Geodesics

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Directed reading project

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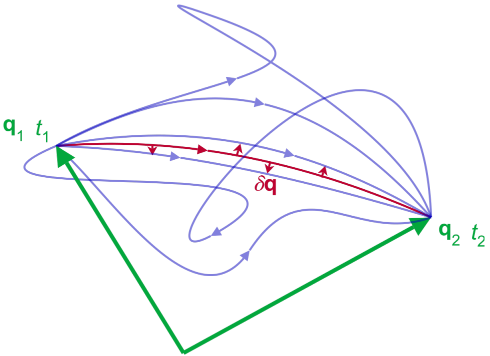
Least action principle

Particles move in the path that results in the least "effort"



How to measure action

We need some sort of "energy function" to help measure the amount of "action" of a certain path.
This is accomplished by the **Lagrangian function**, denoted by L, such that S defined by

$$S[x^i(t)] = \int_{t_1}^{t_2} L(x^i(t), \dot{x}^i(t))dt$$

is a functional that calculates the action.
One example of a Lagrangian would be the kinetic energy function in three dimensions:

$$L(x^i(t), \dot{x}^i(t)) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Calculus of Variations

We are interested in finding local minimums of the functional S. Given some function/path $x^i(t)$ that starts at t_1 and ends at t_2, we can shift it infinitesimally. Consider some $\eta(t)$ such that

$$\eta(t_1) = \eta(t_2) = 0$$

Scale η by some small ϵ to get $\epsilon\eta(t)$, and then use this to variate $x^i(t)$ slightly:

$$x^i(t) + \epsilon\eta(t)$$

Local minimum

Suppose $x^i(t)$ is some local minimum of S, then

$$F(\epsilon) = S[x^i(t) + \epsilon\eta(t)]$$

has local minimum at $\epsilon = 0$.
So

$$\left.\frac{dF}{d\epsilon}\right|_0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial \epsilon} dt = 0$$

Denote $y = x^i(t) + \epsilon\eta(t)$. Use chain rule to get

$$0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \epsilon} dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial y}\eta(t) + \frac{\partial L}{\partial \dot{y}}\eta'(t) dt$$

Some more derivation

Since we evaluate the derivative at $\epsilon = 0$, $y = x^i$

$$0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial y}\eta(t) + \frac{\partial L}{\partial \dot{y}}\dot{\eta}(t)dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial x^i}\eta(t) + \frac{\partial L}{\partial \dot{x}^i}\dot{\eta}(t)dt$$

Split the integral (use integration by parts to go from (1) to (2))

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial x^i}\eta(t)dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}^i}\dot{\eta}(t)dt \quad (1)$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial x^i}\eta(t)dt + \frac{\partial L}{\partial \dot{x}^i}\eta(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta(t)\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}dt \quad (2)$$

$$= \int_{t_1}^{t_2} \eta(t)(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i})dt = 0 \quad (3)$$

The Euler-Lagrange Equation

By the "the Fundamental Lemma of Calculus of Variations", we conclude that

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$$

This is called the Euler-Lagrange equation.

Newton's first law

If we consider the Lagrangian given by the kinetic energy,

$$L(x^i(t), x^{i\prime}(t)) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

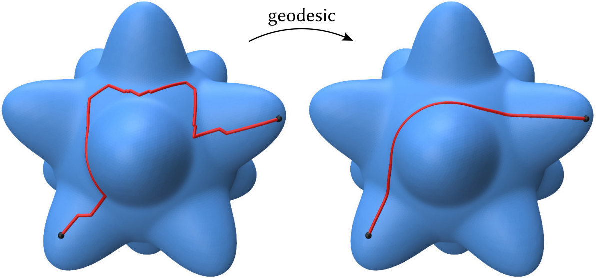
and plug it into the Euler-Lagrange equation, we get (through some chain rule) that

$$0 - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = -m\ddot{x}^i = 0$$

This is saying that particles would move in constant velocity in a straight line.

Geodesics

Intuitively, the geodesic is the notion of a straight line in a curved surface.
"The locally shortest path from point A to B".



https://geometry-central.net/surface/algorithms/flip$_g$*eodesics*/

Curved Space

Spaces can be curved, and in this case, the geodesics might not be straight lines.
Describe curved space by defining local distance via **metric tensor** (basically a bilinear form).
The metric tensor can be expressed as a symmetric matrix (g_{ij}).
Given some point x^i and some infinitesimal dx^i,

$$ds^2 = g_{ij}(x)dx^i dx^j$$

Calculates the distance between x^i and $x^i + dx^i$

The Geodesic Equation

Generalizing the kinetic energy Lagrangian to curved space:

$$L = \frac{m}{2} g_{ij}(x^i)\dot{x}^i \dot{x}^j$$

Can then plug this into the Euler-Lagrange Equation to get the geodesic equation:

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j \dot{x}^k = 0$$

Where

$$\Gamma^i_{jk}(x) = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

is called the Christoffel symbols.

Example

For a 2-Sphere of radius one, we have the metric

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & sin^2\theta \end{pmatrix}$$

Using Euler-Lagrange Equation, we can get that the geodesics on the 2-Sphere are the great circles.

References

David Tong: Lectures on General Relativity:
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