Excess Intersection in Enumerative Geometry

Presenter: Zhong Zhang
Mentor: Thomas Brazelton

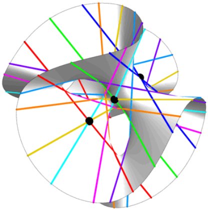
DRP, April 2022

What do we study?

We are interested in counting intersections of algebraic structures, when the expected answer is finite.

Examples

How many lines are there on a cubic surface?



General set up of problems

Definition

An n-dimensional complex projective space is $\mathbb{C}^{n+1} \setminus \{0\}/\sim$ where $[x_0, x_1, \ldots, x_n] \sim [\lambda x_0, \lambda x_1, \ldots, \lambda x_n]$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition

A homogeneous degree-n polynomial is a polynomial with only degree-n terms.

Bezout's Theorem

Theorem

In $\mathbb{CP}^n$, if X_i is the zero locus of a degree-d_i homogeneous polynomial for $1 \leq i \leq n$. If $\cap_i X_i$ is finite, then

$$\text{number of intersections} = d_1 \ldots d_n$$

Excess Intersection

In fact, it turns out that we might have over-counted.

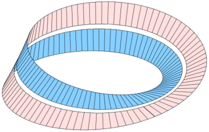
Complex Vector Bundle

Definition

A n-dim complex vector bundle $E \to X$ is an assignment of n-dim complex vector space to every $x \in X$, in a continuous way.

Definition

A section is a function from $X \to E$, which associates a vector in the fiber to every point in X, in a continuous way.



Remark

If we let the section be homogeneous polynomials that define our algebraic structures, then the intersection points exactly correspond to the zero loci of the section.

Indices

Suppose $Z(s) = \sqcup_i Z_i$. Then,

$$e(X) = \sum_i index_{Z_i}(s)$$

Poincaré Hopf Theorem

Theorem

Suppose $E \to X$ is a rank n oriented vector bundle. Suppose s is a section such that $Z(s)$ is a collection of isolated points, then

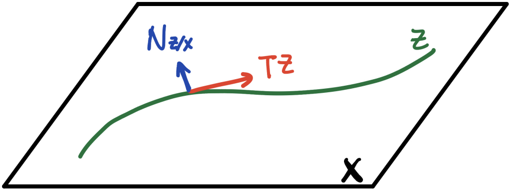
$$e(X) = \sum_{p \in Z(s)} deg_p(s)$$

Normal Bundle and Excess Bundle

Definition

Normal bundle is defined to be tangent directions in X that don't come from tangent spaces in Z.

$$0 \to TZ \to i^* TX \to N_{Z/X} \to 0$$



Definition

Excess bundle is defined as the quotient $E|_Z \big/ N_{Z/X}$.

Chern Class and Chern Polynomial

Definition

The Chern classes are characteristic classes that encode information about complex vector bundles. The Chern polynomial of a rank-n bundle packages information of chern classes into a polynomial
$c_H(E) = 1 + c_1(E)H + \cdots + c_n(E)H^n$

Facts

$c(A \oplus B) = c(A)c(B)$
$c(T\mathbb{P}^n) = (1 + H)^{n+1}$
$c(\mathcal{O}_{\mathbb{P}^2}(n)) = 1 + nH$
$c_i(\text{rank-}n \text{ bundle}) = 0$ for $i > n$

We can define the index as the k-th chern class of the excess bundle, where k is the dimension of Z.

Example: Two Plane Conics Containing the Same Component

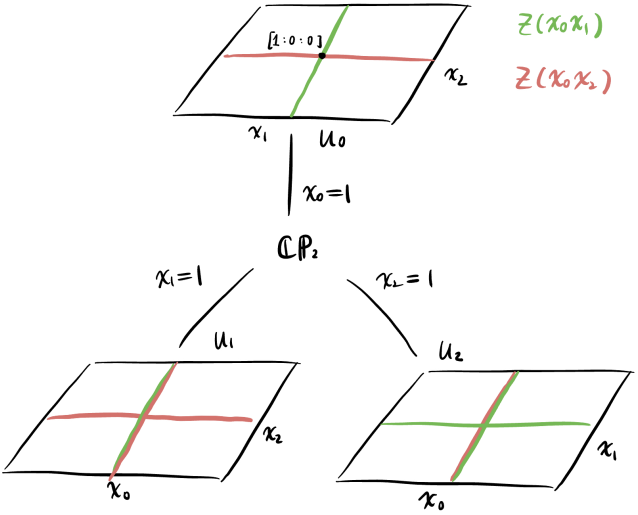
Suppose we want to intersect two plane conics (zero loci of degree-2 polynomials in $\mathbb{CP}^2$) that contain the same component.
Let $C_1 = Z(x_0x_1), C_2 = Z(x_0x_2)$.
Then, set theoretically,

$$C_1 \cap C_2 = [1 : 0 : 0] \cup \{[0 : a : b] \in \mathbb{CP}^2\}$$

Take the section from X to E to be $s = (x_0x_1, x_0x_2)$

$$Z(x_0x_1, x_0x_2) = [1 : 0 : 0] \cup \mathbb{P}^1$$

Charts



Intersection from the point

The point $[1 : 0 : 0]$ is a transverse intersection, so it contributes 1 to the answer.

$$\mathit{index}_{[1,0,0]}(s) = 1$$

Intersections from the line

The Chern polynomial of the normal bundle:

$$c(N_{\mathbb{P}^1/\mathbb{P}^2}) = \frac{c(T\mathbb{P}^2)}{c(T\mathbb{P}^1)} = \frac{(1+H)^3}{(1+H)^2} = 1 + H$$

The Chern polynomial of E:

$$c(E) = c(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2)) = c(\mathcal{O}_{\mathbb{P}^2}(2))^2 = (1+2H)^2 = 1 + 4H + 4H^2$$

Then, the Chern polynomial of the excess bundle:

$$c(F) = \frac{c(E)}{c(N_{Z/X})} = (1 + 4H + 4H^2)(1 - H) = 1 + 3H$$

$\implies c_1(F) = index_Z(s) = 3$

Reference

- Sheldon Katz, Enumerative Geometry and String Theory