Stochastic Models in Queuing Theory with Single and Multiple Servers

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Single server queuing model (M/M/1)

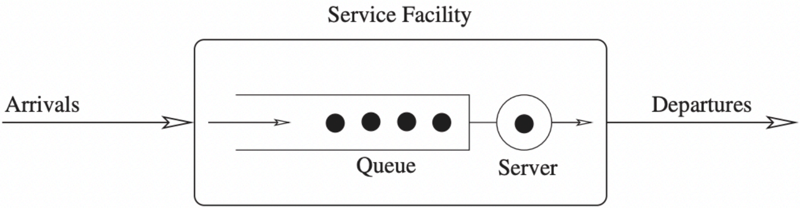


Figure 1: Queuing model with single server (Stewart 2009)

Markov chain representation (M/M/1)

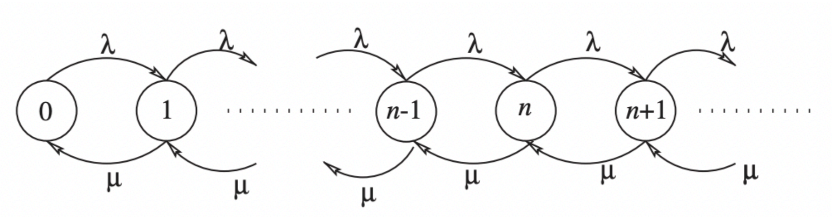


Figure 2: Continuous-time Markov chain with one server and infinite states (Stewart 2009)

M/M/1 Steady State

If the steady state exists, then $\frac{dp_n(t)}{dt} = 0$, where $p_n(t)$ denotes the probability of being in state n at time t.

M/M/1 Steady State, cont.

If the steady state exists, then $\frac{dp_n(t)}{dt} = 0$, where $p_n(t)$ denotes the probability of being in state n at time t.

Recall that the rate of transition from state $n-1$ to n is λ, and the rate of transition from state $n+1$ to n is μ.

M/M/1 Steady State, cont.

If the steady state exists, then $\frac{dp_n(t)}{dt} = 0$, where $p_n(t)$ denotes the probability of being in state n at time t.

Recall that the rate of transition from state $n - 1$ to n is λ, and the rate of transition from state $n + 1$ to n is μ.

Assume that the steady state exists. Thus we can write:

$$0 = -\lambda p_0 + \mu p_1 \rightarrow \mu p_1 = \lambda p_0$$

M/M/1 Steady State, cont.

If the steady state exists, then $\frac{dp_n(t)}{dt} = 0$, where $p_n(t)$ denotes the probability of being in state *n* at time *t*.

Recall that the rate of transition from state $n-1$ to n is λ, and the rate of transition from state $n+1$ to n is μ.

Assume that the steady state exists. Thus we can write:

$$0 = -\lambda p_0 + \mu p_1 \rightarrow \mu p_1 = \lambda p_0$$
$$0 = -(\lambda+\mu)p_n + \mu p_{n+1} + \lambda p_{n-1} \rightarrow (\lambda+\mu)p_1 = \lambda p_0 + \mu p_2$$
$$(\lambda+\mu)p_1 = \mu p_1 + \mu p_2$$

M/M/1 Steady State, cont.

If the steady state exists, then $\frac{dp_n(t)}{dt} = 0$, where $p_n(t)$ denotes the probability of being in state n at time t.

Recall that the rate of transition from state $n-1$ to n is λ, and the rate of transition from state $n+1$ to n is μ.

Assume that the steady state exists. Thus we can write:

$$\begin{aligned}
\mu p_1 &= \lambda p_0 \\
(\lambda + \mu) p_1 &= \lambda p_0 + \mu p_2 \\
(\lambda + \mu) p_1 &= \mu p_1 + \mu p_2 \\
p_2 &= \frac{\lambda}{\mu} p_1 \\
p_2 &= \left(\frac{\lambda}{\mu}\right)^2 p_0
\end{aligned}$$

M/M/1 Steady State, cont.

If the steady state exists, then $\frac{dp_n(t)}{dt} = 0$, where $p_n(t)$ denotes the probability of being in state *n* at time *t*.

Recall that the rate of transition from state $n-1$ to n is λ, and the rate of transition from state $n+1$ to n is μ.

Assume that the steady state exists. Thus we can write:

$$p_2 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

By induction, the steady state solution is:

$$p_i = \left(\frac{\lambda}{\mu}\right)^i p_0$$

Multi server queuing model (M/M/C)

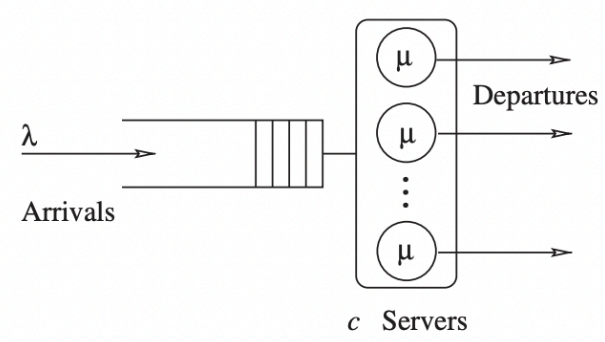


Figure 3: Queuing model with *c* servers (Stewart 2009)

Markov chain representation (M/M/C)

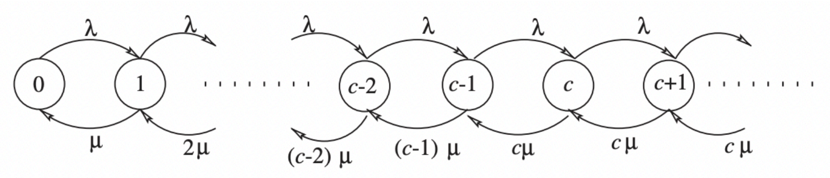
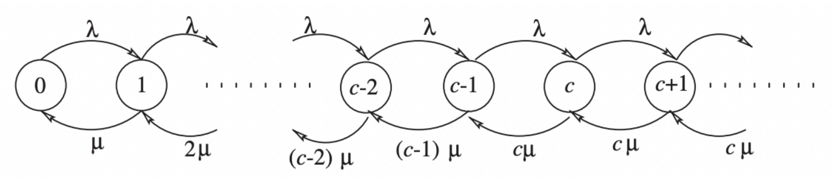


Figure 4: Continuous-time Markov chain with *c* servers and infinite states (Stewart 2009)

Markov chain representation (M/M/C)



Define the transition rates λ and μ:

$$\lambda_n = \lambda, \ \forall n$$
$$\mu_n = n\mu, 1 \leq n \leq c$$
$$\mu_n = c\mu, n \geq c$$

M/M/C Steady State

Define the transition rates λ and μ:

$$\lambda_n = \lambda, \; \forall n$$
$$\mu_n = n\mu, 1 \leq n \leq c$$
$$\mu_n = c\mu, n \leq c$$

Recall that for the M/M/1 queue, the steady state solution was:

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda_{n-1}}{\mu_i}$$

M/M/C Steady State, cont.

Recall that for the M/M/1 queue, the steady solution was:

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda_{n-1}}{\mu_i}$$

For $1 \leq n \leq c$, where $\lambda_n = \lambda$ and $\mu_n = n\mu$:

$$\begin{aligned} p_n &= p_0 \prod_{i=1}^{n} \frac{\lambda}{i\mu} \\ &= p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \end{aligned}$$

M/M/C Steady State, cont.

Recall that for the M/M/1 queue, the steady solution was:

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda_{n-1}}{\mu_i}$$

For $n \geq c$, where $\lambda_n = \lambda$ and $\mu_n = c\mu$:

$$\begin{aligned} p_n &= p_0 \prod_{i=1}^{c} \frac{\lambda}{i\mu} \prod_{i=c+1}^{n} \frac{\lambda}{c\mu} \\ &= p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!} \left(\frac{1}{c}\right)^{n-c} \end{aligned}$$

M/M/C Steady State, cont.

Steady state equations for multiple servers:

$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}, 1 \leq n \leq c$$
$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!} \left(\frac{1}{c}\right)^{n-c}, n \geq c$$

M/M/C Steady State, cont.

Steady state equations derived for multiple servers:

$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}, 1 \leq n \leq c$$

$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!} \left(\frac{1}{c}\right)^{n-c}, n \geq c$$

Let $\rho = \frac{\lambda}{c\mu}$:

$$p_n = p_0 \frac{(c\rho)^n}{n!}, 1 \leq n \leq c$$

$$p_n = p_0 \frac{(c\rho)^n}{c^{n-c} c!} = p_0 \frac{\rho^n c^c}{c!}, n \geq c$$

Applications in traffic flow modeling

- Model roadway capacity under stochastic conditions
- Quantify the effect of bus arrivals / departures on delays and wait time (Gu et al., 2015)
- Evaluate delays of bus stop configurations and prioritize optimal stopping mechanisms (Wang et al. 2018)
- Leads to more efficient and resilient transit infrastructure

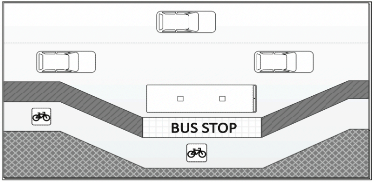


Figure 5: Single-berth bus stop (Wang et al. 2018)

References

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Thank you!

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