numbers*, via the equation

$$E_{n,k}(y, q) = A_{n,k-1}(y, q, \mathbb{T}_n).$$

We have the following easy lemma about the algebraic q, y-Eulerian numbers.

Lemma 4.1.1. *We have*

$$E_{n,k}(y, q) = [y + k - 1]E_{n-1,k}(y, q) = q^{y+k-2}[n - k + 1]E_{n-1,k-1}(y, q)$$

for $n, k \in \mathbb{N}$.

Proof. Let $B = \mathbb{T}_n$ in Lemma 3.1.2. $\square$

4.2 The combinatorial q, y-Eulerian numbers

Our goal in this section is to combinatorially define a q, y-version of the $E_{n,k}(q)$, involving des and maj, which reduces to (4.1) when $y = 1$. To this end, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. If $\sigma_{j_1} = 1$, let y_1 be the cycle $(\sigma_1\cdots\sigma_{j_1})$. If α is the smallest integer not contained in y_1, and $\sigma_{j_2} = \alpha$, let y_2 be the cycle $(\sigma_{j_1+1}\cdots\sigma_{j_2})$, etc. If the result of the above procedure is the product of cycles $y_1y_2\cdots y_p$, we will let $p = \ell rmin(\sigma)$, called the number of *left-to-right minima* of σ. For example, the permutation 72165834 breaks into cycles under the above procedure as (721)(6583)(4), so $\ell rmin(72165834) = 3$. We can now define the *combinatorial* q, y*-Eulerian numbers* by the equation

$$\tilde{E}_{n,k}(y,q) = \sum_{\sigma\in S_n,\ des(\sigma)=k-1} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)}$$

and prove the following.

Proposition 4.2.1. *For any* $n,\ k \in \mathbb{N}$ *we have that*

$$\tilde{E}_{n,k}(y,q) = [y+k-1]\tilde{E}_{n-1,k}(y,q) + q^{y+k-2}[n-k+1]\tilde{E}_{n-1,k-1}(y,q).$$

Proof. We mimic the well known proof of the $y = 1$ case (that is, the regular q-Eulerian numbers $E_{n,k}(q)$). Any permutation in S_n with $k-1$ descents can be built from one in S_{n-1} with either $k-1$ or $k-2$ descents in the following way.

First suppose $\sigma \in S_{n-1}$ has $k-1$ descents, occurring at positions $i_1, i_2, \ldots i_{k-1}$. Thus $\sigma = \sigma_1\cdots\sigma_{i_1}\cdots\sigma_{i_{k-1}}\cdots\sigma_{n-1}$, where

$$\sigma_1 < \sigma_2 \cdots < \sigma_{i_1} > \sigma_{i_1+1} < \cdots < \sigma_{i_{k-1}} > \sigma_{i_{k-1}+1} < \cdots < \sigma_{n-1}.$$

This permutation will contribute

$$[y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}$$

to $\tilde{E}_{n-1,k}(y,q)$.

We can place n in any of the $k-1$ positions of σ where a descent occurs, thereby creating a new permutation σ' in S_n which still has only $k-1$ descents. If we place n in the (i_1+1)th position, all the descents are moved one position to the right, thus increasing maj by $k-1$. Here we see that $\ell rmin(\sigma) = \ell rmin(\sigma')$, since there will clearly be a number to the right of where we have placed n which is smaller than n. However, we have increased the number of letters in the permutation from $n-1$ to n. Thus

$$[y]^{\ell rmin(\sigma')} q^{(n-\ell rmin(\sigma'))(y-1)+maj(\sigma')} =$$

$$\{q^{(y-1)+(k-1)}\} \times [y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}.$$

Next we see that if we place n in the (i_2+1)th position, this time maj will increase by $k-2$, and again $\ell rmin(\sigma') = \ell rmin(\sigma)$ but the number of letters in the permutation increases by one. Therefore in this case, we gain a factor of $q^{(y-1)+(k-2)}$.

Continuing in this manner we proceed from left to right. Placing n in the $(i_{k-1}+1)$th position gives a factor of $q^{(y-1)+1}$, so the sum of all of these factors is $q^{y+k-2} + q^{y+k-3} + \cdots + q^{y+1} + q^y$. There is one last position where we can place n and not increase des, and that is the nth position. This will also not increase maj, however $\ell rmin(\sigma')$ will now be $\ell rmin(\sigma)+1$. We have also increased the total number of letters from $n-1$ to n, but since $\ell rmin(\sigma') = \ell rmin(\sigma)+1$ we have that $(n-1) - \ell rmin(\sigma) = n - \ell rmin(\sigma')$. Thus this last placement of n just contributes $[y]$, and summing over all positions for n which do not increase $des(\sigma)$ gives $[y] + q^y + q^{y+1} + \cdots + q^{y+k-2}$, which is equal to $[y+k-1]$. Summing again, over all $\sigma \in S_{n-1}$ with $k-1$ descents yields the first term in the recurrence.

Now suppose $\sigma \in S_{n-1}$ has $k-2$ descents, occurring at positions $i_1, i_2, \ldots i_{k-2}$. Thus $\sigma = \sigma_1 \cdots \sigma_{i_1} \cdots \sigma_{i_{k-2}} \cdots \sigma_{n-1}$, where

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{i_1} > \sigma_{i_1+1} < \cdots < \sigma_{i_{k-2}} > \sigma_{i_{k-2}+1} < \cdots < \sigma_{n-1}.$$

This permutation will contribute

$$[y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}$$

to $\tilde{E}_{n-1,k-1}(y,q)$.

We can place n in any of the $n-(k-1)$ positions which will create an additional descent in our new permutation σ'. If we place n in the first position, this new descent will add 1 to maj, and it will move each of the $k-2$ descents to the right of it one position to the right, adding another $k-2$ to maj. Thus maj will increase by a total of $k-1$. As argued in the above case, $\ell rmin(\sigma') = \ell rmin(\sigma)$, but since we have increased the number of letters in the permutation from $n-1$ to n, the quantity $n - \ell rmin(\sigma') = \{(n-1) - \ell rmin(\sigma)\} + 1$. Thus we also obtain an extra q^{y-1}, and hence

$$[y]^{\ell rmin(\sigma')} q^{(n-\ell rmin(\sigma'))(y-1)+maj(\sigma')} =$$

$$\{q^{(y-1)+(k-1)}\} \times [y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}.$$

Continuing in this manner until the first descent at position i_1, we obtain factors of $q^{(y-1)+(k-1)}$, $q^{(y-1)+k}$, $\ldots q^{(y-1)+k-2+i_1}$. We do not place n in the (i_1+1)th position, as this will not create a new descent. Instead, we skip over this position and move to the (i_1+2)th position. The new descent created will contribute i_1+2 to maj. Now there will be only $k-3$ descents to the right of where we have placed n, which will each be moved one position to the right increasing maj by $k-3$. As argued in the previous paragraph, we will gain a factor of $q^{(y-1)+k-3+i_1+2}$.

We continue the above placement scheme, skipping over positions where descents are already in σ. The last position will contribute $q^{(y-1)+n-1}$, and the sum over all positions for n in σ which increase des yields $q^{y+k-2}+q^{y+k-1}+\cdots+q^{y+n-2} = q^{y+k-2} \times \{1+q+q^2+\cdots+q^{n-k}\} = q^{y+k-2}[n-k+1]$. Now summing over all $\sigma \in S_{n-1}$ with $k-2$ descents yields the second term in the recurrence.

□

We can now prove the following theorem.

Theorem 4.2.2. *For any $n, k \in \mathbb{N}$ we have that $\tilde{E}_{n,k}(y,q) = E_{n,k}(y,q)$.*

Proof. It is clear that $\tilde{E}_{1,1}(y,q) = [y]$, and it is easy to check by definition of $A_{1,1}(y,q,\mathbb{T}_1)$ that $E_{1,1}(y,q) = [y]$. Thus the $\tilde{E}_{n,k}(y,q)$ and the $E_{n,k}(y,q)$ satisfy the same initial conditions, and by Lemma 4.1.1 and Proposition 4.2.1 they also satisfy the same recurrence. Thus $\tilde{E}_{n,k}(y,q) = E_{n,k}(y,q)$ for all $n,k \in \mathbb{N}$ as desired. □

A number of algebraic identities about the $A_{n,k}(y,q,B)$ for B a regular Ferrers board are proven in [11]. Thus in light of Theorem 4.2.2, we obtain the following identities for the $\tilde{E}_{n,k}(y,q)$.

Corollary 4.2.3. *For $n, k \in \mathbb{N}$ and $1 \le k \le n$ we have that*

$$\tilde{E}_{n,k}(y,q) = \sum_{j=0}^{k-1} \begin{bmatrix} n+y \\ k-1-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-1-j} q^{\binom{k-1-j}{2}} [j+y]^n.$$

Proof. Let $x = y$ and $B = \mathbb{T}_n$ in Lemma 5.1 of [11]. □

Corollary 4.2.4. *Let $(w)_\infty$ denote the product $\prod_{i=0}^{\infty}(1-wq^i)$. Then*

$$\frac{(z)_\infty}{(q^{y+n})_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} y+k-1 \\ k \end{bmatrix} [y+k]^n z^k = \sum_{k=1}^{n} \tilde{E}_{n,k}(y,q) z^{k-1}.$$

Proof. Let $x = y$ and $B = \mathbb{T}_n$ in the equation in [11, p. 455], and note that $A_{n,n}(y,q,\mathbb{T}_n) = 0$. □

4.3 Cycle-Euler-Mahonian permutation statistics

Another direct corollary of Theorem 4.2.2 is the following.

Proposition 4.3.1. *The pair* $(\ell rmin(-), (n - \ell rmin(-))(y-1) + maj(-))$ *is cycle-Mahonian.*

Proof. By definition,

$$\sum_{\sigma \in S_n} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)} = \sum_{k=1}^{n} \tilde{E}_{n,k}(y,q)$$

By Theorem 4.2.2,

$$\sum_{k=1}^{n} \tilde{E}_{n,k}(y,q) = \sum_{k=1}^{n} E_{n,k}(y,q).$$

Again by definition,

$$\sum_{k=1}^{n} E_{n,k}(y,q) = \sum_{k=1}^{n} A_{n,k-1}(y,q,\mathbb{T}_n),$$

which is equal to $[y][y+1]\cdots[y+n-1]$ by (3.4) (since $A_{n,n}(y,q,\mathbb{T}_n) = 0$). $\square$

Note we can bijectively associate the set of permutations in S_n with k descents to the set of placements of n rooks on SQ_n such that exactly k rooks lie off $\mathbb{T}_n$ in the following way (first noted in [14]). Suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ has k descents. First, we find the product $y_1 y_2 \cdots y_p$ of cycles as was done when computing $\ell rmin(-)$ in Section 4.2. Then we place a rook on square (i,j) of SQ_n if and only if i follows j in one of the cycles y_ℓ. It is easy to verify that this placement will have exactly k rooks off $\mathbb{T}_n$, and that this procedure can be reversed. This placement is the *descent graph* of σ, which we will denote $DG(\sigma)$. For example, $DG(72165834)$ is pictured in Figure 4.1. Note that $des(72165834) = 4$, which is the number of rooks off $\mathbb{T}_n$ in $DG(72165834)$. Note now that by Theorem 3.3.1 and the above discussion, we have that

$$E_{n,k}(y,q) = \sum_{\sigma \in S_n,\ des(\sigma)=k-1} [y]^{cyc(DG(\sigma))} q^{(n-cyc(DG(\sigma))(y-1)+b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))}.$$

We can now prove the following.

Theorem 4.3.2. *We have*

$$\sum_{\sigma \in S_n,\ des(\sigma)=k,\ cyc(DG(\sigma))=\ell} q^{b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))} = \sum_{\sigma \in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}$$

for any $n, k, \ell \in \mathbb{N}$.

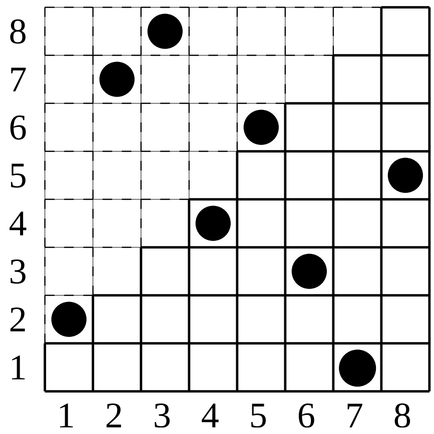

Figure 4.1: The descent graph for 72165834.

Proof. We know by the above discussion that

$$\sum_{\sigma\in S_n,\ des(\sigma)=k} [y]^{cyc(DG(\sigma))} q^{(n-cyc(DG(\sigma))(y-1)+b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))} = E_{n,k+1}(y,q). \tag{4.2}$$

By Theorem 4.2.2, (4.2) is equal to $\tilde{E}_{n,k+1}(y,q)$, where

$$\tilde{E}_{n,k+1}(y,q) = \sum_{\sigma\in S_n,\ des(\sigma)=k} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)},$$

and hence

$$\sum_{\sigma\in S_n,\ des(\sigma)=k} [y]^{cyc(DG(\sigma))} q^{(n-cyc(DG(\sigma))(y-1)+b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))} =$$

$$\sum_{\sigma\in S_n,\ des(\sigma)=k} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)}. \tag{4.3}$$

If we let $z=[y]q^{-(y-1)}$ in (4.3), then we have that

$$q^{n(y-1)} \sum_{\sigma\in S_n,\ des(\sigma)=k} z^{cyc(DG(\sigma))} q^{b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))} =$$

$$q^{n(y-1)} \sum_{\sigma\in S_n,\ des(\sigma)=k} z^{\ell rmin(\sigma)} q^{maj(\sigma)}.$$

Thus

$$\sum_{\sigma\in S_n,\ des(\sigma)=k} z^{cyc(DG(\sigma))} q^{b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))}$$

and

$$\sum_{\sigma\in S_n, des(\sigma)=k} z^{\ell rmin(\sigma)} q^{maj(\sigma)}$$

are equal polynomials in the variable z over $\mathbb{N}[q]$, and hence equal powers of z must have equal coefficients. In particular the coefficient of z^ℓ in each must be equal. That is

$$\sum_{\sigma\in S_n,\ des(\sigma)=k,\ cyc(DG(\sigma))=\ell} q^{b_{n,\mathbb{T}_n}(DG(\sigma))+E(DG(\sigma))} = \sum_{\sigma\in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}$$

as desired.

□

Recall that a permutation statistic s on S_n is called *Euler-Mahonian* if the pairs (des, s) and (des, maj) have the same distribution on S_n, that is,

$$\sum_{\sigma\in S_n,\ des(\sigma)=k} q^{s(\sigma)} = \sum_{\sigma\in S_n,\ des(\sigma)=k} q^{maj(\sigma)}$$

for all values of k. Theorem 4.3.2 leads us to define the following generalization. We will say a pair of permutation statistics $(s_1(-), s_2(-, y))$ is *cycle-Euler-Mahonian* if it is cycle-Mahonian as defined in Section 3.3, and

$$\sum_{\sigma\in S_n,\ des(\sigma)=k,\ s_1(\sigma)=\ell} q^{s_2(\sigma,1)} = \sum_{\sigma\in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}. \tag{4.4}$$

This definition generalizes that of Euler-Mahonian, because if $(s_1(-), s_2(-, y))$ satisfies (4.4) then

$$\sum_{\sigma\in S_n,\ des(\sigma)=k} q^{s_2(\sigma,1)} = \sum_{\ell}\left\{\sum_{\sigma\in S_n,\ des(\sigma)=k,\ s_1(\sigma)=\ell} q^{s_2(\sigma,1)}\right\} =$$

$$\sum_{\ell}\left\{\sum_{\sigma\in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}\right\} = \sum_{\sigma\in S_n,\ des(\sigma)=k} q^{maj(\sigma)}.$$

Thus if $(s_1(-), s_2(-, y))$ is cycle-Euler-Mahonian, this implies that $s_2(-, 1)$ is Euler Mahonian.

By Corollary 3.3.2 $(cyc(DG(-)), (n - cyc(DG(-)))(y - 1) + b_{n,\mathbb{T}_n}(DG(-)) + E(DG(-)))$ is cycle-Mahonian, and by Theorem 4.3.2 we see that

$$(des(-), cyc(DG(-)), b_{n,\mathbb{T}_n}(DG(-)) + E(DG(-)))$$

and

$$(des(-), \ell rmin(-), maj(-))$$

have the same distribution. Thus

$$(cyc(DG(-)), (n - cyc(DG(-)))(y-1) + b_{n,\mathbb{T}_n}(DG(-)) + E(DG(-)))$$

is an example of a cycle-Euler-Mahonian pair of statistics on S_n.

Chapter 5

Other Rook Theory Models

In this chapter, we present two other models which generalize classical rook theory.

5.1 Cycle-counting p, q-rook theory

In this section, we describe a model which generalizes both the $R_k(y, q, B)$ and the p, q-analog of the rook numbers introduced in [15]. A p, q-analog of a number is an expression such that setting $p = 1$ yields a q-analog of that number. For example, the p, q-analog of the integer n is given by the equation

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}.$$

More generally, the p, q analog of the real number y, denoted $[y]_{p,q}$, is

$$\frac{p^y - q^y}{p - q}.$$

Let $B = B(b_1, \ldots, b_n)$ be a Ferrers board, P a placement of rooks on B. As in [15], we let any rook in P p, q-cancel all squares to its right, and let

1. $\alpha_B(P)$ be the number of uncancelled squares above a rook on B,
2. $\beta_B(P)$ the number of uncancelled squares below a rook on B, and
3. $\epsilon_B(P)$ the number of uncancelled squares in an empty column of B.

We can then define the *kth cycle-counting p, q-rook number* of B by the equation

$$R_k(y, p, q, B) = \sum_{P \in \mathcal{R}_k(B)} [y]_{p,q}^{cyc(P)} q^{\alpha_B(P) + \epsilon_B(P) + E(P)(y-1)} p^{\beta_B(P) - (c_1 + \cdots + c_k) + A(P)(y-1)},$$

where the leftmost rook from P is in column c_1, the second in column c_2, ..., the kth in column c_k, $E(P)$ is as defined in Section 1.3, and $A(P)$ is the number of i such that $b_i \geq i$ and there *is* a rook from P in column i strictly above the square $s_i(P)$.

We will see that these numbers satisfy a version of the recurrence that the q-rook numbers were shown to satisfy in [6]. Let us first prove a useful lemma.

Lemma 5.1.1. *For any $k, j \in \mathbb{N}$ with $j \leq k$, we have that*

$$p^{y+k-1} + p^{y+k-2}q + \cdots + p^{y+k-j}q^{j-1} + p^{k-j}q^j[y]_{p,q} + p^{k-j-1}q^{y+j} + \cdots$$
$$+pq^{y+k-2} + q^{y+k-1} = [y+k]_{p,q}.$$

Proof. Putting the left hand side of the above expression over a common denominator yields

$$\begin{gathered}\{p^{y+k-1}(p-q) + p^{y+k-2}q(p-q) + \cdots \\ +p^{y+k-j}q^{j-1}(p-q) + p^{k-j}q^j(p^y - q^y) + p^{k-j-1}q^{y+j}(p-q) + \cdots \\ +pq^{y+k-2}(p-q) + q^{y+k-1}(p-q)\}/(p-q),\end{gathered}$$

which is

$$\begin{gathered}\{p^{y+k} - p^{y+k-1}q + p^{y+k-1}q - \cdots \\ +p^{y+k-j+1}q^{j-1} - p^{y+k-j}q^j + p^{y+k-j}q^j - p^{k-j}q^{y+j} + p^{k-j}q^{y+j} - \cdots \\ +p^2q^{y+k-2} - pq^{y+k-1} + pq^{y+k-1} - q^{y+k}\}/(p-q).\end{gathered}$$

All the middle terms cancel, and we are left with

$$\frac{p^{y+k} - q^{y+k}}{p-q} = [y+k]_{p,q}$$

as desired. □

Proposition 5.1.2. *Let $B = B(b_1, \ldots, b_{n-1}, b_n)$ be a Ferrers board, and let $B' = B(b_1, \ldots, b_{n-1})$. (a) If $b_n < n$, then*

$$R_k(y,p,q,B) = q^{b_n-k}R_k(y,p,q,B') + p^{-n}[b_n - k + 1]_{p,q}R_{k-1}(y,p,q,B').$$

(b) If $b_n \geq n$, then

$$R_k(y,p,q,B) = q^{b_n-k+y-1}R_k(y,p,q,B') + p^{-n}[b_n - k + y]_{p,q}R_{k-1}(y,p,q,B').$$

Proof. **Case (a):** A placement P of k rooks on B can be broken into either a placement of k rooks on B' and no rooks in the last column of B, or a placement of $k-1$ rooks on B' and one rook in the last column of B.

If P comes from a placement P' of k rooks on B', then we have an additional $b_n - k$ uncancelled squares in the last column. Since the last column does not contain a rook, we have that $\epsilon_B(P) = \epsilon_{B'}(P') + b_n - k$, while $\alpha_B(P) = \alpha_{B'}(B')$ and $\beta_B(P) = \beta_{B'}(P')$. Also, since $b_n < n$ we have that $cyc(P)$, $E(P)$ and $A(P)$ remain unchanged. Thus

$$[y]_{p,q}^{cyc(P)} q^{\alpha_B(P)+\epsilon_B(P)+E(P)(y-1)} p^{\beta_B(P)-(c_1+\cdots+c_k)+A(P)(y-1)}$$

is equal to

$$q^{b_n-k} \times \left\{ [y]_{p,q}^{cyc(P')} q^{\alpha_{B'}(P')+\epsilon_{B'}(P')+E(P')(y-1)} p^{\beta_{B'}(P')-(c_1+\cdots+c_k)+A(P')(y-1)} \right\}.$$

Summing over all P' of the first type above yields the first term in the (a) recurrence.

If P comes from a placement P' of $k-1$ rooks on B', then we have $b_n - k + 1$ available squares where to place the kth rook in the last column. Again since $b_n < n$, $cyc(P) = cyc(P')$, $E(P) = E(P')$, and $A(P) = A(P')$. Putting a rook in any position in the last column will add a factor of p^{-n} (since in this case $c_k = n$).

Putting a rook in the top available square increases $\beta_{B'}(P')$ by $b_n - k$ adding a factor of p^{b_n-k}. A rook in the second available square from the top increases $\alpha_{B'}(P')$ by 1 and $\beta_{B'}(P')$ by $b_n - k - 1$, adding a factor of $p^{b_n-k-1}q$. We continue in this manner until we reach the bottom available square, which does not increase $\beta_{B'}(P')$ and increases $\alpha_{B'}(P')$ by $b_n - k$, giving a factor of q^{b_n-k}. Adding all of these factors together gives

$$p^{b_n-k} + p^{b_n-k-1}q + \cdots + pq^{b_n-k-1} + q^{b_n-k} = [b_n - k + 1]_{p,q},$$

so we see that

$$\sum_{P \in \mathcal{R}_k(B),\ P \cap B' = P'} [y]_{p,q}^{cyc(P)} q^{\alpha_B(P)+\epsilon_B(P)+E(P)(y-1)} p^{\beta_B(P)-(c_1+\cdots+c_k)+A(P)(y-1)}$$

is equal to

$$p^{-n}[b_n - k + 1]_{p,q} \times [y]_{p,q}^{cyc(P')} q^{\alpha_{B'}(P')+\epsilon_{B'}(P')+E(P')(y-1)} p^{\beta_{B'}(P')-(c_1+\cdots+c_{k-1})+A(P')(y-1)}.$$

Summing over all P' of the second type yields the second term in the recurrence.

Case (b): This case is very similar to (a), except now there will be a square in the last column which completes a cycle, and hence cyc, E and A can increase. If P on B comes from a placement P' of k rooks on B', we see that there are again

$b_n - k$ uncancelled squares in the last column, adding $b_n - k$ to $\epsilon_{B'}(P')$. Also since $b_n \geq n$, $E(P) = E(P') + 1$, and A and cyc do not increase. Hence we obtain a factor of $q^{b_n-k+y-1}$, and when we sum over all placements of the first type we get the first term in the (b) recurrence.

If P comes from a placement P' of $k-1$ rooks on B', there are $b_n - k + 1$ places to put a rook in the last column. Again, any such placement adds a factor of p^{-n}. Say the jth available square from the top completes a cycle.

A rook in the top square increases $\beta_{B'}(P')$ by $b_n - k$, $A(P')$ by 1, and everything else stays the same. Hence we gain a factor of $p^{b_n-k+y-1}$. The second position adds a factor of $p^{b_n-k-1+y-1}q$. We proceed in this manner until we reach the jth available square from the top, which increases $cyc(P')$ by 1, $\alpha_{B'}(P')$ by $j-1$, $\beta_{B'}(P')$ by $b_n - k + 1 - j$, and everything else stays the same. Hence we gain a factor of $[y]_{p,q}p^{b_n-k+1-j}q^{j-1}$. The $(j+1)$th square increases $\alpha_{B'}(P')$ by j, $\beta_{B'}(P')$ by $b_n - k - j$, $E(P')$ by 1, and everything else stays the same. Hence here we get a factor of $p^{b_n-k-j}q^{j+y-1}$. We continue in this manner until we reach the bottom available square, which increases $\alpha_{B'}(P')$ by $b_n - k$, $E(P')$ by 1 and everything else stays the same, adding a factor of $q^{b_n-k+y-1}$.

Summing we obtain a total factor of

$$p^{b_n-k+y-1} + p^{b_n-k-1+y-1}q + \cdots + [y]_{p,q}p^{b_n-k+1-j}q^{j-1} +$$
$$p^{b_n-k-j}q^{j+y-1} + \cdots + q^{b_n-k+y-1},$$

which equals $[b_n - k + y]_{p,q}$ by Lemma 5.1.1. Hence

$$\sum_{P\in\mathcal{R}_k(B),\ P\cap B'=P'} [y]_{p,q}^{cyc(P)} q^{\alpha_B(P)+\epsilon_B(P)+E(P)(y-1)} p^{\beta_B(P)-(c_1+\cdots+c_k)+A(P)(y-1)}$$

is equal to

$$p^{-n}[b_n - k + y]_{p,q} \times [y]_{p,q}^{cyc(P')} q^{\alpha_{B'}(P')+\epsilon_{B'}(P')+E(P')(y-1)} p^{\beta_{B'}(P')-(c_1+\cdots+c_{k-1})+A(P')(y-1)},$$

and summing over all P' of this second type yields the second term in the (b) recurrence. □

Note that if we let $p = 1$ in Proposition 5.1.2, we obtain a useful recurrence for the $R_k(y, q, B)$. The p, q, y-rook numbers also satisfy the following version of the factorization theorem proven for the q-rook numbers in [6], the q, y-rook numbers in [5], and the p, q-rook numbers in [1].

Theorem 5.1.3. *Let $B = B(b_1, \ldots, b_n)$ be a Ferrers board. Then*

$$\sum_{k=0}^{n} R_{n-k}(y, p, q, B)p^{kz+\binom{k+1}{2}}[z]_{p,q}[z-1]_{p,q}\cdots[z-k+1]_{p,q} =$$

$$\prod_{i \text{ with } b_i \geq i} [z + b_i - i + y]_{p,q} \prod_{i \text{ with } b_i < i} [z + b_i - i + 1]_{p,q}.$$

Proof. We mimic the proof in [1]. Assume $z \in \mathbb{N}$, $z \geq n$, and let B_z denote the board $B(z + b_1, z + b_2, \dots, z + b_n)$. We show that both sides of the expression count

$$\sum_{P \in \mathcal{R}_n(B_z)} [y]_{p,q}^{cyc(P)} q^{\alpha_{B_z}(P)+E(P)(y-1)} p^{\beta_{B_z}(P)+A(P)(y-1)},$$

as follows.

First way of counting: We place rooks in each column of B_z from left to right. Start in the leftmost column of B_z, where the square cycle square $s_1(P) = (1, 1)$ is on B, assuming $b_1 \geq 1$. If we place a rook in the top position, we will gain a factor of $p^{b_1+z-1+y-1}$, second from the top gives $p^{b_1+z-2+y-1}q$, etc., until we reach the square $(1, 1)$, which gives a factor of $[y]_{p,q}p^{b_1-1}q^z$. Continuing, the first square on $B_z \backslash B$ gives a factor of $p^{b_1}q^{z-1+y-1}$, the second a factor of $p^{b_1-1}q^{z+y-1}$, etc., until we reach the bottom, which gives a factor of $q^{b_1+z-1+y-1}$. Adding all of these factors gives

$$p^{b_1+z-1+y-1} + p^{b_1+z-2+y-1}q + \cdots + [y]_{p,q}p^{b_1-1}q^z + p^{b_1}q^{z-1+y-1} +$$
$$p^{b_1-1}q^{z+y-1} + \cdots + q^{b_1+z-1+y-1},$$

which is equal to $[z + b_1 - 1 + y]_{p,q}$ by Lemma 5.1.1. If $b_1 = 0$, a similar argument shows that we gain a factor of $[z + b_1]_{p,q}$.

Now suppose we are placing rooks in the ith column of B_z. Since there are already $i - 1$ rooks in the previous $i - 1$ columns, there are $z + b_i - i + 1$ available squares in this column of B_z. There are now two cases.

1. $b_i \geq i$, in which case there is a cycle square in this column. Suppose the square $s_i(P)$ is the jth available square from the top of the column. A rook in the top position gives a factor of $p^{z+b_i-i+y-1}$, the second $p^{z+b_i-i-1+y-1}q$, $\dots$, the jth a factor of $[y]_{p,q}p^{z+b_i-i+1-j}q^{j-1}$, $\dots$, the bottom a factor of $q^{z+b_i-i+y-1}$, which when summed gives $[z + b_i - i + y]_{p,q}$.

2. $b_i < i$, in which case there is no cycle square in this column, hence no placement of a rook in this column will affect $cyc(P)$, $E(P)$, or $A(P)$. Placing a rook in the top position gives a factor of $p^{z+b_i-i+1-1}$, the second $p^{z+b_i-i+1-2}q$, $\dots$, the second from the bottom a factor of $pq^{z+b_i-i+1-2}$, the bottom a factor of $q^{z+b_i-i+1-1}$, which when summed gives $[z + b_i - i + 1]_{p,q}$.

Finally, summing over all columns gives

$$\prod_{i \text{ with } b_i \geq i} [z + b_i - i + y]_{p,q} \prod_{i \text{ with } b_i < i} [z + b_i - i + 1]_{p,q}.$$

Second way of counting: In this case we first choose a placement of $n-k$ rooks on B, then place the remaining k rooks on $B_z \backslash B$ from left to right. Let us fix a placement P of $n-k$ rooks on B, and compute

$$\sum_{Q \in \mathcal{R}_n(B_z),\ Q \cap B = P} [y]_{p,q}^{cyc(Q)} q^{\alpha_{B_z}(Q)+E(Q)(y-1)} p^{\beta_{B_z}(Q)+A(Q)(y-1)}. \tag{5.1}$$

For any such Q, it is clear that the contribution to any summand of (5.1) coming from the B part of B_z is just

$$[y]_{p,q}^{cyc(P)} q^{\alpha_B(P)+\epsilon_B(P)+E(P)(y-1)} p^{\beta_B(P)+A(P)(y-1)}.$$

The number of uncancelled cells on $B_z \backslash B$ below a rook from P is $kz + \binom{k+1}{2} - (c_1 + \cdots + c_k)$ as argued in [1], contributing an additional factor of $p^{kz+\binom{k+1}{2}-(c_1+\cdots+c_k)}$ to any summand of (5.1).

Finally, we consider the contribution to (5.1) coming from the remaining k columns of $B_z \backslash B$. It is clear that a rook in one of these positions will not affect cyc, E, or A. There are z available squares below B in the first available column; summing over all possible positions here gives a factor of $[z]_{p,q}$. In the second available column, there are now $z-1$ available squares, giving a factor of $[z-1]_{p,q}$. Continuing in this manner, we see that the last column gives a factor of $[z-k+1]_{p,q}$. Hence

$$\sum_{Q \in \mathcal{R}_n(B_z),\ Q \cap B = P} [y]_{p,q}^{cyc(Q)} q^{\alpha_{B_z}(Q)+E(Q)(y-1)} p^{\beta_{B_z}(Q)+A(Q)(y-1)} =$$
$$[y]_{p,q}^{cyc(P)} q^{\alpha_B(P)+\epsilon_B(P)+E(P)(y-1)} p^{\beta_B(P)-(c_1+\cdots+c_k)+A(P)(y-1)} \times$$
$$p^{kz+\binom{k+1}{2}} [z]_{p,q}[z-1]_{p,q} \cdots [z-k+1]_{p,q}.$$

Summing over all possible P with $n-k$ rooks on B gives

$$R_{n-k}(y,p,q,B) p^{kz+\binom{k+1}{2}} [z]_{p,q}[z-1]_{p,q} \cdots [z-k+1]_{p,q},$$

and summing over all k yields

$$\sum_{k=0}^{n} R_{n-k}(y,p,q,B) p^{kz+\binom{k+1}{2}} [z]_{p,q}[z-1]_{p,q} \cdots [z-k+1]_{p,q}$$

as desired. □

5.2 α-Rook theory

In this model, first introduced in [8], we allow only one rook per column but more than one rook in a given row. We will call an arrangement of rooks, with at most one rook in each column (but possibly more than one rook in a row), an *α-rook placement*. We will denote the set of all α-rook placements of k rooks on a board B by $\mathcal{R}_k^{(\alpha)}(B)$. If there are u rooks in a given row, that row has weight

$$1 \text{ if } 0 \le u \le 1,$$

$$\alpha(2\alpha-1)(3\alpha-2)\cdots((u-1)\alpha-(u-2)) \text{ if } u \ge 2.$$

The weight of a placement P is the product of the weights of all the rows (denoted $wt(P)$), and the *kth α-rook number* of the board B is given by the equation

$$r_k^{(\alpha)}(B) = \sum_{P \in \mathcal{R}_k^{(\alpha)}(B)} wt(P)$$

as in [8]. Note that $r_k^{(0)}(B) = r_k(B)$ for any board B, since the placements with more than one rook in a row are weighted by 0 if $\alpha = 0$.

We can now define the *kth α-hit number* of the board $B \subseteq SQ_n$ by the equation

$$h_{n,k}^{(\alpha)}(B) = \sum_{P \in \mathcal{H}_{n,k}^{(\alpha)}(B)} wt(P),$$

where $\mathcal{H}_{n,k}^{(\alpha)}(B)$ is the set of all α-rook placements of n rooks on SQ_n such that exactly k rooks are on B. As with the rook numbers, we note that $h_{n,k}^{(0)}(B) = h_{n,k}(B)$ for any board $B \subseteq SQ_n$.

The following is proven in [8].

Theorem 5.2.1. *Let $B = B(b_1, \ldots, b_n)$ be a Ferrers board. Then*

$$\sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B) z^{(k,\alpha-1)} = \prod_{i=1}^{n} (z + b_i + (i-1)(\alpha-1))$$

where $z^{(a,b)} = z(z+b)(z+2b)\cdots(z+(a-1)b)$ for $a \in \mathbb{N}$ and $b \in \mathbb{C}$.

The $r_k^{(\alpha)}(B)$ satisfy a version of a recurrence proven for the $R_k(q, B)$ in [6], and for the $R_k(y, p, q, B)$ (Proposition 5.1.2 of this dissertation) when B is a Ferrers board. To prove the α-version of this recurrence, let us first define some notation. If P is some fixed α-rook placement on SQ_n consisting of d rooks, let us denote by $P \cup_{(\alpha)} (i, j)$ the α-rook placement of $d+1$ rooks on SQ_n consisting of all the rooks from P, plus an additional rook on square (i, j). Note we are assuming that there is not a rook from P in column i (otherwise $P \cup_{(\alpha)} (i, j)$ would not be an α-rook placement). More generally for $p \leq n - d$, we will also use $P \cup_{(\alpha)} (i_1, j_1) \cup_{(\alpha)} (i_2, j_2) \cup_{(\alpha)} \cdots \cup_{(\alpha)} (i_p, j_p)$ to denote the obvious α-rook placement of $d + p$ rooks on SQ_n. Let us now prove the following lemma, and then the recurrence.

Lemma 5.2.2. *Let $B = B(b_1, \ldots b_{n-1}, b_n)$ be a Ferrers board, and let $B' = B(b_1, \ldots, b_{n-1})$. Let P' be a fixed α-rook placement of $k - 1$ rooks on B'. Then*

$$\sum_{P \in \mathcal{R}_k^{(\alpha)}(B),\ P \cap B' = P'} wt(P) = ((k-1)\alpha + b_n - k + 1) wt(P').$$

Proof. Let us begin by listing the number of rooks from P' in each row of B. Suppose there are ℓ_1 rooks in row j_1, ℓ_2 rooks in row j_2, … , ℓ_m rooks in row j_m (where each $\ell_p > 0$). Then

$$\ell_1 + \ell_2 + \cdots + \ell_m = k - 1, \tag{5.2}$$

and there must be $b_n - m$ rows in column n of B with no rooks. Note that

$$\sum_{P \in \mathcal{R}_k^{(\alpha)}(B),\ P \cap B' = P'} wt(P) = \sum_{j=1}^{b_n} wt(P' \cup_{(\alpha)} (n, j)).$$

Then we can see that

$$wt(P' \cup_{(\alpha)} (n, j_p)) = wt(P') \times (\ell_p \alpha - (\ell_p - 1))$$

for $1 \leq p \leq m$, and

$$wt(P' \cup_{(\alpha)} (n, j)) = wt(P')$$

for $1 \leq j \leq b_n$, when $j \neq j_p$ for $1 \leq p \leq m$. Thus

$$\sum_{j=1}^{b_n} wt(P' \cup_{(\alpha)} (n, j)) = wt(P') \sum_{p=1}^{m} (\ell_p \alpha - (\ell_p - 1)) + wt(P') \sum_{1 \leq j \leq b_n,\ j \neq j_p,\ 1 \leq p \leq m} 1,$$

which is

$$wt(P') \times \left\{ \ell_1 \alpha - (\ell_1 - 1) + \ell_2 \alpha - (\ell_2 - 1) + \cdots + \ell_m \alpha - (\ell_m - 1) + b_n - m \right\} =$$

$$wt(P') \times \left\{ (\ell_1 + \ell_2 + \cdots + \ell_m)\alpha - (\ell_1 + \ell_2 + \cdots + \ell_m) + m + b_n - m \right\}$$

which by (5.2) is

$$wt(P')\{(k-1)\alpha + b_n - k + 1\}$$

as desired.

□

Theorem 5.2.3. *For $B = B(b_1, \ldots b_{n-1}, b_n)$ and $B' = B(b_1, \ldots, b_{n-1})$ Ferrers boards,*

$$r_k^{(\alpha)}(B) = r_k^{(\alpha)}(B') + ((k-1)\alpha + (b_n - k + 1))r_{k-1}^{(\alpha)}(B')$$

for all $k \in \mathbb{N}$.

Proof. We break up $\mathcal{R}_k^{(\alpha)}(B)$ into two subsets, depending upon whether or not there is a rook in column n of B. The placements from $\mathcal{R}_k^{(\alpha)}(B)$ with no rook in column n of B are in bijection with the placements in $\mathcal{R}_k^{(\alpha)}(B')$, and the corresponding placement has the same weight. Thus these placements contribute $r_k^{(\alpha)}(B')$ to $r_k^{(\alpha)}(B)$. A complete listing of the placements from $\mathcal{R}_k^{(\alpha)}(B)$ with a rook in column n can be built from $\mathcal{R}_{k-1}^{(\alpha)}(B')$ by placing a kth rook in each possible position in column n of B. For a fixed placement $P' \in \mathcal{R}_{k-1}^{(\alpha)}(B')$, we have that

$$\sum_{P \in \mathcal{R}_k^{(\alpha)}(B),\ P \cap B' = P'} wt(P) = ((k-1)\alpha + b_n - k + 1)wt(P')$$

by Lemma 5.2.2. Thus the sum of the weights of all placements in $\mathcal{R}_k^{(\alpha)}(B)$ with a rook in column n of B is equal to

$$((k-1)\alpha + b_n - k + 1) \sum_{P' \in \mathcal{R}_{k-1}^{(\alpha)}(B')} wt(P'),$$

which is $((k-1)\alpha + b_n - k + 1)r_{k-1}^{(\alpha)}(B')$ and we obtain the second term in the recurrence.

□

The $r_k^{(\alpha)}(B)$ and $h_{n,k}^{(\alpha)}(B)$ for any board B are related by a version of (1.2). To prove it, we need the following lemmas.

Lemma 5.2.4. *Suppose P is an α-rook placement of k rooks on SQ_n, and there are no rooks from P in column i of SQ_n. Then*

$$\sum_{j=1}^{n} wt(P \cup_{(\alpha)} (i,j)) = (k\alpha + (n-k))wt(P).$$

Proof. This proof is very similar to the proof of Lemma 5.2.2. Suppose in P there are ℓ_1 rooks in row j_1, ℓ_2 rooks in row j_2, …, ℓ_m rooks in row j_m (where each $\ell_p > 0$). Then

$$\ell_1 + \ell_2 + \cdots + \ell_m = k, \tag{5.3}$$

and there must be $n - m$ rows with no rooks. We see that

$$wt(P \cup_{(\alpha)} (i, j_p)) = wt(P)(\ell_p\alpha - (\ell_p - 1))$$

for $1 \leq p \leq m$, and

$$wt(P \cup_{(\alpha)} (i, j)) = wt(P)$$

for $1 \leq j \leq n$, when $j \neq j_p$ for $1 \leq p \leq m$, so

$$\sum_{j=1}^{n} wt(P \cup_{(\alpha)} (i, j)) =$$

$$wt(P) \times \left\{\ell_1\alpha - (\ell_1 - 1) + \ell_2\alpha - (\ell_2 - 1) + \cdots + \ell_m\alpha - (\ell_m - 1) + n - m\right\} =$$

$$wt(P) \times \left\{(\ell_1 + \ell_2 + \cdots + \ell_m)\alpha - (\ell_1 + \ell_2 + \cdots + \ell_m) - m + n - m\right\} =$$

$$wt(P)(k\alpha + (n - k))$$

by (5.3) as desired.

□

By repeatedly applying the argument of Lemma 5.2.4 we obtain the following.

Lemma 5.2.5. *Suppose P' is an α-rook placement of k rooks on SQ_n. Then*

$$\sum_{P \in \mathcal{R}_n^{(\alpha)}(SQ_n),\ P' \subseteq P} wt(P) =$$

$$(k\alpha + (n - k))((k + 1)\alpha + (n - k - 1)) \cdots ((n - 1)\alpha + 1)wt(P').$$

Theorem 5.2.6. *For B any board,*

$$\sum_{k=0}^{n} h_{n,k}^{(\alpha)}(B)z^k =$$

$$\sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B)((n - k)\alpha + k)((n - k + 1)\alpha + k - 1) \ldots ((n - 1)\alpha + 1)(z - 1)^{n-k}.$$

Proof. Let $z = z + 1$ in the above equation, giving

$$\sum_{k=0}^{n} h_{n,k}^{(\alpha)}(B)(z+1)^k =$$

$$\sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B)((n-k)\alpha + k)((n-k+1)\alpha + k - 1)\ldots((n-1)\alpha+1)z^{n-k}. \quad (5.4)$$

Then the coefficient of z^k on the left side of (5.4) is

$$\sum_{j=k}^{n} \binom{j}{k} h_{n,j}^{(\alpha)}(B), \quad (5.5)$$

and the coefficient of z^k on the right side of (5.4) is

$$r_k^{(\alpha)}(B)(k\alpha + (n-k))((k+1)\alpha + (n-k-1))\cdots((n-1)\alpha+1). \quad (5.6)$$

Our goal is to show that (5.5) and (5.6) represent different ways of organizing the terms in the same weighted count. For a fixed k with $0 \leq k \leq n$, consider the expression

$$\sum_{(P,\pi),\ \pi\subseteq P\cap B,\ |\pi|=k} wt(P). \quad (5.7)$$

Here P is any α-rook placement of n rooks on SQ_n, and π is a subset of k rooks from P which are on B.

It is easy to see that (5.5) is the same as (5.7). To obtain a summand of (5.7), first choose $j \geq k$, then choose an α-rook placement $P \in \mathcal{H}_{n,j}^{(\alpha)}(B)$, and finally choose a k element subset of $P \cap B$. The weight of each such P will appear in (5.7) $\binom{j}{k}$ times, once for each possible $\pi \subseteq P$. The sum of weights over all such P is exactly $h_{n,j}^{(\alpha)}(B)$, hence

$$\sum_{(P,\pi),\ P\in\mathcal{H}_{n,j}^{(\alpha)}(B),\ \pi\subseteq P\cap B,\ |\pi|=k} wt(P) = \binom{j}{k} h_{n,j}^{(\alpha)}(B). \quad (5.8)$$

Finally, when summing over all $j \geq k$ in (5.8), the left side is exactly (5.7) and the right side is exactly (5.5).

To obtain a summand of (5.6), first choose a subset π of k rooks on B and extend to an α-rook placement P of n rooks on SQ_n. Hence (5.7) is also equal to

$$\sum_{\pi\in\mathcal{R}_k^{(\alpha)}(B)} \left\{ \sum_{P\in\mathcal{R}_n^{(\alpha)}(SQ_n),\ \pi\subseteq P} wt(P) \right\},$$

which is the same as

$$(k\alpha + (n-k))((k+1)\alpha + (n-k-1)) \cdots ((n-1)\alpha + 1) \sum_{\pi \in \mathcal{R}_k^{(\alpha)}(B)} wt(\pi) \quad (5.9)$$

by Lemma 5.2.5. Finally, (5.9) is equal to

$$(k\alpha + (n-k))((k+1)\alpha + (n-k-1)) \cdots ((n-1)\alpha + 1) r_k^{(\alpha)}(B)$$

by definition. □

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